



Some recent results in linear scalar quantum field theory in globally hyperbolic asymptotically flat spacetimes.

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Abstract: This work concerns some recent results obtained by the author and collaborators about QFT in asymptotically flat spacetimes at null infinity. In particular it is shown that bosonic QFT can be defined on the null boundary \mathfrak{S}^+ of any asymptotically flat spacetime M . This theory admits a state λ which is uniquely determined from invariance under BMS group and a BMS-energy positivity requirement. There is a nice interplay with bosonic (massless, conformally coupled) QFT defined in the bulk spacetime. In particular, under suitable further requirements, the universal state λ induces in the bulk spacetime M a state, λ_M , which enjoys the following remarkable properties. It reduces to standard Minkowski vacuum whenever M is Minkowski spacetime and in the general case, it is invariant under the group of isometries of the spacetime M , it is a ground state (i.e. it satisfies the positive energy condition) with respect to any timelike Killing time of M without zero-modes, finally λ_M enjoys the global Hadamard property so that it is suitable for locally covariant perturbative renormalization procedures.

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1 Asymptotically Flat Spacetimes and *BMS* group

Asymptotically flat spacetimes – by definition – have a certain *asymptotic structure*. The main motivation of the works in the references [1, 2, 3] has been investigating if that asymptotic structure *determines canonically preferred states in every asymptotically flat spacetime* for (linear) scalar quantum field theory. The properties of those states have been also focused. Let us remind the main definition [4] (See [5] for further discussions and references).

Definition 1. (Asymptotically flat vacuum spacetime at future null infinity) *A four dimensional spacetime (M, g) is called asymptotically flat vacuum spacetime at future null infinity if:*

(a) (M, g) can be seen as an embedded submanifold of a larger spacetime (\tilde{M}, \tilde{g}) with $\tilde{g}|_M = \Omega^2 g$, Ω being a smooth function on \tilde{M} , strictly positive on M .

(b) $\mathfrak{S}^+ := \partial M$, called **future null infinity** of M , is a 3-dim \tilde{M} -submanifold satisfying

(i) $\Omega|_{\mathfrak{S}^+} = 0$ but $d\Omega|_{\mathfrak{S}^+} \neq 0$,

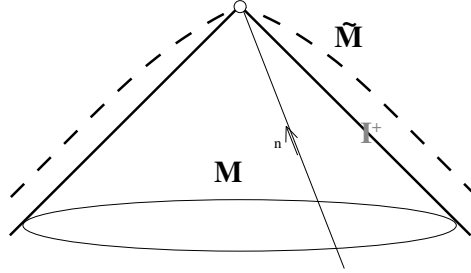
(ii) $\mathfrak{S}^+ \cap \tilde{J}^-(M) = \emptyset$,

(iii) \mathfrak{S}^+ diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$, and it is union of the **null** curves tangent to $n := \nabla\Omega$.

These curves are **complete null geodesics** for a certain choice of Ω .

(c) About \mathfrak{S}^+ , (M, g) is strongly causal and satisfies $\text{Ric}(g) = 0$.

In fact \mathfrak{S}^+ is a 3-dim. null submanifold of \tilde{M} with degenerate metric \tilde{h} induced by \tilde{g} . \mathfrak{S}^+ is the *conic surface* indicated by \mathbf{I}^+ in the figure. The tip is *not* a point of \tilde{M} .



Asimptotically flat spacetime.

There are *gauge transformations* permitted by the definition: $\Omega \rightarrow \omega\Omega$ with $\omega > 0$ in a neighborhood of \mathfrak{S}^+ ,

$$\mathfrak{S}^+ \rightarrow \mathfrak{S}^+, \quad \tilde{h} \rightarrow \omega^2 \tilde{h}, \quad n \rightarrow \omega^{-1} n.$$

For a fixed a.f. spacetime (M, g) , $C_{(M, g)}$ is the class of all triples $\{(\mathfrak{S}^+, \tilde{h}, n)\}$ connected by gauge transformations. $C_{(M, g)}$ encodes the whole geometric extent of \mathfrak{S}^+ .

It is important to stress that there is no physically preferred element of $C_{(M, g)}$ therefore it is mathematical convenient to pass from Ω to $\Omega_B = \omega_B \Omega$ such that $(\mathfrak{S}^+, \tilde{h}_B, n_B)$ reads

$$\mathfrak{S}^+ = \mathbb{R} \times \mathbb{S}^2, \quad \tilde{h}_B = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi, \quad n_B = \frac{\partial}{\partial u}$$

in coordinates $u \in \mathbb{R}$ (future directed), $(\theta, \varphi) \in \mathbb{S}^2$. (u, θ, φ) is a **Bondi frame** on \mathfrak{S}^+ .

Notice that if $(M_1, g_1) \neq (M_2, g_2)$, for a suitable choice of $(\mathfrak{S}_1^+, \tilde{h}_1, n_1) \in C_{(M_1, g_1)}$ and $(\mathfrak{S}_2^+, \tilde{h}_2, n_2) \in C_{(M_2, g_2)}$, \exists diffeomorphism $\gamma : \mathfrak{S}_1^+ \rightarrow \mathfrak{S}_2^+$ with

$$\gamma(\mathfrak{S}_1^+) = \mathfrak{S}_2^+, \quad \gamma^* \tilde{h}_1 = \tilde{h}_2, \quad \gamma^* n_1 = n_2$$

In this sense the class $C = C_{(M, g)}$ is **universal** for all a.f. spacetimes.

We have the following subsequent definition [6]:

Definition 2. (Bondi-Metzner-Sachs (BMS) group.) G_{BMS} is the group of diffeomorphisms $\chi : \mathfrak{S}^+ \rightarrow \mathfrak{S}^+$ preserving the universal structure of \mathfrak{S}^+ , i.e. χ **are isometries up to gauge transformations**:

$$\chi(\mathfrak{S}^+) = \mathfrak{S}^+, \quad \chi^* \tilde{h} = \omega_\chi^2 \tilde{h}, \quad \chi^* n = \omega_\chi^{-1} n \quad \text{for some } \omega_\chi > 0.$$

Let us examine the structure of G_{BMS} . Fix a *Bondi frame* (u, φ, θ) on \mathfrak{S}^+ and pass to complex coordinates on the sphere \mathbb{S}^2 $(u, \varphi, \theta) \leftrightarrow (u, \zeta, \bar{\zeta})$, with $\zeta := e^{i\varphi} \cot(\theta/2) \in \mathbb{C} \cup$

$\{+\infty\}$, the usual Riemann sphere. In this framework G_{BMS} has the structure of a semi-direct product, $G_{BMS} = SO(3, 1)\uparrow \times C^\infty(\mathbb{S}^2)$, where the group composition law is defined in this way:

$$\begin{aligned} G_{BMS} &\ni (\Lambda, f) : (u, \zeta, \bar{\zeta}) \mapsto (u', \zeta', \bar{\zeta}'), \\ u' &:= K_\Lambda(\zeta, \bar{\zeta})(u + f(\zeta, \bar{\zeta})), \quad \zeta' := \frac{a_\Lambda \zeta + b_\Lambda}{c_\Lambda \zeta + d_\Lambda}, \\ K_\Lambda(\zeta, \bar{\zeta}) &:= \frac{(1 + \zeta \bar{\zeta})}{|a_\Lambda \zeta + b_\Lambda|^2 + |c_\Lambda \zeta + d_\Lambda|^2} \quad \text{and} \quad \Pi \begin{bmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{bmatrix} = \Lambda. \end{aligned}$$

Above $SO(3, 1)\uparrow$ is the special orthochronous Poincaré group and $\Pi : SL(2, \mathbb{C}) \rightarrow SO(3, 1)\uparrow$ is the standard covering homomorphism.

It is worth stressing that pairs of Bondi frames are connected by transformations of a subgroup of G_{BMS} . Moreover notice that the decomposition $G_{BMS} = SO(3, 1)\uparrow \times C^\infty(\mathbb{S}^2)$ depends on the used frame, however the subgroup:

$$T^4 := \text{real span of the first 4 spheric harmonics} \subset C^\infty(\mathbb{S}^2)$$

is a **normal** G_{BMS} -subgroup. It is called the subgroup of **4-translations**. Thus T^4 is **BMS-invariant**, i.e. **independent** from used reference (Bondi) frames, and so it may have physical interest. Another important fact is that there is a *Minkowskian decomposition*

$T^4 \ni \alpha = \sum_{\mu=0}^3 \alpha^\mu Y_\mu$ where $\{Y_\mu\}$ is a certain real basis of T^4 . In this framework, if $\alpha, \alpha' \in T^4$

the *Minkowskian scalar product* $(\alpha, \alpha') := \sum_{\mu, \nu} \eta_{\mu\nu} \alpha^\mu \alpha'^\nu$ turns out to be **BMS-invariant**, i.e. invariant under $\alpha \mapsto g \circ \alpha \circ g^{-1}$, $g \in G_{BMS}$ and $\alpha \in C^\infty(\mathbb{S}^2)$. ($\eta = \text{diag}(-1, 1, 1, 1)$.)

We conclude that there is a **BMS-invariant** decomposition of T^4 and thus we have *space-like, (fut./past. directed) timelike, (fut./past. directed) null* 4-translations. Time orientation is induced by that of (M, g) (see [3] for details.)

Remarks.

- (1) A timelike future-directed **direction** $\{\lambda\alpha\}_{\lambda \in \mathbb{R}}$ in T^4 individuates a Bondi frame $(u, \zeta, \bar{\zeta})$ uniquely: that where the action of $\{\lambda\alpha\}_{\lambda \in \mathbb{R}}$ is $u \mapsto u + \lambda$.
- (2) The orthochronous proper Poincaré group $\mathcal{P} := SO(3, 1)\uparrow \times T^4$ is *not* a normal subgroup of G_{BMS} , hence there is **no** physical way to pick out a preferred Poincaré subgroup of G_{BMS} .
- (3) It is known that G_{BMS} encodes the bulk symmetries of each fixed asymptotically flat vacuum at null infinity spacetime (M, g) as well as all the asymptotic symmetries of all bulk spacetimes. Indeed the following result holds true [7].

Theorem 1. *Let (M, g) be asymptotically flat vacuum at null infinity spacetime. The following holds.*

- (a) *If ξ is a Killing vector field of the bulk M , then ξ smoothly extends to a vector field $\tilde{\xi}$, different from the zero vector field and tangent to \mathbb{S}^+ , which generates a one-parameter subgroup of G_{BMS} .*
- (b) *In that way, the isometry group of M is mapped into a subgroup $G_M \subset G_{BMS}$ with:*
 - (i) G_M is isomorphic to a subgroup of a certain Poincaré group $\mathcal{P} \subset G_{BMS}$ where \mathcal{P} generally depends on the particular spacetime M ,
 - (ii) only proper 4-translations are admitted in $G_M \cap C^\infty(\mathbb{S}^2)$.
- (c) $\tilde{\xi}$ generates an one-parameter subgroup of G_{BMS} , if and only if it smoothly extends back

to a field ξ in M with $\Omega^2 \mathcal{L}_\xi g \rightarrow 0$ smoothly approaching \mathfrak{S}^+ .

The item (c) states, in fact, that G_{BMS} contains the so called so-called **asymptotic symmetries** of the physical spacetime M [5]. The BMS group is the group of invariance of any physical theory defined on \mathfrak{S}^+ : mathematical objects defined on \mathfrak{S}^+ may have physical sense only if they are invariant under G_{BMS} .

2 Weyl quantization on \mathfrak{S}^+ and interplay with QFT in the bulk: the state λ_M .

In [1] it has been established that it is possible to define a bosonic QFT for a field defined on \mathfrak{S}^+ . The approach is that algebraic based on Weyl quantization [8]. The ingredients are the following ones. A *real symplectic space*: $(\mathcal{S}(\mathfrak{S}^+), \sigma)$, where σ nondegenerate *symplectic form*, whereas the real vector space $\mathcal{S}(\mathfrak{S}^+)$ is defined as (for a fixed Bondi frame):

$$\mathcal{S}(\mathfrak{S}^+) := \left\{ \psi \in C^\infty(\mathfrak{S}^+) \mid \psi, \partial_u \psi \in L^2(\mathbb{R} \times \mathbb{S}^2, du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \bar{\zeta})) \right\}, \quad (2.1)$$

whereas the explicit form of σ is the following:

$$\sigma(\psi_1, \psi_2) := \int_{\mathbb{R} \times \mathbb{S}^2} \left(\psi_2 \frac{\partial \psi_1}{\partial u} - \psi_1 \frac{\partial \psi_2}{\partial u} \right) du \wedge \epsilon_{\mathbb{S}^2}(\zeta, \bar{\zeta}), \quad (2.2)$$

where $\epsilon_{\mathbb{S}^2}$ is the standard measure on a unit 2-sphere. There is a **weighted representation** of G_{BMS} , $A_g : C^\infty(\mathfrak{S}^+) \rightarrow C^\infty(\mathfrak{S}^+)$, $g \in G_{BMS}$, acting on the considered symplectic space:

$$(A_g \psi)(u, \zeta, \bar{\zeta}) := (K_\Lambda^{-1} \cdot \psi)(g^{-1}(u, \zeta, \bar{\zeta})) \quad (\text{notice the weight } K_\Lambda^{-1}, g = (\Lambda, f)). \quad (2.3)$$

Notice that $A_g|_{\mathcal{S}(\mathfrak{S}^+)} \subset \mathcal{S}(\mathfrak{S}^+)$, moreover, due to the weight K_Λ^{-1} , the G_{BMS} representation A *preserves* the symplectic form of σ , so that, in particular, the mentioned structure does not depend on the used Bondi frame.

With those ingredients one defines the (unique) **Weyl C^* -algebra** $\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$ with generators $W(\psi) \neq 0$, $\psi \in \mathcal{S}(\mathfrak{S}^+)$, satisfying Weyl relations (also known as **CCR**):

$$W(-\psi) = W(\psi)^*, \quad W(\psi)W(\psi') = e^{i\sigma(\psi, \psi')/2} W(\psi + \psi').$$

Moreover the representation A induces a **-automorphism* G_{BMS} -*representation*

$$\alpha : \mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma) \rightarrow \mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma) \quad (2.4)$$

uniquely individuated by the requirement $\alpha_g(W(\psi)) := W(A_{g^{-1}}\psi)$.

In order to find a possible physical meaning of the theory constructed above, a natural question arises: Spacetime physics is BMS-invariant so, *are there BMS-invariant (quasifree) algebraic states on $\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$?*

An answer has been found in [1]. There it has demonstrated that, in fact, there is a (quasifree pure) *BMS*-invariant state. Let us summarize this result. Consider the quasifree pure state λ on $\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$ uniquely induced by linearity and continuity from:

$$\lambda(W(\psi)) = e^{-\mu_\lambda(\psi, \psi)/2}, \quad \mu_\lambda(\psi_1, \psi_2) := -i\sigma(\overline{\psi_{1+}}, \psi_{2+}), \quad \psi \in \mathcal{S}(\mathfrak{S}^+), \quad (2.5)$$

where ψ_+ is the *positive u -frequency part* of ψ , with respect to any (arbitrarily fixed) Bondi frame defined on \mathfrak{S}^+ . The positive frequency part is obtained performing the usual Fourier

transform with respect to the variable u considered as a “time” coordinate (see [1] for details). Then we pass to focus on the GNS representation $(\mathfrak{H}_\lambda, \Pi_\lambda, \Upsilon_\lambda)$. It turns out that the Hilbert space \mathfrak{H}_λ is a bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ with 1-particle Hilbert space $\mathcal{H} \equiv L^2(\mathbb{R}^+ \times \mathbb{S}^2; dE \otimes \epsilon_{\mathbb{S}^2})$ (containing u -Fourier transforms $\widehat{\psi}_+$), the GNS Cyclic vector is the Fock vacuum Υ_λ . Since the GNS representation is a Fock representation, λ is a regular state and *symplectically-smear*d field operators $\Psi(\psi)$ with $\Pi_\lambda(W(\psi)) = e^{-i\overline{\Psi(\psi)}}$ can be defined using Stone theorem directly (see e.g. [8] for the general theory).

In this context we have the following theorem established in [1].

Theorem 2. *Referring to the Weyl algebra $\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$, its GNS representation $(\mathfrak{H}_\lambda, \Pi_\lambda, \Upsilon_\lambda)$, the BMS representation α (2.4), and the state λ (2.5), the following facts are valid.*

- (a) λ is G_{BMS} -invariant: $\lambda(\alpha_g(a)) = \lambda(a)$ if $g \in G_{BMS}$ and $a \in \mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$.
- (b) The unique unitary representation of G_{BMS} leaving Υ_λ invariant (that is $U_g \Upsilon_\lambda = \Upsilon_\lambda$) and implementing α (i.e. $U_g \Pi_\lambda(a) U_g^* = \Pi_\lambda(\alpha_g(a))$) is that induced by:

$$\left(U_{(\Lambda, f)} \widehat{\psi}_+ \right) (E, \zeta, \bar{\zeta}) = \frac{e^{iEK_\Lambda(\Lambda^{-1}(\zeta, \bar{\zeta}))f(\Lambda^{-1}(\zeta, \bar{\zeta}))}}{\sqrt{K_\Lambda(\Lambda^{-1}(\zeta, \bar{\zeta}))}} \widehat{\psi}_+ (EK_\Lambda(\Lambda^{-1}(\zeta, \bar{\zeta})), \Lambda^{-1}(\zeta, \bar{\zeta})) . \quad (2.6)$$

$\widehat{\psi}_+$ being the positive-frequency part of $\psi \in \mathcal{S}(\mathfrak{S}^+)$ in Fourier representation (E being the conjugate variable with u).

- (c) Making G_{BMS} topological equipping $C^\infty(\mathbb{S}^2) \subset G_{BMS}$ with test-function Fréchet topology, consider the representation $G_{BMS} \ni g \mapsto U_g$, then one has:

- (i) it is irreducible and strongly continuous,
- (ii) it is a Wigner-Mackey-like representation associated with a scalar representation of the little group, $\Delta \subset SL(2, \mathbb{C})$, the double covering of the 2D Euclidean group,
- (iii) it is defined on an orbit in space of characters χ with $m_{BMS}^2(\chi) = 0$.

To make a comment to (c) we stress that the Abelian G_{BMS} -subgroup $C^\infty(\mathbb{S}^2)$ is *infinite dimensional and non-locally-compact*, but Mackey machinery *works anyway* as proved in [9]. Moreover, concerning (iii) notice that the characters, χ_β , are labeled by distributions (here $\mathcal{D}'(\mathbb{S}^2)$ is the dual space of $C_0^\infty(\mathbb{S}^2)$) $\beta \in \mathcal{D}'(\mathbb{S}^2)$, $\chi_\beta(\alpha) = e^{i\beta(\alpha)}$, $\forall \alpha \in C^\infty(\mathbb{S}^2)$. Therefore, in the space of characters can be defined a *BMS-invariant mass*: $m_{BMS}^2(\beta) := -\eta^{\mu\nu} \beta(Y_\mu) \beta(Y_\nu)$ which turns out to be invariant with respect to the dual action $(g(\beta))(\alpha) := \beta(g(\alpha))$, $\forall g \in G_{BMS}$. This is a notion of mass which is *a priori* independent from that invariant under the action of Poincaré group.

Now a question arises naturally: *Is there any relation with massless particles propagating in the bulk spacetime?* The answer is positive as established in [1]: the fields ψ on \mathfrak{S}^+ are “extensions” of linear, massless, conformally coupled fields in the bulk spacetime and the action of symmetries on the fields in the bulk is equivalent to the action of G_{BMS} on the associated fields on \mathfrak{S}^+ .

Theorem 3. *Let (M, g) be an asymptotically flat vacuum at future null infinity spacetime with associated unphysical spacetime $(\tilde{M}, \tilde{g} = \Omega^2 g)$. Assume that both M, \tilde{M} are globally hyperbolic. Consider Weyl QFT in (M, g) based on the symplectic space $(\mathcal{S}(M), \sigma_M)$. $\mathcal{S}(M)$ is the space of real smooth, compactly supported on Cauchy surfaces, solutions ϕ of massless, conformally-coupled, K-G equation*

$$\square\phi - \frac{1}{6}R\phi = 0 \quad \text{in } M.$$

with isometry-invariant symplectic form:

$$\sigma_M(\phi_1, \phi_2) := \int_{\Sigma} (\phi_2 \nabla_N \phi_1 - \phi_1 \nabla_N \phi_2) d\mu_g^{(S)},$$

Σ being any Cauchy surface of M . Then:

(a) ϕ vanishes approaching \mathfrak{S}^+ but $\Omega_B^{-1}\phi$ extends to a smooth field $\psi := \Gamma\phi$ on \mathfrak{S}^+ uniquely, Γ being linear.

(b) If $\{g_t\}$ is a 1-parameter group of M -isometries and $\{g'_t\}$ the associated G_{BMS} -subgroup: the action of $\{g_t\}$ on ϕ ($\phi \mapsto \alpha_{g_t}(\phi) := \phi \circ g_{-t}$) is equivalent to the action of $A_{g'_t}$ on $\psi = \Gamma\phi$,

$$A_{g'_t}(\psi) = \Gamma(\alpha_{g_t}(\phi)) \quad \text{if} \quad \psi = \Gamma\phi.$$

Remark. The more usual point of view in considering QFT in globally hyperbolic spacetimes (M, g) is that based on field operators $\Phi(f)$ smeared with smooth compactly supported functions $f \in C_0^\infty(M)$, instead of with solutions of Klein-Gordon equation of $\mathcal{S}(M)$. Actually the two points of view, for linear theories, are *completely equivalent*, see [8] for instance. If $\phi \in \mathcal{S}(M)$, the Weyl generators $W(\phi)$ are to be formally understood as $W(\phi) := e^{-i\sigma_M(\Phi, \phi)}$, where $\sigma_M(\Phi, \psi)$ is the **field operator symplectically smeared** with elements of $\mathcal{S}(M)$. In this context, it turns out that $\Phi(f) := \sigma_M(\Phi, Ef)$ where $E = A - R : C_0^\infty(M) \rightarrow \mathcal{S}(M)$ is the **causal propagator** (or “advanced-minus retarded” fundamental solution) of Klein-Gordon operator [8].

To go on, assume that furthermore $\Gamma : \mathcal{S}(M) \rightarrow \mathcal{S}(\mathfrak{S}^+)$ is an *injective symplectomorphism* i.e. (H1) $\text{Range}[\Gamma] \subset \mathcal{S}(\mathfrak{S}^+)$ (H2) $\sigma_{\mathfrak{S}^+}(\Gamma\phi_1, \Gamma\phi_2) = \sigma_M(\phi_1, \phi_2)$.

In this case one finds that *the field observables of the bulk M can be identified with observables of the boundary \mathfrak{S}^+* . More precisely [1]: $\exists!$ a (isometric) $*$ -homomorphism from the Weyl algebra $\mathcal{W}(\mathcal{S}(M), \sigma_M)$ of field observables of the bulk, to the Weyl-algebra $\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$:

$$\iota_\Gamma : \mathcal{W}(\mathcal{S}(M), \sigma_M) \rightarrow \mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$$

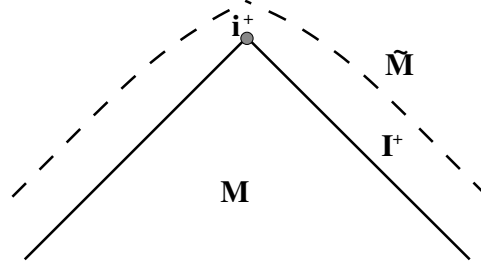
determined by the requirement on Weyl generators:

$$\iota_\Gamma(W_M(\phi)) = W_{\mathfrak{S}^+}(\Gamma\phi).$$

As a consequence of the existence of the $*$ -homomorphism ι_Γ , one may induce a preferred state λ_M on the observables in the bulk M from the natural state λ defined on the observables on the boundary \mathfrak{S}^+ . In other words, *the boundary state λ can be pulled back to a quasifree state λ_M acting on observables for the field ϕ propagating in the bulk spacetime M* :

$$\lambda_M(a) := \lambda(\iota_\Gamma(a)) \quad \text{for all } a \in \mathcal{W}(\mathcal{S}(M), \sigma_M) \quad (2.7)$$

In [1] it has been shown that If (M, g) is Minkowski spacetime (so that (\tilde{M}, \tilde{g}) is Einstein closed universe), hypotheses **H1** and **H2** are fulfilled so that ι_M exists and λ_M coincides with *Minkowski vacuum*. This is not the only case. To illustrate it we recall an important notion. Let (M, g) be an asymptotically flat vacuum at future null infinity spacetime. One says that it admits **future time infinity** i^+ if $\exists i^+ \in \tilde{M} \cap I^+(M)$ ($i^+ \notin \mathfrak{S}^+$) such that the geometric extent of $\mathfrak{S}^+ \cup \{i^+\}$ about i^+ “is the same as that in a region about the tip i^+ of a light cone in a (curved) spacetime”. The rigorous definition has been given by Friedrich [10].



Asymptotically flat spacetime with future time infinity i^+ .

We stress that there are lots of Einstein eq.s solutions admitting i^+ as established by Friedrich (actually he considered the past time infinity i^- , but the extend is completely symmetric). With this notion we may state the following result [2] whose proof is based on fine estimations of the behaviour about i^+ of $\Gamma\phi$. The difficult point is to show that $\text{Range}[\Gamma] \subset \mathcal{S}(\mathfrak{S}^+)$. Then, barring technicalities, $\sigma(\Gamma\phi_1, \Gamma\phi_2) = \sigma_M(\phi_1, \phi_2)$ is a consequence of divergence theorem.

Theorem 4. *If (M, g) is asymptotically flat at future null infinity, (M, g) and (\tilde{M}, \tilde{g}) are both globally hyperbolic and (M, g) admits future time infinity i^+ , then hypotheses **H1** and **H2** are fulfilled so that ν_M exists and λ_M can be induced in the bulk form the BMS-invariant state on \mathfrak{S}^+ , λ .*

3 Uniqueness of λ and remarkable properties of λ_M .

The state λ is *universal*: it does not depend on the particular bulk spacetime M , but it induces a preferred state λ_M on the observables in the bulk in any considered asymptotically flat spacetime (provided that relevant hypotheses on asymptotic flatness and existence of i^+ be fulfilled). Let us investigate some properties of λ and λ_M [2, 3].

Theorem 5. *Assume that hypotheses (H1) and (H2) hold for the globally hyperbolic a.f. spacetime (M, g) (with (\tilde{M}, \tilde{g}) globally hyperbolic as well) so that the state λ_M exists.*

- (a) λ_M is **invariant under** under the (unit component Lie) group of isometries of M , $\{g_t\}$.
- (b) If the Killing generator ξ of $\{g_t\}$ is **timelike and future directed**, then the associated unitary 1-parameter group in the GNS representation of λ_M **admits positive self-adjoint generator** and in the one-particle space there are no zero modes for that generator.

Comments on the proofs.

(a) Let $\{\beta_{g_t}\}$ and $\{\beta_{g'_t}\}$ be, respectively, the groups of $*$ -automorphisms induced by $\{g_t\}$ and $\{g'_t\} \subset G_{BMS}$, acting on bulk and boundary observables $a \in \mathcal{W}(M, \sigma_M)$, $b \in \mathcal{W}(\mathfrak{S}^+, \sigma)$ respectively. The action of $\{\beta_{g_t}\}$ on λ_M is equivalent to the action of $\{\beta_{g'_t}\}$ on λ and λ is G_{BMS} -invariant. Therefore: $\lambda_M(\beta_{g_t}(a)) = \lambda(\beta_{g'_t}(\Gamma(a))) = \lambda(\Gamma(a)) = \lambda_M(a)$.

(b) If the Killing generator of $\{g_t\}$ is *timelike and future directed* then $\{g'_t\} \subset G_{BMS}$ is generated by a single *causal, future directed* 4-translation of T^4 . Then passing to GNS Hilbert spaces, using $\{U_{(\Lambda, \alpha)}\}$, one sees, by direct inspection, that $\{U_{\beta_{g'_t}}\}$ has *positive self-adjoint*

generator. The analog holds in the GNS representation of λ_M .

Positivity of self-adjoint generators of timelike future-directed 4-translation of T^4 holds true also for λ . It is a remnant of **energy positivity condition** in the bulk. Positivity condition on energy is a stability requirement: it guarantees that, under small (external) perturbations, the system does not collapse to lower and lower energy states. Actually positive energy condition determines uniquely λ . In fact the subsequent result is valid [2, 3].

Theorem 6. *In the hypotheses of theorem 5, fix a timelike future-directed 4-translation in T^4 and let $\{g_t\}$ be the generated 1-parameter G_{BMS} -subgroup. Under those hypotheses the following facts are true.*

(a) *If ω is a pure quasifree algebraic state on $\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$ satisfying both:*

(i) *it is invariant under $\{g_t\}$,*

(ii) *$\{g_t\}$ admits positive self-adjoint generator in the GNS representation of ω ,*

*then ω must be **invariant** under the **whole** BMS group and ω must **coincide** with λ .*

(b) *If ω is a pure (not necessarily quasifree) algebraic state on $\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$ and it is invariant under $\{g_t\}$, the folium of ω cannot contain other $\{g_t\}$ -invariant states.*

Comments on the proofs.

(a) consequence of *cluster property*: $\lim_{t \rightarrow +\infty} \omega(a\beta_{g_t}(b)) \rightarrow \omega(a)\omega(b)$, $\forall a, b \in \mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma)$ valid for a $\{g_t\}$ -invariant pure state, and a uniqueness results by B.S.Kay [11].

(b) consequence of *weak asymptotic commutativity* valid for a $\{\beta_{g_t}\}$ -invariant pure state ω : $w\text{-}\lim_{t \rightarrow +\infty} [U(g_t)AU(g_t)^*, B] = 0$, $\forall A \in \Pi_\omega(\mathcal{W}(\mathcal{S}(\mathfrak{S}^+), \sigma))$, $B \in \mathfrak{B}(\mathfrak{H}_\omega)$.

4 The Hadamard property.

Does λ_M satisfy the *Hadamard property*? if the answer is positive, λ_M is a good starting point for generally covariant and local renormalization procedure, in particular it determines a well-behaved renormalized stress-energy tensor [12, 13]. Let ω be a regular state on $\mathcal{W}(M, \sigma_M)$. Let us denote by $\omega(x, y)$ the integral kernel of the two-point function of the state ω :

$$\omega(\Phi(f)\Phi(g)) = \omega(Ef, Eg)$$

where $\Phi(f)$ is the standard *field-operator* smeared with a test function $f \in C_0^\infty(M)$, $E = A - R : C_0^\infty(M) \rightarrow \mathcal{S}(M)$ being the *causal propagator* (or “advanced-minus retarded” fundamental solution) of Klein-Gordon operator [8]. The **global Hadamard property** states that in normal geodesically convex neighborhoods of every point of the spacetime:

$$\omega(x, y) = \Delta(x, y)\sigma(x, y)^{-1} + v(x, y) \ln \sigma(x, y) + \text{regular function}$$

where $\sigma(x, y)$ is the squared geodesic distance and Δ, v depend on the local geometry only. The **global Hadamard property** is similar, but it involves the (complicated) behaviour of the two-point function in a neighborhood of a Cauchy surface of M [12]. This is a condition *very* difficult to check directly for $\omega = \lambda_M$!

Radzikowski found out a micro-local characterization of the (global) Hadamard condition [14]: *If $\omega \in \mathcal{D}'(M \times M)$, that is if ω is a distribution on $M \times M$ thus satisfying continuity*

with respect to the relevant seminorm topologies, the global Hadamard property is **equiv-
alent** to a specific shape of *wave front set* of ω , $WF(\omega)$. More precisely $WF(\omega)$ is made of the elements $(x, \mathbf{p}_x, y, -\mathbf{p}_y) \in T^*(M \times M) \setminus 0$ such that:

- (1) \mathbf{p}_x is future directed and
- (2) there is a null geodesic from x to y having there cotangent vectors \mathbf{p}_x and $-\mathbf{p}_y$ respectively.)

Using Radzikowski framework as far as the item (b) has been concerned, the following final result has been recently obtained [3].

Theorem. *Assume the hypotheses of theorem 5. The following facts are true.*

(a) $\lambda_M \in \mathcal{D}'(M \times M)$ also if there are bad compositions of distributions ($\mathcal{S}(\mathfrak{S}^+)$ nonstandard space of test functions).

(b) λ_M is globally Hadamard on M .

Comments on the proof.

The proof of (b) has been performed establishing first the validity of the local Hadamard condition. Then the global Hadamard property has been reached using a “local-to-global” argument introduced by Radzikowski in the second paper in [14].

5 Final comments.

The unique, positive energy, *BMS*-invariant, quasifree, pure state λ is completely defined using the universal structure of the class of asymptotically flat vacuum spacetimes at null infinity, no reference to any particular spacetime is necessary. In this sense λ is **universal**. On the other hand λ induces a well-behaved quasifree state λ_M in each asymptotically flat spacetime M admitting i^+ . λ_M is quite natural: it coincides with Minkowski vacuum when M is Minkowski spacetime, λ_M is invariant under every isometry of M and fulfills the requirement of energy positivity with respect to every timelike Killing field in M . λ_M may have the natural interpretation of **outgoing scattering vacuum**. Finally λ_M has been showed to verify the Hadamard condition and therefore it may be used as background for perturbative procedures (renormalization in particular), and it provides a natural notion of massless particle also in absence of Poincaré symmetry but in the presence of asymptotic flatness. Indeed all the construction works for *massless* fields (with conformal coupling). What about *massive* fields?

How to connect bulk massive fields to *BMS*-massive fields on \mathfrak{S}^+ and to known unitary representations of G_{BMS} with $m_{BMS} > 0$? [15]. This is an open issue which deserves future investigation

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