A LUSIN TYPE RESULT

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Abstract. By using the property known as Federer-Fleming conjecture (cf. [7, 3.1.17]), recently resolved by B. Bojarski, we prove the following Lusin type result:

Theorem. *Let $A \subset \mathbb{R}^n$ be a measurable set and let k be a nonnegative integer. Assume that to each $x \in A$ corresponds a polynomial $P_x : \mathbb{R}^n \to \mathbb{R}$ of degree less or equal to $k+1$ such that*

$$\operatorname{ap}\lim_{x\to a} \frac{(D^\alpha P_x)(x) - (D^\alpha P_a)(x)}{|x-a|} = 0$$

holds for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$, at a.e. $a \in A$. Then, for each $\varepsilon > 0$, there exists $\varphi \in C^{k+1}(\mathbb{R}^n)$ such that

$$\mathcal{L}^n\left(A \setminus \bigcap_{|\alpha|\leq k+1} \{x \in A : D^\alpha \varphi(x) = (D^\alpha P_x)(x)\}\right) \leq \varepsilon.$$

We will use such a theorem to provide a simple new proof of a well-known property of Sobolev functions.

1. introduction

Let $\mathcal{L}^n$ denote the Lebesgue outer measure on $\mathbb{R}^n$. Then, throughout the paper, the expressions "measurable with respect to $\mathcal{L}^n$" and "almost everywhere with respect to $\mathcal{L}^n$" will be simply referred as "measurable" and "almost everywhere", respectively.

The following result resolves the long-standing Federer-Whitney conjecture [7, 3.1.17]. It has been recently proved by B. Bojarski, compare [1, 2].

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $A \subset \Omega$. Moreover let k be a nonnegative integer and let $\varphi \in C^k(\Omega)$ be such that*

$$\operatorname{ap}\limsup_{x\to a} \frac{|D^\alpha \varphi(x) - D^\alpha \varphi(a)|}{|x-a|} < +\infty$$

for all $a \in A$ and for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k$. Then, for each $\varepsilon > 0$, there exists $\psi \in C^{k+1}(\mathbb{R}^n)$ such that

$$\mathcal{L}^n(A \setminus \{\varphi = \psi\}) \leq \varepsilon.$$

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We will use Theorem 1.1 to prove the following Lusin-type property.

Theorem 1.2 (Main result). *Let $A \subset \mathbb{R}^n$ be a measurable set and let k be a nonnegative integer. Assume that to each $x \in A$ corresponds a polynomial $P_x : \mathbb{R}^n \to \mathbb{R}$ of degree less or equal to $k+1$ such that*

$$\operatorname{ap}\lim_{x\to a} \frac{(D^\alpha P_x)(x) - (D^\alpha P_a)(x)}{|x-a|} = 0 \tag{1.1}$$

holds for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le k$, at a.e. $a \in A$. Then, for each $\varepsilon > 0$, there exists $\varphi \in C^{k+1}(\mathbb{R}^n)$ such that

$$\mathcal{L}^n\left(A \setminus \bigcap_{|\alpha|\le k+1} \{x \in A : D^\alpha\varphi(x) = (D^\alpha P_x)(x)\}\right) \le \varepsilon. \tag{1.2}$$

Observe that assuming (1.1) is (obviously) equivalent to requiring that the function

$$f_\alpha : A \to \mathbb{R}, \quad x \mapsto (D^\alpha P_x)(x)$$

is approximately differentiable at a and $\operatorname{ap} Df_\alpha(a) = D(D^\alpha P_a)(a)$ holds. Thus Theorem 1.2 can be rephrased as follows:

Theorem 1.3 (Main result, second version). *Let $A \subset \mathbb{R}^n$ be a measurable set and let k be a nonnegative integer. Assume that to each $x \in A$ corresponds a polynomial $P_x : \mathbb{R}^n \to \mathbb{R}$ of degree less or equal to $k+1$ such that the following condition is verified: For all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le k$ and for a.e. $a \in A$, the function f_α is approximately differentiable at a and one has $\operatorname{ap} Df_\alpha(a) = D(D^\alpha P_a)(a)$. Then, for each $\varepsilon > 0$, there exists $\varphi \in C^{k+1}(\mathbb{R}^n)$ satisying (1.2).*

In Section 4 we will use Theorem 1.3 to provide a simple new proof of the following well-known Lusin type result for Sobolev functions, which is a special case of Theorem 3.10.5 of [7] (for further developments see also [3]).

Theorem 1.4. *Let Ω be an open subset of $\mathbb{R}^n$, $p \ge 1$ and let k be a nonnegative integer. If $u \in W^{k+1,p}_{loc}(\Omega)$, then for each $\varepsilon > 0$, there exists $\varphi \in C^{k+1}(\mathbb{R}^n)$ such that*

$$\mathcal{L}^n\left(\Omega \setminus \bigcap_{|\alpha|\le k+1} \{x \in \Omega : D^\alpha\varphi(x) = D^\alpha u(x)\}\right) \le \varepsilon.$$

2. General notation and preliminaries

2.1. **General notation.** The standard orthonormal basis of $\mathbb{R}^n$ is denoted by $e_1, \dots, e_n$. The ball of radius r centered at $x \in \mathbb{R}^n$ will be indicated by $B_r(x)$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then we let

$$|\alpha| := \alpha_1 + \ldots + \alpha_n, \quad \alpha! := \alpha_1! \cdots \alpha_n!, \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

If $f \in W^{k,p}_{\text{loc}}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$, with $|\alpha| \le k$, then $D^\alpha f$ denotes the α^{th} weak derivative of f. In the special case when $\alpha_i = 1$ and $\alpha_j = 0$ for all $j \ne i$, we denote $D^\alpha f$ by $D_i f$. The weak gradient of f will simply be denoted by Df. If $\alpha, \beta \in \mathbb{N}^n$ with $\beta \le \alpha$ (that is $\beta_i \le \alpha_i$ for all i), then

$$D^\beta x^\alpha = \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta}. \tag{2.1}$$

2.2. **Points of Lebesgue density.** Recall that $x \in \mathbb{R}^n$ is said to be a point of Lebesgue density of $E \subset \mathbb{R}^n$ if

$$\mathcal{L}^n(B_r(x) \setminus E) = o(r^n) \qquad (\text{as } r \to 0+).$$

The set of points of Lebesgue density of E is denoted by $E^{(n)}$. Observe that if $E, F \subset \mathbb{R}^n$ then

$$E^{(n)} \cap F^{(n)} = (E \cap F)^{(n)}. \tag{2.2}$$

The set $E^{(n)}$ is measurable even if E is not measurable (cf. [4, Proposition 3.1]). A celebrated result by Lebesgue states that if E is measurable then $\mathcal{L}^n(E \Delta E^{(n)}) = 0$ (cf. Corollary 1.5 in Chapter 3 of [9]), hence

$$(E^{(n)})^{(n)} = E^{(n)}. \tag{2.3}$$

As one expects, the tangent cone (cf. [7, 3.1.21]) at a point of Lebesgue density coincides with the whole space. This fact is stated in the following proposition (cf. [5, Proposition 3.4]).

Proposition 2.1. *If $E \subset \mathbb{R}^n$ and $x \in E^{(n)}$, then*

$$\left\{ u \in \mathbb{R}^n : u = \lim_{i\to\infty} \frac{x_i - x}{|x_i - x|} \text{ for some } \{x_i\}_{i=1}^\infty \subset E \setminus \{x\} \text{ with } x_i \to x \right\} = \mathbb{S}^{n-1}.$$

Remark 2.1. Let $\varphi \in C^k(\mathbb{R}^n)$, with $k \ge 1$. If $\alpha \in \mathbb{N}^n$ and $|\alpha| \le k$, denote the set $\{x \in \mathbb{R}^n : D^\alpha\varphi(x) = 0\}$ simply by $\{D^\alpha\varphi = 0\}$. Consider $x \in \{\varphi = 0\}^{(n)}$ and observe that $\varphi(x) = 0$. By Proposition 2.1, there exists $\{x_i\}_{i=1}^\infty \subset \{\varphi = 0\} \setminus \{x\}$ converging to x and such that

$$\lim_{i\to\infty} \frac{x_i - x}{|x_i - x|} = e_1.$$

From

$$\begin{aligned} 0 = \frac{\varphi(x_i) - \varphi(x)}{|x_i - x|} &= \frac{D\varphi(x) \cdot (x_i - x) + o(|x_i - x|)}{|x_i - x|} \\ &= D\varphi(x) \cdot \frac{(x_i - x)}{|x_i - x|} + \frac{o(|x_i - x|)}{|x_i - x|} \quad (\text{as } i \to \infty), \end{aligned}$$

it follows that $D_1\varphi(x) = 0$. The same argument shows that $D_h\varphi(x) = 0$ for all $h = 1, \dots, n$, hence

$$\{\varphi = 0\}^{(n)} \subset \bigcap_{|\alpha|=1} \{D^\alpha\varphi = 0\}. \tag{2.4}$$

If $k \geq 2$ then, by (2.2), (2.3) and (2.4), we obtain

$$\{\varphi = 0\}^{(n)} = (\{\varphi = 0\}^{(n)})^{(n)} \subset \bigcap_{|\alpha|=1} \{D^\alpha \varphi = 0\}^{(n)} \subset \bigcap_{|\alpha|=1} \bigcap_{|\beta|=1} \{D^{\alpha+\beta} \varphi = 0\}$$

that is

$$\{\varphi = 0\}^{(n)} \subset \bigcap_{|\alpha|=2} \{D^\alpha \varphi = 0\}.$$

Replicating this argument, we finally obtain

$$\{\varphi = 0\}^{(n)} \subset \bigcap_{|\alpha|=k} \{D^\alpha \varphi = 0\}.$$

2.3. **Approximate limit, approximately continuous functions, approximately differentiable functions.** Recall from [7, 2.9.12] the definition of approximate upper limit (cf. also [10, Definition 5.9.1] and [6, Section 1.7.2]).

Definition 2.1. *Let $g : A \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ and $x_0 \in A^{(n)}$. Then the approximate upper limit of g at x_0 is defined by*

$$\operatorname{ap}\limsup_{x \to x_0} g(x) := \inf\{t \in \overline{\mathbb{R}} : x_0 \in \{g \leq t\}^{(n)}\}$$

where $\{g \leq t\} := \{a \in A : g(a) \leq t\}$.

Remark 2.2. Consider $g : A \subset \mathbb{R}^n \to \overline{\mathbb{R}}$, $x_0 \in \mathbb{R}^n$ and $B \subset A$ such that $x_0 \in B^{(n)}$ (hence also $x_0 \in A^{(n)}$). For all $t \in \overline{\mathbb{R}}$, one has

$$\{g|_B \leq t\} = \{g \leq t\} \cap B \subset \{g \leq t\}$$

which implies

$$B_r(x_0) \setminus \{g \leq t\} \subset B_r(x_0) \setminus \{g|_B \leq t\} \subset (B_r(x_0) \setminus \{g \leq t\}) \cup (B_r(x_0) \setminus B).$$

Thus

$$\begin{aligned}\mathcal{L}^n(B_r(x_0) \setminus \{g \leq t\}) \leq \mathcal{L}^n(B_r(x_0) \setminus \{g|_B \leq t\}) &\leq \mathcal{L}^n(B_r(x_0) \setminus \{g \leq t\}) \\ &\quad + \mathcal{L}^n(B_r(x_0) \setminus B).\end{aligned}$$

Hence

$$\{t \in \overline{\mathbb{R}} : x_0 \in \{g \leq t\}^{(n)}\} = \{t \in \overline{\mathbb{R}} : x_0 \in \{g|_B \leq t\}^{(n)}\}.$$

We conclude that $\operatorname{ap}\limsup_{x \to x_0} g(x) = \operatorname{ap}\limsup_{x \to x_0} g|_B(x)$.

Remark 2.3. Similarly one can define the approximate lower limit of $g : A \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ at $x_0 \in A^{(n)}$:

$$\operatorname{ap}\liminf_{x \to x_0} g(x) := \sup\{t \in \overline{\mathbb{R}} : x_0 \in \{g \geq t\}^{(n)}\}.$$

If $\operatorname{ap}\liminf_{x \to x_0} g(x) = \operatorname{ap}\limsup_{x \to x_0} g(x) = l \in \overline{\mathbb{R}}$, then the number l is called the approximate limit of g at $x_0 \in A^{(n)}$ and it is denoted by $\operatorname{ap}\lim_{x \to x_0} g(x)$.

We can now define approximate continuity and approximate differentiability (cf. sections 2.9.12 and 3.1.2 in [7]).

Definition 2.2. *We say that* $g : A \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ *is approximately continuous at* $x_0 \in A \cap A^{(n)}$ *if* $\operatorname{ap}\lim_{x\to x_0} g(x) = g(x_0)$.

Definition 2.3. *We say that* $g : A \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ *is approximately differentiable at* $x_0 \in A \cap A^{(n)}$ *if there exists* $v \in \mathbb{R}^n$ *such that*

$$\operatorname{ap}\lim_{x\to x_0} \frac{g(x) - g(x_0) - v \cdot (x - x_0)}{|x - x_0|} = 0.$$

In such a case the vector v *is unique and is denoted by* $\operatorname{ap} Dg(x_0)$. *It is called the approximate derivative of* g *at* x_0.

We will need also the following results (cf. [8, Theorem 7.51] and Theorem 4 in Section 6.1.3 of [6], respectively).

Theorem 2.1. *A function* $g : A \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ *is approximately continuous at* $x \in A \cap A^{(n)}$ *if and only if there exists a measurable set* $E \subset A$ *such that* $x \in E \cap E^{(n)}$ *and* $g|_E$ *is continuous at* x.

Theorem 2.2. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $p \geq 1$. *Then each function in* $W^{1,p}_{loc}(\Omega)$ *is approximately differentiable a.e. in* Ω *and its approximate derivative equals its weak derivative a.e. in* Ω.

3. Proof of Theorem 1.2 (main result)

First of all we state the following remark which will be useful below.

Remark 3.1. Under the assumptions of Theorem 1.2, consider $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$. Since

$$|(D^\alpha P_x)(x) - (D^\alpha P_a)(a)| \leq \frac{|(D^\alpha P_x)(x) - (D^\alpha P_a)(x)|}{|x-a|}|x-a| + |(D^\alpha P_a)(x) - (D^\alpha P_a)(a)|$$

for all $x, a \in A$, then the function $x \mapsto (D^\alpha P_x)(x)$ is approximately continuous at a.e. $a \in A$. Hence $x \mapsto (D^\alpha P_x)(x)$ is also measurable, by Theorem 2.9.13 of [7].

Now we begin the proof of the main result by observing that (throwing away a null subset of A, if necessary) we may assume without loss of generality that (1.1) holds for all $a \in A$ (and for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$). The proof is by induction on k.

STEP 1. Assume $k = 0$. Define

$$f : A \to \mathbb{R}, \quad f(x) := P_x(x)$$

and observe that (for all $a, x \in A$)

$$\frac{|f(x) - f(a)|}{|x-a|} = \frac{|P_x(x) - P_a(a)|}{|x-a|} \leq \frac{|P_x(x) - P_a(x)|}{|x-a|} + \frac{|P_a(x) - P_a(a)|}{|x-a|}. \tag{3.1}$$

Moreover

$$P_a(x) = P_a(a) + (DP_a)(a) \cdot (x-a) + o(|x-a|) \quad (\text{as } x \to a)$$

hence

$$|P_a(x) - P_a(a)| \le |(DP_a)(a)||x-a| + o(|x-a|) \quad (\text{as } x \to a). \tag{3.2}$$

By (3.1), (3.2) and assumption (1.1), we obtain

$$\operatorname{ap}\limsup_{x\to a} \frac{|f(x)-f(a)|}{|x-a|} < +\infty$$

for all $a \in A$. From Theorem 3.1.16 of [7] it follows that, for all $\varepsilon > 0$, there exists $\varphi_1 \in C^1(\mathbb{R}^n)$ such that

$$\mathcal{L}^n(A \setminus A_1) \le \varepsilon \tag{3.3}$$

with

$$A_1 := \{x \in A : \varphi_1(x) = P_x(x)\}.$$

Observe that A_1 is measurable, by Remark 3.1. If $a \in A_1 \cap A_1^{(n)}$, then, by (1.1) and Theorem 2.1 (also taking into account of Remark 3.1), there exists a measurable subset E_1 of A such that $a \in E_1 \cap E_1^{(n)}$ and

$$\lim_{\substack{x\to a\\ x\in E_1}} \frac{P_x(x) - P_a(x)}{|x-a|} = 0. \tag{3.4}$$

By (2.2), one has $a \in A_1^{(n)} \cap E_1^{(n)} = (A_1 \cap E_1)^{(n)}$. Hence and by Proposition 2.1, given $i \in \{1, \dots, n\}$, we can find a sequence $\{a_j\}_{j=1}^{\infty} \subset (A_1 \cap E_1) \setminus \{a\}$ such that

$$a_j \to a, \quad \frac{a_j - a}{|a_j - a|} \to e_i \qquad (\text{as } j \to \infty).$$

Since

$$\varphi_1(a_j) = \varphi_1(a) + D\varphi_1(a) \cdot (a_j - a) + o(|a_j - a|) \quad (\text{as } j \to \infty)$$

we get

$$\frac{\varphi_1(a_j) - \varphi_1(a)}{|a_j - a|} = D\varphi_1(a) \cdot \frac{a_j - a}{|a_j - a|} + \frac{o(|a_j - a|)}{|a_j - a|} \quad (\text{as } j \to \infty)$$

so that

$$\lim_{j\to\infty} \frac{\varphi_1(a_j) - \varphi_1(a)}{|a_j - a|} = D_i\varphi_1(a). \tag{3.5}$$

The same argument shows also that

$$\lim_{j\to\infty} \frac{P_a(a_j) - P_a(a)}{|a_j - a|} = (D_i P_a)(a). \tag{3.6}$$

On the other hand, one has

$$\frac{\varphi_1(a_j) - \varphi_1(a)}{|a_j - a|} = \frac{P_{a_j}(a_j) - P_a(a)}{|a_j - a|} = \frac{P_{a_j}(a_j) - P_a(a_j)}{|a_j - a|} + \frac{P_a(a_j) - P_a(a)}{|a_j - a|}.$$

Hence, by recalling (3.4), (3.5) and (3.6), we obtain

$$D_i\varphi_1(a) = (D_i P_a)(a) \tag{3.7}$$

for all $a \in A_1 \cap A_1^{(n)}$ and $i = 1, \dots, n$. From (3.3) and (3.7) it follows at once that

$$\mathcal{L}^n\left(A \setminus \bigcap_{|\alpha|\le 1} \{x \in A : D^\alpha \varphi_1(x) = (D^\alpha P_x)(x)\}\right) \le \varepsilon$$

which concludes the proof for $k = 0$, with $\varphi := \varphi_1$.

STEP 2. Now let $k \ge 1$ and suppose that:

(i) The assumption (1.1) holds;
(ii) Theorem 1.2 holds for $k - 1$.

Then, for all $\varepsilon > 0$, there exists $\varphi_k \in C^k(\mathbb{R}^n)$ such that

$$\mathcal{L}^n(A \setminus A_k) \le \frac{\varepsilon}{3} \tag{3.8}$$

with

$$A_k := \bigcap_{|\alpha|\le k} \{x \in A : D^\alpha \varphi_k(x) = (D^\alpha P_x)(x)\}$$

Observe that A_k is measurable, by Remark 3.1. If

$$a \in A_k^* := A_k \cap A_k^{(n)}, \quad x \in A_k$$

and $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$, then one has

$$\begin{aligned}
\frac{|D^\alpha \varphi_k(x) - D^\alpha \varphi_k(a)|}{|x-a|} &= \frac{|(D^\alpha P_x)(x) - (D^\alpha P_a)(a)|}{|x-a|} \\
&\le \frac{|(D^\alpha P_x)(x) - (D^\alpha P_a)(x)|}{|x-a|} + \frac{|(D^\alpha P_a)(x) - (D^\alpha P_a)(a)|}{|x-a|} \\
&= \frac{|(D^\alpha P_x)(x) - (D^\alpha P_a)(x)|}{|x-a|} \\
&\quad + \frac{|D(D^\alpha P_a)(a) \cdot (x-a) + o(|x-a|)|}{|x-a|} \\
&\le \frac{|(D^\alpha P_x)(x) - (D^\alpha P_a)(x)|}{|x-a|} \\
&\quad + |D(D^\alpha P_a)(a)| + \frac{o(|x-a|)}{|x-a|} \quad (\text{as } x \to a).
\end{aligned}$$

Since $a \in A_k^{(n)}$ and recalling Remark 2.2, it follows that

$$\begin{aligned}
\operatorname{ap}\limsup_{x\to a}\frac{|D^\alpha\varphi_k(x)-D^\alpha\varphi_k(a)|}{|x-a|} &= \operatorname{ap}\limsup_{x\to a}\left(\frac{|D^\alpha\varphi_k(x)-D^\alpha\varphi_k(a)|}{|x-a|}\right)\Big|_{x\in A_k}\\
&\le \operatorname{ap}\limsup_{x\to a}\left(\frac{|(D^\alpha P_x)(x)-(D^\alpha P_a)(x)|}{|x-a|}\right)\Big|_{x\in A_k}\\
&+ |D(D^\alpha P_a)(a)|\\
&= \operatorname{ap}\limsup_{x\to a}\frac{|(D^\alpha P_x)(x)-(D^\alpha P_a)(x)|}{|x-a|}\\
&+ |D(D^\alpha P_a)(a)|.
\end{aligned}$$

Now, by the assumption (1.1), we obtain

$$\operatorname{ap}\limsup_{x\to a}\frac{|D^\alpha\varphi_k(x)-D^\alpha\varphi_k(a)|}{|x-a|} < +\infty$$

for all $a \in A_k^*$ and for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$. Then, by Theorem 1.1, there exists $\varphi_{k+1} \in C^{k+1}(\mathbb{R}^n)$ such that

$$\mathcal{L}^n(A_k \setminus \{\varphi_k = \varphi_{k+1}\}) = \mathcal{L}^n(A_k^* \setminus \{\varphi_k = \varphi_{k+1}\}) \le \frac{\varepsilon}{3}. \tag{3.9}$$

From (3.8) and (3.9), we get

$$\begin{aligned}
\mathcal{L}^n(A \setminus \{\varphi_k = \varphi_{k+1}\}) &\le \mathcal{L}^n(A \setminus A_k) + \mathcal{L}^n(A_k \setminus \{\varphi_k = \varphi_{k+1}\})\\
&\le \frac{2\varepsilon}{3}.
\end{aligned} \tag{3.10}$$

One has

$$\{\varphi_k = \varphi_{k+1}\}^{(n)} \subset \bigcap_{|\alpha|\le k}\{D^\alpha\varphi_k = D^\alpha\varphi_{k+1}\} \tag{3.11}$$

by Remark 2.1. Define the set

$$B_k := \bigcap_{|\alpha|\le k}\{x \in A : D^\alpha\varphi_{k+1}(x) = (D^\alpha P_x)(x)\}$$

which is measurable, by Remark 3.1. Observe that, by definition of A_k and (3.11), one has

$$B_k \supset A_k \cap \left(\bigcap_{|\alpha|\le k}\{D^\alpha\varphi_k = D^\alpha\varphi_{k+1}\}\right) \supset A_k \cap \{\varphi_k = \varphi_{k+1}\}^{(n)}. \tag{3.12}$$

Consider $a \in B_k \cap B_k^{(n)}$ and let $\beta \in \mathbb{N}^n$ be such that $|\beta| = k$. Then, proceeding similarly as in STEP 1, we can find a measurable set $E_k \subset A$ such that

$$a \in E_k \cap E_k^{(n)}, \quad \lim_{\substack{x\to a\\ x\in E_k}}\frac{(D^\beta P_x)(x)-(D^\beta P_a)(x)}{|x-a|} = 0 \tag{3.13}$$

by assumption (1.1), Remark 3.1 and Theorem 2.1. By (2.2), one has $a \in B_k^{(n)} \cap E_k^{(n)} = (B_k \cap E_k)^{(n)}$. Hence and by Proposition 2.1, given $i \in \{1, \dots, n\}$, we can find a sequence $\{a_j\}_{j=1}^\infty \subset (B_k \cap E_k) \setminus \{a\}$ such that

$$a_j \to a, \quad \frac{a_j - a}{|a_j - a|} \to e_i \qquad (\text{as } j \to \infty).$$

Since

$$D^\beta \varphi_{k+1}(a_j) = D^\beta \varphi_{k+1}(a) + D(D^\beta \varphi_{k+1})(a) \cdot (a_j - a) + o(|a_j - a|) \quad (\text{as } j \to \infty)$$

we get

$$\lim_{j\to\infty} \frac{D^\beta \varphi_{k+1}(a_j) - D^\beta \varphi_{k+1}(a)}{|a_j - a|} = D_i(D^\beta \varphi_{k+1})(a).$$

The same argument shows also that

$$\lim_{j\to\infty} \frac{(D^\beta P_a)(a_j) - (D^\beta P_a)(a)}{|a_j - a|} = D_i(D^\beta P_a)(a).$$

On the other hand, one has

$$\begin{aligned}\frac{D^\beta \varphi_{k+1}(a_j) - D^\beta \varphi_{k+1}(a)}{|a_j - a|} &= \frac{(D^\beta P_{a_j})(a_j) - (D^\beta P_a)(a)}{|a_j - a|} \\ &= \frac{(D^\beta P_{a_j})(a_j) - (D^\beta P_a)(a_j)}{|a_j - a|} + \frac{(D^\beta P_a)(a_j) - (D^\beta P_a)(a)}{|a_j - a|}.\end{aligned}$$

Hence and by (3.13), we obtain

$$D_i(D^\beta \varphi_{k+1})(a) = D_i(D^\beta P_a)(a)$$

for all $a \in B_k \cap B_k^{(n)}$, $i \in \{1, \dots, n\}$ and $\beta \in \mathbb{N}^n$ with $|\beta| = k$. This proves that the set

$$A_{k+1} := \bigcap_{|\alpha| \le k+1} \{x \in A : D^\alpha \varphi_{k+1}(x) = (D^\alpha P_x)(x)\},$$

which is obviously a (measurable, by Remark 3.1) subset of B_k, is actually $\mathcal{L}^n$-equivalent to B_k. Recalling also (3.12), (3.8) and (3.10), it follows that

$$\mathcal{L}^n(A \setminus A_{k+1}) = \mathcal{L}^n(A \setminus B_k) \le \mathcal{L}^n(A \setminus A_k) + \mathcal{L}^n(A \setminus \{\varphi_k = \varphi_{k+1}\}) \le \varepsilon.$$

The conclusion follows by taking $\varphi := \varphi_{k+1}$.

4. Application to Sobolev functions, proof of Theorem 1.4

Let Ω be an open subset of $\mathbb{R}^n$, $p \ge 1$ and let $k \ge 0$ be an integer. Then, given $u \in W^{k+1,p}_{\text{loc}}(\Omega)$, define the $(k+1)$-th degree Taylor polynomial of u at $x \in \Omega$ in the usual way as

$$T^{(k+1)}_{u,x}(y) := \sum_{|\alpha| \le k+1} \frac{u_\alpha(x)}{\alpha!}(y - x)^\alpha, \quad y \in \mathbb{R}^n$$

where α varies in $\mathbb{N}^n$ and u_α denotes the precise representative of $D^\alpha u \in L^p_{\mathrm{loc}}(\Omega)$ (cf. Section 1.7.1 of [6]). Observe that, for all $\beta \in \mathbb{N}^n$ such that $|\beta| \le k+1$, one has

$$(D^\beta T^{(k+1)}_{u,x})(y) = \sum_{|\beta|\le|\alpha|\le k+1} \frac{u_\alpha(x)}{(\alpha-\beta)!}(y-x)^{\alpha-\beta} = \sum_{|\alpha|\le k+1-|\beta|} \frac{u_{\alpha+\beta}(x)}{\alpha!}(y-x)^\alpha$$

for all $x \in \Omega$ and $y \in \mathbb{R}^n$, by (2.1). Since $D^\beta u \in W^{k+1-|\beta|,p}_{\mathrm{loc}}(\Omega)$ and $u_{\alpha+\beta}$ is the precise representative of $D^{\alpha+\beta}u = D^\alpha(D^\beta u)$, this identity shows that

$$D^\beta T^{(k+1)}_{u,x} = T^{(k+1-|\beta|)}_{D^\beta u,x} \tag{4.1}$$

for all $x \in \Omega$. Now, in order to apply Theorem 1.3, define

$$P_x := T^{(k+1)}_{u,x}, \quad x \in \Omega$$

so that

$$f_\alpha(x) = (D^\alpha P_x)(x) = (D^\alpha T^{(k+1)}_{u,x})(x) = u_\alpha(x)$$

for all $x \in \Omega$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k+1$, by (4.1). Then, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$, the function f_α has to be approximately differentiable a.e. in Ω and one has

$$e_i \cdot \mathrm{ap}\, Df_\alpha = u_{\alpha+e_i}, \text{ a.e. in } \Omega \qquad (i = 1, \dots, n) \tag{4.2}$$

by Theorem 2.2. On the other hand, always assuming $|\alpha| \le k$ and $0 \le i \le n$, one has also

$$D_i(D^\alpha P_x) = D_i(D^\alpha T^{(k+1)}_{u,x}) = D^{\alpha+e_i} T^{(k+1)}_{u,x} = T^{(k-|\alpha|)}_{D^{\alpha+e_i}u,x}$$

by (4.1), hence

$$D_i(D^\alpha P_x)(x) = u_{\alpha+e_i}(x) \tag{4.3}$$

for all $x \in \Omega$. From (4.2) and (4.3) we get

$$\mathrm{ap}\, Df_\alpha(x) = D(D^\alpha P_x)(x)$$

for a.e. $x \in \Omega$. The conclusion follows from Theorem 1.3.

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