DISA

Dipartimento di Informatica
e Studi Aziendali

2009/5

On the normalization of a priority vector associated with a reciprocal relation

Michele Fedrizzi, Matteo Brunelli

UNIVERSITÀ DEGLI STUDI
DI TRENTO

DISA WORKING PAPER

DISA

Dipartimento di Informatica
e Studi Aziendali

2009/5

On the normalization of a priority vector associated with a reciprocal relation

Michele Fedrizzi, Matteo Brunelli

UNIVERSITÀ DEGLI STUDI
DI TRENTO

DISA WORKING PAPER

Ricardo Alberto MARQUES PEREIRA
Dipartimento di Informatica e Studi Aziendali
Università degli Studi di Trento
Via Inama 5, TN 38100 Trento ITALIA
Tel +39-0461-882147 Fax +39-0461-882124
E-mail: ricalb.marper@unitn.it

On the normalization of a priority vector associated with a reciprocal relation

Michele Fedrizzi
Department of Computer and Management Science
University of Trento, Via Inama 5, TN 38100 Trento, Italy
e–mail: `michele.fedrizzi@unitn.it`

Matteo Brunelli
IAMSR and Turku Centre for Computer Science
Åbo Akademi University, Joukahainengatan 3-5A, FIN-20520 Åbo, Finland
e–mail: `matteo.brunelli@abo.fi`

Abstract

In this paper we show that the widely used normalization constraint $\sum_{i=1}^{n} w_i = 1$ does not apply to the priority vectors associated with reciprocal relations, whenever additive transitivity is involved. We show that misleading applications of this type of normalization may lead to unsatisfactory results and we give some examples from the literature. Then, we propose an alternative normalization procedure which is compatible with additive transitivity and leads to better results.

Keywords: reciprocal relation; fuzzy preference relation; priority vector; normalization.

Introduction

Vector normalization is a widespread technique used in many fields of mathematics, physics, economics, etc. in order to obtain uniqueness from an infinite set of vectors. A well known example is given by the Eigenvectors of a square matrix. Usually, normalization is obtained by dividing every component w_i of a vector w by a suitable value k. Frequently used values of k are $k = \|w\|$, i.e. the norm of w, and $k = \sum_{i=1}^{n} w_i$. In the first case a unit–norm vector is obtained, $\|w\| = 1$, while in the second case the components of the obtained vector sum up to one,

$$\sum_{i=1}^{n} w_i = 1. \tag{1}$$

Clearly, normalization is meaningful only if all the vectors of the infinite set we are dealing with are equivalent for our purpose, so that the normalized vector can correctly represent the whole vector set. Eigenvectors corresponding to a single eigenvalue are again a suitable example. In the Analytical Hierarchy Process (Saaty, 1980), as well as in other similar methods, the decision maker's judgements a_{ij} estimate the ratios of priorities w_i/w_j. Therefore, priorities (or weights) w_i can be multiplied or divided by the same arbitrarily chosen positive number without changing ratios w_i/w_j. In this framework, normalization (1) is justified and it is usually applied.

Nevertheless, careful attention must be payed in order to avoid misleading applications of (1) in problems where this constraint is not only unnecessary, but leads to unsatisfactory results. More precisely, we show that, as long as reciprocal relations are concerned, constraint (1) is incompatible with additive consistency. Since in many papers on reciprocal relations constraint (1) is imposed, it is important, in our opinion, that researchers are aware of this incompatibility. We cite Lee and Tseng (2006); Lee (2006); Lee *et al.* (2008); Lee and Yeh (2008); Xu (2004, 2007a,b,c); Xu and Chen (2007, 2008a,b,c) as examples, but they are probably not the only ones.

This paper is organized as follows. In section 1 we prove that the normalization (1) conflicts with additive consistency of reciprocal relations and we propose an alternative normalization procedure that can substitute (1) and is compatible with additive consistency. In section 2 we discuss the optimization models proposed in some of the papers mentioned above and we show that better results are obtained if the normalization constraint (1) is substituted with our alternative proposal. Finally, in section 3 we present some comments and conclusions.

1 Reciprocal relations and vector normalization

We assume that the reader is familiar with reciprocal relations on a set of alternatives $\Lambda = \{A_1, A_2, ..., A_n\}$, *RR*s in the following. We only recall that they are nonnegative relations $R : \Lambda \times \Lambda \to [0, 1]$ satisfying additive reciprocity, $r_{ij} + r_{ji} = 1$, $i, j = 1, ..., n$ where $r_{ij} := R(A_i, A_j)$. Therefore, a $n \times n$ matrix $R = (r_{ij})_{n \times n}$ is a suitable way to represent a *RR*. In this domain, *RR*s express preferences according to the following rule:

$$r_{ij} = \begin{cases} 1, & \text{if } A_i \text{ is definitely preferred to } A_j \\ \alpha \in (0.5, 1), & \text{if } A_i \text{ is preferred to } A_j \\ 0.5, & \text{if there is indifference between } A_i \text{ and } A_j \\ \beta \in (0, 0.5), & \text{if } A_j \text{ is preferred to } A_i \\ 0, & \text{if } A_j \text{ is definitely preferred to } A_i. \end{cases}$$

Let us also note that, in literature, *RR*s are often called *fuzzy preference relations*. Despite it, we prefer to distinguish between them and use the former terminology (De Baets *et al.*, 2006).

Tanino (1984) defines additive consistency (transitivity) for a RR as follows,

$$(r_{ih} - 0.5) = (r_{ij} - 0.5) + (r_{jh} - 0.5) \quad i, j, h = 1, \ldots, n \; . \tag{2}$$

He proves also the following proposition, characterizing an additively consistent RR,

Proposition 1. *(Tanino, 1984) A RR $R = (r_{ij})_{n\times n}$ is additively consistent if and only if a non negative vector $w = (w_1, ..., w_n)$ exists with $|w_i - w_j| \leq 1 \;\; \forall i, j$, such that the entries r_{ij} of R are given by*

$$r_{ij} = 0.5 + 0.5(w_i - w_j) \quad i, j = 1, ..., n. \tag{3}$$

Components w_i are unique up to addition of a real constant.

We set the following definition in order to avoid misunderstandings.

Definition 1. *Given an additively consistent RR $R = (r_{ij})$, a vector w is called 'associated' with R if and only if it satisfies (3) as well as the assumptions of Proposition 1. Vector w is said to 'represent' the associated RR.*

Tanino's characterization (3) has been used as optimization criterion in some papers (Lee and Tseng, 2006; Xu, 2004; Xu and Chen, 2008b), but in all of them constraint (1) is imposed. With the following proposition we prove incompatibility of (1) with Tanino's characterization (3).

Proposition 2. *For every positive integer $n \geq 3$, there exists at least an additively consistent RR such that none of its associated weight vectors satisfies the constraint*

$$\sum_{i=1}^{n} w_i < n - 1. \tag{4}$$

Proof. Let us consider the following additively consistent RR

$$\hat{R} = (r_{ij})_{n\times n} = \begin{pmatrix} 0.5 & 0.5 & \cdots & \cdots & 0.5 & 1 \\ 0.5 & 0.5 & \cdots & \cdots & 0.5 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.5 & 0.5 & \cdots & \cdots & 0.5 & 1 \\ 0 & 0 & \cdots & \cdots & 0 & 0.5 \end{pmatrix}. \tag{5}$$

We prove that every vector w associated with (5) cannot satisfy (4). By substituting $r_{in} = 1$ in (3) for $i = 1, ..., n-1$, one obtains

$$w_i = w_n + 1 \quad i = 1, ..., n-1,$$

and therefore

$$\sum_{i=1}^{n} w_i = (n-1)(w_n+1) + w_n = nw_n + n - 1.$$

Since $w_n \geq 0$, inequality (4) is violated and the proposition is proved. □

Proposition 2 can clearly be reformulated in the following way,

Proposition 3. *For every positive integer $n \geq 3$, condition*

$$\sum_{i=1}^{n} w_i \geq n-1. \tag{6}$$

is necessary, in order to represent every additively consistent RR by means of a weight vector w.

The following proposition shows that the bound $n-1$ is tight.

Proposition 4. *For every positive integer $n \geq 3$, every additively consistent RR can be represented by means of a weight vector w satisfying*

$$\sum_{i=1}^{n} w_i \leq n-1\ . \tag{7}$$

Proof. Let us consider an arbitrary additively consistent *RR* $R = (r_{ij})_{n\times n}$. Proposition 1 guarantees the existence of a vector $v = (v_1, ..., v_n)$ representing R, i.e. satisfying (3). Let us assume, without loss of generality, $v_n \leq v_{n-1} \leq \cdots \leq v_1$. Since components of v are unique up to addition of a real constant k (Proposition 1), by choosing $k = -v_n$, it is always possible to represent R by a vector w with $w_n = 0$, obtaining $w = (w_1, \cdots, w_{n-1}, 0)$. From $w_n = 0$ and proposition 1, it follows $0 \leq w_i \leq 1$. Then it is $\sum_{i=1}^{n} w_i \leq n-1$. □

Note that $(1, 1, \cdots, 1, 0)$ is the priority vector representing (5) with the minimum value of the sum of its components and it is $\sum_{i=1}^{n} w_i = n-1$.

One might argue that (5) is a borderline and implausible example, as it corresponds to the case where the first $n-1$ alternatives are strongly preferred to the last one. Let us then briefly consider a very common case, where the preferences on the alternatives are uniformly distributed from the most preferred alternative A_1 to the less preferred A_n. This is perhaps the most simple and frequent reference case and it is represented, for $n = 4$, by the additively consistent *RR*

$$\bar{R} = (\bar{r}_{ij})_{n\times n} = \begin{pmatrix} 3/6 & 4/6 & 5/6 & 6/6 \\ 2/6 & 3/6 & 4/6 & 5/6 \\ 1/6 & 2/6 & 3/6 & 4/6 \\ 0/6 & 1/6 & 2/6 & 3/6 \end{pmatrix}. \tag{8}$$

As it can be easily verified by means of (3), $\bar{w} = (1, \frac{2}{3}, \frac{1}{3}, 0)$ represents (8) and has the minimum value of the sum of the components, $\bar{w}_1 + \bar{w}_2 + \bar{w}_3 + \bar{w}_4 = 2$. Note that the priority vector $\bar{w}$ indicates that it is $A_1 \succ A_2 \succ A_3 \succ A_4$ with uniformly spaced (as the preferences $\bar{r}_{ij}$ are) priority weights.

Example (8) can be extended to the general n–dimensional case,

$$\bar{R} = (\bar{r}_{ij})_{n\times n} = \left(\frac{n-1+j-i}{2n-2}\right)_{n\times n} \tag{9}$$

where the priority vector satisfying (3) and representing (9) with the minimum value of the sum of the components is $\bar{w} = (1, \frac{n-2}{n-1}, \cdots, \frac{2}{n-1}, \frac{1}{n-1}, 0)$, with $\sum_{i=1}^{n} \bar{w}_i = \frac{n}{2}$. Let us prove this result. First, by substituting $\bar{w}_i$ and $\bar{w}_j$ in (3) it can be verified that $\bar{w}$ represents (9). Then, by summing the components of $\bar{w}$ one obtains $\sum_{i=1}^{n} \bar{w}_i = \sum_{i=1}^{n} \frac{n-i}{n-1} = \frac{1}{n-1}(n2 - \frac{n(n+1)}{2}) = \frac{n}{2}$. All other priority vectors associated to (9) have component sum larger than $\frac{n}{2}$, since they are obtained by adding a positive constant to each component of $\bar{w}$. Therefore, also in this case, condition (1) cannot be satisfied and the larger n, the larger the spread between left and right hand side of (1).

Tanino (1984) also considers an alternative kind of consistency for RR which is called *multiplicative*. A RR is multiplicatively consistent if and only if the following condition of transitivity holds

$$\frac{r_{ih}}{r_{hi}} = \frac{r_{ij}}{r_{ji}} \frac{r_{jh}}{r_{hj}} \quad i, j, h = 1, \ldots, n \; . \tag{10}$$

If (10) holds, then a positive vector $v = (v_1, \ldots, v_n)$ exists such that

$$r_{ij} = \frac{v_i}{v_i + v_j} \quad i, j = 1, \ldots, n. \tag{11}$$

Components v_i are unique up to multiplication by a positive constant. Therefore, a priority vector satisfying (11) can be normalized using (1), since the ratio in (11) remains unchanged, as it is in w_i/w_j for Saaty's case. Preference relations used in Saaty's AHP are also called multiplicative preference relation and we recall that, by means of a suitable function, a RR $R = (r_{ij})$ can be transformed into a multiplicative preference relation $A = (a_{ij})$ like those used, for example, in Saaty's AHP (Saaty, 1980). By using function $a_{ij} = 9^{2r_{ij}-1}$ (see Fedrizzi, 1990), additive reciprocity $r_{ij} + r_{ji} = 1$ is transformed into multiplicative reciprocity $a_{ij}a_{ji} = 1$ and additive consistency (2) into multiplicative consistency $a_{ih} = a_{ij}a_{jh}$, while Tanino's characterization (3) corresponds to $a_{ij} = w_i/w_j$.

By using function $a_{ij} = r_{ij}/(1 - r_{ij})$, additive reciprocity $r_{ij} + r_{ji} = 1$ is still transformed into multiplicative reciprocity $a_{ij}a_{ji} = 1$ and multiplicative consistency (10) into multiplicative consistency $a_{ih} = a_{ij}a_{jh}$, while characterization (11) corresponds to $a_{ij} = w_i/w_j$.

To conclude, normalization (1) can be properly applied in the framework of multiplicative preference relations as well as in the framework of *multiplicatively* consistent RRs. In the following section we propose a normalization condition compatible with *additive* consistency for RRs.

1.1 An alternative normalization

Uniqueness of priority vector satisfying (3) can be achieved simply by adding the constant $k = -\min\{w_1, ..., w_n\}$ to each component w_i, thus obtaining a vector with the minimum component equal to zero. Assuming $w_n \leq w_{n-1} \leq \cdots \leq w_1$, it is $k = -w_n$ and the normalized vector becomes

$$w = (w_1, \cdots, w_{n-1}, 0). \tag{12}$$

Contrary to (1), this alternative normalization procedure is compatible with (3) and, as proved above, it guarantees that all the priorities w_i are in the interval [0, 1]. This is a good standard result that also allows an easier and more familiar understanding of the obtained priorities. To summarize, the normalization constraint we propose is

$$\begin{array}{l} \min\{w_1, ..., w_n\} = 0 \\ 0 \leq w_i \leq 1 \quad i = 1, ..., n \end{array} \tag{13}$$

2 Consistency optimization and vector normalization

In the previous section we have considered the case of additively consistent RRs. Let us now consider the case in which additive consistency is not a priori satisfied, but it is the goal of a proposed optimization model. Xu (2004), for instance, considers incomplete RRs $R = (r_{ij})$ and proposes some goal programming models to obtain the priority vector. The author refers to Proposition 1 (see Xu, 2004) and constructs the following multiobjective programming model (denoted by (MOP1))

$$\text{(MOP1)} \qquad \min \varepsilon_{ij} = \delta_{ij} |\, r_{ij} - 0.5(w_i - w_j + 1)| \quad i, j = 1, ..., n$$
$$\text{s.t.}\;\; w_i \geq 0, \;\; i = 1, .., n, \quad \sum_{i=1}^{n} w_i = 1.$$

To solve (MOP1), the author introduces the following goal programming model, denoted by (LOP2); we skip the more general model (LOP1) for brevity (see Xu, 2004),

$$
\text{(LOP2)} \qquad \begin{aligned}
&\min J = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (d_{ij}^{+} + d_{ij}^{-}) \\
&\text{s.t. } \delta_{ij}[r_{ij} - 0.5(w_i - w_j + 1)] - d_{ij}^{+} + d_{ij}^{-} = 0, \;\; i, j \in N, \, i \neq j \\
&w_i \geq 0, \;\; i \in N, \;\; \sum_{i=1}^{n} w_i = 1 \\
&d_{ij}^{+} \geq 0, \; d_{ij}^{-} \geq 0, \;\; i, j \in N, \, i \neq j \, .
\end{aligned}
$$

Optimization models (MOP1) and (LOP2) are clearly based on the idea of moving as close as possible to satisfying (3). The proposal is appropriate and effective but, as proved in the previous section, the normalization constraint (1) required in both (MOP1) and (LOP2) conflicts with the goal.

As a numerical example, let us consider the incomplete RR obtained from (8) by considering r_{14} (and therefore also r_{41}) as missing. Following definition 2.5 of Xu (2004), this RR is called *additively consistent incomplete fuzzy preference relation*. By applying (LOP2), vector $w^* = (\frac{2}{3}, \frac{1}{3}, 0, 0)$ is obtained, and the corresponding value of the objective function is $J(w^*) = \frac{2}{3}$, evidencing that (3) has not been completely fulfilled. Conversely, if the constraint (1) is substituted by (13) in (LOP2), we obtain again vector $\bar{w} = (1, \frac{2}{3}, \frac{1}{3}, 0)$, with $J(\bar{w}) = 0$, so that (3) is completely fulfilled. Note that w^* does not respect preference ordering, as it is $w_3^* = w_4^*$ with $r_{34} > 0.5$. Moreover, while $\bar{w}$ is associated to (8), vector w^* is associated to a different consistent RR, more precisely to

$$
R^* = \begin{pmatrix} 3/6 & 4/6 & 5/6 & 5/6 \\ 2/6 & 3/6 & 4/6 & 4/6 \\ 1/6 & 2/6 & 3/6 & 3/6 \\ 1/6 & 2/6 & 3/6 & 3/6 \end{pmatrix} . \tag{14}
$$

Analogous results are obtained if the goal of the optimization models is still additive consistency for a RRs, but this goal is not fully achievable.

Xu and Chen (2008b) also consider *interval* RRs, represented by square matrices whose entries are real intervals. This approach generalizes the former (Xu, 2004), as each preference is quantified by using an interval $[r_{ij}^{-}, r_{ij}^{+}]$, instead of a single value r_{ij}. Their optimization models (Xu and Chen, 2008b) denoted by (M–1), (M–2), (M–3), (M–4) and (M–5) are still based on the objective of best fulfilment of Tanino's condition (3), but they also contain constraint (1). Therefore, all the arguments exposed above can be repeated also in this case and we do not report, for brevity, a detailed discussion with examples.

Nevertheless, it is necessary to draw the attention to the consequences of imposing (1) in definitions 3 and 4 given by Xu and Chen (2008b) for an 'additive consistent interval fuzzy preference relation' (or 'additive consistent interval *RR*', following our terminology). These definitions extend the well-known case of additively consistent *RR* by requiring that in each entry of the interval matrix a single value $r_{ij} \in [r_{ij}^-, r_{ij}^+]$ can be chosen to form an additively consistent *RR* $R = (r_{ij})$ (i.e. satisfying (3)). In other words, an interval *RR* is called additive consistent if it 'contains' an additively consistent *RR*. [1] By including (1) in definitions 3 and 4, it is implicitly required that Tanino condition (3) must be associated to (1) in order to obtain additive consistency. As we stated above with proposition 3, the two of them are incompatible. Coherence with the definition of additively consistent *RR* can be achieved only by removing (1) or by substituting it with (13). Otherwise, it is easy to check that an interval *RR* obtained simply by adding a small spread to the entries of an additively consistent *RR* could not satisfy the previous definitions and should be classified as inconsistent. This is clearly unacceptable and an example can be constructed by means of (8). It can be verified that the interval *RR* whose entries, for $i \neq j$, are intervals centered in $\bar{r}_{ij}$, i.e. $[r_{ij}^-, r_{ij}^+] = [\bar{r}_{ij} - \varepsilon, \bar{r}_{ij} + \varepsilon]$, does not satisfy the definitions 3 and 4 if $\varepsilon < 0.166$. To be more precise, since all the considered values must remain in the interval [0, 1], we should better define $r_{ij}^- = \max(0, \bar{r}_{ij} - \varepsilon)$ and $r_{ij}^+ = \min(1, \bar{r}_{ij} + \varepsilon)$, but this does not change our conclusion. Definition 3 of Xu and Chen (2008b) is also reported in another work on the same issue (Xu and Chen, 2008c) and in a survey of preference relations (Xu, 2007a), where it is referred to as definition 10.

Without dwelling on their details, we end this section recalling some more contributions which can be improved if (1) is removed or substituted by (13). Xu (2007b) introduces some models to solve multiple-attribute group-decision-making problems with three different preference formats. Xu and Chen (2008a) propose a method for deriving the weight vector from an incomplete *RR*. Intuitionistic preference relations under the form of *RR*s have also been investigated (Xu, 2007c). Some other papers which mainly follow the guidelines of the already cited contributions are those by Lee and Tseng (2006); Lee (2006); Xu and Chen (2007); Lee *et al.* (2008); Lee and Yeh (2008).

[1] An equivalent definition for fuzzy pairwise comparison matrices was given by Fedrizzi and Marques Pereira (1995) in the framework of a fuzzy extension of Saaty's AHP.

3 Comments and conclusions

Given the very frequent use of vector normalization, it is important, in our opinion, that researchers are warned not to consider it as a risk-free routine when they are dealing with RR. Otherwise, interesting proposals can become useless, due to an inadequate choice of the normalization constraint.

References

De Baets, B., De Meyer, H., De Schuymer, B and Jenei, S., 2006. Cyclic evaluation of transitivity of reciprocal relations. *Social Choice and Welfare*, 26, 217–238.

Fedrizzi, M., 1990, On a consensus measure in a group MCDM problem. *In*: J. Kacprzyk and M. Fedrizzi, eds. *Multiperson Decision Making Models using Fuzzy Sets and Possibility Theory*, Dordrecht, Kluwer Academic Publishers, 231–241.

Fedrizzi, M. and Marques Pereira R.A., 1995. Positive fuzzy matrices, dominant eigenvalues and an extension of Saaty's Analytical Hierarchy Process, *Proceedings of the VI International Fuzzy Systems Association World Congress IFSA'95*, São Paulo, Brazil, 245–247.

Lee, H.-S., 2006. A Fuzzy Method for Evaluating Suppliers *In*: L. Wang et al. eds. *FSK 2006, LNAI 4223*. Springer-Verlag Berlin Heidelberg, 1035–1043.

Lee, H.-S. and Tseng, W-K., 2006. Goal Programming Methods for Constructing Additive Consistency Fuzzy Preference Relations, *In*: B. Gabrys, R.J. Howlett and L.C. Jain, eds. *KES 2006, Part II, LNAI 4252*. Springer-Verlag Berlin Heidelberg, 910–916.

Lee, H.-S., Shen, P.-D. and Chyr, W.-L., 2008. Prioritization of Incomplete Fuzzy Preference Relation *In*: I. Lovrek, R.J. Howlett and L.C. Jain, eds. *KES 2008, Part II, LNAI 5178*. Springer-Verlag Berlin Heidelberg, 974–979.

Lee H.-S. and Yeh C.-H., 2008. Fuzzy Multi-criteria Decision Making Based on Fuzzy Preference Relation *In*: I. Lovrek, R.J. Howlett and L.C. Jain, eds. *KES 2008, Part II, LNAI 5178*. Springer-Verlag Berlin Heidelberg, 980–985.

Saaty, T.L., 1980, *The Analytical Hierarchy Process*. New York, McGraw-Hill.

Tanino, T., 1984. Fuzzy preference orderings in group decision making. *Fuzzy sets and systems*, 12 (2), 117–131.

Xu, Z.S., 2004. Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation. *International journal of approximate reasoning*, 36 (3), 261–270.

Xu, Z.S., 2007a. A survey of preference relations. *International journal of general systems*, 36 (2), 179–203.

Xu, Z.S., 2007b. Multiple-attribute group decision making with different formats of preference information on attributes. *IEEE transactions on systems, man, and cybernetics, part B*, 37 (6), 1500–1511.

Xu, Z.S., 2007c. A method for estimating criteria weights from intuitionistic preference relations. *In*: B.-Y. Cao, ed. *Fuzzy information and engineering (ICFIE), ASC 40*, 503–512.

Xu, Z.S. and Chen J., 2007. A subjective and objective integrated method for MAGDM problems with multiple types of exact preference formats. *In*: H. Yin, *et al.*, ed. *IDEAL 2007, LNCS 4881*, 145–154.

Xu, Z.S. and Chen J., 2008a. Group decision-making procedure based on incomplete reciprocal relations. *Soft computing*, 12 (6), 515–521.

Xu, Z.S. and Chen J., 2008b. Some models for deriving the priority weights from interval fuzzy preference relation. *European journal of operational research*, 184 (1), 266–280.

Xu, Z.S. and Chen J., 2008c. MAGDM linear-programming models with distinct uncertain preference structures. *IEEE transactions on systems, man, and cybernetics, part B*, 38 (5), 1356–1370.