UNIVERSITÀ DEGLI STUDI DI TRENTO
Department of Mathematics

Ph.D. in Mathematics
Ciclo XXV

Birational Maps in the Minimal Model Program

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Supervisors: Prof. Marco Andreatta (University of Trento)
Prof. Paolo Cascini (Imperial College London)

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When the shadow of the sash appeared on the curtains it was between seven and eight oclock and then I was in time again, hearing the watch. It was Grandfather's and when Father gave it to me he said I give you the mausoleum of all hope and desire; it's rather excruciating-ly apt that you will use it to gain the reducto absurdum of all human experience which can fit your individual needs no better than it fitted his or his father's. I give it to you not that you may remember time, but that you might forget it now and then for a moment and not spend all your breath trying to conquer it. Because no battle is ever won he said. They are not even fought. The field only reveals to man his own folly, and despair, and victory is an illusion of philosophers and fools.

William Faulkner, The Sound and the Fury.

Dipinte in queste rive
Son dell'umana gente
Le magnifiche sorti e progressive.
Qui mira e qui ti specchia,
Secol superbo e sciocco,
Che il calle insino allora
Dal risorto pensier segnato innanti
Abbandonasti, e volti addietro i passi,
Del ritornar ti vanti,
E proceder il chiami.

Giacomo Leopardi, La ginestra.

To the memory of Professor Giuseppe Vigna Suria, who made me discover the beauty of Math.

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A special acknowledgment goes to my dear friend Marco. He taught me about literature, philosophy and sports. We wasted together the best time of our Ph.D., fighting side by side against productivism, capitalism and the vulgarity of our time (see also the preface of [Piz13]).

Contents

Introduction

The main ambition of Birational Geometry is to classify algebraic varieties up to birational equivalence. Recall that a birational map $f : X \dashrightarrow Y$ is an (algebraic) isomorphism between two open dense subsets of the varieties X and Y.

A first step to reach such a classification is to find, in any birational class, a special element which is simple to study. The intuitive definition of such an element, and the first historical one, is the following. We consider a partial order $\geq$ on the class of varieties, saying that $X \geq Y$ if there exists a birational morphism $f : X \to Y$, and we call X minimal if it is minimal respect to this order, that is, if $X \geq Y$, then $X \cong Y$. It turned out that a better definition is the following: X is a minimal model if K_X is nef, i.e. if K_X intersects non-negatively every curve C on X. It is not difficult to see that if $f : X \to Y$ is a birational morphism between smooth varieties such that K_X is nef, then f is actually an isomorphism.

The simplest birational morphism is the blow-up of a point: if X is a smooth variety of dimension n and $p \in X$ is a point, then the blow-up $f : Y \to X$ replaces p with a divisor $E \cong \mathbb{P}^{n-1} \subset Y$, which is called exceptional divisor, and $f : Y \setminus E \to X \setminus p$ is an isomorphism. Note that any curve C contracted by f, i.e. $C \subset E$, has the property that $K_Y.C < 0$. Therefore, to have a minimal model, we need at least to blow-down exceptional divisors.

The way of finding minimal models goes back to the turn of the nineteenth century, when the Italian school of geometry headed by Castelnuovo, Enriques and Severi, obtained the so called Enriques classification of algebraic surfaces. A (-1)-curve on a smooth surface X is a curve C such that $C \cong \mathbb{P}^1$ and $C^2 = -1$. Castelnuovo's contraction theorem assures us that any (-1)-curve on X is the exceptional divisor of a blow-up. One can also show that if K_X is not nef, then either there exists a (-1)-curve on X or there is a morphism with connected fibres $f : X \to Z$ such that $\dim Z < \dim X$ and such that $-K_X$ is ample on any fibre of f. In the latter case, we say that $\varphi : X \to Z$ is a Mori fibre space and we stop here. In the former case we consider $f : X \to Y$ the blow-down of C, we replace X with Y and we control whether K_Y is nef or not. If K_Y is nef, we have found a minimal model, otherwise we continue as before. This procedure is called *Minimal Model Program.* Denote by $N_1(X)$ the real vector space of 1-cycles modulo numerical equivalence and by $\rho(X) = \dim N_1(X)$ the Picard number of X. Since $\rho(X)$ is a non-negative integer and decreases by 1 at each step, we know that the our procedure terminates either with a minimal model Y (that is K_Y is nef) or with a Mori fibre space.

In higher dimension the situation is far more complicated, but the basic plan is the same. The two main differences are the appearance of singularities and of contractions whose exceptional locus is not a divisor. The latter ones lead to varieties which are too singular to work with. We will explain the main ideas of the MMP in subsection 1.1.2, here we just sketch a picture.

The pioneers in this field have been S. Mori, Y. Kawamata, J. Kollár, M. Reid and V. Shokurov: their intuition was to study the extremal rays of the cone of effective 1-cycles $NE(X) \subset N_1(X)$. A morphism $f : X \to Z$ arising as the contraction of an extremal ray $R \subset \overline{NE}_{K_X<0}$ (i.e. f contracts exactly the curves C whose numerical classes are in R and $K_X.R < 0$) is called Fano-Mori contraction. If K_X is not nef, then, by a deep result of Mori, Kawamata and Shokurov, there exists a Fano-Mori contraction $f : X \to Z$ associated to an extremal ray $R \subset \overline{NE}_{K_X<0}$. If $\dim Z < \dim X$, then $f : X \to Z$ is said to be a Mori fibre space and we stop here, otherwise f is birational. If the exceptional locus of f is a divisor, then we can replace X with Z and go on. If the exceptional locus has codimension greater than 1, then on Z we can not even define the intersection number between K_Z and a curve (K_Z is not $\mathbb{Q}$-Cartier) and we can not proceed the MMP with Z. The solution, proposed by Mori, is to perform a flip and then to go on. A flip is a kind of thorny surgery that replaces the curves contracted by f with curves on which the canonical divisor is positive. The existence of flips in all dimensions has been proven only recently by C. Birkar, P. Cascini, C. Hacon and J. McKernan in [BCHM10] (and in [CL12], [CL13] by P. Cascini-V. Lazić and A. Corti-V. Lazić).

In conclusion, an MMP is a sequence of maps

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m$$

such that each map $\phi_i : X_i \dashrightarrow X_{i+1}$ is a birational morphism or a flip associated to an extremal ray R_i and either K_{X_m} is nef (X_m is a minimal model) or X_m has the structure of a Mori fibre space. The question about the existence of an MMP (that is, the non-existence of infinite sequence of flips) is a very delicate matter. Thanks to [BCHM10] we know that under suitable conditions on X, there exists an MMP which terminates.

Assuming the existence of an MMP, two natural questions are the following: can we describe explicitly each step and the final result? How does X change under an MMP? These two questions are strictly connected, both asking to investigate the properties of Fano-Mori contractions. Chapters 2 and 3 of this dissertation are dedicated to two aspects of these birational maps, whereas chapter 4 focuses on some properties of pluricanonical maps. We now describe the content of the Thesis.

First chapter. In this chapter we develop the basic concepts and results that will be used in the rest of the Thesis. The effort is to make the exposition readable, partially self-contained and to insert our results in the right mathematical context. All the material presented is already known, except for subsection 1.3.2, in which we slightly extend the concept of weighted blow-ups along smooth subvarieties. Roughly speaking, a weighted blow-up is a generalization of the classical blow-up in which the exceptional divisor is a weighted projective space.

Section 1.1 contains useful background material for all the three other chapters. In particular we collect the main properties of log-pairs and the definitions of the singularities that appear in the MMP. We then explain how to run an MMP.

Section 1.2 is preparatory for chapter 2 and summarizes the state of the art about the classification of Fano-Mori contractions with high nef-value. It also introduces the notion of local adjoint contraction, which will be used in the main theorems of chapters 2.

Subsection 1.3.1 is preliminary to chapter 4, while subsection 1.3.2 is necessary for understanding chapter 2.

Second chapter. Let $\varphi : X \to Z$ be a Fano-Mori contraction, where X is a terminal projective variety. Then there are an ample line bundle L and a positive rational number τ such that φ is supported by $K_X + \tau L$, that is φ is given by the linear system $|m(K_X + \tau L)|$ for $m \gg 0$. The content of this chapter is the study of φ depending on τ. For this goal we use methods of adjunction theory, which studies polarized varieties, i.e. pairs (X, L), where L is an ample line bundle on X. The basic idea is to apply induction on a suitable element $X' \in |L|$. By adjunction

$$(K_X + \tau L)_{|X'} = K_{X'} + (\tau - 1)L_{|X'}$$

and we have a Fano-Mori contraction in one dimension less. Then, by induction, we try yo lift properties of (X', L') to (X, L). We will also need the local set-up as developed by Andreatta and Wisniewski, that is, to study $\varphi : X \to Z$ we fix a fibre F and restrict to an affine neighbourhood of the image of the fixed fibre. The global contraction can then be obtained by gluing different local descriptions.

Our first result completes the classification of birational Fano-Mori contractions when $\tau > n - 2$.

Theorem (2.1.1). *Let X be a normal projective variety with $\mathbb{Q}$-factorial terminal singularities and let L be an ample Cartier divisor on X. Let R be an extremal ray in $\overline{NE(X)}_{(K_X+(n-2)L)<0}$ and let $f: X \to Z$ be its associated contraction. Assume that f is birational. Then f is a weighted blow-up of a smooth point with weight $\sigma = (1, 1, b, \ldots, b)$, where b is a positive integer.*

The strategy of the proof is the following. By means of horizontal slicings with good elements of $|L|$, we reduce to the surface case, where we get a Castelnuovo contraction. Then we lift the ideal sheaf of the blow-up to $f: X \to Z$.

Next we classify divisorial contractions associated to extremal rays R such that $R.(K_X + rL) < 0$, where r is a non-negative integer, and the fibres of f have dimension less or equal to $r + 1$:

Theorem (2.1.2). *Let X be a normal projective variety with $\mathbb{Q}$-factorial terminal singularities and let L be an ample Cartier divisor on X. Let R be an extremal ray in $\overline{NE(X)}_{(K_X+rL)<0}$ where $r \in \mathbb{N}$ is a non-negative integer and let $f: X \to Z$ be its associated contraction. Assume that f is divisorial and that all fibres have dimension less or equal to $r + 1$. Let E be the exceptional locus of f and set $C := f(E) \subset Z$.*

1. *Then $\mathrm{codim}_Z C = r + 2$, there is a closed subset $S \subset Z$ of codimension al least 3 such that $Z' = Z \backslash S$ and $C' = C \backslash S$ are smooth, and $f': X' = X \backslash f^{-1}(S) \to Z'$ is a weighted blow-up along C' with weight $\sigma = (1, 1, b, \ldots, b, 0, \ldots, 0)$, where the number of b's is r.*

2. *Let $\mathcal{I}'$ be a σ-weighted ideal sheaf of degree b for $Z' \subset X'$ and let $i: Z' \to Z$ be the inclusion; let also $\mathcal{I} := i_*(\mathcal{I}')$ and $\mathcal{I}^{(m)}$ be the m-th symbolic power of $\mathcal{I}$. Then $X = \mathrm{Proj} \bigoplus_{m \geq 0} \mathcal{I}^{(m)}$.*

These theorems are based on the existence of good elements in the linear system $|L|$. More precisely, Andreatta and Wisniewski ([AW93]) proved that, under our conditions, L is relatively basepoint free. In subsection 2.2.1 we will prove the following.

Theorem (2.2.7). *Let $f: X \to Z$ be a local divisorial Fano-Mori contraction supported by $K_X + \tau L$. Assume that X has log terminal singularities and $n < \tau + 3$. Then $\dim Bs|L| \leq 1$.*

We hope to be able to apply this theorem to understand birational contractions with $\tau > (n - 3) > 0$. In section 2.2 we start the investigation of these contractions, obtaining some partial results.

This chapter collects the contents of the joint paper [AT13] and of a work in progress with M. Andreatta.

Third chapter. Let X be a complex projective manifold of dimension n and let $c_i = c_i(X)$ be its Chern classes. More than fifty years ago, Hirzebruch asked which Chern numbers (product of Chern classes of total degree n) are topological invariants. The problem has been completely settled by Kotschick ([Kot12]), who proved that a rational combination of Chern numbers is invariant if and only if it is multiple of the Euler characteristic c_n.

In this chapter we face the subsequent question posed by Kotschick: does $c_1^3 = -K_X^3$ assumes only finitely many values on the 3-dimensional projective algebraic complex structures with the same underlying 6-manifold?

If X is a variety of dimension n then the volume of X is defined as

$$\mathrm{vol}(X) := \limsup_{m\to+\infty} \frac{n!h^0(X, mK_X)}{m^n},$$

and X is called of general type if $\mathrm{vol}(X) > 0$.

Note that on a minimal model X, we have $K_X^3 = \mathrm{vol}(X)$. Recalling that $\mathrm{vol}(X)$ is a birational invariant and applying a Bogomolov-Miyaoka-Yau inequality we can prove the following.

Theorem (3.1.3)**.** *Let Let X be a smooth projective 3-fold of general type and let*

$$X = X_0 \dashrightarrow \ldots \dashrightarrow X_m$$

be a minimal model program for X. Then

$$\mathrm{vol}(X) = K_{X_m}^3 \leq 64\left(b_1(X) + b_3(X) + \frac{2}{3}b_2(X)\right),$$

where $b_i(X) = \dim H^i(X, \mathbb{Q})$ are the Betti numbers of X. This implies that the volume takes only finitely many values on projective algebraic structures of general type with the same underlying 6-manifold.

The idea is then to bound K_X^3 by some topological invariant of X running an MMP $X \dashrightarrow X_m$ and comparing K_X^3 with $K_{X_m}^3$. It is easy to see that the Betti numbers are in general not enough to bound K_X^3. To any threefold X we can associate an integral cubic form F_X, which comes from the trilinear intersection form on $H^2(X, \mathbb{Z})$. If $f : Y \to X$ is a divisorial contraction to a smooth curve in a the smooth locus of Y, then

$$F_Y(x_0, \ldots, x_n) = ax_0^3 + x_0^2(\sum b_i x_i) + F_X(x_1, \ldots, x_n),$$

where $x_0, x_1, \ldots, x_n$ are coordinates on $H^0(Y, \mathbb{Z}) = \mathbb{Z}[E] \oplus H^0(X, \mathbb{Z})$ and $a = E^3$.

Denote by Δ_F the discriminant of a form F. Our principal technical result is the following.

Theorem (3.2.28)**.** *Let $F \in \mathbb{Z}[x_0, \dots, x_n]$ be a non-degenerate cubic form with integral coefficients such that $\Delta_F \neq 0$. Then, modulo the action of* $\mathrm{SL}(\mathbb{Z}, n)$ *on $(x_1, \dots, x_n)$, there are only finitely many triples $(a, (b_1, \dots, b_n), G)$ such that F can be written as*

$$F = ax_0^3 + x_0^2(\sum b_i x_i) + G(x_1, \dots, x_n).$$

Moreover $\Delta_G \neq 0$.

The proof is composed by two step: first we demonstrate that we can reduce to the case of binary and ternary cubics and then we apply Faltings and Siegel Theorems to show our claim. Roughly speaking, this says that the possible values of E^3 are bounded and, combining everything, we can finally state our main theorem:

Theorem (3.3.5)**.** *Let X be a projective smooth threefold of non-negative Kodaira dimension and let F be its associated cubic. Assume that $\Delta_F \neq 0$ and that there is an MMP for X composed only by divisorial contractions to points and blow-down to smooth curves in smooth loci.*

Then there exists a constant D depending only on the topology of X such that

$$|K_X^3| \leq D.$$

The plan for the future is to extend this result to general divisorial contractions to curves and to flips and flops. Actually, by a recent result of Chen ([Che13]), any MMP of a terminal $\mathbb{Q}$-factorial threefold may be factored into a sequence of flops, blow-downs to smooth curves in smooth loci and divisorial contractions to points (or their inverses). Hence, the main question left is about flops.

All this is the content of a work in progress with P. Cascini.

Fourth chapter. In this chapter we construct varieties of general type with either many vanishing plurigenera, or with many non-birational pluricanonical maps or with assigned volume. The importance of these three objects is that they depend only on the birational properties of X.

Let X be a projective normal variety of general type. For any number r, the dimension of $H^0(X, rK_X)$ is called the r-plurigenera of X. The first question is up to which $r \in \mathbb{Z}$ the space $|H^0(X, rK_X)|$ may be empty. We construct, for every n, smooth varieties of general type of dimension n with the first $\lfloor \frac{n-2}{3} \rfloor$ plurigenera equal to zero.

Then one wants to investigate the pluricanonical map ϕ_r associated to the pluricanonical system $|rK_X|$. Hacon-McKernan, Takayama and Tsuji have recently shown that there are numbers r_n such that for all r greater or equal to r_n, the r-pluricanonical map of every variety of general type of dimension n is birational. Our examples show that r_n grows at least quadratically as a function of n. They also show that the minimal volume of a variety of general type of dimension n is smaller than $3^{n+1}/(n-1)^n$.

In addition we prove that for every positive rational number q there are smooth varieties of general type with volume q and dimension arbitrarily big. For every n we also give an example of a smooth variety X of dimension n with ample canonical divisor such that the kth-canonical map is not birational for $k < n+3$ if n is even and $k < n+2$ if n is odd.

All our examples come from hypersurfaces in weighted projective spaces.

These results are collected in a paper in collaboration with E. Ballico and R. Pignatelli ([BPT13]).

Chapter 1

Preliminaries

We start by fixing some standard notations. We generically refer to [KM98].

- By the term *variety* we mean a separated, reduced, irreducible scheme of finite type over $\mathbb{C}$, which is always our base field.
- Let X, Y be schemes. A rational map $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$, where $U \subset X$ is an open dense subset.

  A rational map with rational inverse is called *birational map*. The exceptional locus $\mathrm{Exc}(f)$ of a birational map $f : X \dashrightarrow Y$ is the set of points of X where f is not an isomorphism.
- Let X be a normal scheme. We denote by $\mathrm{WDiv}(X)$ the group of (Weil) divisors on X, by $\mathrm{Div}(X)$ the group of Cartier divisors and by $\mathrm{Pic}(X)$ the group of invertible sheaves. A $\mathbb{Q}$-divisor is an element of

  $$\mathrm{WDiv}_{\mathbb{Q}} := \mathrm{WDiv} \otimes_{\mathbb{Z}} \mathbb{Q}$$

  and a $\mathbb{Q}$-Cartier divisor is an element of

  $$\mathrm{Div}_{\mathbb{Q}} := \mathrm{Div} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

  A $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier if an integral multiple is Cartier. The symbols $\sim$ and $\sim_{\mathbb{Q}}$ stand for linear equivalence and $\mathbb{Q}$-linear equivalence respectively.
- Let Z_1 be the group of 1-cycles on X, that is the free group generated by irreducible curve on X. If L is a Cartier divisor on X and C is an irreducible curve, we define the intersection $L.C$ as $\deg L_{|C}$. This gives rise to an intersection pairing

  $$\mathrm{Div}(X) \times \mathrm{Z}_1(X) \to \mathbb{Z},$$

  which extends to a product in $\mathbb{R}$

  $$\mathrm{N}^1(X) \times \mathrm{N}_1(X) \to \mathbb{R}$$

where

$$\mathrm{N}^1(X) := (\mathrm{Div}(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R},$$

$$\mathrm{N}_1(X) := (\mathrm{Z}_1(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$$

and $\equiv$ is the equivalence relation induced by the intersection. The vector spaces $\mathrm{N}_1(X)$ e $\mathrm{N}^1(X)$ are dual and have finite dimension, which is called Picard number of X and it is denoted by $\rho(X)$.

- We denote by $NE(X) \subset N_1(X)$ the cone of effective 1-cycles. If L is a $\mathbb{Q}$-Cartier divisor on X, we set $NE(X)_{L\geq 0} := \{z \in NE(X) : L.z \geq 0\}$. An extremal face of a cone $\mathcal{C}$ is a face F such that $x + y \in F$ implies $x, y \in F$ for any $x, y \in \mathcal{C}$.
- A normal variety X is $\mathbb{Q}$-factorial if any $\mathbb{Q}$-divisor D on X is $\mathbb{Q}$-Cartier.
- Let X be a normal variety and let $U \subset X$ be the smooth locus of X. Since the singular locus of X has codimension at least 2, it is well defined the canonical (Weil) divisor class K_X as extension of the canonical class K_U. We refer to K_X as the canonical divisor of X.
- Let D be a $\mathbb{Q}$-Cartier divisor on a projective variety X. D is called pseudoeffective if the numerical class of D is in the closure of the effective cone (i.e. the cone in $N^1(X)$ generated by effective divisors). D is called nef if $D.C \geq 0$ for any curve C in X.
- The volume of $\mathbb{Q}$-Cartier divisor D on a normal variety X is defined as

$$\mathrm{vol}(X; D) := \limsup_{m\to+\infty} \frac{n!h^0(X, mD)}{m^n}$$

  and D is said to be big if $\mathrm{vol}(D) > 0$. A variety X is called of general type if $\mathrm{vol}(X) := \mathrm{vol}(X; K_X) > 0$.
- Let $\Delta = \sum d_i D_i$ be a $\mathbb{Q}$-divisor. Then we define integral part of D as $\lfloor \Delta \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d_i \rfloor$ is the minimum integer less or equal to d_i. Similarly for $\lceil D \rceil$.

1.1 Minimal model program

We refer to [KM98] for details and proofs, if not otherwise stated.

Subsection 1.1.1 is dedicated to the singularities that appear in the context of the MMP, to the notion of log-pair and to related concepts that we will use hereinafter.

In **subsection 1.1.2** we recall the main definitions and theorems needed to run an MMP in the category of klt pairs.

1.1.1 Log-pairs and Singularities

Definition 1.1.1 (SNC divisors)**.** Let X be a non-singular variety of dimension n. A divisor $D = \sum D_i$ (where D_i are prime distinct divisors) is an *SNC divisor* (or D *has simple normal crossings*) if each D_i is smooth and if D is defined in a neighbourhood of any point by an equation in local analytic coordinates of the type

$$z_1 \cdot \ldots \cdot z_k = 0$$

for some $k \leq n$. A $\mathbb{Q}$-divisor $\sum a_i D_i$ has *simple normal crossings support* if $\sum D_i$ is an SNC divisor.

Definition 1.1.2 (Log resolutions)**.** Let X be a variety and let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on X. A *log resolution* of (X, D) is a projective birational morphism

$$\mu : X' \to X$$

with X' non-singular, such that $\mu^*(D) + \mathrm{Exc}(\mu)$ has simple normal crossing support, where $\mathrm{Exc}(\mu)$ denotes the sum of the exceptional divisors of μ.

We recall that, by Hironaka's fundamental result, for any pair (X, Δ) there exists a log resolution.

Definition 1.1.3 (Log pairs)**.** A log pair (X, Δ) consists of a normal variety X together with a Weil $\mathbb{Q}$-divisor $\Delta = \sum d_i D_i$ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Let X be a variety. A *model* of X is a proper birational morphism $\mu : X' \to X$, where X' is a normal variety. Any divisor D on a model of X is called *divisor over* X.

Let (X, Δ) be a log pair and let $\mu : X' \to X$ be a model of X. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. Let E_i be the irreducible exceptional divisors and $\mu_*^{-1}\Delta$ be the birational transform of Δ.

Since the two line bundles

$$\mathcal{O}_{X'}(m(K_{X'} + \mu_*^{-1}\Delta))_{|X'-E} \quad \text{and} \quad \mu^*\mathcal{O}_X(m(K_X + \Delta))_{|X'-E}$$

are naturally isomorphic, there are rational numbers $a(E_i) = a(E_i, X, \Delta)$ such that $ma(E_i)$ are integers, and

$$\mathcal{O}_{X'}(m(K_{X'} + \mu_*^{-1}\Delta)) \cong \mu^*\mathcal{O}_X(m(K_X + \Delta))(\sum_i ma(E_i)E_i).$$

We set $a(\Delta_i, X, \Delta) = -d_i$ and $a(D, X, \Delta) = 0$ for any divisor $D \subset X$ which is different from the D_i.

Using numerical equivalence we can write

$$K_Y + \mu_*^{-1}\Delta \equiv \mu^*(K_X + \Delta) + \sum_{E_i \text{ exceptional}} a(E_i)E_i.$$

The rational number $a(E_i)$ is called discrepancy of E_i with respect to (X, Δ) and does not depend on the map μ, but only on (the valuation induced by) E_i.

A valuation $\nu : k(X) \to \mathbb{Z}$ is called geometric if $\nu = \mathrm{mult_E}$, where E is a prime divisor on a model $Y \to X$. Divisors $E \subset Y \to X$ and $E' \subset Y' \to X$ define the same valuation if and only if the induced birational map $Y \dashrightarrow Y'$ is an isomorphism at the generic point of E and E'. The *centre* on X of a geometric valuation ν associated to a divisor E on a model $f : Y \to X$ is $c_X\nu = c_XE = f(E)$. We often identify a geometric valuation ν and the corresponding divisor E.

We define the *discrepancy* of (X, Δ) as

$$\mathrm{discrep(X, \Delta) := \inf_E\{a(E, X, \Delta) : E \text{ is an exceptional divisor over } X\}.}$$

Definition 1.1.4. Let (X, Δ) be a log pair. We say that (X, Δ) is

$$\begin{cases} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{lc} \end{cases} \quad \text{if discrep(X}, \Delta) \quad \begin{cases} > 0 \\ \geq 0 \\ > -1 \text{ and } \lfloor \Delta \rfloor = 0 \\ \geq -1 \end{cases}$$

Here klt stays for Kawamata log terminal and lc for log canonical.

If $\Delta = 0$ then klt is called lt (log terminal).

The strong motivation behind this definition is that the smallest over-category of smooth varieties in which is possible to run an MMP is that of terminal varieties. Canonical models of smooth varieties has in general canonical singularities. The introduction of log pairs is mainly due to the possibility of applying inductive arguments.

Many important results in birational geometry are in fact attained by induction on suitable subvarieties. A recurring and crucial problem is that of obtaining a global section of a certain linear system $|L|$ on a variety X. Usually one tries, roughly speaking, to produce a section of L on a suitable subvariety $V \subset X$ and then lift this section to X.

The vanishing theorems are fundamental results to succeed in this procedure, since basically we are looking for a surjection $H^0(X, L) \to H^0(V, L_V)$. The archetype is Kodaira vanishing which predicts that if L is an ample line bundle on a smooth variety X, then $H^i(K_X + L) = 0$ for $i > 0$. Starting from this, there have been many generalizations: for big and nef divisor, for singular varieties, for log pairs and for relative settings. We will mainly use Kawamata-Viehweg vanishing.

Theorem 1.1.5 (Relative Kawamata-Viehweg)**.** *Let (X, Δ) be a log terminal pair, $f : X \to S$ be a projective morphism onto a variety S and A be an f-ample $\mathbb{Q}$-divisor on X such that $K_X + \Delta + A$ is an integral divisor. Then*

$$R^i f_* \mathcal{O}_X(K_X + \Delta + A) = 0 \quad \textit{for } i > 0.$$

We will also need the notion of multiplier ideal, for which the most useful vanishing is Nadel vanishing.

Definition 1.1.6 (Multiplier ideals)**.** Let (X, Δ) be a log pair and let D be a $\mathbb{Q}$-Cartier divisor on X and let $\mu : X' \to X$ be a log resolution for (X, D). The multiplier ideal of D (respect to the log pair (X, Δ)) is defined as the sheaf

$$\mathcal{J}(D) = \mathcal{J}(X, \Delta; D) = \mu_* \mathcal{O}_{X'}(K_{X'} - \lceil \mu^*(K_X + \Delta + D) \rceil).$$

It does not depend on the log-resolution chosen.

If Δ and D are effective divisors, then $\mathcal{J}(X, \Delta; D)$ is actually an ideal sheaf.

Theorem 1.1.7 (Nadel)**.** *Let (X, Δ) be a log pair and let D be a $\mathbb{Q}$-Cartier divisor on X.*

1. *If $\mu : X' \to X$ is a log resolution of $(X, \Delta + D)$, then*
$$R^j \mu_* \mathcal{O}_{X'}(K_{X'} - \lceil \mu^*(K_X + \Delta + D) \rceil) = 0 \quad \textit{for } j > 0.$$

2. *If N is an integral Cartier divisor on X such that $N - (K_X + \Delta + D)$ is nef and big, then*
$$H^i(X, \mathcal{O}_X(N) \otimes \mathcal{J}(X, \Delta; D)) = 0 \quad \textit{for } i > 0.$$

The suitable subvarieties from which we want to lift sections arise from the "bad locus" of log pairs (intuitively, if a divisor L has high multiplicity in a point $x \in X$, then we may be able to construct section of L at x).

Definition 1.1.8. Let (X, Δ) be an lc pair. The *non-klt locus* of (X, Δ) is $\mathrm{nklt(X, \Delta) = Supp(\mathcal{O}_X/\mathcal{J}(\Delta))}$.

A *non-klt centre* is a centre W of a geometric valuation such that $W \subset \mathrm{nklt(X, \Delta)}$; therefore $\mathcal{J}(\Delta) \subset \mathcal{I}_W$. Since in our situation (X, Δ) is lc, a non klt-centre W is also called *log canonical centre.* The set of all log canonical centres is denoted by $CLC(X, \Delta)$ and the non-klt locus is also denoted by $LLC(X, \Delta)$.

A non-klt centre W is *isolated* if for any geometric valuation E on X such that $a(E, X, \Delta) = -1$, $c_X E = W$. A non-klt centre W is *exceptional* if it is isolated and if there is unique geometric valuation E on X such that $a(E, X, \Delta) = -1$.

The importance of isolated non-klt centres is given by the following observation.

Lemma 1.1.9. *If W is an isolated non-klt centre, then $\mathcal{J}(\Delta) = \mathcal{I}_W$.*

Note that for any $x \in \mathrm{nklt(X, \Delta)}$ there is a well defined *minimal non-klt centre* through x. In fact, if $\mu : X' \to X$ is a log resolution of (X, Δ) and $E_i \subset Y$ are prime divisors such that $a(E_i, X, \Delta) = -1$ for $i = 1, \dots, k$, then the non-klt centres are the images $\mu(\cap_{i \in I} \mu(E_i))$, where $I \subset \{1, \dots, k\}$. Furthermore, if $W_1, W_2 \in CLC(X, \Delta)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, \Delta)$.

The following is the main tool to isolate non-klt centres (see Proposition 8.7.1 in [Kol97b]).

Proposition 1.1.10 (Tie-breaking)**.** *Let (X, Δ) be a klt pair and D a $\mathbb{Q}$-Cartier divisor on X such that $(X, \Delta + D)$ is lc. Let W be a minimal non-klt centre of $(X, \Delta + D)$ and let A be an ample divisor. Then there are arbitrarily small $\epsilon, \eta > 0$ and a divisor $D' \sim_{\mathbb{Q}} A$ such that W is an exceptional non-klt centre of $(X, \Delta + (1 - \epsilon)D + \eta D')$.*

Finally, we need to know what happens when we restrict to non-klt centres:

Theorem 1.1.11 (Kawamata's subadjunction)**.** *Let (X, Δ) be an lc pair, W an exceptional non-klt centre of (X, Δ) and A an ample $\mathbb{R}$-divisor. Then W is normal and for every $\varepsilon > 0$, there is an effective divisor Δ_W on W such that*

$$(K_X + \Delta + \varepsilon A)_{|W} \sim_{\mathbb{R}} K_W + \Delta_W$$

and (W, Δ_W) is klt.

Definition 1.1.12 (Log canonical threshold)**.** Let (X, Δ) be a log pair and let $x \in X$ be a point. The *log canonical threshold* of (X, Δ) at x is defined as

$$\mathrm{lct(X, \Delta, x) = \sup\{c \in \mathbb{R} : (X, c\Delta) \text{ is lc in the neighbourhood of } x\}.}$$

The log canonical threshold of the pair (X, Δ) is

$$\mathrm{lct(X, \Delta) = \sup\{c \in \mathbb{R} : (X, c\Delta) \text{ is lc}\}.}$$

1.1.2 Running an MMP

The following three results are the basic ingredients to run an MMP in higher dimension. Their proofs were developed by several authors in the 1980s decade, the main contributions are due to Kawamata, Mori and Shokurov.

Theorem 1.1.13 (Kawamata-Shokurov basepoint free)**.** *Let (X, Δ) be a klt pair and let L be a nef and big line bundle on X. Assume that there exists $p > 0$ such that $pL - (K_X + \Delta)$ is nef and big. Then the linear system $|mL|$ is basepoint free for all integers $m \gg 0$.*

Theorem 1.1.14 (Cone Theorem)**.** *Let (X, Δ) be a projective klt pair.*

1. *There are countably many rational curves C_i such that $0 < (K_X + \Delta).C_i \leq 2\dim X$, and*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

2. *For any $\varepsilon > 0$ and ample $\mathbb{Q}$-divisor H,*
$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta+\varepsilon H)\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i].$$

Theorem 1.1.15 (Contraction Theorem)**.** *Let (X, Δ) be a projective klt pair and let $F \subset \overline{NE}(X)$ be $(K_X + \Delta)$-negative extremal face. Then there is a unique morphism $\varphi : X \to Z$ to a projective variety Z such that*

1. *$\varphi_*\mathcal{O}_X = \mathcal{O}_Z$;*

2. *an irreducible curve $C \subset X$ is contracted to a point if and only if $[C] \in F$;*

3. *if L is a Cartier divisor on X such that $L.C = 0$ for every curve C with $[C] \in F$, then $L = \varphi^*L_Z$ for a Cartier divisor L_Z on Z.*

The morphism φ is called Fano-Mori contraction *associated to F (or simply* extremal contraction *of F). If F has dimension 1 then F is called* extremal ray *and φ is called* elementary (extremal) contraction*. In this case we have an exact sequence*
$$0 \to Pic(Z) \to Pic(X) \to \mathbb{Z},$$
where the last map is given by the multiplication by a fixed curve C with $[C] \in R$.

Cone and contraction theorems were first proved by Mori in the smooth threefold case.

Definition 1.1.16. Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair and let $\varphi : X \to Z$ be a contraction of a $(K_X + \Delta)$-negative extremal ray R.

1. If $\dim Y < \dim X$, then $\varphi : X \to Z$ is called a *Mori fibre space* (or simply *Mori fibration*);

2. if $\dim Y = \dim X$ and $\mathrm{codim}_X \mathrm{Exc}(\varphi) = 1$, then $\mathrm{Exc}(\varphi)$ is actually a prime divisor and φ is called *divisorial contraction*;

3. if $\dim Y = \dim X$ and $\mathrm{codim}_X \mathrm{Exc}(\varphi) \geq 2$ (i.e. φ is a small map), then φ is called *flipping contraction*.

Definition 1.1.17. Let (X, Δ) be a log pair.

1. If $K_X + \Delta$ is nef, then (X, Δ) is called *minimal model.*
2. If $K_X + \Delta$ is not nef and there is a Mori fibration $(X, \Delta) \to Z$, then (X, Δ) is called *Mori fibre space.*

From now on X will be assumed to be $\mathbb{Q}$-factorial.

We want to run an MMP, on the model of the surface's case. We have already said that we are forced to admit singular varieties and so let us start, more generally, with a klt pair (X, Δ) such that $K_X + \Delta$ is not nef. By cone and contraction theorems there is a Fano-Mori contraction $\varphi : X \to Z$ associated to an extremal ray R such that $(K_X + \Delta).R < 0$.

If $\varphi : X \to Z$ is a Mori fibre, then we stop. Note that the fibres of φ are Fano varieties.

If $\varphi : X \to Z$ is a divisorial contraction, then it is not difficult to prove that $(Z, \varphi_*\Delta)$ is still a $\mathbb{Q}$-factorial klt pair and hence we go on with the MMP, replacing (X, Δ) with $(Z, \varphi_*\Delta)$. Moreover the Picard number $\rho(X)$ decreases by one, and this assures us that there can not be an infinite sequence of divisorial contractions.

If $\varphi : X \to Z$ is a flipping contraction, then $K_Z + \varphi_*\Delta$ is not $\mathbb{Q}$-Cartier (otherwise it would be $K_X + \Delta = f^*(K_Z + \varphi_*\Delta)$, against the f-negativity of $K_X + \Delta$) and thus we can not proceed. Mori proposed the following solution.

Definition 1.1.18 (Flips)**.** Let $\varphi : (X, \Delta) \to Z$ be a flipping contraction with (X, Δ) klt. A *flip* of φ is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & Z & \end{tikzcd}$$

where

1. φ^+ is small and projective,
2. $K_{X^+} + \Delta^+$ is φ^+-ample, with $\Delta^+ = \varphi^+_*\Delta$,
3. $\rho(X^+/Z) = 1$.

For simplicity, we call flip the induced map $X \dashrightarrow X^+$.

It is not difficult to check that in this case (X^+, Δ^+) is a $\mathbb{Q}$-factorial klt-pair and hence we proceed the MMP, replacing (X, Δ) with (X^+, Δ^+). Two major problems of the past 30 years in birational geometry have been to show that flips exist and that there can not be infinite sequences of flips.

Conjecture 1.1.19 (Existence of flips)**.** *Let $\varphi : (X, \Delta) \to Z$ be a flipping contraction such that (X, Δ) is klt. Then there exists the flip $X \dashrightarrow X^+$.*

Conjecture 1.1.20 (Termination of flips)**.** *There is no infinite sequence of flips.*

In conclusion, if (X, Δ) is klt and existence and termination of flips hold, then there is an $(K_X + \Delta)$-MMP

$$(X_0, \Delta_0) = (X, \Delta^r) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_s, \Delta_s)$$

where each $f_i : X_i \dashrightarrow X_{i+1}$ is a birational map (divisorial contraction or flip) associated to an extremal ray R_i. Any (X_i, Δ_i) is klt and each X_i is $\mathbb{Q}$-factorial. The end result (X_s, Δ_s) is either a minimal model or a Mori fibre space.

The problem about the existence of flips has been completely settled in [BCHM10] and in the papers [CL12], [CL13]. In the tridimensional case, the existence of flips was first proved by Mori ([Mor88]), while the termination is due to Shokurov ([Sho85]).

Theorem 1.1.21. *([BCHM10, Cor. 1.4.1]) Let (X, Δ) be a projective klt pair and let $\varphi : (X, \Delta) \to Z$ be a flipping contraction. Then the flip of φ exists.*

Termination is proved under some conditions in [BCHM10] and in the papers [CL12], [CL13].

Theorem 1.1.22. *([BCHM10, Thm. 1.2, Cor. 1.3.3]) Let (X, Δ) be a projective klt pair such that $K_X + \Delta$ is big (or Δ is big and $K_X + \Delta$ is pseudo-effective). Then there exists a MMP for (X, Δ) which ends with a minimal model.*

Let (X, Δ) be a projective klt pair such that $K_X + \Delta$ is not pseudo-effective. Then there exists a MMP for (X, Δ) which terminates with a Mori fibre space.

1.2 Adjunction theory

We start this section with an introduction to the methods and ideas of adjunction theory. For a detailed treatment from the classical point of view see the book [BS95].

Subsection 1.2.1 develops the notion of local adjoint contraction following [AW93]. We will use this set-up in most of the results of chapter 2.

In **subsection 1.2.2** we sum up the state of the art for what concerns classification of contractions with high nef-value. At the same time we summarize part of the content of the paper [And13], which shows the strict connection between adjunction theory and MMP of klt pairs.

The basic idea of adjunction theory is to understand properties of X via the study of adjoint bundles $K_X + tL$, where t is rational positive number and L is an ample line bundle on X. More precisely, adjunction theory looks at the pair (X, L).

The original setting is that of a variety X and an embedding $\phi : X \hookrightarrow \mathbb{P}^n$ such that $L = \phi^*\mathcal{O}_X(1)$, i.e. L is very ample. The special role of these adjoint bundles became clear already in the past century, when Castelnuovo and Enriques were studying projective surfaces by relating the geometry of a projective surface S to the geometry of its hyperplane sections. In higher dimension the first results in this field may be considered some characterisations of projective spaces.

Definition 1.2.1. A *polarized variety* (or *polarized pair*) is a pair (X, L), where X is a projective variety and L is an ample line bundle.

If L is just big and nef, then we say that (X, L) is a *quasi-polarized variety.*

The following are classical invariants associated to a polarized variety. They are very useful for classification results (see, for example, the book [Fuj90]).

Definition 1.2.2. The Hilbert polynomial of the quasi-polarized pair (X, L) is given by

$$\chi(X, tL) = \sum_{j\geq 0}(-1)^j h^j(X, tL) = \sum_{j=0}^{n} \chi_j t^{[j]}/j!,$$

where $t^{[j]} = t(t+1)\cdots(t+j-1)$, $t^{[0]} = 1$ and $\chi_0, \ldots, \chi_n$ are integers.

By Riemann-Roch Theorem we have that $\chi_n = L^n$ (which is called the degree $d(X, L)$ of (X, L)) and, if X is normal, that $-2\chi_{n-1} = (K_X + (n-1)L)\cdot L^{n-1}$, for a canonical divisor K_X on X.

The **sectional genus** of the pair (X, L) is $g(X, L) = 1 - \chi_{n-1}$.

The **Δ-genus** is defined as $\Delta(X, L) = n + \chi_n - h^0(X, L)$.

Another important invariant associated to a quasi-polarized pair (X, L) is its *nefvalue*. Let $f : X \to Z$ be a projective morphism onto a variety Z. Let L be an f-ample Cartier divisor on X. The f-nef value of (X, L) is defined as

$$\tau(X, L, f) := \inf\{t \in \mathbb{R} : K_X + tL \text{ is } f\text{-nef}\}.$$

Assume now that Z is a point. Then $\tau(X, L, f)$ is simply called nef-value and denoted by $\tau(X, L)$. The following theorem is a very useful instrument to study the nef-value of a pair.

Theorem 1.2.3 (Kawamata's rationality theorem). *Let X be a normal, irreducible projective variety of dimension n with terminal singularities and let $e = \mathrm{index}(X)$ be the index of singularities of X. Let $f : X \to Z$ be a projective morphism onto a variety Z. Let L be an f-ample Cartier divisor on X. If K_X is not f-ample then $\tau = \tau(X, L, f)$ is a rational number. Furthermore, if we write $e\tau = u/v$ with u, v coprime positive integers, we have $u \leq e(b+1)$ where $b = \max_{z \in Z}\{\dim f^{-1}(z)\}$.*

By the basepoint free theorem we know that, for $m \gg 0$, $|m(K_X + \tau L)|$ defines a morphism $g : X \to \mathbb{P}$. Let $g = s \circ \varphi$ be the Stein factorization of g where $\varphi : X \to Y$ has connected fibres and $s : Y \to \mathbb{P}^n$ is a finite-to-one morphism. For m big enough the morphism φ does not depend on m and we call φ the *nefvalue morphism* of (X, L).

By Kleiman's ampleness criterion it is easy to check that τ is the nefvalue of a pair (X, L) if and only if $K_X + \tau L$ is nef but not ample.

Clearly, a nefvalue morphism φ is a Fano-Mori contraction associated to an extremal face F.

Also the viceversa holds. In fact let $\varphi : X \to Z$ be a Fano-Mori contraction of an extremal face $F = H^{\perp} \cap (\overline{NE}(X)\backslash\{0\})$, where H is a nef $\mathbb{Q}$-divisor. Let e be the index of X. Then $L := eaH - eK_X$ is an ample Cartier divisor for some $a > 0$. Assuming that φ is not an isomorphism, we have that $eaH = eK_X + L$ is nef and not ample. Hence, $\tau(X, L) = 1/e$ and φ is the nefvalue morphism of (X, L)

In this context a basic tool is the so-called Apollonius Method, which allows to apply inductive arguments. More precisely we construct an inductive process considering (when possible) a general element $D \in |L|$ to obtain a polarized pair $(D, L_{|D})$ of one dimension less. By adjunction, $K_D + (t-1)L_{|D} = (K_X + tL)_{|D}$. Typical and instructive examples are the following theorems.

Proposition 1.2.4. *Let (X, L) be a polarized variety such that $\dim X = n$, $L^n = 1$ and $h^0(X, L) \geq n + 1$. Then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$.*

Proof. We give a proof assuming that X has at most Cohen-Macaulay singularities.

The proof is by induction on n. If $n = 1$ then it is easy, thus we assume $n \geq 2$. Let $D \in |L|$, then D is a reduced and irreducible divisor (if $L = L_1 + L_2$ then $L^n \geq 2$). Since D is Cartier, it is Cohen-Macaulay, and so it has no embedded component and it is easy to conclude that D is a variety.

By the exact sequence

$$0 \to \mathcal{O}_X(-L) \to \mathcal{O}_X \to \mathcal{O}_D$$

we get $h^0(D, L_D) \geq n$ and by induction $(D, L_D) \cong (\mathbb{P}^{n-1}, \mathcal{O}(1))$ and $h^0(X, L) = n + 1$. Therefore the restriction map $H^0(X, L) \to H^0(D, L_D)$ is surjective and hence L is globally generated. Thus $|L|$ gives a morphism $\phi : X \to \mathbb{P}^n$ which is finite because L is ample. We also have $\deg(\phi) = L^n = 1$ and hence ϕ is birational. By Zariski's main theorem we conclude that ϕ is an isomorphism.

□

Theorem 1.2.5 (Kobayashi-Ochiai). *Let X be an n-dimensional normal projective variety and let L be an ample line bundle on X. Then*

1. $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$ *if* $K_X + (n+1)L \sim \mathcal{O}_X$
2. $(X, L) \cong (\mathcal{Q}, \mathcal{O}(1))$, $\mathcal{Q}$ *quadric in* $\mathbb{P}^{n+1}$, *if* $K_X + nL \sim \mathcal{O}_X$.

Proof. We prove just (1).

We actually prove that if $-(K_X + nL)$ is ample then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$. By Kodaira's vanishing we have $H^i(X, tL) = 0$ for $i > 0$ and $t \geq -n$. Therefore $\chi(t) = \sum_i (-1)^i h^i(X, tL) = 0$ for $-n \leq t < 0$ and $\chi(0) = 1$. Hence $\chi(t) = (t+1) \cdots (t+n)/n!$, from which $L^n = 1$ and $h^0(X, L) = \chi(1) = n+1$. By Proposition 1.2.4 we are done. □

1.2.1 Local adjoint contractions

In this subsection we introduce the notion of *local contraction* as developed in [AW93]. This local set-up will be crucial in our study of Fano-Mori contraction of high length.

Recall that a *contraction* is a surjective morphism, $f : Y \to T$, such that X, T are normal varieties and f has connected fibres.

For a contraction $f : Y \to T$ a $\mathbb{Q}$-Cartier divisor H such that $H = f^*A$ for some ample $\mathbb{Q}$-Cartier divisor on T is called a *supporting divisor* of the contraction.

Let L be an ample Cartier divisor on a normal projective variety X with log terminal singularities. Let r be a positive rational number such that $K_X + rL$ is $\mathbb{Q}$-Cartier and nef, then by the basepoint free Theorem we know that $K_X + rL$ is semiample. In particular the linear system $|m(K_X + rL)|$ for $m \gg 0$ gives rise to a projective contraction $\varphi : X \to Z$ where

$$Z = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(X, m(K_X + rL)).$$

The basic idea of [AW93] to study φ is to fix a fibre and understand the contraction locally, i.e. restricting to an affine neighbourhood of the image of the fixed fibre. The global contraction can then be obtained by gluing different local descriptions. To explain formally this idea, we first analyse the local affine structure of $\varphi : X \to Z$. Set

$$R = R(X, K_X + rL) = \bigoplus_{m \geq 0} H^0(X, m(K_X + rL)).$$

The variety Z has an affine covering given by sets $Z_h = \mathcal{D}_+(h) \cong \mathrm{Spec} R_{(h)}$, where h is a homogeneous form and $R_{(h)}$ denotes the subring of elements of degree 0 in the localization. Let X_h the pull-back of Z_h via φ and φ_h, L_h the restrictions of φ and L to X_h. Then L_h is φ_h-ample and X_h is projective over Z_h. Since the section of $m(K_X + rL)$ associated to multiples of h do not vanish on X_h, we have that $m(K_{X_h} + L_h)$ is a unit in $PicX_h$ and hence, possibly shrinking X_h, we may assume that $K_{X_h} + L_h$ is isomorphic to $\mathcal{O}_{X_h}$. The morphism $\varphi_h : X_h \to Z_h$ is given by the evaluation map $H^0(X_h, K_{X_h} + rL_h) \to (K_{X_h} + L_h)$ and so is described by

$$X_h \ni x \mapsto \text{ideal of sections of } K_{X_h} + rL_h \text{ vanishing at } x.$$

Let us imitate this construction in general. For any scheme X over $\mathbb{C}$ set $\Gamma(X) := \mathrm{Spec} H^0(X, \mathcal{O}_X)$. The evaluation of global functions yields a morphism $\varphi_\Gamma : X \to \Gamma(X)$ defined as follows

$$x \mapsto \text{ideal of global functions vanishing at } x.$$

Lemma 1.2.6. *In the above situation the following statements hold:*

1. *if X is affine then φ_Γ is an isomorphism,*
2. $(\varphi_\Gamma)_* \mathcal{O}_X = \mathcal{O}_{\Gamma(X)}$,
3. *if X is separable over $\mathbb{C}$ then φ_Γ is separable,*
4. *if X is normal then $\Gamma(X)$ is normal.*

Proof. The first two points follows by definition.

The third assertion is a consequence of Corollary 4.6(e) in [Har70].

If X is normal then $H^0(X, \mathcal{O}_X)$ is integrally closed and hence $\Gamma(X)$ is normal.

□

Hence we state the following:

Definition 1.2.7. Let X be a normal variety with at most log terminal singularities and assume that K_X is $\mathbb{Q}$-Cartier. Let L be a Cartier divisor and $\tau \in \mathbb{Q}$ such that $K_X + rL$ has a nowhere vanishing section. The morphism $\varphi_\Gamma : X \to \Gamma(X)$ is called a *local (adjoint) contraction* supported by $K_X + \tau L$.

Let $f : Y \to T$ be a contraction supported by $K_X + \tau L$; fix a fibre F of f and take an open affine $Z \subset T$ such that $f(F) \in Z$ and $\dim f^{-1}(z) \leq \dim F$, for $z \in Z$. Set $X = f^{-1}Z$, then $f : X \to Z$ is a local (adjoint) contraction around F supported by $K_X + \tau L$.

Note that Z is a normal affine variety and the fibres of φ are connected. Let G be a generic non-trivial fibre of f. The dual index of f is

$$d(f) := \dim G - \tau,$$

the character of f is

$$\gamma(f) := \begin{cases} 1 & \text{if } \dim X > \dim Z, \\ 0 & \text{if } \dim X = \dim Z, \end{cases}$$

and the difficulty of f is

$$\Phi(f) = \dim F - \tau.$$

We will say that $(d(f), \gamma(f), \Phi(f))$ is the type of f.

To apply inductive arguments we need the following two slicing results, which are consequences of Bertini's theorem and vanishing theorems.

Theorem 1.2.8 ([AW93], Lemma 2.5)**.** *(Vertical slicing) Let $\varphi : X \to Z$ be a local contraction supported by $K_X + \tau L$, with $\tau \geq -1 + \varepsilon\gamma(\varphi)$ and ε a sufficiently small positive rational number. Assume that X has LT singularities and let h be a general function on Z. Let $X_h = \varphi^*(h)$, then the singularities of X_h are not worse than those of X and any section of L on X_h extends to X.*

Proof. The statement about the singularities is just a version of Bertini's Theorem.

The second assertion follows considering the usual exact sequence

$$0 \to \mathcal{O}_X(L - X_h) \to \mathcal{O}_X(L) \to \mathcal{O}_{X_h}(L) \to 0$$

and noting that $H^1(X, L - X_h) = H^1(X, L) = 0$ by Kawamata-Viehweg vanishing.

□

Theorem 1.2.9 ([AW93], Lemma 2.6)**.** *(Horizontal slicing) Let $\varphi : X \to Z$ be a local contraction around $\{F\}$ supported by $K_X + \tau L$. Let $H_i \in |L|$ be generic divisors and $X_k = \cap_1^k H_i$ be a scheme theoretic intersection; assume that $\dim X_k = n - k > 0$ and that $(\tau - k) \geq 0$; note that since X_k is a complete intersection it is $\mathbb{Q}$-Gorenstein, i.e. K_{X_k} is $\mathbb{Q}$-Cartier.*

i) Let $\varphi_{|X_k} = g \circ \varphi_k$ be the Stein factorization of $\varphi_{|X_k} : X_k \to Z$; then $\varphi_k : X_k \to Z_k$ is a proper morphism with connected fibres, around $\{F \cap (\cap_1^k H_i)\}$, supported by $K_{X_k} + (\tau - k)L_{|X_k}$ and Z_k is affine. In particular, if X_k is normal then φ_k is a local contraction.

Assume that X has LT singularities and, if ε is a sufficiently small positive rational number, that $\tau \geq \varepsilon\gamma(\varphi)$ and $k \leq r + 1 - \varepsilon\gamma(\varphi)$.

ii) Outside of the base locus $Bs|L|$, X_k has singularities which are of the same type of the ones of X and any section of L on X_k extends to a section of L on X.

Proof. The first point is clear.

For the second statement it is enough to show that $H^1(X_i, \mathcal{O}_{X_i}) = 0$ for $i \leq r - \varepsilon\gamma(f)$. Using inductively the exact sequence

$$0 \to \mathcal{O}_{X_i}(-L) \to \mathcal{O}_X X_i \to \mathcal{O}_{X_{i+1}} \to 0$$

we are reduced to prove that $H^j(X, -iL) = 0$ for $i \leq r - \varepsilon\gamma(f)$ and $j > 0$, which follows from Kawamata-Viehweg vanishing.

□

We will often apply the following lower bound on the dimension of a fibre.

Theorem 1.2.10 ([And95], Theorem 2.1)**.** *Let $\varphi : X \to Z$ be a local contraction supported by $K_X + \tau L$ and let F be an irreducible component of a general non-trivial fibre.*

If φ is birational, then $\dim F \geq s := \lfloor \tau \rfloor$ *and equality implies $F \cong \mathbb{P}^s$.*

Corollary 1.2.11 (cfr. [BHN13], Lemma 2.1)**.** *Let $\varphi : X \to Z$ be a birational local contraction supported by $K_X + \tau L$ and let F be an irreducible component of a general non-trivial fibre. Let X_G be the Gorenstein locus of X. If $F \cap X_G \neq \emptyset$, then* $\dim F \geq \tau$.

We finally state a basepoint free theorem, which will be crucial for our inductive arguments.

Theorem 1.2.12. *([AW93]) Let $\varphi : X \to Z$ be a local contraction supported by $K_X + \tau L$. If φ is birational and* $\dim F \leq \tau + 1$*, then L is φ-base point free.*

1.2.2 MMP and reductions

We have already seen how adjunction theory and Mori theory are strictly connected. The relation between them is also highlighted in the recent paper [And13], in which Andreatta shows how the minimal model program with scaling is related to the concepts of zero and first reduction. In this subsection we would like to summarize these results.

We will assume that X is terminal and $\mathbb{Q}$-factorial, if it is not otherwise stated.

Definition 1.2.13. Two quasi-polarized pairs (X_1, L_1) and (X_2, L_2) are said to be **birationally equivalent** if there is a variety Y with birational morphisms $\varphi_i : Y \to X_i$ such that $\varphi_1^* L_1 = \varphi_2^* L_2$.

Lemma 1.2.14. *Let (X, L) be a terminal Q-factorial quasi-polarized pair. Let r be a positive rational number. Then there is an effective $\mathbb{Q}$-divisor $\Delta^r \sim_{\mathbb{Q}} rL$ such that (X, Δ^r) is klt.*

Proof. Since L is big and nef, we may write $rL \sim_{\mathbb{Q}} A + E$, where A is ample and E is effective. Choose $\varepsilon > 0$ small enough so that $(X, \varepsilon E)$ is klt. Then

$$rL \sim_{\mathbb{Q}} (1 - \varepsilon)rL + \varepsilon A + \varepsilon E.$$

Since $(1 - \varepsilon)rL + \varepsilon A$ is ample, the lemma follows by Bertini.

□

By [BCHM10] we can run a $K_X + \Delta^r$-MMP for (X, Δ^r)

$$(X_0, \Delta_0^r) = (X, \Delta^r) \dashrightarrow (X_1, \Delta_1^r) \dashrightarrow \cdots \dashrightarrow (X_s, \Delta_s^r)$$

Each $\varphi_i : X_i \dashrightarrow X_{i+1}$ is a birational map (divisorial contraction or flip) associated to an extremal ray R_i. Note that any (X_i, Δ_i^r) is klt and each X_i is terminal $\mathbb{Q}$-factorial. If $K_X + \Delta^r$ is pseudo-effective then $K_{X_s} + \Delta_s^r$ is nef, otherwise (X_s, Δ_s^r) is a Mori fibre space.

The major question is

Question 1.2.15. *Can we describe explicitly the steps of the MMP and the final result?*

When $r \geq (n - 2)$ (and L is ample), a complete answer to this question is given by Theorem 2.1.1.

We start reviewing the case $r \geq n-1$. We show that, for any $i = 0, \ldots, s$, $\Delta^r.R_i = 0$ and that there exists a nef and big Cartier divisor L_i on X_i such that $\varphi_i^* L_{i+1} = L_i$ and $\Delta_i^r \sim_{\mathbb{Q}} rL_i$.

The proof is by induction on i. Assume by contradiction that $L_i.R_i > 0$, that is L_i is φ_i-ample. By adding to L the pull back of a sufficiently ample line bundle from X_{i+1} we can simply assume that L_i is ample. Let F_i be a generic non-trivial fibre of φ_i, then, by Corollary 1.2.11, we get that F is a divisor and $r < n - 1$, a contradiction. Hence $L_i.R_i = 0$ and by Corollary 3.17 of [KM98] there exists a nef and big Cartier divisor L_{i+1} on X_{i+1} such that $\varphi_i^* L_{i+1} = L_i$. In particular each (X_i, L_i) is a quasi-polarized variety.

Definition 1.2.16. Assume that $r = n - 1$. The pair $(X', L') := (X_s, L_s)$ is called a *zero-reduction* of (X, L).

For the zero-reduction the situation is summarized by Theorem 5.1 in [And13], which we report. We first recall the classification of Mori fibre spaces with nef value higher than $n - 2$.

Theorem 1.2.17 ([And13]). *Let X be a normal variety with terminal $\mathbb{Q}$-factorial singularities and L be a nef and big line bundle on X. Let $\varphi_R : X \to Z$ be a Mori fibre space associated with the extremal ray $R = \mathbb{R}^+[C]$ and r be a positive rational number such that $(K_X + rL)\cdot C < 0$. Note that this implies that $\tau(X, L) > r$ (possibly adding to L the pull-back of a sufficiently ample line bundle from Z).*

A) If $r \geq (n-1)$ then (X, L) is one of the following pairs:

- *$(\mathbb{P}^n, \mathcal{O}(1))$ and $r < (n+1)$,*
- *$(Q, \mathcal{O}(1)_{|Q})$, where $Q \subset \mathbb{P}^{n+1}$ is a quadric and $r < n$,*
- *$C_n(\mathbb{P}^2, \mathcal{O}(2))$, a generalized cone over $(\mathbb{P}^2, \mathcal{O}(2))$, and $r < n$,*
- *φ_R gives to X the structure of a $\mathbb{P}^{n-1}$-bundle over a smooth curve C and L restricted to any fibre is $\mathcal{O}(1)$ and $r < n$.*

B) If $r \geq (n-2)$ then (X, L) is one of the following pairs:

- *one of the pair in the previous list,*
- *a del Pezzo variety, that is $-K_X \sim_{\mathbb{Q}} (n-1)L$ with L ample, $r < (n-1)$,*
- *$(\mathbb{P}^4, \mathcal{O}(2))$,*
- *$(\mathbb{P}^3, \mathcal{O}(3))$,*
- *$(Q, \mathcal{O}(2)_{|Q})$, where $Q \subset \mathbb{P}^4$ is a quadric,*
- *φ_R gives to X the structure of a quadric fibration over a smooth curve C and L restricted to any fibre is $\mathcal{O}(1)_{|Q}$, $r < (n-1)$,*
- *φ_R gives to X the structure of a $\mathbb{P}^{n-2}$-bundle over a normal surface S and L restricted to any fibre is $\mathcal{O}(1)$, $r < (n-1)$,*
- *$n = 3$, Z is a smooth curve, the general fibre of φ_R is $\mathbb{P}^2$ and L restricted to it is $\mathcal{O}(2)$.*

Theorem 1.2.18 ([And13]). *Let (X, L) be a quasi-polarized variety such that X has at most terminal $\mathbb{Q}$-factorial singularities.*

1. *$K_X + (n+1)L$ is pseudo-effective and on a zero-reduction (X', L') the $\mathbb{Q}$-Cartier divisor $K_{X'} + (n+1)L'$ is nef.*
2. *If $K_X + nL$ is pseudo-effective, then on a zero-reduction (X', L') the $\mathbb{Q}$-Cartier divisor $K_{X'} + nL'$ is nef.*

   $K_X + nL$ is not pseudo-effective if and only if any zero-reduction (X', L') is $(\mathbb{P}^n, \mathcal{O}(1))$.

3. *If $K_X + (n-1)L$ is pseudo-effective then on a zero-reduction (X', L') the $\mathbb{Q}$-Cartier divisor $K_{X'} + (n-1)L'$ is nef.*

   $K_X + (n-1)L$ is not pseudo-effective if and only if any zero-reduction (X', L') is one of the pairs in 1.2.17 A).

The zero-reduction is useful to contract the curves C on which $-K_X$ is ample and L is not ample. In fact on a zero-reduction, $K_{X'} + (n+1)L'$ is nef and hence $L'.C > 0$ for any curve $C \subset X'$ such that $K_{X'}.C < 0$.

We go now a step further considering the first reduction. Let (X, L) be a quasi-polarized pair with at most terminal $\mathbb{Q}$-factorial singularities and let (X', L') be the zero-reduction of (X, L). Consider a rational number $r' \geq n-2$ and an effective $\mathbb{Q}$-divisor $\Delta^{r'} \sim_{\mathbb{Q}} r'L'$ such that $(X', \Delta^{r'})$ is klt.
We can run a $K_{X'} + \Delta^{r'}$-MMP

$$(X'_0, \Delta_0^{r'}) = (X', \Delta^{r'}) \dashrightarrow (X'_1, \Delta_1^{r'}) \dashrightarrow \cdots \dashrightarrow (X'_{s'}, \Delta_{s'}^{r'}).$$

By induction and by Theorem 2.1.1, we have that each step $\varphi'_i : X'_i \to X'_{i+1}$ is a weighted blow-up of a smooth point with weight $(1, 1, b, \ldots, b)$ and that we have nef and big divisors L'_i on X'_i such that

$$\varphi_i'^* L'_{i+1} = L'_i + bE_i.$$

Thus for each $i = 0, \ldots, s'$ we have a quasi-polarized pair (X'_i, L'_i) with terminal $\mathbb{Q}$-factorial singularities such that $\Delta_i^{r'} \sim_{\mathbb{Q}} r'L'_i$.

Definition 1.2.19. Assume that $r' = n-2$. The pair $(X'', L'') = (X'_{s'}, L'_{s'})$ is called a *first-reduction* of (X, L).

The conclusion is the following.

Theorem 1.2.20. *([And13, Thm. 5.7]) Let (X, L) be a quasi-polarized variety such that X has at most terminal $\mathbb{Q}$-factorial singularities.*

1. *If $K_X+(n-2)L$ is pseudo-effective then on any first-reduction (X'', L'') the divisor $K_{X''} + (n-2)L''$ is nef.*

2. *$K_X + (n-2)L$ is not pseudo-effective if and only if any first-reduction (X'', L'') is one of the pairs in 1.2.17 A) or B).*

We show a nice application of these results for immersed projective varieties.

Corollary 1.2.21. *([And13]) Let $X \subset \mathbb{P}^N$ be a non degenerate projective variety of dimension $n \geq 3$ and of degree d. Assume that $d < 2\mathrm{codim}_{\mathbb{P}^N}(X)+2$. Then either $(X, \mathcal{O}(1))$ is birationally equivalent to one of the quasi-polarized pair in Proposition 1.2.17 A) or the first-reduction of the resolution of X is one of the quasi-polarized varieties in Proposition 1.2.17 B).*

Proof. Let $\pi : \tilde{X} \to X$ be a resolution of the singularities of X and let $\tilde{L} := \pi^* L$. The divisor $\tilde{L}$ is globally generated and $h^0(\tilde{X}, \tilde{L}) \geq N+1$. Take $L_1, \ldots, L_{n-1}$ general members in $|\tilde{L}|$ and let $C := L_1 \cap \ldots \cap L_{n-1}$; Lemma A.2 in [Mel02] says that $(K_{\tilde{X}} + (n-2)\tilde{L}) \cdot C < 0$. By Theorem 0.2 in [BDPP13] this implies that $K_{\tilde{X}} + (n-2)\tilde{L}$ is not pseudo-effective.

By Theorem 1.2.20 we are done.

□

1.3 Weighted projective spaces and blow-ups

1.3.1 Weighted projective spaces

We collect here some basic definitions and known results about weighted projective space, that will be used later on (mainly in chapter 4). We principally follow [IF00].

Let $a_0, \ldots, a_n$ be positive integers and let $S = S(a_0, \ldots, a_n)$ be the graded polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$, graded by $\deg x_i = a_i$. The *weighted projective space* $\mathbb{P}(a_0, \ldots, a_n)$ is defined as

$$\mathbb{P}(a_0, \ldots, a_n) = \mathrm{Proj} S.$$

Let $\mathbb{A}^{n+1}$ be the affine $n+1$-space and let $x_0, \ldots, x_n$ be coordinates for $\mathbb{A}^{n+1}$. Consider the action of $\mathbb{C}^*$ on $\mathbb{A}^{n+1}$ via

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n) \quad \text{for } \lambda \in \mathbb{C}^*.$$

Then

$$\mathbb{P}(a_0, \ldots, a_n) = (\mathbb{A}^{n+1} \backslash \{0\}) / \mathbb{C}^*$$

and $x_0, \ldots, x_n$ are homogeneous coordinates on $\mathbb{P}(a_0, \ldots, a_n)$.

Definition 1.3.1 (Cyclic quotient singularities)**.** Let $r > 0$, $b_1, \ldots, b_n$ integers and let $x_1, \ldots, x_n$ coordinates on $\mathbb{A}^n$. Consider the action of $\mathbb{Z}_r$ on $\mathbb{A}^n$ given by

$$x_i \mapsto \epsilon^{b_i} x_i$$

where ϵ is a primitive r-th root of unity. Denote the quotient by $\mathbb{A}^n / \mathbb{Z}_r(b_1, \ldots, b_n)$ or simply $\mathbb{A}^n / \mathbb{Z}_r$ if there is no ambiguity. A singularity $Q \in X$ is said to be of type $\frac{1}{r}(b_1, \ldots, b_n)$ if (X, Q) is isomorphic to an analytic neighbourhood of the origin of $(\mathbb{A}^n)/\mathbb{Z}_r$.

For any $i = 1, \ldots, n$ let $U_i = \{x_i \neq 0\} \subset \mathbb{P}(a_0, \ldots, a_n)$. Then

$$U_i = \mathrm{Spec}\mathbb{C}[x_0, \ldots, x_n, x_i^{-1}] \cong \mathrm{Spec}\mathbb{C}[u_0, \ldots, \hat{u}_i, \ldots, u_n]^{\mathbb{Z}_{a_i}} = \mathbb{A}^n / \mathbb{Z}_{a_i}$$

where the group $\mathbb{Z}_{a_i}$ acts via

$$u_j \mapsto \varepsilon^{a_j} u_j$$

for $j \neq i$ and for ε a primitive a_i-th root of unity. The coordinates u_j are given by $u_j = x_j/x_i^{a_j/a_i}$ (see [KSC04] for details).

We say that $\mathbb{P}(a_0, \ldots, a_n)$ is *well-formed* if for each i, $\gcd(a_0, \ldots, \hat{a}_i, \ldots, a_n) = 1$. By the following proposition we may often assume that a weighted projective space is well-formed.

Proposition 1.3.2. *Let $\mathbb{P}(a_0, \ldots, a_n)$ be a weighted projective space. Then there are positive integer $b_0, \ldots, b_n$ such that $\mathbb{P}(a_0, \ldots, a_n) \cong \mathbb{P}(b_0, \ldots, b_n)$ and $\mathbb{P}(b_0, \ldots, b_n)$ is well-formed.*

Definition 1.3.3. Let $I \subset S$ be an ideal generated by a regular sequence $\{f_i\}$ of homogeneous elements of S. Define

$$X_I = \mathrm{Proj} S/I \subset \mathbb{P}(a_0, \ldots, a_n).$$

X_I is called a weighted complete intersection of multidegree $\{d_i = \deg f_i\}$. We denote by $X_{d_1,\ldots,d_k} \subset \mathbb{P}$ a sufficiently general element of the family of weighted complete intersections of multidegree $\{d_i\}$.

Note that if X_d in $\mathbb{P}(a_0, \ldots, a_n)$ is such that $d = a_i$ for some i, then $X_d \cong \mathbb{P}(a_0, \ldots, \hat{a}_i, \ldots, a_n)$.

We now look at the singularities of $\mathbb{P}(a_0, \ldots, a_n)$ from a toric point of view. Let $P_i = [0, \ldots, 0, 1, 0, \ldots, 0]$ be the i-th coordinate point. We indicate with $P_{i_1} \cdots P_{i_k}$ the toric stratum determined by $P_{i_1}, \ldots, P_{i_k}$ and by Δ the fundamental simplex of $\mathbb{P}$ (that is the union of all the hyperplane $P_0 \ldots \hat{P}_i \ldots P_n$).

Singularities only appear on Δ and $\mathrm{codim}_{\mathbb{P}}(\mathbb{P}_{sing}) \geq 2$. The vertex P_i is a singular point of type $\frac{1}{a_i}(a_1, \ldots, \hat{a}_i, \ldots, a_n)$. Each generic point P of an edge P_iP_j has an analytic neighbourhood U which is analytically isomorphic to $(0, Q) \in \mathbb{A}^1 \times Y$, where $Q \in Y$ is a singularity of type $\frac{1}{h_{i,j}}(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_n)$. Similar results hold for higher dimensional strata.

Definition 1.3.4. A subvariety X in $\mathbb{P}(a_0, \ldots, a_n)$ of codimension c is well formed if $\mathbb{P}(a_0, \ldots, a_n)$ is well-formed and X contains no codimension $c+1$ singular stratum of $\mathbb{P}(a_0, \ldots, a_n)$.

Lemma 1.3.5. *The hypersurface $X_d \subset \mathbb{P}(a_0, \ldots, a_n)$ is well-formed if and only if $\mathbb{P}(a_0, \ldots, a_n)$ is well-formed and*

$$\gcd(a_0, \ldots, \hat{a}_i, \ldots, a_j, \ldots, a_n) | d$$

for all distinct i, j.

Let $X \subset \mathbb{P}(a_0, \ldots, a_n)$ be a subvariety and let $p : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}(a_0, \ldots, a_n)$ be the canonical projection. The affine cone C_X over X is defined as the completion of $p^{-1}(C)$ in $\mathbb{A}^{n+1}$.

Definition 1.3.6. A subvariety X in $\mathbb{P}(a_0, \ldots, a_n)$ is quasi-smooth if its affine cone C_X is smooth outside its vertex.

If $X \subset \mathbb{P}$ is a quasi-smooth subvariety, then the singularities of X are due to the $\mathbb{C}^*$-action and hence are quotient singularities.

We will often apply the following criterion by Reid to determine whether a quotient singularity is terminal or not.

Theorem 1.3.7 ([Rei87], Thm. 4.11)**.** *A quotient singularity of type $\frac{1}{m}(a_1, \ldots, a_n)$ is terminal if and only if*

$$\frac{1}{m}\sum_i \overline{ka_i} > 1, \qquad \textit{for } k = 1, \ldots, m-1,$$

where $\bar{t}$ denotes the smallest residue of t mod m.

It is also important to recall that if $X_{d_1,\ldots,d_c} \subset \mathbb{P}(a_0, \ldots, a_n)$ is well-formed and quasi-smooth, then by adjunction $\omega_X \cong \mathcal{O}_X(\sum d_i - \sum a_i)$ (see [IF00, 6.14]).

1.3.2 Weighted blow-ups

In this subsection we define and study weighted blow-ups along smooth subvarieties; they are a straightforward generalization of weighted blow-ups of points as defined, for example, in Section 10 of [KM98] or in Section 3 of [Hay99]. This material appeared in a joint paper with M. Andreatta ([AT13]).

Let $\sigma = (a_1, \ldots, a_k, 0, \ldots, 0) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1, \ldots, a_k) = 1$. Let $M = \mathrm{lcm}(a_1, \ldots, a_k)$. We denote by $\mathbb{P}(a_1, \ldots, a_k)$ the weighted projective space with weight $(a_1, \ldots, a_k)$. Let $X = \mathbb{A}^n = \mathrm{Spec}\mathbb{C}[x_1, \ldots, x_n]$ and $Z = \{x_1 = \ldots = x_k = 0\} \subset X$.

Consider the rational map

$$\varphi : \mathbb{A}^n \to \mathbb{P}(a_1, \ldots, a_k)$$

given by $(x_1, \ldots, x_n) \mapsto (x_1^{a_1} : \ldots : x_k^{a_k})$.

Definition 1.3.8. The *weighted blow-up* of X along Z with weight σ is defined as the closure $\overline{X}$ in $\mathbb{A}^n \times \mathbb{P}(a_1, \ldots, a_k)$ of the graph of φ, together with the morphism $\pi : \overline{X} \to X$ given by the projection on the first factor.

The map π is birational and contracts an exceptional irreducible divisor E to Z. Moreover for any point $z \in Z$ we have $\pi^{-1}(z) = \mathbb{P}(a_1, \ldots, a_k)$.

We now describe a convenient affine covering for $\bar{X}$.

One can see that

$$\begin{aligned} U_i &\cong \mathrm{Spec}\mathbb{C}[\bar{x}_1, \ldots, \bar{x}_n]/\mathbb{Z}_{a_i}(-a_1, \ldots, \overset{i\text{-th}}{1}, \ldots, -a_k, 0, \ldots, 0) \\ &\cong \left(\mathrm{Spec}\mathbb{C}[\bar{x}_1, \ldots, \bar{x}_k]/\mathbb{Z}_{a_i}(-a_1, \ldots, \overset{i\text{-th}}{1}, \ldots, -a_k)\right) \times \mathbb{A}^{n-k}. \end{aligned}$$

and

$$\pi_{|U_i} : U_i \ni (\bar{x}_1, \ldots, \bar{x}_n) \mapsto (\bar{x}_1\bar{x}_i^{a_1}, \ldots, \overset{i\text{-th}}{\bar{x}_i^{a_i}}, \ldots, \bar{x}_k\bar{x}_i^{a_k}, \bar{x}_{k+1}, \ldots, \bar{x}_n) \in X.$$

In the affine set U_i, E is defined by $\{\bar{x}_i = 0\}/\mathbb{Z}_{a_i}$ and hence ME is a Cartier divisor on $\overline{X}$.

It is worthy to give a toric description of the above construction. Let M be the free abelian group $\mathbb{Z}^n$ and $N = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{M}, \mathbb{Z})$ its dual. By identifying $m = (m_1, \ldots, m_n) \in M$ with $x_1^{m_1} \cdots x_n^{m_n}$, we refer to M as the lattice of monomials and to N as the lattice of weights. Let $\{e_i\}_{i=1,\ldots,n}$ be the standard basis of of $N_{\mathbb{R}}$ and let $\tau = \sum_{i=1}^n \mathbb{R}_{\geq 0} e_i \subset N_{\mathbb{R}}$ be the positive quadrant.

Let $\sigma = (a_1, \ldots, a_k, 0, \ldots, 0) \in N$ such that $a_i > 0$ and $\gcd(a_1, \ldots, a_k) = 1$. Consider the fan $\Delta(\sigma)$ consisting of the cones

$$\tau_i(\sigma) = \sum_{j \neq i} \mathbb{R}_{\geq 0} e_j + \mathbb{R}\sigma$$

for $i = 1, \ldots, k$, together with all their faces. The weighted blow-up of $X = \mathbb{A}^n$ along $Z = \{x_1 = \ldots = x_k = 0\} \subset X$ with weight σ corresponds to the toric variety defined by $\Delta(\sigma)$ together with the associated birational morphism $\phi : \overline{X} \to X$. This morphism is obtained by gluing the morphisms

$$\phi_i : \mathrm{Spec}\mathbb{C}[\tau_i(\sigma)^\vee \cap M] \to \mathbb{A}^n,$$

for $k = 1, \ldots, k$.

To show that this construction gives the weighted blow-up as defined before, it is enough to check that, for any $i = 1, \ldots, k$, we have

$$\mathbb{C}[z_1, \ldots, z_n]^{\mathbb{Z}_{a_i}} \cong \mathbb{C}[\tau_i(\sigma)^\vee \cap M],$$

where $\mathbb{Z}_{a_i}$ acts on $\mathbb{C}[z_1, \ldots, x_x]$ as

$$z_j \mapsto \varepsilon^{-a_j} z_j \text{ for } j \neq i \text{ and } z_i \mapsto \varepsilon z_i,$$

and ε is a primitive a_j-th root of unity. As usual, $\mathbb{C}[z_1, \ldots, z_n]^{\mathbb{Z}_{a_i}}$ denotes the subring of invariants of $\mathbb{C}[z_1, \ldots, z_n]$ under the action of $\mathbb{Z}_{a_i}$.

This follows by the correspondence

$$z_i^{b_i} \prod_{j \neq i} z_j^{b_j} \mapsto x_i^{(b_i - \sum_{j \neq i} b_j a_j)/a_i} \prod_{j \neq i} x_j^{b_j},$$

since

$$\mathbb{C}[z_1, \ldots, z_n]^{\mathbb{Z}_{a_i}} = \mathbb{C}[z_i^{b_i} \prod_{j \neq i} z_j^{b_j} : b_i - \sum_{j \neq i} a_j b_j \equiv 0 \mod a_i].$$

We can look at weighted blow-ups also from a more algebraic point of view. Define the function

$$\sigma\text{-wt} : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{N}$$

as follows. For a monomial $T = x_1^{s_1} \ldots x_n^{s_n}$ we set $\sigma\text{-wt}(T) := \sum_{i=1}^{k} s_i a_i$. For a polynomial $f = \sum_I \alpha_I T_I$, where $\alpha_I \in \mathbb{C}$ and T_I are monomials, we set

$$\sigma\text{-wt} := \min\{\sigma\text{-wt}(T_I) : \alpha_I \neq 0\}.$$

Definition 1.3.9. Let $\sigma = (a_1, \ldots, a_k, 0, \ldots, 0) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1, \ldots, a_k) = 1$. For any $d \in \mathbb{N}$ we define the σ-weighted ideal of degree d as

$$I_{\sigma,d} = \{g \in \mathbb{C}[x_1, \ldots, x_n] : \sigma\text{-wt}(g) \geq d\} = (x_1^{s_1} \cdots x_n^{s_n} : \sum_{j=1}^{k} s_j a_j \geq d).$$

Lemma 1.3.10. *Let $\pi : \overline{X} \to X$ be the weighted blow-up of $X = \mathbb{A}^n$ along $Z = \{x_1 = \ldots = x_k = 0\}$ with weight $\sigma = (a_1, \ldots, a_k, 0, \ldots, 0)$. Then*

$$\pi_* \mathcal{O}_X(-dE) = I_{\sigma,d}.$$

Therefore

$$\overline{X} = \operatorname{Proj} \bigoplus_{d \geq 0} I_{\sigma,d}.$$

Proof. Let $\{U_i\}$ be the standard affine covering of $\overline{X}$.

As the exceptional divisor E is effective, we have that $J := \pi_* \mathcal{O}_X(-dE) \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal. A polynomial $g(x_1, \ldots, x_n)$ is an element of J if and only if for any $1 \leq i \leq k$ we have that $g(\bar{x}_1, \ldots, \bar{x}_n) \in \Gamma(U_i, \mathcal{O}_X(-dE))$. Since E is defined by $\{\bar{x}_i = 0\}/\mathbb{Z}_{a_i}$ on the affine subset U_i, we have that $\bar{x}_i^d$ divides $g(\bar{x}_1, \ldots, \bar{x}_n)$ if and only if $\sigma\text{-wt}(g) \geq d$, and the lemma follows. $\square$

We now look more carefully at the blows-ups we will encounter in Chapter 2.

Proposition 1.3.11. *Let $\pi : \overline{X} \to X$ be the weighted blow-up of $p = (0, \ldots, 0) \in X = \mathbb{A}^n$ with weight $\sigma = (1, a, b, c, \ldots, c)$, where $(a, b) = 1$ and $ab|c$. Then*

$$\pi_* \mathcal{O}_X(-dcE) = \pi_* \mathcal{O}_X(-cE)^d$$

and

$$\overline{X} = \operatorname{Proj} \bigoplus_{d \geq 0} I_{\sigma,c}^d.$$

Proof. Let $c = kab$ with $k \in \mathbb{N}$.

It suffices to prove that for every integer $d \geq 1$ the natural map

$$\pi_*\mathcal{O}_X(-(d-1)cE) \otimes \pi_*\mathcal{O}_X(-cE) \to \pi_*\mathcal{O}_X(-dcE)$$

is surjective. For $d = 1$ there is nothing to prove, so we assume $d \geq 2$.

Let $g = x_1^{s_1} \cdots x_n^{s_n} \in \pi_*\mathcal{O}_X(-dcE) = (x_1^{s_1} \cdots x_n^{s_n} : s_1 + s_2a + s_3b + \sum_{i=4}^k s_ic \geq dc)$. We claim that there exists $h = x_1^{t_1} \cdots x_n^{t_n} \in \pi_*\mathcal{O}_X(-cE)$ such that $t_1 + t_2a + t_3b + \sum_{i=4}^n t_ic = c$ and $t_i \leq s_i$ for all $1 \leq i \leq n$. In fact, if there is $j \in \{4, \ldots, k\}$ such that $s_j \neq 0$, then just set $h = x_j$. If $s_j = 0$ for $j = 4, \ldots, k$, then $s_1 + s_2a \geq c$ or $s_3b \geq c$. In the first we can set $t_2 = \min\{s_2, c/a\}$ and $t_1 = c - t_2a$. In the second case the claim follows setting $h = x_3^{c/b}$.

Let $k = g \cdot h^{-1}$, then $k \in \pi_*\mathcal{O}_X(-(d-1)bE)$ and $g = k \cdot h$.

The second part of the statement is a consequence of the first part and Lemma 1.3.10. □

We remark that the previous Proposition does not hold for any weight σ as the following example shows. Nevertheless, since the algebra

$$\bigoplus_{d\geq 0} I_{\sigma,b}^d$$

is finitely generated, there is always a positive integer L such that

$$I_{\sigma,L}^d = I_{\sigma,dL}$$

for any $d \in \mathbb{N}$.

Example 1.3.12. Let $Z = \{x_1 = x_2 = x_3 = 0\} \subset X = \mathbb{A}^n$ and $\sigma = (10, 14, 35, 0, \ldots, 0)$. Let $\pi : \overline{X} \to X$ be the weighted blow-up of X along Z with weight σ. Consider

$$g = x_1^5 x_2^4 x_3 \in \pi_*\mathcal{O}_X(-2ME),$$

where $M = \mathrm{lcm}(2, 5, 7) = 70$; note that σ-wt$(g) = 141$. It is easy to check that there is no triple $(t_1, t_2, t_3) \in \mathbb{N}^3$ such that $10t_1 + 14t_2 + 35t_1 = 70$ and $t_1 \leq 5$, $t_2 \leq 4$, $t_3 \leq 1$ and hence

$$\pi_*\mathcal{O}_X(-140E) \neq \pi_*\mathcal{O}_X(-70E)^2.$$

The following is a simple application of Reid's criterion 1.3.7.

Lemma 1.3.13. *Let $\pi : \overline{X} \to X$ be the weighted blow-up of $p = (0, \ldots, 0) \in X = \mathbb{A}^n$ with weight $\sigma = (1, a, b, c, \ldots, c)$, where $(a, b) = 1$ and $ab|c$. Then $\overline{X}$ has terminal singularities.*

Proof. By the description of the standard open cover $\{U_i\}_{i=1,\dots,n}$ (see for instance [KM98, 10.3]) we need to check that the following quotient singularities are terminal:

$$\frac{1}{a}(-1,1,-b,-c\dots,-c),\quad \frac{1}{b}(-1,-a,1,-c\dots,-c),\quad \frac{1}{c}(-1,-a,-b,1,-c\dots,-c).$$

We apply Reid's criterium 1.3.7. The first two cases are immediate. For the third case note that

$$\frac{1}{c}(\overline{-k}+\overline{-ka}+\overline{-kb}+\overline{k}) = 1+\frac{1}{c}(\overline{-ka}+\overline{-kb}) \geq 1 \qquad \text{for } k=1,\dots,c-1$$

and the equality is impossible: in fact we have equality only if $c|ka$ and $c|kb$ and, since $c = sab$ for a positive integer s, this implies $sa|k$ and $sb|k$. As a and b are coprime we deduce that $sab|k$, which is a contradiction. □

We now define the symbolic powers of an ideal and check that σ-weighted ideals behave well with respect to symbolic powers.

Definition 1.3.14. Given an ideal $I \subset R$ in a Noetherian ring R and $t \in \mathbb{N}$, the t-th symbolic power $I^{(t)}$ of I is defined as the restriction of $I^t R_S$ to R, where S is the complement of the union of the minimal associated primes of I and R_S is the localization of R at the multiplicative system S.

If I is a prime ideal, then the definition of symbolic power is for instance given in [Laz04, Definition 9.3.4] or in [AM69, Exercise 4.13]. Note that, by definition, $I^t \subset I^{(t)}$; in general the inclusion might be strict.

Lemma 1.3.15. *Let $\sigma = (a_1,\dots,a_k,0,\dots,0) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1,\dots,a_k) = 1$, and let L be a positive integer such that*

$$I^d_{\sigma,L} = I_{\sigma,dL}.$$

for any $d \in \mathbb{N}$, where $I_{\sigma,d}$ is the σ-weighted of degree d. Set $I = I_{\sigma,L}$.

Then for any $t \in \mathbb{N}$ we have $I^t = I^{(t)}$.

Proof. We will use the fact that if $f,g \in R$ then $\sigma\text{-wt}(fg) = \sigma\text{-wt}(f) + \sigma\text{-wt}(g)$.

We first show that I is primary: if $fg \in I$ then $\sigma\text{-wt}(f) \geq 1$ or $\sigma\text{-wt}(g) \geq 1$ and hence $f^m \in I$ or $g^m \in I$ for m big enough. Then the only prime associate to I is its radical ideal $r(I)$, which is $r(I) = (x_1,\dots,x_k)$ since there is always a power of x_i in I for $1 \leq i \leq k$.

Let now $S = R\backslash r(I)$. By Proposition 3.11 in [AM69] we have $I^{(t)} = \bigcup_{s\in S}(I^t : s)$. Using the fact that $\sigma\text{-wt}(s) = 0$, we have that for any $s \in S$

$$(I^t : s) = \{g \in R : \sigma\text{-wt}(gs) \geq tL\} = \{g \in R : \sigma\text{-wt}(g) \geq tL\} = I^t.$$

□

The definition of weighted blow-up in 1.3.8 depends on the local coordinates chosen. To construct a global weighted blow-up along a subvariety Z of a complete variety X one needs to patch together weighted blow-ups defined on a covering of X, in such a way that the local coordinates preserve the weight. We propose the following.

Definition 1.3.16. Let X be a smooth variety and Z a smooth subvariety of codimension k and let $\sigma = (a_1, \dots, a_k, 0, \dots, 0) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1, \dots, a_k) = 1$. Let $\mathcal{I}_{\sigma,d}$ be ideal sheaves on X such that there is an covering $\{U_i \cong \mathbb{D}^n\}_{i \in I}$ on X so that for any $i \in I$ there are coordinates $x_1, \dots, x_n$ on U_i for which $Z \cap U_i = \{x_1 = \dots = x_k = 0\}$ and $\Gamma(U, \mathcal{I}_{\sigma,d}) = \{g \in \mathbb{C}[x_1, \dots, x_n] : \sigma\text{-wt}(g) \geq d\}$. A *weighted blow-up* of X along Z with weight σ is the projectivization

$$\pi : \overline{X} = \operatorname{Proj} \bigoplus_{d \geq 0} \mathcal{I}_{\sigma,d} \to X.$$

We call $\mathcal{I}_{\sigma,d}$ a *σ-weighted ideal sheaf* of degree d for Z in X.

Let $Z \subset X$ be a smooth subvariety of codimension k in the regular locus of a variety X; let also $\sigma = (a_1, \dots, a_k, 0, \dots, 0)$ be a weight. The question about the existence of a *σ-weighted ideal* for Z in X, and therefore of a weighted blow-up of X along Z with weight σ, is not clear. In general it seems a difficult problem to find sufficient conditions for a positive answer.

However, if $Z \subset \mathbb{P}^n$ is a complete intersection, then for any weight $\sigma = (a_1, \dots, a_k, 0, \dots, 0)$ there exists a weighted blow-up along Z with weight σ. In fact, let $F_1, \dots, F_k$ be a regular sequence of homogeneous polynomials generating the ideal $I_Z \subset R = \mathbb{C}[x_0, \dots, x_n]$ of Z. We need the following standard fact to define a weight function σ-wt on $\mathbb{C}[x_0, \dots, x_n]$.

Proposition 1.3.17. *([Mat80, Thm. 27]) Let R be a ring and let $f_1, \dots, f_k \in R$ be a regular sequence. Consider the map*

$$\psi : R[x_1, \dots, x_k] \to \operatorname{gr}_I R = \bigoplus_{j \in \mathbb{N}} I^j / I^{j+1}$$

which sends an homogeneous polynomial $G[x_1, \dots, x_n] \in R[x_1, \dots, x_k]$ of degree j to the image of $G(f_1, \dots, f_n)$ in I^j/I^{j+1}. Then ψ induces a map

$$\phi : (R/I)[x_1, \dots, x_k] \to \operatorname{gr}_I R = \bigoplus_{j \in \mathbb{N}} I^j / I^{j+1}$$

which is an isomorphism of graded rings.

Let now $g \in \mathbb{C}[x_1, \dots, x_n]$. If $g \notin I_Z$ then just set $\sigma\text{-wt}(g) = 0$. If $g \in I_Z$, then consider $\bar{g}$, the image of g in $\operatorname{gr}_I R$, and write

$$\bar{g} = \sum_{\beta \in \mathbb{N}^k} h_\beta \bar{F}_1^{\beta_1} \cdots \bar{F}_k^{\beta_k},$$

where $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{N}^k$, $h_\beta \in R/I$ and $\bar{F}_1, \dots, \bar{F}_k$ are the images of $F_1, \dots, F_k$ in $\mathrm{gr}_I R$. This writing is unique by the previous proposition, and hence

$$\sigma\text{-wt}(g) := \min_\beta \left\{ \sum_{i=1}^k a_i \beta_i : h_\beta \neq 0 \right\}$$

is a well defined function and it is a valuation on $\mathbb{C}[x_0, \dots, x_k]$, that is $\sigma\text{-wt}(gh) = \sigma\text{-wt}(g) + \sigma\text{-wt}(h)$ and $\sigma\text{-wt}(g+h) \geq \min\{\sigma\text{-wt}(g), \sigma\text{-wt}(h)\}$.

Then, finally, for any $d \in \mathbb{N}$, we may define the σ-weighted ideal sheaf $\mathcal{I}_{\sigma,d}$ setting

$$\Gamma(U, \mathcal{I}_{\sigma,d}) = \left\{ \frac{f}{g} : f, g \text{ are homogeneous, } g_{|U} \not\equiv 0 \text{ and } \sigma\text{-wt}(f) - \sigma\text{-wt}(g) \geq d \right\}$$

for any open subset $U \subset \mathbb{P}^n$.

Chapter 2

Classification of contractions

This chapter collects the results of the joint paper [AT13] and of a work in progress with M. Andreatta.

Let X be a terminal $\mathbb{Q}$-factorial projective variety and let L be an ample Cartier divisor on X. We recall that contractions of fibre type with nef value higher than $n-2$ are completely classified (see 1.2.17), as well contractions supported by $K_X + (n-2)L$ (see [Mel97]).

In the **first section** of this chapter we will complete the description of the extremal rays $R = \mathbb{R}^+[C]$ contained in the cone $\overline{NE(X)}_{(K_X+(n-2)L)<0}$ whose associated contractions are birational (Theorem 2.1.1). Then we will classify divisorial contractions associated to extremal rays R such that $R.(K_X + rL) < 0$, where r is a non-negative integer, and the fibres of f have dimension less or equal to $r+1$ (Theorem 2.1.2).

In section 2.2 we study birational Fano-Mori contractions $f : X \to Z$ supported by $K_X + \tau L$, where $\tau > n-3$. Let F be an irreducible component of a general non trivial fibre of f. By Corollary 1.2.11 there are four possibilities:

1. f is a divisorial contraction to a point $p \in Z$,
2. f is a divisorial contraction to a curve $C \subset Z$, F is normal, $\dim F = n-2$ and $\Delta(F, L) = 0$,
3. f is a small contraction to a point, F is normal, $\dim F = n-2$ and $\Delta(F, L) = 0$,
4. f is a small contraction to a point, F is contained in the singular locus of X and $F \cong \mathbb{P}^{n-3}$.

In the first case, we can prove Theorem 2.2.11, which is an application of Theorem 2.2.7 and in which we show how to extend Kawakita contractions in higher dimension under suitable hypotheses. This is the content of **subsections 2.2.1** and **2.2.2**.

The second case is already included in Theorem 2.1.2.

In **subsection 2.2.3** we will start the study of the third case. It is plausible that the forth case does not happen, but we do not have a proof of this fact.

In the **last subsection** we see that an MMP for a klt pair (X, Δ) such that $\Delta \sim rL$, L is nef and big and r is close enough to $n-2$ is just the classical second reduction of (X, L).

2.1 Contractions with $\tau > n - 2$

The results of this subsection are the content of a joint paper with M. Andreatta ([AT13]).

Theorem 2.1.1. *([AT13, Thm. 1.1]) Let X be a normal projective variety with $\mathbb{Q}$-factorial terminal singularities and let L be an ample Cartier divisor on X. Let R be an extremal ray in $\overline{NE(X)}_{(K_X+(n-2)L)<0}$ and let $f : X \to Z$ be its associated contraction. Assume that f is birational. Then f is a weighted blow-up of a smooth point with weight $\sigma = (1, 1, b, \dots, b)$, where b is a positive integer (see Definition 1.3.8).*

If $n = 3$ the Theorem follows from the results in [And13] and the main Theorem in [Kaw01]; our proof is however independent of [Kaw01].

Proof. Let F_1 be a non trivial fibre of f. We pass to a local set-up, i.e. we assume that $f : X \to Z$ is a local F-M contraction around F_1 supported by $K_X + \tau L$, where τ is the nef value of the pair (X, L), a positive rational number greater than $n - 2$. Let F be a component of F_1. By Corollary 1.2.11, we get that $\dim F \geq \tau > n - 2$; this means that $\dim F = n - 1$ and hence the exceptional locus of f has codimension 1. Since the exceptional locus of a divisorial contraction is irreducible we conclude that F is the exceptional divisor, and f is the contraction of F to a point $p \in Z$.

Since f is a K_X-negative contraction, Z is terminal and $\mathbb{Q}$-factorial (see [KM98, Corollary 3.43]). Proposition 3.6 of [And13] says that p is smooth. For the reader's convenience we recall that proof. More precisely, by induction on $n \geq 2$, we prove that p is smooth in Z and that

$$K_X = f^*K_Z + ((n-2)b + 1)F \quad \text{and} \quad \tau = n - 2 + \frac{1}{b} \tag{2.1}$$

for a positive integer b such that $f^*f_*L = L + bF$.

If $n = 2$, X is smooth and in this case we can apply Castelnuovo's theorem, which says that f is the contraction of a (-1)-curve F to a smooth surface Z, therefore $K_X = f^*K_Z + F$. Note that $L_1 := f_*L$ is a Cartier divisor on Z and there is a positive integer b such that $L = f^*L_1 - bF$. From $0 = (K_X + \tau L).F = (K_X - \tau bF).F = -1 + \tau b$, we get $\tau = 1/b$.

Let $n \geq 3$ and pick a general member $X' \in |L|$: by Theorem 1.2.12 and Bertini's theorem it has terminal $\mathbb{Q}$-factorial singularities. Consider the restricted morphism $f' := f_{|X'} : X' \to Z'$; by Lemma 1.2.9 it is a divisorial contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$. By inductive assumption, $p \in Z'$ is smooth; by [Mel97, Lemma 1.7] we conclude that $p \in Z$ is smooth, $L_1 := f_*L$ is Cartier and $L = f^*L_1 - bF$ for a positive integer b. Denoting by F' the exceptional divisor of f', by induction we have

$$K_{X'} = f^*K_{Z'} + ((n-3)b+1)F' \quad \text{and} \quad \tau - 1 = n - 3 + \frac{1}{b},$$

from which (2.1) follows.

Let $X' \in |L|$ be again a general element and $f' := f_{|X'} : X' \to Z'$ be the restricted morphism. Since Z and $Z' = f_*X'$ are smooth at p we may choose local coordinates $x_1, \dots, x_n$ around p such that $Z \cong \mathbb{D}^n$ and $f_*X' = \{x_n = 0\}$.

Note that $\mathcal{O}_X(-bF)$ is f-ample and that the map f is proper; thus we have that

$$X = \mathrm{Proj}(\oplus_{d \geq 0} f_*\mathcal{O}_X(-dbF)).$$

By Lemma 1.3.10, X will be the weighted blow-up we are looking for if

$$f_*\mathcal{O}_X(-dbF) = I_{\sigma,d} = (x_1^{s_1} \cdots x_n^{s_n} : s_1 + s_2 + \sum_{j=3}^{n} bs_j \geq db).$$

The proof of this is by double induction on n and d, starting with $n = 2$ and $d = 0$.

Consider the exact sequence

$$0 \to \mathcal{O}_X(-L - dbF) \to \mathcal{O}_X(-dbF) \to \mathcal{O}_{X'}(-dbF) \to 0.$$

Note that

$$-L - dbF \sim_f -(d-1)bF \sim_f K_X + (n - 3 + d + \frac{1}{b})L.$$

Hence, pushing down to Z the above exact sequence and applying the relative Kawamata-Viehweg Vanishing, we have

$$0 \to f_*\mathcal{O}_X(-(d-1)bF) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dbF) \to f_*\mathcal{O}_{X'}(-dbF) \to 0. \qquad (2.2)$$

By induction on n, we can assume that

$$f_*\mathcal{O}_{X'}(-dbF) = (x_1^{s_1} \cdots x_{n-1}^{s_{n-1}} : s_1 + s_2 + \sum_{j=3}^{n-1} bs_j \geq db)$$

where $s_j \in \mathbb{N}$. The case $n = 2$ follows from Castelnuovo's theorem. By induction on d, we can also assume that

$$f_*\mathcal{O}_X(-(d-1)bF) = (x_1^{s_1} \cdots x_n^{s_n} : s_1 + s_2 + \sum_{j=3}^{n} bs_j \geq (d-1)b),$$

the case $d = 0$ being trivial.

Let $g = x_1^{s_1} \cdots x_n^{s_n} \in f_*\mathcal{O}_X(-dbF)$ be a monomial.

If $s_n \geq 1$ then g, looking at the sequence (2.2), comes from $f_*\mathcal{O}_X(-(d-1)bF)$ by the multiplication by x_n; therefore

$$s_1 + s_2 + \sum_{j=3}^{n-1} s_j b + s_n b \geq (d-1)b + s_n b \geq db.$$

If $s_n = 0$, then $g \in f_*\mathcal{O}_{X'}(-dbF)$ and so

$$s_1 + s_2 + \sum_{j=3}^{n} s_j b = s_1 + s_2 + \sum_{j=3}^{n-1} s_j b \geq db.$$

The non-monomial case follows immediately.

□

We can include the above Theorem in a more general statement regarding a ray $R = \mathbb{R}^+[C]$ contained in the cone $\overline{NE(X)}_{(K_X+rL)<0}$, where r is a non-negative integer, whose associated contraction is divisorial (i.e. it is birational and its exceptional locus is a divisor) with all fibres of dimension less or equal to $r+1$. That is we prove the following Theorem, which is a generalization of Theorem 4.9 in [KM92] (see also Theorem 3.2 in [And95]).

Theorem 2.1.2. *([AT13, Thm. 1.2]) Let X be a normal projective variety with $\mathbb{Q}$-factorial terminal singularities and let L be an ample Cartier divisor on X. Let R be an extremal ray in $\overline{NE(X)}_{(K_X+rL)<0}$ where $r \in \mathbb{N}$ is a non-negative integer and let $f : X \to Z$ be its associated contraction. Assume that f is divisorial and that all fibres have dimension less or equal to $r+1$. Let E be the exceptional locus of f and set $C := f(E) \subset Z$.*

1. *Then $\mathrm{codim}_Z C = r+2$, there is a closed subset $S \subset Z$ of codimension al least 3 such that $Z' = Z\backslash S$ and $C' = C\backslash S$ are smooth, and $f' : X' = X\backslash f^{-1}(S) \to Z'$ is a weighted blow-up along C' with weight $\sigma = (1, 1, b, \ldots, b, 0, \ldots, 0)$, where the number of b's is r (see Definitions 1.3.8 and 1.3.16).*

2. *Let $\mathcal{I}'$ be a σ-weighted ideal sheaf of degree b for $Z' \subset X'$ (see Definition 1.3.16) and let $i : Z' \to Z$ be the inclusion; let also $\mathcal{I} := i_*(\mathcal{I}')$ and $\mathcal{I}^{(m)}$ be the m-th symbolic power of $\mathcal{I}$ (see Definition 1.3.14). Then $X = \mathrm{Proj} \bigoplus_{m \geq 0} \mathcal{I}^{(m)}$.*

To prove Theorem 2.1.2 we need some preliminary lemmas.

The following is a local version of Theorem 2.1.2 around a general fibre.

Lemma 2.1.3. *Let $f : X \to Z$ be a local contraction supported by $K_X + \tau L$, where X is terminal $\mathbb{Q}$-factorial and L is an f-ample Cartier divisor. Assume that f is divisorial and let E be the exceptional divisor. Set $C = F(E) \subset Z$. Assume also that there exists a positive integer r such that $\tau > r$ and that any fibre of f has dimension less or equal to $r+1$. Let F be a general non-trivial fibre. Then $f(F)$ is a smooth point and we may assume that locally in $f(F)$ there are analytic coordinates $x_1, \dots, x_n$ such that $C = \{x_1 = \dots = x_{r+2} = 0\}$ and f is a weighted blow-up along C with weight $\sigma = (1, 1, b, \dots, b, 0 \dots, 0)$, where b is a positive integer and the number of b's is r.*

Proof. By the assumptions and by Corollary 1.2.11, we gain that $\dim F = r+1$; therefore $\dim C = n - r - 2$.

First we prove that $p = f(F)$ is a smooth point of Z. Take $n-r-2$ general functions $h_j \in H^0(X, \mathcal{O}_X)$, $j = 1, \dots, n-r-2$ and let $X_j \subset X$ be the divisor defined by h_j. By Lemma 1.2.8, setting $X" = \bigcap_{j=1}^{n-r+2} X_j$, we have that $f" := f_{|X"} : X" \to Z"$ is a local contraction supported by $K_{X"} + \tau L_{X"}$, it is birational, it contracts a divisor F to the point $p = C \cap Z"$ and $\tau > r = \dim X" - 2$. Note that p is general in C and hence F is a general non-trivial fibre of f. The contraction $f" : X" \to Z"$ satisfies the assumption of Theorem 2.1.1 and hence we have that $f"$ is a weighted blow-up of a smooth point with weight $(1, 1, b, \dots, b)$, where b is a positive integer.

Therefore we may assume that $Z" = f"(X")$ is smooth at p. Since $Z"$ is an intersection of Cartier divisors in Z, we conclude that Z is smooth at p.

The proof is now by induction on the dimension of F, i.e. on $\dim F = r+1$; for this we apply Lemma 1.2.9.

Assume $\dim F = 1$. Since X has terminal singularities, which are in codimension 3, F is contained in the smooth locus of X, and hence, in the local set-up, we may assume that X is smooth. Therefore X is a smooth blow-up (see for instance Corollary 4.11 in [AW93]): i.e. f is, locally around p, the blow-up of $C = \{x_1 = x_2 = 0\}$ with weights $(1, 1, 0 \dots, 0)$; in particular we have that

$$K_X = f^* K_Z + E.$$

If $\dim F = r + 1 \geq 2$, let X' be a general element in $|L|$. By Theorem 1.2.12 and Bertini's theorem, X' has terminal $\mathbb{Q}$-factorial singularities. Consider the contraction $f' := f_{|X'} : X' \to Z'$; by Lemma 1.2.9, it is a divisorial contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$ with fibres of dimension equal to r. We have already proved that Z and Z' smooth in p, therefore, by induction, we may assume that locally $C = \{x_1 = \dots = x_{r+2} = 0\} \subset Z' = \{x_{r+2} = 0\} \subset Z$ and that f' is the smooth blow-up along C.

Let L_1 be the Cartier divisor f_*L; we have $L = f^*L_1 - bE$ for a positive integer b and bE is a Cartier divisor.

Reasoning as in the proof of Theorem 2.1.1, by horizontal slicing and by induction on r, we get the formulae

$$K_X = f^*K_Z + (rb+1)E \quad \text{and} \quad \tau = r + \frac{1}{b}.$$

Since $\mathcal{O}_X(-bE)$ is f-ample we have

$$X = \mathrm{Proj} \oplus_{d\geq 0} f_*\mathcal{O}_X(-dbE).$$

By Lemma 1.3.10 we have to show that

$$f_*\mathcal{O}_X(-dbE) = I_{\sigma,d} = (x_1^{s_1}\cdots x_n^{s_n} : s_1 + s_2 + \sum_{j=3}^{r+2} s_j b \geq db).$$

The proof is by double induction on $r \geq 0$ and $d \geq 0$, and it is similar to the proof of Theorem 2.1.1.

Consider the exact sequence

$$0 \to \mathcal{O}_X(-L - dbE) \to \mathcal{O}_X(-dbE) \to \mathcal{O}_{X'}(-dbE) \to 0.$$

Note that

$$-L - dbE \sim_f K_X + (r + (d-1) + \frac{1}{b})L$$

and hence, by the Relative Kawamata-Viehweg Vanishing, we have

$$0 \to f_*\mathcal{O}_X(-(d-1)bE) \overset{\cdot x_{r+2}}{\to} f_*\mathcal{O}_X(-dbE) \to f_*\mathcal{O}_{X'}(-dbE) \to 0. \quad (2.3)$$

By induction on r we can assume that

$$f_*\mathcal{O}_{X'}(-dbE) = (x_1^{s_1}\cdots x_{r+1}^{s_{r+1}} x_{r+3}^{s_{r+3}} \cdots x_n^{s_n} : s_1 + s_2 + \sum_{j=3}^{r+1} s_j b \geq db),$$

where $s_j \in \mathbb{N}$. We have already treated above the case $r = 0$. By induction on $d \geq 0$, we can assume that

$$f_*\mathcal{O}_X(-(d-1)bE) = (x_1^{s_1}\cdots x_n^{s_n} : s_1 + s_2 + \sum_{j=3}^{r+2} s_j b \geq (d-1)b),$$

the case $d = 0$ being trivial.

Let $g = x_1^{s_1}\cdots x_n^{s_n} \in f_*\mathcal{O}_X(-dbE)$.

If $s_{r+2} \geq 1$ then, looking at the sequence (2.3), g comes from $f_*\mathcal{O}_X(-(d-1)dE)$ by the multiplication by x_{r+2} and so

$$s_1 + s_2 + \sum_{j=3}^{r+1} s_j b + s_{r+2} b \geq (d-1)b + s_{r+2} \geq db.$$

Otherwise $g \in f_*\mathcal{O}_{X'}(-bdE)$ and it satisfies

$$s_1 + s_2 + \sum_{j=3}^{r+1} s_j b = s_1 + s_2 + \sum_{j=3}^{r} s_j b \geq db.$$

□

Proof of Theorem 2.1.2. First notice that, as in the first line of the proof of 2.1.1, $\dim C = n - r - 2$.

We proceed as in [KM92, Theorem 4.9].

By Lemma 2.1.3, $C \cap \mathrm{Sing}(Z) \subsetneq C$; moreover since Z has terminal singularities, $\mathrm{Sing}(Z)$ has codimension at least three. Therefore we can find a codimension three closed subset $S \subset Z$ such that $Z' = Z \backslash S$ is smooth and $f^{-1}(S)$ has codimension at least two in X. Moreover, by Lemma 2.1.3, we may assume that $X' = X \backslash f^{-1}(S)$ is a weighted blow-up of Z' along $C' = C \backslash S$ with weight $\sigma = (1, 1, b, \ldots, b, 0, \ldots, 0) \in \mathbb{N}^n$. This proves (i).

To prove (ii), since $X = \mathrm{Proj} \bigoplus_{m \geq 0} f_*\mathcal{O}_X(-mbE)$, we need to show that

$$f_*\mathcal{O}_X(-mbE) = \mathcal{I}^{(m)}. \tag{2.4}$$

Note that by Proposition 1.3.11 we have

$$f_*(\mathcal{O}_X(-mbE))_{|Z'} = (\mathcal{I}_{|Z'})^m.$$

By definition of symbolic power, and using the fact proved in Lemma 1.3.15 that $(\mathcal{I}_{|Z'})^m = (\mathcal{I}_{|Z'})^{(m)} = (\mathcal{I}^{(m)})_{|Z'}$, we obtain

$$i_*((\mathcal{I}^{(m)})_{|Z'}) = \mathcal{I}^{(m)}.$$

Therefore 2.4 follows by

$$i_*(f_*(\mathcal{O}_X(-mbE))_{|Z'}) = f_*(\mathcal{O}_X(-mbE)),$$

which is a consequence of the following general fact.

Lemma 2.1.4. *Let $f : U \to V$ be a proper morphism. Let $S \subset V$ be a closed subset such that the codimension of $f^{-1}(S)$ in U is at least two. Let $\mathcal{F}$ be a sheaf that satisfies Serre's condition S_2 (e.g. U is normal and $\mathcal{F}$ is reflexive). Then $f_*\mathcal{F} = i_*(f_*\mathcal{F}_{|V \backslash S})$, where $i : V \backslash S \to V$ is the injection.*

□

2.2 Contractions with $\tau > n-3$

The results of this section are the content of a work in progress with M. Andreatta.

2.2.1 Existence of good sections

In this subsection X is lt and $f : X \to Z$ is a local adjoint contraction around a fibre F, supported by $K_X + \tau L$ where L is an ample Cartier divisor on X and τ is a rational number.

The following lemma is an immediate consequence of the Nadel vanishing theorem.

Lemma 2.2.1. *Let $D \equiv_f \beta L$ be a $\mathbb{Q}$-divisor such that (X, D) is lc and let $W \in CLC(X, D)$ be a minimal centre. Assume that $\tau - \beta > -1$, or that $\tau - \beta \geq -1$ if f is birational. Then the restriction map*

$$H^0(X, L) \to H^0(W, L_{|W})$$

is surjective.

Proof. By the tie-breaking technique (see [Kol97b, Prop. 8.7.1]), we may assume that W is an exceptional lc centre and hence $I_W = \mathcal{J}(D)$, where I_W is the ideal sheaf of W and $J(D)$ is the multiplier ideal of D. Consider the exact sequence

$$0 \to \mathcal{O}_X(L) \otimes \mathcal{I}_W \to \mathcal{O}_X(L) \to \mathcal{O}_W(L_{|W}) \to 0.$$

Since $L - (K_X + D) \equiv (1 + \tau - \beta)L$ is nef and big, we can apply Nadel vanishing [Laz04, Thm. 9.4.17] to obtain

$$H^1(\mathcal{O}_X(L) \otimes \mathcal{I}_W) = 0,$$

from which the conclusion follows. □

Lemma 2.2.2. *Let $D \equiv_f \beta L$ be a $\mathbb{Q}$-divisor such that (X, D) is lc and let $W \in CLC(X, D)$ be a minimal centre. Assume that $\tau - \beta > -1$, or that $\tau - \beta \geq -1$ if f is birational; assume also that one of the following conditions is satisfied:*

(i) $\dim W \leq 2$,

ii) $\dim W \geq 3$ *and* $\tau - \beta > \dim W - 3$.

Then $H^0(W, L_{|W}) \neq 0$.

Proof. By subadjunction formula (see Theorem 1.2 of [FG12]), there is an effective $\mathbb{Q}$-divisor D_W such that (W, D_W) is klt and

$$K_W + D_W \equiv (K_X + D)_{|W} \equiv -(\tau - \beta)L_{|W}.$$

If $\dim W \leq 2$, then we conclude by Theorem 3.1 of [Kaw00].

If $\dim W \geq 3$, then (W, D_W) is a log Fano variety of index $i(W, D_W) > \dim W - 3$ and the result follows by the main Theorem in [Amb99].

□

The next is an immediate corollary of the above two lemmas; it is the first step for proving the existence of a good section in the linear system $|L|$.

Corollary 2.2.3. *Let $D \equiv_f \beta L$ be a $\mathbb{Q}$-divisor such that (X, D) is lc and let $W \in CLC(X, D)$ be a minimal centre. Assume that $\tau - \beta > -1$ or that $\tau - \beta \geq -1$ if f is birational; assume also that one of the following conditions is satisfied:*

(i) $\dim W \leq 2$,

ii) $\dim W \geq 3$ *and* $\tau - \beta > \dim W - 3$.

Then there exists a section of $|L|$ not vanishing identically on W.

Proposition 2.2.4. *If $\tau > -1$ and $\dim F < \tau + 3$, then there exists a section of $|L|$ not vanishing identically along F.*

Proof. Let $\{h_i\} \in H^0(Z, \mathcal{O}_Z)$ be general functions vanishing at $f(F)$ and let $D = \sum f^*(h_i)$ such that (X, D) is not lc. Let $\gamma = \operatorname{lct}(X, D)$ and let $W \in CLC(X, \gamma D)$ a minimal lc centre such that $W \subset F$. Since $\gamma D \equiv_f 0$, Corollary 2.2.3 implies that there exists a section of $|L|$ not vanishing identically on W. □

The following proposition is a generalization of Proposition 3.3 in [Mel99].

Proposition 2.2.5. *Assume that* $\dim F < \tau + 3$*, F is irreducible and $\tau \geq 0$. Then the general element of $|L|$ has lt singularities, except possibly when $\tau = 0$ and f is a contraction to point.*

If $\dim F < \tau + 2$*, then the same holds without the assumption that F is irreducible.*

Proof. Let $S \in |L|$ be general and assume by contradiction that S has singularities worse than lt singularities. Then, by Proposition 7.3.2 of [Kol97b], (X, S) is not plt. Set $\gamma = \operatorname{lct}(X, S) \leq 1$ and $V = LLC(X, \gamma S)$.

We claim that, modulo a vertical slicing, we may assume $V \subset F$. In fact, let $c = \dim f(V)$. Consider $h_1, \ldots, h_c$ general functions on Z. Set

$X_{h_i} = f^*h_i$ and $X' = \cap X_{h_i}$. By vertical slicing ([AW93, Lemma 2.5]), we get a local contraction $f' : X' \to Z'$ around a fibre $F' = F \cap X'$, supported by $K_{X'} + \tau L'$ where $L' = L_{|X'}$. Set $S' = S \cap X'$. Possibly shrinking Z', we have $LLC(X, \gamma S') = V \cap X' \subset F'$ and the claimed is proved.

Note that, by Bertini, $LLC(X, S) \subset Bs|L|$.

If $\tau > 0$, let $W \in CLC(X, \gamma H)$ be a minimal lc centre. We want to show that there is a section of $|L|$ not vanishing identically on W, obtaining in this way a contradiction. By Proposition 2.2.4, $W \subsetneq F$; thus $\dim W \leq \dim F - 1 < \tau + 2$. If $\dim W \geq 3$, then $\tau - \gamma > \dim W - 3 \geq 0$ and we can apply point (i) of Corollary 2.2.3. If $\dim W \leq 2$, then the contradiction follows by point (ii) of Corollary 2.2.3.

Assume that $\tau = 0$ and f is not a contraction to a point. Let $H = \varepsilon f^*(h)$, where h is a general function on Z vanishing at $f(F)$ and $0 < \varepsilon << 1$. Set $D = S + H$ and $\delta = \mathrm{lct}(X, S) < 1$. For ε small enough, we still get $LLC(X, \delta D) \subset F$ and, by Bertini, $LLC(X, \delta D) \subset Bs|L|$. Hence, we can consider a minimal centre $W \in CLC(X, \delta D)$ and reason as before.

□

Proposition 2.2.6. *Assume that* $\dim F < \tau + 3$*,* F *is irreducible and* $\tau \geq 1$*. Let* $S \in |L|$ *be a general element. If* X *has canonical singularities, then* S *has canonical singularities. If* X *has terminal singularities, then* S *has terminal singularities, except possibly when* $\tau = 1$ *and* f *is a contraction to a point.*

If $\dim F < \tau + 2$*, then the same holds without the assumption that* F *is irreducible.*

Proof. Let S be a general element of $|L|$; by Proposition 2.2.5, S has lt singularities. Let $\mu : Y \to X$ a log resolution of the pair (X, S). We can write

$$\mu^* S = \overline{S} + \sum_i r_i E_i$$

$$K_Y = \mu^* K_X + \sum_i a_i E_i$$

$$K_Y + \overline{S} = \mu^*(K_X + S) + \sum_i (a_i - r_i) E_i$$

where $\overline{S} = \mu_*^{-1} S$ is the strict transform of S and $|\overline{S}|$ is basepoint free. Note that since L is Cartier the r_i are integers.

Assume that S has not canonical singularities (resp. terminal singularities); after reordering we can assume that $a_0 < r_0$ (resp. $a_0 \leq r_0$). Since S is generic, by Bertini we can assume that $\mu(E_i) \subset Bsl|L|$, for all i such that $r_i > 0$.

Let $D = S + S_1$, where S_1 is another generic section in $|L|$; note that μ is a log resolution also for the pair (X, D). (X, D) is not LC since $a_0 + 1 < r_0 + r_0^1$

$(a_0 + 1 \leq r_0 + r_0^1)$, where $r_0^1 \geq 1$ is the multiplicity of S_1 at the centre of valuation associated to E_0. Now we can reason as in the proof of Proposition 2.2.5.

□

We can finally state the main result of this section.

Theorem 2.2.7. *Assume that X has log terminal singularities, $\tau > 0$ and $\dim F = n - 1 < \tau + 2$. Then $\dim Bs|L| \leq 1$.*

Proof. By vertical slicing we may assume that $Bs|L| \subset F$.

We start by proving that $|L|$ has no fixed components. Suppose, by contradiction, that there is a component V of $Bs|L|$ of dimension $n-1$. Let $H \in |L|$ be a general element and set $c = \mathrm{lct}(X, H)$. If $c < 1$, then $LCC(X, cH) \subset Bs|L|$ and by Corollary 2.2.3 we get a contradiction. If $c = 1$, then $V \subset |L|$ is an lc centre of (X, H) and we conclude again by Corollary 2.2.3.

The proof of the theorem is now by induction on $n \geq 3$. If $n = 3$, we have just proved it.

Assume $n > 3$. Let $X' \in |L|$ general. Since $|L|$ has no fixed component, by Bertini we get that X' does not contain any irreducible component of F (and that it is irreducible and reduced). Moreover, by Proposition 2.2.5, we have that X' is log terminal. Hence, by horizontal slicing, $f : X' \to Z'$ is a divisorial contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$ around a fibre $F' = F \cap X'$. It also follows that $\dim Bs|L| \leq \dim Bs|L'|$, because any section of L' lifts to a section of L. By induction, we are done.

□

Corollary 2.2.8. *Assume that X has terminal singularities and that f is a divisorial contraction of an irreducible $\mathbb{Q}$-Cartier divisor E such that $\dim X < \tau + 3$ and $\tau > 0$. For $i = 1, \ldots, n-3$, let $H_i \in |L|$ be general divisors and set $X' = \cap H_i$. Then X' is a terminal threefold and $f' : X' \to Z'$ is a divisorial contraction of an irreducible $\mathbb{Q}$-Cartier divisor E'.*

Proof. By horizontal slicing, $f' : X' \to Z'$ is a Fano-Mori contraction. The fact that X' is terminal follows by Proposition 2.2.6. By Theorem 2.2.7, the base locus of $|L|$ has dimension at most one, and hence E' is an irreducible divisor. Finally, E' is the intersection of $\mathbb{Q}$-Cartier divisors and hence it is Q-Cartier. □

In the case of Gorenstein singularities we can prove a stronger result.

Proposition 2.2.9. *Assume that $\dim F < \tau + 3$, F is irreducible and $\tau \geq 0$. If X has canonical Gorenstein singularities, then the general element of $|L|$ has canonical singularities, except possibly when f is a contraction to a point and $\tau = 0$.*

Proof. In the notation of the proof of Proposition 2.2.6, assume by contradiction that S is not canonical. Then $a_i - r_i < 0$ for some i; since a_i and r_i are integers, we get $a_i - r_i \leq -1$ and hence (X, S) is not plt. Set $\gamma = \mathrm{lct}(X, S) \leq 1$. Now, as in the proof of Proposition 2.2.5, we derive a contradiction. □

2.2.2 Lifting of contractions

Let X be a $\mathbb{Q}$-factorial terminal threefold and let $f : X \to Z$ be a divisorial Fano-Mori contraction of an extremal ray R to a smooth point $p \in Z$. By an important result of Kawakita (see [Kaw01]) we know that f is a weighted blow-up with weight $(1, a, b)$.

We ask the following.

Question 2.2.10. *Let $f : X \to Z$ be a Fano-Mori contraction of an extremal ray $R \subset \overline{NE}(X)_{(K_X+(n-3)L)<0}$ on a terminal $\mathbb{Q}$-factorial variety X with L an ample Cartier divisor. Assume that f contracts a divisor to a smooth point $p \in Z$. Are there local coordinates for p such that f is the weighted blow-up of p?*

Thanks to the results of the previous section we can give a positive answer under suitable (strong) assumptions.

Theorem 2.2.11. *Let $f : X \to Z$ be a local contraction supported by $K_X + \tau L$, where X is n-dimensional terminal $\mathbb{Q}$-factorial variety and L is an f-ample $\mathbb{Q}$-Cartier divisor. For $i = 1, \dots, n-3$, let $H_i \in |L|$ be general divisors and set $X'' = \cap H_i$. By Theorem 2.2.7, $f'' := f_{|X''} : X'' \to Z''$ is birational. Assume that f'' contracts a divisor E'' to a smooth point $p := f(E'')$.*

Then $L = f^* f_* L - cE$ for a positive integer c and f is a weighted blow-up of a smooth point with weight $(1, a, b, c, \dots, c)$, where a, b are positive integers, $(a, b) = 1$ and $ab|c$.

Proof. Let $X' \in |L|$ be a general element. By Theorem 1.2.12 and by Lemma 1.2.9, $f' = f_{|X'} : X' \to Z'$ is a local contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$. By a repeated application of [Mel97, Lemma 1.7], f' contracts a divisor E' to the smooth point $p := f(E') \in Z'$. Again by [Mel97, Lemma 1.7], $p \in Z$ is smooth.

Let $x_1, \dots, x_n$ local coordinates for p and note that $L_1 = f_* L$ is a Cartier divisor and we have $L = f^* L_1 - cE$, where c is a positive integer. We may also assume that $f_*(X') = \{x_n = 0\}$.

If $\dim X = 4$ then, by Kawakita's result, f' is a weighted blow-up of p with weight $(1, a, b)$. If $\dim X > 4$ then, by induction on n, f' is a weighted blow-up of p with weight $(1, a, b, c, \dots, c)$.

By induction we have the formulae

$$L = f^*(L_1) - cE, \quad \tau = (n-3) + \frac{a+b}{c} \quad \text{and} \quad K_X = f^*(K_Z) + (a+b+(n-3)c)E.$$

Note that $\mathcal{O}_X(-cE)$ is f-ample and that the map f is proper; so we have that

$$X = \mathrm{Proj}(\oplus_{d\geq 0} f_*\mathcal{O}_X(-dcE)).$$

By Lemma 1.3.10, X we need to prove that

$$f_*\mathcal{O}_X(-dcE) = (x_1^{s_1}\cdots x_n^{s_n} : s_1 + s_2 a + s_3 b + \sum_{j=4}^{n} cs_j \geq dc).$$

The proof of this is by double induction on n and d, starting with $n = 4$ and $d = 0$.

Consider the exact sequence

$$0 \to \mathcal{O}_X(-L - dcE) \to \mathcal{O}_X(-dcE) \to \mathcal{O}_{X'}(-dcE) \to 0.$$

Note that

$$-L - dcE \sim_f -(d-1)cE \sim_f K_X + (n - 3 + d - 1 + \frac{a+b}{c})L,$$

Hence, pushing down to Z the above exact sequence and applying the relative Kawamata-Viehweg Vanishing, we have

$$0 \to f_*\mathcal{O}_X(-(d-1)cE) \stackrel{\cdot x_n}{\to} f_*\mathcal{O}_X(-dcE) \to f_*\mathcal{O}_{X'}(-dcE) \to 0. \qquad (2.5)$$

By induction on n, we can assume that

$$f_*\mathcal{O}_{X'}(-dcE) = (x_1^{s_1}\cdots x_{n-1}^{s_{n-1}} : s_1 + s_2 a + s_3 b + \sum_{j=4}^{n-1} cs_j \geq dc),$$

where $s_j \in \mathbb{N}$. The case $n = 3$ follows from Kawakita's Theorem. By induction on d, we can also assume that

$$f_*\mathcal{O}_X(-(d-1)bE) = (x_1^{s_1}\cdots x_n^{s_n} : s_1 + s_2 a + s_3 b + \sum_{j=4}^{n} cs_j \geq (d-1)c),$$

the case $d = 0$ being trivial.

Let $g = x_1^{s_1}\cdots x_n^{s_n} \in f_*\mathcal{O}_X(-dcE)$ be a monomial. If $s_n \geq 1$ then g, looking at the sequence (2.5), comes from $f_*\mathcal{O}_X(-(d-1)cE)$ by the multiplication by x_n; therefore

$$s_1 + s_2 a + s_3 b + \sum_{j=4}^{n-1} s_j c + s_n c \geq (d-1)c + s_n c \geq dc.$$

If $s_n = 0$, then $g \in f_*\mathcal{O}_{X'}(-dcE)$ and so

$$s_1 + s_2 a + s_3 b + \sum_{j=4}^{n} s_j c = s_1 + s_2 a + s_3 b + \sum_{j=4}^{n-1} s_j c \geq dc.$$

The non-monomial case follows immediately. $\square$

2.2.3 Flipping contractions

We now start the investigation of small contractions with nef value bigger than $(n-3)$. We begin describing the exceptional locus in the case of index 2 singularities.

Proposition 2.2.12. *Let X be a terminal $\mathbb{Q}$-factorial variety of dimension n. Let $f : X \to Z$ be a local contraction around F supported by $K_X + \tau L$, where L is an ample Cartier divisor.*

Assume that $\tau > (n-3)$, that $\dim F = n-2$ and that f is birational and small; assume also that X has only points of index 1 and 2.

Then

- $\tau = \frac{2n-5}{2}$ *and* $(F, L) = (\mathbb{P}^{n-2}, \mathcal{O}(1))$ *or*
- $n = 4$, $\tau = \frac{5}{4}$ *and* $(F, L) = (\mathcal{Q}^{n-2}, \mathcal{O}(1))$.

Proof. By Theorem 2.1 of [And95] we get that $\dim F = n-2$, where F is any fibre of f. By the rationality theorem we have

$$2\tau = \frac{u}{v} \leq \frac{2(n-1)}{v}$$

and thus

$$n - 3 < \tau = \frac{u}{2v} \leq \frac{n-1}{v}.$$

If $n = 4$ this gives the possibilities $v = 1, 2$ and so $u = 3$ and $u = 5$ respectively (since $u \leq 6$).

If $n > 4$ then $v = 1$ and $n - 3 < \frac{u}{2} < n - 2$ with the only possibility $u = 2n - 5$.

By [AW93, Thm. 5.1] we can suppose that L is globally generated. Pick $n-3$ general members $H_i \in |L|$ $(1 \leq i \leq n-3)$ and let $X' = \cap H_i$ be the scheme intersection. Note that $\dim X' = 3$ and X' has terminal 2-factorial singularities: in fact $2K_{X'} = 2(K_X + (n-3)L)_{|X'}$ is Cartier. Consider the restricted morphism $f' := f_{|X'} : X' \to Z'$, it is a small contraction supported by $K_{X'} + (\tau - n + 3)L_{|X'}$ (see Theorem 1.2.9). Let $C = \cap H_i \cap F$. At this point the situation is described in [KM92], Theorem 4.2. In particular $C \cong \mathbb{P}^1$, so we deduce that F is irreducible. Moreover $-K_{X'}.C = \frac{1}{2}$.

By [And95, Thm. 2.1] we have that $\Delta(F, L) = 0$ and there are two possibilities.

If $\tau = \frac{2n-5}{2}$ then $L_{|X'}.C = 1$ and thus $L_{|F}^{n-2} = 1$. By the classification of varieties with Δ-genus equal to zero, we get that $(F, L) = (\mathbb{P}^{n-2}, \mathcal{O}(1))$.

If $n = 4$ and $\tau = \frac{5}{4}$, then $L_F^{n-2} = 2$ and so $(F, L) = (\mathcal{Q}^{n-2}, \mathcal{O}(1))$.

□

2.2.4 Towards a second reduction

Classically (see [BS95]), the second reduction is defined as follows.

Definition 2.2.13. Let (X, L) be a quasi-polarized pair such that $K_X + (n-2)L$ is nef and big and let $\phi : X \to Z$ be the morphism associated to $|m(K_X + (n-2)L)|$ for $m \gg 0$. Consider $D = (\phi_* L)$. The pair (Z, D) with the morphism ϕ is called *second reduction* of (X, L).

One of the problem is that D may not be a Cartier divisor. To define a more general second reduction in the spirit of Andreatta's zero and first reductions (see subsection 1.2.2) we need to consider a quasi-polarized pair (X, L) and then study the minimal model program for a klt pair (X, Δ) where $\Delta \sim_{\mathbb{Q}} rL$ and $r \geq n-3$. In the last three subsections we have seen that we are just at the beginning of the job.

In this subsection we observe that if r is close enough to $n-2$, then we fall in the case of the classical second reduction, already studied by M. Mella (see [Mel97]).

Lemma 2.2.14. *Let $n \geq 3$ and e be positive integers. For any positive rational numbers $\alpha < \beta$ the set*

$$N_{\alpha,\beta} := \{\alpha < \tau(X, L, \pi) < \beta \mid \dim X = n, \mathrm{index}(X) = e\}$$

is finite.

Moreover, any element of $N_{\alpha,\beta}$ can be written in the form c/d for coprime positive integers c and d such that $d < e(n+1)/\alpha$.

In particular, setting $\varepsilon = \frac{1}{e}$, we have that

$$N_{n-2-\varepsilon,n-2} = N_{\frac{n-2-\varepsilon}{2},\frac{n-2}{2}} = \emptyset.$$

Proof. Let $\tau \in N_{\alpha,\beta}$. By the rationality theorem we have that $\tau = c/d$ for coprime positive integers c, d and that $e\tau = u/v$ for coprime positive integers u, v with $u \leq e(n+1)$. This implies

$$e\alpha < \frac{u}{v} \leq \frac{e(n+1)}{v},$$

hence $v < (n+1)/\alpha$ and $d < e(n+1)/\alpha$. Finally u may assume only a finite number of values since v does. □

Theorem 2.2.15. *Let (X, L) be the first reduction of a quasi-polarized variety $(\mathcal{X}, \mathcal{L})$, with $\mathcal{X}$ a terminal variety of dimension $n \geq 4$. Let $\Delta \sim_{\mathbb{Q}} (n-2-\varepsilon)L$ be an effective divisor such that (X, Δ) is a klt pair with $0 < \varepsilon < \frac{1}{e}$, where e is the index of singularity of X. Consider a $(K_X + \Delta)$-MMP*

$$(X_0, \Delta_0) = (X, \Delta) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_s, \Delta_s).$$

Then each $\phi_i : X_i \dashrightarrow X_{i+1}$ is a contraction of a divisor E, such that $(E, E_{|E})$ is isomorphic to one of the following:

1. *($\mathbb{P}^{n-1}, \mathcal{O}(-2)$),*

2. *($\mathbb{Q}^{n-1}, \mathcal{O}(-1)$),*

3. *a singular hyperquadric in $\mathbb{P}^n$ with $E_{|E} = \mathcal{O}(-1)$,*

4. *($\mathbb{P}(1,1,1,2,\dots,2), \mathcal{O}(-1/2)$)*

5. *ϕ_i is the blowing up of an irreducible reduced curve on X_{i+1}.*

Moreover the composition $\phi : X \to X_s$ is exactly the second reduction of the pair (X, L) (so it is the contraction of disjoint divisors as in the list).

Proof. We may assume that $K_X + \Delta$ is not nef, otherwise there is nothing to prove. Since (X, L) is a first reduction, $K_X + (n-1)L$ is nef and hence L is positive on any extremal ray R such that $K_X.R < 0$. For any i set $L_i := \phi_{i*}(L_{i-1})^{**}$.

We prove by induction on $i = 0, \dots, s$, that any extremal ray R on X_i supported on $K_{X_i} + (n-2)L_i$ defines a contraction ϕ_R which is one of our list.

By Lemma 2.2.14 the case $i = 0$ follows by Theorem 2.3 of [Mel97].

So consider $\phi_i : X_{\to} X_{i+1}$, with $i \geq 1$. By induction and by Corollary 2.4 of [Mel97] at any step the index of singularity is not worse than $\max\{2, e\}$ and L_k is 2-Cartier.

Let R_{i-1} be the extremal ray of the map ϕ_{i-1}, let $R_i = R^+[C]$ be the extremal ray associate to ϕ_i and ϕ_{R_i} be the associated contraction. Since L_i is ϕ_i-ample we can apply Lemma 2.2.14 to obtain that

$$(K_{X_i} + (n-2)L_i).R_i = 0.$$

Let $\tilde{C}$ be a strict transform of C. Let E be the divisor contracted by ϕ_{i-1} and write

$$K_{X_{i-1}} = \phi_{i-1}^*(K_{X_i}) + aE \quad , \quad L_{i-1} = \phi_{i-1}^*(L_i) - bE,$$

where $a > 0$ and $b > 0$. Since

$$0 = (K_{X_{i-1}} + (n-2)L_{i-1}).R_{i-1}$$

we have that $a = (n-2)b$ and hence

$$(K_{X_{i-1}} + (n-2)L_{i-1}).\tilde{C} = 0.$$

The composition $\phi_i \circ \phi_{i-1}$ is given by a 2-dimensional face F, which contains R_{i-1} and $\tilde{R} = \mathbb{R}^+[\tilde{C}]$ and so by induction we are done. □

Chapter 3

Chern numbers of threefolds

This chapter collects the results of a joint work in progress with P. Cascini. The aim of this work is to answer the following question posed by Kotschick: does c_1^3 assumes only finitely many values on the projective algebraic structure with the same underlying 6-manifold? We obtain a partial result (Theorem 3.3.5).

We start recalling the definition of Chern numbers.

Definition-Lemma 3.0.16. *Let X be a topological space. For any complex vector bundle E on X of dimension k there exist elements $c_i(E) \in H^{2i}(X, \mathbb{Z})$, called* Chern classes *of E, which are uniquely determined by the following axioms:*

1. $c_0(E) = 1$ *and* $c_i(E) = 0$ *if* $i > k$,
2. *if* $f : Y \to X$ *is a continuous map, then* $c_i(f^*E) = f^*c_i(E)$,
3. *if* $0 \to F \to E \to G$ *is an exact sequence of vector bundles, then* $c_i(E) = \sum_{j=0}^{i} c_j(F) \cdots c_{i-j}(G)$,
4. $\sum_{j=1}^{k} c_j(E) = e(E_{\mathbb{R}})$, *where* $e(E_{\mathbb{R}})$ *is the Euler class of the underlying real vector bundle.*

When X is a complex manifold, then $c_i(X) = c_i(T_X)$ are simply called Chern classes of X. If $\dim X = n$, any product of Chern classes of total degree n is called Chern number *of X.*

The study of Chern numbers is a very classical topic, in particular Hirzebruch raised the following question in 1954: which linear combinations of Chern numbers are topological invariant? This has been completely settled by Kotschick in the projective case.

Theorem 3.0.17 ([Kot12])**.** *A rational linear combination of Chern numbers is a homeomorphism invariant of smooth complex projective varieties if and only if it is a multiple of the Euler characteristic.*

The ensuing question posed by Kotschick is whether a Chern number can be bounded by topological invariants.

In general let

$$\chi_p = \chi(\Omega_X^p) = \sum_{q=0}^{n} (-1)^q h^{p,q},$$

where $h^{p,q}$ is the dimension of $H^q(X, \Omega_X^p)$. By the Hirzebruch-Riemann-Roch theorem, χ_p is a linear combination of Chern numbers. On the other side, by the Hodge decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^q(X, \Omega^p)$$

they are bounded above and below by linear combinations of Betti numbers.

It turns out that this is the only possibility:

Theorem 3.0.18 ([Kot12])**.** *A rational linear combination of Chern numbers of smooth complex projective varieties can be bounded in terms of Betti numbers if and only if it is a linear combination of the χ_p.*

Let us fix $\dim X = 3$. Then there are three Chern numbers: c_3, c_1c_2 and c_1^3. By the Hirzebruch-Riemann-Roch theorem we have

$$\frac{1}{24} c_1 c_2 = 1 - h^{1,0} + h^{2,0} - h^{3,0},$$

thus c_1c_2 is bounded from below and above by linear combinations of the Betti numbers.

On the other hand one can not expect to be able, in general, to bound $c_1(X)^3 = K_X^3$ just using Betti numbers (see, for instance, example 3.2.21). Hence we try to study the arithmetic properties of the integral cubic form F_X associated to the intersection cup on $H^2(X, \mathbb{Z})$. In fact, F_X takes special forms in the case of divisorial contractions to points and divisorial contractions to LCI curves (we call them *reduced forms*, see definition 3.2.2) and this allows us to bound K_X^3 when we are dealing with these types of maps.

In the **first section** we show that the volume of X is bounded by a combination of Betti numbers.

Section 3.2 is devoted to a detailed study of some arithmetic properties of cubic forms. The main result is Theorem 3.2.28, in which we show that for an integral cubic form $F(x_0, \dots, x_n)$ such that $\Delta_F \neq 0$, there are just a finite number of possible reduced forms.

In the **last section** we apply these results to prove our main Theorem 3.3.5, which states that if X is a smooth threefold and there is an MMP for X constituted only by divisorial contractions to points and divisorial contractions to smooth curves in smooth loci, then K_X^3 is bounded by a topological invariant of X.

By a recent result of Chen ([Che13]), any MMP of a terminal $\mathbb{Q}$-factorial threefold may be factored into a sequence of flops, blow-downs to a smooth curve in a smooth 3-fold and divisorial contractions to points (or their inverses). Hence, the main question left is about flops.

3.1 Bounding the volume

Let $P \in X$ be a threefold terminal singularity. Then there is a small deformation of P into $k \geq 1$ terminal cyclic quotient singularities $P_1, \dots, P_k$. The number $aw(P \in X) := k$ is called the axial weight of $P \in X$. We set

$$aw(X) := \sum_{P \in \mathrm{Sing(X)}} aw(P \in X).$$

We may assume that the singularities P_i have type $\frac{1}{r_i}(1, -1, b_i)$, where $0 < b_i \leq r/2$. The collection $\{P_1, \dots, P_k\}$ is known as *fictitious singularities of* P or as the basket of singularities of (X, P) and it can be written as

$$\mathcal{B}(P \in X) = \{n_i \times (b_i, r_i) \mid i \in I, n_i \in \mathbb{Z}^+\},$$

where n_i denotes the number of times that a point P_i representing a singularity $\frac{1}{r_i}(1, -1, b_i)$ appears.

To a given variety X, we can associate the basket of singularities

$$\mathcal{B}(X) = \bigcup_{P \in \mathrm{Sing(X)}} \mathcal{B}(P \in X)$$

and we can define the following invariants:

$$\Xi(P \in X) = \sum_{i=1}^{aw(P \in X)} r(P_i), \quad \Xi(X) = \sum_{P \in \mathrm{Sing(X)}} \Xi(P \in X).$$

By [CZ12, Proposition 3.3] we can control the singularities of a minimal model by the topology of the initial threefold.

Proposition 3.1.1. *Let X be a smooth projective threefold and assume that*

$$X = X_0 \dashrightarrow \dots \dashrightarrow X_k = Y$$

is a sequence of steps for a minimal model program for X.

Then

$$\Xi(Y) \leq 2b_2(X).$$

In particular, the inequality holds if Y is the minimal model of X.

Recently Tian and Wang have announced the following result.

Theorem 3.1.2. *If Y is a minimal projective n-fold of general type with canonical singularities and which is smooth in codimension 2, then*

$$(K_Y^2 - 2\frac{n+1}{n}c_2(Y)) \cdot K_Y^{n-2} \leq 0.$$

Let X be a smooth projective 3-fold of general type. We prove that $\mathrm{vol}(X)$ is bounded by some constant which depends only on the topological Betti numbers of X.

Theorem 3.1.3. *Let X be a smooth projective 3-fold of general type. Then*

$$\mathrm{vol}(X, K_X) \leq 64\left(b_1(X) + b_3(X) + \frac{2}{3}b_2(X)\right).$$

Proof. Let $X \dashrightarrow Y$ be the minimal model of X. Then Y admits only terminal singularities, and in particular it is smooth outside a finite number of points. In addition,

$$\mathrm{vol}(X, K_X) = \mathrm{vol}(Y, K_Y) = K_Y^3.$$

By the singular version of Riemann-Roch [Kaw86, Rei87], we have that

$$\chi(Y, \mathcal{O}_Y) = \frac{1}{24}(-K_Y \cdot c_2(Y) + e)$$

where

$$e = \sum_{p_\alpha}\left(r(p_\alpha) - \frac{1}{r(p_\alpha)}\right),$$

and the sum is taken over all the points of all the baskets $\mathcal{B}(Y, p)$ of singularities of Y. Note that $e \leq \Xi(Y)$. Thus,

$$\begin{aligned}
\mathrm{vol}(X, K_X) = K_Y^3 &\leq \frac{8}{3}K_Y \cdot c_2(Y) \\
&= -\frac{8}{3}(24(\chi(Y, \mathcal{O}_Y) - e)) \\
&= 64(\sum_{i=0}^{3}(-1)^{i+1}h^i(X, \mathcal{O}_X) + \frac{1}{3}e) \\
&\leq 64(b_1(X) + b_3(X) + \frac{1}{3}\Xi(Y)).
\end{aligned}$$

The result follows now by Proposition 3.1.1. □

Corollary 3.1.4. *The volume only takes finitely many values on projective algebraic structures of general type with the same underlying 6-manifold.*

Proof. The volume $\mathrm{vol}(X, K_X)$ is a rational number whose denominator is bounded by the index of the minimal model of X. Thus, the claim follows immediately from Proposition 3.1.1 and Theorem 3.1.3. □

3.2 Cubic forms

The aim of this section is to investigate elementary facts about cubic polynomials, which will be useful in the study of the intersection form on a threefold.

For any polynomial $P \in \mathbb{C}[x_0, \dots, x_n]$, we denote by $\partial_i P(x)$ the partial derivative of P with respect to x_i at the point x. Given a cubic $F \in \mathbb{C}[x_0, \dots, x_n]$, i.e. an homogeneous polynomial of degree 3, let $\mathcal{H}_F(x) = (\partial_i \partial_j F(x))_{i,j}$ be the Hessian of F at the point $x \in \mathbb{C}^{n+1}$. Note that, for any $x \in \mathbb{C}$ and for any $\lambda \neq 0$, the rank of $\mathcal{H}_F$ at the point λx is constant with respect to λ and therefore we will denote, by abuse of notation, $\operatorname{rk} \mathcal{H}_F(p)$ to be the rank of $\mathcal{H}_F$ at any point in the class of p. We say that F is *non-degenerate* if $\operatorname{rk} \mathcal{H}_F$ is maximal at the general point of $\mathbb{P}^n$.

3.2.1 Invariants

If $P(x_0, \dots, x_n) \in \mathbb{Z}[x_0, \dots, x_n]$ is a polynomial we denote by $\operatorname{ct}(P)$ the *content* of P, that is the gcd of the coefficients of P. As in the case of one variable, it is easy to see that the content is multiplicative.

Let $F(x_0, \dots, x_n) = \sum_I c_I x^I$ be a form of degree d. Then the discriminant Δ_F of F is the unique (up to sign) irreducible polynomial with integral coefficients in the variables c_I such that $\operatorname{ct}(\Delta_\mathrm{F}) = 1$ and such that $\Delta_F = 0$ if and only if the hypersurface $\{F = 0\} \subseteq \mathbb{P}^n_\mathbb{C}$ is singular.

We will need the following observation about discriminants.

Lemma 3.2.1. *Let*

$$F = ax_0^3 + x_0^2 (\sum_{j=1}^n b_j x_j) + G(x_1, \dots, x_n)$$

be a cubic form, where $G = \sum_I c_I x^I$. Then Δ_G divides Δ_F.

Proof. Let $R = \mathbb{C}[a, b_j, c_I]$ and let $Z(F), Z(G) \subseteq \mathbb{P}^N_\mathbb{C} = \operatorname{Proj} R$ be the closed subsets defined by Δ_F and Δ_G respectively. Note that $Z(G) \subseteq Z(F)$ because if $\{G = 0\}$ has a singular point $z = [z_1, \dots, z_n]$ then $[0, z_1, \dots, z_n]$ is a singular point of $\{F = 0\}$. Since Δ_G is irreducible over $\mathbb{Q}$ by definition, and hence $Z(G)$ is reduced over $\mathbb{C}$, we deduce that $\Delta_F = \Delta_G H$ where $H \in R$.

We must show that $H \in \mathbb{Z}[a, b_j, c_I]$. We start assuming by contradiction that $H \notin \mathbb{Q}[a, b_j, c_I]$. Fix an order on R and consider the maximal monomial M in H such that its coefficient is not rational. Consider now the product between M and the highest monomial in Δ_G to get a contradiction. Hence $H \in \mathbb{Q}[a, b_j, c_I]$. Now using the fact that the content of Δ_G is 1 and that the content is multiplicative it is easy to conclude. □

3.2.2 Reduced forms

Given a cubic form $F \in \mathbb{C}[x_0, \dots, x_n]$ and a matrix $T \in \mathrm{SL}(n+1, \mathbb{C})$, we will denote by $T \cdot F$ the cubic given by

$$T \cdot F(x) = F(T \cdot x).$$

We set

$$W_F = \{p \in \mathbb{P}^n \mid \operatorname{rk} \mathcal{H}(p) \leq 1\}$$

and

$$V_F = \{p \in \mathbb{P}^n \mid \operatorname{rk} \mathcal{H}(p) \leq 2\}.$$

Definition 3.2.2. Let R be a subring of a number field K and let $F \in R[x_0, \dots, x_n]$ be a non-degenerate cubic form. Let $a \in R$ and $b = (b_1, \dots, b_n) \in R^n$. We say that F is in *reduced form* if we may write

$$F = ax_0^3 + x_0^2 \cdot \sum_{i=1}^{n} b_i x_i + G(x_1, \dots, x_n), \tag{3.1}$$

where $G \in R[x_1, \dots, x_n]$ is a non-degenerate cubic form. For simplicity, we will denote (3.1) as

$$F = (a, b, G).$$

We say that two reduced forms (a, b, G) and (a', b', G'), are *equivalent* if $a = a'$ and there is an element $M \in \mathrm{SL}(n, R)$ such that $b' = M \cdot b$ and $G' = M \cdot G$.

Note that if $F \in \mathbb{C}[x_0, \dots, x_n]$ is a cubic in reduced form, and $p = [1, 0, \dots, 0]$, then $p \in V_F$. In addition, we can associate to F the hyperplane

$$H_F = \{\sum_{i=1}^{n} b_i x_i = 0\}.$$

Clearly $p \in H_F$.

Viceversa assume that $p \in V_F$ if such that $F(p) \neq 0$ and $p \notin W_F$, and assume that there exists $M \in SL(n+1, R)$ such that $M \cdot p = [1, 0, \dots, 0]$ and $M \cdot F$ is in reduced form, then it is easy to check that the hyperplane H_F is uniquely determined.

Lemma 3.2.3. *Let $F \in \mathbb{Z}[x_0, \dots, x_n]$ be a cubic and let $p \in V_F$ such that $F(p) \neq 0$.*

Then there are only finitely many non-equivalent reduced forms $T \cdot F$ such that $T \cdot p = [1, 0, \dots, 0]$.

Proof. We may assume that $p = [1, 0, \dots, 0]$ and that $F = (a, b, G)$ is in reduced form, for some $a \in \mathbb{Z}$, $b \in \mathbb{Z}^n$ and $G \in \mathbb{Z}[x_1, \dots, x_n]$. We consider all the matrices $T \in SL(n+1,\mathbb{Z})$ such that $T\cdot p = p$ and $M\cdot F = (a_T, b_T, G_T)$ is in reduced form, for some $a_T \in \mathbb{Z}$, $b_T \in \mathbb{Z}^n$ and $G_T \in \mathbb{Z}[x_1,\dots,x_n]$.

If we write $T = (t_{ij})_{i,j=0,\dots,n}$ with $t_{ij} \in \mathbb{Z}$, then, since $T\cdot p = p$, we have $t_{i0} = 0$ for $1 \le i \le n$. Thus, $t_{00} = \pm 1$ and in particular $a_T = \pm a$.

Now we need to show that the G_T are in finite number. Acting on $(x_1,\dots,x_n)$ with $\mathrm{SL}(n,\mathbb{Z})$ we may assume that $b = (b_1, 0, \dots, 0)$ and $b_T = (b_1', 0\dots, 0)$.

Looking at the coefficients of $x_0^2 x_i$ and $x_0 x_i^2$ we obtain the equations

$$\begin{aligned} &3at_{0i} + b_1 t_{1i} = 0 \qquad \text{for } i = 2,\dots,n \text{ and} \\ &3at_{0i}^2 + 2b_1 t_{0i} t_{1i} = 0 \qquad \text{for } i = 1,\dots,n. \end{aligned} \tag{3.2}$$

Note that $a \neq 0$ or $b_1 \neq 0$. We divide the last part of the proof in three cases.

If $b_1 = 0$ then $t_{0i} = 0$ for $i = 2,\dots,n$. By (3.2), we have

$$3at_{01}^2 = 0$$

which implies $t_{01} = 0$ and we are done.

If $a = 0$ then by (3.2) we have $t_{1i} = 0$ for $i = 2,\dots,n$ and hence $t_{11} = \pm 1$, so we can look at $x_0 x_1 x_i$ for $i = 1,\dots,n$ to get the equations

$$b_1 t_{0i} t_{11} = 0$$

to deduce that $t_{0i} = 0$ for $i = 1,\dots,n$ and we are done again.

Finally if $a, b \neq 0$ then by (3.2) we have that $t_{0i} = t_{1i} = 0$ for $2 \le i \le n$. This implies that $t_{11} = \pm 1$ and looking at $x_0 x_1^2$ we gain that there are just a finite number of possible t_{01}. This means that also the possible forms G_T are finite. □

We have:

Lemma 3.2.4. *Let $F \in \mathbb{C}[x_0,\dots,x_n]$ be a cubic such that there exists a point $p \in \mathbb{P}^n$ for which $\operatorname{rk} \mathcal{H}_F(p) = 0$ (i.e. $\mathcal{H}_F(p)$ is the trivial matrix).*

Then after a suitable coordinate change, F depends on at most n variables. In particular, $\det \mathcal{H}_F$ vanishes identically on $\mathbb{P}^n$.

Proof. Euler's formula for homogeneous polynomials implies that

$$F(p) = \partial_i F(p) = 0 \qquad \text{for all } i = 0,\dots,n.$$

After a suitable coordinate change, we may assume that $p = (1, 0, \dots, 0)$. Let $f(y_1,\dots,y_n) = F(1, y_1,\dots,y_n)$. By Taylor's formula, f is an homogeneous polynomial of degree 3. Thus, $F(x_0,\dots,x_n) = f(x_1,\dots,x_n)$ and the claim follows. □

Proposition 3.2.5. *Let $F \in \mathbb{C}[x_0, \dots, x_n]$ be a cubic such that* $\operatorname{rk} \mathcal{H}_F$ *is maximal at the generic point of* $\mathbb{P}^n$*.*

Then W_F is a finite set.

Remark 3.2.6. Note that the same result does not hold under the weaker assumption that $\operatorname{rk} \mathcal{H}_F(p) \geq 1$ for any $p \in \mathbb{P}^n$, e.g. consider

$$F(x_0, \dots, x_4) = x_4 x_3^2 + x_3 x_1 x_0 + x_2 x_1^2.$$

Proof. Let $W'_F = W_F \cap \{F = 0\}$. We first show that W'_F is a finite set. Assume by contradiction that there exist an irreducible curve C inside W'_F and let $p \in C$. We say that an hyperplane $H \subseteq \mathbb{P}^n$ is *associated* to p if:

1. $\det \mathcal{H}_F$ vanishes along H,
2. $p \in H$, and
3. if $G = F_{|H}$ then $\mathcal{H}_G(p)$ is trivial.

We first show that there exists exactly one hyperplane H_p associated to p. Lemma 3.2.4 implies that $\operatorname{rk} \mathcal{H}_F(p) = 1$. After taking a suitable coordinate change, we may assume that $p = [0, \dots, 0, 1]$. In particular

$$F(x_0, \dots, x_n) = x_n^2 \cdot L_1 + x_n \cdot Q_1 + R_1$$

, for some homogeneous polynomials $L_1, Q_1, R_1 \in \mathbb{C}[x_0, \dots, x_{n-1}]$ of degree $1, 2$ and 3 respectively. Since $p \in W_F$, it follows that $L_1 = 0$. By assumption, Q_1 is not zero. Using again the fact that $p \in W_F$ it follows that, after taking a suitable coordinate change in $x_0, \dots, x_{n-1}$, we may assume that $Q_1 = x_{n-1}^2$. We may write

$$R_1(x_0, \dots, x_{n-1}) = x_{n-1}^2 \cdot L + x_{n-1} \cdot Q + R,$$

for some homogeneous polynomials $L \in C[x_0, \dots, x_{n-1}]$ and $Q, R \in \mathbb{C}[x_0, \dots, x_{n-2}]$ of degree $1, 2$ and 3 respectively. After replacing x_n by $x_n + L$, we may assume that $L = 0$. Thus, we have

$$F(x_0, \dots, x_n) = x_n \cdot x_{n-1}^2 + x_{n-1} \cdot Q + R.$$

Let $H_p = \{x_{n-1} = 0\}$. An easy computation shows that H_p is an hyperplane associated to p. We now show that such an hyperplane is unique. Assume that $H' \subseteq \mathbb{P}^n$ is an hyperplane associated to p. Since $p \in H'$ we have $H' = \{\ell = 0\}$ for some linear function $\ell \in \mathbb{C}[x_0, \dots, x_{n-1}]$. If $H' \neq H_p$, after a suitable change of coordinates in $x_0, \dots, x_{n-2}$, we may assume that

$$\ell = x_0 - \alpha x_{n-1}$$

for some $\alpha \in \mathbb{C}$. Thus if $G' = F_{|H'}$, we may write

$$G(x_1, \dots, x_n) = x_n x_{n-1}^2 + x_{n-1} Q(\alpha x_{n-1}, \dots, x_{n-2}) + R(\alpha x_{n-1}, \dots, x_{n-2})$$

and it follows that

$$\partial_{n-1}\partial_{n-1}G(0,\dots,0,1)\neq 0,$$

which contradicts (3).

Now let $q\in C$ be a point such that $H_p=H_q$. We want to show that $q=p$. If $R=0$ then if follows easily that $W'_F=\{p\}$. Thus, by Lemma 3.2.4, after a suitable change in coordinates in $x_0,\dots,x_{n-2}$, we may assume that $R=R(x_0,\dots,x_k)$ for some $k\geq 0$ and that there is no point $z\in\mathbb{P}^k$ such that $\mathcal{H}_R(z)$ is trivial. If $q=[y_0,\dots,y_n]$, it follows by (3) that

$$y_0=\cdots=y_k=0.$$

Since $\operatorname{rk}\mathcal{H}_F(q)=1$, it follows the that the minor spanned by the i-th and $(n-i)$-th rows and columns of $\mathcal{H}_F(p)$ must have determinant equal to zero for any $i=0,\dots,n-2$ and in particular, since $y_{n-1}=0$ and $\mathcal{H}_R(y_0,\dots,y_{n-2})$ is trivial, it follows that $\partial_i Q(y_0,\dots,y_n)=0$. It is easy to show that this implies that if $q\neq p$ then $\det\mathcal{H}_R$ vanishes identically, a contradiction.

Since by assumption $\det\mathcal{H}_F$ is a non-trivial function, there exist only finitely many hyperplanes on which $\det\mathcal{H}_F$ vanishes and (1) implies that $H_p=H_q$ for infinitely many pair of points $p,q\in C$, a contradiction. Thus, W'_F is a finite set.

Now let $p\in W_F$ be a point such that $F(p)\neq 0$. After a suitable change of coordinates, we may assume that $p=[0,\dots,0,1]$ and that

$$F(x_0,\dots,x_n)=x_n^3+x_n^2\cdot L+x_n\cdot Q+R$$

for some homogeneous polynomials $L,Q,R\in\mathbb{C}[x_0,\dots,x_n]$ of degree 1, 2 and 3 respectively. After replacing x_n by $x_n+\frac{1}{3}L$ we may assume that $L=0$. Since $p\in W_F$ it follows that $Q=0$. Let $q=[z_0,\dots,z_n]\in W_F$. Then either $q=p$ or $z_n=0$ and $[z_0,\dots,z_{n-1}]\in W_R$. Thus, the result follows by induction on n. $\square$

Fix a positive integer n and let ℓ and k be non-negative integers such that $n\geq\ell+2k+1$. We will denote:

$$I_{\ell,k}=\{\ell+2i+1\mid i=0,\dots,k\}\cup\{\ell+2k+2,\dots,n\}.$$

Given a finite subset $I\subseteq\mathbb{N}$, we will also denote by $\mathbb{C}[x_I]$ the algebra of polynomials in x_i with $i\in I$.

Theorem 3.2.7. *Let $F\in\mathbb{C}[x_0,\dots,x_n]$ be a cubic form such that $\operatorname{rk}\mathcal{H}_F$ is maximal at the generic point of $\mathbb{P}^n$. Let $C\subseteq V_F$ be a curve such that $F(p)\neq 0$ at the general point of C.*

Then, there exist non-negative integers ℓ,k such that, after a suitable change of coordinates, we may write

$$F=\sum_{i=0}^{\ell}G_i+\sum_{i=1}^{k}(x_{\ell+2i+1}^2+M_i)\cdot x_{\ell+2i}+R_{\ell+k+1}$$

where

1. $G_i \in \mathbb{C}[x_i, x_{i+1}]$ *is a cubic for any* $i = 0, \dots, \ell$ *with*
$$G_0 = x_0^3 + x_0x_1^2;$$
2. $M_i = \delta_i x_{\ell+1}^2$ *for any* $i = 1, \dots, k$ *with* $\delta_i \in \mathbb{C}$*;*
3. $R_{\ell+k+1} \in \mathbb{C}[x_{I_{\ell,k}}]$ *is a cubic;*
4. $C \subseteq \bigcap_{i \in I_{\ell,k+1}} \{x_i = 0\}$.

Moreover if $C \not\subseteq \{x_{l+2k+2} = 0\}$ *we may write*
$$R_{\ell+k+1} = M_{k+1} \cdot x_{\ell+2k+2} + R_{l+k+2}$$
where

5. $R_{\ell+k+2} \in \mathbb{C}[x_{I_{\ell,k+1}}]$ *is a cubic and* $M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \dots, x_{\ell+2k+1}]$ *is a quadric.*

Proof.

Step 1. By Proposition 3.2.5 there exists $p \in C$ such that $F(p) \neq 0$ and $\operatorname{rk} \mathcal{H}_F(p) = 2$. Since $F(p) \neq 0$, after a suitable change of coordinates we may assume that $p = [1, 0, \dots, 0]$ and
$$F = x_0^3 + x_0^2 L + x_0 Q + R$$
for some homogeneous polynomials $L, Q, R \in \mathbb{C}[x_1, \dots, x_n]$ of degree 1,2 and 3 respectively. After replacing x_0 by $x_0 - \frac{1}{3}L$ we may assume that $L = 0$. Since $\operatorname{rk} \mathcal{H}_F(p) = 2$, after a suitable change of coordinates in $x_1, \dots, x_n$, we may assume that $Q = x_1^2$. Thus, we have
$$F = G_0 + R_1,$$
where $G_0 = x_0^3 + x_0x_1^2$ and $R_1 = R \in \mathbb{C}[x_1, \dots, x_n]$. We distinguish two cases. If C is contained in the hyperplane $\{x_1 = 0\}$, then we set $k = \ell = 0$ and we continue to Step 3. Otherwise, we set $\ell = 1$ and we proceed to Step 2.

Step 2. We are assuming that
$$F = \sum_{i=0}^{\ell-1} G_i + R_\ell$$
where $G_i \in \mathbb{C}[x_i, x_{i+1}]$ and $R_\ell \in \mathbb{C}[x_\ell, \dots, x_n]$ are cubics, and C is not contained in the hyperplane $\{x_\ell = 0\}$. We claim that after a suitable change of coordinates in $x_\ell, \dots, x_n$, we may write
$$R_\ell = G_\ell + R_{\ell+1}$$

where $G_\ell \in \mathbb{C}[x_\ell, x_{\ell+1}]$ and $R_{\ell+1} \in \mathbb{C}[x_{\ell+1}, \dots, x_n]$ are cubics. Assuming the claim, if C is contained in the hyperplane $\{x_{\ell+1} = 0\}$ we set $k = 0$ and we proceed to Step 3. Otherwise, we replace ℓ by $\ell + 1$ and we repeat Step 2.

We now prove the claim. By assumption, there exists $q \in C$ such that $q \notin \{x_\ell = 0\}$. After a suitable change of coordinates in $x_\ell, \dots, x_n$, we may assume that

$$q = [z_0, \dots, z_{\ell-1}, 1, 0, \dots, 0],$$

for some $z_0, \dots, z_{\ell-1} \in \mathbb{C}$. We may write

$$R_\ell = \alpha_\ell x_\ell^3 + L_\ell x_\ell^2 + Q_\ell x_\ell + R_{\ell+1},$$

for some homogeneous polynomials $L_\ell, Q_\ell, R_\ell \in \mathbb{C}[x_{\ell+1}, \dots, x_n]$ of degree 1,2 and 3 respectively. Since $\operatorname{rk} \mathcal{H}_F(q) \leq 2$, after a suitable change of coordinates, we may write $L_\ell = \beta_\ell x_{\ell+1}$ and $Q_\ell = \gamma_\ell x_{\ell+1}^2$ for some $\beta_\ell, \gamma_\ell \in \mathbb{C}$. We may define

$$G_\ell = \alpha_\ell x_\ell^3 + \beta_\ell x_\ell^2 \cdot x_{\ell+1} + \gamma_\ell x_\ell \cdot x_{\ell+1}^2$$

and the claim follows.

Step 3. We are assuming that

$$F = \sum_{i=0}^{\ell} G_i + \sum_{i=1}^{k} (x_{\ell+2i+1}^2 + M_i) \cdot x_{\ell+2i} + R_{\ell+k+1}$$

where G_i, M_i and $R_{\ell+k+1}$ satisfy (1), (2) and (3) and

$$C \subseteq \{x_{\ell+1} = x_{\ell+3} = \cdots = x_{\ell+2k+1} = 0\}.$$

If we also have that

$$C \subseteq \{x_{\ell+2k+2} = \cdots = x_n = 0\}$$

then we are done. In particular, if $n < \ell + 2k + 2$, then we are done. Otherwise, after a suitable change of coordinates in $x_{\ell+2k+2}, \dots, x_n$ we may assume that there exists

$$q = [z_0, \dots, z_n] \in C$$

such that $z_{\ell+2k+2} \neq 0$ and $z_{\ell+2k+3} = \cdots = z_n = 0$. Since

$$\det(\partial_i \partial_j F(p))_{i,j=0,1} \neq 0,$$

we may assume that the same inequality holds for q. We may write

$$R_{\ell+k+1} = \alpha_{\ell+k+1} x_{\ell+2k+2}^3 + x_{\ell+2k+2}^2 \cdot L_{\ell+k+1} + x_{\ell+2k+2} \cdot Q_{\ell+k+1} + R_{\ell+k+2}$$

where $\alpha_{\ell+k+1} \in \mathbb{C}$, and $L_{\ell+k+1}, Q_{\ell+k+1}, R_{\ell+k+2} \in \mathbb{C}[x_{I_{\ell,k+1}}]$ are homogeneous polynomials of degree 1, 2 and 3 respectively.

We first assume that $\alpha_{\ell+k+1} \neq 0$. After replacing $x_{\ell+2k+2}$ by $x_{\ell+2k+2} - \frac{1}{3\alpha_{\ell+k+1}} L_{\ell+k+1}$, we may assume that $L_{\ell+k+1} = 0$. Since $q \in V_F$, we get a contradiction by considering the minor

$$(\partial_i \partial_j F(q))_{i,j=0,1,\ell+2k+2}.$$

We now assume that $\alpha_{\ell+k+1} = 0$. Since $z_{\ell+2k+2} \neq 0$ and $q \in V_F$ it follows that $L_{\ell+k+1} = 0$ and that after a suitable change of coordinates, $Q_{\ell+k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \dots, x_{\ell+2k+3}]$. We may write

$$Q_{\ell+k+1} = \beta_k x_{\ell+2k+3}^2 + x_{\ell+2k+3} \cdot \ell_k + M_k$$

where $\beta_k \in \mathbb{C}$ and $\ell_k, M_k \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \dots, x_{\ell+2k+1}]$ are homogeneous polynomials of degree 1 and 2 respectively. If $\beta_k \neq 0$ then, after a suitable change of coordinates, we may assume $\beta_k = 1$ and $\ell_k = 0$. By considering the minor

$$(\partial_i \partial_j F(q))_{i,j=0,\ell+2k+2,\ell+2k+3}$$

it follows that $C \subseteq \{x_{\ell+2k+3} = 0\}$. Thus, we may proceed to Step 4.

If $\beta_k = 0$, then since $q \in V_F$ it follows that $\ell_k = 0$. In case C is contained in $\{x_{\ell+2k+3} = \dots = x_n = 0\}$ we are done, so we may assume that there exists a point

$$q' = [z'_0, \dots, z'_n] \in C \cap \bigcap_{i \in J} \{x_i = 0\}$$

such that $z'_0 \neq 0$ and $z'_{\ell+2k+3} \neq 0$, where, $J = I_{\ell,k+1} \setminus \{\ell+2k+3\}$. Proceeding as above, we may write

$$R_{\ell+k+2} = x_{\ell+2k+3} \cdot Q_{\ell+k+2} + R_{\ell+k+3},$$

where $Q_{\ell+k+2} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \dots, x_{\ell+2k+1}, x_{\ell+2k+4}]$ and $R_{\ell+k+3} \in \mathbb{C}[x_J]$ are homogeneous polynomials of degree 2 and 3 respectively. We may write

$$Q_{\ell+k+2} = \beta_{k+1} x_{\ell+2k+4}^2 + x_{\ell+2k+4} \cdot \ell_{k+1} + M_{k+1}$$

where $\beta_{k+1} \in \mathbb{C}$ and $\ell_{k+1}, M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \dots, x_{\ell+2k+1}]$ are homogeneous polynomials of degree 1 and 2 respectively.

If $\beta_{k+1} = 0$ then $\ell_{k+1} = 0$ because $q' \in V_F$. Denoting by $\mathcal{H}_F^i$ the i-th column of $\mathcal{H}_F$, it follows that the vectors $\mathcal{H}_F^{\ell+2}, \mathcal{H}_F^{\ell+4} \dots, \mathcal{H}_F^{\ell+2k+2}$ and $\mathcal{H}_F^{\ell+2k+3}$ are linearly dependent. Thus, $\mathcal{H}_F$ does not have maximal rank which contradicts the assumptions.

Hence we have $\beta_{k+1} \neq 0$. After a suitable change of coordinates, we may assume that $\beta_{k+1} = 1$ and $\ell_{k+1} = 0$. By considering the minor

$$(\partial_i \partial_j F(q'))_{i,j=0,\ell+2k+3,\ell+2k+4}$$

it follows that $C \subseteq \{x_{\ell+2k+4} = 0\}$. Thus we first exchange $x_{\ell+2k+3}$ and $x_{\ell+2k+4}$, then we exchange $x_{\ell+2k+2}$ and $x_{\ell+2k+4}$. So we may write

$$R_{\ell+k+1} = x_{\ell+2k+2} \cdot (x_{\ell+2k+3}^2 + M_{k+1}) + R_{\ell+k+2}$$

where $M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}]$ is a quadric, $R_{\ell+k+2} \in \mathbb{C}[x_{I_{\ell,k+1}}]$ is a cubic and $C \subset \{x_{\ell+2k+3}\}$. We also may write

$$R_{\ell+k+2} = x_{\ell+2k+4} \cdot M_{k+2} + R_{\ell+k+3}$$

where $M_{k+2} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}]$, $R_{\ell+k+3} \in \mathbb{C}[x_{I_{\ell,k+2}}]$ are homogeneous polynomials of degree 2 and 3 respectively.

Moreover we have a point

$$q' = [z_0', \ldots, z_n'] \in C \cap \bigcap_{i \in J} \{x_i = 0\}$$

such that $z_0' \neq 0$ and $z_{\ell+2k+2}' \neq 0$, where $J = I_{\ell,k+1} \setminus \{\ell+2k+4\}$. Replacing $x_{\ell+2k+4}$ by $x_{\ell+2k+4} + \frac{z_{\ell+2k+4}'}{z_{\ell+2k+2}'} x_{\ell+2k+2}$ we get a point

$$q = [z_0, \ldots, z_n] \in C \cap \bigcap_{i \in I_{l,k+1}} \{x_i = 0\}$$

such that $z_0 \neq 0$, $z_{\ell+2k+2} \neq 0$ and we may proceed to Step 4.

Step 4. We are assuming that

$$F = \sum_{i=0}^{\ell} G_i + \sum_{i=1}^{k} (x_{\ell+2i+1}^2 + M_i) \cdot x_{\ell+2i} + R_{\ell+k+1}$$

where G_i, M_i and $R_{\ell+k+1}$ satisfy (1), (2) and (3) and

$$C \subseteq \{x_{\ell+1} = x_{\ell+3} = \cdots = x_{\ell+2k+1} = 0\}.$$

By Step 3 we also have that

$$R_{\ell+k+1} = x_{\ell+2k+2} \cdot (x_{\ell+2k+3}^2 + M_{k+1}) + R_{\ell+k+2}$$

where $M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}]$ is homogeneous of degree 2 and $C \subset \{x_{\ell+2k+3} = 0\}$. Moreover there is a point $q = [z_0, \ldots, z_n]$ such that $z_0 \neq 0$, $z_{\ell+2k+2} \neq 0$ and

$$q \in C \cap \bigcap_{i \in I_{l,k+1}} \{x_i = 0\}.$$

We show that we may assume

$$M_{k+1} = \delta_{k+1} x_{\ell+1}^2$$

where $\delta_k \in \mathbb{C}$.

Since $q \in C$ and $z_{\ell+2k+2} \neq 0$ we have $\det(\partial_i \partial_j F(q))_{i,j=0,1} = 0$. Considering the minors

$$(\partial_i \partial_j F(q))_{i=0,h,\ell+2k+3}^{i=0,m,\ell+2k+3}$$

for $h, m = 1, \dots, n$, $(h, m) \neq (\ell + 2k + 3, \ell + 2k + 3)$ we deduce that $\partial_h \partial_m F(q) = 0$ and so, since by induction $M_i = \delta_i x_{\ell+1}$ for $i = 1, \dots k$, we have

$$M_{k+1} = \sum_{j=0}^{k} \gamma_k^j x_{\ell+2j+1}^2,$$

where $\gamma_k^j \in \mathbb{C}$. Since $M_j = \delta_j x_{\ell+1}$ for $j = 1, \dots k$ to conclude it is enough to replace $x_{\ell+2j}$ with $x_{\ell+2j} - \gamma_k^j x_{\ell+2k+2}$ for $j = 1, \dots, k$. In this way we get

$$M_{k+1} = \delta_{k+1} x_{\ell+1}^2$$

where $\delta_{k+1} = \gamma_k^0 - \sum_{i=1}^{k} \gamma_k^i \delta_i$.

After replacing k by $k + 1$, we may repeat Step 3. □

Theorem 3.2.8. *Let $F \in \mathbb{C}[x_0, \dots, x_n]$ be a cubic such that $\operatorname{rk} \mathcal{H}_F$ is maximal at the generic point of $\mathbb{P}^n$.*

Then the set of points $p \in V_F$ such that $F(p) \neq 0$ is a finite union of points, lines, plane conics and plane cubics.

Proof. We may assume that there is an irreducible component $C \subset V_F$ such that $\dim C \geq 1$ and $F(p) \neq 0$ at the general point p of C, otherwise we are done. By Theorem 3.2.7 we may write

$$F = \sum_{i=0}^{\ell} G_i + \sum_{i=1}^{k} (x_{\ell+2i+1}^2 + M_i) \cdot x_{\ell+2i} + R_{\ell+k+1}$$

where G_i, M_i and $R_{\ell+k+1}$ are as in 3.2.7 and

$$C \subseteq \{x_{\ell+1} = x_{\ell+3} = \dots = x_{\ell+2k+1} = 0\}.$$

By the proof of 3.2.7 we may also assume that for any $i = 1, \dots, k$ there is a point $q_i \in C$ such that $q_i \notin \{x_0 = 0\}$, $q_i \notin \{x_{\ell+2i} = 0\}$ and $q_i \in \bigcap_{j=2i+1}^{n} \{x_{\ell+j} = 0\}$.

We distinguish two cases: $C \subset \{x_1 = 0\}$ and $C \not\subset \{x_1 = 0\}$.

If $C \subset \{x_1 = 0\}$ then $\ell = 0$. Let $z = [z_0, \dots, z_n] \in C$ be a general point in C.

If $C \subset \{x_{2k+2} = 0\}$ then considering

$$(\partial_i \partial_j F(z))_{i=0,1,2k+1}^{i=0,1,2k+1}$$

we immediately get a contradiction because $\det(\partial_i\partial_j F(z))_{i,j=0,1} \neq 0$ and $z_{2k} \neq 0$.

So let $C \not\subset \{x_{2k+2} = 0\}$. Then we may write

$$R_{\ell+k+1} = M_{k+1} \cdot x_{\ell+2k+2} + R_{l+k+2}$$

as in 5 of 3.2.7. Assume that $k > 2$. Then we have

$$\begin{aligned} &\det(\partial_i\partial_j F)_{i=0,1,2k+1}^{j=0,3,2k+1} = \\ &= 6x_0 \cdot (2\gamma_{1,3}x_{2k}x_{2k+2} + \gamma_{1,3}\gamma_{2k+1,2k+1}x_{2k+2}^2 - \gamma_{1,2k+1}\gamma_{3,2k+1}x_{2k+2}^2 + Q) \end{aligned}$$

where $Q \in \mathbb{C}[x_1, \dots, x_n]$ is a quadratic form such that $C \subset \{Q = 0\}$ (because $C \subseteq \bigcap_{i \in I_{\ell,k+1}} \{x_i = 0\}$) and where $\gamma_{i,j}$ is the coefficient of x_{2k+2} in $\partial_i\partial_j F$. Note that $\gamma_{1,3} \neq 0$ (because $\partial_3\partial_3 F(z) \neq 0$, being this last inequality true for q_2).

Since $z_0 \neq 0$ and $z_{\ell+2k} \neq 0$ we conclude that $C \subset \{2\gamma_{1,3}x_{2k} + (\gamma_{1,3}\gamma_{2k+1,2k+1} - \gamma_{1,2k+1}\gamma_{3,2k+1})x_{2k+2} = 0\}$, which contradicts $q_k \in C$. Hence we conclude that $k \leq 2$. Now it is easy to see that C is a line or a plane conic.

Assume now that $C \not\subset \{x_1 = 0\}$. Then $\ell \geq 1$. Note that for $j = 3, \dots, n$ we have $\partial_1\partial_j F = 0$, hence for a general point $z = [z_0, \dots, z_n] \in C$, for $h = 2, \dots, n$ and for $m = 3, \dots, n$ we may consider

$$(\partial_i\partial_j F(z))_{i=0,1,h}^{j=0,1,m}$$

to conclude that $\partial_h\partial_m F(z) = 0$ (because $\det(\partial_i\partial_j F(z))_{i,j=0,1} \neq 0$). This implies easily that we may assume $k = 0$. By Step 2. of the proof of 3.2.7 for any $i = 1, \dots, \ell$ there is a point $p_i \in C$ such that $p_i \notin \{x_0 = 0\}$, $p_i \notin \{x_i = 0\}$ and $p_i \in \bigcap_{j=i+1}^n \{x_j = 0\}$.

Assume first that $C \subset \{x_{\ell+2} = 0\}$ so we may write

$$F = \sum_{i=0}^{\ell} G_i + R_{\ell+1}$$

where $G_i \in \mathbb{C}[x_i, x_{i+1}]$, $R_{\ell+1} \in \mathbb{C}[x_{\ell+1}, \dots, x_n]$ are cubics and $C \subset \bigcap_{i=\ell+1}^n \{x_i = 0\}$.

Suppose that $\ell > 2$. Since $\partial_3\partial_3 F(p_2) = 0$, $\partial_2\partial_3 F(p_2) = 0$ and $\partial_3\partial_3 F(p_3) = 0$ we see that the monomials $x_2x_3^2$, $x_2^2x_3$ and x_3^3 do not appear in F. The same holds for $x_3x_4^2$ and $x_3^2x_4$ which gives a contradiction. Hence $\ell \leq 2$ and it is easy to conclude.

If $C \not\subset \{x_{\ell+2} = 0\}$ then we may write

$$F = \sum_{i=0}^{\ell} G_i + x_{\ell+1}^2 \cdot x_{\ell+2} + R_{\ell+1}.$$

where $G_i \in \mathbb{C}[x_i, x_{i+1}]$ and $R_{\ell+1} \in \mathbb{C}[x_{I_{\ell,1}}]$.

Suppose $\ell \geq 2$. Since $\partial_{\ell+1}\partial_{\ell+1}F(p_\ell) = 0$ we see that $x_{\ell+1}^2 x_\ell$ does not appear in F and this implies, considering $\partial_{\ell+1}\partial_{\ell+1}F(z)$, that also $x_{\ell+1}^2 x_{\ell+2}$ does not appear in F, which is a contradiction. Thus $\ell < 2$ and we are done. □

Remark 3.2.9. Note that in general V_F might contain surfaces, e.g. if

$$F(x_0, \dots, x_n) = x_n^3 + x_{n-1}^2 x_n + x_{n-1} \cdot \sum_{i=0}^{n-2} x_i^2$$

then $\dim V_F = n - 2$.

For our purposes we need to refine Theorems 3.2.7 and 3.2.8 in the case of cubic forms with integral coefficients. If $C \subseteq \{x_1 = 0\}$ then it is straightforward to reduce F to a special form acting with $\mathrm{SL}(n-1, \mathbb{Z})$. If $C \not\subseteq \{x_1 = 0\}$ we have to use an $\mathrm{SL}(\mathbb{Q}, n)$ transformation, but if $\Delta_F \neq 0$ we can isolate an integral ternary cubic that gives the curve C; in this way we can apply the results on ternary cubics. If $C \not\subseteq \{x_1 = 0\}$ and $\Delta_F = 0$ then we obtain a very special form.

Corollary 3.2.10. *Let*

$$F(x_0, \dots, x_n) = a x_0^3 + b x_0^2 x_1 + G(x_1, \dots, x_n)$$

be a non-degenerate cubic form with integral coefficients. Let $C \subseteq V_F$ be an irreducible variety of positive dimension such that $C \not\subseteq \{F = 0\}$, $p = [1, 0, \dots, 0] \in C$ and $C \subseteq \{x_1 = 0\}$. Assume that C contains infinitely many rational points. Then acting on $(x_2, \cdots, x_n)$ with $\mathrm{SL}(n-1, \mathbb{Z})$ we may assume that either

- $C = \{x_1 = x_3 = x_4 = \ldots = x_n = 0\}$ *is a line and*

  $$F = a x_0^3 + b x_0^2 x_1 + c x_1^2 x_2 + R$$

  where $c \in \mathbb{Z}$ and $R \in \mathbb{Z}[x_1, x_3, x_4, \dots, x_n]$ is a cubic or

- $C \subseteq \Pi = \{x_1 = x_3 = x_5 = x_6 = \ldots = x_n = 0\}$ *is a conic and*

  $$F = a x_0^3 + b x_0^2 x_1 + x_2 M_1 + x_4 M_2 + R$$

  where $M_1, M_2 \in \mathbb{Z}[x_1, x_3]$ are quadric forms and $R \in \mathbb{Z}[x_1, x_3, x_5, x_6, \dots, x_n]$ is a cubic.

Proof. The proof follows the same ideas as the proof of Theorem 3.2.7. Note that by assumption $a \neq 0$. Acting on $(x_2, \dots, x_n)$ with $\mathrm{SL}(n-1, \mathbb{Z})$ we may assume that there exists a point $q = [z_0, 0, z_2, 0, \ldots, 0] \in C$ such that $z_0, z_2 \neq 0$. We may also assume that

$$\det(\partial_i \partial_j F(q))_{i,j=0,1} \neq 0$$

since this is true for $p = [1, 0, \ldots, 0]$.

Write

$$F = ax_0^3 + bx_0^2 x_1 + \alpha_1 x_2^3 + x_2^2 \cdot L_1 + x_2 \cdot Q_1 + R_2$$

where $\alpha_1 \in \mathbb{Z}$, and $L_1, M_1, R_2 \in \mathbb{Z}[x_1, x_3, x_4, \ldots, x_n]$ are homogeneous polynomials of degree 1, 2 and 3 respectively. We claim that $\alpha_1 = 0$. Assume by contradiction that $\alpha_1 \neq 0$. After replacing x_2 by $x_2 - \frac{1}{3\alpha_1} L_1$, we may assume $L_1 = 0$. Since $q \in V_F$, we get a contradiction by considering the minor

$$(\partial_i \partial_j F(q))_{i,j=0,1,2}.$$

Hence $\alpha_1 = 0$ and since $q \in V_F$ we also get $L_1 = 0$. It is straightforward to check that $\operatorname{rk} M_1 \leq 2$ and that acting on $(x_3, \ldots, x_n)$ with $\mathrm{SL}(n-2, \mathbb{Z})$ we may assume $M_1 \in \mathbb{Z}[x_1, x_3]$.

If C is a line then $C = \{x_1 = x_3 = x_4 = \ldots x_n = 0\}$ and it is easy to check that $M_1 = cx_1^2$.

If C is not a line then acting with $\mathrm{SL}(n-3, \mathbb{Z})$ on $(x_4, \ldots, x_n)$ we get a new point $q_1 = [y_0, 0, y_2, 0, y_4, 0, \ldots, 0]$ such that $y_0, y_2, y_4 \neq 0$ and we can proceed as in the proof of Theorem 3.2.7. Thus, the claim follows. □

Corollary 3.2.11. *Let*

$$F(x_0, \ldots, x_n) = ax_0^3 + x_0^2(bx_1 + cx_2) + G(x_1, \ldots, x_n)$$

be a non-degenerate cubic form with integral coefficients such that $b \neq 0$. *Assume that the line* $C = \{x_2 = x_3 = \ldots = x_n = 0\}$ *is contained inside* V_F.

Then there exists $T = (t_{ij})_{i,j=0,\ldots,n} \in SL(n+1, \mathbb{Q})$ *such that*

$$T \cdot F = ax_0^3 + bx_0^2 x_1 + c_1 x_1^3 + R(x_2, \ldots, x_n)$$

where $c_1 \in \mathbb{Z}$ *and* $R \in \mathbb{Q}[x_2, \ldots, x_n]$ *is a cubic form. Moreover we may choose* T *such that* $t_{00} = t_{11} = 1$, $t_{0i} = t_{i0} = 0$ *for* $i = 1, \ldots, n$, $t_{ij} = 0$ *for* $i = 2, \ldots, n$ *and* $j = 1$

Proof. After replacing x_1 by $x_1 - cx_2/b$, we may write

$$F = ax_0^3 + bx_0^2 x_1 + c_1 x_1^3 + Lx_1^2 + Qx_1 + R$$

where $c_1 \in \mathbb{Z}$ and $L, Q, R \in \mathbb{Q}[x_2, \ldots, x_n]$ are homogeneous polynomials of degree 1,2 and 3 respectively. Acting on $(x_2, \ldots, x_n)$ we may also assume that $L = c_2 x_2$, for some $c_2 \in \mathbb{Q}$. Let $q = [0, 1, 0 \ldots, 0] \in C$. We distinguish two cases: $c_1 \neq 0$ and $c_1 = 0$.

If $c_1 \neq 0$ then, since $b \neq 0$ and $\operatorname{rk} \mathcal{H}_F(q) \leq 2$, we see that $Q = c_3 x_2^2$ for some $c_3 \in \mathbb{Q}$ and

$$|(\partial_i \partial_j F(q))_{i=1,2}| = 0.$$

Thus, $|(\partial_i \partial_j F(z))_{i=1,2}| = 0$ for any $z \in C$. Looking at $(\partial_i \partial_j F(z))_{i=0,1,2}$, it follows that $c_2 = 0$ and therefore $c_3 = 0$. Thus, $L = Q = 0$ and we are done.

If $c_1 = 0$ then since $b \neq 0$ and $\operatorname{rk} \mathcal{H}_F(q) \leq 2$, it follows that $c_2 = 0$. Since $\operatorname{rk} \mathcal{H}_F(z) \leq 2$ at the general point $z \in C$, we have $Q = 0$ and we are done again. Note that in this case $\Delta_F = 0$. □

Corollary 3.2.12. *Let*

$$F(x_0, \ldots, x_n) = a x_0^3 + x_0^2(b x_1 + c x_3) + G(x_1, \ldots, x_n)$$

be a non-degenerate cubic form with integral coefficients such that $b \neq 0$.

Let $C \subseteq V_F$ *be an irreducible variety of positive dimension such that* $C \not\subseteq \{F = 0\}$, $p = [1, 0, \ldots, 0] \in C$ *and* $C \not\subseteq \{x_1 = 0\}$. *Assume that* C *contains infinite many rational points. Assume moreover that* $C \subseteq \Pi = \{x_3 = \ldots = x_n = 0\}$ *and* C *is not a line.*

Then there exists $T = (t_{ij})_{i,j=0,\ldots,n} \in SL(n+1, \mathbb{Q})$ *such that* $t_{00} = 1$, $t_{i0} = t_{0i} = 0$ *for* $i = 1, \ldots, n$, $t_{ij} = 0$ *for* $i = 3, \ldots, n$ *and* $j = 1, 2$, $(t_{ij})_{i,j=0,1,2} \in \mathrm{SL}(3, \mathbb{Z})$ *and such that one of the following holds:*

(a) there exist $R \in \mathbb{Z}[x_1, x_2]$ *and* $S \in \mathbb{Q}[x_3, \ldots, x_n]$ *cubic forms such that*

$$T \cdot F = a x_0^3 + b x_0^2 x_1 + R(x_1, x_2) + S(x_3, \ldots, x_n);$$

or

(b)

$$T \cdot F = a x_0^3 + b x_0^2 x_1 + c_1 x_1^3 + \alpha x_1^2 x_3 + \beta x_1 x_3^2 + \gamma x_2 x_3^2 + S(x_3, \ldots, x_n)$$

where $c_1 \in \mathbb{Z}$, $\alpha, \beta, \gamma \in \mathbb{Q}$ *and* $S \in \mathbb{Q}[x_3, \ldots, x_n]$ *is a cubic form (note that* $\Delta_F = 0$*).*

Proof. We may assume that there is a point $q = [z_0, z_1, 0, \ldots, 0] \in C$ such that $z_0, z_1 \neq 0$. Indeed for a general $m \in \mathbb{Z}$ we may intersect the line $\{m x_1 + x_2 = 0\} \cap \Pi$ with C to obtain a point of the form $[z_0, 1, -m, 0, \ldots, 0]$ where $z_0 \neq 0$. Then replacing x_2 with $x_2 + m x_1$ we get our point q. Note that this transformation is in $\mathrm{SL}(n+1, \mathbb{Z})$.

Then we replace x_1 with $x_1 - c/b x_3$ and write

$$F(x_0, \ldots, x_n) = a x_0^3 + b x_0^2 x_1 + c_1 x_1^3 + c_2 x_1^2 x_2 + c_3 x_1 x_2^2 + c_4 x_2^3 + x_1^2 L + x_1 Q + S(x_2, \ldots, x_n)$$

where $c_i \in \mathbb{Z}$ and $L \in \mathbb{Q}[x_3, \ldots, x_n]$, and $Q, G \in \mathbb{Q}[x_2, \ldots, x_n]$ are forms of degree 1,2 and 3 respectively such that the coefficient of x_2^2 in Q and the coefficient of x_2^3 in S are zero.

If $c_2 \neq 0$ then, possibly replacing x_2 with $x_2 - L/c_2$, we may assume $L = 0$. Since $b \neq 0$ and $q \in V_F$, it follows that $Q = 0$. Now considering a general point $z \in C\{x_3 = \ldots = x_n = 0\}$ it is easy to see that S does not depend on x_2 and we are done.

Assume that $c_2 = 0$ and $L = 0$. Then the Hessian $c_3x_2^2 + Q$ has rank not greater than 1, which means that

$$c_3x_2^2 + Q = c_3(x_2 + L_1)^2$$

for some $L_1 \in \mathbb{Q}[x_3, \cdots, x_n]$ of degree 1. Hence, replacing x_2 with $x_2 - L_1$ we may assume that $Q = 0$. It is now immediate to conclude.

Finally assume that $c_2 = 0$ and $L \neq 0$. Acting on $(x_3, \ldots, x_n)$ with $\mathrm{SL}(n-2, \mathbb{Q})$ we may write $L = \alpha x_3$, where $\alpha \neq 0$. Note that the columns $\mathcal{H}_F^0(q)$ and $\mathcal{H}_F^1(q)$ are linearly independent, which implies that $c_3 = 0$ and $Q = \beta x_3^2$. Considering now a general point $y = [y_0, y_1, y_2, 0, \ldots, 0] \in C$ it is immediate to see that $c_4 = 0$ and that x_2 appears just in $\gamma x_2 x_3^2$. Since $[0, 0, 1, 0, \ldots, 0]$ is a singular point of F we conclude that $\Delta_F = 0$ and we get (b). □

3.2.3 Binary and ternary cubics

We now study reduced forms of non-degenerate binary and ternary cubics over the integers. If F is a binary cubic the question is easily settled: there are just a finite number of reduced forms (see Proposition 3.2.15). Let

$$F = ax^3 + bx^2y + G(y, z)$$

be an integral ternary cubic.

We may summarize the results of this section as follows:

- if $\Delta_F \neq 0$ then there are only finitely many reduced forms for F (see Proposition 3.2.18);
- if $\Delta_F = \Delta_G = 0$ and $S \neq 0$ then there may be an infinite number of reduced forms for F (see example 3.2.19);
- if $\Delta_F = \Delta_G = 0$ and $S = 0$ then we have the finiteness of a_i and G_i for the possible reduced forms (a_i, b_i, G_i) of F (see Lemma 3.2.22);

Let $F \in \mathbb{C}[x, y, z]$ be a cubic form. We denote by S_F and T_F the two $\mathrm{SL}(3, \mathbb{C})$-invariants of F as defined in 4.4.7 and 4.5.3 of [Stu]. Recall that the discriminant of F satisfies

$$\Delta_F = T_F^2 - 64S_F^3.$$

If there is no ambiguity we will drop the subscript F.

We recall the following result (see, for instance, Proposition 7 in [OVdV95]):

Proposition 3.2.13. *Let $\Delta \neq 0$ be an integer. Then there are only finitely many classes of binary and ternary cubics over $\mathbb{Z}$ with discriminant Δ.*

Let K be a number field, i.e. a finite field extension of $\mathbb{Q}$. If W is a finite set of valuations on K we indicate with R_W the ring of W-integers of K, that is

$$R_W = \{k \in K \ : \ \nu(k) \geq 0 \text{ for all } \nu \notin W\}.$$

We recall two celebrated theorems that will be very useful in the current section.

Theorem (Siegel)**.** *Let R_W be a ring of integers in the number field K and let C be an affine smooth curve defined over K such that $g(C) \geq 1$. Then there are only finitely many R_W-integral points on C.*

Theorem (Faltings)**.** *Let K be a number field and let C be a smooth algebraic curve defined over K such that $g(C) \geq 2$. Then C has only a finite number of K-rational points.*

We start to investigate binary cubics.

Lemma 3.2.14. *Let $F(x,y) = ax^3 + bx^2y + cy^3 \in R_W[x,y]$ be a binary cubic on a ring of integers R_W such that $c \neq 0$. Then there are only finitely many constants $a', b' \in R_W$ such that F can be written in reduced form with associated triple (a', b', cy^3).*

Proof. The discriminant D of F is given by

$$D = 4b^3c + 27a^2c^2.$$

Assume first that $D \neq 0$ and set $d := D/c$. We are looking for the integral points on the affine plane curve C given, in coordinates (s,t), by the equation

$$4s^3 + 27ct^2 - d = 0.$$

Since $d \neq 0$ we have that C is a smooth curve of genus 1. By Siegel's Theorem the integral points of C are finitely many and so we are done.

Assume now that $D = 0$ (the following proof actually works also for $D \neq 0$). Consider an $\mathrm{SL}(2,\mathbb{Z})$ coordinate change

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' \\ y &= \mu_1 x' + \mu_2 y'. \end{aligned}$$

such that $F(x,y) = a'x'^3 + b'x'^2y' + cy'^3$, for $a', b' \in \mathbb{Z}$. Note that $F(\lambda_2, \mu_2) = c$ and, since $c \neq 0$, the equation $F(x,y) = c$ gives a smooth plane curve of genus 1. By Siegel's Theorem the are only a finite number of solutions. For (λ_1, μ_1) we have the two linear equations

$$\begin{aligned}1 &= \lambda_1\mu_2 - \lambda_2\mu_1,\\ 0 &= (3a\lambda_2^2 \ + 2b\lambda_2\mu_2)\lambda_1 + (b\lambda_2^2 + 3c\mu_2^2)\mu_1\end{aligned}$$

and we can easily conclude.

□

Proposition 3.2.15. *Let $F(x, y) = ax^3 + bx^2y + cy^3 \in \mathbb{Z}[x, y]$ be a binary integral cubic with $c \neq 0$. Then there are only finitely many integers $a', b', c' \in \mathbb{Z}$ such that $c' \neq 0$ and F can be written in reduced form with associated triple $(a', b', c'y^3)$.*

Proof. By Lemma 3.2.14 it is enough to show that there is only a finite number of possible $c' \in \mathbb{Z}$.

If the discriminant $D = 4b^3c + 27a^2c^2$ is not zero, then $c'|D$ and we are done.

Assume that $D = 0$. We may also assume that a, b and c do not have a common factor (otherwise just consider the cubic obtained erasing this common factor). Suppose that there are a cubic form $F'(x', y') = a'x'^3 + b'x'^2y' + c'y'^3$ and a $\mathrm{SL}(2, \mathbb{Z})$ coordinate change

$$\begin{aligned}x' &= \lambda_1 x + \lambda_2 y\\ y' &= \mu_1 x + \mu_2 y.\end{aligned}$$

such that $F'(x', y') = ax^3 + bx^2y + cy^3 = F(x, y)$ and $c' \neq 0$. We have

$$a = a'\lambda_1^3 + b'\lambda_1^2\mu_1 + c'\mu_1^3, \tag{3.3}$$
$$b = 3a'\lambda_1^2\lambda_2 + b'\lambda_1^2\mu_2 + 2b'\lambda_1\lambda_2\mu_1 + 3c'\mu_1^2\mu_2, \tag{3.4}$$
$$0 = 3a'\lambda_1\lambda_2^2 + b'\lambda_2^2\mu_1 + 2b'\lambda_1\lambda_2\mu_2 + 3c'\mu_1\mu_2^2, \tag{3.5}$$
$$c = a'\lambda_2^3 + b'\lambda_2^2\mu_2 + c'\mu_2^3 \tag{3.6}$$

and $GCD(a', b', c') = 1$.

Let p be a prime factor of c' such that $p \neq 2, 3$ and let α be a positive integer such that $p^\alpha|c'$. Then, since $D = 0$, it follows that $p^{\alpha/3}$ divides b'. By (3.5) we deduce that either $p^{\alpha/3}|\lambda_1$ or $p^{\alpha/6}|\lambda_2$ (recalling that $\gcd(\lambda_1, \lambda_2) = 1$). In the first case $p^\alpha|a$ by (3.3), in the second case $p^{\alpha/2}|c$ by (3.6). Hence p^α is bounded because $a, c \neq 0$ are fixed. The same holds for the powers of 2 and 3. Hence c' is bounded.

□

We pass now to analyse the case of ternary cubics.

Proposition 3.2.16. *Let R be a subring of a number field K, $F \in R[x, y, z]$ be a cubic form and $d \in R$ such that $d \neq 0$. Assume that $\Delta \neq 0$ and $S \neq 0$.*

Let $G(y,z) = dy^3 + z^3$. Then there are only finitely many $a \in R$ and $(b,c) \in R^2$ such that F may be written in reduced form with associated triple $(a,(b,c),G)$.

Proof. We write

$$F = ax^3 + (by + cz)x^2 + dy^3 + z^3.$$

Then, it is easy to compute that

$$S = dbc \text{ and } T = 27a^2d^2 + 4b^3d + 4c^3d^2.$$

We consider the curve $C \subseteq \mathbb{P}^3$ given by the ideal

$$I = (Sx_3^2 - dx_1x_2, Tx_3^3 - 27d^2x_0^2x_3 - 4dx_1^3 - 4d^2x_2^3).$$

We claim that the K-rational points $[a,b,c,1]$ on C are in finite number and hence the claim follows.

It is not difficult to see that C is an irreducible complete intersection. Thus, by adjunction, the arithmetic genus of C is

$$p_a(C) = \frac{1}{2}(2 \cdot 3)(2 + 3 - 4) + 1 = 4.$$

The only singular point of C is $P = [1,0,0,0]$ and the tangent cone at P is the union of the two lines $s = \{x_1 = x_3 = 0\}$ and $t = \{x_2 = x_3 = 0\}$. Therefore P is a node and by Plücker formula we get the geometric genus

$$g(C) = 3.$$

Our claim follows now by Faltings Theorem. □

In the case $S_F = 0$ we can prove something weaker, which is a consequence of Lemma 3.2.14.

Lemma 3.2.17. *Let R_W be a ring of W-integers, $F \in R_W[x,y,z]$ be a cubic form and $d \in R_W$ such that $d \neq 0$. Assume that $\Delta_F \neq 0$ and $S_F = 0$. Let $G(y,z) = dy^3 + z^3$. Then there are only finitely many $a \in R_W$ and $(b,c) \in R_W^2$ such that F may be written in reduced form with associated triple $(a,(b,c),G)$.*

First proof. Writing

$$F = ax^3 + (by + cz)x^2 + dy^3 + z^3$$

we have

$$S = 0 = dbc \text{ and } T = 27a^2d^2 + 4b^3d + 4c^3d^2.$$

This implies $b = 0$ or $c = 0$. By symmetry assume $c = 0$. Then consider the curve in $\mathbb{A}^2$ given by

$$27x_0^2d^2 + 4x_1^3d - T = 0.$$

By Siegel's Theorem we are done. □

Second proof. Writing

$$F = ax^3 + (by + cz)x^2 + dy^3 + z^3$$

we have

$$S = 0 = dbc.$$

This implies $b = 0$ or $c = 0$. By symmetry assume $c = 0$. Let $M \in SL(3, R_W)$ be a transformation such that

$$M \cdot F = a_1x^3 + (b_1y)x^2 + dy^3 + z^3$$

where $a_1, b_1 \in R_W$.

Note $q = [0, 0, 1] \in W_F$ and $L = \{z = 0\} \subseteq V_F$. Hence we may consider q and L fixed by T. Thus, Lemma 3.2.14 implies the claim. □

As a consequence of the previous two results we immediately obtain the following proposition.

Proposition 3.2.18. *Let $F \in \mathbb{Z}[x, y, z]$ be a cubic form such that $\Delta_F \neq 0$. Then there are only finitely many non-equivalent triples (a, b, G) such that F can be written in reduced form with associated triple (a, b, G).*

Proof. Let $(a_i, b_i, G_i)_{i\in I}$ be non-equivalent reduced forms of F.

By Lemma 3.2.1 we have that $\Delta_{G_i}|\Delta_F$ and hence there are just a finite number of possible values for Δ_{G_i}. Moreover since $\Delta_F \neq 0$ we also have $\Delta_{G_i} \neq 0$. This implies, by Proposition 3.2.13, that the number of non-equivalent G_i is finite.

Hence we may consider $G_i = G$ fixed and we may consider a ring of integers R_W such that acting with $\mathrm{SL}(2, R_W)$ on (x_1, x_2) we may write

$$F_i = a_ix_0^3 + (B_ix_1 + C_ix_2) + dx_1^3 + x_2^2$$

where $B_i, C_i, d \in R_W$. Now we conclude applying Proposition 3.2.16 and Lemma 3.2.17. □

Note that Proposition 3.2.16 does not hold if the discriminant of G is zero, neither considering $R = \mathbb{Z}$ as the following example shows.

Example 3.2.19. Let

$$F = ax^3 + bx^2y + x^2z - 3y^2z$$

where $a, b \in \mathbb{Z}$. Consider Pell's equation

$$s^2 - 3t^2 = 1. \tag{3.7}$$

For any solution $(\alpha, \beta) \in \mathbb{Z}^2$ of (3.7) we define the matrix

$$M = \begin{pmatrix} \alpha & 3\beta & 0 \\ \beta & \alpha & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}$$

where $m_{31} = \beta(3b\beta^2 + 9a\alpha\beta + 2b\alpha^2)$ and $m_{32} = 3\beta^2(3a\beta + b\alpha)$.

Then $M \in SL(3, \mathbb{Z})$ and setting $(x, y, z) = M \cdot (X, Y, Z)^T$ and acting with M on F we get

$$F \cdot M = AX^3 + BX^2Y + X^2Z - 3Y^2Z.$$

where

$$A = 3b\alpha^2\beta + 3b\beta^3 + a\alpha^3 + 9a\alpha\beta^2 \text{ and } B = 9a\beta^3 + 9b\alpha\beta^2 + 9a\alpha^2\beta + b\alpha^3.$$

Since (3.7) has infinitely many integral solutions we conclude that there are infinite many ways to write F in reduced form.

Nevertheless all the counter-examples over $\mathbb{Z}$ are of this type and we can state the following

Lemma 3.2.20. *Let*

$$F(x, y, z) = ax^3 + (by + cz)x^2 + y^2(c_1y + c_2z) = ax^3 + (by + cz)x^2 + G(y, z)$$

be a cubic form with integral coefficients such that $S_F \neq 0$. Then there are finitely many $C \in \mathbb{Z}$ such that we may write F in reduced form with associated triple $(A, (B, C), G)$. Moreover for any such triple there is a solution $(\alpha, \beta) \in \mathbb{Z}^2$ of the equation

$$Cs^2 + c_2t^2 = C \tag{3.8}$$

such that

$$A = a\alpha(4\alpha^2 - 3) + b\beta(4\alpha^2 - 1) - \frac{Cc_1\beta(4\alpha^2 - 1)}{c_2}$$

and

$$B = -\frac{ac_2\beta(4\alpha^2 - 1)}{C} + b\alpha(4\alpha^2 - 3) + \frac{Cc_1(3\alpha - 4\alpha^3 + 1)}{c_2}.$$

Proof. If F may be written in reduced form with associated triple $(A, (B, C), G)$ then

$$S_F = C^2c_2^2$$

and hence we can consider $c = C$ fixed. (Note that $\Delta = 0$ and hence we do not get any more information from the invariants of F)

Let $M = (m_{ij})_{i,j=1,2,3} \in SL(3,\mathbb{Z})$ such that, setting $(x,y,z) = M \cdot (X,Y,Z)^T$ and acting with M on F we get

$$F \cdot M = AX^3 + (BY + CZ)X^2 + G(Y,Z).$$

Let $C \subseteq \mathbb{P}^2$ be the cubic curve given by the equation $F(x,y,x) = 0$. Note that $P = (0:0:1)$ is a singular point of C and thus we may consider P fixed by M, which implies $m_{13} = m_{23} = 0$ and $m_{33} = 1$.

Working out the equations

$$\begin{aligned} m_{11}m_{22} - m_{12}m_{21} &= 1, & c_2 m_{21}^2 + c m_{11}^2 &= c,\\ c m_{12}^2 + c_2 m_{22}^2 &= c_2, & c m_{11} m_{12} + c_2 m_{21} m_{22} &= 0 \end{aligned}$$

we get $m_{11} = m_{22}$, $m_{12} = -m_{21}c_2/c$ and

$$c_2 m_{21}^2 + c m_{11}^2 = c. \tag{3.9}$$

Finally, looking at the coefficients of Y^3 and XY^2, we have

$$\begin{aligned} m_{13} &= -\frac{b m_{12}^2 m_{21} + 3a m_{11} m_{12}^2 + 2b m_{11} m_{12} m_{22} + 3c_1 m_{21} m_{22}^2}{c_2}\\ m_{23} &= -\frac{a m_{12}^3 + c_1 m_{22}^3 + b m_{12}^2 m_{22} - c_1}{c_2}. \end{aligned}$$

Hence for any solution $(\alpha,\beta) \in \mathbb{Z}^2$ of (3.9) we obtain the claimed expressions for A and B. $\square$

Note that the examples above are nodal cubics. It is worth showing that a nodal cubic of that form (with $c \cdot c_2 < 0$) is indeed realizable as the cubic form associated to a smooth threefold. Indeed we can construct a series of examples coming from the blow-up of rational curves.

Example 3.2.21. Let $W = \mathbb{P}^3$, h the hyperplane class and C a line. Note that $\deg N_{C/W} = 2$. Let $\pi : X \to W$ be the blow-up of W along C and set $H = \pi^* h$. Let $\{L_1, L_2\}$ be the basis of $H^2(X,\mathbb{Z})$ given by

$$L_1 = H \quad \text{ and } L_2 = H - E$$

where E is the exceptional of π. We have $H.E^2 = -1$ and hence

$$L_1^3 = 1, \quad L_1^2.L_2 = 1, \quad L_1.L_2^2 = 0, \quad \text{and} L_2^3 = 0.$$

Thus the intersection cubic form on $H^2(X,\mathbb{Z})$ is

$$G(y,z) = y^3 + 3y^2 z.$$

Let b and c be positive integers. Assume that there is an irreducible smooth rational curve $C' \subseteq \mathbb{P}^3$ such that C' has degree d and $C' \cap C$ consists of exactly c points. Then $D = p_*^{-1}C' \equiv bH^2 - cH \cdot E$ and blowing-up X along D we get a threefold Y with associated cubic form

$$F(x, y, z) = ax^3 - 3(by + cz)x^2 + y^3 + 3y^2 z.$$

We may ensure the existence of such a curve as C' applying Theorem 2.15 of [EF01].

If $\Delta_G = 0$ and $S_F = 0$ then the possible reduced forms (a_i, b_i, G_i) for F may be infinitely many, but the a_i and G_i are in finite number, as the following Lemma shows. This is enough for our purposes.

Lemma 3.2.22. *Let*

$$F(x, y, z) = ax^3 + (by + cz)x^2 + G(y, z)$$

be an integral non-degenerate cubic form such that $S_F = 0$ and G is a non-degenerate cubic with $\Delta_G = 0$. Let $(a_i, (b_i, c_i), H_i)_{i \in I}$ be reduced forms for F where $\Delta_{H_i} = 0$. Then $c = c_i = 0$ and $\{a_i\}_{i \in I}$, $\{G_i\}_{i \in I}$ are finite sets.

Proof. Since $\Delta_G = 0$ we act on (y, z) with $\mathrm{SL}(2, \mathbb{Z})$ so that we may assume

$$G(y, z) = c_1 y^3 + c_2 y^2 z$$

where $c_1, c_2 \in \mathbb{Z}$. If F may be written in reduced form with associated triple $(A, (B, C), H)$ for the same reason we may assume that

$$G(Y, Z) = d_1 Y^3 + d_2 Y^2 Z$$

where $d_1, d_2 \in \mathbb{Z}$. Moreover we have

$$0 = S_F = c^2 c_2^2 = C^2 d_2^2$$

and hence we may assume $c = C = 0$ (because G and H are non-degenerate and hence $c_2, d_2 \neq 0$). Moreover, since we are always allowed to act with $\mathrm{SL}(2, \mathbb{Z})$ on (y, z), we may replace z with $z - ky$ where k is the smallest nonnegative residue of c_1 modulo c_2 (the smallest nonnegative residue of d_1 modulo d_2 respectively) so that $0 \leq c_1 < c_2$ and $0 \leq d_1 < d_2$.

Let $T = (t_{ij})_{i,j=1,2,3} \in SL(3, \mathbb{Z})$ such that, setting $(x, y, z) = M \cdot (X, Y, Z)^T$ and acting with T on F we get

$$F \cdot T = AX^3 + (BY)X^2 + H(Y, Z).$$

Note that $P = [0, 0, 1]$ is a singular point of $\{F(x, y, x) = 0\} \subseteq \mathbb{P}^2$ and thus we may consider P fixed by T, which implies $t_{13} = t_{23} = 0$ and $t_{33} = \pm 1$. We may assume $t_{33} = 1$.

Working out the equations

$$t_{11}t_{22} - t_{12}t_{21} = 1, \qquad c_2 t_{21}^2 + c t_{11}^2 = c,$$
$$c t_{12}^2 + c_2 t_{22}^2 = c_2, \qquad c t_{11} t_{12} + c_2 t_{21} t_{22} = 0$$

we get $t_{21} = 0$ and we may assume $t_{11} = t_{22} = 1$. This implies $A = a$ and $d_2 = c_2$. This tells us that there are just a finite number of possible values for c_1 and d_1, and hence there are only finitely many G_i and the Lemma is proved.

Note that we have

$$F \cdot M = ax^3 + (3a + t_{12})x^2 y + (c_2 t_{31} + 2bt_{12} + 3at_{12}^2)xy^2 + (at_{12}^3 + c_1 + c_2 t_{32} + bt_{12}^2)y^3 + c_2 y^2 z$$

and hence there might be infinitely many different b_i depending on t_{12}. □

Lemma 3.2.23. *Let X be a smooth threefold such that $K_X^3 \neq 0$ and let F be the cup-form of X. Let $f : Y \to X$ be the blow-up of a smooth curve $C \subseteq X$. Then we may write*

$$F(x, y, z) = ax^3 + (by + cz)x^2 + G(y, z)$$

where G is the cup-form of X, $a = -\deg(N_{C/X})$ and there is a non-trivial linear relation among a,b and c depending only on G and K_X^3.

Proof. By Proposition 13 in [OVdV95] we need just to prove the existence of a linear relation among a,b and c. By Proposition 16 in [OVdV95] we know that $G(y, z)$ is not a cube, hence the equation $G(y, z) = K_X^3$ has only a finite number of solutions in $\mathbb{Z}^2$ (apply Thue's Theorem if G is irreducible, otherwise it is trivial). Let (α, β) be such a solution. We have $-3K_X.C = \alpha b + \beta c$. On the other hand we know that

$$a = -\deg(N_{C/X}) = K_X.C + 2 - 2g(C)$$

and we are done. □

3.2.4 General cubics

The next proposition shows that we can handle the cubics coming from Corollay 3.2.10.

Proposition 3.2.24. *Let*

$$F(x_0, \ldots, x_n) = ax_0^3 + bx_0^2 x_1 + G(x_1, \ldots, x_n)$$

be a non-degenerate cubic form with integral coefficients. Let $C \subseteq V_F$ be an irreducible component of positive dimension such that $p = [1, 0, \ldots, 0] \in C$, $C \subseteq \{x_1 = 0\}$ and $C \nsubseteq \{F = 0\}$.

Let $(F_i)_{i\in I}$ be non-equivalent reduced forms for F with associated triple $(a_i,(b_i,0),G_i)_{i\in I}$ and let $(T_i)_{i\in I}\in \mathrm{SL}(n+1,\mathbb{Z})$ such that $T_i\cdot F=F_i$. Assume that for any $i\in I$, $[1,0,\dots,0]\in T_i(C)$.

Then $\{a_i\}_{i\in I}$ and $\{G_i\}_{i\in I}$ are finite sets.

Proof. We assume by contradiction that I is infinite. By Lemma 3.2.3 the set $\{T_i^{-1}([1,0,\dots,0])\}_{i\in I}$ is infinite. In particular, C admits infinitely many rational points. Moreover $b\neq 0$ otherwise we get a contradiction Proposition 3.2.5 and Lemma 3.2.3 because $p\in W_F$.

By Corollary 3.2.10 there are two cases: C is a line or C spans a 2-dimensional plane. Fix $i\in I$ and set $T=T_i$. Write $T=(t_{hk})_{h,k=0,\dots,n}$.

Assume first that C is a line. By Corollary 3.2.10 we may act on $(x_2,\dots,x_n)$ with an element of $\mathrm{SL}(n-1,\mathbb{Z})$ so that $C=\{x_1=x_3=x_4=\ldots=x_n=0\}$ and

$$F=ax_0^3+bx_0^2x_1+cx_1^2x_2+H(x_1,x_3,x_4,\dots,x_n)$$

where $H\in\mathbb{Z}[x_1,x_3,x_4,\dots,x_n]$ is a cubic form and $c\in\mathbb{Z}$. Since reduced forms are considered modulo the action of $\mathrm{SL}(n,\mathbb{Z})$ on $(x_1,\dots,x_n)$ we may assume that

$$F_i=a_ix_0^3+b_ix_0^2x_1+c_ix_1^2x_2+H_i(x_1,x_3,x_4,\dots,x_n)$$

where $H_i\in\mathbb{Z}[x_1,x_3,x_4,\dots,x_n]$ is a cubic form, $c_i\in\mathbb{Z}$ and that $T(C)=C$.

Denote by $S=(t_{hk})_{h,k=1,\dots,n}$ the submatrix of T obtained removing the first row and column.

Since C is fixed by T, we have $t_{hk}=0$ for $h=1,3,4,\dots,n$ and $k=0,2$. Since x_2 does not appear in degree 3 or 2 in F and F_i, it is easy to check that we must have $t_{02}=0$.

Hence $\det T=t_{00}\det S$ and we may assume $t_{00}=\det S=1$. Since

$$a_i=F(t_{00},t_{10},\dots,t_{n0})$$

we conclude that $a_i=a$

Now we need to prove that there are only finitely many non-equivalent G_i. We must have $t_{1j}=$ for $j\geq 0$, because x_2 appears only in the monomial $x_1^2x_2$. This implies also that $t_{11}=1$ and $c_i=c$. It is immediate to see that in this situation $t_{0j}=0$ for $j\ ge2$. To conclude just note that, replacing x_2 with x_2+kx_1 for $k\in\mathbb{Z}$, we can keep the coefficient of x_1^3 between 0 and c.

If C spans a plane Π then by Corollary 3.2.10 we may write

$$F=ax_0^3+bx_0^2x_1+x_2M_1+x_4M_2+H$$

where $H\in\mathbb{Z}[x_1,x_3,x_5,x_6,\dots,x_n]$ is a cubic form and $M_1,M_2\in\mathbb{Z}[x_1,x_3]$ are quadric forms. We may also assume that

$$F_i=a_ix_0^3+b_ix_0^2x_1+x_2N_1+x_4N_2+H_i$$

where $H_i \in \mathbb{Z}[x_1, x_3, x_5, x_6, \ldots, x_n]$ is a cubic form and $N_1, N_2 \in \mathbb{Z}[x_1, x_3]$ are quadric forms.

Hence $\Pi = \{x_1 = x_3 = x_5 = x_6 = \ldots = x_n = 0\}$ and we may consider Π fixed by any T_i. Set $T = T_i$ and denote by $S = (t_{hk})_{h,k=1,\ldots,n}$ the submatrix of T obtained removing the first row and column.

Since Π is fixed by T it is immediate to see that $t_{hk} = 0$ for $h = 1, 3, 5, 6, \ldots, n$ and $k = 0, 2, 4$. Moreover since $M_1 \neq 0$ and $M_2 \neq 0$ it easy to check, reasoning as before, that $t_{0k} = 0$ for $k = 2, 4$.

Since $\det T = t_{00} \det S$ we may assume $t_{00} = 1$. Note also that

$$a_i = F(t_{00}, t_{10}, \ldots, t_{n0})$$

and hence $a_i = a$.

Now we need to prove that the non equivalent G_i are a finite number. The idea is similar as that for the case C line, but the calculation is a bit more involved.

First note that $(t_{22}t_{44} - t_{24}t_{42})$ divides $\det T$ and hence, acting on (x_2, x_4) with $SL(2\mathbb{Z})$, we may assume $t_{22} = t_{44} = 1$ and $t_{24} = t_{42} = 0$. This implies that $T \cdot M = M_1$ and $T \cdot N = N_1$. Hence, looking at $x_0^2 x_k$ and $x_0 x_k^2$ for $k = 5, \ldots, n$ we get the equations

$$3at_{0k} + bt_{1k} = 0 \text{ and } 3at_{0k}^2 + 2bt_{0k}t_{1k} = 0$$

which imply $t_{1k} = 0$ and $t_{0k} = 0$ (if $a = 0$, then look at $x_0x_1x_k$) for $k = 5, \ldots, n$.

Now $(t_{11}t_{33} - t_{13}t_{31})$ divides $\det T$ and, acting on (x_1, x_3), we may assume that $t_{11} = t_{33} = 1$ and $t_{13} = t_{31} = 0$. This implies that $M_1 = N_1$ and $M_2 = N_2$.

To conclude we need to bound the coefficients of the monomials x_1^3, $x_1^2 x_3$, $x_1 x_3^2$ and x_3^3 (which are the only ones that may change the equivalence class of G_i). For this it is enough to use transformations of the type $x_2 + d_1 x_1 + d_3 x_3$ and $x_4 + f_1 x_1 + f_3 x_3$ where $d_i, f_i \in \mathbb{Z}$ (generalize the argument of the previous case).

□

In the next Lemma we show that under the action of the transformations given by Corollaries 3.2.11 and 3.2.12 we may control the last part of a reduced form.

Lemma 3.2.25. *Let $s \in \{1, 2\}$ and let $F, F_1 \in \mathbb{Q}[x_0, \ldots, x_n]$ be non-degenerate cubic forms such that*

$$F = ax_0^3 + bx_0^2 x_1 + R(x_1, x_s) + H(x_{s+1}, \ldots, x_n)$$

and

$$F_1 = a_1 x_0^3 + b_1 x_0^2 x_1 + R_1(x_1, x_s) + H_1(x_{s+1}, \ldots, x_n)$$

where $b, b_1 \neq 0$.

Assume that there exists $T = (t_{hk})_{h,k=0,\dots,n} \in SL(n+1,\mathbb{Q})$ such that $T \cdot F = F_1$, $t_{hk} = 0$ for $h = s+1,\dots,n$ and $k = 0,\dots,s$ and $\det(t_{hk})_{h,k=0,\dots,s} = 1$. Then H and H_1 are $\mathrm{SL}(n-s-1,\mathbb{Q})$-equivalent.

Proof. We prove the case $s = 2$, the case $s = 1$ is similar and easier.

We will show that $t_{hk} = 0$ for $h = 0,1,2$ and $k = 3,\dots,n$, which implies our statement.

Set $S = (t_{hk})_{h,k=0,1,2}$ and consider the transformation $\overline{T} = (\overline{t_{hk}})_{h,k=0,\dots,n} \in SL(n+1,\mathbb{C})$ constructed as follows:

$$(\overline{t_{hk}})_{h,k=0,1,2} = S^{-1} \quad (\overline{t_{hk}})_{h,k=3,\dots,n} = I_{n-2},$$

$$(\overline{t_{hk}})_{h=3,\dots,n}^{k=0,1,2} = 0, \quad (\overline{t_{hk}})_{h=0,1,2}^{k=3,\dots,n} = 0$$

where $I_{n-2} \in SL(n-2,\mathbb{C})$ is the identity matrix.

Note that the composition $M = (m_{ij})_{i,j=0,\dots,n} = \overline{T} \cdot T$ acts on F giving as result a cubic form $\overline{F_1}$ which is in reduced form with associated triple $(a,(b,0),R+H_1)$. Moreover

$$(m_{hk})_{h,k=0,1,2} = I_3 \quad (m_{hk})_{h=0,1,2}^{k=3,\dots,n} = (t_{hk})_{h=0,1,2}^{k=3,\dots,n}$$

$$(m_{hk})_{h=3,\dots,n}^{k=0,1,2} = 0.$$

If $a \neq 0$ and $b \neq 0$ then for any $k = 3,\dots,n$, looking at the monomials $x_0x_k^2$ and $x_0^2x_k$ in $\overline{F_1}$, we get the equations

$$3at_{0k} + bt_{1k} = 0 \ \text{ and } \ 3at_{0k}^2 + 2bt_{0k}t_{1k} = 0$$

which give $t_{0k} = t_{1k} = 0$. Let us write

$$R(x_1,x_2) = c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3.$$

Then looking at $x_1^2x_k$, $x_1x_k^2$ and $x_2^2x_k$ in $\overline{F_1}$ we get the conditions

$$c_2t_{2k} = 0, \quad c_3t_{2k}^2 = 0, \quad c_4t_{2k} = 0$$

from which $t_{2k} = 0$ and we are done.

If $a = 0$ then $t_{1k} = 0$ for $k = 3,\dots,n$. Moreover for $k = 3,\dots,n$ considering the monomial $x_0x_1x_k$ we get $t_{0k} = 0$. Now we can conclude as before. □

Proposition 3.2.26. *Let*

$$F(x_0,\dots,x_n) = ax_0^3 + bx_0^2x_1 + G(x_1,\dots,x_n)$$

be a cubic form with integral coefficients such that $\Delta_F \neq 0$. Let $C \subseteq V_F$ be an irreducible component of positive dimension such that $p = [1,0,\dots,0] \in C$, $C \not\subseteq \{F = 0\}$ and $C \not\subseteq \{x_1 = 0\}$. Let $(F_i)_{i\in I}$ be non-equivalent reduced forms for F with associated triple $(a_i,(b_i,0),G_i)_{i\in I}$ and let $(T_i)_{i\in I} \in SL(n+1,\mathbb{Z})$ such that $T_i \cdot F = F_i$. Assume that for any $i \in I$, $[1,0,\dots,0] \in T_i(C)$. Then I is finite.

Proof. We assume by contradiction that I is infinite. By Lemma 3.2.3 the set $\{T_i^{-1}([1,0,\dots,0])\}_{i\in I}$ is infinite. In particular, C admits infinitely many rational points. Moreover $b \neq 0$ otherwise we get a contradiction by Proposition 3.2.5 and Lemma 3.2.3 because $p \in W_F$.

We assume first that C is a line. Acting on $(x_1,\dots,x_n)$ with $\mathrm{SL}(n,\mathbb{Z})$ we may assume that $C = \{x_2 = x_3 = x_4 = \ldots = x_n\}$ and we may write

$$F = ax_0^3 + (bx_1 + cx_2)x_0^2 + G(x_1,\dots,x_n)$$

where $b,c \in \mathbb{Z}$, $b \neq 0$ and $G \in \mathbb{Z}[x_1,\dots,x_n]$ is a cubic form. Since reduced forms are considered modulo the action of $\mathrm{SL}(n,\mathbb{Z})$ on $(x_1,\dots,x_n)$ we may assume that this holds for any $i \in I$, that is

$$F = a_ix_0^3 + (b_ix_1 + c_ix_2)x_0^2 + G_i(x_1,\dots,x_n)$$

and $T_i(C) = \{x_2 = x_3 = x_4 = \ldots = x_n\}$.

Let $i \in I$ and set $T = T_i$. Write $T = (t_{hk})_{h,k=0,\dots,n}$. Since $\{x_2 = x_3 = x_4 = \ldots = x_n\}$ is fixed by T we have $t_{hk} = 0$ for $h = 2,\dots,n$ and $k = 0,1$. By $\det T = 1$ we may assume $\det(t_{h,k})_{h,k=0,1} = 1$.

We may find elements $M, M_i \in SL(n,\mathbb{Q})$ as in Corollary 3.2.11 such that

$$\hat{F} = M \cdot F = ax_0^3 + bx_0^2x_1 + dx_1^3 + H(x_2,\dots,x_n)$$

and

$$\hat{F}_i = M_i \cdot F_i = a_ix_0^3 + b_ix_0^2x_1 + d_ix_1^3 + H_i(x_2,\dots,x_n)$$

where $d, d_i \in \mathbb{Z}$ and $H, H_i \in \mathbb{Q}[x_2,\dots,x_n]$ are cubic forms.

Moreover, setting $(\hat{t}_{hk}) = \hat{T} = M^{-1} \cdot T \cdot M_i$, we have that $\hat{T} \cdot \hat{F} = \hat{F}_i$. Set

$$U := (\hat{t}_{hk})_{h,k=0,1}.$$

Note that, by the structure of T, M and M_i we have that $\hat{t}_{hk} = 0$ for $h = 2,\dots,n$ and $k = 0,1$ and $U \in SL(3,\mathbb{Z})$. Setting

$$F' = \hat{F}_{|C} = ax_0^3 + bx_0^2x_1 + dx_1^3 \text{ and } F_i' = \hat{F}_{i|C} = a_ix_0^3 + b_ix_0^2x_1 + d_ix_1^3$$

we have that $U \cdot F' = F_i'$ and in particular $\Delta_{F'} = \Delta_{F_i'}$.

Recall that $\Delta_F = 0$ if and only if the hypersurface $\{F = 0\} \subseteq \mathbb{P}^n$ is singular. If $\Delta_{F'} = 0$ then we would also have a singular point of F and hence $\Delta_{F'} \neq 0$. Since F', F_i' have integer coefficients we can apply Lemma 3.2.14 to conclude that there are finitely many a_i, b_i and d_i.

By Lemma 3.2.25 we also get that H and H_i are $\mathrm{SL}(n-2, \mathbb{Q})$-equivalent, which implies that we have a finite number of possible G_i and if C is a line we are done.

Assume now that C is not a line. By Theorem 3.2.8 we know that C spans a plane Π. Acting on $(x_1, \ldots, x_n)$ with an element of $\mathrm{SL}(n, \mathbb{Z})$ we may assume $\Pi = \{x_3 = x_4 = \ldots = x_n = 0\}$ and we may write

$$F = ax_0^3 + x_0^2(bx_1 + cx_3) + G(x_1, \ldots, x_n)$$

where $b, c \in \mathbb{Z}$, $b \neq 0$ and $G \in \mathbb{Z}[x_1, \ldots, x_n]$ is a cubic.

Since reduced forms are considered modulo the action of $\mathrm{SL}(n, \mathbb{Z})$ on $(x_1, \ldots, x_n)$ we may assume that this holds for any $i \in I$, that is

$$F_i = a_i x_0^3 + x_0^2(b_i x_1 + c_i x_3) + G_i(x_1, \ldots, x_n)$$

and $T_i(C) \subseteq \Pi = \{x_3 = x_4 = \ldots = x_n = 0\}$.

Let $i \in I$ and set $T = T_i$. Write $T = (t_{hk})_{h,k=0,\ldots,n}$. Since we may consider $\Pi = \{x_3 = \ldots = x_n = 0\}$ fixed by T we have $t_{hk} = 0$ for $h = 3, \ldots, n$ and $k = 0, 1, 2$. By $\det T = 1$ we may assume $\det(t_{h,k})_{h,k=0,1,2} = 1$.

We may find elements $M, M_i \in SL(n, \mathbb{Q})$ as in Corollary 3.2.12 such that

$$\hat{F} = M \cdot F = ax_0^3 + bx_0^2 x_1 + R(x_1, x_2) + H(x_3, \ldots, x_n)$$

and

$$\hat{F}_i = M_i \cdot F_i = a_i x_0^3 + b_i x_0^2 x_1 + R_i(x_1, x_2) + H_i(x_3, \ldots, x_n)$$

where $R, R_i \in \mathbb{Z}[x_1, x_2]$ and $H, H_i \in \mathbb{Q}[x_3, \ldots, x_n]$ are cubic forms. Moreover, setting $(\hat{t}_{hk}) = \hat{T} = M^{-1} \cdot T \cdot M_i$, we have that $\hat{T} \cdot \hat{F} = \hat{F}_i$. Set

$$U := (\hat{t}_{hk})_{h,k=0,1,2}.$$

Note that, by the structure of T, M and M_i we have that $\hat{t}_{hk} = 0$ for $h = 3, \ldots, n$ and $k = 0, 1, 2$ and $U \in SL(3, \mathbb{Z})$. Setting

$$F' = \hat{F}_{|\Pi} = ax_0^3 + bx_0^2 x_1 + R(x_1, x_2) \text{ and } F_i' = \hat{F}_{i|\Pi} = a_i x_0^3 + b_i x_0^2 x_1 + R_i(x_1, x_2)$$

we have that $U \cdot F' = F_i'$ and in particular $\Delta_{F'} = \Delta_{F_i'}$. Recall that $\Delta_F = 0$ if and only if the hypersurface $\{F = 0\} \subseteq \mathbb{P}^n$ is singular. If $\Delta_{F'} = 0$ then we would also have a singular point of F and hence $\Delta_{F'} \neq 0$. Since F', F_i' have integer coefficients we can apply Proposition 3.2.18 to conclude that there are finitely many a_i, b_i and R_i.

By Lemma 3.2.25 we also get that H and H_i are $\mathrm{SL}(n-2, \mathbb{Q})$-equivalent, which implies that we have a finite number of possible G_i and the claim follows. □

Lemma 3.2.27. *Let*

$$F = bx_0^2x_1 + G(x_1, \ldots, x_n)$$

be a non-degenerate cubic form with complex coefficients such that $\Delta_F \neq 0$. *Let* $p = [1, 0, \ldots, 0]$ *and let* $p \in C \subseteq V_F$ *be an irreducible component. Assume that* $C \subseteq \{F = 0\}$. *Then* $C = p$.

Proof. Since F is non-degenerate, it follows that $b \neq 0$. Assume by contradiction that $\dim C \geq 1$. There are two cases: $C \subseteq \{x_1 = 0\}$ and $C \nsubseteq \{x_1 = 0\}$.

Let us start with the case $C \subseteq \{x_1 = 0\}$ and write

$$F = bx_0^2x_1 + c_1x_1^3 + Lx_1^2 + Qx_1 + R$$

where $c_1 \in \mathbb{C}$ and $L, Q, R \in \mathbb{C}[x_2, \ldots, x_n]$ are homogeneous polynomials of degree 1,2 and 3 respectively. Since $\partial_0\partial_0 F(z) = 0$ for any $z \in C$ we have $\mathcal{H}_R(z) = 0$ and by Lemma 3.2.4, acting on $(x_2, \ldots, x_n)$, we may assume that R does not depend on x_2. If x_2^2 does not appear in Q, the point $[0, 0, 1, 0, \ldots, 0]$ is a singular point of F, in contradiction with $\Delta_F \neq 0$. Hence we may assume that Q contains the monomial x_2^2 and since

$$\frac{\partial F}{\partial x_1} = bx_0^2 + 3c_1x_1^3 + 2x_1L + Q$$

it is easy to get a point $[s, 0, t, 0 \ldots, 0]$ which is singular for F. This prove that $C = p$.

Assume now that $C \nsubseteq \{x_1 = 0\}$. Acting on $(x_1, x_2, \ldots, x_n)$ we may assume that there is point $q = [q_0, q_1, 0, \ldots, 0] \in C$ such that $q_0, q_1 \neq 0$ and that $L = c_2x_2$. Note that since $C \subseteq \{F = 0\}$, we have that C is not a line. Thus, we may assume that $Q = c_3x_2^2$ and we may write

$$F = bx_0^2x_1 + c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3 + R$$

where $c_i \in \mathbb{C}$ and $L, Q, R \in \mathbb{C}[x_2, \ldots, x_n]$ are homogeneous polynomials of degree 1,2 and 3 respectively and the monomial x_2^3 does not appear in R. It is easy to see that $\partial_i\partial_j H_F(z) = 0$ for $i = 2, \ldots, n$, $j = 2, \ldots, n$, $(i, j) \neq (2, 2)$ and for any point $z \in C$. If $C \subset \{x_2 = 0\}$ then, acting on $(x_3, \ldots, x_n)$, we may assume that there is a point $r = [r_0, r_1, 0, r_3, 0, \ldots, 0] \in C$ such that $r_3 \neq 0$. Since $\partial_i\partial_j H_F(r) = 0$ for $i = 2, \ldots, n$, $j = 2, \ldots, n$, $(i, j) \neq (2, 2)$, it is immediate to check that

$$R = \alpha x_2^2x_3 + R_1(x_2, x_4, \ldots, x_n),$$

where $\alpha \in \mathbb{C}$ and $R_1 \in \mathbb{C}[x_4, \ldots, x_n]$ is a cubic form. Then $[0, 0, 0, 1, 0 \ldots, 0]$ is a singular point of F, which is a contradiction. Hence $C \not\subset \{x_2 = 0\}$

and we may assume that there is a point $s = [s_0, s_1, s_2, 0, \ldots, 0]$ such that $s_2 \neq 0$. Since $\partial_i \partial_j H_F(s) = 0$ for $i = 2, \ldots, n$, $j = 2, \ldots, n$, $(i, j) \neq (2, 2)$, we get that R does not depend on x_2 and therefore $\mathcal{H}_R(z) = 0$ for any $z \in C$. By Lemma 3.2.4, we have that $C \subseteq \{x_3 = \ldots = x_n = 0\}$.

If $c_4 = 0$ then F is singular and again we get a contradiction, and so we may assume that $c_4 \neq 0$. Then C is a plane cubic given, in the space $\{x_3 = \ldots = x_n = 0\}$, by the 3×3 minor $\mathcal{H}_F{}_{i=0,1,2}^{j=0,1,2}$. That is C is the Hessian curve of a curve D given by

$$F_1 = bx_0^2 x_1 + c_1 x_1^3 + c_2 x_1^2 x_2 + c_3 x_1 x_2^2 + c_4 x_2^3 = 0$$

in the plane $\{x_3 = \ldots = x_n = 0\}$. Since $\Delta_F \neq 0$ we have that D is smooth and by $C \subset \{F = 0\}$ we conclude that $C \subset D$, which is a contradiction.

□

Now we can state the main result of this section.

Theorem 3.2.28. *Let $F \in \mathbb{Z}[x_0, \ldots, x_n]$ be a non-degenerate cubic form with integral coefficients such that $\Delta_F \neq 0$. Then there are only finitely many non-equivalent triples (a, B, G) such that F can be written in reduced form with associated triple (a, B, G). Moreover $\Delta_G \neq 0$.*

Proof. We may assume that

$$F = ax_0^3 + bx_0^2 x_1 + G$$

where $a, b \in \mathbb{Z}$ and $G \in \mathbb{Z}[x_1, \ldots, x_n]$ is an integral cubic form.

Let $(a_i, B_i, G_i)_{i \in I}$ be non-equivalent reduced forms for F. Let $T_i \in SL(n+1, \mathbb{Z})$ such that $T_i \cdot F = F_i$. Acting on $(x_1, \ldots, x_n)$ with $\mathrm{SL}(n, \mathbb{Z})$ we may assume that $B_i = (b_i, 0, \ldots, 0)$.

Let $p = [1, 0, \ldots, 0]$. We suppose by contradiction that I is infinite. By Lemma 3.2.3 we may assume that the set $T_i^{-1}([1, 0, \ldots, 0])$ is infinite for any $i \in I$.

Let $C \subseteq V_F$ be an irreducible component of positive dimension such that $p \in C$. We claim that we may assume $p \in T_i(C)$ for any $i \in I$. Let $C_1 = C, C_2, \ldots, C_k$ be the irreducible components of V_F with positive dimension. For any $j = 1, \ldots, n$ let $I_j = \{i \in I : p \in T_i(C_j)\}$ and fix $i_j \in I_j$. Note that for any $j = 1, \ldots, k$, $\{F_i\}_{i \in I_j}$ are cubic forms such that $F_i = T_i \cdot T_{i_j}^{-1} \cdot F_{i_j}$ and $p \in T_i(C_j)$. Hence, reasoning j by j, we get the claim.

If $C \subset \{F = 0\}$ then $b = 0$ and by by Lemma 3.2.27 we get a contradiction.

If $C \subseteq \{x_1 = 0\}$, then by Proposition 3.2.10, $\Delta_F = 0$, which contradicts our assumptions. Hence $C \not\subseteq \{x_1 = 0\}$ and we may apply Proposition 3.2.26 to conclude. □

Definition 3.2.29. Let $F \in \mathbb{Z}[x_0, \ldots, x_n]$ be an integral non-degenerate cubic. We say that F is *decomposable* with *decomposition pair* (H_1, H_2) if there is an element $T \in SL(n+1, \mathbb{Z})$ such that

$$T \cdot F = G + H$$

where $G \in \mathbb{Z}[x_0, \ldots, x_k]$ and $H \in \mathbb{Z}[x_{k+1}, \ldots, x_n]$ are non-degenerate cubics.

Two decompositions (H, G) and (H', G') are equivalent if there is an element $T \in SL(n+1, \mathbb{Z})$ such that $T \cdot H = H'$ and $T \cdot G = G'$.

Proposition 3.2.30. *Let $F \in \mathbb{Z}[x_0, \ldots, x_n]$ be a non-degenerate cubic. Then F has only a finite number of non equivalent decompositions.*

Proof. Let $\{(G_i, H_i)\}_{i \in I}$ be non equivalent decompositions of F and $T_i \in SL(n+1, \mathbb{Z})$ such that $T_i \cdot F = F_i = G_i + H_i$. We may assume that $G_i \in \mathbb{Z}[x_0, \ldots, x_k]$ with k fixed, $0 \leq k \leq n-1$.

Let $V^m_{F_i} = \{p \in \mathbb{P}^n \mid \operatorname{rk} \mathcal{H}_{F_i}(p) \leq m\}$. Let $L = \{x_0 = \ldots = x_k = 0\} \subset \mathbb{P}^n$ and $M = \{x_{k+1} = \ldots, x_n = 0\} \subset \mathbb{P}^n$. Note that for any $i \in I$, M is an irreducible component of $V^{k+1}_{F_i}$. In fact, assume by contradiction that, for some $i \in I$, there is an irreducible component $S \subset V^{k+1}_{F_i}$ such that $M \subset\neq S$. Then the general point $s = [s_0, \ldots, s_n] \in S$ is such that $s_j \neq 0$ for $j = 0, \ldots, k$ and (possibly reordering) $s_{k+1} \neq 0$. Since G_i and H_i are non degenerate, $\operatorname{rk} \mathcal{H}_{G_i}(s) = k+1$, which implies $\operatorname{rk} \mathcal{H}_{H_i}(s) = 0$, a contradiction by Lemma 3.2.4. The same holds for L respect to $V^{n-k}_{F_i}$.

Since the irreducible components of $V^m_{F_i}$ are finitely many, we may assume that L and M are fixed by any T_i. Fix $T = T_i$. Writing $T = (t_{lm})_{l,m=0,\ldots,n}$ as a matrix it is easy to see that $t_{lm} = t_{ml} = 0$ for $l = k+1, \ldots, n$ and $m = 0, \ldots, k$. This implies the Theorem.

□

We can finally classify the possible cubics with infinitely many reduced forms in the case $C \not\subset \{F = 0\}$.

Theorem 3.2.31. *Let $F \in \mathbb{Z}[x_0, \ldots, x_n]$ be a cubic form such that there are infinitely many reduced forms (a_i, b_i, G_i) associated to F. Assume that no positive dimensional component of V_F is contained inside $\{F = 0\}$. Then there exists $T = (t_{ij})_{i,j=0,\ldots,n} \in SL(n+1, \mathbb{Q})$ such that $t_{00} = 1$, $t_{10} = t_{01} = 0$ for $i = 1, \ldots, n$, $t_{ij} = 0$ for $i = 3, \ldots, n$ and $j = 0, 1, 2$, $(t_{ij})_{i,j=0,1,2} \in \mathrm{SL}(3, \mathbb{Z})$ such that*

1. *either*

$$T \cdot F = a x_0^3 + b x_0^2 x_1 + R(x_1, x_2) + S(x_3, \ldots, x_n) = F_0(x_0, x_1, x_2) + S(x_3, \ldots, x_n)$$

where $R \in \mathbb{Z}[x_1, x_2]$ and $S \in \mathbb{Q}[x_3, \ldots, x_n]$ are cubic forms and $\Delta_{F_0} = 0$;

2. or

$$T \cdot F = ax_0^3 + bx_0^2 x_1 + \alpha x_1^2 x_3 + \beta x_1 x_3^2 + \gamma x_2 x_3^2 + S(x_3, \dots, x_n)$$

where $\alpha, \beta, \gamma \in \mathbb{Q}$ and $S \in \mathbb{Q}[x_3, \dots, x_n]$ is a cubic form.

Proof. By Theorem 3.2.28 we know that $\Delta_F = 0$.

We may assume that

$$F = ax_0 + bx_0^2 x_1 + G(x_1, \dots, x_n)$$

is in reduced form and that, as usual, $b \neq 0$. Let C be a component of V_F passing through $p = [1, 0, \dots, 0]$. By Lemma 3.2.3 we may assume that $\dim C \geq 1$. If $C \subset \{x_1 = 0\}$ then F has only a finite number of reduced forms by Proposition 3.2.24. So $C \not\subset \{x_1 = 0\}$.

If C is a line then we apply Corollary 3.2.11 to obtain $T \in SL(n+1, \mathbb{Z})$ such that

$$T \cdot F = ax_0^3 + bx_0^2 x_1 + cx_1^3 + R(x_2, \dots, x_n)$$

where $c \in \mathbb{Z}$ and $R \in \mathbb{Q}[x_2, \dots, x_n]$ is a cubic form. We choose T such that $t_{00} = t_{11} = 1$, $t_{0i} = t_{i0} = 0$ for $i = 1, \dots, n$, $t_{ij} = 0$ for $i = 2, \dots, n$ and $j = 1$.

Note that $c \neq 0$, otherwise $\mathcal{H}_G = 0$. If F_1 is is an integral cubic form associated to F we may assume that

$$F_1 = a_1 x_0^3 + b_1 x_0^2 x_1 + c_1 x_1^3 + R_1(x_2, \dots, x_n)$$

and that $ax_0^3 + bx_0^2 x_1 + cx_1^3$ is $SL(2, \mathbb{Z})$-equivalent to $a_1 x_0^3 + b_1 x_0^2 x_1 + c_1 x_1^3$, where $a_1, b_1, c_1 \in \mathbb{Z}$ and $R_1 \in \mathbb{Q}[x_2, \dots, x_n]$ is a cubic form. By Proposition 3.2.15 and Lemma 3.2.25 the reduced forms associated to F are in finite number, which contradicts the hypothesis.

Hence C is not a line and we are in the situation (b) of Corollary 3.2.12, i.e there is $T \in SL(n, \mathbb{Z})$ such that $T = (t_{ij})_{i,j=0,\dots,n} \in SL(n+1, \mathbb{Q})$ such that $t_{00} = 1$, $t_{i0} = t_{0i} = 0$ for $i = 1, \dots, n$, $t_{ij} = 0$ for $i = 3, \dots, n$ and $j = 1, 2$, $(t_{ij})_{i,j=0,1,2} \in \mathrm{SL}(3, \mathbb{Z})$ and such that

$$T \cdot F = ax_0^3 + bx_0^2 x_1 + cx_1^3 + \alpha x_1^2 x_3 + \beta x_1 x_3^2 + \gamma x_2 x_3^2 + R(x_3, \dots, x_n)$$

where $c \in \mathbb{Z}$, $\alpha, \beta, \gamma \in \mathbb{Q}$ and $R \in \mathbb{Q}[x_3, \dots, x_n]$ is a cubic form. If F_1 is is an integral cubic form associated to F we may assume that

$$F_1 = a_1 x_0^3 + b_1 x_0^2 x_1 + c_1 x_1^3 + \alpha_1 x_1^2 x_3 + \beta_1 x_1 x_3^2 + \gamma_1 x_2 x_3^2 + R_1(x_3, \dots, x_n).$$

Let $M = (m_{ij})_{i,j=0,\dots,n} \in SL(n+1, \mathbb{Q})$ such that $M \cdot F = F_1$ and that $(m_{ij})_{i,j=0,1,2} \in SL(3, \mathbb{Z})$. We may assume that $\Pi = \{x_3 = \ldots = x_n = 0\}$

is fixed by M and hence $m_{hk} = 0$ for $h = 3, \ldots, n$ and $h = 0, 1, 2$. Since $q = [0, 0, 1, 0, \ldots, 0] \in W_F$ we may also assume that q is fixed by M and so $m_{02} = m_{12} = 0$. This implies $m_{22} = 1$ and $(m_{ij})_{i,j=0,1} \in SL(2, \mathbb{Z})$.

If $c \neq 0$, then the result follows by Proposition 3.2.15.

If $c = 0$, we are in case (2).

□

3.3 Bounding $c_1(X)^3$

Let X be a terminal $\mathbb{Q}$-factorial threefold. By $H^i(X, \mathbb{Z})$ we mean the i-th cohomology singular group of X modulo its torsion subgroup. We denote by $b_i(X) = \operatorname{rk} H^i(X, \mathbb{Z}) = \dim H^i(X, \mathbb{Q})$ the i-th Betti number of X. Since a $\mathbb{Q}$-factorial terminal threefold is a rational homology manifold (see [Kol91]), the singular cohomology coincides with the intersection cohomology and, in particular Poincaré and Lefschetz dualities hold.

For any Cartier divisor D on X it is well defined its first Chern class $c_1(D) \in H^2(X, \mathbb{Z})$. Moreover given Cartier divisors D_1, D_2, D_3, their intersection number $D_1 \cdot D_2 \cdot D_3 \in \mathbb{Z}$ corresponds to the cup products of their first Chern classes.

Let $\underline{h} = (h_1, \ldots, h_n)$ be a basis of $H^2(X, \mathbb{Z})$. The intersection cup product induces a symmetric trilinear form

$$\phi_X : H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to \mathbb{Z}.$$

Hence we may define a cubic homogeneous polynomial $F_X \in \mathbb{Z}[x_1, \ldots, x_n]$ as

$$F_X = \sum_{\substack{\underline{i}=(i_1,\ldots,i_n): \\ i_1+\ldots+i_n=3}} \binom{3}{\underline{i}} \underline{h}^{\underline{i}} \underline{x}^{\underline{i}}.$$

Since X is a rational homology manifold, the proof of Proposition 16 in [OVdV95] works in our case and hence F_X is non-degenerate.

Definition 3.3.1. We call F_X *the cubic form associated to* X. Let

$$S_X := \sup\{|a| : F_X \text{ may be written in reduced form with triple } (a, b, G)\},$$

where we set $S_X = 0$ if there are no reduced forms for F_X. We call S_X the *Skansen number* of X.

Note that S_X is a topological invariants of X since F_X is an invariant modulo the action of $GL(n, \mathbb{Z})$.

First we show that we can control K_X^3 in the case of divisorial contractions to points.

Proposition 3.3.2. *Let X_0 be a smooth projective threefold and let*

$$X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_{k-1} \dashrightarrow X_k$$

be a sequence of some steps for a minimal model program of X_0. Assume that $f : X_{k-1} \to X_k$ is a divisorial contraction to a point. Then

$$0 < K^3_{X_{k-1}} - K^3_{X_k} \leq 2^8 b_2^{2b_2},$$

where $b_2 = b_2(X_0)$ is the second Betti number of X_0.

Proof. Set $X = X_k$ and $Y = X_{k-1}$, so we are considering a divisorial contraction to a point $f : Y \to X$.

Let n be the index of K_X, that is the smallest positive integer such that nK_X is Cartier. Let a/n be the discrepancy of f, which is defined by the relation $K_Y = f^*(K_X) + (a/n)E$ and which is positive since X is terminal.

The possible values of $(a/n)E^3$ are listed in Table 1 and 2 of [Kaw05] and we have

$$0 < \frac{a}{n}E^3 \leq 4$$

and hence $a/n \leq 4/E^3$. Let R be the least common multiple of the indices of the fictitious singularities of Y. By [Kaw05, Lemma 2.3] we know that $E^3 \geq 1/R$ and we conclude that

$$0 < \left(\frac{a}{n}E\right)^3 \leq 64R^2.$$

We now give a very rough explicit bound for R depending on $b_2 = b_2(X_0)$. Of course R is not greater than the product of all the indices of the singularities of Y and the number of these singularities is not greater than $\Xi = \Xi(Y)$. Hence, by the arithmetic-geometric mean inequality we have

$$R \leq \Xi^{\Xi}.$$

Applying Proposition 3.1.1 we get

$$\left(\frac{a}{n}E\right)^3 \leq 2^8 b_2^{2b_2}$$

Since $K_Y^3 - K_X^3 = (a/n)^3E^3$, we are done.

□

Recall that $f : Y \supset E \to X \supset C$ is said to be a blow-down to a LCI curve C, if C is a local complete intersection in the smooth locus of X and f is the blow-up of the ideal sheaf of C.

We now show how cubic forms of threefolds change under divisorial contractions.

Lemma 3.3.3. *Let $f : Y \to X$ be a divisorial contraction to a point or a blow-down to a LCI curve with exceptional divisor E and let r be the smallest positive integer for which rE is Cartier. Set $e = c_1(rE)$. Then $H^2(Y, \mathbb{Z}) \cong \mathbb{Z}[e] \bigoplus H^2(X, \mathbb{Z})$.*

Let $x_0, \dots, x_n$ be coordinates on $H^2(Y, \mathbb{Z})$ respect to a basis $e, e_1, \dots, e_n$, where the e_i are the pullback of elements h_i of a basis on $H^2(X, \mathbb{Z})$.

If f contracts E to a point, then

$$F_Y(x_0, \dots, x_n) = ax_0^3 + F_X(x_1, \dots, x_n),$$

where $a = e^3$.

If f is the blow-down to a LCI curve C, then

$$F_Y(x_0, \dots, x_n) = ax_0^3 + 3x_0^2(\sum_{i=1}^{n} b_i x_i) + F_X(x_1, \dots, x_n),$$

where $a = e^3$ and $b_i = -h_i.C$.

Proof. If $f : Y \to X$ is a blow-down to a LCI curve C, then C is contained in the smooth locus of X and E is just the projectivization of the normal bundle $N_{C/X}$ of C in X: the result is hence standard (see, for instance, Proposition 14 of [OVdV95]).

Assume now that $f : Y \to X$ contracts E to a point $p \in X$.

Let $X_1 \subset X$ be a Stein and contractible representative of the germ of $p \in X$. Set $X_2 = X \setminus \{p\}$ and let $\tilde{X}_1 = f^{-1}(X_1)$ and $\tilde{X}_2 = f^{-1}(X_2)$. The Mayer-Vietoris sequence reads

$$\to H^1(\tilde{X}_1 \cap \tilde{X}_2) \to H^2(Y) \to H^2(\tilde{X}_1) \bigoplus H^2(\tilde{X}_2) \to H^2(\tilde{X}_1 \cap \tilde{X}_2) \to$$

and

$$H^1(\tilde{X}_1 \cap \tilde{X}_2) = H^1(X_1 \setminus \{p\}) = 0 = H^2(X_1 \setminus \{p\}) = H^2(\tilde{X}_1 \cap \tilde{X}_2).$$

Moreover $H^2(\tilde{X}_2) = H^2(X)$. So $H^2(Y) = H^2(X) \oplus H^2(\tilde{X}_1)$. By the proof of Lemma 4.3 in [Cai05] , $H^2(\tilde{X}_1) \cong Pic\tilde{X}_1$. On the other hand, we have the exact sequence

$$0 \to \mathbb{Z}[E] \to Cl\tilde{X}_1 \to ClX \to 0.$$

By $\mathbb{Q}$-factoriality we get that $Pic\tilde{X}_1$ is generated over $\mathbb{Z}$ by e. The statement about the cubic form follows immediately by the projection formula. □

Proposition 3.3.4. *Let $f : Y \supset E \to X \supseteq C$ be a blow-up of a smooth curve C contained in the smooth locus of X. Then*

$$|K_Y^3 - K_X^3| \leq 2S_Y + 6(b_3(Y) + 1)$$

where S_Y is the Skansen number of Y.

Proof. Since C is a local complete intersection, the exceptional divisor E is the projectivization of the normal bundle of C; call $\pi : E \to C$ the fibration morphism and let $\xi := c_1(\mathcal{O}_E(1))$ be a section. Then we have the following relations:

$$N_{E/Y} = E_{|E} = \mathcal{O}_E(-1) = -\xi, \quad E^3 = -\deg(N_{C/X}) = K_X.C + 2 - 2g$$

$$K_Y^2.E = -K_X.C + 2 - 2g, \quad K_Y^3 = K_X^3 - 2K_X.C + 2 - 2g.$$

Moreover, considering the cubic F_Y associate to Y and applying Lemma 3.3.3, we have that $|E^3| \leq S_Y$. Hence

$$\begin{aligned} |K_Y^3 - K_X^3| &= |-2(E^3 - 2 + 2g) + 2 - 2g| = |-2E^3 + 6 - 6g| \\ &\leq 2S_Y + 6(b_3(Y) + 1). \end{aligned}$$

□

We can finally state the main result of this section.

Theorem 3.3.5. *Let X be a projective smooth threefold of non-negative Kodaira dimension and let F_X be its associated cubic. Assume that $\Delta_{F_X} \neq 0$ and that there is an MMP for X composed only by divisorial contractions to points and blow-down to smooth curves in smooth loci.*

Then there exists a constant D_X depending only on the topology of X such that

$$|K_X^3| \leq D_X.$$

Proof. Let

$$X = X_0 \to X_1 \to \ldots \to X_k$$

be an MMP for X such that each $f_i : X_i \to X_{i+1}$ is a divisorial contraction to a point or to a smooth curve in a smooth locus.

Denote with F_i the cubic form associated to X_i and with S_i the Skansen number of X_i.

By induction on $i = 0, \ldots, k$, by Lemma 3.3.3 and by Theorem 3.2.28 it is easy to see that, for any $i = 0, \ldots, k$, $\Delta(F_i) \neq 0$, $S_i < +\infty$ and that each S_i depends only on F_0 (and, as consequence, only on the topology of X).

Set

$$D_k = 64\left(b_1(X) + b_3(X) + \frac{2}{3}b_2(X)\right)$$

and for $i < k$ set

$$D_i = D_{i+1} + \max\{2^8 b_2(X)^{2b_2(X)}, 2S_i + 6(b_3(X_i) + 1)\}.$$

We claim that

$$|K_{X_i}^3| \leq D_i.$$

for any $i = 0, \ldots, k$.

The proof is by descending induction on $i = k, \dots, 0$. If $i = k$ the result follows by Theorem 3.1.3.

Assume $i < k$ and $|K^3_{X_{i+1}}| \leq D_{i+1}$. The claim follows now combining Propositions 3.3.2 and 3.3.4.

To conclude it is enough to observe that $b_3(X_i) \leq b_3(X)$ for any $i = 1, \dots, k$ by the following exact sequence (see Proposition 3.2 of [Cai05]):

$$0 \to H^k(X_{i+1}, \mathbb{Q}) \to H^k(X_i, \mathbb{Q}) \oplus H^k(f_i(E_i), \mathbb{Q}) \to H^k(E_i, \mathbb{Q}) \to 0.$$

$\square$

Chapter 4

Hypersurfaces with assigned volume

In this chapter we show examples of weighted hypersurfaces of general type with either many vanishing plurigenera, with many non-birational pluricanonical maps or with assigned volume. It collects the results of a joint paper with E. Ballico and C. Fontanari (see [BPT13]).

Let X be a normal projective variety. A natural object attached to X is its *canonical ring*

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, mK_X).$$

If $R(X, K_X)$ is finitely generated then we may define the canonical model of X,

$$X_{can} = \mathrm{Proj} R(X, K_X),$$

which is a birational invariant of X.

Definition 4.0.6 (Kodaira dimension)**.** The dimension of X_{can} is called Kodaira dimension of X and indicated by $\kappa(X)$. If $|mK_X|$ is empty for every m then we set $\kappa(X) = -\infty$.

Note that if $\kappa(X) = \dim X$, then X is of general type.

For any $m \in \mathbb{N}$ consider the rational map $\phi_m : X \to Z_m$ given by the m-pluricanonical system $|mK_X|$; it is called the m-pluricanonical map. Moreover set

$$R_m = H^0(X, mK_X) \quad \text{and} \quad R^{(m)} = \bigoplus_{j \geq 0} R_{mj}.$$

Since there exists an integer r such that $R^{(r)}$ is generated by R_r, we may realize X_{can} as the image of ϕ_r. Hence any pluricanonical maps factors as

$$\phi_m : X \dashrightarrow X_{can} \dashrightarrow \mathbb{P}(R_m)$$

and $\kappa(X)$ is $\max_{m>0}\{\dim \phi_m(X)\}$.

When $\kappa(X) \geq 0$ we expect the following picture:

$$X \dashrightarrow X_{min} \to X_{can}$$

where the first birational map should come from the minimal model program and the second morphism is given by a multiple of K_X. This second part is known as abundance conjecture and it is one of the major open questions in birational geometry:

Conjecture 4.0.7 (Abundance)**.** *Let X be a terminal variety such that K_X is nef. Then K_X is semi-ample.*

Abundance conjecture is known to hold in dimension less or equal to 3.

From now on we assume that X is of general type, that is $\kappa(X) = n$ or equivalently $\mathrm{vol}(X) > 0$.

It is natural to ask how "small" can the Hilbert function of $R(X)$ be. There are many different possible definition of "small"; we are mainly interested in two of them. First, we would like to consider varieties of small canonical volume. Making the volume small is the same as making small the asymptotical behaviour of the Hilbert function. Second, we would like to understand which plurigenera $P_m := h^0(X, mK_X)$ may be zero.

For curves of general type $vol(X) \geq 2$ and K_X is effective; for surfaces $vol(X) \geq 1$ and $P_2 \neq 0$, while there are surfaces of general type with $P_1 = 0$. For threefolds the record from both points of view is attained by an example of Iano-Fletcher (see 15.1 of [IF00]) with volume $\frac{1}{420}$ and $P_1 = P_2 = P_3 = 0$. See also [CC10b] and [CC10a] for related results in dimension 3.

In higher dimension we are able to prove the following.

Theorem 4.0.8. *Let $n \geq 5$ be an integer. There exists a smooth variety of general type X of dimension n such that $H^0(X, mK_X) = 0$ for $0 < m < \lfloor \frac{n+1}{3} \rfloor$ and $\mathrm{vol}(X) < \frac{3^{n+1}}{(n-1)^n}$.*

Recently the following result, generalization of a famous theorem of [Bom73], was proven in [HM06], [Tak06] and [Tsu07].

Theorem. *For any positive integer n there exists an integer r_n such that if X is a smooth variety of general type and dimension n, then*

$$\phi_r : X \dashrightarrow \mathbb{P}(H^0(X, rK_X))$$

is birational onto its image for all $r \geq r_n$.

A consequence of this theorem is that for any smooth variety of general type X of dimension n we have

$$\operatorname{vol}(K_X) \geq \frac{1}{r_n^n}.$$

and that the set of the volumes of the manifolds of general type of dimension n has a minimum $v_n > 0$.

An instant consequence of Theorem 4.0.8 is the following

Corollary 4.0.9. *Let v_n be the minimal volume of an n-dimensional smooth variety of general type. Then $\lim_{n\to\infty} v_n = 0$.*

A natural problem is to estimate r_n. It is well known that $r_1 = 3$ and $r_2 = 5$. By the mentioned example of Iano-Fletcher we know that $r_3 \geq 27$ (see 15.1 of [IF00]). Let x'_n be the minimal positive integer such that for every n-dimensional smooth variety X of general type there is an integer $t \leq x'_n$ such that ϕ_t is generically finite; obviously $r_n \geq x'_n$.

The examples of Theorem 4.0.8 provide a lower bound for x'_n (and therefore for r_n) which is quadratic in n. More precisely

Theorem 4.0.10. *For any integer $n \geq 7$ we have*

$$r_n \geq x'_n \geq \frac{n(n-3)}{9},$$

In particular

$$\lim_{n\to+\infty} r_n = \lim_{n\to+\infty} x'_n = +\infty$$

The canonical system of these varieties is not ample. In view of Fujita's conjecture, smooth varieties with ample canonical system should not give anything better than a linear bound. We show

Theorem 4.0.11. *For any positive integer $n > 0$ there is a smooth variety X of dimension n such that K_X is ample and $\phi_{|tK_X|}$ is not birational for $t < n+3$ if n is even or $t < n+2$ if n is odd.*

The idea of this example is taken from [Kaw00], Example 3.1 (2).

These bounds are optimal up to dimension 3 (for the three dimensional case see [CCZ07]). Note that the bound for n even is the same predicted by Fujita's conjecture for the very ampleness, while for n odd is one less.

Let r'_n be the minimal positive integer such that for every n-dimensional smooth variety X of general type there is an integer $r \leq r'_n$ such that $|rK_X|$ induces a birational map. Let x_n be the minimal integer such that for every n-dimensional smooth variety of general type and every integer $t \geq x_n$

the map induced by $|tK_X|$ is generically finite. Of course $r_n \geq r'_n \geq x'_n$, $r_n \geq x_n \geq x'_n$. It is also natural to study the behaviour of these numbers.

Taking $X = Y \times C$ with C a smooth curve of genus 2 and Y a smooth variety of general type we get $r_n \geq r_{n-1}$ and $r'_n \geq r'_{n-1}$ for all $n \geq 2$.

Question 4.0.12. *Is $r_{n+1} > r_n$ for all n? Is $r'_{n+1} > r'_n$ for all n?*

All our examples satisfy $h^i(X, \mathcal{O}_X) = 0$ for all $1 \leq i \leq n-1$. Manifolds X with the property $h^i(X, \mathcal{O}_X) = 0$ for all $1 \leq i \leq n-1$ are very special, but it may be worthwhile to fix the integer $q := h^1(X, \mathcal{O}_X)$ and study the integers $r_n(q)$, $r'_n(q)$, $x_n(q)$, $x'_n(q)$ ($r_n(\geq q)$, $r'_n(\geq q)$, $x_n(\geq q)$, $x'_n(\geq q)$ and $r_n(\leq q)$, $r'_n(\leq q)$, $x_n(\leq q)$, $x'_n(\leq q)$) obtained taking only manifolds with irregularity q (resp. $\geq q$, resp. $\leq q$). Taking $X = Y \times D$ with D a curve of genus $x \geq 2$ we get $r_n(q) \geq r_{n-1}(q-x)$ for all integers $n \geq 2$ and q, x such that $2 \leq x \leq q$ (and similarly for the other integers $r'_n(q)$).

Our last result is that every positive rational number is a canonical volume:

Theorem 4.0.13. *Let $q = r/s$ be a rational number with $r, s > 0$ and $(r, s) = 1$. There are infinite positive integers n such that there is a smooth variety of dimension n with*

$$\mathrm{vol}(X) = \frac{r}{s}.$$

4.1 The proofs

We will need the following lemma.

Lemma 4.1.1. *Let $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ be a well-formed weighted projective space. If the coordinate points are canonical singularities then all the singularities of $\mathbb{P}$ are canonical.*

Proof. Let P be a singular point of $\mathbb{P}$. Define $U := \{0, 1, \ldots, n\}$ and let $S \subset U$ be the subset of the variables nonzero at P. By hypothesis, if $\#S = 1$, then P is a canonical singularity. Assume then $\#S > 1$.

We denote by h_S the highest common factor of the set $\{a_i | i \in S\}$. Then $h_S > 1$ and, chosen a $k \in S$, P is a cyclic quotient singularity of type

$$\frac{1}{h_S}(a_0, \ldots, \hat{a}_k, \ldots, a_n).$$

By the Reid's criterium 1.3.7, P is a canonical singularity if and only if

$$\frac{1}{h_S} \sum_{i=0}^{n} \overline{ja_i}^S \geq 1 \qquad \text{for all } 1 \leq j \leq h_S - 1$$

where $\overline{a}^S$ denotes the smallest (non negative) residue of a mod h_S.

We argue by contradiction. Suppose

$$\frac{1}{h_S}\sum_{i=0}^{n}\overline{ja_i}^S < 1 \qquad \text{for some } 1 \le j \le h_S - 1.$$

Take a $k \in S$. Then $a_k = mh_S$ for a positive integer m. Note that $mj < a_k$. Then we have

$$\frac{1}{a_k}\sum_{i=0}^{n}\overline{mja_i}^{\{k\}} = \frac{1}{mh_S}\sum_{i=0}^{n}\overline{mja_i}^{\{k\}} = \frac{1}{h_S}\sum_{i=0}^{n}\overline{ja_i}^S < 1,$$

which contradicts the hypothesis. □

Proposition 4.1.2. *Let $k \ge 2$ and $l \ge 0$ be integers. Consider the weighted projective space*

$$\mathbb{P} := \mathbb{P}(k^{(k+2)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)}).$$

Then the general hypersurface X_d in $\mathbb{P}$ of degree $d := (l+3)k(k+1)$ has at worst canonical singularities, $K_{X_d} \sim \mathcal{O}_{X_d}(1)$, $\dim X_d = 3k + l - 1$ and

$$\operatorname{vol}(X_d) = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}.$$

Proof. The weighted projective space $\mathbb{P}$ is well-formed since $k \ge 2$. We use Reid's criterium 1.3.7, to control that the singularities of $\mathbb{P}$ are canonical. By the previous lemma it is enough to look at the coordinates points. They are of three types.

1. The singularities of type
$$\frac{1}{k}(k^{(k+1)}, (k+1)^{(2k-1)}, (k(k+1))^{(l)}).$$
We have to check that
$$\frac{1}{k}(2k-1)\overline{j(k+1)} \ge 1$$
for $1 \le j \le k-1$, where $\bar{\ }$ denotes the smallest (non negative) residue mod k. This is trivial since $\overline{j(k+1)} = j \ge 1$ for $1 \le j \le k-1$.

2. The singularities of type
$$\frac{1}{k+1}(k^{(k+2)}, (k+1)^{(2k-2)}, (k(k+1))^{(l)}).$$
We have to check that
$$\frac{1}{k+1}(k+2)\overline{jk} \ge 1$$
for $1 \le j \le k$, where $\bar{\ }$ denotes the smallest (non negative) residue mod $k+1$. This is trivial since $\overline{jk} \ge 1$ for $1 \le j \le k$.

3. The singularities of type
$$\frac{1}{k(k+1)}(k^{(k+2)},(k+1)^{(2k-1)},(k(k+1))^{(l-1)}),$$
when $l \geq 1$. We have to check that
$$\frac{1}{k(k+1)}((k+2)\overline{ik}+(2k-1)\overline{i(k+1)}) \geq 1$$
for $1 \leq i \leq k(k+1)-1$, where $\bar{}$ denotes the smallest (non negative) residue mod $k(k+1)$. This follows since $k \not| j$ then $\overline{j(k+1)} \geq k+1$ and if $(k+1) \not| j$ then $\overline{jk} \geq k$.

Now note that $\mathcal{O}_{\mathbb{P}}(d)$ is base point free (since d is a multiple of every weight) and locally free (by Lemma 1.3 of [Mor75]).

Then we can apply a Kollár-Bertini theorem (Proposition 7.7 of [Kol97a], see also Theorem 1.3 of [Rei80]) to conclude that the general hypersurface X_d of degree d is canonical (and obviously well-formed and quasi-smooth).

Finally, by adjunction (6.14 of [IF00]), $K_{X_d} \sim \mathcal{O}_X(1)$ and so
$$\mathrm{vol}(X_d) = \frac{(l+3)k(k+1)}{k^{k+2}(k+1)^{2k-1}(k(k+1))^l} = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}}.$$
□

Proof of theorem 4.0.8. Write $n = 3k+l-1$, with integers $k \geq 2$ and $0 \leq l \leq 2$, so $k = \lfloor \frac{n+1}{3} \rfloor$. We can apply the previous proposition to obtain a canonical variety X_d of dimension n in the projective space
$$\mathbb{P} := \mathbb{P}(k^{(k+2)},(k+1)^{(2k-1)},(k(k+1))^{(l)}).$$

By Theorem 3.4.4 (and the proof of the lemma above) in [Dol82] we deduce that
$$H^0(X, mK_{X_d}) = 0$$
for $0 < m < k$.

Moreover
$$\mathrm{vol}(X_d) = \frac{(l+3)}{k^{k+1+l}(k+1)^{2k-2+l}} < \frac{(l+3)}{k^{n+l}} = \frac{l+3}{3k^l}\cdot\frac{3}{k^n} \leq \frac{3^{n+1}}{(n-1)^n}.$$

Take as X any desingularization of X_d. □

Proof of theorem 4.0.10. Let $n = 3k+l-1$, with integers $k \geq 2$ and $2 \leq l \leq 4$ so $k = \lfloor \frac{n-1}{3} \rfloor$. We can apply the proposition 4.1.2 to obtain a canonical variety X_d of dimension n in the projective space
$$\mathbb{P} := \mathbb{P}(k^{(k+2)},(k+1)^{(2k-1)},(k(k+1))^{(l)}).$$

We denote the coordinates of this space z_i for $0 \leq i \leq n+1$. Recall that $K_{X_d} \sim \mathcal{O}_{X_d}(1)$.

Fix $t < k(k+1)$. Thanks to Proposition 3.3 of [Mor75] we get

$$|\mathcal{O}_X(t)| = |\mathcal{O}_{\mathbb{P}}(t)|,$$

but the last two variable z_n and z_{n+1} can not appear in an element of $|\mathcal{O}_{\mathbb{P}}(t)|$ for degree's reasons.

Take as X any desingularization of X_d. Then the map induced by $|tK_X|$ is not generically finite. In particular

$$x'_n \geq k(k+1) \geq \frac{n(n-3)}{9}.$$

□

Proof of theorem 4.0.11. We first consider the case n even. Let $d = n+3$ and let $\mathbb{P}$ the weighted projective space $\mathbb{P}(1^{(n)}, 2, d)$. The coherent sheaf $\mathcal{O}_{\mathbb{P}}(2d)$ is a line bundle ([Mor75], Lemma 1.5). We call the coordinates of this space z_i for $0 \leq i \leq n+1$. Since d is odd the general weighted hypersurface X of degree $2d$ do not meet the singularities of $\mathbb{P}$ and it is smooth. Note that its equation is of the form

$$z_{n+1}^2 = P(z_0, \ldots, z_n)$$

where P is a polynomial of weighted degree $2d$. Moreover we have $K_X \sim \mathcal{O}_X(1)$.

If we take a positive integer $t < d$, then the linear system $|tK_X|$ does not induce a birational map. Indeed by Proposition 3.3 of [Mor75] we have

$$|\mathcal{O}_X(t)| = |\mathcal{O}_{\mathbb{P}}(t)|,$$

but the variable z_{n+1} can not appear in an element of $|\mathcal{O}_{\mathbb{P}}(t)|$ for degree's reasons and so the induced map has at least degree 2.

If $n = 1$ we use a smooth curve of genus 2. If n is odd and $n \geq 3$ we consider a manifold Y of dimension $n-1$ such that $r_n \geq n+2$ and define $X := Y \times C$ where C is smooth curve of genus 2. Alternatively, for n odd, take $d = n+2$ and a general hypersurface of degree $2d$ in $\mathbb{P}(1^{(n+1)}, d)$. □

Proof of theorem 4.0.13. Let b be a positive integer such that

$$br \equiv 1 \quad mod\ s,$$

and write

$$br - 1 = ts.$$

Let a be a positive integer such that $(a, s) = (a, b) = 1$.

We set

$$n := rab + 1 - a - s - b, \qquad d := n - 1 + a + s + b = rab$$

$$\mathbb{P} = \mathbb{P}(1^{(n-2)}, a, s, b).$$

Observe that choosing a and b we can have n arbitrarily large. Hence we may assume $n \geq 3$.

Note that $(a, s) = (a, b) = (s, b) = 1$, hence the only singularities of $\mathbb{P}$ are $P = (0^{(n-2)}, 0, 1, 0)$, $Q = (0^{(n-2)}, 1, 0, 0)$ and $R = (0^{(n-2)}, 0, 0, 1)$.

Now consider a general weighted hypersurface X_d of degree d in $\mathbb{P}$. The sheaf $\mathcal{O}_{\mathbb{P}}(1)$ is locally free and spanned outside P. By Bertini's theorem applied to $\mathbb{P} \setminus \{P\}$ we get that X_d is smooth outside P. Since $s \not| d$, $a|d$ and $b|d$, so $P \in X_d$ while $Q, R \notin X_d$. Since $n \geq 3$ and X_d has a unique singular point, it is well-formed.

We will show that for a and b large enough, X_d is a (well-formed) quasi-smooth variety with at most a terminal singularity in P. Then we would have finished. Indeed note that by adjunction (6.14 of [IF00]) $K_X \sim \mathcal{O}_X(1)$ is ample and therefore

$$\operatorname{vol}(X_d) = K_X^n = \mathcal{O}_X(1)^n = \frac{d}{asb} = \frac{r}{s}.$$

To control the quasi-smoothness we use criterium 8.1 of [IF00].

Let $z_0, \ldots, z_{n+1}$ be the coordinates of $\mathbb{P}$. For every $I \subset \{0, \ldots, n+1\}$ except $I = \{n\}$ the condition 2.a of 8.1 in [IF00] is satisfied because the only variable whose degree does not divide d is z_n. In the case $I = \{n\}$ we can use 2.b since $d = tas + a$ and we can take the monomial

$$z_n^{ta} z_{n-1}.$$

Since X_d is quasi-smooth its singularities are induced by those of $\mathbb{P}$ and so we have only to control that X_d is terminal in P.

Let $f = 0$ be an equation of X_d. We can write

$$f = z_n^{ta} z_{n-1} + \ldots$$

We consider the affine piece $(z_n = 1)$. The point $P \in X_d$ looks like

$$(\tilde{f} = f(z_0, \ldots, z_{n-1}, 1, z_{n+1}) = z_{n-1} + \ldots = 0) \subset \mathbb{A}^{n+1}/\epsilon$$

where ϵ is a primitive s-th root of unity and acts via

$$z_i \mapsto \epsilon z_i \quad 0 \leq i \leq n-2,$$

$$z_{n-1} \mapsto \epsilon^a z_{n-1}$$

and

$$z_{n+1} \mapsto \epsilon^b z_{n+1}.$$

Note that $\partial f/\partial z_{n-1} \neq 0$ in P, hence, by the Inverse Function Theorem, z_i are local coordinates for P in X_d for $i \neq n-1, n$. This gives a quotient singularity of type

$$\frac{1}{s}(1^{(n-2)}, b).$$

By Reid's criterium 1.3.7, if $n - 2 \geq s$ then P is terminal.

Now you can simply take a desingularization of X_d. $\square$

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