# Anomalous behaviour of the correction to the Central Limit Theorem for a model of Random Walk in Random media

**Luca Di Persio**

Dipartimento di Matematica - Povo

## 1. – Introduction

The aim of this paper is to develop some new results about a well studied model of random walk in random environment.

We consider a particle moving in a $\nu$-dimensional infinite lattice which evolves stochastically in discrete time t. The environment is described by a random field $\xi \equiv \{\xi_t(x) : x \in \mathbb{Z}^\nu, t \in \mathbb{Z}^+\}$ that is the result of i.i.d. copies of the same random variable taking value in some finite set $\mathscr{S}$.

The one step transition probability from a site $x$ at time $t$ to a site $y$ at following time $t+1$ for a given the realization of the environment $\xi$ is:

$$P(X_{t+1} = y \mid X_t = x, \xi) = P_0(y-x) + \varepsilon c(y-x; \xi)$$

this is a sum of two terms, a free homogeneous random walk $P_0(\cdot)$ and a random perturbation $c(\cdot, \cdot)$ in which the coupling intensity is given by the parameter $\varepsilon$.

A finite range assumption on $P_0(\cdot)$ and $c(\cdot;\cdot)$ is made. Under some others standard technical conditions the results in [1], [2] and [3] include the Central Limit Theorem (CLT) for the displacement $X_t - X_0$ almost everywhere in the realization of the environment and in any dimension.

Moreover in [2] is proved that the time behaviour of the corrections to the CLT for the RW $X_t$, in dimension $\nu \geq 3$, depends on the environment and the traditional expansion in inverse powers of $T^{\frac{1}{2}}$ is reduced to only a finite number of terms, in fact it holds up to the term of order $T^{\frac{-k}{2}}$, where $k \leq [\frac{(\nu-1)}{2}]$ is the largest integer smaller than $\frac{\nu}{2}$.

Analogous conclusions are shown, again in dimension $\nu \geq 3$, in [2] for the cumulants of the first and second order. In [4] is proved that, in dimension $\nu = 1$, the correction to the CLT is a term of order $T^{\frac{-1}{4}}$, depending on the environment which, if normalized, tends, as $T \to \infty$, to a random gaussian variable.

We find the correct term of normalization in dimension $\nu = 2$, which is $\sqrt{\frac{T}{\ln T}}$, proving that the correction to the CLT tends to a limiting centered gaussian variable the dispersion of which we can write as an explicit integral.

Moreover we prove that similar anomalous behaviour happens for the corrections to the average and to the covariance matrix in dimension $\nu = 1, 2$.

In the first section we will describe the model and we will enunciate the main results which will be proved in the second section using a specific sort of cluster expansion. Some details of the proofs will be shifted in the appendix.

## 2. – Definitions and main results

We denote by $X_t \in \mathbb{Z}^\nu$, with $t \in \mathbb{Z}$, the position of a particle which is moving in a $\nu$-dimensional infinite lattice. Time is discrete and the particle's probability to jump from one site to another depends on the state of environment.

More precisely we put independent copies of the same discrete random variable on each site of the grid, this variable takes values in a finite set $\mathscr{S} \equiv \{s_1, \ldots, s_n\}$ with a non degenerate probability $\pi$. We will define $\hat{\Omega} \equiv \mathscr{S}^{\mathbb{Z}^{\nu+1}}$ as the set of all possible configurations of the environment equipped with the natural product measure $\Pi \equiv \pi^{\mathbb{Z}^{\nu+1}}$.

In the following $\langle \cdot, \cdot \rangle$ and $\mathbb{E}(\cdot)$ indicate expectations with respect to the distribution $\Pi_0$ (or to the measure $\pi$ for a single point $(x,t) \in \mathbb{Z}^{\nu+1}$) and over the trajectories $\{X_t\}$ respectively.

Once a configuration $\xi \in \hat{\Omega}$ of the environment is fixed and for $\varepsilon < 1$, we define one step transition probabilities as follows:

$$P(X_{t+1} = y \mid X_t = x, \xi) \equiv P_0(y-x) + \varepsilon c(y-x; \xi_t(x)). \tag{1}$$

hence they are defined as a sum made of an homogeneous random walk $P_0(u)$ plus a random term $c(u,s)$ which, with no loss of generality, is supposed to have zero average (i.e. $\langle c(u; \cdot) \rangle = 0$ ) and to be such that $\sum_{u \in \mathbb{Z}^2} c(u; s) = 0$.
Further assumptions are the following:

(1) $0 \leq P_0\,(u) + \varepsilon c\,(u,s) < 1$

(2) $\exists D \geq 1 : P_0\,(u) = c\,(u,s) = 0 \quad \forall u \in \mathbb{Z}^2 \ : \parallel u \parallel_2 > D, \forall s \in \mathscr{S}$

(3) The characteristic function associated to $P_0$:

$$\tilde{p}_0(\lambda) = \sum_{u \in \mathbb{Z}^\nu} P_0\,(u)\, e^{i(\lambda,u)}, \quad \lambda \equiv (\lambda_1, \ldots, \lambda_\nu) \in T^\nu$$

where $T^\nu$ is the usual $\nu-$dimensionale torus, satisfies:

(3a) $\mid \tilde{p}_0\,(\lambda) \mid < 1, \quad \forall \lambda \neq 0$

As a consequence of (2) and (3a) we also have that the quadratic term which appears in the following Taylor expansion:

$$\ln \tilde{p}_0\,(\lambda) = \imath \sum_{k=1}^{\nu} b_k \lambda_k - \frac{1}{2} \sum_{i,j=1}^{\nu} \mathfrak{c}_{ij} \lambda_i \lambda_j + \cdots$$

around $\lambda = 0$, is strictly positive for $\lambda \neq 0$.

We want to prove that, in dimension $\nu = 2$, an anomalous correction to the CLT for the displacement $X_t - X_0$ appears.

In fact if we define:

$$Q_T(x \mid \xi) \equiv P(X_T = x \mid X_0 = 0; \xi) - P_0^T(x)$$

where $\mathbf{b} = (b_1, \ldots, b_\nu)$ represents the drift ,see (3b), the following theorem holds:

**Theorem 2.1.** *If $\varepsilon$ is sufficiently small and for all function $f \in \mathscr{C}^{2,lim}$ the sequence of functionals:*

$$\hat{\mathscr{Q}}_T(f \mid \xi) \equiv \sqrt{\frac{T}{\ln T}} \sum_{x\in\mathbb{Z}^2} Q_T(x \mid \xi) f\left(\frac{x - \mathbf{b}T}{\sqrt{T}}\right) \tag{2}$$

*tends, in distribution for $T \to \infty$ and some constants $\tilde{c}_0$, $\mathfrak{M}_{ij}$ $(i, j = 1, 2)$, to a centered gaussian variable with dispersion:*

$$\frac{\tilde{c}_0}{2} \sum_{ij=1}^{2} \mathfrak{M}_{ij} \left(\int K_C(1, v) f_i(v) dv\right) \left(\int K_C(1, v) f_j(v) dv\right) \tag{3}$$

*which depends only on the position reached by the particle at the final time and where:*

$$K_C(s, v) \equiv \frac{\sqrt{C}}{2\pi s} \cdot e^{-\frac{\mathcal{A}(v)}{2s}} \quad , \quad f_i \equiv \frac{\partial f}{\partial x_i} \tag{4}$$

*with $\mathcal{A} = \{\mathfrak{c}_{ij}\}^{-1}$ which defines, $\forall v \in \mathbb{R}^\nu$, the quadratic form $\mathcal{A}(v) \equiv \sum_{i,j=1}^{\nu} \mathfrak{a}_{ij} v_i v_j$.*

The same techniques used to prove previous result allow us to investigate the growth of the correction to the average and to the covariance matrix in dimension $\nu = 1, 2$. In fact if we set for the average vector components

$$\mathscr{E}_i^{(T)}(\xi) \equiv \mathbb{E}\left((X_t)_i \mid X_0 = 0, \xi)\right) - b_i T$$

where $\mathbf{b} \equiv (b_1, \ldots, b_\nu)$ and defining the covariance matrix elements:

$$\mathscr{C}_{ij}^{(T)}(\xi) \equiv \mathbb{E}\left((X_T - \mathbf{b}T)_i (X_T - \mathbf{b}T)_j \mid X_0 = 0, \xi\right) - \mathfrak{c}_{ij} T$$

for $i, j = 1, \ldots, \nu$ and $\nu = 1, 2$, we have the following results:

**Theorem 2.2.** *For $\nu = 1$, if $\varepsilon$ is small enough and setting $S_T \equiv \langle (\mathscr{E}^{(T)})^2 \rangle^{\frac{1}{2}}$, the sequence:*

$$\frac{\mathscr{E}^{(T)}(\xi)}{S_T}$$

*converges in distribution, for $T \to \infty$, to a standard gaussian variable. Moreover we have: $S_T \asymp T^{\frac{1}{4}}$.*

**Theorem 2.3.** *For $\nu = 2$, if $\varepsilon$ is small enough and setting $S_T \equiv \langle (\mathscr{E}^{(T)})^2 \rangle^{\frac{1}{2}}$, the sequence:*

$$\frac{\mathscr{E}^{(T)}(\xi)}{S_T}$$

*converges in distribution, for $T \to \infty$, to a centered gaussian variable with covariance matrix:*

$$\Sigma \equiv \{\mathfrak{b}_{ij}\} = \{\langle \mathfrak{b}_i(\cdot) \mathfrak{b}_j(\cdot) \rangle\}$$

*where $\mathfrak{b}_i(\cdot) \equiv \sum_{u\in\mathbb{Z}^2} u_i c(u; \cdot)$. Moreover $S_T \asymp (\ln T)^{\frac{1}{2}}$.*

**Theorem 2.4.** *For $\nu = 1$, if $\varepsilon$ is small enough and setting $\tilde{S}_T \equiv \langle (\mathscr{C}^{(T)})^2 \rangle^{\frac{1}{2}}$, the sequence:*

$$\frac{\mathscr{C}^{(T)}(\xi)}{\tilde{S}_T}$$

*converges in distribution, for $T \to \infty$, to a standard gaussian variable. Moreover $\tilde{S}_T \asymp T^{\frac{3}{4}}$.*

**Theorem 2.5.** *For $\nu = 2$, if $\varepsilon$ is small enough and setting $\tilde{S}_{ij}^{(T)} \equiv \langle (\mathscr{C}_{ij}^{(T)})^2 \rangle^{\frac{1}{2}}$, the sequence:*

$$\frac{\mathscr{C}_{ij}^{(T)}(\xi)}{\tilde{S}_{ij}^{(T)}}$$

*converges in distribution, for $T \to \infty$, to a standard gaussian variable. Moreover $\tilde{S}_{ij}^{(T)} \asymp T^{\frac{1}{2}}$.*

## 3. – Proofs

Our model is characterized by a space-time invariance so there is no loss of generality in assuming that the random walk always starts at the origin at time $t = 0$.

We can rewrite (2) as:

$$Q_T(x \mid \xi) = \sum_{0 \le t_1 \le t_2 \le T-1} \sum_{y_1, y_2 \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_*(t_2 - t_1, y_2 - y_1; \xi_{(t_1, y_1)}) \times$$
$$\times\, h^{T-t_2}(x - y_2; \xi_{t_2}(y_2))$$

where:

$$h^t(y; s) \equiv \sum_{u \in \mathbb{Z}^2} c(u; s) P_0^{t-1}(y - u)$$

$$M_*(t, y; \xi) \equiv \sum_{B:(0,0) \to (t,y)} \varepsilon^{|B|} M_B^*(\xi), \quad M_B^*(\xi) \equiv \prod_{i=0}^{n-1} h^{\tau_i}(z_i; \xi_{t_i}(y_i))$$

and $\xi_{(t,y)}$ is the shifted environment, i.e. :

$$\xi_{(t,y)}(z, \tau) \equiv \xi_{\tau - t}(z - y)$$

Sums in the definition of $M_*(t, y; \xi)$ are over all possible subsets of points $B = \{(t_1, y_1), \ldots, (t_n, y_n)\}$ from $(0, 0)$ to $(t, y)$.

The quantities $\tau_i$ and the positions $z_i$ are defined as $\tau_i \equiv t_{i+1} - t_i$ , $z_i \equiv y_{i+1} - y_i$ and we assume $P_0^0(y) \equiv \delta_{y0}$ and $M_*(0, y; \xi) \equiv \varepsilon \delta_{y0}$.

Setting $\mathfrak{b}(s) \equiv \sum_{u \in \mathbb{Z}^2} u c(u; s)$ and indicating with $\mathscr{H}_f(x)$ the Hessian matrix of the function $f$ calculated at a certain point $x \in \mathbb{R}^2$, we obtain that for all $y \in \mathbb{R}^2$ there exists $\zeta \in \mathbb{R}^2$ with $\|\zeta\|_2 \le D$ such that:

$$\sum_{u \in \mathbb{Z}^2} c(u; s) f\left(\frac{y + u}{\sqrt{T}}\right) = \frac{1}{\sqrt{T}} \left(\mathfrak{b}(s) \nabla f\left(\frac{y}{\sqrt{T}}\right)\right) + r_T(y; s)$$

where:

$$r_T(y,s) \equiv \frac{1}{2T}\sum_{u\in\mathbb{Z}^2} c(u;s)\mathscr{H}_f(\zeta)\cdot(u,u).$$

In the following we will work only with function $f \in \mathcal{C}^{2,lim}(\mathbb{R}^2)$ with a norm defined by:

$$\| f \| \equiv \| f \|_\infty + \| \nabla(f) \|_\infty + \| \mathscr{H}_f \|_\infty$$

where:

$$\| \nabla(f) \|_\infty \equiv \max_{x\in\mathbb{R}^2}\left\{\left|\frac{\partial f}{\partial x_1}(x)\right|, \left|\frac{\partial f}{\partial x_2}(x)\right|\right\}$$

$$\| \mathscr{H}_f \|_\infty \equiv \max_{x\in\mathbb{R}^2}\left\{\left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| : i,j=1,2\right\}$$

We have:

(1) $\langle r_T(y;\cdot)\rangle = 0,\ \forall y\in\mathbb{R}^2$

(2) $\mid r_T(y;s)\mid \leq \frac{cost\|\mathscr{H}_f\|_\infty}{T} \xrightarrow{T\to\infty} 0,\ \forall (y,s)\in\mathbb{R}^2\times\mathscr{S}$

Let be:

$$\begin{aligned}\delta_T(t,y;s) &\equiv \sum_{x\in\mathbb{Z}^2} h^t(x;s) f\left(\frac{y+x}{\sqrt{T}}\right) - \frac{\mathfrak{b}(s)}{\sqrt{T}}\cdot\sum_{z\in\mathbb{Z}^2} P_0^{t-1}(z)\nabla f\left(\frac{y+z}{\sqrt{T}}\right) = \\ &= \sum_{z\in\mathbb{Z}^2} P_0^{t-1}(z)\left[\sum_{u\in\mathbb{Z}^2} c(u;s) f\left(\frac{y+u+z}{\sqrt{T}}\right) - \frac{\mathfrak{b}(s)}{\sqrt{T}}\nabla f\left(\frac{y+z}{\sqrt{T}}\right)\right]\end{aligned} \tag{5}$$

then $\delta_T(t,y;s)$ has zero average and satisfy $\forall (s,y)\in\mathscr{S}\times\mathbb{R}^2$:

$$\begin{aligned}|\delta_T(t,y;s)| &= \left|\sum_{z\in\mathbb{Z}^2} P_0^{t-1}(z)\left\{\sum_{u\in\mathbb{Z}^2} c(u;s)\left[f\left(\frac{y+z}{\sqrt{T}}\right) + \right.\right.\right. \\ &\left.\left.\left. + \nabla f\left(\frac{y+z}{\sqrt{T}}\right)\frac{u}{\sqrt{T}} + \frac{\mathscr{H}f(\zeta)\cdot(u,u)}{T}\right] - \frac{\mathfrak{b}(s)}{\sqrt{T}}\nabla f\left(\frac{y+z}{\sqrt{T}}\right)\right\}\right| = \\ &= \left|\sum_{z\in\mathbb{Z}^2} P_0^{t-1}(z) r_T(y+z,s)\right| \leq \frac{cost \,\|\, \mathscr{H}_f \,\|_\infty}{T} \xrightarrow{T\to\infty} 0\end{aligned}$$

The proof of our main theorem, i.e. Th. (2.1), will be split into several results stated as lemmas and propositions. In the course of the proof the notation *const* will denote several constants, which may depend on the parameter $\varepsilon$.

**Proof:** (of Th. (2.1)) If we define the vector $M_\sharp(t,y \mid \xi) \equiv M_*(t,y\mid\xi)\mathfrak{b}(\xi_t(y))$, then we have:

**Proposition 3.1.** *For $i=1,2$ and $\varepsilon$ small enough there exists a positive constant $C=C(\varepsilon)$ such that:*

$$\sum_{y\in\mathbb{Z}^2}\langle\left((M_\sharp(t,y\mid\cdot))_i\right)^2\rangle \leq \frac{\varepsilon^2 C(\varepsilon)}{(t+1)^2}$$

**Proof:** If two subsets of points don't coincide, in space and time, their contribution is equal to zero by the definition of the $c(u;s)$ term, then setting:

$$b_i \equiv \max_{s\in\mathscr{S}} \mid (\mathfrak{b}(s))_i \mid , i = 1, 2$$

we have:

$$\begin{aligned} &\sum_{y\in\mathbb{Z}^2} \langle \left( (M_\sharp(t, y \mid \cdot))_i \right)^2 \rangle = \sum_{y\in\mathbb{Z}^2} \langle \left( \sum_{B:(0,0)\to(t,y)} \varepsilon^{|B|} M_B^*(\cdot) \, (\mathfrak{b}(\cdot))_i \right)^2 \rangle \le \\ &\le (\varepsilon b_i)^2 \sum_{y\in\mathbb{Z}^2} \sum_{n=1}^{t} \varepsilon^{2n} \sum_{\substack{t_1+\cdots+t_n=t \\ \min\{t_2,\ldots,t_n\}>0}} \sum_{x_1,\ldots,x_n} \prod_{i=1}^{n} \langle (h^{t_i}(x_i;\cdot))^2 \rangle \end{aligned} \tag{6}$$

Remembering that:

$$\int_{\mathbb{R}^\nu} e^{-c\frac{(y-\mathbf{b}t)^2}{t}} dy = \int_{\mathbb{R}^\nu} t^{\frac{\nu}{2}} e^{-cx^2} dx \le cost \cdot t^{\frac{\nu}{2}} \tag{7}$$

by appendix A of [2], we have:

$$\max_{s\in\mathscr{S}} \sum_{y\in\mathbb{Z}^2} (h^t(y;s))^2 \le \sum_{y\in\mathbb{Z}^2} A_1 \frac{e^{-\alpha\frac{(y-\mathbf{b}t)^2}{t}}}{(t+1)^3} \le \frac{cost}{t^2}$$

then the quantity on the second line of (6) is bounded by:

$$\begin{aligned} &(\varepsilon b_i)^2 \sum_{y\in\mathbb{Z}^2} \sum_{n=1}^{t} \varepsilon^{2n} \sum_{\substack{t_1+\cdots+t_n=t \\ \min\{t_2,\ldots,t_n\}>0}} \sum_{x_1,\ldots,x_n} \prod_{i=1}^{n} A_1 \frac{e^{-\alpha\frac{(x_i-\mathbf{b}t_i)^2}{t_i}}}{(t_i+1)^3} \le \\ &\le (\varepsilon b_i)^2 \sum_{n=1}^{t} \varepsilon^{2n} \sum_{\substack{t_1+\cdots+t_n=t \\ \min\{t_2,\ldots,t_n\}>0}} \frac{cost}{\prod_{i=1}^{n}(t_i+1)^2}. \end{aligned}$$

finally, iterating the following inequality:

$$\sum_{t_1=1}^{T-1} \frac{1}{t_1^a} \cdot \frac{1}{(T-t_1)^a} \le \frac{K(a)}{T^a}$$

which is valid for all $a > 1$ and some constant $K(a) > 0$, by the small randomness condition, i.e. $\varepsilon < 1$, we can sum over $n$ to obtain the result.

□

Let be:

$$\mathscr{Q}_T^{(1)}(f \mid \xi) \equiv$$
$$\frac{1}{\sqrt{T}} \sum_{\substack{t_1+t_2+t_3=T-1 \\ x,y_1,y_2\in\mathbb{Z}^2}} P_0^{t_1}(y_1) M_\sharp(t_2, y_2 - y_1 \mid \xi_{(t_1,y_1)}) P_0^{t_3}(x-y_2) \cdot \nabla f \left( \frac{x - \mathbf{b}T}{\sqrt{T}} \right)$$

assuming:

$$\hat{\mathscr{Q}}_T^{(1)}(f \mid \xi) \equiv \sqrt{\frac{T}{\ln T}} \mathscr{Q}_T^{(1)}(f \mid \xi)$$

then for the sequence of functionals $\hat{\mathscr{Q}}_T(f \mid \xi)$ defined in (2) we can prove:

**Lemma 3.1.** *For $\varepsilon$ small enough:*

$$\langle \left( \hat{\mathscr{Q}}_T(f \mid \cdot) - \hat{\mathscr{Q}}_T^{(1)}(f \mid \cdot) \right)^2 \rangle \xrightarrow{T\to\infty} 0$$

**Proof:** By the definition of $\delta_T(t, y; s)$ in (5), we have that the following difference between functionals:

$$\hat{\mathscr{Q}}_T(f \mid \cdot) - \hat{\mathscr{Q}}_T^{(1)}(f \mid \cdot)$$

can be written as:

$$\left(\frac{T}{\ln T}\right)^{\frac{1}{2}} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \sum_{y_1, y \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_*(t_2, y - y_1 \mid \xi_{(t_1,y_1)}) \delta_T \left(T - t, y - \mathbf{b}T; \xi_t(y)\right)$$

By $L^2(\Pi)$ orthogonality of the terms:

$$M_*(t_2, y - y_1 \mid \xi_{(t_1,y_1)}) \delta_T(T - t, y - \mathbf{b}T; \xi_t(y))$$

and using the result contained in (3) for the quantity $\delta_T(t, y; s)$, we find:

$$\begin{aligned} &\langle \left( \hat{\mathscr{Q}}_T(f \mid \cdot) - \hat{\mathscr{Q}}_T^{(1)}(f \mid \cdot) \right)^2 \rangle \le \\ &\le \frac{cost \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \sum_{y_1, y_2 \in \mathbb{Z}^2} (P_0^{t_1}(y_1))^2 \langle M_*^2(t_2, y_2 \mid \cdot) \rangle \end{aligned} \tag{8}$$

Using again the inequality (7) and Prop. (3.1) we have that the right side of (8) is bounded by:

$$\frac{cost \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \frac{C(\varepsilon)\varepsilon^2}{(t_1+1)(t_2+1)^2} \le \frac{cost(\varepsilon) \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=1}^{T-1} \frac{1}{t}$$

as:

$$\sum_{t_1+t_2=t} \frac{1}{t_1+1} \frac{1}{(t_2+1)^2} \le \frac{cost}{t}$$

and:

$$\sum_{t=1}^{T} \frac{1}{t} \asymp \log T$$

then:

$$\frac{cost(\varepsilon) \parallel \mathscr{H}_f \parallel_\infty^2}{T \ln T} \sum_{t=1}^{T-1} \frac{1}{t} \le \frac{cost(\varepsilon) \parallel \mathscr{H}_f \parallel_\infty^2}{T} \xrightarrow{T\to\infty} 0$$

□

To determine the constants $\mathfrak{M}_{ij}$ introduced before in (3) and indicating with $\mathfrak{b}_i(\xi_t(y))$ the i-th component of the vector $\mathfrak{b}$ for $i = 1, 2$, we can prove:

**Proposition 3.2.** *For $i, j = 1, 2$, if $\varepsilon$ is small enough, the sequence:*

$$\mathfrak{C}_{ij}^{(T)}(\xi) \equiv \sum_{t=0}^{T-1} \sum_{y\in\mathbb{Z}^2} (M_*(t, y; \xi)) \left(\mathfrak{b}(\xi_t(y))\right)_i \left(\mathfrak{b}(\xi_t(y))\right)_j \tag{9}$$

*converges, for $T \to \infty$, to a limiting functional $\mathfrak{C}_{ij}$ both in $L^2$ as well as $\Pi - a.e.$ .*

**Proof:** By Prop. (3.1) and using the $L^2(\Pi)$ orthogonality of the terms $M_*(t, y; \xi)$, for $T' > T$ we have:

$$\langle \left( \mathfrak{C}_{ij}^{T'}(\cdot) - \mathfrak{C}_{ij}^{(T)}(\cdot) \right)^2 \rangle \leq \sum_{t=T}^{T'-1} \frac{cost}{(t+1)^2} \leq cost \left( \frac{1}{T} - \frac{1}{T'} \right)$$

and we can conclude the proof using the result contained in the appendix A of [3].

□

We will show later that the constants $\mathfrak{M}_{ij}$, which appear in (3), are exactly the second moments of the limiting functionals $\mathfrak{C}_{ij}$.
We now define the following quantities:

$$T_1 \equiv [T^\beta],\ \beta \in (0, 1), \quad T_* \equiv [\ln_+ T], \quad \ln_+ T \equiv \max\{1, \ln T\}$$

and let be:

$$H_T(t, y) = \sum_{z\in\mathbb{Z}^2} P_0^{T-t-1}(z) \nabla f \left( \frac{y + z - \mathbf{b}T}{\sqrt{T}} \right)$$

then the functional:

$$\hat{\mathscr{Q}}_T^{(2)}(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1 y_2\in\mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2 - t_1, y_2 - y_1 \mid \xi_{(t_1,y_1)}) \cdot H_T(t_2, y_2)$$

is obtained from removing those terms that are relative to large $t_1$ and large differences $t_2 - t_1$ and for $\hat{\mathscr{Q}}_T^{(1)}$ the following result holds:

**Lemma 3.2.** *For $\varepsilon$ small enough we have:*

$$\langle \left( \hat{\mathscr{Q}}_T^{(1)}(f \mid \cdot) - \hat{\mathscr{Q}}_T^{(2)}(f \mid \cdot) \right)^2 \rangle \xrightarrow{T\to\infty} 0$$

**Proof:** First, considering the large $t_1$ values, we define:

$$\bar{\mathscr{Q}}(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{t_1=T-T_1+1}^{T-1} \sum_{t_2=t_1}^{T-1} \sum_{y_1 y_2\in\mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2 - t_1, y_2 - y_1 \mid \xi_{(t_1,y_1)}) \cdot H_T(t_2, y_2)$$

Proceeding as in the proof of lemma (3.1) we have:

$$\langle (\bar{\mathscr{Q}}(f \mid \cdot))^2 \rangle \leq \frac{cost}{T \ln T} \parallel \nabla f \parallel_\infty^2 \sum_{t=1}^{T} \frac{1}{t} = \frac{cost}{T} \parallel \nabla f \parallel_\infty^2 \xrightarrow{T\to\infty} 0$$

The contribution for large differences $t_2 - t_1$ can be rewritten as:

$$\widetilde{\mathscr{Q}}(f \mid \xi) \equiv$$
$$\frac{1}{\sqrt{\ln T}} \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1+T_*+1}^{T-1} \sum_{y_1 y_2 \in \mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2 - t_1, y_2 - y_1 \mid \xi_{(t_1,y_1)}) \cdot H_T(t_2, y_2)$$

so we obtain:

$$\langle (\widetilde{\mathscr{Q}}(f \mid \cdot))^2 \rangle \leq \frac{cost \parallel \nabla f \parallel_\infty^2}{\ln T} \sum_{t_1=0}^{T-T_*} \frac{1}{(t_1+1)} \sum_{t'=T_*}^{T-t_1} \frac{1}{(t'+1)^2}$$
$$\leq \frac{cost \parallel \nabla f \parallel_\infty^2}{T_* \cdot \ln T} \sum_{t_1=1}^{T} \frac{1}{t_1} \leq \frac{cost \parallel \nabla f \parallel_\infty^2}{T_*} \xrightarrow{T\to\infty} 0$$

□

**Lemma 3.3.** *For $\varepsilon$ small enough we have:*

$$\langle (\hat{\mathscr{Q}}_T^{(2)}(f \mid \cdot))^2 \rangle \xrightarrow{T\to\infty} \frac{\tilde{c}_0}{2} \sum_{ij=1}^{2} \mathfrak{M}_{ij} \left( \int K_C(1,v) f_i(v) dv \right) \left( \int K_C(1,v) f_j(v) dv \right)$$

*where $\mathfrak{M}_{ij} = \langle (\mathfrak{C}_{ij}(\cdot))^2 \rangle$, $\mathfrak{C}_{ij}$ are the same limiting functionals that appear in Prop. (3.2) and $K_C$ is the heat kernel defined in (4).*

**Proof:** Hypotheses on our model imply (see for example [5]) that, around the point $\lambda = 0$ in the $\nu-$dimensional torus, the Taylor expansion of the characteristic function of $P_0$ is:

$$\ln \tilde{p}_0 (\lambda) = \imath \sum_{k=1}^{\nu} b_k \lambda_k - \frac{1}{2} \sum_{i,j=1}^{\nu} \mathfrak{c}_{ij} \lambda_i \lambda_j + \cdots$$

In the bidimensional case the Local Limit Theorem (LLT) implies:

$$P_0^t(x) = \frac{\sqrt{C}}{(2\pi t)} e^{-\frac{1}{2}\mathcal{A}(\frac{x-\mathbf{b}t}{\sqrt{t}})} \cdot \left( 1 + O\left( \frac{1}{\sqrt{t}} \right) \right)$$

then we have, for all $f \in \mathcal{C}^{2,lim}(\mathbb{R}^2)$, that:

$$\left| \sum_z \left( P_0^t(z) - \frac{\sqrt{C} e^{-\frac{1}{2}\mathcal{A}(\frac{z-\mathbf{b}t}{\sqrt{t}})}}{2\pi t} \right) \cdot f\left( \frac{y+z-\mathbf{b}T}{\sqrt{T}} \right) \right| =$$

$$= \left| \sum_z \frac{\sqrt{C} e^{-\frac{1}{2}\mathcal{A}(\frac{z-\mathbf{b}t}{\sqrt{t}})}}{2\pi t} \cdot O\left(\frac{1}{\sqrt{t}}\right) \cdot f\left(\frac{y+z-\mathbf{b}T}{\sqrt{T}}\right) \right| \le \frac{cost \cdot \parallel f \parallel_\infty}{\sqrt{t}}$$

Now we want to control the error that occurs replacing sums with integrals. In other words we want to calculate the asymptotic of the following Riemann sum:

$$\frac{1}{t} \sum_{z \in \mathbb{Z}^2} e^{-\frac{1}{2}\mathcal{A}\left(\frac{z-\mathbf{b}t}{\sqrt{t}}\right)} f\left(\frac{y-\mathbf{b}(T-t)}{\sqrt{T}} + \sqrt{\frac{t}{T}} \frac{z-\mathbf{b}t}{\sqrt{t}}\right)$$

If $Q_t(z)$ is the square centered in $\frac{z-\mathbf{b}t}{\sqrt{t}}$, with sides parallel to the cartesian axes and of length $t^{-\frac{1}{2}}$, defining $R \equiv \frac{y-\mathbf{b}(T-t)}{\sqrt{T}}$ we have:

$$\int_{\mathbb{R}^2} e^{-\frac{1}{2}\mathcal{A}(x)} f\left(R + \sqrt{\frac{t}{T}} x\right) - \frac{1}{t} \sum_{z \in \mathbb{Z}^2} e^{-\frac{1}{2}\mathcal{A}\left(\frac{z-\mathbf{b}t}{\sqrt{t}}\right)} f\left(R + \sqrt{\frac{t}{T}} \frac{z-\mathbf{b}t}{\sqrt{t}}\right) \tag{10}$$

Let be:

$$G(x) \equiv e^{-\frac{1}{2}\mathcal{A}(x)} f\left(R + \sqrt{\frac{t}{T}} x\right)$$

the integral over $Q_t(z)$ can be written as:

$$\Delta(z) \equiv \int_{Q_t(z)} \left[G(x) - G\left(\frac{z-\mathbf{b}t}{\sqrt{t}}\right)\right] dx \tag{11}$$

If we write the second order Taylor expansion of $G(x)$ in (11) around a point $x = \frac{z-\mathbf{b}t}{\sqrt{t}}$ we have that the term of zero order is cancelled by $G\left(\frac{z-\mathbf{b}t}{\sqrt{t}}\right)$. For the first order term we have:

$$\nabla G\left(\frac{z-\mathbf{b}t}{\sqrt{t}}\right) \cdot \int_{Q_t(z)} \left(x - \frac{z-\mathbf{b}t}{\sqrt{t}}\right) dx = 0$$

is zero by symmetry and the first term that survives is the second order one:

$$| \Delta(z) | = \frac{1}{2} \left| \int_{Q_t(z)} dx \sum_{j,k=1,2} \left[\frac{\partial^2 G}{\partial x_j \partial x_k}\right]_{x=\bar{x}} \left(x - \frac{z-\mathbf{b}t}{\sqrt{t}}\right)_j \left(x - \frac{z-\mathbf{b}t}{\sqrt{t}}\right)_k \right| \le$$
$$\le \max_{x \in Q_t(z)} \max_{j,k=1,2} \left| \frac{\partial^2 G(x)}{\partial x_j \partial x_k} \right| \frac{const}{t^2}$$

Now we recognize that:

$$\frac{\partial^2 G(x)}{\partial x_j \partial x_k} = \frac{t}{T} f_{jk}\left(R + \sqrt{\frac{t}{T}} x\right) e^{-\frac{\mathcal{A}(x)}{2}} + \sqrt{t} T f_j \left(R + \sqrt{\frac{t}{T}} x\right) \frac{\partial e^{-\frac{\mathcal{A}(x)}{2}}}{\partial x_k} +$$
$$+ \sqrt{\frac{t}{T}} f_k \left(R + \sqrt{\frac{t}{T}} x\right) \frac{\partial e^{-\frac{\mathcal{A}(x)}{2}}}{\partial x_j} + f\left(R + \sqrt{\frac{t}{T}} x\right) \frac{\partial^2 e^{-\frac{\mathcal{A}(x)}{2}}}{\partial x_j \partial x_k}$$

so if we indicate with $z_t^*$ the point in which the function $Q_t(z)$ reaches his maximum then we can write:

$$\left| \frac{\partial^2 G(x)}{\partial x_j \partial x_k} \right| \le cost \parallel f \parallel \max_{x \in Q_t(z)} e^{-\frac{\mathcal{A}(x)}{4}} = cost \parallel f \parallel e^{-\frac{\mathcal{A}(z_t^*)}{4}}$$

But the sum:

$$\sum_{z\in\mathbb{Z}^2}\frac{1}{t}e^{-\frac{\mathcal{A}(z_t^*)}{4}}$$

is a bounded Riemann sum then we have that the difference in (10) can be bounded in the following way:

$$\left|\int_{\mathbb{R}^2} e^{-\frac{1}{2}\mathcal{A}(x)} f\left(R+\sqrt{\frac{t}{T}}x\right) - \frac{1}{t}\sum_{z\in\mathbb{Z}^2} e^{-\frac{1}{2}\mathcal{A}\left(\frac{z-\mathbf{b}t}{\sqrt{t}}\right)} f\left(R+\sqrt{\frac{t}{T}}\frac{z-\mathbf{b}t}{\sqrt{t}}\right)\right| \le cost\frac{\parallel f \parallel}{t} \tag{12}$$

Again by the LLT we have:

$$H_T(t,y) = \sum_{z\in\mathbb{Z}^2} P_0^{T-t}(z)\nabla f\left(\frac{y-\mathbf{b}t+z-\mathbf{b}(T-t)}{\sqrt{T}}\right) =$$
$$= \sum_{z\in\mathbb{Z}^2}\sqrt{C}\frac{e^{-\frac{1}{2}\mathcal{A}\left(\frac{z-\mathbf{b}(T-t)}{\sqrt{T-t}}\right)}}{2\pi t}\nabla f\left(\frac{y-\mathbf{b}t+z-\mathbf{b}(T-t)}{\sqrt{T}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{T-t}}\right)$$

then, using (12), we have:

$$\sum_{z\in\mathbb{Z}^2}\sqrt{C}\frac{e^{-\frac{1}{2}\mathcal{A}\left(\frac{z-\mathbf{b}(T-t)}{\sqrt{T-t}}\right)}}{2\pi t}\nabla f\left(\frac{y-\mathbf{b}t+z-\mathbf{b}(T-t)}{\sqrt{T}}\right) =$$
$$= \int \frac{\sqrt{C}e^{-\frac{1}{2}\mathcal{A}(x)}}{2\pi}\nabla f\left(\frac{y-\mathbf{b}t}{\sqrt{T}} + \sqrt{1-\frac{t}{T}x}\right)dx + \mathcal{O}\left(\frac{1}{T-t}\right)$$

if we change variable in the last integral, defining $v \equiv x\sqrt{1-\frac{t}{T}}$, and setting:

$$H_T^*(t,y,\nabla f) \equiv \int_{\mathbb{R}^2} K_C\left(1-\frac{t}{T},v\right)\nabla f\left(\frac{y-\mathbf{b}t}{\sqrt{T}}+v\right)dv$$

so that:

$$H_T^*(t,y,\nabla f) = \left(H_T^*(t,y,\frac{\partial f}{\partial y_1}), H_T^*(t,y,\frac{\partial f}{\partial y_2})\right) = (H_T^*(t,y,f_1), H_T^*(t,y,f_2))$$

we obtain:

$$H_T(t,y) = H_T^*(t,y,\nabla f) + \mathcal{O}\left(\frac{1}{\sqrt{T-t}}\right) \tag{13}$$

where $K_C$ is the 2-dimensional heat kernel:

$$K_C(s,v) \equiv \frac{\sqrt{C}}{2\pi s}\cdot e^{-\frac{\mathcal{A}(v)}{2s}}$$

In $\hat{\mathscr{Q}}_T^{(2)}(f|\xi)$ the contribution for $t_1 \le T_1$ is given by:

$$\bar{\mathscr{Q}}''(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}}\sum_{t_1=0}^{T_1}\sum_{t_2=t_1}^{t_1+T_*}\sum_{y_1,y_2} P_0^{t_1}(y_1)M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_2,y_2)})\cdot H_T(t_2,y_2)$$

and it can be neglected, in fact we have:

$$\langle (\bar{\mathscr{Q}}''_T(f \mid \xi))^2 \rangle \leq cost \frac{(\| \nabla(f) \|_\infty)^2}{\sqrt{T} \cdot \ln T} \cdot \sum_{t_1=1}^{T_1} \frac{1}{t_1} \xrightarrow{T\to\infty} 0$$

then , using the approximation result in (13), we are left with the asymptotic of the quantity:

$$\left(\frac{1}{\ln T}\right) \sum_{t_1=T_1+1}^{T-T_1} \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1,y_2} (P_0^{t_1}(y_1))^2 \langle \left(M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_2,y_2)}) \cdot H_T^*(t_2, y_2, \nabla f)\right)^2 \rangle \tag{14}$$

We can start taking in account the diagonal component of index $(1,1)$. By the short range condition we obtain:

$$\mid H_T^*(t_2, y_2, f_1) - H_T^*(t_1, y_1, f_2) \mid \leq cost(\| f \|_\infty + \| \nabla(f) \|_\infty) \cdot \frac{T_*}{\sqrt{T}}$$

hence we can replace $H_T^*(t_2, y_2, f_1)$ with $H_T^*(t_1, y_1, f_1)$ and sum over $(t_1, y_1)$. Then, using the definition of the functionals $\mathfrak{C}_{ij}^{(T)}$ given in (9), the asymptotic of (14) in the first spatial coordinate of $H_T^*$ is the same as:

$$\frac{\langle \left(\mathfrak{C}_{11}^T(\cdot)\right)^2 \rangle}{\ln T} \sum_{t=T_1-1}^{T-T_1} \sum_y (P_0^t(y))^2 F_1\left(\frac{t}{T}, \frac{y-\mathbf{b}t}{\sqrt{T}}\right) \tag{15}$$

where:

$$F_1\left(\frac{t}{T}, \frac{y-\mathbf{b}t}{\sqrt{T}}\right) \equiv \left(\int K_c\left(1-\frac{t}{T}, v\right) f_1\left(\frac{y-\mathbf{b}t}{\sqrt{T}} + v\right) dv\right)^2$$

Taking the first order Taylor expansion of $F_1$, in the space variable, we can rewrite (15), for an appropriate point $y^\star = y^\star(y)$, as:

$$\frac{1}{\ln T} \left\{ \sum_{t=T_1}^{T-T_1} \sum_{y\in\mathbb{Z}^2} \left(P_0^t(y)\right)^2 \left[ F_1\left(\frac{t}{T}, 0\right) + \nabla_y\, F_1\left(\frac{t}{T}, y^\star(y)\right) \cdot \left(\frac{y-\mathbf{b}t}{\sqrt{T}}\right)\right]\right\} \tag{16}$$

But, using the first inequality of (A.1) in appendix A of [2], we have:

$$\frac{1}{\sqrt{T}\ln T} \sum_{t\geq 1} \sum_{y\in\mathbb{Z}^2} \left(P_0^t(y)\right)^2 \| y-\mathbf{b}t \| \leq \frac{cost}{\sqrt{T}\ln T} \sum_{t\geq 1} \frac{1}{t} \sum_{y\in\mathbb{Z}^2} P_0^t(y) \| y-\mathbf{b}t \| \asymp$$

$$\asymp \frac{cost}{\sqrt{T}\ln T} \sum_{t\geq 1} \frac{\sqrt{t}}{t} \asymp \frac{cost}{\ln T} \xrightarrow{T\to\infty} 0$$

then in (16) we have only to control the behaviour of the first addendum, for doing this we have to determine the asymptotic of a quantity of the following type:

$$I_T(f) \equiv \frac{1}{\ln T} \sum_{t=0}^{T-1} f\left(\frac{t}{T}\right) \sum_{y\in\mathbb{Z}^2} \left(P_0^t(y)\right)^2 \tag{17}$$

where $f$ is a sufficiently smooth function in $[0,1]$. First we will find the asymptotic of the following quantity:

$$J(T) \equiv \sum_{t=0}^{T-1} \sum_{y \in \mathbb{Z}^2} \left(P_0^t(y)\right)^2 \tag{18}$$

If we indicate with $\tilde{p}_0(\lambda)$ the characteristic function of $P_0$ then its centered version will be $\hat{p}(\lambda) = e^{-i(\lambda,\mathbf{b})}\tilde{p}_0(\lambda)$ , hence we can rewrite (18) as:

$$J\left(T\right) = \sum_{t=0}^{T-1} \int_{\Gamma^2} \mid \hat{p}(\lambda) \mid^{2t} dm(\lambda) = \int_{T^2} \frac{1- \mid \hat{p}(\lambda) \mid^{2T}}{1- \mid \hat{p}(\lambda) \mid^2} dm(\lambda)$$

Splitting the above integral in two parts we have:

$$J^{'}(T) \equiv \int_{1-|\hat{p}(\lambda)|^2<\delta} \frac{1- \mid \hat{p}(\lambda) \mid^{2T}}{1- \mid \hat{p}(\lambda) \mid^2} dm(\lambda$$

$$J^{''}(T) \equiv \int_{1-|\hat{p}(\lambda)|^2\geq\delta} \frac{1- \mid \hat{p}(\lambda) \mid^{2T}}{1- \mid \hat{p}(\lambda) \mid^2} dm(\lambda)$$

The $J^{''}(T)$ term remains bounded for $T \to \infty$, hence its asymptotic in (17) is equal to zero. In $J^{'}(T)$ we perform the coordinate change $1- \mid \hat{p}(\lambda) \mid^2 = u^2$ and we indicate its Jacobian with $C(u) \equiv \sum_{k=0}^{\infty} c_k(u)$, where $c_k(u)$ are homogeneous function of degree k, we obtain:

$$J^{'}(T) = \int_{u^2<\delta} C(u) \frac{1-\left(1-u^2\right)^{2T}}{u^2} du$$

If we pass to polar coordinates $(u_1,u_2) \to (\rho,\theta)$ then the previous jacobian will be rewritten as $C(\rho,\theta) = \sum_{k\geq 0} \rho^k \hat{c}_k(\theta)$ and if $k$ is odd then we have $\int \hat{c}_k(\theta)d\theta = 0$ and assuming $\tilde{c}_k \equiv \int \hat{c}_k(\theta)d\theta$, we obtain:

$$J^{'}(T) = \sum_{k\geq 0} \tilde{c}_{2k} \int_o^{\delta} \rho^{2k-1}(1-(1-\rho^2)^{2T})d\rho$$

For $k \geq 1$ our sum gives a constant, hence we are left with '$k=0$'-*case* in the limt for $T \to \infty$. Let be $\rho = \frac{z}{\sqrt{2T}}$, we have:

$$\tilde{c}_0 \int_0^{\delta} \frac{1-(1-\rho^2)^{2T}}{\rho} d\rho = \tilde{c}_0 \int_0^{\sqrt{2T}\delta} \left[1-\left(1-\frac{z^2}{2T}\right)^{2T}\right] \frac{dz}{z} =$$

$$= \mathscr{O}(1) + \tilde{c}_0 \int_0^{\sqrt{2T}\delta} \frac{1-e^{-z^2}}{z} = \mathscr{O}(1) + \frac{\tilde{c}_0}{2} \ln T \Rightarrow J(T) = \mathscr{O}(1) + \frac{\tilde{c}_0}{2} \ln T$$

Assuming $J(0)=0$, the quantity in (17) can be rewritten as:

$$I_T(f) = \frac{1}{\ln T}\left[f\left(\frac{T-1}{T}\right)J(T) - \sum_{t=1}^{T}\left(f\left(\frac{t}{T}\right) - f\left(\frac{t-1}{T}\right)\right)J(t)\right]$$

but we know that:

$$f\left(\frac{T-1}{T}\right)\frac{J(T)}{\ln T}\xrightarrow{T\to\infty} f(1)\frac{\tilde{c}_0}{2}$$

and according with the asymptotic of $J(T)$ it remains to control the quantity:

$$\frac{-\tilde{c}_0}{2\ln T}\sum_{t=0}^{T-1}\left(f\left(\frac{t+1}{T}\right)-f\left(\frac{t}{T}\right)\right)\ln(t+1)$$

for which we have:

$$-\frac{\tilde{c}_0}{2\ln T}\sum_{t=0}^{T-1}\left(f\left(\frac{t+1}{T}\right)-f\left(\frac{t}{T}\right)\right)\ln(t+1)\asymp-\frac{\tilde{c}_0}{2T\ln T}\sum_{t=0}^{T-1}f'\left(\frac{t}{T}\right)\left(\ln\frac{t}{T}+\ln T\right)=$$
$$=-\frac{\tilde{c}_0}{2T\ln T}\sum_{t=0}^{T-1}\left[f'\left(\frac{t}{T}\right)\ln\frac{t}{T}\right]-\frac{\tilde{c}_0}{2T}\sum_{t=0}^{T-1}f'\left(\frac{t}{T}\right)\asymp-\frac{\tilde{c}_0}{2\ln T}\int_0^1 f'(x)\ln x\,dx+$$
$$-\frac{\tilde{c}_0}{2}\int_0^1 f'(x)dx=\frac{\tilde{c}_0}{2}\left(f(0)-f(1)\right)$$

finally for the asymptotic of (17) we obtain:

$$I_T(f)\xrightarrow{T\to\infty}\frac{\tilde{c}_0}{2}f(0)$$

then in our case and for $i=1,2$, we have:

$$f(0)=F_i(0,0)=\left(\int K_C(1,v)f_i(v)dv\right)^2$$

The same argument can be applied to the mixed terms, hence we have the result.

□

Given the previous results it is sufficient to show the CLT for the sequence of functionals $\hat{\mathscr{Q}}^2_T(f\mid\xi)$. Let be:

$$\mathcal{E}^{(T)}(t_1\mid\xi)\equiv\sum_{t_2=t_1}^{t_1+T_*}\sum_{y_1y_2\in\mathbb{Z}^2}P_0^{t_1}(y_1)M_\sharp(t_2-t_1,y_2-y_1\mid\xi_{(t_1,y_1)})\cdot H_T(t_2,y_2)$$

then:

$$\hat{\mathscr{Q}}^2_T(f\mid\xi)=\frac{1}{\sqrt{\ln T}}\sum_{t=0}^{T-T_1}\mathcal{E}^{(T)}(t\mid\xi)\tag{19}$$

and observing that for $t_1<t_1'$ e $t_1'-t_1>T_*$ the quantities $\mathcal{E}^{(T)}(t_1\mid\xi)$ and $\mathcal{E}^{(T)}(t_1'\mid\xi)$ are independent it is natural to apply the Bernstein method (see for example [6]). We begin by defining:

$$0<\delta<\gamma<1,\quad r\equiv[T^\gamma],\quad s\equiv[T^\delta],\quad \mathscr{K}(T)\equiv\left[\frac{T}{T^\gamma+T^\delta}\right]$$

the intervals $I_k$:

$$I_k\equiv\left[(k-1)(r+s),kr+(k-1)s-1\right],\quad k=1,\dots,\mathscr{K}(T)$$

the *corridors* $J_k$:

$$J_k \equiv [kr + (k-1)s), k(r+s) - 1]\,, \quad k = 1, \dots, \mathscr{K}(T)$$

and:

$$R \equiv [\mathscr{K}(r+s), T-1]$$

which may be empty. If we consider the quantity:

$$\hat{\mathscr{Q}}''_T(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{t \in \cup_{k=1}^{\mathscr{K}} J_k \cup R} \mathcal{E}^{(T)}(t \mid \xi)$$

then the following result holds:

**Lemma 3.4.** *If $\varepsilon$ is small enough then:*

$$\langle (\hat{\mathscr{Q}}''_T(f \mid \cdot))^2 \rangle \xrightarrow{T \to \infty} 0$$

**Proof:** The estimates done in the proof of lemma (3.2) imply:

$$\langle (\mathcal{E}^{(T)}(t \mid \cdot))^2 \rangle \leq const \frac{\| \nabla f \|_\infty^2}{t}$$

so that:

$$\langle (\sum_{t \in J_k} \mathcal{E}^{(T)}(t \mid \cdot))^2 \rangle \leq const \| \nabla f \|_\infty^2 \sum_{t = kr + (k-1)s}^{k(r+s)-1} \frac{1}{t} \leq$$

$$\leq const \| \nabla f \|_\infty^2 \left[ \ln(k(r+s) - 1) - \ln(kr + (k-1)s) \right] \leq \frac{s \cdot cost \| \nabla f \|_\infty^2}{kr}$$

Summing over $k$ from 1 to $\mathscr{K}$ we have, at numerator, a factor that grows like the logarithm of $\mathscr{K}$ and it can be bounded by $\ln T$. Hence the behaviour of the numerator is compensated by the factor $\frac{1}{\ln T}$ which appears in $\hat{\mathscr{Q}}_T^2$, see equation (19), hence the quantity which we are interested in, including the contribute due to summing over the interval R, tends to zero at least like $\frac{1}{T^{\gamma - \delta}}$.

□

Lemma (3.4) implies that the limiting distribution of $\hat{\mathscr{Q}}_T$ is the same as that of the difference $\hat{\mathscr{Q}}'_T \equiv \hat{\mathscr{Q}}_T^{(2)} - \hat{\mathscr{Q}}''_T$, which can be written as a sum of independent variables:

$$\hat{\mathscr{Q}}'_T(f \mid \xi) \equiv \frac{1}{\sqrt{\ln T}} \sum_{j=1}^{\mathscr{K}} \mathscr{A}_T^{(j)}(\xi), \quad \mathscr{A}_T^{(j)} \equiv \sum_{t \in I_j} \mathcal{E}^{(T)}(t \mid \xi)$$

Hence to finish the proof of Th. (2.1) is enough to establish a Lyapunov condition and for which we want an $L^4$-estimate for functionals of the type:

$$\mathscr{A}_{\tau_1, \tau_2}^{(T)}(\xi) \equiv \sum_{t = \tau_1}^{\tau_2} \mathcal{E}^{(T)}(t \mid \xi), \qquad \tau_2 + T_* < T$$

in fact the results holds, see for example [5], if we can show:

$$\frac{1}{(\ln T)^2}\sum_{j=1}^{\mathscr{K}(T)}\langle\left(\mathscr{A}_T^{(j)}(\xi)\right)^4\rangle \xrightarrow{T\to\infty} 0$$

Remembering that:

$$r\equiv[T^\gamma] \quad , \quad s\equiv[T^\delta] \quad , \quad \mathscr{K}\equiv[\frac{T}{r+s}]$$

if $I_j=[\tau_{1_j},\tau_{2_j}]$ then:

$$\mathscr{A}_T^{(j)}(\xi)=\sum_{t=\tau_{1_j}}^{\tau_{2_j}}\mathcal{E}^{(T)}(t\mid\xi)$$

and we have $\tau_{1_j}=(j-1)(r+s)$ e $\tau_{2_j}=jr+(j-1)s-1$, so $\tau_{2_j}-\tau_{1_j}+T_*=r-j+1+T_*\le c\cdot(r+T_*)$, and $\tau_{1_j}=(j-1)(r+s)\ge j\cdot r$ so, using (A.1) for $n=2$ and the Lagrange Theorem, we obtain:

$$\frac{1}{(\ln T)^2}\sum_{j=1}^{\mathscr{K}(T)}\langle(\mathscr{A}_T^{(j)}(\cdot))^4\rangle\le\frac{C(\varepsilon,2)}{(\ln T)^2}\sum_{j=1}^{\mathscr{K}(T)}\left(\frac{r+T_*}{j\cdot r}\right)^3\xrightarrow{T\to\infty}0$$

and this concludes the proof of Theorem (2.1).

□

From now on we will work to prove our results about the behaviour of cumulants in dimension $\nu=1,2$, i.e. theorems (2.2),(2.3),(2.4),(2.5). We begin by defining:

$$\mathscr{E}^{(T)}(\xi)\equiv\mathbb{E}(X_T\mid\xi)-\mathbf{b}T=\sum_{t=0}^{T-1}\sum_{y\in\mathbb{Z}^1}M(y,t\mid\xi)\mathfrak{b}(\xi_t(y))$$

which can be written as:

$$\sum_{0\le t_1\le t_2\le T-1}\ \sum_{y_1,y_2\in\mathbb{Z}^1}P_0^{t_1}(y_1)M_\sharp(t_2-t_1,y_2-y_1,\xi_{(t_1,y_1)})$$

**Proposition 3.3.** *In dimension $\nu=1$, for $\varepsilon$ small enough, there exists a constant $C=C(\varepsilon)$ such that:*

$$\sum_{y\in\mathbb{Z}^1}\langle M_\sharp^2(t,y\mid\cdot)\rangle\le\frac{C(\varepsilon)\varepsilon^2}{(t+1)^{\frac{3}{2}}}$$

**Proof:** Let be:

$$b\equiv\max_{s\in\mathscr{S}}\mid\mathfrak{b}(s)\mid\ ,\ k^t(x)\equiv\max_{s\in\mathscr{S}}\mid h^t(x;s)\mid$$

using the orthogonality of $M_B{}^\sharp(\xi)$, the inequalities of lemma (A.1) in appendix A of [2] and iterating the following estimate, which holds for $a>1$ and some constant $K=K(a)$:

$$\sum_{t_1=1}^{T-1}[t_1(T-t_1)]^{-a}\le K(a)T^{-a}$$

we can choose $\varepsilon$ small enough such that, for a certain constant $C = C(\varepsilon)$, the following estimate holds:

$$\sum_{y\in\mathbb{Z}^1} \langle M_\sharp^2(t,y \mid \cdot)\rangle \leq (\varepsilon b)^2 \sum_{n=1}^{t} \varepsilon^{2n} \sum_{\substack{t_1+\cdots+t_n=t \\ \min\{t_2,\ldots,t_n\}>0}} \sum_{x_1,\ldots,x_n\in\mathbb{Z}^1} \prod_{i=1}^{n} (k^{t_1}(x_i))^2 \leq \frac{\varepsilon^2 C(\varepsilon)}{(t+1)^{\frac{3}{2}}} \tag{20}$$

**Proposition 3.4.** *In dimension $\nu = 1$, for $\varepsilon$ small enough, we have:*

$$(S_T)^2 \equiv \langle (\mathscr{E}^{(T)})^2 \rangle \asymp \sqrt{T}$$

**Proof:** Using (20) and again the estimates about $P_0^t(y)$ contained in appendix A of [2] we can write:

$$(S_T)^2 \leq \sum_{t=0}^{T} \sum_{t_1+t_2=t} \frac{1}{t_1^{\frac{1}{2}} t_2^{\frac{3}{2}}} \leq \sum_{t=1}^{T} \frac{cost}{t^{\frac{1}{2}}} \leq cost\sqrt{T}$$

besides, if $B = \{(y,t)\}$ is a certain set of points, by the LLT about $P_0$ we obtain:

$$(S_T)^2 \geq cost \sum_{t=0}^{T-1} \sum_{y:|y-\mathbf{b}t|>o(t^{\frac{2}{3}})} \left(P_0^T(y)\right)^2 \asymp \sqrt{T}$$

□

Proceeding like in Prop. (3.4) one can prove that, in dimension $\nu = 2$, if $\varepsilon$ is small enough we have:

$$(S_T)^2 \equiv \langle (\mathscr{E}^{(T)})^2 \rangle \asymp \ln(T)$$

Let be $T_1 \equiv [T^\beta]$, for $\beta \in (0,1)$, and $T_* \equiv [\log_+ T]$, where $\log_+ T \equiv \max\{1, \log T\}$, and consider the functional:

$$\hat{\mathscr{E}}^{(T)}(\xi) \equiv \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1 y_2\in\mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_1,y_1)})$$

which differs from $\mathscr{E}^{(T)}(\xi)$ in that terms with large $t_1$ and large differences $t_2 - t_1$ have been removed, then, reproducing the same arguments seen in the proof of Lemma (3.2), the following result holds:

**Proposition 3.5.** *In dimension $\nu = 2$ if $\varepsilon$ is small enough then:*

$$\lim_{T\to\infty} \frac{1}{(S_T)^2} \langle \left(\mathscr{E}^{(T)}(\xi) - \hat{\mathscr{E}}^{(T)}(\xi)\right)^2 \rangle = 0$$

Hence the proof of the Th.(2.2) is reduced to prove the CLT for $\frac{1}{S_T}\hat{\mathscr{E}}_T(\xi)$. Using again the Bernstein method, we divide the axis of time in intervals $I_k$ and *corridors* $J_k$. Let be:

$$\hat{\mathcal{E}}^{(T)}(t_1 \mid \xi) \equiv \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1,y_2\in\mathbb{Z}^2} P_0^{t_1}(y_1) M_\sharp(t_2-t_1, y_2-y_1 \mid \xi_{(t_1,y_1)})$$

and set:

$$\hat{\mathscr{E}}'_T(\xi) \equiv \frac{1}{S_T} \sum_{t \in \cup_{k=1}^{\mathscr{K}} J_k \cup R} \hat{\mathcal{E}}^{(T)}(t \mid \xi)$$

for which, see the proof of Lemma (3.4), the following holds:

**Lemma 3.5.** *In dimension $\nu = 2$, if $\varepsilon$ is small enough, we have:*

$$\langle \left( \hat{\mathscr{E}}'_T(\xi) \right)^2 \rangle \xrightarrow{T \to \infty} 0$$

From Lemma(3.5) we deduce that the limit distribution of $\mathscr{E}^{(T)}(\xi)$ is the same as that of: $\hat{\mathscr{E}}''_T \equiv \hat{\mathscr{E}}^{(T)} - \hat{\mathscr{E}}'_T$, which can be written as a sum of independent variables:

$$\hat{\mathscr{E}}''_T(\xi) \equiv \sum_{j=1}^{\mathscr{K}} \hat{\mathscr{A}}_T^{(j)}(\xi), \quad \hat{\mathscr{A}}_T^{(j)} \equiv \sum_{t \in I_j} \hat{\mathcal{E}}^{(T)}(t \mid \xi)$$

To prove the CLT for the quantity $\hat{\mathscr{E}}''_T$ is sufficient to establish a Lyapunov condition. Therefore we need an $L^4$-estimate for quantities of the type:

$$\hat{\mathscr{A}}_{t_1,t_2}^{(T)} \equiv \sum_{t=t_1}^{t_2} \hat{\mathcal{E}}^{(T)}(t \mid \xi)$$

this result is proved in [4] using a technique of graphs summation and it implies:

**Proposition 3.6.** *In dimension $\nu = 1$ if $\varepsilon$ is small enough then there exists a positive constant $K = K(\varepsilon)$ such that:*

$$\langle \left( \hat{\mathscr{A}}_{\tau 1,\tau 2}^{(T)} \right)^4 \rangle \leq \varepsilon^4 K(\varepsilon) \left( \sqrt{t_2} - \sqrt{t_1} \right)^2$$

Proposition (3.6) implies:

$$\frac{1}{(S_T)^4} \sum_{j=1}^{(T)} \langle \left( \hat{\mathscr{A}}_T^{(j)}(\xi) \right)^4 \rangle \leq \frac{cost}{(S_T)^4} \sum_{j=1}^{(T)} \frac{r^2}{j(r+s)} \leq cost \frac{T^\gamma \ln T}{(S_T)^4} \xrightarrow{T \ \to \infty} 0$$

hence Th. (2.2) is proved.

Now it is easy to prove theorems (2.3), ( 2.4) and (2.5).
**Proof** (of Th. (2.3)): Fixed a generic vector $\mathbf{v} \in \mathbb{R}^2$ we define:

$$\mathscr{E}^{(T)^{\mathbf{v}}} \equiv (\mathscr{E}^{(T)}) \cdot \mathbf{v} = \sum_{y \in \mathbb{Z}^2} M(y, t \mid \xi_t(y)) \mathfrak{b}(\xi_t(y)) \cdot \mathbf{v}$$

By Prop. (3.4) we have:

$$(S_{\mathbf{v}}^T)^2 \equiv \langle \left( \mathscr{E}^{(T)^{\mathbf{v}}} \right)^2 \rangle \asymp \ln T$$

With the same arguments used in the proof of Th. (2.1) and by the results contained in Prop. (A.1) if we define the following matrix:

$$\Sigma \equiv \{\mathfrak{b}_{ij}\} = \{\langle \mathfrak{b}_i(\cdot)\mathfrak{b}_j(\cdot)\rangle\}$$

where $\mathfrak{b}_i(\cdot) \equiv \sum_{u\in\mathbb{Z}^2} u_i c(u;\cdot)$, then we have:

$$\frac{\mathscr{E}^{(T)\mathbf{v}}}{S^T_{\mathbf{v}}} \xrightarrow{\mathscr{D}} \mathscr{N}(0,\Sigma)$$

or equivalently:

$$\frac{\mathfrak{b}\cdot\mathbf{v}}{\sum_{t=0}^{T-1}\sum_{y\in\mathbb{Z}^2}\langle (M(y,t\mid\cdot)^2\rangle}\mathscr{E}(\xi) \xrightarrow{\mathscr{D}} \mathscr{N}(0,\mathbf{b}\Sigma\mathbf{b}^T)$$

where the matrix $\Sigma$ is non degenerate if and only if:

$$\mathfrak{b}_{11}\mathfrak{b}_{22} \neq \mathfrak{b}_{12}^2$$

□

**Proofs** (of Theorems 2.4 e 2.5): In dimension $\nu = 1,2$ the corrections to covariance matrix are:

$$\mathscr{C}_{ij}^{(T)}(\xi) \equiv \hat{\mathscr{C}_{ij}^{(T)}}(\xi) + \mathscr{E}_{ij}^{(T)}(\xi) - \mathscr{E}_i^{(T)}(\xi)\mathscr{E}_j^{(T)}(\xi)$$

where we have placed:

$$\hat{\mathscr{C}_{ij}^{(T)}} \equiv \sum_{t=0}^{T-1}\sum_{y\in\mathbb{Z}^\nu} M(y,t\mid\xi)\left[(y_i - b_i t)\mathfrak{b}_i(\xi_t(y)) + (y_j - b_j t)\mathfrak{b}_j(\xi_t(y))\right]$$

$$\mathscr{E}_{ij}^{(T)}(\xi) \equiv \sum_{t=0}^{T-1}\sum_{y\in\mathbb{Z}^\nu} M(y,t\mid\xi)\sum_{u\in\mathbb{Z}^\nu}(u_i - b_i)(u_j - b_j)c(y;\xi_t(y))$$

By the short range condition and the results used in the proof of theorem (2.2) it is easy to see that the asymptotic behaviour of $\mathscr{E}_{ij}^{(T)}$ is the same as that of $\mathscr{E}^{(T)}$. Now we want to consider the $\mathscr{C}_{ij}^{(T)}(\xi)$ term. In [2] is showed that the following inequality holds:

$$\langle (M(y,t\mid\cdot))^2\rangle \leq \frac{cost}{t^\nu}e^{-\beta\frac{(y-\mathbf{b}t)^2}{2t}}$$

for some positive constants $\beta$, and for all $(y,t)\in\mathbb{Z}^{\nu+1}$. Therefore setting $\mathbf{b}$ the drift of our model, we have:

$$\sum_{y\in\mathbb{Z}^\nu}\langle (M(y,t\mid\cdot))^2\rangle (y-\mathbf{b}t)^2 \leq \frac{cost}{t^{\frac{\nu}{2}-1}}$$

and:

$$\sum_{t=1}^{T-1}\frac{1}{t^{\frac{\nu}{2}-1}} \asymp T^{2-\frac{\nu}{2}}$$

hence setting:

$$\tilde{S}_{ij}^{(T)} \equiv \langle \left( \hat{\mathscr{C}}_{ij}^{(T)} \right)^2 \rangle^{\frac{1}{2}}$$

and reproducing the same arguments used in Prop. (3.4), we find:

$$\left( \tilde{S}_{ij}^{(T)} \right)^2 \asymp T^{2-\frac{\nu}{2}}$$

so that:

$$\frac{\mathscr{E}_{ij}^{(T)}(\xi)}{\tilde{S}_{ij}^{(T)}} \xrightarrow{\mathscr{D}} 0 \quad , \quad \frac{\mathscr{E}_{i}^{(T)} \mathscr{E}_{j}^{(T)}}{\tilde{S}_{ij}^{(T)}} \xrightarrow{\mathscr{D}} 0$$

and the only term that still has an importance in the asymptotic of the correction $\mathscr{C}_{ij}^{(T)}(\xi)$ is $\hat{\mathscr{C}}_{ij}^{(T)}(\xi)$, but we know its limit in dimension $\nu = 1$ as well as in dimension $\nu = 2$. In fact, analogously to what we have seen in the proofs of theorems (2.2) and (2.3), in dimension $\nu = 1$ we can use the results contained in [4], while in dimension $\nu = 2$ we have Prop. (A.1).

□

**Appendix**

**A.** –

Under our assumptions on the model and in dimension $\nu = 2$ we want to prove the following proposition:

**Proposition A.1.** *Let be $n \geq 1$, if $\varepsilon$ is small enough, there exists a positive constant $C = C(\varepsilon, n)$ such that*

$$\langle \left( \mathscr{A}^{(T)}_{\tau 1, \tau 2} \right)^{2n} \rangle \leq C(\varepsilon, n) \cdot (\ln(\tau_2 + T_*) - \ln(\tau_1))^{2n-1}$$

**Proof:** We have that:

$$M_\sharp(t_2 - t_1, y_2 - y_1 \mid \xi_{(t_1,y_1)}) = \sum_{B:(t_1,y_1)\to(t_2,y_2)} \varepsilon^{|B|} \cdot M^\sharp_B(\xi) \tag{A.1}$$

and being:

$$M^\sharp_B(\xi) \equiv \prod_{i=1}^{|B|} h^{\tau_i}(z_i, s_i) \cdot \mathfrak{b}(\xi_{t_f(B)}(y_f(B)))$$

the moments of the type $\langle \prod_{k=1}^{2n} M^\sharp_{B_k} \rangle$ are zero, unless the sets $\{B_1, \ldots, B_{2n}\}$ *cover each other*, i.e. the relation:

$$B_j \in \bigcup_{\substack{i=1,\ldots,2n \\ i \neq j}} B_i, \quad 1, \ldots, 2n$$

holds, we will call this property *covering*. Hence we can define the following class of sets:

$$\mathscr{B}_{2n} \equiv \{\mathscr{B} = \{B_1, \ldots, B_{2n}\} \mid \mathscr{B} \textit{ has the covering property}\}$$

An element $\mathscr{B} = \{B_1, \ldots, B_{2n}\} \in \mathscr{B}_{2n}$ is identified by a finite subset of points in $\mathbb{Z}^{2+1}$:

$$B \equiv \bigcup_{j=1}^{2n} B_j \subset \mathbb{Z}^{2+1}.$$

Any point $v \in B$ can be equipped with the following specification :

$$l_v \equiv \{j \mid v \in B_j\}$$

which is a collection of labels each of one represents exactly the set which $v$ holds on. We are interested only in that collections of sets which have the *covering* property so it must be $| \, l_v \, | \geq 2$ for all vertex $v \in \mathscr{B}$. If we define $S \equiv \{l_v \mid v \in B\}$, then there is a one-to-one correspondence between elements $\mathscr{B} \subset \mathscr{B}_{2n}$ and the pairs (B,S) obtained by imposing the following conditions:

(i) If two distinct points have the same time coordinate then the correspondent sets $l_v$ are disjoint

(ii) Each label must appear at least once , i.e. :

$$\bigcup_{v\in B} l_v = \{1, 2, \dots, 2n\}.$$

We can associate to each element in $(B, S) \in \mathscr{B}_{2n}$ a graph:

$$\mathscr{G} \equiv (B_0, \mathscr{L})$$

where $B_0 \equiv B \bigcup \{0\}$ is the set of vertexes while $\mathscr{L}$ is the set of bonds obtained by the union of two subsets of bonds $\mathscr{L}_*, \mathscr{L}'$ which is determined as follows. For each vertex $v = (t, x) \in B$ and each $j \in l_v$, we consider the class of vertexes: $\nu_j \equiv \{v' = (t', x') \mid j \in l_{v'}, t' > t\}$. If $\nu_j \neq \emptyset$ we draw a bond from $v$ to the vertex $v_* \in \nu_j$ with minimal time coordinate (which is unique by condition (i) ), this method complete the construction of $\mathscr{L}'$. To construct $\mathscr{L}_*$ we simply draw a bond connecting the origin to the initial point of each $B_j$.

Denoting by $\mathscr{B}_{2n}^{\tau_1,\tau_2+T_*}$ the subset of $\mathscr{B}_{2n}$ made by all and only those collections of trajectories $\mathscr{B} = \{B_1, \dots, B_{2n}\}$ for which we have:

$$t_f(B_j) \in \{\tau_1, \dots, \tau_2 + T_*\},\ j = 1, \dots, 2n$$

setting:

$$N(\mathscr{B}) \equiv \sum_{j=1}^{2n} \mid B_j \mid \quad , \quad b \equiv \max_{s\in\mathscr{S}} \parallel \sum_{u\in\mathbb{Z}^2} uc(u; s) \parallel$$

and remembering what we have seen in $(A.1)$, we have:

$$\begin{aligned} &\langle(\mathscr{A}^{(T)}_{\tau_1,\tau_2}(\cdot))^{2n}\rangle = \langle(\sum_{t_1=\tau_1}^{\tau_2} \mathcal{E}(t_1 \mid \xi))^{2n}\rangle = \\ &= \langle(\sum_{t_1=\tau_1}^{\tau_2} \sum_{t_2=t_1}^{t_1+T_*} \sum_{y_1,y_2} P_0^{t_1}(y_1) M_\sharp(t_2 - t_1, y_2 - y_1 \mid \cdot) H(t_2, y_2))^{2n}\rangle \leq \\ &\leq b^{2n} \parallel \nabla f \parallel_\infty^{2n} \sum_{\mathscr{B}\in\mathscr{B}_{2n}^{\tau 1,\tau_2+T_*}} \varepsilon^{N(\mathscr{B})} \cdot S(\mathscr{G}_\mathscr{B}) \end{aligned} \tag{A.2}$$

where $\mathscr{G}_\mathscr{B}$ is the graph associated to the particular choice of $\mathscr{B} \in \mathscr{B}_{2n}^{\tau_1,\tau_2+T_*}$ and:

$$S(\mathscr{G}_\mathscr{B}) \equiv \prod_{b\in\mathscr{L}_*} \pi_*(b) \cdot \prod_{b\in\mathscr{L}'} \pi(b)$$

with weights $\pi_*(b), \pi(b)$ which are defined as follows ( b=(v,v'), with v=(t,x) e v'=(t',x') ):

$$\pi(b) \equiv \max_{s\in\mathscr{S}} \mid h^t(y; s) \mid = \max_{s\in\mathscr{S}} \mid \sum_u c(u; s) \cdot P_0^{t-1}(y - u) \mid$$

while if $b \in \mathscr{L}_*$ with b=(0,v) and v=(t,x) then $\pi_*(b) \equiv P_0^t(x)$. For every set of points $B = \{(y_1,t_1),\ldots,(y_n,t_n)\}$ we define the following quantity:

$$N_0(B) \equiv P_0^{t_1}(y_1) \prod_{i=1}^{n-1} \max_{s\in\mathscr{S}} \mid h^{t_{i+1}-t_i-1}(s, y_{i+1}-y_i) \mid$$

so we can rewrite the last row in $(A.2)$ as:

$$b^{2n} \cdot \|\nabla f\|_\infty^{2n} \cdot \sum_{(\mathscr{B}_1,\ldots,\mathscr{B}_{2n})\in\mathscr{B}_{2n}^{\tau 1,\tau_2+T_*}} \prod_{i=1}^{2n} \varepsilon^{|B_i|} N_0(B_i)$$

Using the results in appendix A of [2] about certain $Lp$ inequalities we obtain:

$$\langle(\mathscr{A}_{\tau_1,\tau_2}^{(T)}(\cdot))^{2n}\rangle \le \sum_{k=1}^{2n-1} \sum_{\substack{n_1,\ldots,n_k\ge 1\\ \max_{n_j>1,\sum_j n_j=2n}}} c(n_1,\ldots,n_k) \prod_{j=1}^{k} \sum_{t_j=\tau_1}^{\tau_2+T_*} \frac{1}{(t_j+1)^{m_j}}$$

where $c(n_1,\ldots,n_k)$ is constant that depends on $\varepsilon$ while the exponent $m_j$ is defined as $m_j \equiv \max\{1, n_j-1\}$, hence we have:

$$\langle\big((\mathscr{A}_{\tau_1,\tau_2}^{(T)}(\cdot)\big)^{2n}\rangle \le C(\varepsilon,n)\cdot(\ln(\tau_2+T_*)-\ln(\tau_1))^{2n-1}$$

and this concludes the proof.

□


## Acknowledgments

I owe gratitude to Prof. A. Pellegrinotti for his deep and constant support and to Prof. C. Boldrighini for various fruitful suggestions.

## References

[1] Bernabei M.S., Boldrighini C., Minlos R.A., Pellegrinotti A. : Almost-sure central limit theorem for a model of random walk in fluctuating random environment. Markov Processes. Related Fields 4 (1998), pp. 381-393

[2] Boldrighini C., Minlos R.A., Pellegrinotti A.: Almost-sure central limit theorem for a Markov model of random walk in dynamical random environment. Probability Theory and realted fields 109, Springer Verlag (1997), pp. 245-273

[3] Boldrighini C., Minlos R.A., Pellegrinotti A.: Central limit theorem for a random walk in dynamical environment: integral and local. Theory of Stochastic Processes, vol. 5 (21). n.3-4 (1999), pp. 16-28

[4] Boldrighini C., Pellegrinotti A. : $T^{-1/4}$ noise for a random walks in dynamic environment on Moscow Mathematical Journal, Volume 1, Number 3, July-September 2001, pp. 1-16

[5] Gihman I.I., Skorohod, A.V. : The theory of stochastic processes I. Grundlhren der methematischen wissenschaften 210 Springer-Verlag (1974)

[6] Ibraghimov I.A., Linnik Yu. V. : Independent and stationary sequences of random variables. Ed. by J. F. C. Kingman. - Groningen, Wolters-Noordhoff (1971)