# A RESULT ABOUT $C^3$-RECTIFIABILITY OF LIPSCHITZ CURVES AN APPLICATION IN GEOMETRIC MEASURE THEORY

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ABSTRACT. Let $\gamma_0 : [a,b] \to \mathbf{R}^{1+k}$ be Lipschitz. Our main result provides a sufficient condition, expressed in terms of further accessory Lipschitz maps, for the $C^3$-rectifiability of $\gamma_0([a,b])$. Such a condition finds a natural interpretation in the context of Gauss maps of curves and in fact an application to one-dimensional generalized Gauss graphs is given.

## 1. INTRODUCTION

The main goal of this paper is to prove the following result.

**Theorem 1.1.** *Let be given three Lipschitz maps*

$$\gamma_0, \gamma_1 : [a,b] \to \mathbf{R}^{1+k} \qquad \text{and} \qquad \gamma_2 = (\gamma_{2\top}, \gamma_{2\perp}) : [a,b] \to \mathbf{R}^{1+k} \times \mathbf{R}^{1+k}$$

*and a function $\omega : [a,b] \to \{\pm 1\}$ such that the following equalities*

$$\gamma_0'(t) = \omega(t)\|\gamma_0'(t)\|\gamma_1(t) \tag{1.1}$$

*and*

$$(\gamma_0'(t), \gamma_1'(t)) = \omega(t)\|(\gamma_0'(t), \gamma_1'(t))\|\gamma_2(t) \tag{1.2}$$

*hold at almost every $t \in [a,b]$. Then $\gamma_0([a,b])$ is a $C^3$-rectifiable set.*

The proof moves from the $C^2$-rectifiability of $\gamma_0([a,b])$ which is provided by the condition (1.1), as we showed in [7]. Hence one is easily reduced to prove that $\gamma_0([a,b])$ intersects the graph of any map of class $C^2$

$$f : \mathbf{R} \to (\mathbf{R}u)^\perp \qquad (u \in \mathbf{R}^{1+k}, \|u\| = 1)$$

in a $C^3$-rectifiable set (Section 2). From the up-to-second-order derivatives of $f$ expressed in terms of the $\gamma_i$, one obtains a second order Taylor-type formula for $f$ with the remainder expressed in terms of the $\gamma_i$ (Section 3). Finally, Theorem 1.1 follows by the Whitney extension Theorem, also involving a Lusin-type argument (Section 4).

In the special case when $\gamma_0$ is smooth (class $C^2$ is enough) and regular, the conditions (1.1) and (1.2) with $\omega := 1$ say that $\gamma_1(t)$ and $\gamma_2(t)$ are, respectively, the unit tangent vector of $\gamma_0$ at $t$ and the unit tangent vector of $(\gamma_0, \gamma_1)$ at $t$. In other words, the map

$$(\gamma_0, \gamma_1, \gamma_2) : [a,b] \to \mathbf{R}^{4(1+k)}$$

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parametrizes the "graph of the Gauss map to the graph of the Gauss map to $\gamma_0$". Despite the ugliness of its description in quotes, this kind of objects looks quite natural from the geometric point of view and can easily be extended to the context of geometric measure theory, through the machinery of generalized Gauss graphs (e.g. see at [5] for the main definitions). The last two sections of the present work are devoted to this aim. First of all the absolute curvature for a one-dimensional $C^2$-rectifiable set $P$ is defined and it is proved to be approximately differentiable almost everywhere, whenever $P$ is $C^3$-rectifiable (Section 5). Then the notion of "2-storey tower of one-dimensional generalized Gauss graphs" is provided and some main properties are proved (Section 6). Particular attention is paid to the case $k = 1$, where a formula expressing the orientation of a tower in terms of the absolute curvature and its approximate differential is given.

## 2. Reduction to graphs

By virtue of the main result stated in [7], the equality (1.1) implies that $\gamma_0([a, b])$ is $C^2$-rectifiable. As a consequence, there must be countably many unit vectors

$$u_j \in \mathbf{R}^{1+k}$$

and maps of class $C^2$

$$f_j : \mathbf{R} \to (\mathbf{R}u_j)^\perp$$

such that

$$\mathcal{H}^1\left(\gamma_0([a, b]) \backslash \cup_j G_{f_j}\right) = 0$$

where

$$G_{f_j} := \{(xu_j, f_j(x)) \mid x \in \mathbf{R}\}.$$

Hence we are reduced to show that the sets $\gamma_0([a, b]) \cap G_{f_j}$ are $C^3$-rectifiable. In other words, Theorem 1.1 becomes an immediate corollary of the following result.

**Theorem 2.1.** *Let $\gamma_0, \gamma_1, \gamma_2$ be as in Theorem 1.1 and consider a map*

$$f : \mathbf{R} \to (\mathbf{R}u)^\perp \qquad (u \in \mathbf{R}^{1+k}, \|u\| = 1)$$

*of class $C^2$. If $G_f := \{(xu, f(x)) \mid x \in \mathbf{R}\}$, then the set $G_f \cap \gamma_0([a, b])$ is $C^3$-rectifiable.*

In this section we just do a first step toward the proof of Theorem 2.1, which will be concluded later in §4.

Let us define

$$L := \gamma_0^{-1}(G_f) \cap \{t \in [a, b] \mid \gamma_0'(t) \text{ and } \gamma_1'(t) \text{ exist, } \gamma_0'(t) \neq 0, \text{ (1.1) and (1.2) hold}\}.$$

By the Lusin Theorem, for any given real number $\varepsilon > 0$, there exists a closed subset $L_\varepsilon$ of $L$ such that

$$\gamma_0'|L_\varepsilon \text{ and } \omega|L_\varepsilon \text{ are continuous and } \mathcal{L}^1(L\backslash L_\varepsilon) \leq \varepsilon. \tag{2.1}$$

If $L_\varepsilon^*$ denotes the set of the density points of $L_\varepsilon$, then

$$L_\varepsilon^* \subset L_\varepsilon \tag{2.2}$$

in that $L_\varepsilon$ is closed. The following equality also holds

$$\mathcal{L}^1(L_\varepsilon \backslash L_\varepsilon^*) = 0 \tag{2.3}$$

by a celebrated Lebesgue's result. In the special case that $L$ has measure zero, we take $L_\varepsilon := \emptyset$, hence $L^*_\varepsilon := \emptyset$.

Now observe that

$$G_f \cap \gamma_0([a,b]) \backslash \gamma_0(L^*_\varepsilon) \subset \gamma_0 \left( \gamma_0^{-1}(G_f) \cap [a,b] \backslash L^*_\varepsilon \right)$$

hence

$$\begin{aligned}
\mathcal{H}^1 \left( G_f \cap \gamma_0([a,b]) \backslash \gamma_0(L^*_\varepsilon) \right) &\le \mathcal{H}^1 \left( \gamma_0 \left( \gamma_0^{-1}(G_f) \cap [a,b] \backslash L^*_\varepsilon \right) \right) \\
&\le \int_{\gamma_0^{-1}(G_f) \cap [a,b] \backslash L^*_\varepsilon} \|\gamma_0'\| \\
&= \int_{L \backslash L^*_\varepsilon} \|\gamma_0'\| \\
&\le \varepsilon \, Lip(\gamma_0)
\end{aligned}$$

which implies

$$\mathcal{H}^1 \left( G_f \cap \gamma_0([a,b]) \backslash \cup_{j=1}^\infty \gamma_0(L^*_{1/j}) \right) = 0.$$

As a consequence, in order to prove Theorem 2.1, it will be enough to verify that

$$\gamma_0(L^*_\varepsilon) \text{ is } C^3\text{-rectifiable} \tag{2.4}$$

for all $\varepsilon > 0$.

## 3. Second order Taylor formula and estimates

First of all, we will state formulas for the up-to-second-order derivatives of $f$ in terms of the $\gamma_i$ (Proposition 3.1). Hence a suitable second order Taylor formula will be obtained (Proposition 3.1).

Throughout this section we shall assume $\mathcal{L}^1(L) > 0$. Notice that

$$\gamma_{2\top}(s) \neq 0, \text{ for all } s \in L \tag{3.1}$$

by (1.2), thus the map

$$\mu : \{t \in [a,b] \,|\, \gamma_{2\top}(t) \neq 0\} \to \mathbf{R}^{1+k}, \qquad \mu(t) := \frac{\gamma_{2\perp}(t)}{\|\gamma_{2\top}(t)\|}$$

is well-defined in $L$.

**Lemma 3.1.** *Let $A, B, u \in \mathbf{R}^{1+k}$, with $\|u\| = 1$. Then*

$$(A \wedge B) \llcorner u = (A \cdot u)B - (B \cdot u)A.$$

*Proof.* Let $\{e_j\}$ be an orthonormal basis of $\mathbf{R}^{1+k}$ such that $e_1 = u$. One has

$$\begin{aligned}
[(A \wedge B) \llcorner u] \cdot e_i &= \langle A \wedge B, u \wedge e_i \rangle \\
&= \sum_{\substack{j,l \\ j<l}} (A_j B_l - A_l B_j) \langle e_j \wedge e_l, e_1 \wedge e_i \rangle \\
&= A_1 B_i - A_i B_1 \\
&= [(A \cdot u)B - (B \cdot u)A] \cdot e_i
\end{aligned}$$

for all $i = 1, 2, \dots, 1+k$. $\square$

**Proposition 3.1.** *Set*

$$x(t) := \gamma_0(t) \cdot u, \qquad t \in \mathbf{R}.$$

*Then, for all*

$$s \in L^*_\varepsilon$$

*one has*

$$x'(s) = \gamma_0'(s) \cdot u \neq 0 \qquad i.e. \qquad \gamma_1(s) \cdot u \neq 0 \tag{3.2}$$

*and*

$$f'(x(s)) = \frac{\gamma_1(s)}{\gamma_1(s) \cdot u} - u. \tag{3.3}$$

*Moreover*

$$f''(x(s)) = \frac{[\gamma_1(s) \wedge \mu(s)] \llcorner u}{[\gamma_1(s) \cdot u]^3}. \tag{3.4}$$

*Proof.* Observe that

$$f(x(t)) = \gamma_0(t) - [\gamma_0(t) \cdot u]u = \gamma_0(t) - x(t)u$$

for all $t \in \gamma_0^{-1}(G_f)$. The members of this equality are both differentiable in $L^*_\varepsilon$ and since each point in $L^*_\varepsilon \subset \gamma_0^{-1}(G_f)$ is a limit point of $L_\varepsilon \subset \gamma_0^{-1}(G_f)$, the derivatives have to coincide in $L^*_\varepsilon$. Then

$$x'(s)f'(x(s)) = \gamma_0'(s) - [\gamma_0'(s) \cdot u]u = \gamma_0'(s) - x'(s)u \tag{3.5}$$

for all $s \in L^*_\varepsilon$. We obtain the formula (3.2), by recalling that $\gamma_0'(s) \neq 0$ at all $s \in L^*_\varepsilon$.

As for (3.3), note that it follows at once from (3.5) and (1.1).

By virtue of (3.2), the members of (3.3) are both differentiable in $L^*_\varepsilon$. Moreover the derivatives must coincide in $L^*_\varepsilon$, in that each point of $L^*_\varepsilon$ is a limit point of $L^*_\varepsilon$. By also recalling Lemma 3.1, we get

$$x'(s)f''(x(s)) = \frac{[\gamma_1(s) \cdot u]\gamma_1'(s) - [\gamma_1'(s) \cdot u]\gamma_1(s)}{[\gamma_1(s) \cdot u]^2} = \frac{[\gamma_1(s) \wedge \gamma_1'(s)] \llcorner u}{[\gamma_1(s) \cdot u]^2}$$

for all $s \in L^*_\varepsilon$. The formula (3.4) finally follows from (3.2), (1.1) and (1.2). $\square$

Now, in order to state the second order Taylor formula, we have to introduce some more notation. First of all, set

$$\Delta_s(t) := \gamma_0(t) - \gamma_0(s), \qquad s, t \in [a, b].$$

Then observe that the map

$$\Sigma_s(t) := \Delta_s(t) - [\Delta_s(t) \cdot \gamma_1(s)]\, \gamma_1(s) - \frac{[\Delta_s(t) \cdot u]^2}{2[\gamma_1(s) \cdot u]^2} \mu(s), \qquad t \in [a, b]$$

is well-defined for any given $s \in L^*_\varepsilon$, by Proposition 3.1.

If $s \in L^*_\varepsilon$, hence $s \in (a, b)$ and (3.1) holds, one has

$$\|\gamma_{2\top}(\sigma)\| \geq \frac{1}{2}\|\gamma_{2\top}(s)\| > 0, \text{ for all } \sigma \in I_s$$

where $I_s$ denotes a certain non trivial open interval centered at $s$ and included in $[a,b]$, existing by the continuity of $\gamma_{2\top}$. In particular, this inequality shows that $\mu|I_s$ is Lipschitz, hence the map

$$\Psi_s(\sigma) := \mu(\sigma) - [\mu(\sigma)\cdot\gamma_1(s)]\gamma_1(s) - \frac{\mu(s)}{[\gamma_1(s)\cdot u]^2}\Big([\gamma_1(\sigma)\cdot u]^2 + [\Delta_s(\sigma)\cdot u][\mu(\sigma)\cdot u]\Big), \qquad \sigma\in I_s$$

is well-defined and Lipschitz too, provided $s\in L_\varepsilon^*$. One also has

$$\Psi_s(s) = 0$$

as it follows at once from (1.2) and from the following simple result.

**Proposition 3.2.** *If $s\in L_\varepsilon^*$ then $\gamma_1(s)\cdot\gamma_1'(s) = 0$.*

*Proof.* Let $\{s_j\}$ be a sequence in $L_\varepsilon$ converging to $s$, with $s_j\neq s$ for all $j$. Since

$$\|\gamma_1(s_j)\| = \|\gamma_1(s)\| = 1, \quad \text{for all } j$$

by (1.1) and (2.2), then we have

$$0 = \frac{\|\gamma_1(s_j)\|^2 - \|\gamma_1(s)\|^2}{s_j - s} = \frac{\gamma_1(s_j)-\gamma_1(s)}{s_j-s}\cdot[\gamma_1(s_j)+\gamma_1(s)]$$

hence the conclusion follows by letting $j\to\infty$. □

We are finally ready to state and prove the announced Taylor formula.

**Theorem 3.1.** *Let $s\in L_\varepsilon^*$. Then*

(1) *For all $t\in\gamma_0^{-1}(G_f)$ one has*

$$\begin{aligned}
f(x(t)) - f(x(s)) - f'(x(s))[x(t)-x(s)] - \frac{f''(x(s))}{2}[x(t)-x(s)]^2 &= \\
= \frac{1}{\gamma_1(s)\cdot u}\,(\gamma_1(s)\wedge\Sigma_s(t)) \llcorner u;&
\end{aligned} \tag{3.6}$$

(2) *For all $t\in I_s$, one has*

$$\Sigma_s(t) = \int_s^t \omega(\rho)\|\gamma_0'(\rho)\|\left(\int_s^\rho \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\,d\sigma\right)d\rho.$$

*Proof.* (1) By recalling Proposition 3.1 and Lemma 3.1, we get

$$\begin{aligned}
f(x(t)) - f(x(s)) - f'(x(s))[x(t)-x(s)] - \frac{f''(x(s))}{2}[x(t)-x(s)]^2 &= \\
&= \gamma_0(t) - x(t)u - (\gamma_0(s) - x(s)u) - \left(\frac{\gamma_1(s)}{\gamma_1(s)\cdot u} - u\right)[x(t)-x(s)] + \\
&\quad - \frac{[\gamma_1(s)\wedge\gamma_{2\perp}(s)]\llcorner u}{2\|\gamma_{2\top}(s)\|[\gamma_1(s)\cdot u]^3}[x(t)-x(s)]^2 \\
&= \Delta_s(t) - \frac{\gamma_1(s)}{\gamma_1(s)\cdot u}[\Delta_s(t)\cdot u] - \frac{[\gamma_1(s)\wedge\gamma_{2\perp}(s)]\llcorner u}{2\|\gamma_{2\top}(s)\|[\gamma_1(s)\cdot u]^3}[\Delta_s(t)\cdot u]^2 \\
&= \frac{1}{\gamma_1(s)\cdot u}\left([\gamma_1(s)\wedge\Delta_s(t)]\llcorner u - \frac{[\gamma_1(s)\wedge\gamma_{2\perp}(s)]\llcorner u}{2\|\gamma_{2\top}(s)\|[\gamma_1(s)\cdot u]^2}[\Delta_s(t)\cdot u]^2\right)
\end{aligned}$$

that is just (3.6), by recalling the definition of $\Sigma_s(t)$.

(2) Since $\Delta_s$ is Lipschitz and $\Delta_s(s) = 0$, one has

$$\begin{aligned}\Sigma_s(t) &= \int_s^t \gamma_0'(\rho) - [\gamma_0'(\rho)\cdot\gamma_1(s)]\gamma_1(s) - \frac{\mu(s)}{2[\gamma_1(s)\cdot u]^2}\frac{d}{d\rho}[\Delta_s(\rho)\cdot u]^2 d\rho \\ &= \int_s^t \gamma_0'(\rho) - [\gamma_0'(\rho)\cdot\gamma_1(s)]\gamma_1(s) - \frac{\mu(s)}{[\gamma_1(s)\cdot u]^2}[\Delta_s(\rho)\cdot u][\gamma_0'(\rho)\cdot u]d\rho\end{aligned}$$

namely

$$\Sigma_s(t) = \int_s^t \omega(\rho)\|\gamma_0'(\rho)\|\Phi_s(\rho)\, d\rho \tag{3.7}$$

by (1.1), where $\Phi_s$ is the Lipschitz map defined as follows

$$\Phi_s(\rho) := \gamma_1(\rho) - [\gamma_1(\rho)\cdot\gamma_1(s)]\gamma_1(s) - \frac{\mu(s)}{[\gamma_1(s)\cdot u]^2}[\Delta_s(\rho)\cdot u][\gamma_1(\rho)\cdot u], \qquad \rho\in[a,b].$$

Observe that

$$\|\gamma_0'(\sigma)\|\gamma_{2\perp}(\sigma) = \|(\gamma_0'(\sigma),\gamma_1'(\sigma))\|\|\gamma_{2\top}(\sigma)\|\gamma_{2\perp}(\sigma) = \omega(\sigma)\|\gamma_{2\top}(\sigma)\|\gamma_1'(\sigma)$$

for a.e. $\sigma\in[a,b]$, by (1.2). Hence

$$\gamma_1'(\sigma) = \omega(\sigma)\|\gamma_0'(\sigma)\|\frac{\gamma_{2\perp}(\sigma)}{\|\gamma_{2\top}(\sigma)\|} = \omega(\sigma)\|\gamma_0'(\sigma)\|\mu(\sigma)$$

for a.e. $\sigma\in[a,b]$ such that $\gamma_{2\top}(\sigma)\neq 0$, e.g. for a.e. $\sigma\in I_s$. By recalling the definition of $\Psi_s$, it follows at once that

$$\Phi_s'(\sigma) = \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma) \tag{3.8}$$

for a.e. $\sigma\in I_s$. We conclude by (3.7), (3.8) and noting that $\Phi_s(s)=0$. $\square$

As a consequence, we get the following integral representation of $\Sigma_s'$ and the related first order Taylor formula for $f'$.

**Corollary 3.1.** *Let $s\in L_\varepsilon^*$ and $t\in L_\varepsilon^*\cap I_s$. Then*

(1) *The map $\Sigma_s$ is differentiable at $t$ and the following formula holds*

$$\Sigma_s'(t) = \omega(t)\|\gamma_0'(t)\|\int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\, d\sigma;$$

(2) *One has*

$$\begin{aligned}&f'(x(t)) - f'(x(s)) - f''(x(s))[x(t)-x(s)] = \\ &\qquad = \frac{1}{[\gamma_1(t)\cdot u]\,[\gamma_1(s)\cdot u]}\left(\gamma_1(s)\wedge\int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)d\sigma\right)\llcorner u\end{aligned}$$

*Proof.* (1) First of all, observe that

$$t+h\in I_s\subset(a,b)$$

provided $|h|$ is small enough. Then, by (2) of Theorem 3.1, one has

$$\frac{\Sigma_s(t+h)-\Sigma_s(t)}{h}=\frac{1}{h}\int_t^{t+h}\omega(\rho)\|\gamma_0'(\rho)\|\left(\int_s^\rho\omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\,d\sigma\right)d\rho=I_1(h)+I_2(h)$$

for all small enough value of $|h|$, where (with an innocent abuse of notation and recalling that $\omega|L_\varepsilon^*$ is continuous, by (2.1) and (2.2))

$$I_1(h):=\frac{\omega(t)}{h}\int_{[t,t+h]\cap L_\varepsilon^*}\|\gamma_0'(\rho)\|\left(\int_s^\rho\omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)d\sigma\right)d\rho$$

and

$$I_2(h):=\frac{1}{h}\int_{[t,t+h]\setminus L_\varepsilon^*}\omega(\rho)\|\gamma_0'(\rho)\|\left(\int_s^\rho\omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)d\sigma\right)d\rho.$$

The following equality holds

$$\begin{aligned}I_1(h)=\frac{\omega(t)}{h}&\int_{[t,t+h]\cap L_\varepsilon^*}(\|\gamma_0'(\rho)\|-\|\gamma_0'(t)\|)\left(\int_s^\rho\omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\,d\sigma\right)d\rho+\\&+\frac{\omega(t)\|\gamma_0'(t)\|}{h}\int_{[t,t+h]\cap L_\varepsilon^*}\left(\int_s^t\omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)d\sigma+\int_t^\rho\omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)d\sigma\right)d\rho.\end{aligned}$$

Recalling that

(i) $\gamma_0'|L_\varepsilon^*$ is continuous, by (2.1) and (2.2)
(ii) $\gamma_0$ is Lipschitz and $\Psi_s$ is bounded (in fact it is Lipschitz!)
(iii) $t$ is a density point of $L_\varepsilon$ (hence of $L_\varepsilon^*$, by (2.3))

it follows immediately that

$$\lim_{h\to 0}I_1(h)=\omega(t)\|\gamma_0'(t)\|\int_s^t\omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma)\,d\sigma.$$

The conclusion follows now by observing that, as an easy consequence of (ii) and (iii), one also has

$$\lim_{h\to 0}I_2(h)=0.$$

(2) The two members of (3.6) are differentiable at $t$, by (1). Since $t$ is a limit point of $L_\varepsilon\subset\gamma_0^{-1}(G_f)$ the derivatives have to coincide, by Theorem 3.1(1), namely

$$\left(f'(x(t))-f'(x(s))-f''(x(s))[x(t)-x(s)]\right)x'(t)=\frac{1}{\gamma_1(s)\cdot u}\left(\gamma_1(s)\wedge\Sigma_s'(t)\right)\llcorner u.$$

We conclude by recalling Proposition 3.1, the statement (1) and (1.1). □

## 4. Back to the proof of Theorem 1.1: conclusion

In order to conclude the proof of Theorem 2.1, hence of Theorem 1.1, we have to verify (2.4). To this aim, for $i=1,2,\dots$, let us define $\Gamma^{(i)}$ as the set of the points $s\in L_\varepsilon^*$ satisfying the estimates

$$\begin{aligned}\left\|f(x(t))-f(x(s))-f'(x(s))[x(t)-x(s)]-\frac{f''(x(s))}{2}[x(t)-x(s)]^2\right\|&\le\\\le i|x(t)-x(s)|^3\end{aligned}\tag{4.1}$$

$$\|f'(x(t)) - f'(x(s)) - f''(x(s))[x(t) - x(s)]\| \le i|x(t) - x(s)|^2 \tag{4.2}$$

$$\|f''(x(t)) - f''(x(s))\| \le i|x(t) - x(s)| \tag{4.3}$$

for all $t \in L^*_\varepsilon$ such that $|t - s| \le (b - a)/i$.

Obviously one has

$$\Gamma^{(i)} \subset \Gamma^{(i+1)} \; (\subset L^*_\varepsilon)$$

for all $i$. Moreover, we can verify that

$$\cup_i \, \Gamma^{(i)} = L^*_\varepsilon. \tag{4.4}$$

Indeed, if

$$s \in L^*_\varepsilon$$

then, by Theorem 3.1 and since the Lipschitz function $\Psi_s$ satisfies $\Psi_s(s) = 0$, we have

$$\begin{aligned}\left\| f(x(t)) - f(x(s)) - f'(x(s))[x(t) - x(s)] - \frac{f''(x(s))}{2}[x(t) - x(s)]^2 \right\| &\le \frac{\|\Sigma_s(t)\|}{|\gamma_1(s) \cdot u|} \\ &\le \frac{\mathrm{Lip}(\gamma_0)^2\,\mathrm{Lip}(\Psi_s)}{|\gamma_1(s) \cdot u|} \left| \int_s^t \left( \int_s^\rho |\sigma - s| d\sigma \right) d\rho \right| \\ &= A(s)\,|t - s|^3\end{aligned}$$

for all $t \in L^*_\varepsilon \cap I_s$, where

$$A(s) := \frac{\mathrm{Lip}(\gamma_0)^2\,\mathrm{Lip}(\Psi_s)}{6\,|\gamma_1(s) \cdot u|}.$$

Since

$$\frac{x(t) - x(s)}{t - s} \to x'(s) \qquad (\text{as } t \to s)$$

and $x'(s) \ne 0$ by Proposition 3.1, it follows that

$$\left| \frac{x(t) - x(s)}{t - s} \right| \ge \frac{|x'(s)|}{2} > 0 \tag{4.5}$$

provided $|t - s|$ is small enough. Then

$$\begin{aligned}\left\| f(x(t)) - f(x(s)) - f'(x(s))[x(t) - x(s)] - \frac{f''(x(s))}{2}[x(t) - x(s)]^2 \right\| \le \\ \le \frac{8A(s)}{|x'(s)|^3} |x(t) - x(s)|^3\end{aligned} \tag{4.6}$$

whenever $t \in L^*_\varepsilon$ and $|t - s|$ is small enough.

Analogously, from the statement (2) of Corollary 3.1 we get

$$\begin{aligned}\|f'(x(t)) - f'(x(s)) - f''(x(s))[x(t) - x(s)]\| &\le \frac{1}{|\gamma_1(t) \cdot u|\,|\gamma_1(s) \cdot u|} \left\| \int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|\Psi_s(\sigma) d\sigma \right\| \\ &\le \frac{\mathrm{Lip}(\gamma_0)\,\mathrm{Lip}(\Psi_s)}{|\gamma_1(t) \cdot u|\,|\gamma_1(s) \cdot u|} \left| \int_s^t |\sigma - s| d\sigma \right| \\ &= \frac{B(s)}{|\gamma_1(t) \cdot u|} |t - s|^2\end{aligned}$$

for all $t \in L^*_\varepsilon \cap I_s$, where

$$B(s) := \frac{\mathrm{Lip}(\gamma_0)\,\mathrm{Lip}(\Psi_s)}{2\,|\gamma_1(s)\cdot u|}.$$

Since

$$\gamma_1(t) \to \gamma_1(s) \qquad (\text{as } t \to s)$$

and $\gamma_1(s)\cdot u \neq 0$ by Proposition 3.1, one also has

$$|\gamma_1(t)\cdot u| \geq \frac{|\gamma_1(s)\cdot u|}{2} > 0 \tag{4.7}$$

provided $|t-s|$ is small enough. Recalling (4.5), we obtain that the following inequality holds

$$\|f'(x(t)) - f'(x(s)) - f''(x(s))[x(t)-x(s)]\| \leq \frac{8\,B(s)}{|\gamma_1(s)\cdot u|\,|x'(s)|^2}|x(t)-x(s)|^2 \tag{4.8}$$

on condition that $t \in L^*_\varepsilon$ and $|t-s|$ is small enough.

Since $\mu|I_s$ is Lipschitz and by (4.7), it follows that the map

$$t \mapsto \frac{[\gamma_1(t)\wedge\mu(t)]\llcorner u}{[\gamma_1(t)\cdot u]^3}$$

is Lipschitz in a neighborhood of $s$. Then, by also recalling Proposition 3.1, a number $C(s)$ has to exist such that

$$\|f''(x(t)) - f''(x(s))\| \leq C(s)\,|t-s|$$

provided $t \in L^*_\varepsilon$ and $|t-s|$ is small enough. By (4.5) one has

$$\|f''(x(t)) - f''(x(s))\| \leq \frac{2\,C(s)}{|x'(s)|}\,|x(t)-x(s)| \tag{4.9}$$

whenever $t \in L^*_\varepsilon$ and $|t-s|$ is small enough.

Now (4.6), (4.8) and (4.9) imply that $s \in \Gamma^{(i)}$, for $i$ big enough. Hence (4.4) follows.

As a consequence of (4.4), we are reduced to verify that

$$\gamma_0(\Gamma^{(i)}) \text{ is } C^3\text{-rectifiable (for all } i). \tag{4.10}$$

In order to prove such an assertion, let us firstly set

$$\begin{aligned} a_j^{(i)} &:= a + \frac{(b-a)j}{i} \qquad (j = 0,\dots,i);\\ \Gamma_j^{(i)} &:= \Gamma^{(i)} \cap [a_j^{(i)}, a_{j+1}^{(i)}] \qquad (j = 0,\dots,i-1);\\ F_j^{(i)} &:= \overline{x(\Gamma_j^{(i)})} \qquad (j = 0,\dots,i-1). \end{aligned}$$

If consider any couple

$$\xi, \eta \in F_j^{(i)}$$

then there are two sequences $\{s_h\}, \{t_h\} \subset \Gamma_j^{(i)}$ such that

$$\lim_{h\to\infty} x(s_h) = \xi, \qquad \lim_{h\to\infty} x(t_h) = \eta.$$

Since (4.1), (4.2) and (4.3) hold with

$$s = s_h, \qquad t = t_h$$

by letting $h \to \infty$ we get

$$\left\| f(\eta) - f(\xi) - f'(\xi)(\eta - \xi) - \frac{f''(\xi)}{2}(\eta - \xi)^2 \right\| \le i\,|\eta - \xi|^3;$$

$$\|f'(\eta) - f'(\xi) - f''(\xi)(\eta - \xi)\| \le i\,|\eta - \xi|^2;$$

$$\|f''(\eta) - f''(\xi)\| \le i\,|\eta - \xi|.$$

Then $f|F_j^{(i)}$ can be extended to a map of class $C^{2,1}$

$$f_j^{(i)} : \mathbf{R} \to (\mathbf{R}u)^\perp$$

by invoking the Whitney extension Theorem [13, Ch. VI, §2.3].

Finally, the Lusin type result [9, §3.1.15] implies that $\gamma_0(\Gamma_j^{(i)})$ has to be $C^3$-rectifiable (compare [1, Proposition 3.2]). Hence (4.10) follows.

## 5. Approximately differentiable absolute curvature of a one-dimensional $C^3$-rectifiable set

First of all, we are going to extend the notion of absolute curvature to any one-dimensional $C^2$-rectifiable subset $P$ of $\mathbf{R}^{1+k}$. To this aim, let us consider a "$C^2$-covering of $P$", namely a countable family

$$\mathcal{A} = \{C_i\}$$

where the $C_i$ are compact curves of class $C^2$, embedded in the base space and such that

$$\mathcal{H}^1\left(P \backslash \cup_i C_i\right) = 0.$$

The assertion (1) in the following proposition and the related Remark 5.1 provide the argument proving the well-posedness of Definition 5.1 below.

**Proposition 5.1.** *Let $\varphi, \psi : \mathbf{R} \to \mathbf{R}^{1+k}$ be maps of class $C^2$ and $x_0$ be a density point of*

$$F := \{x \in \mathbf{R} \,|\, \varphi(x) = \psi(x)\}.$$

*Then*

(1) *One has $\varphi'(x_0) = \psi'(x_0)$ and $\varphi''(x_0) = \psi''(x_0)$;*
(2) *In the particular case when $\varphi$ and $\psi$ are of class $C^3$, also $\varphi'''(x_0) = \psi'''(x_0)$ holds.*

*Proof.* Let $F^*$ denote the set of the density points of $F$. The statement follows at once, by observing that $F^* \subset F$ and $\mathcal{L}^1(F \backslash F^*) = 0$, hence every point in $F^*$ is a limit point of $F^*$. $\square$

*Remark* 5.1. Based on Proposition 5.1(1), we can easily verify that:

- If $x$ is a density point of both $P \cap C_i$ and $P \cap C_j$, then the absolute curvatures of $C_i$ and $C_j$ coincide at $x$. Hence, denoting by $(P \cap C_i)^*$ the set of the density points of $P \cap C_i$, the following function results to be well-defined:

$$\alpha_P^{\mathcal{A}} : \cup_i (P \cap C_i)^* \to \mathbf{R}, \quad x \mapsto \text{ the absolute curvature of } C_{i(x)} \text{ at } x$$

where $i(x)$ is any index such that $x \in (P \cap C_{i(x)})^*$. Also observe that

$$\mathcal{H}^1\left(P \setminus \cup_i (P \cap C_i)^*\right) = \mathcal{H}^1\left(P \setminus \cup_i (P \cap C_i)\right) = \mathcal{H}^1\left(P \setminus \cup_i C_i\right) = 0$$

by a well-known Lebesgue's result.

- If $\mathcal{B}$ is another $C^2$-covering of $P$, then $\alpha_P^{\mathcal{A}}$ and $\alpha_P^{\mathcal{B}}$ are representatives of the same measurable function, with domain $P$.

**Definition 5.1.** *The measurable real-valued function with domain $P$ and having $\alpha_P^{\mathcal{A}}$ as a representative (compare Remark 5.1) is said to be the "absolute curvature" of $P$ and is denoted by $\alpha_P$.*

As one expect, when $P$ is $C^3$-rectifiable, the following result holds.

**Proposition 5.2.** *If $P$ is $C^3$-rectifiable, then $\alpha_P$ is approximately differentiable, namely:*

(1) *For any given $C^3$-covering $\mathcal{A} = \{C_i\}$ of $P$, the function $\alpha_P^{\mathcal{A}}$ is approximately differentiable at every point in $(P \cap C_i)^*$, for all $i$;*
(2) *If $\mathcal{A}$ and $\mathcal{B}$ are $C^3$-coverings of $P$, then one has*

$$apD\alpha_P^{\mathcal{A}} = apD\alpha_P^{\mathcal{B}}, \text{ a.e. in } P.$$

*Proof.* (1) Let us consider any point

$$a \in (P \cap C_{i_0})^*.$$

Without loss of generality, we can assume that $C_{i_0}$ is the graph of a function of class $C^3$

$$h : I \to \mathbf{R}^k$$

where $I$ is a closed interval centered at 0 and with $a = (0, h(0))$. Then set

$$U := I^\circ \times \mathbf{R}^k$$

and let $g : U \to \mathbf{R}$ be defined as the function mapping $(t, v) \in U$ to the absolute curvature of $C_{i_0}$ at $(t, h(t))$, that is

$$g(t,v) = \frac{(\|h''(t)\|^2 + \|h'(t)\|^2\|h''(t)\|^2 - [h'(t) \cdot h''(t)]^2)^{1/2}}{(1 + \|h'(t)\|^2)^{3/2}}, \quad (t,v) \in U \tag{5.1}$$

as it follows at once from the formulas (6.3) and (6.4) below, with $\gamma(t) := (t, h(t))$. Obviously, the function $g$ is differentiable at $a$. Moreover, since

$$(P \cap C_{i_0})^* \subset E := \left\{x \in \cup_i (P \cap C_i)^* \,\middle|\, \alpha_P^{\mathcal{A}}(x) = g(x)\right\}$$

by the definition of $\alpha_P^{\mathcal{A}}$, the set $E$ has density 1 at $a$. According to [9, §3.2.16], the function $\alpha_P^{\mathcal{A}}$ is approximately differentiable at $a$ and one has

$$\mathrm{ap}D\alpha_P^{\mathcal{A}}(a) = Dg(a)|\mathbf{R}\tau, \text{ with } \tau := (1, h'(0)). \tag{5.2}$$

(2) The statement follows easily from (5.1) and (5.2), by recalling Proposition 5.1. $\square$

## 6. An application: 2-storey towers of one-dimensional generalized Gauss graphs

First of all, recall from [2, 5] that a "one-dimensional generalized Gauss graph (based in $\mathbf{R}^N$)" is an integral current (see [9, 11, 12])

$$T \in \mathbf{I}_1(\mathbf{R}^N \times \mathbf{R}^N)$$

such that:

(i) The carrier $G$ of $T$ is equivalent in measure to a subset of $\mathbf{R}^N \times \mathbf{S}^{N-1}$, i.e.

$$\mathcal{H}^1\left(G \backslash (\mathbf{R}^N \times \mathbf{S}^{N-1})\right) = 0;$$

(ii) If $\varphi$ denotes the following 1-form in $\mathbf{R}^N \times \mathbf{R}^N$

$$(x, y) \mapsto \sum_{j=1}^{N} y_j dx_j$$

and $*$ is the usual Hodge star operator in $\mathbf{R}^N$, then one has:

- $T(*\varphi \llcorner \omega) = 0$ for all smooth $(N-2)$-forms with compact support in $\mathbf{R}^N \times \mathbf{R}^N$;
- $T(g\varphi) \geq 0$ for all nonnegative smooth functions $g$ with compact support in $\mathbf{R}^N \times \mathbf{R}^N$.

Now we can introduce the notion giving the title to this section.

**Definition 6.1.** *A "2-storey tower of one-dimensional generalized Gauss graphs (based in $\mathbf{R}^{1+k}$)" is a one-dimensional generalized Gauss graph $T$ based in $\mathbf{R}^{1+k} \times \mathbf{R}^{1+k}$ such that pushing forward $T$ by the projection map*

$$\mathbf{R}^{1+k} \times \mathbf{R}^{1+k} \times \mathbf{R}^{1+k} \times \mathbf{R}^{1+k} \to \mathbf{R}^{1+k} \times \mathbf{R}^{1+k}, \qquad (x, y, z, w) \mapsto (x, y)$$

*produces a one-dimensional generalized Gauss graphs based in $\mathbf{R}^{1+k}$.*

*Example (the smooth case).* The situation to keep in mind, in order to understand the meaning of Definition 6.1, is the following one. Consider a regular 1-1 curve of class $C^3$

$$\gamma : [a, b] \to \mathbf{R}^{1+k}$$

and set

$$\Gamma := (\gamma_0, \gamma_1, \gamma_2) : [a, b] \to \mathbf{R}^{4(1+k)}$$

where

$$\gamma_0 := \gamma, \qquad \gamma_1 := \frac{\gamma_0'}{\|\gamma_0'\|} = \frac{\gamma'}{\|\gamma'\|}$$

and

$$\gamma_2 := (\gamma_{2\top}, \gamma_{2\perp}) := \frac{(\gamma_0', \gamma_1')}{\|(\gamma_0', \gamma_1')\|} = \frac{(\gamma', (\gamma'/\|\gamma'\|)')}{\|(\gamma', (\gamma'/\|\gamma'\|)')\|}. \tag{6.1}$$

We can define the multiplicity-one current

$$T := [\![G, \eta, 1]\!] \in \mathbf{I}_1\left(\mathbf{R}^{4(1+k)}\right)$$

with the carrier

$$G := \Gamma([a, b])$$

and the orientation

$$\eta : G \to \mathbf{R}^{4(1+k)}, \qquad \eta(Q) := \Gamma'(\Gamma^{-1}(Q))/\|\Gamma'(\Gamma^{-1}(Q))\|.$$

Then $T$ is a 2-storey tower of one-dimensional generalized Gauss graphs based in $\mathbf{R}^{1+k}$.

Observe that:

- The equalities (1.1) and (1.2), with $\omega := 1$, are obviously satisfied;
- Since $(*\gamma') \llcorner \gamma' = 0$, one has

$$(*\gamma_1) \llcorner \frac{\gamma_{2\perp}}{\|\gamma_{2\top}\|} = \frac{1}{\|\gamma'\|^2}(*\gamma') \llcorner (\gamma'/\|\gamma'\|)' = \frac{1}{\|\gamma'\|^3}(*\gamma') \llcorner \gamma'' \tag{6.2}$$

  where $*$ denotes the Hodge star operator in $\mathbf{R}^{1+k}$. Also, if

$$u, v \in \mathbf{R}^{1+k}, \quad \|u\| = 1$$

  and $\{e_j\}$ is an orthonormal basis of $\mathbf{R}^{1+k}$, then

$$(*u) \llcorner v = (e_2 \wedge \cdots \wedge e_{1+k}) \llcorner \sum_{j=1}^{1+k} v_j e_j = \sum_{j=2}^{1+k} v_j (e_2 \wedge \cdots \wedge e_{1+k}) \llcorner e_j$$

  where $v_j := v \cdot e_j$. Hence

$$\|(*u) \llcorner v\|^2 = \sum_{j=2}^{1+k} v_j^2 = \|v\|^2 - (v \cdot u)^2 = \|v \wedge u\|^2. \tag{6.3}$$

  By recalling the formula (8.4.13.1) of [3], we then obtain the following expression for the absolute curvature $\alpha_\gamma$ of $\gamma$

$$\alpha_\gamma = \frac{\|\gamma' \wedge \gamma''\|}{\|\gamma'\|^3} = \frac{\|(*\gamma') \llcorner \gamma''\|}{\|\gamma'\|^3} \tag{6.4}$$

  which can be written in terms of $\gamma_1$ and $\gamma_2$, as it follows from (6.2):

$$\alpha_\gamma = \frac{\|(*\gamma_1) \llcorner \gamma_{2\perp}\|}{\|\gamma_{2\top}\|}.$$

  In the particular case when $k = 1$, (6.2) provides the following formula for the signed curvature $\kappa_\gamma$ of $\gamma$ (compare [8, §1-5, Exercise 12]):

$$\kappa_\gamma = \frac{\gamma'' \cdot (*\gamma')}{\|\gamma'\|^3} = \frac{\gamma_{2\perp} \cdot (*\gamma_1)}{\|\gamma_{2\top}\|}.$$

*Remark* 6.1. In the smooth case the following representation formula for $\eta$ holds.

**Proposition 6.1.** *Let $\gamma$ be as in Example above and $k = 1$. Then one has*

$$\frac{\eta}{\|\eta_0\|} \circ \Gamma = \Big(\gamma_1, \kappa_\gamma(*\gamma_1), \frac{\kappa_\gamma}{(1+\kappa_\gamma^2)^{3/2}}\Big[(1+\kappa_\gamma^2)(*\gamma_1) - \frac{\kappa_\gamma'}{\|\gamma'\|}\gamma_1\Big],$$
$$\frac{1}{(1+\kappa_\gamma^2)^{3/2}}\Big[\frac{\kappa_\gamma'}{\|\gamma'\|}(*\gamma_1) - (1+\kappa_\gamma^2)\kappa_\gamma^2\gamma_1\Big]\Big).$$

*Proof.* Denote by $\lambda$ the reparametrization of $\gamma([a,b])$ by arc length satisfying $\lambda(0)=\gamma(a)$. Then, by adopting the same notation as in Example, we have

$$\lambda_0' = \lambda_1, \qquad \lambda_1' = \kappa_\lambda(*\lambda_1).$$

Moreover

$$\lambda_2 = \frac{(\lambda',\lambda'')}{\|(\lambda',\lambda'')\|} = \frac{(\lambda_1,\kappa_\lambda(*\lambda_1))}{(1+\kappa_\lambda^2)^{1/2}}$$

by (6.1), hence

$$\begin{aligned}\lambda_2' &= \frac{(1+\kappa_\lambda^2)^{1/2}(\lambda_1', \kappa_\lambda'(*\lambda_1)+\kappa_\lambda(*\lambda_1')) - (1+\kappa_\lambda^2)^{-1/2}\kappa_\lambda\kappa_\lambda'(\lambda_1,\kappa_\lambda(*\lambda_1))}{1+\kappa_\lambda^2}\\ &= \frac{(\kappa_\lambda(1+\kappa_\lambda^2)(*\lambda_1)-\kappa_\lambda\kappa_\lambda'\lambda_1\,,\,(1+\kappa_\lambda^2)\kappa_\lambda'(*\lambda_1)-(1+\kappa_\lambda^2)\kappa_\lambda^2\lambda_1-\kappa_\lambda^2\kappa_\lambda'(*\lambda_1))}{(1+\kappa_\lambda^2)^{3/2}}\end{aligned}$$

namely

$$\lambda_{2\top}' = \frac{\kappa_\lambda}{(1+\kappa_\lambda^2)^{3/2}}\left((1+\kappa_\lambda^2)(*\lambda_1)-\kappa_\lambda'\lambda_1\right)$$

and

$$\lambda_{2\perp}' = \frac{1}{(1+\kappa_\lambda^2)^{3/2}}\left(\kappa_\lambda'(*\lambda_1)-(1+\kappa_\lambda^2)\kappa_\lambda^2\lambda_1\right).$$

By recalling the equalities

$$\lambda=\gamma\circ\tau, \qquad \kappa_\lambda=\kappa_\gamma\circ\tau$$

where $\tau$ denotes the inverse function of $t\mapsto\int_a^t\|\gamma'\|$, it follows immediately that

$$\lambda_0' = \frac{\gamma'}{\|\gamma'\|}\circ\tau=\gamma_1\circ\tau, \qquad \lambda_1'=\kappa_\lambda(*\lambda_1)=[\kappa_\gamma(*\gamma_1)]\circ\tau,$$

$$\lambda_{2\top}' = \left[\frac{\kappa_\gamma}{(1+\kappa_\gamma^2)^{3/2}}\left((1+\kappa_\gamma^2)(*\gamma_1)-\frac{\kappa_\gamma'}{\|\gamma'\|}\gamma_1\right)\right]\circ\tau$$

and

$$\lambda_{2\perp}' = \left[\frac{1}{(1+\kappa_\gamma^2)^{3/2}}\left(\frac{\kappa_\gamma'}{\|\gamma'\|}(*\gamma_1)-(1+\kappa_\gamma^2)\kappa_\gamma^2\gamma_1\right)\right]\circ\tau.$$

From

$$\frac{\eta}{\|\eta_0\|}\circ\Gamma\circ\tau=(\lambda_0',\lambda_1',\lambda_{2\top}',\lambda_{2\perp}')$$

we finally get the conclusion. □

The following result summarizes some properties of a 2-storey tower of one-dimensional generelized Gauss graphs based in $\mathbf{R}^{1+k}$ (shortly referred as "tower", in the sequel). In particular it proves that the carrier is projected to the base space into a $C^3$-rectifiable set and extends the representation formula stated in Proposition 6.1 to the case of a tower. Recall from [9, §4.2.25] that an indecomposable one-dimensional integral current has always multiplicity one.

**Proposition 6.2.** *For a tower $T=[\![G,\eta,\theta]\!]$, the following facts hold.*

(1) *There exists a finite sequence of indecomposable towers* $T_j = [\![G_j, \eta_j, 1]\!]$ *such that*

$$T = \sum_j T_j$$

*and*

$$\mathbf{M}(T) = \sum_j \mathbf{M}(T_j), \qquad \mathbf{M}(\partial T) = \sum_j \mathbf{M}(\partial T_j). \tag{6.5}$$

*Moreover one has*

$$G = \cup_j G_j, \qquad \eta | G_j = \eta_j, \qquad \theta(x) = \#\{j \mid x \in G_j\} \tag{6.6}$$

*where the equality sign has to be intended "modulo null-measure sets".*

(2) *The projection of* $G$ *to the base space* $\mathbf{R}^{1+k}$ *is a* $C^3$*-rectifiable one-dimensional set.*

(3) *If* $T$ *is indecomposable, then there exists a Lipschitz map*

$$\Gamma := (\gamma_0, \gamma_1, \gamma_{2\top}, \gamma_{2\perp}), \qquad \gamma_0, \gamma_1, \gamma_{2\top}, \gamma_{2\perp} : [0, \mathbf{M}(T)] \to \mathbf{R}^{1+k}$$

*such that*

(i) $\Gamma | [0, \mathbf{M}(T))$ *is injective,* $\Gamma_\# [\![0, \mathbf{M}(T)]\!] = T$ *and* $\|\Gamma'(t)\| = 1$ *for a.e.* $t \in [0, \mathbf{M}(T)]$.

(ii) *The equalities (1.1) and (1.2), with* $\omega := 1$*, are satisfied at a.e.* $t \in [0, \mathbf{M}(T)]$.

*Proof.* (1) Recalling [9, §4.2.25], we can find a sequence of indecomposable currents $T_j \in \mathbf{I}_1(\mathbf{R}^{4(1+k)})$ such that

$$T = \sum_j T_j, \qquad \mathbf{N}(T) = \sum_j \mathbf{N}(T_j).$$

The number of the $T_j$ has to be finite, in that

$$\mathbf{N}(T_j) \geq \begin{cases} \mathbf{M}(T_j) \geq 2\pi & \text{if } \partial T_j = 0, \\ \mathbf{M}(\partial T_j) = 2 & \text{if } \partial T_j \neq 0 \end{cases}$$

by [4, Theorem 4.1] and [9, §4.2.25]. Moreover from

$$\sum_j \mathbf{N}(T_j) = \mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T) \leq \sum_j \mathbf{M}(T_j) + \sum_j \mathbf{M}(\partial T_j) = \sum_j \mathbf{N}(T_j)$$

and

$$\mathbf{M}(T) \leq \sum_j \mathbf{M}(T_j), \qquad \mathbf{M}(\partial T) \leq \sum_j \mathbf{M}(\partial T_j)$$

we get at once (6.5). Now the equalities (6.6) follow from [6, Proposition 4.2]. As a consequence of such equalities, $T_j$ inherits from $T$ the geometric properties characterizing a tower, compare [5, Proposition 4.1], hence each $T_j$ has to be itself a tower.

(2) Let $\{T_j\}$ be as in (1) and indicate with $p$ the projection to the base space, i.e.

$$p : \mathbf{R}^{4(1+k)} \to \mathbf{R}^{1+k}, \qquad p(x, y, z, w) := x.$$

From the first equality in (6.6), we get

$$pG = \cup_j pG_j$$

where each $pG_j$ has to be $C^3$-rectifiable, by the assertion (3) and Theorem 1.1.

(3) Assertion (i) follows from the structure theorem in [9, §4.2.25], while (ii) is a consequence of [5, Proposition 4.1]. □

**Proposition 6.3.** *Let $T$ be a tower and $T_j$ be any fixed indecomposable tower of the sequence mentioned in Proposition 6.2(1). Denote with $P$ (resp. $P_j$) the projection of the carrier of $T$ (resp. $T_j$) to the base space and consider a $C^3$-covering $\mathcal{A}$ of $P$ (it exists by Proposition 6.2(2)!). Then one has*

(1) *$P_j \subset P$ (modulo null-measure sets) and $\alpha_P^{\mathcal{A}} | P_j = \alpha_{P_j}^{\mathcal{A}}$.*

*Moreover, if*

$$\Gamma := (\gamma_0, \gamma_1, \gamma_{2\top}, \gamma_{2\perp}), \qquad \gamma_0, \gamma_1, \gamma_{2\top}, \gamma_{2\perp} : [0, \mathbf{M}(T_j)] \to \mathbf{R}^{1+k}$$

*is a Lipschitz parametrization of $T_j$ with the properties stated in Proposition 6.2(3), then the following equalities*

(2) $\alpha_P^{\mathcal{A}} \circ \gamma_0 = \frac{\|(*\gamma_1) \llcorner \gamma_{2\perp}\|}{\|\gamma_{2\top}\|}$

(3) $\langle (apD\alpha_P^{\mathcal{A}}) \circ \gamma_0 \, , \, \gamma_0' \rangle = \left( \frac{\|(*\gamma_1) \llcorner \gamma_{2\perp}\|}{\|\gamma_{2\top}\|} \right)'$

*hold almost everywhere in*

$$E := \{ t \in [0, \mathbf{M}(T_j)] \,|\, \gamma_0'(t) \text{ exists and } \gamma_0'(t) \neq 0 \}.$$

*Proof.* (1) The inclusion $P_j \subset P$ (modulo null-measure sets) follows trivially from (6.6). Hence $\mathcal{A}$ covers $P_j$ too and the conclusion follows from the definition of absolute curvature given in §5.

Now on, we will concentrate on a (arbitrarily chosen) curve of $\mathcal{A}$. Without affecting the generality of our argument, we will assume that such a curve is the graph of a function of class $C^3$

$$f : I \to (\mathbf{R}u)^\perp, \qquad (u \in \mathbf{S}^k, \, I \text{ compact interval}).$$

Preliminary to proving (2) and (3), we have to show that

$$\gamma_0(t) \in [G_f \cap \gamma_0 ([0, \mathbf{M}(T_j)])]^* \, , \text{ for a.e. } t \in \gamma_0^{-1}(G_f) \cap E. \tag{6.7}$$

In order to prove (6.7), let us first observe that

$$\gamma_0 \left( \gamma_0^{-1}(G_f) \cap E \right) \sim G_f \cap \gamma_0 ([0, \mathbf{M}(T_j)])$$

where $\sim$ denotes the equivalence relation of measurable sets (with respect to the measure $\mathcal{H}^1$). Also one has

$$G_f \cap \gamma_0 ([0, \mathbf{M}(T_j)]) \sim [G_f \cap \gamma_0 ([0, \mathbf{M}(T_j)])]^*$$

by [10, Theorem 16.2]. As a consequence, setting

$$Z := \left\{ t \in \gamma_0^{-1}(G_f) \cap E \,|\, \gamma_0(t) \notin [G_f \cap \gamma_0 ([0, \mathbf{M}(T_j)])]^* \right\}$$

we find

$$\gamma_0(Z) = \gamma_0 \left( \gamma_0^{-1}(G_f) \cap E \right) \setminus [G_f \cap \gamma_0 ([0, \mathbf{M}(T_j)])]^* \sim \emptyset.$$

Since $\gamma_0' \neq 0$ everywhere in $Z$, it follows that $Z$ is a measure zero set. This concludes the proof of (6.7).

(2) We are reduced to prove that

$$\frac{\|f''\|^2(1+\|f'\|^2)-(f'\cdot f'')^2}{(1+\|f'\|^2)^3}\bigg|_{\gamma_0\cdot u}=\frac{\|(*\gamma_1)\llcorner\gamma_{2\perp}\|^2}{\|\gamma_{2\top}\|^2}\tag{6.8}$$

a.e. in $\gamma_0^{-1}(G_f)\cap E$, where $G_f$ denotes the graph of $f$, i.e.

$$G_f:=\{(xu,f(x))\,|\,x\in I\}.$$

Indeed the left hand side of (6.8) is just the square of the curvature of $G_f$ at its point of abscissa $\gamma_0\cdot u$, as one easily obtains from (6.4) with $\gamma(x)=(x,f(x))$ by also recalling (6.3).

Define $\omega:=1$ and

$$L:=\gamma_0^{-1}(G_f)\cap\{t\in[0,\mathbf{M}(T_j)]\,|\,\gamma_0'(t)\text{ and }\gamma_1'(t)\text{ exist, }\gamma_0'(t)\neq0,\text{ (1.1) and (1.2) hold}\}\,.$$

Then, by recalling the notation and the arguments in §2 and §3, we will prove the following facts which immediately allow to conclude:

Fact 1: The formula (6.8) holds in

$$L^*:=\cup_{\varepsilon>0}L^*_\varepsilon;$$

Fact 2: One has

$$L^*\subset\gamma_0^{-1}(G_f)\cap E\quad\text{ and }\quad\mathcal{H}^1\left(\gamma_0^{-1}(G_f)\cap E\backslash L^*\right)=0.$$

*Proof of Fact 1.* First of all observe that the right hand side member of (6.8) makes sense, in that $L^*\subset L$ and (3.1) holds. Moreover, with a standard computation based on Proposition 3.1, we find the following formula in $L^*$:

$$\frac{\|f''\|^2(1+\|f'\|^2)-(f'\cdot f'')^2}{(1+\|f'\|^2)^3}\bigg|_{\gamma_0\cdot u}=\frac{\|(\gamma_1\wedge\gamma_{2\perp})\llcorner u\|^2-([(\gamma_1\wedge\gamma_{2\perp})\llcorner u]\cdot\gamma_1)^2}{(\gamma_1\cdot u)^2\|\gamma_{2\top}\|^2}.$$

Therefore, by also recalling (6.3), we remain to show that

$$\frac{\|[(\gamma_1\wedge\gamma_{2\perp})\llcorner u]\wedge\gamma_1\|}{|\gamma_1\cdot u|}=\|\gamma_1\wedge\gamma_{2\perp}\|,\ \text{ a.e. in } L^*.\tag{6.9}$$

In order to prove (6.9), assume $\gamma_1\wedge\gamma_{2\perp}\neq0$ (otherwise there is nothing to prove!) and denote by $\{\varepsilon_1,\varepsilon_2\}$ an orthonormal basis of $\operatorname{span}\{\gamma_1,\gamma_{2\perp}\}$ such that

$$\varepsilon_1=\frac{\tilde u}{\|\tilde u\|}$$

where $\tilde u$ is the projection of $u$ to $\operatorname{span}\{\gamma_1,\gamma_{2\perp}\}$ ($\tilde u\neq0$, by Proposition 3.1). Then one has

$$\|[(\gamma_1\wedge\gamma_{2\perp})\llcorner u]\wedge\gamma_1\|=\|[(\gamma_1\wedge\gamma_{2\perp})\llcorner\tilde u]\wedge\gamma_1\|=\|\tilde u\|\|\gamma_1\wedge\gamma_{2\perp}\|\|[(\varepsilon_1\wedge\varepsilon_2)\llcorner\varepsilon_1]\wedge\gamma_1\|$$

where

$$[(\varepsilon_1\wedge\varepsilon_2)\llcorner\varepsilon_1]\wedge\gamma_1=\varepsilon_2\wedge\gamma_1=(\gamma_1\cdot\varepsilon_1)\,\varepsilon_2\wedge\varepsilon_1.$$

It follows that

$$\|[(\gamma_1\wedge\gamma_{2\perp})\llcorner u]\wedge\gamma_1\|=\|\tilde u\|\|\gamma_1\wedge\gamma_{2\perp}\||\gamma_1\cdot\varepsilon_1|=\|\gamma_1\wedge\gamma_{2\perp}\||\gamma_1\cdot\tilde u|$$

hence (6.9).

*Proof of Fact 2.* One has

$$L^*=\cup_{\varepsilon>0}L^*_\varepsilon\subset L\subset\gamma_0^{-1}(G_f)\cap E$$

and

$$\mathcal{H}^1\left(\gamma_0^{-1}(G_f)\cap E\setminus L\right)=0$$

by Proposition 6.2(3). Then

$$\mathcal{H}^1\left(\gamma_0^{-1}(G_f)\cap E\setminus L^*\right)=\mathcal{H}^1\left(\gamma_0^{-1}(G_f)\cap E\setminus L\right)+\mathcal{H}^1(L\backslash L^*)\leq\mathcal{H}^1(L\backslash L^*_\varepsilon)\leq\varepsilon$$

for all $\varepsilon$. The conclusion follows from the arbitrariness of $\varepsilon$.

(3) For simplicity set

$$\Omega:=\{t\in L^*\,|\,\gamma_0(t)\in[G_f\cap\gamma_0([0,\mathbf{M}(T_j)])]^*,\ \alpha_P^{\mathcal{A}}\circ\gamma_0(t)=\rho(t),\ \rho'(t)\text{ exists}\}$$

with

$$\rho:=\frac{\|(*\gamma_1)\llcorner\,\gamma_{2\perp}\|}{\|\gamma_{2\top}\|}$$

and observe that

$$\Omega\sim L^*\sim\gamma_0^{-1}(G_f)\cap E \tag{6.10}$$

by (6.7), Fact 2 and (2) above. Then, given

$$t\in\Omega\subset L^*$$

and racalling that $L^*$ has density one at $t$, we can find $\{t_n\}\subset\Omega$ such that

$$t_n\to t,\qquad t_n\neq t\text{ for all }n.$$

We get

$$\rho'(t)=\lim_n\frac{\rho(t_n)-\rho(t)}{t_n-t}=\lim_n\frac{\alpha_P^{\mathcal{A}}\circ\gamma_0(t_n)-\alpha_P^{\mathcal{A}}\circ\gamma_0(t)}{t_n-t}$$

where

$$\alpha_P^{\mathcal{A}}\circ\gamma_0(t_n)-\alpha_P^{\mathcal{A}}\circ\gamma_0(t)=\langle\mathrm{ap}D\alpha_P^{\mathcal{A}}(\gamma_0(t)),\gamma_0(t_n)-\gamma_0(t)\rangle+o(\gamma_0(t_n)-\gamma_0(t))$$

by the definition of $\mathrm{ap}D\alpha_P^{\mathcal{A}}$. Thus the formula (3) holds in $\Omega$, hence almost everywhere in $\gamma_0^{-1}(G_f)\cap E$, by (6.10). □

In the final proposition, devoted to the case $k=1$, we extend the representation formula for the orientation of a smooth tower (given in Proposition 6.1) to the case of a general tower.

**Proposition 6.4.** *Let $T=[\![G,\eta,\theta]\!]$ be a tower based in $\mathbf{R}^2$ (i.e. we are assuming $k=1$) and adopt for the components of $\eta$ the notation as we used for the components of $\Gamma$, namely*

$$\eta=(\eta_0,\eta_1,\eta_{2\top},\eta_{2\perp}):G\to\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2\times\mathbf{R}^2.$$

(1) *If $T$ is indecomposable and*

$$\Gamma:=(\gamma_0,\gamma_1,\gamma_{2\top},\gamma_{2\perp}),\qquad\gamma_0,\gamma_1,\gamma_{2\top},\gamma_{2\perp}:[0,\mathbf{M}(T)]\to\mathbf{R}^2$$

*is as in Proposition 6.2(3), then almost everywhere in*

$$E:=\{t\in[0,\mathbf{M}(T)]\,|\,\gamma_0'(t)\text{ exists and }\gamma_0'(t)\neq0\}$$

*one has* $\gamma_{2\top} \neq 0$ *and*

$$\frac{\eta}{\|\eta_0\|} \circ \Gamma = \Bigg(\gamma_1, \kappa_\Gamma(*\gamma_1), \frac{\kappa_\Gamma}{(1+\kappa_\Gamma^2)^{3/2}}\Big[(1+\kappa_\Gamma^2)(*\gamma_1) - \frac{\kappa'_\Gamma}{\|\gamma'_0\|}\gamma_1\Big], \frac{1}{(1+\kappa_\Gamma^2)^{3/2}}\Big[\frac{\kappa'_\Gamma}{\|\gamma'_0\|}(*\gamma_1) - (1+\kappa_\Gamma^2)\kappa_\Gamma^2\gamma_1\Big]\Bigg)$$

*where*

$$\kappa_\Gamma := \frac{\gamma_{2\perp}\cdot(*\gamma_1)}{\|\gamma_{2\top}\|}.$$

(2) *If* $P$ *denotes the projection of* $G$ *to the base space* $\mathbf{R}^2$, $\mathcal{A}$ *is a* $C^3$*-covering of* $P$ *(existing by Proposition 6.2(2)) and define*

$$\sigma(y,w) := sign(w\cdot(*y)), \qquad y,w \in \mathbf{R}^2$$

*then the following formulae hold at* $\mathcal{H}^1 \llcorner \|\eta_0\|$*-a.e.* $(x,y,z,w)$ *in* $G$*:*

$$\frac{\eta_0}{\|\eta_0\|}(x,y,z,w) = y,$$
$$\frac{\eta_1}{\|\eta_0\|}(x,y,z,w) = \sigma(y,w)\alpha_P^{\mathcal{A}}(x)\,(*y),$$
$$\frac{\eta_{2\top}}{\|\eta_0\|}(x,y,z,w) = \frac{\sigma(y,w)\alpha_P^{\mathcal{A}}(x)}{\big[1+\alpha_P^{\mathcal{A}}(x)^2\big]^{3/2}}\Big(\Big[1+\alpha_P^{\mathcal{A}}(x)^2\Big](*y) + - \sigma(y,w)\langle apD\alpha_P^{\mathcal{A}}(x), y\rangle y\Big),$$
$$\frac{\eta_{2\perp}}{\|\eta_0\|}(x,y,z,w) = \frac{1}{\big[1+\alpha_P^{\mathcal{A}}(x)^2\big]^{3/2}}\Big(\sigma(y,w)\langle apD\alpha_P^{\mathcal{A}}(x), y\rangle(*y) + - \Big[1+\alpha_P^{\mathcal{A}}(x)^2\Big]\alpha_P^{\mathcal{A}}(x)^2\, y\Big).$$

*Proof.* First of all, observe that

$$\gamma_{2\top} \neq 0 \ \text{ a.e. in } E$$

by (ii) of Proposition 6.2(3). Moreover, one has

$$\eta\circ\Gamma = \Gamma' = (\gamma'_0, \gamma'_1, \gamma'_{2\top}, \gamma'_{2\perp}) \ \text{ a.e. in } [0, \mathbf{M}(T)]$$

by (i) of Proposition 6.2(3), hence

$$\frac{\eta}{\|\eta_0\|}\circ\Gamma = \Big(\frac{\gamma'_0}{\|\gamma'_0\|}, \frac{\gamma'_1}{\|\gamma'_0\|}, \frac{\gamma'_{2\top}}{\|\gamma'_0\|}, \frac{\gamma'_{2\perp}}{\|\gamma'_0\|}\Big) = \Big(\gamma_1, \frac{\gamma'_1}{\|\gamma'_0\|}, \frac{\gamma'_{2\top}}{\|\gamma'_0\|}, \frac{\gamma'_{2\perp}}{\|\gamma'_0\|}\Big)$$

holds a.e. in $E$, by (ii) of Proposition 6.2(3). Thus we are reduced to prove that, for $k=1$, the following formulas

$$\frac{\gamma'_1}{\|\gamma'_0\|} = \kappa_\Gamma(*\gamma_1) \tag{6.11}$$

$$\frac{\gamma'_{2\top}}{\|\gamma'_0\|} = \frac{\kappa_\Gamma}{(1+\kappa_\Gamma^2)^{3/2}}\Big[(1+\kappa_\Gamma^2)(*\gamma_1) - \frac{\kappa'_\Gamma}{\|\gamma'_0\|}\gamma_1\Big] \tag{6.12}$$

$$\frac{\gamma'_{2\perp}}{\|\gamma'_0\|} = \frac{1}{(1+\kappa_\Gamma^2)^{3/2}}\left[\frac{\kappa'_\Gamma}{\|\gamma'_0\|}(*\gamma_1) - (1+\kappa_\Gamma^2)\kappa_\Gamma^2\gamma_1\right] \tag{6.13}$$

hold a.e. in $E$.

Proof of (6.11). By (1.2) and Proposition 3.2

$$\gamma_{2\perp}\cdot\gamma_1 = 0, \text{ a.e. in } E \tag{6.14}$$

hence, by recalling the definition of $\kappa_\Gamma$, one finds

$$\frac{\gamma'_1}{\|\gamma'_0\|} = \frac{\gamma_{2\perp}}{\|\gamma_{2\top}\|} = \left[\frac{\gamma_{2\perp}}{\|\gamma_{2\top}\|}\cdot(*\gamma_1)\right](*\gamma_1) \;= \kappa_\Gamma\,(*\gamma_1), \text{ a.e. in } E.$$

Proof of (6.12). From the definition of $\kappa_\Gamma$ and (6.14), it follows that

$$|\kappa_\Gamma| = \frac{\|\gamma_{2\perp}\|}{\|\gamma_{2\top}\|}, \text{ a.e. in } E \tag{6.15}$$

hence

$$1+\kappa_\Gamma^2 = \frac{1}{\|\gamma_{2\top}\|^2}, \text{ a.e. in } E. \tag{6.16}$$

As a consequence, we easily obtain

$$\kappa_\Gamma\kappa'_\Gamma = -\frac{\gamma_{2\top}\cdot\gamma'_{2\top}}{\|\gamma_{2\top}\|^4}, \text{ a.e. in } E. \tag{6.17}$$

Define

$$Z := \{t\in E \,|\, \kappa_\Gamma(t) = 0\} \subset E$$

and observe that

- One has

$$\gamma'_{2\top}\cdot\gamma_1 = 0, \text{ a.e. in } Z \tag{6.18}$$

  by the formulas (1.1), (1.2) and (6.17).
- From the definition of $\kappa_\Gamma$ it follows that

$$\gamma_{2\perp}\cdot(*\gamma_1) = 0, \text{ in } Z$$

  hence

$$\gamma'_1\cdot(*\gamma_1) = 0, \text{ a.e in } Z$$

  by (1.2). Recalling Proposition 3.2, we get

$$\gamma'_1 = 0, \text{ a.e in } Z.$$

  Then, since

$$\gamma'_{2\top}\cdot(*\gamma_1) = [\gamma_{2\top}\cdot(*\gamma_1)]' - \gamma_{2\top}\cdot(*\gamma'_1), \text{ a.e in } [0,\mathbf{M}(T)]$$

  and

$$\gamma_{2\top}\cdot(*\gamma_1) = 0, \text{ a.e in } E \tag{6.19}$$

  by (1.1) and (1.2), we conclude

$$\gamma'_{2\top}\cdot(*\gamma_1) = 0, \text{ a.e. in } Z. \tag{6.20}$$

By virtue of (6.18) and (6.20), we obtain

$$\gamma_{2\top}' = 0, \text{ a.e. in } Z$$

hence the formula (6.12) has to hold a.e. in $Z$.

It remains to prove that the formula holds a.e. in $E\backslash Z$. To this aim, let us invoke (6.17) which yields the identity

$$\kappa_\Gamma' = -\frac{\gamma_{2\top}\cdot\gamma_{2\top}'}{\|\gamma_{2\top}\|^3[\gamma_{2\perp}\cdot(*\gamma_1)]}, \text{ a.e. in } E\backslash Z. \tag{6.21}$$

Then, a.e. in $E\backslash Z$, one has

$$\begin{aligned}
\frac{\kappa_\Gamma}{(1+\kappa_\Gamma^2)^{3/2}}\left[(1+\kappa_\Gamma^2)(*\gamma_1) - \frac{\kappa_\Gamma'}{\|\gamma_0'\|}\gamma_1\right] &= \frac{\gamma_{2\perp}\cdot(*\gamma_1)}{\|\gamma_{2\top}\|}\|\gamma_{2\top}\|^3\left[\frac{*\gamma_1}{\|\gamma_{2\top}\|^2} + \frac{(\gamma_{2\top}\cdot\gamma_{2\top}')\gamma_1}{\|\gamma_0'\|\|\gamma_{2\top}\|^3[\gamma_{2\perp}\cdot(*\gamma_1)]}\right]\\
&= [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma_1) + \left(\frac{\gamma_{2\top}'}{\|\gamma_0'\|}\cdot\frac{\gamma_{2\top}}{\|\gamma_{2\top}\|}\right)\gamma_1\\
&= [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma_1) + \left(\frac{\gamma_{2\top}'}{\|\gamma_0'\|}\cdot\gamma_1\right)\gamma_1\\
&= [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma_1) - \left[\frac{\gamma_{2\top}'}{\|\gamma_0'\|}\cdot(*\gamma_1)\right](*\gamma_1) + \frac{\gamma_{2\top}'}{\|\gamma_0'\|}
\end{aligned}$$

by (1.1), (1.2) and (6.16). Therefore, in order to conclude, it is enough to show that

$$\frac{\gamma_{2\top}'}{\|\gamma_0'\|}\cdot(*\gamma_1) = \gamma_{2\perp}\cdot(*\gamma_1), \text{ a.e. in } E\backslash Z.$$

But such an equality is a consequence of (1.1), (1.2) and (6.19). Indeed

$$\begin{aligned}
\frac{\gamma_{2\top}'}{\|\gamma_0'\|}\cdot(*\gamma_1) &= \frac{[\gamma_{2\top}\cdot(*\gamma_1)]' - \gamma_{2\top}\cdot(*\gamma_1')}{\|\gamma_0'\|} = -\gamma_{2\top}\cdot\left[*\left(\frac{\gamma_1'}{\|\gamma_0'\|}\right)\right]\\
&= -\gamma_{2\top}\cdot\left[*\left(\frac{\gamma_{2\perp}}{\|\gamma_{2\top}\|}\right)\right] = -\gamma_1\cdot(*\gamma_{2\perp})\\
&= \gamma_{2\perp}\cdot(*\gamma_1)
\end{aligned}$$

a.e. in $E$.

Proof of (6.13). As in the proof of (6.12), we begin by showing that the formula holds a.e. in $Z$.

Observe that

$$\|\gamma_{2\top}\| = 1, \text{ a.e. in } Z \tag{6.22}$$

by (1.2) and (6.15). Moreover

$$\gamma_{2\top}\cdot\gamma_{2\perp} = 0, \text{ a.e. in } E$$

by (1.1), (1.2) and Proposition 3.2, hence

$$\gamma_{2\top}'\cdot\gamma_{2\perp} + \gamma_{2\top}\cdot\gamma_{2\perp}' = 0, \text{ a.e. in } E.$$

By recalling (6.15) again, we get

$$\gamma_{2\perp}'\cdot\gamma_{2\top} = 0, \text{ a.e. in } Z$$

that is

$$\gamma_{2\perp}'\cdot\gamma_1 = 0, \text{ a.e. in } Z \tag{6.23}$$

by (1.1) and (1.2).

From the definition of $\kappa_\Gamma$

$$\|\gamma_{2\top}\|\kappa_\Gamma = \gamma_{2\perp}\cdot(*\gamma_1),\ \text{a.e. in } E$$

it follows that

$$\|\gamma_{2\top}\|'\kappa_\Gamma + \|\gamma_{2\top}\|\kappa'_\Gamma = \gamma'_{2\perp}\cdot(*\gamma_1) + \gamma_{2\perp}\cdot(*\gamma'_1),\ \text{a.e. in } E.$$

Then, by also invoking (6.15) and (6.22), we obtain

$$\kappa'_\Gamma = \gamma'_{2\perp}\cdot(*\gamma_1),\ \text{a.e. in } Z.$$

This identity, along with (6.23), imply

$$\gamma'_{2\perp} = (\gamma'_{2\perp}\cdot\gamma_1)\gamma_1 + [\gamma'_{2\perp}\cdot(*\gamma_1)](*\gamma_1) = \kappa'_\Gamma(*\gamma_1),\ \text{a.e. in } Z$$

namely (6.13) holds a.e. in $Z$.

The following computation follows from (1.1), (6.15), (6.16) and (6.21). It is valid a.e. in $E\backslash Z$.

$$\begin{aligned}\frac{1}{(1+\kappa_\Gamma^2)^{3/2}}\left[\frac{\kappa'_\Gamma}{\|\gamma'_0\|}(*\gamma_1) - (1+\kappa_\Gamma^2)\kappa_\Gamma^2\gamma_1\right] &= \|\gamma_{2\top}\|^3\left[-\frac{(\gamma_{2\top}\cdot\gamma'_{2\top})(*\gamma_1)}{\|\gamma'_0\|\|\gamma_{2\top}\|^3[\gamma_{2\perp}\cdot(*\gamma_1)]} - \frac{\|\gamma_{2\perp}\|^2\gamma_1}{\|\gamma_{2\top}\|^4}\right]\\ &= -\frac{\gamma_{2\top}\cdot\gamma'_{2\top}}{\|\gamma'_0\|[\gamma_{2\perp}\cdot(*\gamma_1)]}(*\gamma_1) - \frac{\|\gamma_{2\perp}\|^2}{\|\gamma_{2\top}\|}\gamma_1.\end{aligned}$$

Hence we have to prove that

$$-\frac{\gamma_{2\top}\cdot\gamma'_{2\top}}{\gamma_{2\perp}\cdot(*\gamma_1)} = \gamma'_{2\perp}\cdot(*\gamma_1),\ \text{a.e. in } E\backslash Z \tag{6.24}$$

and

$$-\frac{\|\gamma'_0\|\|\gamma_{2\perp}\|^2}{\|\gamma_{2\top}\|} = \gamma'_{2\perp}\cdot\gamma_1,\ \text{a.e. in } E\backslash Z. \tag{6.25}$$

First, since

$$\|\gamma_{2\top}\|^2 + \|\gamma_{2\perp}\|^2 = 1,\ \text{a.e. in } E$$

by (1.2) and recalling Proposition 3.2, we get

$$-\gamma_{2\top}\cdot\gamma'_{2\top} = \gamma_{2\perp}\cdot\gamma'_{2\perp} = [(\gamma_{2\perp}\cdot\gamma_1)\gamma_1 + [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma_1)]\cdot\gamma'_{2\perp} = [\gamma_{2\perp}\cdot(*\gamma_1)][\gamma'_{2\perp}\cdot(*\gamma_1)]$$

a.e. in $E\backslash Z$, that is just (6.24).

Since

$$\gamma_{2\perp} = (\gamma_{2\perp}\cdot\gamma_1)\gamma_1 + [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma_1) = [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma_1),\ \text{a.e. in } E$$

by (1.2) and Proposition 3.2, one has

$$\begin{aligned}\gamma'_{2\perp} &= [\gamma'_{2\perp}\cdot(*\gamma_1) + \gamma_{2\perp}\cdot(*\gamma'_1)](*\gamma_1) + [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma'_1)\\ &= [\gamma'_{2\perp}\cdot(*\gamma_1)](*\gamma_1) + [\gamma_{2\perp}\cdot(*\gamma_1)](*\gamma'_1)\end{aligned}$$

a.e. in $E$, by (1.2). Once again invoking (1.2) and Proposition 3.2, we finally obtain

$$\begin{aligned}\gamma'_{2\perp}\cdot\gamma_1 &= [\gamma_{2\perp}\cdot(*\gamma_1)][(*\gamma'_1)\cdot\gamma_1]\\ &= -\|\gamma'_0\|[\gamma_{2\perp}\cdot(*\gamma_1)]\left[\frac{\gamma'_1}{\|\gamma'_0\|}\cdot(*\gamma_1)\right]\\ &= -\frac{\|\gamma'_0\|}{\|\gamma_{2\top}\|}[\gamma_{2\perp}\cdot(*\gamma_1)]^2\\ &= -\frac{\|\gamma'_0\|\|\gamma_{2\perp}\|^2}{\|\gamma_{2\top}\|}\end{aligned}$$

a.e. in $E$, namely (6.25).

(2) It can be easily derived from the statement (1), Proposition 6.2(1) and Proposition 6.3. □

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