ON THE NON-DEFECTIVITY OF SEGRE-VERONESE EMBEDDINGS

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ABSTRACT. We prove a theorem which implies that all Segre-Veronese varieties of multidegree $(d_1, \dots, d_k)$ and format $(n_1, \dots, n_k)$ with $n_1 \geq \cdots \geq n_k > 0$ are not defective if $d_1 \geq 3$, $d_2 \geq 3$ and $d_i \geq 2$ for all $i > 2$. As a particular case we prove the non-defectivity of any Segre-Veronese variety with at least 2 factors and $d_i \geq 3$ for all i, extending to the case $k > 2$ a theorem of Galuppi and Oneto. Our general result also shows that many Segre-Veronese varieties with 2 factors are not secant defective if they are embedded in bidegree $(x, 2)$, $x \geq 4$.

1. INTRODUCTION

In this paper we study the vector space, V, of the partially symmetric tensors of format $(n_1 + 1) \times \cdots (n_k + 1)$ and multidegree $(d_1, \dots, d_k)$ and for each positive integer a the dimension of the subsets of V formed by the partially symmetric tensors with prescribed rank or with prescribed border rank.

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Fix a positive integers a. Take $q \in \mathbb{P}^r$. The point q is said to have X-rank a if a is the minimal integer such that q is contained in the linear span of a points of X. Let $\sigma_a^0(X)$ denote the set of all points of $\mathbb{P}^r$ with X-rank a and let $\sigma_a(X)$ denote its closure in $\mathbb{P}^r$. Over $\mathbb{C}$ it is known that we may take the closure in the euclidean topology or in the Zariski topology and get the same variety, $\sigma_a(X)$. Elements of $\sigma_a(X)$ are said to have border rank a. It is known that $\sigma_a(X)$ is an irreducible variety of dimension at most $\min\{r, -1 + a(1 + \dim X)\}$. The integer $\min\{r, -1 + a(1 + \dim X)\}$ is the expected dimension of $\sigma_a(X)$. X is said to be *not secant defective* or just *not defective* if for each positive integer a the a-secant variety $\sigma_a(X)$ has dimension $\min\{r, a(\dim X + 1) - 1\}$. Suppose we know the dimension, x, of $\sigma_a(X)$. It is known that $\sigma_a^0(X)$ is constructible in the sense of Chevalley, i.e. it is a finite union of locally closed subsets of $\mathbb{P}^r$ for the Zariski topology of $\mathbb{P}^r$, and hence that it contains a non-empty Zariski open subset of its closure, $\sigma_a(X)$ ([21, Ex. II.3.18 and Ex. II.3.19]). Thus we know the dimension, x, of $\sigma_a^0(X)$. Hence if X is not secant defective we know the dimensions of all $\sigma_a(X)$ and all $\sigma_a^0(X)$.

Take a multiprojective space $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let $\nu : Y \longrightarrow \mathbb{P}^r$, $r := -1 + \prod_{i=1}^{k} \binom{n_i + d_i}{n_i}$, be the embedding of Y of multidegree $(d_1, \dots, d_k)$. Set $X := \nu(Y)$. The projective space $\mathbb{P}^r$ in which X sits is $\mathbb{P}(V)$, with V the vector space of all partially symmetric tensors of format $(n_1 + 1) \times \cdots (n_k + 1)$ and multidegree

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$(d_1, \dots, d_k)$. Take $v \in V$, $v \neq 0$. The partially symmetric rank of v is the X-rank of the point $[v] \in \mathbb{P}(V)$ and hence v has border rank $a \geq 2$ if and only $[v] \in \sigma_a(X) \setminus \sigma_{a-1}(X)$. This is the main motivation for the study of the dimensions of the secant varieties of Segre-Veronese varieties. On this topic there are many recent results ([1, 2, 3, 4, 5, 12, 13, 14, 15, 16, 17, 18, 20, 23, 24]). See the references of the quoted papers and [25] and its references for many other older papers.

We prove the following result.

Theorem 1.1. *Fix integers $k \geq 3$, $n_1 \geq \cdots \geq n_k > 0$ and $d_i \geq 2$, $1 \leq i \leq k$, such that $d_1 \geq 3$ and $d_2 \geq 3$. Then the Segre-Veronese embedding of multidegree $(d_1, \dots, d_k)$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is not secant defective.*

As a particular case of Theorem 1.1 we get the following result.

Theorem 1.2. *Let X be a Segre-Veronese embedding of multidegree $(d_1, \dots, d_k)$ of a multiprojective space with $k \geq 2$ factors. If $d_i \geq 3$ for all i, then X is not secant defective.*

The case $k = 2$ of Theorem 1.2 was conjectured by Abo and Brambilla ([4, Conjecture 5.6] and proved by Galuppi and Oneto ([20]). For Theorem 1.2 we need the case $k = 2$ proved in [20] (we do not have another proof). With our inductive method we could not get Theorem 1.2 without proving Theorem 1.1 For the case $k = 1$ there are the exceptional cases in the Alexander-Hirschowitz list ([8, 9, 10]), i.e. $(\dim X, d_1) \in \{(2,4), (3,4), (4,3), (4,4)\}$. For 2-factors Segre-Veronese embeddings we prove the following result, certainly not optimal, but in our opinion very interesting.

Theorem 1.3. *Fix integers $y \geq 2$, $m \geq 2$ and $x \geq 4$ such that $\binom{y+x}{x} > 2(y+m)^2$ and $(m+1)(2m-1) \geq (y+m)^2(y+m+3)$. If $y < m$ assume $2ym + m^2 - 2y - 3m - 1)\binom{y+x}{x} \geq 2(y+m+1)(y+m-1)^2$. Then the embedding of $\mathbb{P}^y \times \mathbb{P}^m$ in bidegree $(x, 2)$ is not secant defective.*

All defective Segre-Veronese embeddings of $\mathbb{P}^y \times \mathbb{P}^1$ are known ([3]). By [20] all embeddings of $\mathbb{P}^y \times \mathbb{P}^m$ in bidegrees (x, t), $x > 2$, $t > 2$, are not secant defective. Thus in the statement of Theorem 1.3 we only state the bidegree $(x, 2)$. It would be very interesting to know for which (y, m) the embedding of $\mathbb{P}^y \times \mathbb{P}^m$ in bidegree $(x, 2)$, $x \leq 3$, is not secant defective.

Theorem 1.4. *Fix an integer $k \geq 2$ and positive integers n_i, d_i such that $d_k \geq 2$. Set $n := n_1 + \cdots + n_k$, $m := n_k$, $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{k-1}}$, $X := Y \times \mathbb{P}^{n_k}$ and $\alpha := \prod_{i=1}^{k-1} \binom{n_i+d_i}{n_i}$. Assume $\alpha > 2n^2$, $n \geq m+2$ and either $m = 1$ or $(m+1)(2m-1)\alpha \geq n^2(n+3)$. If $m > n/2$ assume*

$$(2nm - m^2 - 2n - m - 1)\alpha \geq 2(n+1)(n-1)^2 \tag{1}$$

Assume that the Segre-Veronese embedding of Y of multidegree $(d_1, \dots, d_{k-1})$ is not secant defective. Then the Segre-Veronese embedding of X of multidegree $(d_1, \dots, d_k)$ is not secant defective.

Recursively we get the non-secant defectivity of many Segre-Veronese varieties of multidegree $(d_1, \dots, d_k)$ with many (but not all) $d_i = 1$. There are many defective cases for multidegrees $(d_1, \dots, d_k)$ with many (or all) $d_i = 1$ ([2, 3, 4, 5]). There is a general method (called *unbalanced* in [4, 5]) to construct defective Segre-Veronese varieties). A few cases are listed in [4].

Since the Segre-Veronese varieties are homogeneous, for a fixed format $(n_1, \dots, n_k)$ and a fixed multidegree $(d_1, \dots, d_k)$ only a small number of secant varieties need to be checked ([16]), but these are the hardest. Indeed, for any non-degenerate variety $X \subset \mathbb{P}^r$ if a secant variety $\sigma_a(X) \subset \mathbb{P}^r$, $a \geq 2$, has dimension $a(\dim X + 1) - 1$, then $\dim \sigma_x(X) = x(\dim X + 1) - 1$ for all $1 \leq x < a$, while if $\sigma_a(X) = \mathbb{P}^r$, then $\sigma_y(X) = \mathbb{P}^r$ for all $y > a$.

Assume that $X \subset \mathbb{P}^r$ is not secant defective. A. Massarenti and M. Mella proved a fundamental result ([26, Th. 1.5]) which gives the identifiability of a general element of the secant variety $\sigma_\alpha(X) \neq \mathbb{P}^r$, except at most for one α, under very mild assumptions. We spell out it in the case of Theorems 1.1 but it is well-known that [26, Th. 1.5] applies to almost all linear algebra problems related to secant varieties of homogeneous projective algebraic varieties. See [26, Remark 3.10] for an example in which its assumption does not hold and identifiability fails.

Corollary 1.5. *Fix integers $k \geq 3$, $n_1 \geq \cdots \geq n_k > 0$ and $d_i \geq 2$, $1 \leq i \leq k$, such that $d_1 \geq 3$ and $d_2 \geq 3$. Set $n := n_1 + \cdots + n_k$ and $r := -1 + \prod_{i=1}^{k} \binom{n_i + d_i}{n_i}$. Fix a positive integer α such that $(\alpha + 1)(n + 1) \leq r + 1$. Then a general rank α partially symmetric tensor of format $(n_1, \dots, n_k)$ and multidegree $(d_1, \dots, d_k)$ has an essentially unique representation as a sum of α rank 1 partially symmetric tensors.*

It seems out of reach to get a test that given a format and multidegree says whether or not the corresponding Segre-Veronese is defective. A few classes are perfectly understood, e.g. $n_i = 1$ for all i ([24]) or $n_i \leq 2$ for all i with mild restrictions on the multidegrees ([14]) or $k = 2$ and $n_2 = 1$ ([4]). However, the papers [13, 14] handle (in the cases they list) a more general problem: start with a non-defective pair $(Y, \mathcal{L})$ (in which Y is not assumed to be a multiprojective space).

Question 1.6. Is it possible to give the full list of all $k \geq 2$, formats $(n_1, \dots, n_k)$, multidegrees $(d_1, \dots, d_k)$ and a such that $d_i \geq 2$ for all i and the a-secant variety of $\mathcal{O}_{\mathbb{P}^{n_1}} \times \cdots \times \mathcal{O}_{\mathbb{P}^{n_k}}(d_1, \dots, d_k)$ is defective?

Question 1.7. Is it possible to extend [13, 14] to the case $Y \times \mathbb{P}^m$ with respect to the line bundle $p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^m}(t))$, $t \geq 2$, starting from a non-defective pair $(Y, \mathcal{L})$ only knowing that $(Y, \mathcal{L})$ is not-defective and knowing the integers $\dim Y$ and $h^0(\mathcal{L})$, just imposing that $h^0(\mathcal{L})$ is large enough with respect to the integers $\dim Y$ and m?

In some small cases it is even possible to assume less than the non-defectivity of $\mathcal{L}$, just that it is has at most one defective secant variety, adding a stronger assumption on the integer $h^0(\mathcal{L})$ ([13, Th. 3]). We tried with no luck Question 1.7. We tried to mix the Differential Horace Lemma ([8, 10]) with the Blowing Up Horace Method ([9]).

In this paper all our proofs we only use the Differential Horace Lemma (see Lemma 2.4 for its classical formulation and Lemma 2.6 for a stronger version also used in [13, 14]). By the Terracini Lemma ([6, Cor. 10], [7, Lemma 2.1]) computing the dimension of the a-secant variety of an embedded variety is translated in the computation of the Hilbert function of a general union of a double points of the variety. It is to this translated problem that the Horace Method and its variations are applied. We use in an essential way joins of different varieties and [6, Cor. 10] and [7, Lemma 2.1] are stated and proved in this more general setting.

Section 2 (resp. Section 3) gives the preliminaries and notation not related (resp. related) to the multiprojective spaces and their Segre-Veronese embeddings. In particular they contain the fine tuning of [7] that we need for our proofs.

Section 4 contains the proof of Theorem 1.4.

Section 5 contains the proof of Theorems 1.1 and 1.3.

Section 6 contains some suggestions for further works.

We work over an algebraically closed field of characteristic 0.

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2. General preliminaries

For any subscheme E of a projective space let $\langle E\rangle$ denote its linear span. For any smooth variety M and any $p \in M$ let $(2p, M)$ (or just $2p$) denote the closed subscheme of M with $(\mathcal{I}_{p,M})^2$ as its ideal sheaf. For any variety D let D_{reg} denote the set of its smooth points.

We first state a general result due to B. Ådlandsvik ([7, Prop. 2.1(ii)] in a slightly stronger form that the one contained in [16].

Theorem 2.1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Let $W \subset \mathbb{P}^r$ be an integral variety (we allow the case $W = \emptyset$ and the case W degenerate). Set $n := \dim X$ and $w := \dim W$, with the convention $w = -1$ if $W = \emptyset$. For all integers $i \geq 0$ set $b_i(X, W) := \dim[\sigma_i(X), W]$ and $a_i(X, W) = b_{i+1}(X, W) - b_i(X, W)$. Assume that no $\sigma_i(X) \neq \mathbb{P}^r$ is a cone. Let $c \geq 0$ be the maximal integer such that $\dim[\sigma_i(X), W]) = w + ni$. Let g be the minimal integer such that $[\sigma_g(X); W] = \mathbb{P}^r$. Assume $c < g$ and set $k := g - c$ and $s := r - \dim W - 1$. Then:*

(1) *We have $0 \leq k \leq n$.*
(2) *We have $a_{i+1}(X, W) < a_i(X, W)$ for all $i = c - 1, \dots, g - 2$.*
(3) *We have $c \geq \frac{s+1}{n+1} - \frac{nk}{n+1} + \frac{k(k-1)}{2n+2}$.*
(4) *We have $g \leq \frac{s+1}{n+1} + k - \frac{k(k-1)}{2n+2}$.*

Fix an integer $a \geq 0$. Let $B \subset W$ be a general double point of W with the convention $B = \emptyset$ if $W = \emptyset$. Let $A \subset X$ be a general union of a double points of X. Then

(i) $r - \dim\langle A \cup B\rangle \leq \max\{0, r - an - w - n(n+1)/2\}$.
(ii) *If $w + (n+1)a \leq r - (n+1)n/2$, then $\dim\langle A \cup B\rangle = w + (n+1)a$.*
(iii) *If $w + (n+1)a \geq r + (n+1)n/2$, then $\langle A \cup B\rangle = \mathbb{P}^r$.*

Proof. The integer g is well-defined, because X is non-degenerate. By the definition of g we have $a_g(X, W) = 0$. Since $a_g(X, W) = 0$, we have $c \leq g$. Part (2) is [7, Prop. 2.1(ii)]. By assumption $c \leq g$, i.e. $k \geq 0$. Part (2) gives part (1). By the definition of c we have $a_i(X, W) = n + 1$ if $i < c$ and $a_c(X, W) \leq n$. Thus (2) implies that the non-zero $a_i(X, W) \neq n+1$ are among the integers $\{n, \dots, 1\}$ with each integer $a_i(X, W)$ appearing at most once. Thus each of the non-zero integers $\dim \sigma_i(X, W) - i(n + 1) - w + 1$ appears at most once in the set $\{n, \dots, 1\}$. We have $\sum_{i=1}^{n} i = n(n + 1)/2$. Thus (1), (3) and (4) hold. Part (i), and hence parts (ii) and (iii), follows from (3) and (4) and the Terracini Lemma ([6, Cor. 1.10], [7, Lemma 1.1]). □

Remark 2.2. Take $X \subset \mathbb{P}^r$ as in Theorem 2.1. If there is an algebraic group G acting on X and the embedding $X \subset \mathbb{P}^r$ is induced by a complete linear system $|\mathcal{R}|$ such that G acts on $H^0(\mathcal{R})$ and $H^0(\mathcal{R})$ is an irreducible G-representation, then no proper secant variety of X is a cone and hence we may apply Theorem 2.1. In

our case $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $G = GL(n_1+1) \times \cdots GL(n_k+1)$. However, to use the Differential Horace Method we also need the case $W \neq \emptyset$ and this is the reason to state Theorem 2.1. In our applications to the proof of Theorem 1.4 and the other results listed in the introduction, W will be a secant variety $\sigma_b(H)$ with H a divisor of X. In our case H is homogeneous for the action of a smaller group $G_1 := GL(n_1+1) \times \cdots \times GL(n_{k-1}+1) \times GL(n_k)$ and we know a decomposition of $H^0(\mathcal{R})$ into irreducible representations (Remark 3.2). To apply, as here, the Differential Horace Lemma, to the homogeneous embeddings of the homogeneous variety X it is necessary to apply Theorem 2.1 with as W the b-secant variety of an effective divisor D of X with $b \leq n-1$. Since b is very small, it is easy to prove that $\sigma_b(D) = bn - 1$ and hence we may take $w = bn - 1$ (see Lemma 3.1 for our application to the proof of Theorem 1.4).

In several preliminary versions to get better bounds in the case $m = 2$ we used this result.

Lemma 2.3. *([14, Theorem 1]). Take $X = Y \times \mathbb{P}^2$, $(Y, \mathcal{L})$ not secant defective and set $y := \dim Y$. Assume*

(1) $\alpha \geq 60$ *for* $y = 2, 3$;
(2) $\alpha \geq 98$ *for* $y = 4$;
(3) $\alpha \geq 133$ *for* $y = 5$;
(4) $\alpha \geq 176$ *for* $y = 6$;
(5) $\alpha \geq 231$ *for* $y = 7$;
(6) $\alpha \geq (27y^3 + 14y^2 + 210y + 79)/81$ *for* $y \geq 8$.

Then for all $t \geq 2$ the line bundle $\mathcal{L}[t]$ is not secant defective.

Note that Lemma 2.3 is much more than the case of a multiprojective space, since $(Y, \mathcal{L})$ is any secant non-defective pair (with mild numerical conditions only related to the integers $\dim Y$ and $h^0(Y, \mathcal{L})$).

Let M be an integral projective variety, $D \subset M$ an effective divisor and $Z \subset M$ a zero-dimensional scheme. The residual scheme $\mathrm{Res}_D(Z)$ of Z with respected to D is the zero-dimensional subscheme of M with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. Fix $p \in M_{\mathrm{reg}}$. If $p \in D_{\mathrm{reg}}$, then $D \cap (2p, M) = (2p, D)$ and $\mathrm{Res}_D((2p, M)) = \{p\}$. If $p \notin D$, then $\mathrm{Res}_D(2p, M)) = (2p, M)$. For any line bundle $\mathcal{R}$ the following sequence

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{R}(-D) \longrightarrow \mathcal{I}_Z \otimes \mathcal{R} \longrightarrow \mathcal{I}_{Z \cap D, D} \otimes \mathcal{R}_{|D} \longrightarrow 0 \tag{2}$$

is exact.

We often use the following Differential Horace Lemma ([8, 9, 11]).

Lemma 2.4. *([8, 9, 11]) Let M be an integral projective variety, $D \subset M$ and irreducible effective Cartier divisor, $\mathcal{R}$ a line bundle on M and $Z \subset M$ a zero-dimensional scheme. Fix a positive integer e. Let $A \subset X$ be a general union of e double points of M. Let $S \subset D_{\mathrm{reg}}$ be a general subset of cardinality e. To prove that $h^1(M, \mathcal{I}_{Z \cup A} \otimes \mathcal{R}) = 0$ (resp. $h^0(M, \mathcal{I}_{Z \cup A} \otimes \mathcal{R}) = 0$) it is sufficient to prove $h^1(D, \mathcal{I}_{(W \cap D) \cup S} \otimes \mathcal{R}_{|D}) = 0$ and $h^1(M, \mathcal{I}_{\mathrm{Res}_D(Z) \cup (2S, D)} \otimes \mathcal{R}(-D)) = 0$ (resp. $h^0(D, \mathcal{I}_{(W \cap D) \cup S} \otimes \mathcal{R}_{|D}) = 0$ and $h^0(M, \mathcal{I}_{\mathrm{Res}_D(Z) \cup (2S, D)} \otimes \mathcal{R}(-D)) = 0$).*

Remark 2.5. Let $Z \subset W$ be zero-dimensional subschemes of the projective variety M and $\mathcal{R}$ a line bundle on M. Obviously, $h^0(M, \mathcal{I}_Z \otimes \mathcal{R}) \geq h^0(M, \mathcal{I}_W \otimes \mathcal{R})$. Now assume $h^0(\mathcal{I}_W \otimes \mathcal{R}) = h^0(\mathcal{R}) - \deg(W)$ (resp. $h^1(M, \mathcal{I}_W \otimes \mathcal{R}) = 0$). Since

$\dim W = 0$, we have $h^1(W, \mathcal{I}_{Z,W} \otimes \mathcal{R}_{|W}) = 0$. Thus the long cohomology exact sequence of the ideal sequence of Z in W gives $h^0(M, \mathcal{I}_Z \otimes \mathcal{R}) = h^0(\mathcal{R}) - \deg(Z)$ (resp. $h^1(M, \mathcal{I}_Z \otimes \mathcal{R}) = 0$).

Lemma 2.6. *Let M be an integral projective variety, $e \in \mathbb{N}$, $D \subset M$ an integral Cartier divisor of M, $W \subset M$ a zero-dimensional scheme and $S \subset M$ a general subset of D with $\#S = e$. Assume $h^0(M, \mathcal{I}_W \otimes \mathcal{R}) = h^0(M, \mathcal{R}) - \deg(W)$.*

(a) We have $h^0(M, \mathcal{I}_{W\cup S} \otimes \mathcal{R}) = h^0(M, \mathcal{R}) - \deg(W) - e$ if and only if $h^0(M, \mathcal{I}_{\mathrm{Res}_D(W)} \otimes \mathcal{R}(-D)) \le h^0(M, \mathcal{R}) - \deg(W) - e$.

(b) We have $h^0(M, \mathcal{I}_{W\cup S} \otimes \mathcal{R}) = 0$ if and only if $h^0(M, \mathcal{I}_W \otimes \mathcal{R}) \le e$ and $h^0(M, \mathcal{R}(-D)) = 0$.

Proof. We first prove part (a).

Obviously, $h^0(M, \mathcal{I}_{W\cup S} \otimes \mathcal{R}) \ge h^0(M, \mathcal{R}) - \deg(W) - e$. The lemma is trivial if $e = 0$. Thus we may assume $e > 0$ and that the lemma is true for a smaller number of general points of D. Fix $p \in S$ and set $A := S \setminus \{p\}$. By the inductive assumption $h^0(M, \mathcal{I}_{W\cup A} \otimes \mathcal{R}) = h^0(M, \mathcal{R}) - \deg(W) - e + 1$. Assume $h^0(M, \mathcal{I}_{W\cup S} \otimes \mathcal{R}) > h^0(M, \mathcal{R}) - \deg(W) - e$ and hence $h^0(M, \mathcal{I}_{W\cup S} \otimes \mathcal{R}) = h^0(M, \mathcal{I}_{W\cup A} \otimes \mathcal{R})$. Note that $\mathrm{Res}_D(W \cup A) = \mathrm{Res}_D(A)$. Since p is general in D, we get that D is contained in the base locus of $H^0(M, \mathcal{I}_{W\cup A} \otimes \mathcal{R})$ and hence $h^0(M, \mathcal{R}) - \deg(W) - e + 1 = h^0(M, \mathcal{I}_{W\cup A} \otimes \mathcal{R}) = h^0(M, \mathcal{I}_{\mathrm{Res}_D(W)} \otimes \mathcal{R}(-D))$, a contradiction. The proof also give the (never used) "only if" part.

Part (b) follows from part (a) taking $e' := h^0(M, \mathcal{R}) - \deg(W) - e$ and $S' \subseteq S$ such that $\#S' = e'$. □

Remark 2.7. Let Y be a projective variety. Set $X := Y \times \mathbb{P}^m$, $m \ge 1$. Let $p_Y : X \longrightarrow Y$ and $p_2 : X \longrightarrow \mathbb{P}^m$ denote the projections. For any line bundle $\mathcal{L}$ on Y and any $t \in \mathbb{Z}$ set $\mathcal{L}[t] := p_Y^*(\mathcal{L}) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^n}(t))$. The map $(\mathcal{L}, t) \longrightarrow \mathcal{L}[t]$ induces an isomorphism between $\mathrm{Pic}(Y) \times \mathbb{Z}$ and $\mathrm{Pic}(X)$. Now assume $t \ge 0$. The Künneth formula gives $h^0(X, \mathcal{L}[t]) = h^0(Y, \mathcal{L})\binom{m+t}{m}$ and $h^1(X, \mathcal{L}[t]) = h^1(Y, \mathcal{L})\binom{m+t}{m}$. We will always use $(Y, \mathcal{L})$ such that $h^1(\mathcal{L}) = 0$. If $\mathcal{L}$ is spanned, then $\mathcal{L}[t]$ is spanned. If $\mathcal{L}$ is very ample and $t > 0$, then $\mathcal{L}[t]$ is very ample. Let $Z \subset X$ be a zero-dimensional scheme. Let $p_Y(Z)$ denote the scheme-theoretic image of Z. For instance, if Z is a double point of X, then $p_Y(Z)$ is a double point of Y. The definition of $\mathcal{L}[0]$ gives $h^0(\mathcal{I}_Z \otimes \mathcal{L}[0]) = h^0(Y, \mathcal{I}_{p_Y(Z)} \otimes \mathcal{L})$.

3. Preliminaries on the multiprojective spaces

Take $X := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \ge 2$, $n_i > 0$ for all i. Let $\pi_i : X \longrightarrow \mathbb{P}^{n_i}$ denote the i-th projection. Set $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{k-1}}$. Note that $X = Y \times \mathbb{P}^{n_k}$. Let $\pi_Y : X \longrightarrow Y$ denote the projection.

For all $(a_1, \dots, a_k) \in \mathbb{Z}^k$ set $\mathcal{O}_X(a_1, \dots, a_k) := \otimes_{i=1}^k \pi_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(a_i))$. Every line bundle on X is isomorphic to some $\mathcal{O}_X(a_1, \dots, a_k)$. The line bundle $\mathcal{O}_X(a_1, \dots, a_k)$ is globally generated if and only if $a_i \ge 0$ for all i. Assume $a_i \ge 0$ for all i The Künneth formula gives $h^0(\mathcal{O}_X(a_1, \dots, a_k)) = \prod_{i=1}^k \binom{n_i+a_i}{n_i}$ and $h^1(\mathcal{O}_X(a_1, \dots, a_k)) = 0$. The line bundle $\mathcal{O}_X(d_1, \dots, d_k)$ is very ample if and only if $d_i > 0$ for all i.

We write ν or $\nu_{d_1,\dots,d_k}$ for the embedding of X in $\mathbb{P}^r$, $r := -1 + \prod_{i=1}^k \binom{n_i+a_i}{n_i}$ for the embedding induced by $|\mathcal{O}_X(d_1, \dots, d_k)|$ (the Segre-Veronese embedding of X of multidegree $(d_1, \dots, d_k)$).

Set $\varepsilon_k := (a_1, \dots, a_k) \in \mathbb{N}^k$ with $a_i = 0$ for $i < k$ and $a_k = 1$. Fix $H \in |\mathcal{O}_X(\varepsilon_k)|$. The variety H is a divisor of X and $H \cong Y \times \mathbb{P}^{n_k-1}$. Note that $\nu_{|H}$ is the Segre-Veronese embedding of H of multidegree $(d_1, \dots, d_k)$. If $n_k = 1$, then $H = Y \times \{p\}$ for some $p \in \mathbb{P}^1$.

For a fixed multidegree $(d_1, \dots, d_k)$ and all $(a, b) \in \mathbb{N}^2$ let $\sigma_{a,b}(X, H)$ denote the join of a copies of $\nu(X)$ and b copies of $\nu(H)$.

Lemma 3.1. *Assume $\alpha \geq n$. Let $B \subset H$ be a general union of $n-1$ double points of H. Then $h^1(H, \mathcal{I}_{B,H} \otimes \mathcal{L}[1]) = 0$.*

Proof. Since $\deg(B) = n(n-1)$ and $h^0(H, \mathcal{L}[1]_{|H}) = n\alpha \geq n^2$, it is sufficient to prove that $h^1(H, \mathcal{I}_{B,H} \otimes \mathcal{L}[1]_{|H}) = 0$. If $m = 1$ the result is true, because $(Y, \mathcal{L})$ is not secant defective. Thus we may assume $m > 1$ and use induction on m. Take $H_1 \in |\mathcal{O}_H(\varepsilon_k)|$, specialize one connected component of B to a double point of H with reduction contained in H_1 and apply the residual exact sequence of $H_1 \subset H$, the inductive assumption on m and that $h^1(\mathcal{I}_p \otimes \mathcal{R}) = 0$ for every spanned line bundle $\mathcal{R}$ on X. □

Remark 3.2. Take $H \in |\mathcal{O}_X(\varepsilon_k)|$. The group $G_1 := GL(n_1+1) \times \cdots \times GL(n_k)$ acts transitively on H. For each $(a_1, \dots, a_k) \in \mathbb{N}^k$ the vector space $H^0(H, \mathcal{O}_H(a_1, \dots, a_k))$ is an irreducible G_1-representation. The inclusion $H \subset X$ induces an inclusion of $GL(n_k)$ into $GL(n_k + 1)$. This inclusion induces an injective map from G_1 to $G := GL(n_1+1) \times \cdots \times GL(n_k+1)$ and we use this injective map to see G as a subgroup of G_1. Now assume $a_i > 0$ for all i. The surjection $H^0(\mathcal{O}_X(a_1, \dots, a_k)) \longrightarrow H^0(H, \mathcal{O}_H(a_1, \dots, a_k))$ allows as to see $\nu_{a_1,\dots,a_k;H}(H)$ as a subvariety of $\nu_{a_1,\dots,a_k}(X) \subset \mathbb{P}(H^0(\mathcal{O}_X(a_1, \dots, a_k))$. Since G is reductive, the G-module $H^0(\mathcal{O}_X(a_1, \dots, a_k))$ is a direct sum of irreducible G-representations. The restriction of the $GL(n_k + 1)$-representation $H^0(\mathcal{O}_{\mathbb{P}^{n_k}}(a_k))$ to $GL(n_k)$ is the direct sum of $a_k + 1$ representations isomorphic to the $GL(n_k)$-representation $H^0(\mathcal{O}_{\mathbb{P}^{n_k-1}}(c))$, $0 \leq c \leq a_k$ (use the expansion of a homogeneous polynomial with respect to the last variable or use Pieri's formula ([19, p. 80]). Thus $H^0(H, \mathcal{O}_H(a_1, \dots, a_k))$ splits as a direct sum of $a_k + 1$ irreducible G-representations $H^0(H, \mathcal{O}_H(a_1, \dots, c))$, $0 \leq c \leq a_k$. Since these $a_k + 1$ irreducible representations are pairwise not isomorphic, $H^0(\mathcal{O}_X(a_1, \dots, a_k))$ has only finitely many G_1-subrepresentations and each of these G_1-invariant linear subspaces is equal (not just isomorphic) to a sum of some of these representations and the sum is a direct one.

Proposition 3.3. *Assume $\alpha \geq n$. Fix a positive integer t and $(a, b) \in \mathbb{N}^2$ such that $b \leq n-1$. Let $Z \subset X$ be a general union of a double points of X and b double points of H. Then $h^0(\mathcal{I}_Z \otimes \mathcal{L}[t]) \leq \max\{0, \binom{m+t}{m}\alpha - (n+1)a - nb + n(n/2+1)\}$.*

Proof. Let ν be the Segre-Veronese embedding of X induced by $|\mathcal{L}[t]|$. Since $\nu(X)$ is homogeneous and non-degenerate, no proper secant variety of $\nu(X)$ is a cone. Set $W := \sigma_b(H) \subset |\mathcal{L}[t]|^\vee$ with the convention $W = \emptyset$ if $b = 0$. Since $b < n$, Lemma 3.1 gives $w := \dim W = bn - 1$. Apply the Terracini Lemma and the last part of Theorem 2.1. □

Remark 3.4. Fix the integer $n - m$ and then take $m \gg 0$. The assumption $\alpha \geq n$ made in Proposition 3.3 is not satisfied.

Lemma 3.5. *Let $W \subset X$ be multiprojective spaces with X depending on k factors. Fix $(a_1, \dots, a_k) \in \mathbb{N}$. Then the restriction map $\rho : H^0(\mathcal{O}_X(a_1, \dots, a_k)) \longrightarrow H^0(W, \mathcal{O}_W(a_1, \dots, a_k))$ is surjective.*

Proof. Note that $\mathcal{O}_W(a_1,\dots,a_k)$ is well-defined as the restriction of $\mathcal{O}_X(a_1,\dots,a_k)$ to W. By induction on the integer $\dim X - \dim W$ we reduce to the case $\dim W = \dim X - 1$. Thus there is $i \in \{1,\dots,k\}$ such that $W \in |\mathcal{O}_X(\varepsilon_i)|$. Since $a_i - 1 \geq -1$, the Künneth formula gives $h^1(\mathcal{O}_X(a_1,\dots,a_k) - \varepsilon_i)) = 0$. The defining exact sequence of W as divisors of X gives that ρ is surjective. □

Corollary 3.6. *Assume $\alpha \geq n$ and $0 \leq b \leq n-1$. The join $\sigma_{a,b}(X,H)$ has dimension $(n+1)a + nb - 1$ (resp. $r = -1 + (m+1)\alpha$) with respect to $\mathcal{L}[1]$ if $(n+1)a + nb - 1 \leq (m+1)\alpha - n(n+1)/2$ (resp. $(n+1)a + nb \geq \alpha + n(n+1)/2$).*

Proof. Since $b \leq n-1$, we may apply Lemma 3.1 and Proposition 3.3. □

4. Proof of Theorem 1.4 and related results

In this section we prove Theorem 1.4 by induction on t. First we prove the case $t = 2$ (Theorem 4.1), then the case $t = 3$ using the proved case $t = 2$ (Theorem 4.3) and then the cases $t \geq 4$ by induction on t.

Set $m := n_k$ and fix $H \in |\mathcal{O}_X(\varepsilon_k)|$. Note that $H \cong Y \times \mathbb{P}^{m-1}$.

Theorem 4.1. *Assume $\alpha > 2n^2$, $n \geq m+2$ and either $m = 1$ or $(m+1)(2m-1)\alpha \geq n^2(n+3)$. If $m > n/2$ assume (1). Then $\mathcal{L}[2]$ is not secant defective.*

Proof. The case $m = 1$ is true by [13].

Thus we may assume $m > 1$. In step (c) we will prove that we may omit the assumption (1) if $m \leq n/2$.

Set $u := \lfloor \binom{m+2}{2}\alpha/(n+1) \rfloor$, $u_1 := \binom{m+2}{2}\alpha - (n+1)u$, and $v := \lceil \binom{m+2}{2}\alpha/(n+1) \rceil$. Let $Z \subset X$ (resp. $W \subset X$) be a general union of u (resp. v) double points of X. By Remark 2.5 to prove the theorem it is sufficient to prove that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[2]) = 0$ and $h^0(\mathcal{I}_W \otimes \mathcal{L}[2]) = 0$. Set $e := \lfloor \binom{m+1}{2}\alpha/n \rfloor$ and $e_1 := \binom{m+2}{2}\alpha - ne$. We have $0 \leq e_1 \leq n-1$.

(a) In this step we prove that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[2]) = 0$. Let $Z' \subset X$ be a general union of $u - e - e_1$ double points of X. Take a general $(S, S') \subset H \times H$ such that $\#S = e$ and $\#S' = e_1$. By the inductive assumption on m we have $h^1(H, \mathcal{I}_{Z'',H} \otimes \mathcal{L}[2]_{|H}) = 0$, where Z'' is a general union of e double points of H. Thus $h^1(H, \mathcal{I}_{Z'',H} \otimes \mathcal{L}[2]_{|H}) = e_1$. Since S' is general in H, $h^i(H, \mathcal{I}_{Z'' \cup S',H} \otimes \mathcal{L}[2]_{|H}) = 0$, $i = 0, 1$. We specialize e of the connected components of Z to double points of X with a different point of S as their reduction. By the Differential Horace Lemma (Lemma 2.4) to prove that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[2]) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z' \cup S \cup (2S',H)} \otimes \mathcal{L}[1]) = 0$. Since $(n+1)u \leq h^0(\mathcal{L}[2])$, we have $\deg(Z' \cup S \cup (2S', H)) \leq h^0(\mathcal{L}[1])$. The generality of Z' gives $Z' \cap H = \emptyset$. Since S' is general in H, to prove that $h^1(\mathcal{I}_{Z' \cup S \cup (2S',H)} \otimes \mathcal{L}[1]) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z' \cup (2S',H)} \otimes \mathcal{L}[1]) = 0$ and $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[0]) \leq h^0(\mathcal{L}[1]) - \deg(Z') - \deg(S) - \deg((2S', H)) = (m+1)\alpha - \deg(Z') - e - ne_1 + u_1$ (Lemma 2.6).

(a1) In this step we prove that $h^1(\mathcal{I}_{Z' \cup (2S',H)} \otimes \mathcal{L}[1]) = 0$. By Proposition 3.3 and the inequality $e_1 \leq n-1$, it is sufficient to prove that $(n+1)(u-e-e_1) + ne_1 \leq (m+1)\alpha - n(n+1)/2$, i.e. $(n+1)u - (n+1)e - e_1 \leq (m+1)\alpha - n(n+1)/2$. Since $e = (\binom{m+1}{2}\alpha - e_1)/n$ and $(n+1)u \leq \binom{m+2}{2}\alpha$, we have $(n+1)u - (n+1)e - e_1 \leq \binom{m+2}{2}\alpha - (n+1)\binom{m+1}{2}\alpha/n - e_1/n$. Hence $(n+1)u - (n+1)e - e_1 \leq \binom{m+2}{2}\alpha - (n+1)\binom{m+1}{2}\alpha/n$. We have $(m+2)n - (n+1)m = 2n - m$ and $(m+1)(2n-m) - 1 = 2nm - m^2 - 2n - m - 1$. We have $u_1 \geq 0$ and $\binom{m+2}{2}\alpha - (n+1)\binom{m+1}{2}\alpha/n = (m+1)(2m-1)\alpha/2n$. Thus $(n+1)u - (n+1)e - e_1 - (m+1)\alpha \leq (m+1)(2m-1)\alpha/2n$. Hence it it

sufficient to assume $(m+1)(2m-1)\alpha \geq n^2(n+1)$. In step (b1) we will need the stronger inequality $(m+1)(2m-1)\alpha \geq n^2(n+3)$.

(a2) Here we prove that $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[0]) \leq (m+1)\alpha - \deg(Z') - e - ne_1$. We have $\deg(Z') + e + ne_1 = (n+1)u - ne - e_1$ with $ne + e_1 = \binom{m+1}{2}\alpha$, $(n+1)u = \binom{m+2}{2}\alpha - u_1$ and $0 \leq u_1 \leq n$. Thus $h^0(\mathcal{L}[1]) - \deg(Z') - \deg(S) - \deg((2S', H)) = u_1 \geq 0$. Hence it is sufficient to prove that $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[0]) \leq u_1$. Since $\mathcal{L}$ is not defective and $\pi_Y(Z')$ is a general union of $u - e - e_1$ double points of Y, we have $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[0]) = \max\{0, \alpha - (n-m+1)(u-e-e_1)\}$. Thus we need to prove that $\alpha \geq (n-m+1)(u-e-e_1) - u_1 = -u_1 + (n-m+1)u - (n-m+1)e - (n-m+1)e_1$. We have $u = \binom{m+2}{2}\alpha/(n+1) - u_1/(n+1)$ and $e = \binom{m+1}{2}\alpha/n - e_1/n$. Thus $-u_1 + (n-m+1)u - (n-m+1)e - (n-m+1)e_1 = -u_1 + (n-m-1)(\binom{m+2}{2}\alpha - u_1)/(n+1) - (n-m+1)(\binom{m+1}{2}\alpha - e_1)/n - (n-m+1)e_1$. Since $u_1 \geq 0$ and $(n-m+1)/(n+1) \geq 0$, it is sufficient to prove that $(n-m-1)(\binom{m+2}{2}\alpha - u_1)/(n+1) - (n-m+1)(\binom{m+1}{2}\alpha - e_1)/n - (n-m+1)e_1 \leq \alpha$. Since $e_1 \leq n-1$, we have $(n-m+1)e_1 - (n-m+1)e_1/n = e_1(n-m+1)(n-1)/n \leq (n-m+1)(n-1)^2/n$. We have $(m+2)n - (n+1)m = 2n - m$ and $(m+1)(2n-m) - 1 = 2nm - m^2 - 2n - m - 1$. Thus it is sufficient to assume (1).

(b) In this step we prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}[2]) = 0$. Let $W' \subset X$ be a general union of $v - e - e_1$ double points of X. We take (S, S') as in step (a). Thus we reduce the proof that $h^0(\mathcal{I}_W \otimes \mathcal{L}[2]) = 0$ to the proof that $h^0(\mathcal{I}_{W' \cup S \cup (2S'H)} \otimes \mathcal{L}[1]) = 0$. Since S is general in H, it is sufficient to prove that $h^0(\mathcal{I}_{W' \cup (2S', H)} \otimes \mathcal{L}[1]) \leq e$ and that $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[0]) = 0$ (Lemma 2.6).

(b1) We first check that $h^0(\mathcal{I}_{W' \cup (2S', H)} \otimes \mathcal{L}[1]) \leq e$. In step (a1) we proved that $h^1(\mathcal{I}_{Z' \cup S \cup (2S', H)} \otimes \mathcal{L}[1]) = 0$ and hence $h^0(\mathcal{I}_{Z' \cup (2S', H)} \otimes \mathcal{L}[1]) = e + u_1 < e + n + 1$. Thus it is sufficient to prove that $h^1(\mathcal{I}_{W' \cup (2S, H)} \otimes \mathcal{L}[1]) = 0$. We adapt step (a1) with $v \in \{u, u+1\}$. If $v = u$, i.e. if $u_1 = 0$, then we proved it in step (a1). Now assume $u_1 > 0$. We have $(n+1)v = \binom{m+2}{2}\alpha + n + 1 - u_1$ with $n + 1 - u_1 \leq n$. Thus it is sufficient to use $\binom{m+2}{2}\alpha + n$ instead of $\binom{m+2}{2}\alpha$. Hence it is sufficient to assume $(m+1)(2m-1)\alpha \geq n^2(n+3)$.

(b2) Now we prove that $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[0]) = 0$. Since $\mathcal{L}$ is not defective and $\pi_Y(W')$ is a general union of $v - e - e_1$ double points of Y, we have $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[0]) = \max\{0, \alpha - (n-m+1)(v-e-e_1)\}$. Thus we need to prove that $(n-m+1)(v-e-e_1) \geq \alpha$. We have $ne + e_1 = \binom{m+1}{2}\alpha$ and hence $(n-m+1)v - (n-m+1)e - (n-m+1)e_1 \geq (n-m+1)\binom{m+2}{2}\alpha/(n+1) - (n-m+1)\binom{m+1}{2}\alpha/n - (n-m+1)e_1 + (n-m+1)e_1/n$. We have $v = (\binom{m+2}{2}\alpha + (n+1-u_1))/(n+1)$ and $e = (\binom{m+1}{2}\alpha - e_1)/n$. We have $(n-m+1)(n+1-u_1)/(n+1) > 0$. Since $e_1 \leq n-1$, we have $(n-m+1)e_1 - (n-m+1)e_1/n = e_1(n-m+1)(n-1)/n \leq (n-m+1)(n-1)^2/n$. We have $(m+2)n - (n+1)m = 2n - m$ and $(m+1)(2n-m) - 1 = 2nm - m^2 - 2n - m - 1$. Hence it is sufficient to assume the inequality (1).

(c) Now assume $m \leq n/2$. For fixed n and α call $f(m)$ the difference between the left hand side and the right hand side of (1). We have $f'(x) = 2nx - 2x - 1 > 0$ for all $n/2 \leq x \leq n$. Thus it is sufficient to prove that $f(n/2) \geq 0$, i.e. $(3n^2/4 - 3n/2 - 1)\alpha \geq 2(n+1)(n-1)^2$, which is true by the assumption $\alpha > 2n^2$ and the inequality $n \geq m + 2 \geq 4$. □

Lemma 4.2. *Assume $\alpha > 2n^2$, $n \geq m+2 \geq 4$ and $(m+1)(2m-1)\alpha \geq n^2(n+3)$. If $m > n/2$, assume (1). Set $u := \lfloor \binom{m+3}{3}\alpha/(n+1) \rfloor$, $e := \lfloor \binom{m+2}{3}\alpha/(n+1) \rfloor$ and*

$e_1 := \binom{m+2}{2}\alpha-(n+1)e$. *Let $Z' \subset X$ be a general union of $u-e-e_1$ double points of X. Take a general $S' \subset H$ such that $\#S' = ge_1$. Then $h^1(\mathcal{I}_{Z'\cup(2S',H)} \otimes \mathcal{L}[2]) = 0$.*

Proof. Set $u_1 := \binom{m+3}{3}\alpha-(n+1)u$. Thus $0 \le u_1 \le n$. Since $ne+e_1 = \binom{m+2}{3}\alpha$, we have $\deg(Z')+\deg((2S',H)) = h^0(\mathcal{L}[2])-e-u_1 \le h^0(\mathcal{L}[2])-e$. Set $c := \lfloor\binom{m+1}{2}\alpha/n\rfloor$ and $c_1 := \binom{m+1}{2}\alpha - nc$. Since $e_1 \le n-1$, the assumption of α gives $c \ge e_1$. We specialize Z' to a general union of $u-e-c$ double points and $e-c_1$ double points of X with reduction contained in H. To prove the lemma using part (a) of the proof of Theorem 4.1 it is sufficient to prove that $e \ge e_1+c_1$. We have $e_1+c_1 \le 2n-2$. By the definition of e, it is sufficient to prove that $n+(n+1)(2n-2) \le \binom{m+2}{3}\alpha$, an obvious inequality for $m \ge 2$ by the assumption $\alpha > 2n^2$. □

Theorem 4.3. *Assume $\alpha > 2n^2$, $n \ge m+2$ and either $m = 1$ or $(m+1)(2m-1)\alpha \ge n^2(n+3)$. If $m > n/2$ assume (1). Then $\mathcal{L}[3]$ is not secant defective.*

Proof. The case $m = 1$ is true by [13]. Thus we may assume $m > 1$ and use induction on m. Set $u := \lfloor\binom{m+3}{3}\alpha/(n+1)\rfloor$ and $v := \lceil\binom{m+3}{3}\alpha/(n+1)\rceil$. Let $Z \subset X$ (resp. $W \subset X$) be a general union of u (resp. v) double points of X. To prove the theorem it is sufficient to prove that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[3]) = 0$ and $h^0(\mathcal{I}_W \otimes \mathcal{L}[3]) = 0$. Set $e := \lfloor\binom{m+2}{3}\alpha/n\rfloor$ and $e_1 := \binom{m+2}{2}\alpha - ne$. We have $0 \le e_1 \le n-1$.

(a) In this step we prove that $h^1(\mathcal{I}_Z\otimes\mathcal{L}[3]) = 0$. Let $Z' \subset X$ be a general union of $u-e-e_1$ double points of X. Take a general $(S,S') \subset H \times H$ such that $\#S = e$ and $\#S' = e_1$. By the inductive assumption we have $h^1(H,\mathcal{I}_{Z'',H} \otimes \mathcal{L}[3]_{|H}) = 0$, where Z'' is a general union of e double points of H. Thus $h^1(H,\mathcal{I}_{Z'',H} \otimes \mathcal{L}[3]_{|H}) = e_1$. Since S' is general in H, $h^i(H,\mathcal{I}_{Z''\cup S',H} \otimes \mathcal{L}[3]_{|H}) = 0$, $i = 0,1$. We specialize e of the connected components of Z to double points of X with a different point of S as their reduction. By the Differential Horace Lemma (Lemma 2.4) to prove that $h^1(\mathcal{I}_{Z'} \otimes \mathcal{L}[3]) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z'\cup S\cup(2S',H)} \otimes \mathcal{L}[2]) = 0$. Since $(n+1)u \le h^0(\mathcal{L}[3])$ and $ne+e_1 = \binom{m+2}{3}\alpha$, we have $\deg(Z'\cup S\cup(2S',H)) \le h^0(\mathcal{L}[2])$. The generality of Z' gives $Z' \cap H = \emptyset$. Since S' is general in H, to prove that $h^1(\mathcal{I}_{Z'\cup S\cup(2S',H)} \otimes \mathcal{L}[2]) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z'\cup(2S',H)} \otimes \mathcal{L}[2]) = 0$ and $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[1]) \le h^0(\mathcal{L}[2]) - \deg(Z') - \deg(S) - \deg((2S',H))$ (Lemma 2.6). Lemma 4.2 gives $h^1(\mathcal{I}_{Z'\cup(2S',H)} \otimes \mathcal{L}[2]) = 0$.

Now we prove that $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[1]) \le h^0(\mathcal{L}[2]) - \deg(Z') - \deg(S) - \deg((2S',H))$. We have $h^0(\mathcal{L}[2]) - \deg(Z') - \deg(S) - \deg((2S',H)) = u_1 \ge 0$.

Recall that $\#S = e$ and $\deg((2S',H)) = ne_1$. Since $ne+e_1 = \binom{m+1}{2}\alpha$ and $(n+1)u \le \binom{m+3}{3}\alpha$, we have $\binom{m+2}{2}\alpha - \deg(Z') \ge e+ne_1$. By Proposition 3.3 to get $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[1]) = 0$ it is sufficient to prove the inequality $(n+1)(u-e-e_1) \ge (m+1)\alpha + n(n+1)/2$.

We have $(n+1)u \ge \binom{m+3}{3}\alpha - n$. We have $ne+e_1 = \binom{m+2}{3}\alpha$ and hence $(n+1)(e+e_1) = (n+1)\binom{m+2}{3}\alpha/n+ne_1$. Thus $(n+1)(e+e_1) = (n+1)\binom{m+2}{3}\alpha/n+ne_1$. Since $e_1 \le n-1$, we get $(n+1)(u-e-e_1) \ge \binom{m+3}{3}\alpha-(n+1)\binom{m+2}{3}\alpha/n-n^2+n$.

We have $n(m+3)-(n+1)m = 3n-m$ and hence $\binom{m+3}{3}\alpha-(n+1)\binom{m+2}{3}\alpha/n = (3n-m)(m+2)(m+1)\alpha/6n$. Since $(3n-m)(m+2)-6n = 3nm-m^2-2m$, we have $(2n-m-1)(m+2)(m+1)\alpha-6n(m+1)\alpha = \alpha(m+1)m(3n-m-2)$. We have $n^2-n+n(n+1)/2 = n(2n-1)/2$. Thus it is sufficient to have

$$m(m+1)(3n-m-2)\alpha \ge 3n^2(3n-1) \tag{3}$$

Since $n \geq m+2$ and $\alpha > 2n^2$, (3) is satisfied if $m \geq 3$. Now assume $m = 2$. In this case (3) is the inequality $2(3n-4)\alpha \geq 3n^2(3n-1)$, which is true by the assumptions $\alpha > 2n^2$ and $n \geq 4$.

(b) Now we prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}[3]) = 0$. Since $Z = W$ if $u_1 = 0$, we may assume $W \neq Z$ and hence $\deg(W) = (n+1) - u_1 + \binom{m+3}{3}\alpha \geq 1 + \binom{m+3}{3}\alpha$. Let $W' \subset X$ be a general union of $v - e - e_1$ double points of X. As in step (a) take a general $(S, S') \subset H \times H$ such that $\#S = e$ and $\#S' = e_1$. As in step (a) by Lemma 2.6 to conclude the proof it is sufficient to prove that $h^0(\mathcal{I}_{W' \cup (2S', H)} \otimes \mathcal{L}[2]) \leq e$ and that $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[1]) = 0$.

Lemma 4.2 gives $h^1(\mathcal{I}_{Z' \cup (2S', H)} \otimes \mathcal{L}[2]) = 0$ and hence $h^0(\mathcal{I}_{Z' \cup (2S', H)} \otimes \mathcal{L}[2]) = e + u_1$. The same proof (or step (b1) of the proof of Theorem 4.1) with W' instead of Z' gives $h^0(\mathcal{I}_{W' \cup (2S', H)} \otimes \mathcal{L}[2]) \leq e$.

Now we prove that $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[1]) \leq h^0(\mathcal{L}[2]) - \deg(W') - \deg(S) - \deg((2S', H))$. We have $h^0(\mathcal{L}[2]) - \deg(Z') - \deg(W') - \deg((2S', H)) = n + 1 - u_1 \geq 1$. In step (a) we proved that $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[1]) = 0$ and hence $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[1]) = 0$. □

Proof of Theorem 1.4: The case $m = 1$ is true by [13]. Thus we assume $m > 1$ and use induction on m.

We also use induction on t. The case $t = 2$ is true by Theorem 4.1. The case $t = 3$ is true by Theorem 4.3. Thus we may assume $t > 3$ and that $\mathcal{L}[t-1]$ and $\mathcal{L}[t-2]$ are not secant defective.

Set $u := \lfloor \binom{t+m}{m}\alpha/(n+1) \rfloor$, $u_1 := \binom{m+t}{m}\alpha - (n+1)u$ and $v := \lceil \binom{t+m}{m}\alpha/(n+1) \rceil$. Let $Z \subset X$ (resp. $W \subset X$) be a general union of u (resp. v) double points of X. By Remark 2.5 it is sufficient to prove that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[t]) = 0$ and $h^0(\mathcal{I}_W \otimes \mathcal{L}[t]) = 0$. Set $e := \lfloor \binom{m+t-1}{m-1}\alpha/n \rfloor$ and $e_1 := \binom{m+t-1}{m-1}\alpha - ne$. We have $0 \leq e_1 \leq n$. Take a general $(S, S') \subset H \times H$ such that $\#S = e$ and $\#S' = e_1$.

(a) In this step we prove that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[t]) = 0$. Let $Z' \subset X$ be a general union of $u - e - e_1$ double points of X. By the inductive assumption on m we have $h^1(H, \mathcal{I}_{Z'',H} \otimes \mathcal{L}[t]_{|H}) = 0$, where Z'' is a general union of e double points of H. Thus $h^1(H, \mathcal{I}_{Z'',H} \otimes \mathcal{L}[t]_{|H}) = e_1$. Since S' is general in H, $h^i(H, \mathcal{I}_{Z'' \cup S', H} \otimes \mathcal{L}[t]_{|H}) = 0$, $i = 0, 1$. We specialize e of the connected components of Z to double points of X with a different point of S as their reduction. By the Differential Horace Lemma (Lemma 2.4) to prove that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[t]) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z' \cup S \cup (2S', H)} \otimes \mathcal{L}[t-1]) = 0$. Since $(n+1)u \leq h^0(\mathcal{L}[t])$ and $ne + e_1 = \binom{m+t-1}{m-1}\alpha$, we have $\deg(Z' \cup S \cup (2S', H)) \leq h^0(\mathcal{L}[t-1])$. The generality of Z' gives $Z' \cap H = \emptyset$. Since S' is general in H, to prove that $h^1(\mathcal{I}_{Z' \cup S \cup (2S', H)} \otimes \mathcal{L}[t-1]) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z' \cup (2S', H)} \otimes \mathcal{L}[t-1]) = 0$ and $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[t-2]) \leq h^0(\mathcal{L}[t-1]) - \deg(Z') - \deg(S) - \deg((2S', H))$ (Lemma 2.6). Since $e_1 \leq n-1$ and $e \geq \binom{m+t-1}{m-1}/n - 1$, we have $e \geq c_1$.

(a1) Here we check that $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[t-2]) \leq h^0(\mathcal{L}[t-1]) - \deg(Z') - \deg(S) - \deg((2S', H))$. Since $(n+1)u \leq \binom{m+t}{m}\alpha$ and $ne + e_1 = \binom{m+t-1}{m-1}\alpha$, we have $h^0(\mathcal{L}[t-1]) - \deg(Z') - \deg(S) - \deg((2S', H)) \geq 0$. Since $\mathcal{L}[t-2]$ is not secant defective, to prove that $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[t-2]) = 0$ it is sufficient to prove that $(n+1)(u - e - e_1) \geq \binom{m+t-2}{m}\alpha$. We have $(n+1)u \geq \binom{m+t}{m}\alpha - n$ and $(n+1)(e+e_1) = e + ne_1 + \binom{m+t-1}{m-1}\alpha$. Thus $(n+1)(u - e - e_1) \geq -n + \binom{m+t-1}{m}\alpha - e - ne_1$ with $e \leq \binom{m+t-1}{m-1}\alpha/n$ and $e_1 \leq n-1$. Thus it is sufficient to prove that $\binom{m+t-1}{m}\alpha \geq \binom{m+t-1}{m-1}\alpha/n + n^2$.

We have $\binom{m+t-1}{m} - \binom{m+t-1}{m-1}/n = \binom{m+t-1}{m-1}[t/m - 1/n]$. Since $t \ge 3$, $m \ge 2$ and $n \ge m+2$, it is sufficient to use that $\alpha > n^2$.

(a2) Here we prove that $h^1(\mathcal{I}_{Z' \cup (2S',H)} \otimes \mathcal{L}[t-1]) = 0$. Set $c := \lfloor \binom{m+t-2}{m-1}\alpha/n \rfloor$ and $c_1 = \binom{m+t-2}{m-1}\alpha - nc$. Since $\binom{m+t-2}{m-1}\alpha \ge n(n-1)$ and $e_1 \le n-1$, we have $c \ge e_1$. Let $Z'' \subset X$ be a general union of $u - e - c$ double points of X. Let $S'' \subset H$ be a general subset of H with cardinality $c - e_1$. Set $Z_1 := Z'' \cup 2S'' \cup (2S', H)$. By the semicontinuity theorem for cohomology to prove that $h^1(\mathcal{I}_{Z' \cup (2S',H)} \otimes \mathcal{L}[t-1]) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Z_1} \otimes \mathcal{L}[t-1]) = 0$. We have $\mathrm{Res}_H(Z_1) = Z'' \cup S''$. Since $nc \le \binom{m+t-2}{m-1}\alpha$ and $\mathcal{L}[t-1]_{|H}$ is not secant defective, $h^1(H, \mathcal{I}_{Z_1 \cap H} \otimes \mathcal{L}[t-1]_{|H}) = 0$. Hence it is sufficient to prove that $h^1(\mathcal{I}_{Z'' \cup S''} \otimes \mathcal{L}[t-2]) = 0$. We have $\deg(Z'' \cup S'') = \deg(Z) - \binom{m+t-1}{m-1}\alpha - \binom{m+t-2}{m-1}\alpha - e + c_1$. Since $c_1 \le n-1 \le e$, we have $\deg(Z'' \cup S'') \le h^0(\mathcal{L}[t-2])$. Since $\mathcal{L}[t-2]$ is not secant defective, we have $h^1(\mathcal{I}_{Z''} \otimes \mathcal{L}[t-2]) = 0$ and $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[t-2]) \ge c$. By Lemma 2.6 to conclude the proof that $h^1(\mathcal{I}_Z \otimes \mathcal{L}[t]) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[t-3]) \le \binom{m+t-1}{m} - \deg(Z'') - c$. We proved that the right hand side of the last inequality (to be proved) is ≥ 0. We have $\deg(Z'') = \binom{m+t-2}{m}\alpha - u_1 - e - c$. Assume for the moment $t \ge 5$, so that $\mathcal{L}[t-3]$ is not defective. In this case either $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[t-3]) = 0$ or $h^1(\mathcal{I}_{Z''}[t-3]) = 0$ and hence $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[t-3]) = \binom{m+t-3}{m}\alpha - \deg(Z'')$. In the latter case it is sufficient to use that $\binom{m+t-3}{m-1}\alpha \ge c$, because $m + t - 2 \le n(t-2)$ and $nc \le \binom{m+t-2}{m-1}$.

Now assume $t = 4$. Since $h^1(\mathcal{I}_{Z''} \otimes \mathcal{L}[2]) = 0$, we have $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[2]) = \binom{m+2}{2}\alpha - \deg(Z'')$. By Proposition 3.3 we have $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[1]) \le \max\{0, (m+1)\alpha - \deg(Z'') + n(n+1)/2\}$. Thus to conclude the step proving that $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[1]) = 0$. it is sufficient to prove that $\binom{m+2}{2}\alpha \ge (m+1)\alpha + c + n(n+1)/2$, i.e. $\binom{m+1}{2}\alpha \ge c + n(n+1)/2$. We have $c \le \binom{m+2}{3}/n$. Thus it is sufficient to prove the inequality $n\binom{m+1}{2}\alpha \ge \binom{m+2}{3}\alpha + n^2(n+1)/2$. This inequality is true for $m = 2$, because $n \ge 4$. We claim that for $m \ge 3$ the inequality is true by induction on m with n and α fixed. Indeed, $n\binom{m+2}{2} - n\binom{m+1}{2} = n(m+1)$, while $\binom{m+3}{3} - \binom{m+2}{3} = \binom{m+2}{2}$.

(b) In this step we prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}[t]) = 0$. If $u_1 = 0$, i.e. if $u = v$, this is true by step (a), because $h^0(\mathcal{I}_Z \otimes \mathcal{L}[t]) = h^1(\mathcal{I}_Z \otimes \mathcal{L}[t])$ if $u_1 = 0$. Thus we may assume $u_1 > 0$. Set $v_1 := n + 1 - u_1$. Let $W' \subset X$ be a general union of $v - e - e_1$ double points of X. By the inductive assumption on m we have $h^1(H, \mathcal{I}_{Z'',H} \otimes \mathcal{L}[t]_{|H}) = 0$, where Z'' is a general union of e double points of H. Thus $h^1(H, \mathcal{I}_{Z'',H} \otimes \mathcal{L}[t]_{|H}) = e_1$. Since S' is general in H, $h^i(H, \mathcal{I}_{Z'' \cup S',H} \otimes \mathcal{L}[t]_{|H}) = 0$, $i = 0, 1$. We specialize e of the connected components of W to double points of X with a different point of S as their reduction. By the Differential Horace Lemma to prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}[t]) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{W' \cup S \cup (2S',H)} \otimes \mathcal{L}[t-1]) = 0$. Since $(n+1)v \ge \binom{m+t}{m}\alpha$ and $W' \cap H = \emptyset$, Lemma 2.6 shows that to prove step (b) it is sufficient to prove that $h^0(\mathcal{I}_{W' \cup (2S,H)} \otimes \mathcal{L}[t-1]) \le e$ and that $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[t-2]) = 0$.

(b1) Since $\mathcal{L}[t-2]$ is not secant defective, to prove that $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[t-2]) = 0$ it is sufficient to check that $\deg(W') \ge \binom{m+t-2}{m}\alpha$. Since $e_1 \le n-1$ and $ne = \binom{m+t-1}{m}\alpha - e_1$, we have $(n+1)(e+e_1) = (n+1)\binom{m+t-1}{m-1}(n+1)/n + e_1(n^2-1)/n \le (n+1)\binom{m+t-1}{m-1}/n + (n^2-1)(n-1)/n$. Since $(n+1)v \ge \binom{m+t+m}{m}\alpha$, to prove that $h^0(\mathcal{I}_{W'} \otimes \mathcal{L}[t-2])) = 0$ it is sufficient to check that $n\binom{m+t}{m}\alpha - (n+1)\binom{m+t-1}{m-1}\alpha \ge (n^2-1)(n-1)$. By our assumption on α it is sufficient to prove that $c(n,m) := n\binom{m+t}{m} - (n+1)\binom{m+t-1}{m-1} \ge 2$. We have $c(n,2) = n(t+2)(t+1)/2 - (n+1)(t+$

1) ≥ 2, because $t \geq 4$. Now assume $m > 2$ and that $c(n, m-1) \geq 2$. We have $d(n,m) := c(n,m) - c(n,m-1) = n\binom{m+t-1}{m} - (n+1)\binom{m+t-2}{m-1}$. We have $m!d(n,m) = \prod_{i=0}^{m-2}(m+t-1-i)[n(m+t-1) - (n+1)m] \geq 0$.

(b2) Here we check that $h^0(\mathcal{I}_{W'\cup(2S,H)} \otimes \mathcal{L}[t-1]) \leq e$. In step (a2) we proved that $h^1(\mathcal{I}_{Z'\cup(2S,H)} \otimes \mathcal{L}[t-1]) = 0$, i.e. $h^0(\mathcal{I}_{Z'\cup(2S,H)} \otimes \mathcal{L}[t-1]) = u_1 + e$, which is not enough to conclude, unless u_1 is very low. Since $n+1 \geq n+1-u_1$, it is sufficient to prove that $h^1(\mathcal{I}_{W'\cup(2S',H)} \otimes \mathcal{L}[t-1]) = 0$. We take c and S'' as in step (a2). We have $c \geq e_1$. Let $W'' \subset X$ be a general union of $v - e - c$ double points of X. We specialize $c - e_1$ of the connected components of W' to different points of S'. Since $\mathcal{L}[t-1]_{|H}$ is not defective, a residual exact sequence shows that it is sufficient to prove that $h^1(\mathcal{I}_{W''\cup S''} \otimes \mathcal{L}[t-2]) = 0$.

Observation 1: As in step (a2) we see that $\deg(W'') + c - e_1 = \binom{m+t-2}{m}\alpha + v_1 + c_1 - e$. Since $e \geq 2n-1$, we get $\deg(W'' \cup S'') \leq \binom{m+t-2}{m}\alpha$

By Lemma 2.6 to conclude the proof of the theorem it is sufficient to prove that $h^1(\mathcal{I}_{W''} \otimes \mathcal{L}[t-2]) = 0$ and that $h^0(\mathcal{I}_{W''} \otimes \mathcal{L}[t-3]) \leq \binom{m+t-2}{m}\alpha - \deg(W'') - (c - e_1)$.

Since $\mathcal{L}[t-2]$ is not secant defective, Observation 1 gives $h^1(\mathcal{I}_{W''} \otimes \mathcal{L}[t-2]) = 0$. If $t \geq 5$, $\mathcal{L}[t-3]$ is not secant defective. Thus $h^0(\mathcal{I}_{W''} \otimes \mathcal{L}[t-3]) = \max\{0, \binom{m+t-3}{m}\alpha - \deg(W'')\}$ and we saw that $c - e_1 \leq \binom{m+t-2}{m}\alpha - \binom{m+t-3}{m}\alpha$.

Now assume $t = 4$. We proved in step (a2) that $h^0(\mathcal{I}_{Z''} \otimes \mathcal{L}[1]) = 0$. Hence $h^0(\mathcal{I}_{W''} \otimes \mathcal{L}[1]) = 0$. □

5. Proofs of Theorems 1.1 and 1.3 and Corollary 1.5

Proof of Theorem 1.3: The theorem is a particular case of Theorem 4.1. □

Proof of Theorem 1.1: We set $m := n_k$ and $n := n_1 + \cdots + n_k$. By [20] we may assume $k \geq 3$ and hence $m \leq n/k$. It is sufficient to prove that the assumptions of Theorem 1.4 are satisfied. In steps (a) and (b) we assume $m > 1$.

(a) Assume $k = 3$. We have $\alpha = \binom{d_1+n_1}{n_1}\binom{d_2+n_2}{n_2}$. Thus α is an increasing function of d_1 and d_2. Thus it is sufficient to check the case $d_1 = d_2 = 3$. First assume $n_1 = n_2 = n_3$ and hence $n = 3m$ and $\alpha = (m+3)^2(m+2)^2(m+1)^2/36$. We have $2n^2 = 18m^3$. If $m \geq 6$ we have $(m+3)^2(m+2) \geq 36 \cdot 18$ and hence $\alpha > 2n^2$. If $m = 5$ we have $\alpha = 56^2 > 2250 = 18m^3$. If $m = 4$ we have $\alpha = 35^2 > 18 \cdot 64$. If $m = 3$ we have $\alpha = 400 > 2 \cdot 81$. For $m = 2$ we have $\alpha = 100 > 2 \cdot 6^2$. We have $n^2(n+3) = 27(m+1)m^2$ and hence $(m+1)(2m-1)\alpha \geq n^2(n+3)$ for all $m \geq 2$. Now assume again $(d_1, d_2) = (3,3)$, but we allow the case $n_1 \geq n_3$. We write $\alpha(n_1, n_2)$ instead of α and of course $n = n_1 + n_2 + m$. We have $\alpha(n_1+1, n_2)/\alpha(n_1, n_2) = \binom{n_1+4}{3}/\binom{n_1+3}{3} = (n_1+4)/n_1 = 1 + \frac{4}{n_1}$. We have $2(n_1+1+n_2+m)^2/2(n_1+n_2+m) = (1 + \frac{1}{n_1+n_2+m})^2 < 1 + \frac{4}{n_1}$. In the same way we see that passing from (n_1, n_2, m) to (n_1, n_2+1, m) only improve that inequality. Now we fix m and consider the inequality

$$(m+1)(2m-1)\alpha \geq (n_1+n_2+m)^2(n_1+n_2+m+3). \tag{4}$$

For a fixed m call $a(n_1, n_2)$ (resp. $b(n_1, n_2)$) the left hand side (resp. the right hand side) of the inequality 4. We saw that $a(n_1+1, n_2)/(a(n_1, n_2) = 1 + \frac{4}{n_1}$. We have $b(n_1+1, n_2)/b(n_1, n_2) = (1 + \frac{1}{n_1+n_2+m})^2(1 + \frac{1}{n_1+n_2+m+3}) < 1 + \frac{4}{n_1}$. In the same way we see that $a(n_1, n_2+1)/a(n_1, n_2) > b(n_1, n_2+1)/b(n_1, n_2)$, concluding the proof of the case $k = 3$.

(b) Assume $k > 3$. Since α is an increasing function of the variables $d_1, \dots, d_{k-1}$, it is sufficient to do the case $d_1 = d_2 = 3$ and $d_i = 2$ for $i = 3, \dots, k-1$.

First assume $n_i = m$ for all i and hence $\alpha = \binom{m+3}{3}^2 \binom{m+2}{2}^{k-3}$. We have $n = km$ and hence $2n^2 = 2k^k 2m^2$. For a fixed m we write α_k instead of α. We have $2(k+1)^2m^2/2k^2m^2 = (1+\frac{1}{k})^2$. We have $\alpha_{k+1}/\alpha_k = \binom{m+2}{2} \geq 3$. Thus we conclude by the case $k = 3$ by induction on k. We also have $(m+1)(2m-1)\alpha_{k+1}/(m+1)(2m-1)\alpha_k = \binom{m+2}{2} \geq 3 > (1+\frac{1}{k})^2 (+\frac{1}{k+1})$. Then we fix k and, as in step (a) pass from the inequality for $(n_1, \dots, n_{k-1})$ to the inequality for $(m_1, \dots, m_{k-1})$ with $m_i \geq n_i$ for all i and $m_1 + \cdots + m_{k-1} = n_1 + \cdots + n_{k-1} + 1$ with $\alpha(m_1, \dots, m_{k-1})/\alpha(n_1, \dots, n_{k-1}) = 1 + \frac{3}{n_i}$ if $m_i = n_i + 1$ for $3 \leq i \leq k-1$. We have $(n_1 + \cdots + n_{k-1} + m + 1)^2/(n_1 + \cdots n_{k-1} + m)^2 = (1 + \frac{1}{n_1+\cdots+n_{k-1}+m})^2 (1 + \frac{1}{n_1+\cdots+n_{k-1}+m+1})^2 < 1 + \frac{3}{n_i}$ and $(n_1 + \cdots + n_{k-1} + m + 1)^2 (n_1 + \cdots + n_{k-1} + m + 4)/(n_1 + \cdots + n_{k-1} + m)^2 (n_1 + \cdots + n_{k-1} + m + 3) = (1 + \frac{1}{n_1+\cdots+n_{k-1}+m})^2 (1 + \frac{1}{n_1+\cdots+n_{k-1}+m+3})$. Since $n_i \leq (n_1 + n_2 + n_i)/3$ for all $i > 2$, if $i > 2$ we have $1 + \frac{3}{n_i} > (1 + \frac{1}{n_1+\cdots+n_{k-1}+m})^2 (1 + \frac{1}{n_1+\cdots+n_{k-1}+m+3})$.

(c) Now assume $m = 1$. If $n_1 = 1$, then we quote [24]. Now assume $n_1 \geq 2$. We have $n \geq 4$ and $\alpha > n^2$. Thus we may use [13], starting with the projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i}$, where i is the maximal integer such that $n_i > 1$, unless $n_2 = 1$. If $n_2 = 1$, we use $\mathbb{P}^{n_1} \times \mathbb{P}^1$ in bidegree (d_1, d_2) to start the induction. □

Proof of Corollary 1.5: We apply [26, Th. 1.5] to the Segre-Veronese embedding $X \subset \mathbb{P}^r$ of multidegree $(d_1, \dots, d_k)$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We only need to check the assumption "X not Gaussian defective" of [26, Th.1.5]. This check is obvious in all interesting cases for linear algebra. In our case it is true, because X is not ruled by lines ([22, p. 173]). □

6. Related open problems

Of course, if we can improve the numerology of the proof of Theorem 1.4, then more cases can be done. The key step would be to weaken the numerical assumptions in the statement of Theorem 4.1, since the numerical inequalities to be checked for the proof of Theorem 4.3 are weaker (we need the ones of Theorem 4.1, because we use $\mathcal{L}[2]$ in the proof of of Theorem 4.3). It would also improve Theorem 1.3. Even easier is the inductive proof of Theorem 1.4 for $t \geq 5$, with only small checks for the case $t = 4$. This is of course old news, spelled out explicitly in [4] and implicitly since the earlier papers by J. Alexander and A. Hirschowitz ([8, 9, 10, 11] and earlier papers).

We want to stress some open possibilities of extensions.

(a) Take a homogeneous variety X, a very ample homogeneous line bundle $\mathcal{R}$ on X and an integral effective divisor D such that all line bundles $\mathcal{R}(tD)$, $t \geq 1$, are homogeneous. We do not require that D is G-homogenous for the same algebraic group G such that $X = G/P$, because usually no such D exists, but we require that $\mathcal{R}(tD)$ is G-homogeneous. Set $n := \dim X$. Call j the embedding induced by $|\mathcal{R}(D)|$. Call $\sigma_{a,b}(X, D) \subseteq \mathbb{P}H^0(\mathcal{R}(D))^\vee$ the join of a copies of $j(X)$ and b copies of $j(D)$. If we know that $\sigma_{0,b}(X, D)$ has the expected dimensions for all $0 \leq b \leq n-1$, we apply Theorem 2.1 to all $\sigma_{a,b}(X, D)$ and then we are in the set-up of the use of the Differential Horace Lemma for all $\mathcal{R}(tD)$, $t \geq 2$. Each $(X, \mathcal{R})$ should require different numerical assumptions.

(b) Take X, $\mathcal{R}$ and D as in (a) with $(X, \mathcal{R})$ homogenous with respect to an algebraic group G_1 and D homogeneous for a proper subgroup $G \subset G_1$ for which we know the decomposition of $H^0(\mathcal{R}(tD))$ into irreducible G-representations. We tried to use it with $D := H$ (Remark 3.2), but we got worst bounds on α to prove that $\mathcal{L}[2]$ is not secant defective.

(c) In the set-up of (a) consider for $|\mathcal{R}(tD)|$ the dimensions of all joins of a copies of the image of X and b copies of the images of D for all possible a and b. It is easy in the step $\mathcal{R}(tD) \Longrightarrow \mathcal{R}((t+1)D)$ to increase both a and b. By the Terracini Lemma we get the equivalence of this problem and the interpolation problem with the union of a general double points of X and a general double points of D.

(d) Theorem 1.4 is proved by induction on t, the key step being to prove the case $t = 2$ using a far weaker statement for $t = 1$ coming from Theorem 2.1 for $t = 1$. To get the case $t = 3$ it is sufficient to have a weaker statement for $t = 2$ (but far better than the one coming applying Theorem 2.1 for $t = 2$). And so on, of course, if the result for $t = 2$ is not sufficient to get for $t = 3$ the non-defectivity of all secant varieties.

7. Competing interest

The author has no competing interest.

References

[1] H. Abo, On non-defectivity of certain Segre-Veronese varieties, J. Symbolic Comput 45 (2010), no. 12, 1254–1269.

[2] H. Abo and M. C. Brambilla, Secant varieties of Segre-Veronese varieties $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}(1, 2)$, Experiment. Math. 18 (2009), no. 3, 369–384.

[3] H. Abo and M. C. Brambilla, New examples of defective secant varieties of Segre-Veronese varieties, Collect. Math. 63 (2012), no. 3, 287–297.

[4] H. Abo and M. C. Brambilla, On the dimensions of secant varieties of Segre-Veronese varieties, Ann. Mat. Pura Appl. 192 (2013), no. 1, 61–92.

[5] H. Abo, C. Peterson, and G. Ottaviani, Induction for secant varieties of Segre varieties, Trans. Amer. Math. Soc. 361 (2009), no. 2, 767–792.

[6] B. Ådlandsvik, Joins and higher secant varieties. Math. Scand. 1987, 61, 213–222. https://doi.org/10.7146/math.scand.a-12200.

[7] B. Ådlandsvik, Varieties with an extremal number of degenerate higher secant varieties, J. Reine Angew. Math. 392 (1988), 16–26.

[8] J. Alexander and A. Hirschowitz, Un lemme d'Horace différentiel: application aux singularité hyperquartiques de $\mathbb{P}^5$, J. Algebraic Geom. 1 (1992), 411–426.

[9] J. Alexander and A. Hirschowitz, La méthode d'Horace éclatée: application à l'interpolation en degrée quatre. Invent. Math. 107 (1992), no. 3, 585–602.

[10] J. Alexander and A. Hirschowitz, Polynomial interpolation in several variables. J. Algebraic Geom. 4(1995), no. 2, 201–222.

[11] J. Alexander and A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points, Invent. Math. 140 (2000), 303–325. https://doi.org/10.1007/s002220000053

[12] C. Aranjo, A. Massarenti, and R. Richter, On non-secant defectivity of Segre-Veronese varieties of Segree varieties, Trans. Amer. Math. Soc. 371 (2019), no. 4, 2255–2278.

[13] E. Ballico, Partially symmetric tensors and the non-defectivity of secant varieties of products with a projective line as a factor, Vietnam Journal of Mathematics (to appear)

[14] E. Ballico, A. Bernardi, and T. Mańdzuk, Tensoring by a plane preserves secant-regularity in degree at least two, ArXiv:2312.01933.

[15] A. Bernardi, E. Carlini, and M. V. Catalisano, Higher secant varieties of $\mathbb{P}^n \times \mathbb{P}^m$ embedded in bidegree $(1, d)$, J. Pure, Appl. Algebra 215 (2011), no. 12, 2853–2858.

[16] A. T. Blomenhofer and A. Casarotti, Nondefectivity of invariant secant varieties, ArXiv:2312.12335

[17] M. V. Catalisano, A. V. Geramita, and A. Gimigliano, Higher secant varieties of Segre-Veronese varieties, in Projective Varieties with unexpected varieties, pp. 81–107, Walter de Gruyter, Berlin, 2005, MR 2202248
[18] M. V. Catalisano, A. V. Geramita, and A. Gimigliano, Secant varieties of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n times) are not defective for $n \geq 5$, J. Algebraic Geom. 20 (2011), no. 2, 295–327.
[19] W. Fulton and J. Harris, Theory of representations, A first course, Springer-Verlag, Berlin, 1991.
[20] F. Galuppi and A. Oneto, A. Secant non-defectivity via collisions of fat points, Adv. Math. 2022, 409, 108657, https://doi.org/10.1016/j.aim.2022.108657
[21] R. Hartshorne, Algebraic Geometry, Springer, Berlin, 1977.
[22] S. L. Kleiman, Tangency and duality, Proceedings of the 1984 Vancouver conference in algebraic geometry, 163–225, CMS Conf. Proc., 6, Amer. Math. Soc., Providence, RI, 1986.
[23] A. Laface, A. Massarenti, and R. Richter, On secant defectiveness and identifiability of Segre-Veronese varieties, Rev. Mt. Iberoam. 38 (2022), no. 5, 1605–1635.
[24] A. Laface and E. Postinghel, Secant varieties of Segre-Veronese embeddings of $(\mathbb{P}^1)^r$. Math. Ann. (2013) 356, 1455–1470.
[25] J. Landsberg, Tensors: Geometry and applications, Providence, RI: American Mathematical Society (AMS) (2012).
[26] A. Massarenti and M. Mella, Bronowski's conjecture and the identifiability of projective varieties, ArXiv:2210.13524v3 (to appear on Duke Math. J.).

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