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The harmonic driven pendulum as a teaching example of integration of laboratory and computation

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Abstract. We describe an activity developed as laboratory at home during the pandemic period by which it is possible to investigate numerically and experimentally a harmonically driven pendulum with a variable frequency. We realized a prototype using LEGO® bricks and EV3 motor. Because the pendulum undergoes large amplitude oscillations, the motion has been studied by implementing the non-linear numerical model in Python. Students can acquire the experimental data with the software of video analysis and compare them with the numerical model. This approach allows to integrate, at varying levels of complexity, computational and experimental skills into the laboratory practice. The activity also allows a connection with the recent topic of gravitational waves detectors that use pendulums in cascade, as mechanical filters to attenuate the seismic noise.

1. Introduction

The harmonically driven pendulum (in the small angle approximation) is a well-known topic in undergraduate physics courses (see, for example, [1, 2]) that can also be examined in high school context. At the same time, when the pendulum cannot be treated anymore as a linear harmonic oscillator by using the approximation $\sin\theta \approx \theta$, the system can exhibit a non-linear dynamic and has been extensively studied as a model for chaotic motion [3].

Proposing it in a laboratory allows students to learn and experiment with important concepts such as resonance and observe that for frequencies higher than the resonant frequency the pendulum behaves like a mechanical low-pass filter. This feature is one of the basic ideas for attenuating seismic noise in gravitational wave detectors [4,5]. This allows to connect a standard physics topic to the most recent research and, if presented as a question at the beginning of the exploration, it is a “need to know question” that motivates and engages students to understand and experiment [6]. The activity could also be used as a hands-on proposal in the educational field on gravitational waves. Nobel laureate Prof. Rainer Weiss used a simple hand-moved pendulum as a demonstration of this principle at the conference announcing the discovery of gravitational waves, at the National Science Foundation on 11 February 2016 [7]. Of course, the Super Attenuator used in Virgo is much more complex: it consists of six cascaded pendulums and other passive and active mechanisms [8].

Because the pendulum, approaching the resonance frequency, undergoes large amplitude oscillations, its motion has been studied by implementing and solving the non-linear numerical model in Python. During the pandemic period, where it was possible to work only at home, we realized a prototype using LEGO® bricks and LEGO® MINDSTORMS EV3 motor. To obtain a harmonic motion we used the available LEGO® gears. After checking that the motion of the suspension point is harmonic, we acquired the experimental data at various speeds of the motor with the Tracker video analysis software [9] and compared them with the numerical model. In this way we can answer the main research question of this work, that is: "Is it possible to build a harmonic driven pendulum realized through cheap material like LEGO bricks and able to be used for a quantitative approach to the phenomena?"

We have analysed the three important cases: low angular velocity, near resonance, high angular velocity. Acquiring data also for other angular frequency values we have built a resonance curve to be compared to the numerical one. The three cases are useful in explaining the behaviour of the pendulum as a low-pass mechanical filter. At angular frequencies high relative to the resonant frequency, it is observed that the bob has a very small amplitude of oscillation relative to that of the suspension point. In the linear approximation regime, for underdamped oscillator, the solution is a combination of a transient term and a stationary term at the driving frequency, with the transient term decaying and not relevant for the long-time behaviour. Because our system is very weakly damped (in the air), it would take many minutes to get the transient term cancelled [1].

There is growing evidence that activity focusing on skills can be more effective than lab activity focusing on reinforcing concepts [10]. The activity we propose here can be positioned within this line of research, allowing to integrate, at varying levels of complexity, computational and experimental skills (with the use of video analysis [11]) into the laboratory practice. The introduction of computation elements can in fact help students to learn physics in a more creative and exploratory way [12,13] and it is also a skill highlighted in the AAPT Lab Recommendations and AAPT Conference Report (2019) [14,15]. The PICUP group is working also on proposals for high school and introductory physics [16]. Finally, the activity can be proposed as part of lessons on oscillations and creates a link with the most recent research in physics.

2. Classical derivation and numerical model

We briefly derive the equation of motion of the simple pendulum subject to a harmonic suspension point force. We will derive it in the Newtonian framework, alternatively one could start from the Lagrangian of the system.

If the pivot point oscillates with equation: $x_P = x_0 \cos \omega t$, with simple steps (see Annex), after simplifying the mass term to both members, we get:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{x_0\omega^2}{L}\cos\theta\cos\omega t \qquad (1)$$

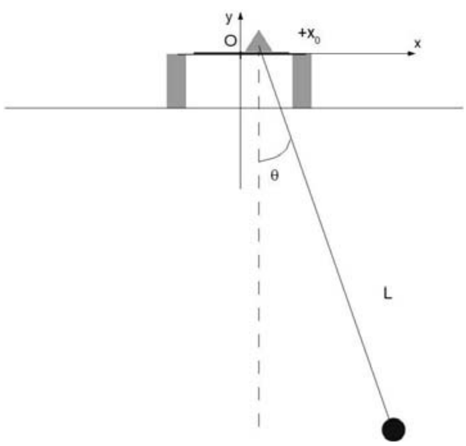

Figure 1. Scheme of experimental apparatus.

In the presence of viscous damping, with a viscous force proportional to the bob's speed, i.e. $f = -bv$, in the equation appears also the term $2\beta\dot{\theta}$, with $\beta = \frac{b}{2m}$. This is a non-linear second order differential equation in the variable θ that it is possible to solve by standard numerical methods (We did use the odeint method of the Python Scipy library, for example by writing it as a system of two first-order equations:

$$\begin{cases} y = \dot{\theta} \\ \dot{y} = -2\beta\dot{\theta} - \dfrac{g}{L}\sin\theta + \dfrac{x_0\omega^2}{L}\cos\theta\cos\omega t \end{cases} \quad (2)$$

We have, as a check, compared the numerical solution with the exact solution in the linear case for small oscillations. As is well known (for example in [1], [2]), in the linear case, i.e., a harmonic oscillator with a sinusoidal driving force, the differential equation is:

$$\ddot{\theta} + 2\beta\dot{\theta} + {\omega_0}^2\theta = f_0\cos\omega t \quad (3)$$

The well-known solution in the "under damped" case is:

$$\theta(t) = A(\omega)\cos(\omega t - \delta) + Be^{-\beta t}\cos(\omega_1 t + \alpha) \quad (4)$$

where ω is the driving frequency of the motor, ω_0 is the natural frequency of the oscillator, and ω_1 is the lower frequency due to the viscous force, given by: $\omega_1 = \sqrt{{\omega_0}^2 - \beta^2}$. The factor A is given by:

$$A(\omega) = \frac{f_0}{\sqrt{({\omega_0}^2-\omega^2)^2+4\beta^2\omega^2}} \quad (5)$$

Equation (1) of the pendulum with sinusoidal driving term in the case of small oscillations can be traced to (3) with ${\omega_0}^2 \to \frac{g}{L}$, $f_0 \to \frac{x_0\omega^2}{L}$ so having:

$$A_1(\omega) = \frac{x_0\omega^2}{L\sqrt{({\omega_0}^2-\omega^2)^2+4\beta^2\omega^2}} \quad (6)$$

Equation (4) is characterized by a stationary term that depends on driving frequency ω and a transient term that depends on the viscous friction coefficient $e^{-\beta t}$. After a certain number of drive cycles, the motion is indistinguishable from cosine term, oscillating at exactly the driving frequency. Because β in air is very small the transient term can persist for several minutes.

As a first step, we have realized an interactive 3D simulation of the system in VPython using Glowscript or Trinket (for example see Rhett Allain's many simulations with VPython proposed on Wired [17]). Subsequently, for more accurate results we have implemented the model in a Jupiter Notebook in Python. Jupiter Notebooks are programming environments that allow to create hybrid texts with written parts, live code and images [18]. It easily allows the study of solutions also for high school students that don't have all the skills. By implementing the notebook on Google Colab (available with Google account) students also don't need to install anything and can work collaboratively together.

3. Experimental setup and Measurements

From the experimental point of view, we have studied the system by building a device with LEGO® bricks integrated with gears, a MINDSTORMS® EV3 servo motor (the apparatus could be even cheaper by using an Arduino-driven motor that can be connected with LEGO®axles and wheels), and an EV3 Intelligent Brick to adjust the motor speed by setting the power percentage (Figure 2). The equipment has a low cost, and its realization allows students to get a brief introduction to basic robotics and to think how to realize a mechanical system with gears to produce a horizontal harmonic motion. The resulting motion has been filmed with a smartphone photo camera and then analysed with the Tracker video analysis tool. To adjust the frequency ω of the suspension point, a two-wheeled mechanism has been built; in addition, some work and inventiveness is also required to find solutions to extend the harmonic motion of the suspension point as much as possible. The maximum amplitude we were able to achieve with our device, ensuring at the same time, a smooth movement is x_0=0.022 m.

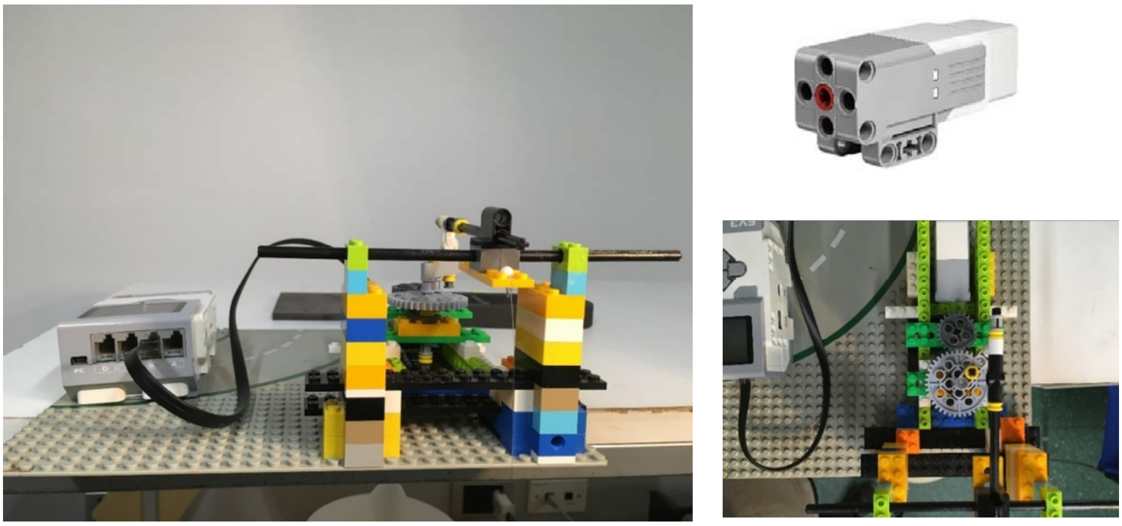

Figure 2. From left to right, top to bottom: Experimental apparatus for generating harmonic motion with *LEGO®* Mindstorms components; EV3 motor; top view of apparatus.

It is possible to film the whole system composed by the suspension point and the pendulum, at a couple of meters, or to film the motion of the pendulum bob in a closer position (from above, see Figure 3) thus improving the accuracy of tracking (the majority of videos were filmed in slow motion at 240 fps) and in a second time to film the motion of the suspension point (from which we estimated the driving frequency). From the tracking of the video of simple pendulum (no driving force) in the air it is possible to get an estimation of experimental ω_0, $\omega_{exp} = 3.056\ \mathrm{rad/s}$ (from theory, $\omega_0 = \sqrt{\frac{g}{L}} = 3.06\ \mathrm{rad/s}$). It is possible to vary the frequency of the LEGO® Mindstorms motor via the app on the computer. In the following paragraphs we explore in more detail the following three cases, the most important in our opinion from an educational point of view: low angular frequency (with respect to ω_0), near resonance, high angular frequency.

3.1 Example 1 - Lowest driving frequency

The lowest driving angular frequency attainable with a smooth movement of our device is ω= 0.89 rad/s as estimated through Figure 3, which shows the *x*-displacement of the suspension point of the pendulum as obtained by Tracker software. With this procedure we can also verify the harmonic motion of the suspension point. Both visually with naked eye and from the graph shown, we have a good accordance with the theory: the pendulum (bob) follows the suspension point's displacement at the top. The ratio of the amplitudes of the two motions tends to one, while the angle θ to the vertical goes to zero (Figure 3).

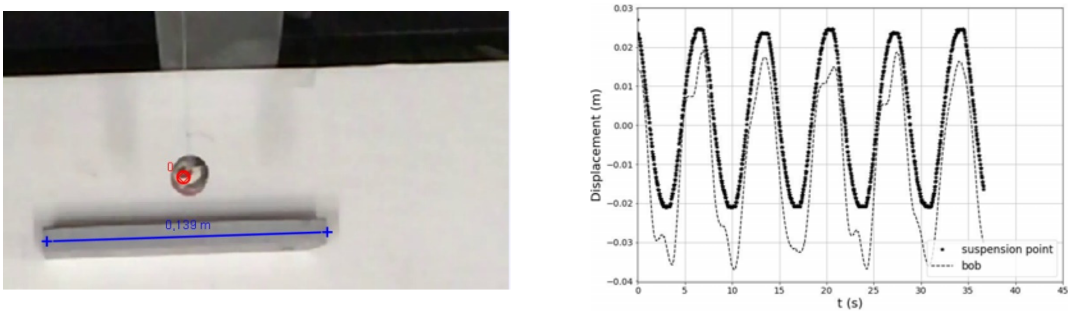

Figure 3. From left to right: Experimental set up to film the bob with smartphone camera located above. θ is measured by the custom variable introduced in Tracker: $\theta = \mathrm{atan}(x/L)$; displacement x of the suspension point and the bob in the case of low frequency (experimental data).

3.2 Example 2 - Near the resonance frequency

When the motor is set to 18% of its power, ω of the device is 2.95 rad/s (obtained by analysing the motion with Tracker - see Figure 4 from which we can verify the harmonic motion of the suspension point). By plotting the motion of the pendulum and comparing it to the numerical model (Figure 4, right), we obtain a good agreement between experimental data and the numerical model. We observed that for values higher than the resonance frequency, the pendulum shows a beating behaviour, i.e., the amplitude periodically increases to then become null.

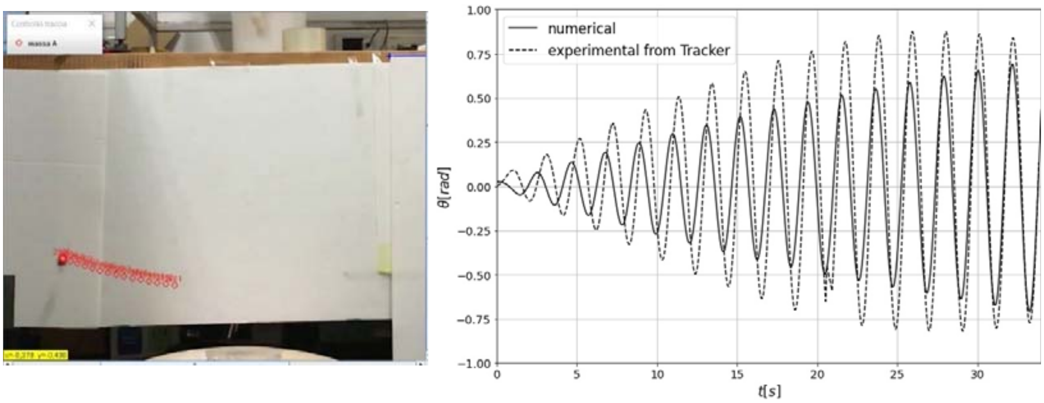

Figure 4. From left to right: Excerpt of pendulum motion with Tracker. Numerical solution and experimental data for ω= 2.95 rad/s.

3.3 Example 3 - high driving frequency

By setting the motor to the maximum frequency (100% on the laptop' app) the driving frequency is ω=11.77 rad/s. The motion of the pendulum is not characterized only by the driving frequency ω, because the transient term is still present (see the initial section) due to the low viscosity of the air (as further confirmation by applying the Fourier transform to the experimental data we find that a component of the angular frequency ω_0 is present in addition to the driving frequency ω). The displacement of the pendulum is, as expected, of much smaller amplitude than that of the suspension point (Figure 5). The pendulum behaves approximately like a low-pass filter, the crucial feature used to filter seismic noise in gravitational wave detectors.

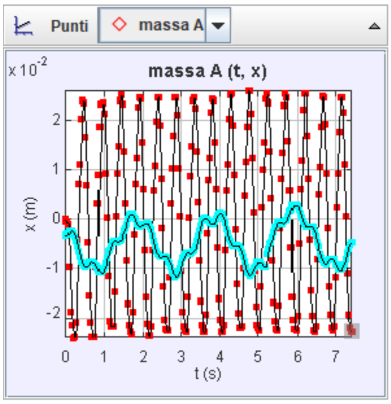

Figure 5. (Tracker screenshot) Displacement x of the suspension point (red dots) and displacement (in light blue) of the bob in the case of maximum reached ω. The bob has a much smaller displacement x.

3.4 Theoretical and experimental resonance curve

Apart from the three cases highlighted above, we filmed the motion as the driving frequency of the suspension point varies and by analysing the data obtained by Tracker with a script in Google Colab, have extracted the values of experimental maximum amplitude θ for each ω, that can be compared with the maximum amplitude from the numerical model when ω varies in small steps (Figure 6). We estimated the parameter β equal to $\beta = 0.003$ s^{-1}. In fact, considering that the amplitude of motion of the damped pendulum is decreasing according to $e^{-\beta t}$, it is possible to determine β by performing an exponential fit to the maximum amplitude points as time varies. The maximum amplitude is obtained at a frequency slightly smaller than the theoretical ω_0, as is well known (for example see [2] or this recent article [19]).

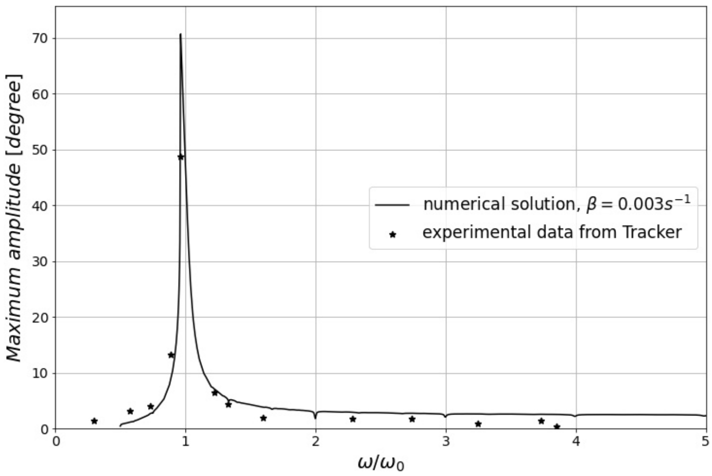

Figure 6. Resonance curve with numerical model (L=1.05 m) and viscous friction and experimental data from Tracker.

In the experimental data there are various possible sources of uncertainty; in addition to the usual ones about length measurement, neglected friction, motor vibration, non-point mass, there are uncertainties due to the video recording with a smartphone camera (e.g. parallax errors) and the tracking done with Tracker. To check the uncertainty, we have repeated the experiment at the same conditions multiple times and did look at the variations in the results of the auto tracking procedure, verifying the consistency of data across different trials.

3.5 *A further exploration*

As we already pointed out, in the motion of pendulum during the time interval of our observations, the effects of the transient part at the ω_0 frequency (we do not distinguish between ω_0 and ω_1) is still visible (equation (3) for the linear approximation).

To make the time to reaching the stationary solution shorter, when the motion of the pendulum is expected to be characterized by the frequency of the driving force, we immersed the pendulum in water (Figure 7).

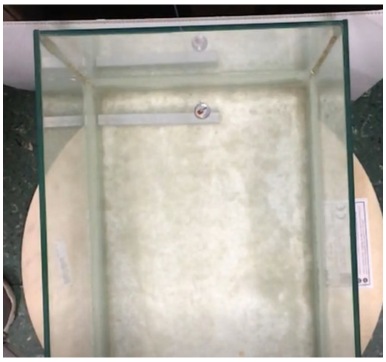

Figure 7. The pendulum immersed in water to study high frequency motion.

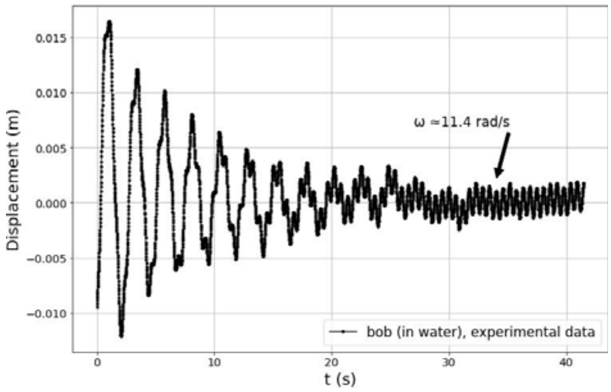

Figure 8. Motion of the pendulum in water with Tracker.

As before, the value of the driving frequency ω can be obtained by tracking the motion of the suspension point: $\omega = 10.8$ rad/s. From experimental pendulum motion data between instants 30 s and 38 s (Figure 8), we get an estimated value of ω equal to 11.4 rad/s. It was interesting to note that it was possible to "hear" qualitatively the bob going at the same frequency as the motor. However, it is beyond the scope of this paper to analyse the dynamics of the bob in the water [20].

The maximum angular frequency of the device is limited by the motor, the structure and assigned features of the LEGO® devices and components.

4. Discussion and Conclusions

In this paper we have presented the design of an activity concerning the harmonic driven pendulum realized with a low-cost device built with LEGO® bricks and we have shown that it is possible to get a good agreement between the experimental data and the numerical model implemented in Python. The designed activity, which can be carried out also as a remote laboratory, brings into play various experimental skills and could be proposed, in all or in parts, both in high school and introductory university laboratory courses. Going into more details, at a basic level, the instructor can provide videos realized with the harmonic driven pendulum LEGO® device to students who must analyse them with video analysis software (e.g. Tracker). It is assumed that the students have already worked with this kind of software, for example, in previous mechanics activities.

Important hints and warnings in this part of the activity are that students must orient a set of axes, correcting so the tilt of the video camera and set a length scale based on the reference showed in the video. Then they must try to use the auto tracking tool of Tracker, that not always works properly, for example when the pendulum is going too fast or not distinct enough from the bottom.

The instructor can also provide the script in Python so that students can calculate the numerical solutions and compare them with the experimental data obtained from their video analysis. At a more advanced level, students can make the videos themselves in the lab, confronting the various difficulties of video analysis (typical hints are: guarantee the visibility of the pendulum in relation to the background, select an appropriate distance, pose a length reference in the video, etc.) or work completely from home building their own device in a creative way; if they have the availability of components, they can realise the mechanism for a harmonic motion of the suspension point (which, as mentioned, requires some attention and time).

In cases where computational elements have already been introduced in the course, the instructor may ask the students to develop the code themselves to numerically solve the model equation, or only provide them with incomplete code. As regards computation, the learning objectives can be, in an increasing difficulty, that students are able to read and execute the code provided by the teacher, applying only simple modifications (setting of parameters, initial conditions, import data files, etc.) to modify and create new code, also passing through the process of debugging. Other learning objectives, at a more advanced level, are that students have to interpret the numerical solutions and compare them with experimental data and, if needed, modify the numerical model.

The activity allows to investigate various characteristics of the driven pendulum not least the influence of viscous friction on the achievement of the stationary solution (in the linear approximation). The activity then lends itself to be used to demonstrate the property of mechanical filtering for seismic noise in gravitational wave detectors.

Supplementary material, with list of main LEGO components, example of a video and a notebook in python are available at the following link: https://github.com/etufino/harmonic-driven-pendulum

5. Acknowledgments

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7. Appendix on deriving the equation of motion (1)

Since we have not found a reference in which the deduction of the equation of motion was explicitly described, in this appendix, for the convenience of the reader, we draw a schematic deduction of (1) from Newton's laws of motion. The system consists in a wire from which an object of mass m is suspended. The suspension point P moves with harmonic motion (see Figure 9). The forces acting on the object are tension force **T** and the weight **W**. We neglect here the viscous friction of the air.

Assuming that: $\overrightarrow{PB} \equiv (L\sin\theta, -L\cos\theta)$, $\overrightarrow{OP} \equiv (x_0 \cos\omega t\,, 0) \equiv (X, 0)$

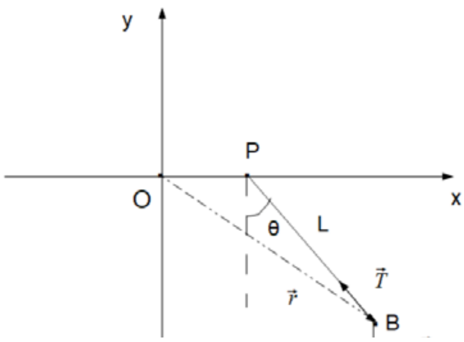

Figure 9. Force diagram for the harmonically driven pendulum.

$$\vec{r} = \overrightarrow{OB} = \overrightarrow{PB} + \overrightarrow{OP} \equiv (X + L\sin\theta, -L\cos\theta)$$

Differentiating once with respect to time we get:

$$\dot{r_x} = \dot{X} + L\cos\theta\,\dot{\theta} \;\; ; \;\; \dot{r_y} = L\sin\theta\,\dot{\theta} \qquad \text{(A.1)}$$

And similarly, we can proceed to calculate $\ddot{r_x}$ and $\ddot{r_y}$:

$$\ddot{r_x} = \ddot{X} - L\sin\theta\,\dot{\theta}^2 + L\cos\theta\,\ddot{\theta} \;\; ; \;\; \ddot{r_y} = L\cos\theta\,\dot{\theta}^2 + L\sin\theta\,\ddot{\theta} \qquad \text{(A.2)}$$

The equations of motion are derived using Newton's second law of motion $\Sigma\vec{F_i} = m\vec{a}$:

$$\begin{cases} m\ddot{r_x} = -T\sin\theta \\ m\ddot{r_y} = T\cos\theta - mg \end{cases} \qquad \text{(A.3)}$$

$$\begin{cases} \ddot{X} - L\sin\theta\dot{\theta}^2 + L\cos\theta\,\ddot{\theta} = -\frac{T}{m}\sin\theta \\ L\cos\theta\dot{\theta}^2 + L\sin\theta\ddot{\theta} + g = \frac{T}{m}\cos\theta \end{cases} \qquad \text{(A.4)}$$

By taking the ratio between the two equations:

$$\tan\theta = \frac{-\ddot{X} + L\sin\theta\dot{\theta}^2 - L\cos\theta\,\ddot{\theta}}{L\cos\theta\dot{\theta}^2 + L\sin\theta\ddot{\theta} + g} \qquad \text{(A.5)}$$

which can be rearranged as follows:

$$g\tan\theta + L\frac{\sin^2\theta}{\cos\theta}\ddot{\theta} + L\dot{\theta}^2\sin\theta = -\ddot{X} + L\sin\theta\;\dot{\theta}^2 - L\cos\theta\ddot{\theta} \qquad \text{(A.6)}$$

After simplifying we get:

$$L\ddot{\theta} + g\sin\theta = -\ddot{X}\cos\theta \qquad \text{(A.7)}$$

Considering that $X = x_0\cos\omega t$, the equation (1) is obtained:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{x_0\omega^2}{L}\cos\theta\cos\omega t \qquad \text{(A.8)}$$

8. References

[1] J H Taylor 2004 *Classical mechanics*, United States, University Science Books
[2] A P French *Vibrations and waves* (New York: Norton & Company)
[3] G Baker and J Gollub 1996 *Chaotic Dynamics, An Introduction*, New York, Cambridge University Press
[4] B P Abbott et al. 2016 Observation of Gravitational Waves from a Binary Black Hole Merger" *Phys. Rev. Lett.* **116**, 061102
[5] A Giazotto, E Campani, D Passuello, and A Stefanini 1986 Performance of an active pendulum with interferometric sensing. In *Proceedings of the fourth Marcel Grossmann meeting on General Relativity, part B* pp 1329–1334
[6] D T Brookes, E Ektina, G Planinsic 2020 Implementing an epistemologically authentic approach to student-centered inquiry learning, *Phys. Rev. Phys. Educ. Res* **16** 020148
[7] Press conference announcing the first direct detection of gravitational waves - online available from https://www.youtube.com/watch?v=aEPIwEJmZyE

[8] M Bassan (Ed.) 2014 *Advanced Interferometers and the Search for Gravitational Waves*, Berlin, Germany, Springer
[9] Tracker Video Analysis and Modeling Tool retrieved online from https://www.compadre.org/osp/items/detail.cfm?ID=7365
[10] B Wilcox and H J Lewandowski 2017 "Developing skills versus reinforcing concepts in physics labs: Insight from a survey of students' beliefs about experimental physics," *Phys. Rev. Phys. Educ. Res.* **13** 010108
[11] M Monteiro et al 2021 Smartphone sensors and video analysis: two allies in the physics laboratory battle field *J. Phys.: Conf. Ser.* **1929** 012038
[12] R Chabay and B Sherwood 2008 Computational physics in the introductory calculus-based course, *Am. J. Phys.* **76** p 307
[13] J Weber and T Wilhelm 2020 The benefit of computational modelling in physics teaching: a historical overview, *Eur. J. Phys.* **41** 034003
[14] AAPT Recommendations for Computational Physics in the Undergraduate Physics Curriculum UCTF report 2016 retrieved online from https://www.aapt.org/resources/upload/aapt_uctf_compphysreport_final_b.pdf
[15] Advancing Interdisciplinary Integration of Computational Thinking in Science 2019 retrieved online from https://www.aapt.org/Resources/upload/Computational_Thinking_Conference_Report_Final_200212.pdf
[16] M D Caballero, M N Chonacky, L Engelhardt et al. 2019 PICUP: A Community of Teachers Integrating Computation into Undergraduate Physics Courses, *Phys. Teach.* **57** p 397
[17] R Allain available from https://www.wired.com/author/rhett-allain/
[18] T O B Odden and A Malthe-Sørenssen 2021 Using computational essays to scaffold professional physics practice, *Eur. J. Phys.* **42** 015701
[19] R Cross 2021 Observations of a driven pendulum at low Amplitudes, *Eur. J. Phys.* **42**
[20] J C Leme and O Oliveira 2017 Pendulum Underwater – An Approach for Quantifying Viscosity, *Phys. Teach.* **55** p 555