Wick powers in stochastic PDEs: an introduction.

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1 Introduction

Consider the following stochastic differential equation in $L^2(0,2\pi)$ (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$),

$$dX = \left[\tfrac{1}{2}\,(X_{\xi\xi} - X) - X^3\right] dt + dW(t), \quad X(0) = x \in L^2(0,2\pi), \tag{1.1}$$

where $\xi \in [0,2\pi]$, X is 2π–periodic and $W(t)$ is a cylindrical Wiener process (defined below) and $X_{\xi\xi}$ denotes the second derivative of X with respect to ξ.

Denote by $(e_k)_{k\in\mathbb{Z}}$ the complete orthonormal system of $L^2(0,2\pi)$,

$$e_k(\xi) = \tfrac{1}{\sqrt{2\pi}}\, e^{ik\xi},\ \ \xi \in [0,2\pi],\ k \in \mathbb{Z}$$

and define

$$W(t) = \sum_{k\in\mathbb{Z}} \beta_k(t) e_k,$$

where $(\beta_k(t))_{k\in\mathbb{Z}}$ is a family of standard Brownian motions mutually independent in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Let us write equation (1.1) in the following mild form

$$X(t) = e^{tA}x - \int_0^t e^{(t-s)A} X^3(s)\, ds + W_A(t), \tag{1.2}$$

where [1]

[1] $H^2(0,2\pi)$ is the usual Sobolev space.

$$Ax = \tfrac{1}{2}\,(x_{\xi\xi} - x), \quad x \in \{y \in H^2(0, 2\pi) :\ y(0) = y(2\pi),\ y_\xi(0) = y_\xi(2\pi)\}$$

and

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s) = \sum_{k\in\mathbb{Z}} \int_0^t e^{-\frac{1}{2}\,(t-s)(1+|k|^2)} d\beta_k(s). \tag{1.3}$$

It is easy to see that the *stochastic convolution* $W_A(t)$ is a Gaussian random variable in $L^2(0, 2\pi)$ with mean 0 and covariance operator

$$C(t) = C(1 - e^{tA}), \qquad t \geq 0$$

where

$$C = -\tfrac{1}{2}\, A^{-1}.$$

Notice that

$$Ce_k = \frac{1}{1 + |k|^2}\, e_k, \qquad k \in \mathbb{Z},$$

so that $C(t)$ is a trace class operator. Moreover, one can see that the probability measure (on $L^2(0, 2\pi)$)

$$\nu(dx) = \frac{\exp\{-\frac{1}{2}\int_0^{2\pi} x^4(\xi)\, d\xi\}}{\displaystyle\int_{L^2(0,2\pi)} \exp\{-\frac{1}{2}\int_0^{2\pi} y^4(\xi)\, d\xi\}\ \mu(dy)}\ \mu(dx), \tag{1.4}$$

where μ is the Gaussian measure with mean 0 and covariance operator C, is the invariant measure of the Markov semigroup associated to the process $X(t)$.

It is not difficult to solve equation (1.2) by a fixed point argument, see e.g. [14].

Try now to generalize this result to the two dimensional case by considering the equation

$$dX = \left[\frac{1}{2}\,(\Delta_\xi - X) - X^3\right] dt + dW(t), \quad X(0) = x \tag{1.5}$$

in the space $L^2((0, 2\pi)^2)$. Proceeding as before we consider the complete orthonormal system $(e_k)_{k\in\mathbb{Z}^2}$ in $L^2((0, 2\pi)^2)$,

$$e_k(\xi) = \tfrac{1}{2\pi}\, e^{i\langle k,\xi\rangle}, \quad k = (k_1, k_2) \in \mathbb{Z}^2,\ \xi \in [0, 2\pi]^2$$

and define

$$W(t) = \sum_{k\in\mathbb{Z}^2} \beta_k(t) e_k,$$

where $(\beta_k(t))_{k\in\mathbb{Z}^2}$ is a family of standard Brownian motions mutually independent in $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Again, we write equation (1.5) in mild form

$$X(t) = e^{tA}x - \int_0^t e^{(t-s)A} X^3(s)\, ds + W_A(t), \tag{1.6}$$

where

$$Ax = \tfrac{1}{2}\,(\Delta_\xi x - x), \quad x \in \{y \in H^2((0,2\pi)^2):\ y, y_{\xi_1}, y_{\xi_2} \text{ periodic in } \xi_1, \xi_2\}$$

and

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s) = \sum_{k\in\mathbb{Z}^2} \int_0^t e^{-\frac{1}{2}\,(t-s)(1+|k|^2)} d\beta_k(s). \tag{1.7}$$

But in this case the operator

$$C = -\tfrac{1}{2}\, A^{-1}$$

is not of trace class. In other words the stochastic convolution $W_A(t)$ is not a well defined random variable with values in $L^2((0,2\pi)^2)$. One can easily see that it is well defined and Gaussian in every Sobolev space $H^{-\varepsilon}((0,2\pi)^2)$ with $\varepsilon > 0$; thus, it is natural to try to solve equation (1.6) in this space. However, a problem will arise since the nonlinear term x^3 is not well defined in $H^{-\varepsilon}((0,2\pi)^2)$ which is a distributional space.

For this reason the function x^3 is replaced by the following one

$$:x^3:= \lim_{N\to\infty} \left([x_N]^3 - 3\rho_N^2 x_N\right),$$

where

$$x_N = \sum_{|k|\leq N} \langle e_k, x\rangle e_k$$

and

$$\rho_N = \frac{1}{2\pi} \left[\sum_{|k|\leq N} \frac{1}{1+|k|^2}\right]^{1/2}$$

and the limit exists in $L^2(\mathcal{H}, \mu)$ where $\mathcal{H}$ is a suitable extension of the space H and μ is a Gaussian measure of covariance C, see the section 3 below for details. In this way we have changed the original problem with the following one

$$dX = \left[\tfrac{1}{2}\,(\Delta_\xi X - X) -\, :X^3:\right] dt + dW(t), \quad X(0) = x. \tag{1.8}$$

This is the so called *renormalization* procedure. This choice is physically justified in quantum field theory and somebody believes that it is natural even in other situations as: reaction diffusion and Ginzburg-Landau equations, see e.g. [6].

In the past few years, some attention has been payed to the so called *stochastic quantization*, see G. Parisi and Y.S. Wu [25], in order to compute integrals of the form

$$\int_H f(x)\,\nu(dx)$$

where ν is the invariant measure of (1.8) defined as (1.4), using the ergodic theorem

$$\int_H f(x)\,\nu(dx) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))\,dt.$$

The renormalization has a long story, also in connection with the constructive field theory in the euclidean framework, see J. Glimm-A. Jaffe [17], B. Simon [27] and references therein.

In this paper we shall describe the renormalization of the power and the Nelson estimate, following essentially the ideas in B. Simon [27]. We shall proceed similarly as in [11], where we presented a reformulation of the theory in the space $H^{-1}((0,2\pi)^2)$, but here we prefer to enlarge the space $L^2((0,2\pi)^2)$ introducing the product space

$$\mathcal{H} = \times_{k\in\mathbb{Z}^2} \mathbb{R}_k, \quad \mathbb{R}_k = \mathbb{R},$$

identifying H with $\ell^2(\mathbb{Z}^2) \subset (\mathbb{R}^2)^\infty$ and setting

$$\mu = \times_{k\in\mathbb{Z}^2} N_{(1+|k|^2)^{-1}},$$

where $N_{(1+|k|^2)^{-1}}$ represents the one–dimensional Gaussian measure with mean 0 and variance $(1+|k|^2)^{-1}$. This is essentially equivalent to work in the space of distributions, but it avoids for example the use of the Minlos theorem.

§2 is devoted to adapt some basic results on Gaussian measures to the product space $\mathcal{H}$.

In §3 we shall define for every integer n the Wick product $:\phi^n:$ with respect to the Gaussian measure μ. As shown here, this definition corresponds, roughly speaking, to subtract to ϕ^n some divergent term.

In §4 we present the Nelson estimate which allows to define the measure

$$\nu(d\phi) = \frac{\exp\{-\frac{1}{2}\langle 1, :\phi^4:\rangle\}}{\int_{\mathcal{H}} \exp\{-\frac{1}{2}\langle 1, :\psi^4:\rangle\}\,\mu(d\psi)}\,\mu(d\phi), \tag{1.9}$$

In §5 we construct the Dirichlet form corresponding to ν using ideas in S. Albeverio-M. Röckner [4]. In that paper this result was used to find a weak solution of (1.8) through the infinite dimensional generalization of the Fukushima theory (see [15]), due to [3].

In §6 we solve the Kolmogorov equation in $L^1(\mathcal{H},\nu)$, corresponding to a modified form of (1.8), namely

$$dX = -\tfrac{1}{2}\, C^{\varepsilon-1} X dt - C^{\varepsilon} :X^3: dt + C^{\varepsilon/2} dW(t), \tag{1.10}$$

where $\varepsilon > 0$.

Finally, §7 is devoted to some generalization of the renormalization method in higher dimension. We show in particular that the Wick product $:\phi^3:$ cannot be defined in dimension 3. A final remark is devoted to show that the Kardar–Parisi model too cannot be treated in this framework.

We recall further results which will not be reviewed in this paper.

- Equation (1.10) with $\varepsilon > \frac{9}{10}$ was solved in [20] by a suitable extension of the Girsanov formula. For further interesting developments of this theory the reader can look at [21]. Other contributions in this direction can be found in [7] and [16].

- Existence of a martingale solution of (1.8), was proved in [23], [16].

- A construction of the measure ν in infinite volume (instead of the box $[0,2\pi]^2$) in dimension 2 can be found in Glimm-Jaffe [17], Simon [27] and references therein.

- The method of renormalization in dimension 2 does not extend in a straightforward way to dimension 3; a further subtraction of an infinite term is needed, see [17] and [5].

Finally, we recall that existence and uniqueness of the strong solution of equation (1.8) by a fixed point argument in suitable Besov spaces was proved in [9] for equation (1.8) and in [8] for the 2-D Navier–Stokes equation, see also [1], [2]. Notice that in the case of 2-D Navier–Stokes equation the renormalized problem coincides with the original one; the renormalization procedure is a useful tool for the proof.

2 Gaussian measures in product spaces

Let $H = L^2(\mathcal{O})$ (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), where $\mathcal{O}$ is the square $[0,2\pi]^2$. We denote by $(e_k)_{k\in\mathbb{Z}^2}$ the complete orthonormal system of H

$$e_k(\xi) = \frac{1}{2\pi}\, e^{i\langle k,\xi\rangle},\;\; \xi\in\mathcal{O},\;\; k=(k_1,k_2)\in\mathbb{Z}^2,$$

where $\langle k,\xi\rangle = k_1\xi_1+k_2\xi_2$, and by H_0 the linear (not closed) span of $(e_k)_{k\in\mathbb{Z}^2}$. For any $x\in H$ we set

$$\langle x, e_k\rangle = x_k, \quad \text{for all } k\in\mathbb{Z}^2.$$

We shall identify H with the space $\ell^2(\mathbb{Z}^2)$ of all square summable sequences $(x_k)_{k\in\mathbb{Z}^2}\subset\mathbb{R}$ through the isomorphism

$$x\in H\mapsto (x_k)_{k\in\mathbb{Z}^2}\in\ell^2(\mathbb{Z}^2).$$

A basic rôle in the construction of the Wick products will be played by the following linear bounded operator in H,

$$Ce_k=\frac{1}{1+|k|^2}\,e_k,\ \ k\in\mathbb{Z}^2.$$

Notice that $C=(1-\Delta)^{-1}$ (where Δ is the realization of the Laplace operator in $L^2(\mathcal{O})$ with periodic boundary conditions) and that $\operatorname{Tr} C=+\infty$, so that C is not the covariance operator of a Gaussian measure in H. For this reason we shall introduce a larger space $\mathcal{H}$, namely the product space,

$$\mathcal{H}=\bigtimes_{k\in\mathbb{Z}^2}\mathbb{R}_k,\quad \mathbb{R}_k=\mathbb{R}$$

and we shall consider H (identified with $\ell^2(\mathbb{Z}^2)$) as a subspace of $(\mathbb{R}^2)^\infty$. We shall denote by $x,y,z,\dots$ elements in H and by $\phi,\psi,\zeta,\dots$ elements in $\mathcal{H}$.

Next we define the Borel product measure μ on $\mathcal{H}$ (endowed with the product topology),

$$\mu=\bigtimes_{k\in\mathbb{Z}^2}N_{(1+|k|^2)^{-1}},$$

where $N_{(1+|k|^2)^{-1}}$ represents the one–dimensional Gaussian measure with mean 0 and variance $(1+|k|^2)^{-1}$. Notice that $\mu(H^{-\varepsilon}(\mathcal{O}))=1$ for all $\varepsilon>0$, where $H^{-\varepsilon}(\mathcal{O})$ is the usual Sobolev space with negative exponent.

The following duality between H_0 and $\mathcal{H}$ is important in what follows. For any $x\in H_0$ and any $\phi\in\mathcal{H}$ we define

$$\langle x,\phi\rangle=\sum_{k\in\mathbb{Z}^2}x_k\phi_k.$$

Now, we can extend without any difficulty the usual definition of *white noise*. First for any $z\in H_0$ we define a function W_z in $L^2(\mathcal{H},\mu)$ setting (note that the sum below is finite),

$$W_z(\phi)=\langle C^{-1/2}z,\phi\rangle=\sum_{k\in\mathbb{Z}^2}\sqrt{1+|k|^2}\;z_k\phi_k,\quad \phi\in\mathcal{H}.$$

It is easy to check that

$$\int_{\mathcal{H}} W_z(\phi)W_{z'}(\phi)\mu(d\phi) = \langle z, z'\rangle, \quad z, z' \in H_0. \tag{2.1}$$

Therefore the mapping

$$H_0 \to L^2(\mathcal{H}, \mu), \ z \mapsto W_z,$$

is an isometry and consequently it can be extended to the whole H. Thus W_z is a well defined element of $L^2(\mathcal{H}, \mu)$ for any $z \in H$.

Proposition 2.1 *For any $z \in H$, W_z is a real Gaussian random variable with mean 0 and variance $|z|^2$.*

Proof. Let first $z \in H_0$. Then

$$W_z(\phi) = \langle C^{-1/2}z, \phi\rangle = \sum_{k\in\mathbb{Z}^2} \sqrt{1+|k|^2}\, z_k\phi_k.$$

Thus W_z is the sum of a finite number of independent Gaussian random variables and consequently is Gaussian with mean 0 and covariance,

$$\sum_{k\in\mathbb{Z}^2}(1+|k|^2)\, z_k^2 \int_{\mathcal{H}} \phi_k^2 d\mu = |z|^2.$$

Let finally $z \in H$ be arbitrary and let $(z_n) \subset H_0$ be such that $z_n \to z$ in H. Then

$$\lim_{n\to\infty} W_{z_n} = W(z) \quad \text{in} \quad L^2(\mathcal{H}, \mu),$$

as easily checked. So, W_z is Gaussian $N_{|z|^2}$-distributed as claimed. □

2.1 Wiener chaos

Let us recall the definition of *Hermite polynomials*. Let F denote the function

$$F(t,\xi) = e^{-\frac{t^2}{2}+t\xi}, \quad t, \xi \in \mathbb{R}.$$

Since F is analytic, there exists a sequence of functions $(H_n)_{n\in\{0\}\cup\mathbb{N}}$ such that

$$F(t,\xi) = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}}\, H_n(\xi), \quad t, \xi \in \mathbb{R}. \tag{2.2}$$

Proposition 2.2 *For any $n \in \{0\} \cup \mathbb{N}$ the following identity holds*

$$H_n(\xi) = \frac{(-1)^n}{\sqrt{n!}}\, e^{\frac{\xi^2}{2}}\, D_\xi^n \left(e^{-\frac{\xi^2}{2}}\right),\ \xi \in \mathbb{R}. \tag{2.3}$$

Proof. We have in fact

$$\begin{aligned} F(t,\xi) &= e^{\frac{\xi^2}{2}} e^{-\frac{1}{2}(t-\xi)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, e^{\frac{\xi^2}{2}}\, D_t^n \left(e^{-\frac{1}{2}(t-\xi)^2}\right)\Big|_{t=0} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\, (-1)^n\, e^{\frac{\xi^2}{2}} D_\xi^n \left(e^{-\frac{\xi^2}{2}}\right). \end{aligned}$$

Thus the conclusion follows. □

By the proposition we see that for any $n \in \mathbb{N} \cup \{0\}$, H_n is a polynomial of degree n having a positive leading coefficient. H_n are called *Hermite* polynomials.

We have in particular

$$H_0(\xi) = 1,\ H_1(\xi) = \xi,\ \ H_2(\xi) = \frac{1}{\sqrt{2}}\,(\xi^2 - 1),$$

$$H_3(\xi) = \frac{1}{\sqrt{6}}\,(\xi^3 - 3\xi),\ \ H_4(\xi) = \frac{1}{2\sqrt{6}}\,(\xi^4 - 6\xi^2 + 3).$$

In the following proposition some important properties of the Hermite polynomials are collected. The corresponding proofs are straightforward, they are left to the reader.

Proposition 2.3 *For any $n \in \mathbb{N}$ we have*

$$\xi H_n(\xi) = \sqrt{n+1}\ H_{n+1}(\xi)\ + \sqrt{n}\ H_{n-1}(\xi), \quad \xi \in \mathbb{R}, \tag{2.4}$$

$$D_\xi H_n(\xi) = \sqrt{n}\ H_{n-1}(\xi), \quad \xi \in \mathbb{R}, \tag{2.5}$$

$$D_\xi^2 H_n(\xi) - \xi D_\xi H_n(\xi) = -n H_n(\xi), \quad \xi \in \mathbb{R}. \tag{2.6}$$

Identity (2.5) shows that the derivation D_ξ acts as a shift operator with respect to the system $(H_n)_{n\in\{0\}\cup\mathbb{N}}$. Moreover by (2.6) it follows that the Hermite operator

$$T\varphi := \frac{1}{2}\, D_\xi^2 \varphi - \frac{1}{2}\, \xi D_\xi,$$

is diagonal with respect to $(H_n)_{n\in\{0\}\cup\mathbb{N}}$.

Now we define Hermite polynomials in $\mathcal{H}$. They will be useful for costructing a complete orthonormal system on $L^2(\mathcal{H}, \mu)$. To this end we need the following

Lemma 2.4 *Let $h, g \in H$ with $|h| = |g| = 1$ and let $n, m \in \mathbb{N} \cup \{0\}$. Then we have:*

$$\int_{\mathcal{H}} H_n(W_h) H_m(W_g) d\mu = \delta_{n,m} [\langle h, g \rangle]^n. \tag{2.7}$$

Proof. For any $t, s \in \mathbb{R}$ we have

$$\int_{\mathcal{H}} F(t, W_h) F(s, W_g) d\mu = e^{-\frac{t^2+s^2}{2}} \int_{\mathcal{H}} e^{tW_h + sW_g} \, d\mu$$

$$= e^{-\frac{t^2+s^2}{2}} \int_{H} e^{W_{th+sg}} \, d\mu = e^{-\frac{t^2+s^2}{2}} e^{\frac{1}{2}|th+sg|^2} = e^{ts\langle h,g \rangle},$$

because $|h| = |g| = 1$. It follows that

$$e^{ts\langle h,g \rangle} = \sum_{m,n=0}^{\infty} \frac{t^n s^m}{\sqrt{n!m!}} \int_{\mathcal{H}} H_n(W_h) H_m(W_g) \, d\mu,$$

which clearly implies (2.7). □

We are now ready to define a complete orthonormal system in $L^2(\mathcal{H}, \mu)$. Let Γ be the set of all mappings

$$\gamma : \mathbb{Z}^2 \to \{0\} \cup \mathbb{N}, \; n \mapsto \gamma_n,$$

such that

$$|\gamma| := \sum_{k \in \mathbb{Z}^2} \gamma_k < +\infty.$$

Note that if $\gamma \in \Gamma$ then $\gamma_n = 0$ for all n, except at most a finite number. For any $\gamma \in \Gamma$ we define the *Hermite polynomial*,

$$H_\gamma(\phi) = \prod_{k \in \mathbb{Z}^2} H_{\gamma_k}(W_{e_k}(\phi)), \quad \phi \in \mathcal{H}.$$

This definition is meaningful since all factors, with the exception of at most a finite number, are equal to $H_0(W_{e_k}(\phi)) = 1, \; \phi \in \mathcal{H}$.

We can now prove the result.

Theorem 2.5 *System $(H_\gamma)_{\gamma \in \Gamma}$ is orthonormal and complete on $L^2(\mathcal{H}, \mu)$.*

Proof. Orthonormality. Let $\gamma, \eta \in \Gamma$, then we have, taking into account Lemma 2.4, and recalling that the random variables W_{e_n} are mutually independent,

$$\int_{\mathcal{H}} H_\gamma H_\eta d\mu = \int_{\mathcal{H}} \prod_{n \in \mathbb{Z}^2} H_{\gamma_n}(W_{e_n}) H_{\eta_n}(W_{e_n}) d\mu$$

$$= \prod_{n \in \mathbb{Z}^2} \int_{\mathcal{H}} H_{\gamma_n}(W_{e_n}) H_{\eta_n}(W_{e_n}) d\mu = \delta_{\eta,\gamma},$$

where $\delta_{\eta,\gamma} = \prod_{n\in\mathbb{Z}^2} \delta_{\eta_n,\gamma_n}$. So the system $(H_\gamma)_{\gamma\in\Gamma}$ is orthonormal.

Completeness. Let $\psi \in L^2(\mathcal{H},\mu)$ be such that

$$\int_{\mathcal{H}} \psi H_\gamma d\mu = 0, \ \ \forall\, \gamma \in \Gamma. \tag{2.8}$$

We have to show that $\psi = 0$.

By (2.8) it follows in particular that

$$\int_{\mathcal{H}} \psi H_k(W_{e_1}) d\mu = 0, \ \ \forall\, k \in \mathbb{Z}^2,$$

that implies by (2.2)

$$\int_{\mathcal{H}} \psi F(t_1, W_{e_1}) d\mu = 0, \ \ \forall\, t_1 \in \mathbb{R}.$$

In a similar way we obtain

$$\int_{\mathcal{H}} \psi F(t_1, W_{e_1}) F(t_2, W_{e_2}) ... F(t_n, W_{e_n}) d\mu = 0,$$

for all $n \in \mathbb{N},\ t_1, t_2, ..., t_n \in \mathbb{R}$, that yields

$$\int_{\mathcal{H}} \psi\ e^{\sum_{k=1}^n \alpha_k \phi_k} d\mu = 0, \ \ \forall\, n \in \mathbb{N},\ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}.$$

Since the linear span of the set of functions

$$\left\{ \exp\left\{ \sum_{k=1}^n \alpha_k \phi_k \right\} : \ \ n \in \mathbb{N},\ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R} \right\},$$

is dense in $L^2(\mathcal{H},\mu)$, it follows that $\psi = 0$ as required. □

Now, we define the *Itô–Wiener decomposition.* For all $n \in \{0\} \cup \mathbb{N}$ we denote by $L^2_n(\mathcal{H},\mu)$ the closed subspace of $L^2(\mathcal{H},\mu)$ spanned by

$$\{H_n(W_f) : \ f \in H,\ |f| = 1\}.$$

In particular, $L^2_0(\mathcal{H},\mu)$ is the set of all constant functions in $L^2(\mathcal{H},\mu)$ and $L^2_1(\mathcal{H},\mu)$ is given by

$$L^2_1(\mathcal{H},\mu) = \{W_f : \ f \in H\}.$$

We shall denote by Π_n the orthogonal projector onto $L^2_n(\mathcal{H},\mu),\ n \in \{0\} \cup \mathbb{N}$. Arguing as in the proof of Theorem 2.5 we see that

$$L^2(\mathcal{H},\mu) = \bigoplus_{n=0}^{\infty} L^2_n(\mathcal{H},\mu).$$

We give now a characterization of $L^2_n(\mathcal{H},\mu),\ n \in \{0\} \cup \mathbb{N}$.

Proposition 2.6 *For any $n \in \{0\} \cup \mathbb{N}$ the space $L^2_n(\mathcal{H}, \mu)$ coincides with the closed subspace of $L^2(\mathcal{H}, \mu)$ spanned by*

$$V_n := \{H_\gamma : \ |\gamma| = n\}.$$

Proof. It is enough to show that if $n, N \in \mathbb{N}$, $f \in H$ with $|f| = 1$, $k_1, ..., k_N \in \mathbb{N}$, and $k_1 + ... + k_N \neq n$, we have

$$\int_{\mathcal{H}} H_{k_1}(W_{e_1})...H_{k_N}(W_{e_N})H_n(W_f)d\mu = 0. \tag{2.9}$$

We have in fact

$$\begin{aligned} I: &= \int_{\mathcal{H}} F(t_1, W_{e_1})...F(t_N, W_{e_N})F(t_{N+1}, W_f)d\mu \\ &= e^{-\frac{1}{2}(t_1^2+...+t_{N+1}^2)} \int_{\mathcal{H}} e^{W_{t_1e_1+...+t_Ne_N+t_{N+1}f}} d\mu \\ &= e^{t_{N+1}(t_1f_1+...+t_Nf_N)}. \end{aligned}$$

On the other hand we have

$$I = \sum_{k_1,...,k_{N+1}=0}^{\infty} \frac{t_1^{k_1}...t_{N+1}^{k_{N+1}}}{\sqrt{k_1!...k_{N+1}!}} \int_{\mathcal{H}} H_{k_1}(W_{e_1})...H_{k_N}(W_{e_N})H_{N+1}(W_f)d\mu,$$

and the conclusion follows. □

We now prove an important property of the projection Π_n.

Proposition 2.7 *Let $f \in H$ such that $|f| = 1$, and let $n \in \mathbb{N}$. Then we have*

$$\Pi_n(W_f^n) = \sqrt{n!}\ H_n(W_f), \tag{2.10}$$

Proof. Since $\sqrt{n!}\ H_n(W_f) \in L^2_n(\mathcal{H}, \mu)$ by definition, it is enough to show that for all $g \in H$ such that $|g| = 1$, we have

$$\int_{\mathcal{H}} [W_f^n - \sqrt{n!}\ \ H_n(W_f)]H_n(W_g)d\mu = 0,$$

or, equivalently,

$$\int_{\mathcal{H}} W_f^n H_n(W_g)d\mu = \sqrt{n!}\ [\langle f, g\rangle]^n. \tag{2.11}$$

Now (2.11) follows easily from the identity

$$I := \int_{\mathcal{H}} e^{sW_f} H_n(W_g)d\mu = \frac{1}{\sqrt{n!}}\ s^n e^{\frac{s^2}{2}}\ [\langle f, g\rangle]^n, \tag{2.12}$$

(by differentiating n times with respect to s and then setting $s = 0$), that we shall prove now. We have, taking into account (2.7),

$$I = e^{\frac{s^2}{2}} \int_{\mathcal{H}} F(s, W_f) H_n(W_g) d\mu$$

$$= e^{\frac{s^2}{2}} \sum_{k=0}^{\infty} \frac{s^k}{\sqrt{k!}} \int_{\mathcal{H}} H_k(W_f) H_n(W_g) d\mu$$

$$= \frac{1}{\sqrt{n!}} \, s^n e^{\frac{s^2}{2}} \, [\langle f, g \rangle^n],$$

that yields (2.12). □

Now we can compute easily the projections of an exponential function.

Corollary 2.8 *Let $f \in H$ with $|f| = 1$. Then we have*

$$\Pi_n \left(e^{sW_f} \right) = \frac{1}{\sqrt{n!}} \, s^n e^{\frac{s^2}{2}} \, H_n(W_f). \tag{2.13}$$

Proof. We have in fact

$$\Pi_n \left(e^{sW_f} \right) = \sum_{k=0}^{\infty} \frac{s^k}{k!} \, \Pi_n(W_f^k) = s^n e^{\frac{s^2}{2}} \, H_n(W_f).$$

□

2.2 The Sobolev space $W^{1,2}(\mathcal{H}, \mu)$

We denote by $\mathcal{FC}_0^\infty(\mathcal{H}\ )$ the set of all functions $u = u(\phi)$, depending only on a finite number of variables ϕ_k, which are of class C_0^∞. We set $D_h = D_{\phi_h},\ h \in \mathbb{Z}^2$.

We need the following integration by parts formula.

Proposition 2.9 *Let $u, v \in \mathcal{FC}_0^\infty(\mathcal{H}\ )$. Then for any $k \in \mathbb{Z}^2$ we have,*

$$\int_{\mathcal{H}} D_k u \; v \; d\mu = -\int_{\mathcal{H}} D_k v \; u \; d\mu + (1 + |k|^2)^{1/2} \int_{\mathcal{H}} \phi_k \; uv \; d\mu. \tag{2.14}$$

Proof. Assume that u, v depend only on $\phi_k,\ |k| \le N$. Let H_N be the span of $(e_k),\ |k| \le N$. H_N is obviously a finite dimensional Hilbert space with coordinates $\phi^{(N)} = (\phi_k)_{|k| \le N}$ and with Lebesgue measure $d\phi^{(N)}$; note that the marginal measure of μ on H_N has a density $\rho_N(\phi^{(N)})$ with respect to the Lebesgue measure $d\phi^{(N)}$ given by

$$\rho_N(\phi^{(N)}) = c_N \exp\left\{-\frac{1}{2}\sum_{|k|\le N}(1+|k|^2)^{1/2}\phi_k^2\right\}.$$

whith

$$c_N = (2\pi)^{N/2}\prod_{|k|\le N}(1+|k|^2)^{1/4}.$$

Then we have (for $|k|\le N$)

$$\int_{\mathcal{H}} D_k u\ v\ d\mu = c_N \int_{H_N} D_k u(\phi^{(N)})\ v(\phi^{(N)})\ \rho_N(\phi^{(N)})d\phi^{(N)}$$

$$= -\int_{\mathcal{H}} u\ D_k v\ d\mu + (1+|k|^2)^{1/2}\int_{\mathcal{H}} u\ v\ d\mu.$$

□

Proposition 2.10 *For any $k\in\mathbb{Z}^2$ the operator D_k is closable.*

Proof. Let $k\in\mathbb{Z}^2$, $(u_n)\subset \mathcal{FC}_0^\infty(\mathcal{H}\ )$ and $v\in L^2(\mathcal{H},\mu)$ be such that

$$u_n\to 0,\quad D_k u_n\to v\quad \text{in } L^2(\mathcal{H},\nu).$$

We have to show that $v=0$. If $w\in\mathcal{FC}_0^\infty(\mathcal{H}\ )$, then by (2.14) we have

$$\int_{\mathcal{H}} D_k u_n\ w\ d\nu = -\int_{\mathcal{H}} u_n\ D_k w\ d\nu + (1+|k|^2)^{1/2}\int_{\mathcal{H}}\phi_k\ u_n\ w\ d\mu.$$

As $n\to\infty$ the first integral tends to $\int_{\mathcal{H}} vwd\mu$, the second and the third integral tend to 0, since $\phi_k w$ belongs to $L^2(\mathcal{H},\mu)$. Therefore $\int_{\mathcal{H}} vwd\mu = 0$ for all $\zeta\in\mathcal{FC}_0^\infty(\mathcal{H}\ )$, so that $v=0$ as required. □

We shall still denote by D_k the closure of D_k on $L^2(\mathcal{H},\mu)$. If φ belongs to the domain of D_k we say that $D_k\varphi$ belongs to $L^2(\mathcal{H},\mu)$.

We now define the space $W^{1,2}(\mathcal{H},\mu)$ as the linear space of all functions $u\in L^2(\mathcal{H},\mu)$ such that $D_k u\in L^2(\mathcal{H},\mu)$ for all $k\in\mathbb{Z}^2$ and

$$\sum_{k\in\mathbb{Z}^2}\int_{\mathcal{H}}|D_k u(\phi)|^2\mu(d\phi) < +\infty.$$

$W^{1,2}(\mathcal{H},\mu)$, endowed with the inner product,

$$\langle u,v\rangle_{W^{1,2}(\mathcal{H},\mu)} = \langle u,v\rangle_{L^2(\mathcal{H},\mu)} + \sum_{k\in\mathbb{Z}^2}\int_{\mathcal{H}}(D_k u)(D_k v)d\mu,$$

is a Hilbert space.

If $u \in W^{1,2}(\mathcal{H},\mu)$ we set

$$Du(\phi) = \sum_{k\in\mathbb{Z}^2} D_k u(\phi)e_k,\ \mu-\text{a.e. in }\ \mathcal{H}.$$

Since

$$|Du(\phi)|^2 = \sum_{k\in\mathbb{Z}^2}|D_k u(\phi)|^2,\ \mu-\text{a.e. in }\ \mathcal{H},$$

the series is convergent for almost all $\phi\in\mathcal{H}$. We call $Du(\phi)$ the *gradient* of u at ϕ. Notice that

$$Du \in L^2(\mathcal{H},\mu;H).$$

3 Renormalization of the power

We fix here $n\in\mathbb{N}$ and $x\in H$. Our goal in this section is to give a meaning to the function on $\mathcal{H}$,

$$\langle x,\phi^n\rangle = \int_{\mathcal{O}} x(\xi)\phi^n(\xi)d\xi.$$

For this we shall proceed as follows. Given $\phi\in\mathcal{H}$ and $\xi\in\mathcal{O}$ we set

$$\phi_N(\xi) = \sum_{k\in\mathbb{Z}^2}\langle e_k,\phi\rangle e_k(\xi).$$

Notice that $\phi_N\in C^\infty(\mathcal{O})$.

Since (as one can check) $\phi_N^n(\xi)$ does not converge as $N\to\infty$ in $L^2(\mathcal{H},\mu)$ we shall replace $\phi_N^n(\xi)$ by its projection on the n^{th} Wiener chaos $L^2_n(\mathcal{H},\mu)$ setting,

$$:\phi_N^n:(\xi) = \Pi_n(\phi_N^n(\xi)). \tag{3.1}$$

To compute $\Pi_n(\phi_N^n(\xi))$ we shall use (2.10). For this it is useful to express the function $\phi\to\phi_N^n(\xi)$ in terms of the white noise function. We can write obviously

$$\phi_N(\xi) = \left\langle \sum_{|k|\le N}\frac{\overline{e_k(\xi)}}{\sqrt{1+|k|^2}}\ e_k, C^{-1/2}\phi\right\rangle.$$

and so

$$\phi_N(\xi) = \rho_N W_{\eta_N(\xi)}(\phi), \quad \xi \in \mathcal{O},\ N \in \mathbb{N}, \tag{3.2}$$

where

$$\eta_N(\xi) = \frac{1}{\rho_N} \sum_{|k|\le N} \frac{\overline{e_k(\xi)}}{\sqrt{1+|k|^2}}\, e_k, \tag{3.3}$$

and

$$\rho_N = \frac{1}{2\pi} \left[\sum_{k\in\mathbb{Z}^2} \frac{1}{1+|k|^2}\right]^{1/2}. \tag{3.4}$$

Notice that $|\eta_N(\xi)| = 1$. Now we can prove the following result

Proposition 3.1 *We have*

$$:\phi_N^n:(\xi) = \sqrt{n!}\ \rho_N^n\ H_n\left(\frac{\phi_N(\xi)}{\rho_N}\right) = \sqrt{n!}\ \rho_N^n\ H_n(W_{\eta_N(\xi)}),\ \ \xi \in \mathcal{O}, \tag{3.5}$$

$:\phi_N^n:(\xi)$ is called the *renormalization* of $\phi_N^n(\xi)$.

Note in particular that

$$:\phi_N^1:(\xi) = \phi_N(\xi),$$

$$:\phi_N^2:(\xi) = [\phi_N(\xi)]^2 - \rho_N^2,$$

$$:\phi_N^3:(\xi) = [\phi_N(\xi)]^3 - 3\rho_N^2\phi_N(\xi),$$

$$:\phi_N^4:(\xi) = [\phi_N(\xi)]^4 - 3\rho_N^2[\phi_N(\xi)]^2 + 6\rho_N^4.$$

So, for any $n \in \mathbb{N}$, $:\phi_N^n:$ is equal to ϕ_N^n up to lower order terms which are divergent as $N \to \infty$.

The following asymptotic behavior of ρ_N is basic in what follows

$$\rho_N^2 = O(\log N). \tag{3.6}$$

It can be seen from

$$\frac{1}{(2\pi)^2} \sum_{|h|\le N} \frac{1}{1+|h|^2} \sim \int_0^N \frac{r}{1+r^2}\, dr = \frac{1}{2}\log(1+N^2).$$

The main result of this section is to prove that for any fixed $x \in H$, there exists the limit

$$\langle x, :\phi^n:\rangle \overset{\text{def}}{=} \lim_{N\to\infty} \int_{\mathcal{O}} :\phi_N^n:(\xi)\, x(\xi) d\xi \quad \text{in } L^2(\mathcal{H},\mu). \tag{3.7}$$

This will be done in §3.1 below. To understand the spirit of the proof it is useful to see first that $\sup_N I_N < \infty$ where

$$I_N := \int_{\mathcal{H}} \left| \int_{\mathcal{O}} :\phi_N^n:(\xi)\, x(\xi) d\xi \right|^2 \mu(d\phi). \tag{3.8}$$

We have in fact

$$I_N = \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)d\xi_1 d\xi_2 \int_{\mathcal{H}} :\phi_N^n:(\xi_1)\ :\phi_N^n:(\xi_2) d\mu$$

$$= n!\rho_N^{2n} \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)d\xi_1 d\xi_2 \int_{\mathcal{H}} H_n(W_{\eta_N(\xi_1)})H_n(W_{\eta_N(\xi_2)})d\mu$$

$$= n!\rho_N^{2n} \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)\ \langle \eta_N(\xi_1), \eta_N(\xi_2)\rangle^n\ d\xi_1 d\xi_2.$$

To compute the last integral note that

$$\langle \eta_N(\xi_1), \eta_N(\xi_2)\rangle = \frac{1}{\rho_N^2}\ \gamma_N(\xi_1 - \xi_2), \quad \xi_1, \xi_2 \in \mathcal{O}, \ \ N \in \mathbb{N}, \tag{3.9}$$

where

$$\gamma_N = \sum_{|k|\le N} \frac{1}{1+|k|^2}\ e_k, \ \ N \in \mathbb{N}. \tag{3.10}$$

Then we have

$$I_N = \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)\gamma_N^n(\xi_1 - \xi_2)d\xi_1 d\xi_2. \tag{3.11}$$

To compute the supremum of I_N we have to find the behaviour of γ_N^n as $N \to +\infty$. For this it is useful to define

$$\gamma = \sum_{k\in\mathbb{Z}^2} \frac{1}{1+|k|^2}\ e_k, \ \ N \in \mathbb{N}. \tag{3.12}$$

It is interesting to notice that $\gamma \in L^2(\mathcal{O})$ and coincides with the kernel of C, that is

$$Cx(\xi) = \int_{\mathcal{O}} \gamma(\xi - \xi_1)x(\xi_1)d\xi_1 = \gamma * x(\xi), \ x \in H,$$

as easily checked. Notice that γ is not bounded but it belongs to $L^p(\mathcal{O})$ for all $p \ge 1$. We have in fact the result

Proposition 3.2 *For all $p \geq 1$ we have*

$$|\gamma|_{L^p(\mathcal{O})} \leq (2\pi)^{\frac{p-2}{2}} \left[\sum_{h\in\mathbb{Z}^2} \left(\frac{1}{1+|h|^2} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}. \tag{3.13}$$

Proof. Let us consider the mapping

$$\Gamma : \{\lambda_j\}_{j\in\mathbb{Z}^2} \to \sum_{h\in\mathbb{Z}^2} \lambda_h e_h.$$

Then

$$\begin{aligned} \Gamma : \ell^1(\mathbb{Z}^2) &\to L^\infty(\mathcal{O}), \text{ with norm } (2\pi)^{-1}, \\ \Gamma : \ell^2(\mathbb{Z}^2) &\to L^2(\mathcal{O}), \text{ with norm } 1. \end{aligned}$$

By the Riesz–Thorin theorem if $p > 2$ and $q = \frac{p}{p-1}$ we have

$$\Gamma : \ell^q(\mathbb{Z}^2) \to L^p(\mathcal{O}), \text{ with norm less or equal to } (2\pi)^{\frac{p-2}{2}},$$

and the conclusion follows. □

Therefore, since $\gamma \in L^n$ it follows by (3.11) that $\sup_{N\in\mathbb{N}} I_N < +\infty$.

3.1 Existence of the limit (3.7)

We need a lemma.

Lemma 3.3 *If $p \geq 2$, we have*

$$|\gamma - \gamma_N|_{L^p(\mathcal{O})} \leq b_p N^{-\frac{2}{p}}, \tag{3.14}$$

where $b_p = (p-1)(2\pi)^{\frac{p-2}{2}}$.

Proof. We have in fact

$$\begin{aligned} |\gamma - \gamma_N|_{L^p(\mathcal{O})} &\leq (2\pi)^{\frac{p-2}{2}} \sum_{|h|\geq N} \left(\frac{1}{1+|h|^2} \right)^{\frac{p}{p-1}} \\ &\leq (2\pi)^{\frac{p-2}{2}} \int_N^{+\infty} \frac{2r}{(1+r^2)^{\frac{p}{p-1}}} dr \\ &= (p-1)(2\pi)^{\frac{p-2}{2}} (1+N^2)^{-\frac{1}{p-1}} \leq (p-1)(2\pi)^{\frac{p-2}{2}} N^{-\frac{2}{p}}. \end{aligned}$$

□

We shall use the following straightforward identity. If $M \geq N$ we have

$$\langle \eta_N(\xi), \eta_M(\xi') \rangle = \frac{1}{\rho_N \rho_M} \gamma_N(\xi - \xi'), \quad \xi, \xi' \in D, \ \ N \in \mathbb{N}. \tag{3.15}$$

Lemma 3.4 *For any $z \in H$ we have*

$$\int_{\mathcal{H}} |\langle z, :\phi_N^n:\rangle|^2 \mu(dx) = n! \langle \gamma_N^n * z, z\rangle. \tag{3.16}$$

Proof. Set

$$L_N = \int_{\mathcal{H}} |\langle z, :\phi_N^n:\rangle|^2 \mu(d\phi) = \int_{\mathcal{H}} \left| \int_{\mathcal{O}} \langle :\phi_N^n(\xi):, z(\xi)\rangle d\xi \right|^2 \mu(d\phi).$$

Then we have

$$\begin{aligned} L_N &= n! \rho_N^{2n} \int_{\mathcal{O}\times\mathcal{O}} z(\xi)\overline{z(\xi_1)} d\xi d\xi_1 \\ &\times \int_{\mathcal{H}} H_n(W_{\eta_N(\xi)}(x)) H_n(W_{\eta_N(\xi_1)}(x)) \mu(d\phi) \\ &= n! \rho_N^{2n} \int_{\mathcal{O}\times\mathcal{O}} z(\xi)\overline{z(\xi_1)} \left[\langle \eta_N(\xi), \eta_N(\xi_1)\rangle\right]^n d\xi d\xi_1 \\ &= n! \int_{\mathcal{O}\times\mathcal{O}} \gamma_N^n(\xi - \xi_1) z(\xi)\overline{z(\xi_1)} d\xi d\xi_1 \\ &= n! \langle \gamma_N^n * z, z\rangle. \end{aligned}$$

□

Theorem 3.5 *Let $M > N$ and $x \in H$. Then we have*

$$\int_{\mathcal{H}} |\langle x, :\phi_N^n: - :\phi_M^n:\rangle|^2 \mu(d\phi) = n! \langle (\gamma_M^n - \gamma_N^n) * x, x\rangle. \tag{3.17}$$

Moreover there exists $c_n > 0$ such that

$$\int_{\mathcal{H}} |\langle x, :\phi_N^n: - :\phi_M^n:\rangle|^2 \mu(d\phi) \le \frac{c_n}{N} |x|^2. \tag{3.18}$$

Therefore there exists the limit

$$\lim_{N\to\infty} \langle x, :\phi_N^n:\rangle := \langle :\phi^n:, x\rangle, \text{ in } L^2(\mathcal{H}, \mu). \tag{3.19}$$

Proof. Let $N > M$, and set

$$L_{N,M} = \int_{\mathcal{H}} |\langle x, :\phi_N^n:\rangle - \langle x, :\phi_M^n:\rangle|^2 \mu(d\phi).$$

Then we have

$$
\begin{aligned}
L_{N,M} &= n!\rho_M^n \int_{\mathcal{O}\times\mathcal{O}} x(\xi)\overline{x(\xi_1)}d\xi d\xi_1 \\
&\times \int_{\mathcal{H}} [\rho_M^n H_n(W_{\eta_M(\xi)}(\phi) - \rho_N^n H_n(W_{\eta_N(\xi)}(\phi)] \\
&\times [\rho_M^n H_n(W_{\eta_M(\xi_1)}(\phi)) - \rho_N^n H_n(W_{\eta_N(\xi_1)}(\phi))]\mu(d\phi) \\
&\times n! \int_{\mathcal{O}\times\mathcal{O}} z(\xi)\overline{z(\xi_1)} \Bigg\{ \rho_M^{2n}\left[\langle \eta_M(\xi), \eta_M(\xi_1)\rangle\right]^n - \rho_M^n\rho_N^n\left[\langle \eta_M(\xi), \eta_N(\xi_1)\rangle\right]^n \\
&- \rho_M^n\rho_N^n\left[\langle \eta_M(\xi), \eta_N(\xi)\rangle\right]^n - \rho_N^{2n}\left[\langle \eta_N(\xi), \eta_N(\xi_1)\rangle\right]^n \Bigg\} d\xi d\xi_1 \\
&= n! \int_{\mathcal{O}\times\mathcal{O}} [\gamma_M^n(\xi-\xi_1) - 2\gamma_N^n(\xi-\xi_1) + \gamma_N^n(\xi-\xi_1)]\phi(\xi)\overline{\phi(\xi_1)}d\xi d\xi_1 \\
&= n!\langle(\gamma_N^n - \gamma_M^n) * \phi, \phi\rangle.
\end{aligned}
$$

Therefore (3.17) is proved.

It remains to prove (3.18). We have in fact

$$
|\gamma_M^n - \gamma_N^n|_{L^1(\mathcal{O})} \le \sum_{j=0}^{n-1} \int_{\mathcal{O}} (\gamma_M - \gamma_N)\gamma_M^j \gamma_N^{n-1-j} d\xi.
$$

Using the Hölder estimate, and taking into account (3.13), we obtain

$$
\begin{aligned}
|\gamma_M^n - \gamma_N^n|_{L^1(\mathcal{O})} &\le \sum_{j=0}^{n-1} |\gamma_M^n - \gamma_N^n|_{L^2(\mathcal{O})} |\gamma_M^n - \gamma_N^n|^j_{L^{4j}(\mathcal{O})} |\gamma_M^n - \gamma_N^n|^{n-1-j}_{L^{4(n-1-j)}(\mathcal{O})} \\
&= \frac{2b_2}{N} \sum_{j=0}^{n-1} a_{4j}^j \, a_{4(n-1-j)}^{n-1-j}.
\end{aligned}
$$

The proof is complete. □

Remark 3.6 $:\phi^n:$ does not belong to $L^2(\mathcal{H}, \mu; H)$. In fact by (3.16) we have

$$
\int_{\mathcal{H}} |:\phi^n:|^2 \mu(d\phi) = \sum_{k\in\mathbb{Z}^2} \int_{\mathcal{H}} |\langle e_k, :\phi^n:\rangle|^2 \mu(d\phi) = n! \text{ Tr } [C^{\otimes n}] = +\infty.
$$

However we are able to define $C^\varepsilon :\phi^n:$ as an element of $L^2(\mathcal{H},\mu;H)$ for any $\varepsilon > 0$, as the next proposition shows.

The following result can be proved as Proposition 3.5.

Proposition 3.7 *Let $M > N$ and $z \in H$. Then we have*

$$\int_{\mathcal{H}} |C^\varepsilon :\phi_N^n: - C^\varepsilon :\phi_M^n: |^2 \mu(d\phi) = n! \left(\sum_{k\in\mathbb{Z}^2} \frac{1}{(1+|k|^2)^{1+2\varepsilon}} \right)^n. \tag{3.20}$$

Thus there exists the limit

$$\lim_{N\to\infty} C^\varepsilon :\phi_N^n := C^\varepsilon :\phi^n:, \text{ in } L^2(\mathcal{H},\mu;H). \tag{3.21}$$

4 The Nelson estimate

We shall need some hypercontractivity estimates. We shall present here a proof based on purely combinatorial arguments, following Simon [27, Lemma (I.18)]. For a modern proof based on log-Sobolev inequality one look at [18], see also Nualart [24].

4.1 Hypercontractivity estimates

Let $\varphi \in L^2_n(\mathcal{H},\mu)$. We want to prove that φ belongs to $L^{2m}(\mathcal{H},\mu)$ for any $m \in \mathbb{N}$.

We need two lemmas.

Lemma 4.1 *Let $N \in \mathbb{N}$, $Q_1,\dots,Q_N$ be countable sets (of indices). Set, for $l = 1,2,\dots,N$,*

$$\widehat{i_l} = \{i_1,\dots,i_{l-1},i_{l+1},\dots,i_N\},$$

and

$$\widehat{Q_l} = \{Q_1\times\cdots\times Q_{l-1}\times Q_{l+1}\times\cdots\times Q_N\}.$$

and, for $l = 1,2,\dots,N$, let $a^{(l)}$ be a mapping

$$a^{(l)} : \widehat{Q_l} \to \mathbb{R},\ (i_1,\cdots,i_{l-1},i_{l+1},\cdots,i_N) \to a^{(l)}_{i_1,\cdots,i_{l-1},i_{l+1},\cdots,i_N}.$$

Then we have

$$\left| \sum_{\widehat{i_l}\in\widehat{Q_l},l=1,..,N} a^{(1)}_{\widehat{i_1}} \cdots a^{(N)}_{\widehat{i_N}} \right|^2 \le \prod_{k=1}^N \sum_{\widehat{i_k}\in\widehat{Q_k}} \left(a^{(k)}_{\widehat{i_k}}\right)^2. \tag{4.1}$$

Proof. The proof follows by using several times Hölder's estimate. □

Lemma 4.2 *Let $f \in H$ such that $|f| = 1$, and let $m \in \mathbb{N}$. Set*

$$I_{k_1,\dots,k_{2m}} = \int_E \prod_{i=1}^{2m} H_{k_i}(W_f) d\mu, \tag{4.2}$$

where $k_1, \dots, k_{2m} \in \mathbb{N} \cup \{0\}$.

Let moreover $\mathcal{M}_{k_1,\dots,k_{2m}}$ the set (possibly empty) of all finite sequences

$$p = \{p_{i,j} \in \mathbb{N} \cup \{0\},\ i < j,\ i, j = 1, \dots, 2m\},$$

such that

$$\begin{cases} p_{1,2} + p_{1,3} + \cdots + p_{1,2m} = k_1 \\ p_{1,2} + p_{2,3} + \cdots + p_{2,2m} = k_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_{1,2m} + p_{2,2m} + \cdots + p_{2m-1,2m} = k_{2m}. \end{cases} \tag{4.3}$$

Then we have

$$I_{k_1,\dots,k_{2m}} = \begin{cases} \displaystyle\sum_{p \in \mathcal{M}_{k_1,\dots,k_{2m}}} \frac{\left[\prod_{i=1}^{2m} k_i!\right]^{1/2}}{\prod_{i<j=1}^{2m} p_{i,j}!} \ \text{if } \mathcal{M}_{k_1,\dots,k_{2m}} \neq \emptyset \\ 0 \ \ \text{if } \mathcal{M}_{k_1,\dots,k_{2m}} = \emptyset. \end{cases} \tag{4.4}$$

Moreover

$$I_{k_1,\dots,k_{2m}} \leq (2m-1)^{\frac{k_1+\cdots+k_{2m}}{2}}. \tag{4.5}$$

Proof. First notice that

$$\int_{\mathcal{H}} \prod_{i=1}^{2m} F(t_i, W_f) d\nu = e^{-\frac{1}{2}\sum_{i=1}^{2m} t_i^2} \int_{\mathcal{H}} e^{\sum_{i=1}^{2m} t_i W_f} d\mu$$

$$= e^{\sum_{i<j=1}^{2m} t_i t_j}.$$

Now (4.4) follows easily. Let us prove (4.5). We set for brevity

$$\mathcal{M}_{k_1,\dots,k_{2m}} = \mathcal{M}$$

We have

$$\sum_{p\in\mathcal{M}} \frac{\left[\prod_{i=1}^{2m} k_i!\right]^{1/2}}{\prod_{i<j}^{2m} p_{i,j}!}$$

$$= \sum_{p\in\mathcal{M}} \left[\frac{k_1!}{p_{1,2}!p_{1,3}!\cdots p_{1,2m}!}\right]^{1/2} \cdots \left[\frac{k_{2m}!}{p_{1,2m}!p_{2,2m}!\cdots p_{2m-1,2m}!}\right]^{1/2}.$$

Taking into account Lemma 4.1 it follows that

$$\sum_{p\in\mathcal{M}} \frac{\left[\prod_{i=1}^{2m} k_i!\right]^{1/2}}{\prod_{i<j}^{2m} p_{i,j}!} \leq$$

$$\left(\sum_{p\in\mathcal{M}} \frac{k_1!}{p_{1,2}!p_{1,3}!\cdots p_{1,2m}!}\right)^{1/2} \cdots \left(\sum_{p\in\mathcal{M}} \frac{k_{2m}!}{p_{1,2m}!p_{2,2m}!\cdots p_{2m-1,2m}!}\right)^{1/2}.$$

Since for any $l = 1, \ldots, 2m$,

$$\sum_{p\in\mathcal{M}} \frac{k_l!}{p_{l,1}!\cdots p_{l-1,l}!p_{l,l+1}!\cdots p_{l,2m}!} \leq (2m-1)^{k_l}$$

the conclusion follows. □

We are now ready to prove the main result of this section.

Theorem 4.3 *Let $n, m \in \mathbb{N}$, and $u \in L^2_n(\mathcal{H}, \nu)$. Then we have*

$$\left(\int_{\mathcal{H}} |u(\phi)|^{2m} \mu(d\phi)\right)^{\frac{1}{2m}} \leq (2m-1)^{\frac{n}{2}} \left(\int_{\mathcal{H}} |u(\phi)|^2 \mu(d\phi)\right)^{\frac{1}{2}}. \tag{4.6}$$

Proof. Since

$$\varphi = \sum_{\alpha\in\Gamma_n,} \varphi_\alpha H_\alpha,$$

we have

$$\varphi^{2m} = \sum_{\alpha^{(1)}\cdots\alpha^{(2m)}\in\Gamma_n} \varphi_{\alpha^{(1)}}\cdots\varphi_{\alpha^{(2m)}} H_{\alpha^{(1)}}\cdots H_{\alpha^{(2m)}}.$$

By integrating on E it follows

$$\int_E |\varphi(x)|^{2m} \nu(dx) = \sum_{\alpha^{(1)}\cdots\alpha^{(2m)}\in\Gamma_n} \varphi_{\alpha^{(1)}}\cdots\varphi_{\alpha^{(2m)}} \prod_{l=0}^{\infty} I_{\alpha_l^{(1)}\cdots\alpha_l^{(2m)}}.$$

Therefore, by Lemma 4.2,

$$\int_E |\varphi(x)^{2m}\nu(dx) \le (2m-1)^{mn} \sum_{\alpha^{(1)}\cdots\alpha^{(2m)}\in\Gamma_n} c_{\alpha^{(1)}\cdots\alpha^{(2m)}}\varphi_{\alpha^{(1)}}\cdots\varphi_{\alpha^{(2m)}}, \quad (4.7)$$

where

$$c_{\alpha^{(1)}\cdots\alpha^{(2m)}} = \begin{cases} 1 \text{ if } \mathcal{M}_{\alpha_l^{(1)}\cdots\alpha_l^{(2m)}} \ne \emptyset,\ \forall\, l\in\mathbb{N}, \\ \\ 0 \text{ otherwise.}\end{cases}$$

Assume now that $\alpha^{(1)}\cdots\alpha^{(2m)}$ is such that

$$\mathcal{M}_{\alpha_l^{(1)}\cdots\alpha_l^{(2m)}} \ne \emptyset,\ \forall\, l\in\mathbb{N}.$$

In order to estimate the sum in (4.7), it is worth to change notation in the following way. Any element γ in Γ_n has all but a finite number of components that are zero, say $\gamma_k = 0$ for all k not in $\{k_1, k_2, \ldots, k_r\}$; it is natural to indicate the γ itself by the following n-uple of integers

$$(\underbrace{k_1, k_1, \ldots, k_1}_{\gamma_{k_1}}, \underbrace{k_2, k_2, \ldots, k_2}_{\gamma_{k_2}}, \cdots, \underbrace{k_r, k_r, \ldots, k_r}_{\gamma_{k_r}})$$

Hence Γ_n can be viewed as the set of all n-uples of integers $(i_1, i_2, \ldots, i_n)$ with $i_1 \le i_2 \le \ldots \le i_n$.

The fact that the coefficient $c_{\alpha^{(1)}\cdots\alpha^{(2m)}}$ is 1 is equivalent to say that, for any $l\in\mathbb{N}$ that appears in some of the $2m$ n-uples associated respectively to $\alpha^{(1)}, \cdots \alpha^{(2m)}$, there exists a solution $p_{i,j}^{(l)}$: $i<j=1,\ldots,2m,\ l\in\mathbb{N}$ to

$$\begin{cases} p_{1,2}^{(l)} + p_{1,3}^{(l)} + \cdots + p_{1,2m}^{(l)} = \alpha_l^{(1)} \\ \\ p_{1,2}^{(l)} + p_{2,3}^{(l)} + \cdots + p_{2,2m}^{(l)} = \alpha_l^{(2)} \\ \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \\ p_{1,2m}^{(l)} + p_{2,2m}^{(l)} + \cdots + p_{2m-1,2m}^{(l)} = \alpha_l^{(2m)}. \end{cases} \quad (4.8)$$

Therefore we are able, for each of the α^j's, considered as n-uple, to find a partition $(\pi_{1,j}, \pi_{2,j}, \cdots, \pi_{j-1,j}, \pi_{j,j+1}, \cdots, \pi_{j,2m})$ of the set of the n integers of the n-uple, in such a way that

$$\alpha^{(1)} \leftrightarrow (\pi_{1,2}, \pi_{1,3}, \cdots, \pi_{1,2m})$$

$$\alpha^{(2)} \leftrightarrow (\pi_{1,2}, \pi_{2,3}, \cdots, \pi_{2,2m})$$

$$\alpha^{(3)} \leftrightarrow (\pi_{1,3}, \pi_{2,3}, \cdots, \pi_{3,2m})$$

$$\vdots$$

$$\alpha^{(2m)} \leftrightarrow (\pi_{1,2m}, \pi_{2,2m}, \cdots, \pi_{2m-1,2m})$$

where $\pi_{i,j}$ in $\alpha^{(i)}$ and in $\alpha^{(j)}$ is the same set of integers.

Then the sum in (4.7) can be written as

$$\sum_{\pi_{i,j}, i<j=1,\ldots,2m} \varphi_{\pi_{1,2},\pi_{1,3},\ldots,\pi_{1,2m}} \varphi_{\pi_{1,2},\pi_{2,3},\ldots,\pi_{2,2m}} \cdots \varphi_{\pi_{1,2m},\pi_{2,2m},\cdots \pi_{2m-1,2m}}. \tag{4.9}$$

Now the conclusion follows from Lemma 4.1. □

4.2 The Nelson estimate

We fix here, once and for all, an *even* integer $n \in \mathbb{N}$ and set

$$U(\phi) = \langle : \phi^n :, 1 \rangle, \;\; U_N(\phi) = \langle : \phi_N^n :, 1 \rangle, \quad \phi \in \mathcal{H}.$$

By Theorem 3.5 there exists $a > 0$ such that

$$\|U - U_N\|_{L^2(\mathcal{H},\mu)} \leq \frac{a}{\sqrt{N}} \tag{4.10}$$

Since $U, U_N \in L^2_n(\mathcal{H}, \mu)$, by the Theorem 4.3 it follows that

$$\|U - U_N\|_{L^p(\mathcal{H},\mu)} \leq \frac{ap^n}{\sqrt{N}} \tag{4.11}$$

Moreover let $c_n > 0$ be such that $H_n(\theta) \leq -c_n$. Then there exists $b > 0$ such that

$$U_N(\phi) \geq -b(\log N)^n, \quad \phi \in \mathcal{H}. \tag{4.12}$$

Proposition 4.4 *For any $p \geq 1$ we have $e^{-U} \in L^p(\mathcal{H}, \mu)$.*

Proof. It is enough to prove the proposition for $p = 1$. We first note that,

$$\int_{\mathcal{H}} e^{-U} d\mu = \int_0^{+\infty} \mu(e^{-U} > t)dt = \int_0^{+\infty} \mu(U < -\log t)dt. \tag{4.13}$$

Set

$$F(t) = \mu(U < -\log t), \quad t \geq 0,$$

and notice that if $u(\phi) < -\log t$ we have

$$U(\phi) \leq -\log t < -\log t + 1 \leq -b(\log N(t))^n \leq U_{N(t)}(\phi), \tag{4.14}$$

provided $N(t)$ is chosen such as

$$-b(\log N(t))^n \geq -\log t + 1,$$

that is

$$N(t) = \exp\left\{ \left(\frac{\log t - 1}{b} \right)^{1/n} \right\}. \tag{4.15}$$

Now, by (4.14) it follows by the Markov inequality that for any $p \geq 2$,

$$F(t) = \mu(U \leq -\log t) \leq \mu\left(|U - U_{N(t)}| \geq 1\right) \leq \|U - U_{N(t)}\|^p_{L^p(\mathcal{H},\mu)}.$$

By (4.11) and (4.15)

$$F(t) \leq a^p p^{np} N(t)^{-p/2} \leq a^p p^{np} \exp\left\{ -\frac{p}{2} \left(\frac{\log t - 1}{b} \right)^{\frac{1}{n}} \right\}.$$

Finally, we choose $p = p(t)$ such that for some $M, \lambda > 0$,

$$F(t) = \mu(U < -\log t) \leq M t^{-(\lambda+1)}, \quad t > 0, \tag{4.16}$$

and so, by (4.13), we see that $\int_{\mathcal{H}} e^{-U} d\mu < +\infty$. □

Proposition 4.5 *We have*

$$\lim_{N\to\infty} \int_M e^{U_N} \mu(dx) = \int_M e^U \mu(dx). \tag{4.17}$$

Proof. Let $N_0 \in \mathbb{N}$ be fixed and set

$$V(x) = \min\{U, U_{N_0}\}, \quad V_N(x) = \min\{U_N, U_{N_0}\}.$$

Then we have

$$\|V - V_N\|_{L^2(\mathcal{H},\mu)} \le \|U - U_N\|_{L^2(\mathcal{H},\mu)},$$

and

$$V_N(x) \ge -b(\log N)^n.$$

Now, arguing as in the proof of Proposition 4.4 (see (4.16)), we find

$$\int_{\mathcal{H}} e^{-V_{N_0}} d\mu \le \int_{\mathcal{H}} e^{-V} d\mu \le 1 + \frac{M}{\lambda},$$

and the conclusion follows. □

5 Construction of the dynamic by variational method

We fix $n \in \mathbb{N}$ and for any $z \in H$ we set

$$U_z(\phi) = \langle z, :\phi^n: \rangle, \quad \mu\text{-a.e. in } \mathcal{H}.$$

Lemma 5.1 *For any $z \in H$ and almost any $\phi, \psi \in \mathcal{H}$ we have*

$$\frac{d}{d\epsilon} U_z(\phi + \epsilon\psi)\Big|_{\epsilon=0} = n\langle z, :\phi^{n-1}: \rangle. \tag{5.1}$$

Proof. Let $\psi \in \mathcal{H}$, $N \in \mathbb{N}$. Set

$$g(\epsilon) = U_z(\phi + \epsilon\psi) = \langle z, :(\phi + \epsilon\psi)^n: \rangle$$

and

$$g_N(\epsilon) = \langle z, :(\phi + \epsilon\psi)^n_N: \rangle.$$

Then

$$g'_N(\epsilon) = n\langle z, :(\phi + \epsilon\psi)^{n-1}_N: \rangle.$$

Consequently, by Theorem 3.5, it follows that

$$\lim_{N\to\infty} g_N(\epsilon) = g(\epsilon)$$

and

$$\lim_{N\to\infty} g'_N(\epsilon) = n\langle z, :\phi^{n-1}: \psi\rangle.$$

Thus, the conclusion follows. □

For any $z \in H$ and any $\phi, \psi \in \mathcal{H}$ we define

$$DU_z(\phi)\psi = \frac{d}{d\epsilon} F_z(\phi + \epsilon\psi)\Big|_{\epsilon=0} = n\langle z, :\phi^{n-1}: \rangle.$$

5.1 The Sobolev space $W^{1,2}(\mathcal{H},\nu)$

We define the following probability measure in $L^2(\mathcal{H},\nu)$,

$$\nu(d\phi) = ae^{-\frac{1}{2}\langle 1,:\phi^4:\rangle}\mu(d\phi),$$

where

$$a^{-1} = \int_{\mathcal{H}} e^{-\frac{1}{2}\langle 1,:\psi^4:\rangle}\mu(d\psi).$$

We set

$$\rho(\phi) = ae^{-\frac{1}{2}\langle 1,:\phi^4:\rangle}, \quad \phi \in \mathcal{H}$$

and

$$\rho_N(\phi) = ae^{-\frac{1}{2}\langle 1,:\phi_N^4:\rangle}, \quad \phi \in \mathcal{H}.$$

We start with an integration by parts formula.

Proposition 5.2 *Let $u, v \in \mathcal{FC}_0^\infty(\mathcal{H}\ )$, where $D_h = D_{\phi_h},\ h \in \mathbb{Z}^2$. Then we have,*

$$\begin{aligned}\int_{\mathcal{H}} D_h u\ v\ d\nu &= -\int_{\mathcal{H}} D_h v\ u\ d\nu \\ &+2\int_{\mathcal{H}} uv\langle e_h, :\phi^3:\rangle d\nu + (1+|k|^2)^{1/2}\int_{\mathcal{H}} \phi_h\ uv\ d\nu.\end{aligned} \tag{5.2}$$

Proof. Let $u, v \in \mathcal{FC}_0^\infty(\mathcal{H}\ )$ and $N \in \mathbb{N}$. By (2.14) we have

$$\begin{aligned}\int_{\mathcal{H}} D_h u\ v\ \rho_N\ d\mu &= -\int_{\mathcal{H}} D_h v\ u\ \rho_N\ d\mu \\ &-\int_{\mathcal{H}} v\ u\ D_h\rho_N\ d\mu + (1+|k|^2)^{1/2}\int_{\mathcal{H}} \phi_k\ uv\ d\mu.\end{aligned}$$

Taking into account Lemma 5.1 we can write

$$\begin{aligned}\int_{\mathcal{H}} D_h u\ v\ \rho_N\ d\mu &= -\int_{\mathcal{H}} D_h v\ u\ \rho_N\ d\mu \\ &+2\int_{\mathcal{H}} v\ u\langle e_h, :\phi_N^3:\rangle\rho_N\ d\mu + (1+|k|^2)^{1/2}\int_{\mathcal{H}} \phi_k\ uv\ d\mu.\end{aligned}$$

Now, the conclusion follows letting $n \to \infty$. $\square$

Proposition 5.3 *For any $h \in \mathbb{Z}^2$ the operator D_h is closable in $L^2(\mathcal{H},\nu)$.*

Proof. Let $k \in \mathbb{Z}^2$, $(u_n) \subset \mathcal{FC}_0^\infty(\mathcal{H}\,)$ and $v \in L^2(\mathcal{H},\nu)$ be such that

$$u_n \to 0, \;\; D_k u_n \to v \quad \text{in } L^2(\mathcal{H},\nu).$$

We have to show that $v = 0$. If $w \in \mathcal{FC}_0^\infty(\mathcal{H}\,)$, then by (5.2) we have that

$$\int_{\mathcal{H}} D_k u_n \; w \; d\nu = -\int_{\mathcal{H}} D_k u_n \; w \; d\nu$$

$$+2\int_{\mathcal{H}} u_n \; w\langle e_k, :\phi^3 :\rangle d\nu + (1+|k|^2)^{1/2}\int_{\mathcal{H}} \phi_k \; u_n \; w \; d\nu.$$

Letting $n \to \infty$ we find $\int_H vwd\mu = 0$ for all $w \in \mathcal{E}\,(H)$, so that $v = 0$ as required. □

We shall still denote by D_k the closure of D_k on $L^2(\mathcal{H},\nu)$. If φ belongs to the domain of D_k we say that $D_k\varphi$ belongs to $L^2(\mathcal{H},\nu)$.

We now define the space $W^{1,2}(\mathcal{H},\nu)$ as the linear space of all functions $u \in L^2(\mathcal{H},\nu)$ such that $D_k u \in L^2(H,\mu)$ for all $k \in \mathbb{Z}^2$ and

$$\sum_{k\in\mathbb{Z}^2}\int_{\mathcal{H}} |D_k u(\phi)|^2 \nu(d\phi) < +\infty.$$

$W^{1,2}(\mathcal{H},\nu)$, endowed with the inner product,

$$\langle u,v\rangle_{W^{1,2}(\mathcal{H},\nu)} = \langle u,v\rangle_{L^2(\mathcal{H},\nu)} + \sum_{k\in\mathbb{Z}^2}\int_{\mathcal{H}} D_k u \; D_k v \; d\nu,$$

is a Hilbert space.

If $u \in W^{1,2}(\mathcal{H},\nu)$ we set

$$Du(\phi) = \sum_{k\in\mathbb{Z}^2} D_k u(\phi) e_k, \quad \nu-\text{a.e. in } \mathcal{H}.$$

Since

$$|Du(\phi)|^2 = \sum_{k\in\mathbb{Z}^2} |D_k u(\phi)|^2, \quad \nu-\text{a.e. in } \mathcal{H},$$

the series is convergent for almost all $\phi \in \mathcal{H}$. We call $Du(\phi)$ the *gradient* of u at ϕ. Notice that $Du \in L^2(\mathcal{H},\nu;H)$.

We set now $V = W^{1,2}(\mathcal{H},\nu)$ and define the Dirichlet form,

$$a(u,v) = \int_{\mathcal{H}} \langle Du, Dv\rangle d\nu. \tag{5.3}$$

Clearly, a is continuous in $V \times V$ and coercive. So, by the Lax–Milgram Theorem there exists a linear bounded operator A mapping from V into its dual V'. Moreover, the operator,

$$A_1 u = Au, \quad u \in D(A_1) = \{u \in V : \ Au \in V\},$$

is self–adjoint in $L^2(\mathcal{H}, \nu)$ and

$$P_t u(\phi) = e^{tA_1} u(\phi),$$

defines a symmetric strongly continuous semigroup in $L^2(\mathcal{H}, \nu)$ having ν as invariant measure.

Remark 5.4 The Dirichlet form approach here presented was introduced in [4]. Here existence and uniqueness of a weak solution (in the sense of Fukushima) of (1.8) was also proved.

6 Essential m–dissipativity of the Kolmogorov operator in $L^1(\mathcal{H}, \mu)$

The Kolmogorov operator corresponding to the stochastic differential equation (1.10) is the following

$$K_0 u(\phi) = \frac{1}{2} \ \mathrm{Tr} \ [C^\varepsilon D^2 u] - \frac{1}{2} \ \langle Du, C^{\varepsilon - 1} \phi \rangle - \langle C^{\varepsilon/2} Du, C^{\varepsilon/2} : \phi^3 : \rangle, \qquad (6.1)$$

where $\varepsilon > 0$ and $u \in \mathcal{FC}_0^\infty(\mathcal{H})$. We notice that the condition $\varepsilon > 0$ is essential in what follows.

Proposition 6.1 *The following statements hold.*

(i) The measure ν is invariant for K_0, that is

$$\int_{\mathcal{H}} K_0 u(\phi) \nu(d\phi) = 0, \quad u \in \mathcal{FC}_0^\infty(\mathcal{H}). \qquad (6.2)$$

(ii) We have

$$\int_{\mathcal{H}} K_0 u(\phi) \ u(\phi) \ \nu(d\phi) = -\frac{1}{2} \int_{\mathcal{H}} |C^{\varepsilon/2} Du(\phi)|^2 \nu(d\phi), \quad u \in \mathcal{FC}_0^\infty(\mathcal{H}). \qquad (6.3)$$

Proof. (i) Assume that u depends only on variables ϕ_k with $|k| \le N$. Then we have

$$I: \ = \frac{1}{2} \int_{\mathcal{H}} \operatorname{Tr}\, [C^{\varepsilon} D^2 u] d\nu = \sum_{|k|\le N} \int_{\mathcal{H}} (1+|k|^2)^{-\varepsilon/2} D_k^2 u d\nu.$$

By the integration by parts formula (5.2) (with $D_k u$ replacing u and 1 replacing v) we obtain,

$$I = \sum_{|k|\le N} (1+|k|^2)^{-\varepsilon/2} \left[2 \int_{\mathcal{H}} u \langle e_k, :\phi^3: \rangle d\nu + (1+|k|^2)^{1/2} \int_{\mathcal{H}} \phi_k u d\nu \right]$$

$$= \frac{1}{2} \langle Du, C^{\varepsilon-1}\phi \rangle + \langle C^{\varepsilon/2} Du, C^{\varepsilon/2} : \phi^3 : \rangle,$$

so that (i) follows.

To prove (ii) it is enough to integrate with respect to ν over $\mathcal{H}$ the straightforward identity

$$K_0(u^2) = 2uK_0 u + |C^{\varepsilon/2} Du|^2.$$

□

Consider now the approximating operator, also defined in $\mathcal{FC}_0^\infty(\mathcal{H}\ )$,

$$\begin{aligned} K_N u \ &= \ \frac{1}{2} \sum_{|k|\le N} (1+|k|^2)^{-\varepsilon} D_k^2 u - \frac{1}{2} \sum_{|k|\le N} (1+|k|^2)^{1-\varepsilon} \phi_k D_k u \\ &\quad - \sum_{|k|\le N} (1+|k|^2)^{-\varepsilon} \langle e_k, : \phi_N^3(\xi) : \rangle D_k u, \end{aligned} \tag{6.4}$$

or, equivalently,

$$K_N u(\phi) = \frac{1}{2} \operatorname{Tr}\, [C_N^{\varepsilon} D^2 u] - \frac{1}{2} \langle Du, C_N^{\varepsilon-1}\phi \rangle - \langle C_N^{\varepsilon/2} Du, C_N^{\varepsilon/2} : \phi_N^3 : \rangle, \tag{6.5}$$

where

$$C_N \phi = \sum_{|k|\le N} (1+|k|^2)^{-1} \phi_k e_k.$$

Moreover, let us consider the following approximating equation.

$$\lambda u_N - K_N u_N = f, \tag{6.6}$$

where $\lambda > 0$ and $f \in \mathcal{FC}_0^\infty(\mathcal{H}\ )$. Equation (6.6) has a unique solution u_N by classical results on elliptic nonlinear equations.

It is convenient to write (6.6) in the following form

$$\lambda u_N - K_0 u_N = f + \langle C_N^{\varepsilon/2} D u_N, C_N^{\varepsilon/2}(:\phi^3: - :\phi_N^3:)\rangle. \tag{6.7}$$

We prove now an a priori estimate for u_N.

Lemma 6.2 *There exists a constant $c = c(\|f\|_\infty)$ such that*

$$\int_{\mathcal{H}} |C_N^{\varepsilon/2} D u_N|^2 d\nu \le c(\|f\|_\infty) \int_{\mathcal{H}} |C_N^{\varepsilon/2}(:\phi^3: - :\phi_N^3:)|^2 d\nu. \tag{6.8}$$

Proof. By multiplying both sides of (6.7) by u_n, integrating on ν over $\mathcal{H}$ and taking into account (6.3) yields,

$$\lambda \int_{\mathcal{H}} |u_N|^2 d\nu + \frac{1}{2}\,\lambda \int_{\mathcal{H}} |C_N^{\varepsilon/2} D u_N|^2 d\nu$$

$$= \int_{\mathcal{H}} f u_N d\nu + \int_{\mathcal{H}} \langle C_N^{\varepsilon/2} D u_N, C_N^{\varepsilon/2}(:\phi^3: - :\phi_N^3:)\rangle u_N d\nu.$$

By the Maximum principle we have,

$$\|u_N\|_\infty \le \frac{1}{\lambda}\,\|f\|_\infty.$$

Consequently,

$$\lambda \int_{\mathcal{H}} |u_N|^2 d\nu + \frac{1}{2}\,\lambda \int_{\mathcal{H}} |C_N^{\varepsilon/2} D u_N|^2 d\nu \le \frac{1}{\lambda}\,\|f\|_\infty \|u_N\|_{L^2(\mathcal{H},\nu)}$$

$$+\frac{1}{\lambda}\,\|f\|_\infty \left(\int_{\mathcal{H}} |C_N^{\varepsilon/2} D u_N|^2 d\nu\right)^{1/2} \left(\int_{\mathcal{H}} |C_N^{\varepsilon/2}(:\phi^3: - :\phi_N^3:)|^2 d\nu\right)^{1/2}.$$

Now the conclusion follows from the Gronwall Lemma. □

We are now ready to prove, arguing as in [13], the main result of this section.

Theorem 6.3 *The closure of K_0 in $L^1(\mathcal{H},\nu)$ is an m–dissipative operator.*

Proof. By Lemma 6.2 we deduce that

$$\lim_{N\to\infty} \langle C_N^{\varepsilon/2} D u_N, C_N^{\varepsilon/2}(:\phi^3: - :\phi_N^3:)\rangle = 0 \quad \text{in} \quad L^1(\mathcal{H},\nu).$$

Therefore, by (6.7) we see that the range of $\lambda - K_0$ is dense in $L^1(\mathcal{H},\nu)$, since its closure includes $\mathcal{FC}_0^\infty(\mathcal{H})$. Thus, the closure of K_0 is m–dissipative in $L^1(\mathcal{H},\nu)$ in view of the theorem of Lumer and Phillips, see e.g. [26].

Remark 6.4 It is possible to show that the closure of K_0 in $L^2(\mathcal{H},\nu)$ is m–dissipative or, equivalently, that K_0 is essentially self–adjoint. For this a somewhat tricky estimate for

$$\int_{\mathcal{H}} |C_N^{\varepsilon/2} D u_N|^4 d\nu,$$

is needed, [22], [12].

7 Generalizations

Let $H = L^2(\mathcal{O})$ (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$), where $\mathcal{O} = [0,2\pi]^d$ and $d \in \mathbb{N}$. We denote by $(e_k)_{k\in\mathbb{Z}^d}$ the complete orthonormal system of H,

$$e_k(\xi) = (2\pi)^{-d/2}\, e^{i\langle k,\xi\rangle},\ \ \xi \in \mathcal{O},\ k \in \mathbb{Z}^d$$

and by H_0 the linear span of $(e_k)_{k\in\mathbb{Z}^d}$. For any $x \in H$ we set

$$\langle x, e_k\rangle = x_k, \quad \text{for all } k = (k_1,k_2) \in \mathbb{Z}^d.$$

We shall identify H with the space $\ell^2(\mathbb{Z}^d)$ of all square summable sequences $(x_k)_{k\in\mathbb{Z}^d} \subset \mathbb{R}$ through the isomorphism

$$x \in H \mapsto (x_k)_{k\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^d).$$

In order to construct the Wick products we introduce the following linear bounded operator in H,

$$Ce_k = \lambda_k\, e_k,\ \ k \in \mathbb{Z}^d,$$

where $\{\lambda_k\}_{k\in\mathbb{Z}^d}$ is a fixed suitable sequence of positive numbers.

As in §1 we introduce the product space $\mathcal{H}$,

$$\mathcal{H} = \bigtimes_{k\in\mathbb{Z}^d} \mathbb{R}_k, \quad \mathbb{R}_k = \mathbb{R}$$

and consider H (identified with $\ell^2(\mathbb{Z}^d)$) as a subspace of $(\mathbb{R}^d)^\infty$. We shall denote by $x, y, z....$ elements in H and by $\phi, \psi, \zeta, ...$ elements in $\mathcal{H}$.

Next we define the Borel product measure μ on $\mathcal{H}$ (endowed with the product topology),

$$\mu = \bigtimes_{k\in\mathbb{Z}^d} N_{\lambda_k},$$

where N_{λ_k} represents the one–dimensional Gaussian measure with mean 0 and variance λ_k.

Next we introduce a duality between H_0 and $\mathcal{H}$ as follows. For any $x \in H_0$ and any $\phi \in \mathcal{H}$ we define

$$\langle x, \phi \rangle = \sum_{k \in \mathbb{Z}^d} x_k \phi_k.$$

Moreover, we extend the previous definition of *white noise.* First for any $z \in H_0$ we define a function W_z in $L^2(\mathcal{H}, \mu)$ setting,

$$W_z(\phi) = \langle C^{-1/2} z, \phi \rangle = \sum_{k \in \mathbb{Z}^d} \lambda_k^{-1/2} \, z_k \phi_k, \quad \phi \in \mathcal{H}.$$

Since the mapping

$$H_0 \to L^2(\mathcal{H}, \mu), \; z \mapsto W_z,$$

is an isometry, it can be extended to the whole H. Thus W_z is a well defined element of $L^2(\mathcal{H}, \mu)$ for any $z \in H$.

Different properties of the space $L^2(\mathcal{H}, \mu)$ as the Wiener–Itô decomposition can be proved as in §2.

As in §3 we give a meaning to the function on $\mathcal{H}$,

$$\langle x, \phi^n \rangle = \int_{\mathcal{O}} x(\xi) \phi^n(\xi) d\xi,$$

where $n \in \mathbb{N}$ and $x \in H$.

Given $\phi \in \mathcal{H}$ and $\xi \in \mathcal{O}$ we set

$$\phi_N(\xi) = \sum_{|k| \le N} \langle e_k, \phi \rangle e_k(\xi).$$

Notice that $\phi_N \in C^\infty(\mathcal{O})$. Moreover, we can write,

$$\phi_N(\xi) = \left\langle \sum_{|k| \le N} \lambda_k^{1/2} \, e_k, C^{-1/2} \phi \right\rangle.$$

and so

$$\phi_N(\xi) = \rho_N W_{\eta_N(\xi)}(\phi), \quad \xi \in \mathcal{O}, \; N \in \mathbb{N}, \tag{7.1}$$

where

$$\eta_N(\xi) = \frac{1}{\rho_N} \sum_{|k| \le N} \lambda_k^{1/2} e_k(\xi) \, e_k, \tag{7.2}$$

and

$$\rho_N^2 = \frac{1}{2\pi} \sum_{|k| \le N} \lambda_k. \tag{7.3}$$

Notice that $|\eta_N(\xi)| = 1$. Finally, we set

$$: \phi_N^n : (\xi) = \sqrt{n!}\ \rho_N^n\ H_n\left(\frac{\phi_N(\xi)}{\rho_N}\right) = \sqrt{n!}\ \rho_N^n\ H_n(W_{\eta_N(\xi)}),\ \ \xi \in \mathcal{O}. \quad (7.4)$$

Notice also that

$$\langle \eta_N(\xi_1), \eta_N(\xi_2)\rangle = \frac{1}{\rho_N^2}\ \gamma_N(\xi_1 - \xi_2), \quad \xi_1, \xi_2 \in \mathcal{O},\ \ N \in \mathbb{N}, \quad (7.5)$$

where

$$\gamma_N = \sum_{|k|\le N} \lambda_k\ e_k, \quad N \in \mathbb{N}. \quad (7.6)$$

Our goal is to see whether the limit

$$\lim_{N\to\infty} \int_{\mathcal{O}} :\phi_N^n\!: (\xi)\, x(\xi) d\xi := \langle x, :\phi^n:\rangle \quad \text{in } L^2(\mathcal{H}, \mu), \quad (7.7)$$

exists for any fixed $x \in H$. For this end, it is necessary to check that, setting

$$I_N : = \int_{\mathcal{H}} \left| \int_{\mathcal{O}} :\phi_N^n\!: (\xi)\, x(\xi) d\xi \right|^2 \mu(d\phi), \quad (7.8)$$

the supremum $\sup_{N\in\mathbb{N}} I_N < +\infty$. Namely, we have

$$I_N = \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)d\xi_1 d\xi_2 \int_{\mathcal{H}} :\phi_N^n\!: (\xi_1)\ :\phi_N^n\!: (\xi_2) d\mu$$

$$= n!\rho_N^{2n} \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)d\xi_1 d\xi_2 \int_{\mathcal{H}} H_n(W_{\eta_N(\xi_1)})H_n(W_{\eta_N(\xi_2)})d\mu$$

$$= n!\rho_N^{2n} \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)\, \langle \eta_N(\xi_1), \eta_N(\xi_2)\rangle^n\, d\xi_1 d\xi_2.$$

Consequently,

$$I_N = \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)\gamma_N^n(\xi_1 - \xi_2)d\xi_1 d\xi_2, \quad (7.9)$$

from which we arrive to the following necessary condition to guarantee the existence of the limit (7.7)

$$\sup_{N\in\mathbb{N}} \int_{\mathcal{O}\times\mathcal{O}} x(\xi_1)x(\xi_2)\gamma_N^n(\xi_1 - \xi_2)d\xi_1 d\xi_2 < +\infty. \quad (7.10)$$

This holds provided the sequence $\{\gamma_N\}$ is bounded in $L^n(\mathcal{O})$. Set

$$\gamma = \sum_{k\in\mathbb{Z}^d} \lambda_k\, e_k. \tag{7.11}$$

The following proposition is proved similarly to Proposition 3.2

Proposition 7.1 *For all $n \geq 1$ we have*

$$|\gamma|^{\frac{n}{n-1}}_{L^n(\mathcal{O})} \leq (2\pi)^{\frac{n(n-2)}{2(n-1)}} \sum_{k\in\mathbb{Z}^d} \lambda_k^{\frac{n}{n-1}}. \tag{7.12}$$

Example 7.2 Consider the case of

$$Ce_k = \frac{1}{(1+|k|^2)^2}\, e_k, \qquad k \in \mathbb{Z}^d.$$

This corresponds to $A = -\frac{1}{2}\, C^{-1} = -\frac{1}{2}\,(\Delta - 1)^2$. By Proposition 7.1 we have that $\gamma \in L^n(\mathcal{O})$ if and only if either $d \leq 4$ or if $d > 4$ and $n < \frac{d}{d-4}$.

7.1 Renormalization in $\mathbb{R}^3$

Here we consider equation (1.8) in the square $[0, 2\pi]^3$ with $n = 3$ and set

$$Ce_k = \frac{1}{1+|k|^2}\, e_k, \qquad k \in \mathbb{Z}^3.$$

Then by Proposition 7.1 we see that $\gamma \in L^n(\mathbb{R}^3)$ if and only if $n < 3$. Hence, it is not possible to define $:x^3:$ and consequently to consider the measure ν defined by (1.9).

Nevertheless, because of the physical relevance of measure ν in quantum field theory, Glimm and Jaffe found a measure ν, introducing further suitable subtractions in the exponent of (1.9).

7.2 The Kardar–Parisi–Zhang equation

We take here $d = 1$ and consider the following Burgers equation in $L^2(0, 2\pi)$ (norm $|\cdot|$, inner product $\langle\cdot,\cdot\rangle$),

$$dX = \frac{1}{2}\, \left[(X_{\xi\xi} - X) - D_\xi(X^2)\right] dt + \frac{\partial^2 W(t,\xi)}{\partial t \partial \xi}, \quad X(0) = x \in L^2(0, 2\pi), \tag{7.13}$$

where $\xi \in [0, 2\pi]$, X is 2π–periodic and

$$W(t, \xi) = \sum_{k=1}^{\infty} e_k(\xi)\beta_k(t),$$

where $(\beta_k(t))_{k\in\mathbb{Z}}$ is a family of standard Brownian motions mutually independent in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Equation (7.13) was introduced in [19] as a model of the interface growing in the phase transitions theory.

Let us write equation (7.13) in the following mild form

$$X(t) = e^{tA}x - \int_0^t e^{(t-s)A} D_\xi(X^2) ds + \int_0^t e^{(t-s)A} BdW(s), \tag{7.14}$$

where

$$Ax = \frac{1}{2}\,(x_{\xi\xi} - x), \quad x \in \{y \in H^2(0, 2\pi) : \ y(0) = y(2\pi), \ y_\xi(0) = y_\xi(2\pi)\},$$

$$Bx = D_\xi x \quad x \in \{y \in H^1(0, 2\pi) : \ y(0) = y(2\pi)\}.$$

Now consider the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} BdW(s) = \sum_{k\in\mathbb{Z}} ik \int_0^t e^{-\frac{1}{2}\,(t-s)(1+|k|^2)} d\beta_k(s). \tag{7.15}$$

$W_A(t)$ is a Gaussian random variable in $L^2(0, 2\pi)$ with mean 0 and covariance operator

$$C(t) = C(1 - e^{tA}) \quad t \geq 0.$$

where

$$Ce_k = \frac{k^2}{1 + k^2}\, e_k, \quad k \in \mathbb{Z}.$$

In order to study (7.14), the first step would be to define $:x^2:$ (since $D_\xi : X^2 := D_\xi(X^2)$). However, since C has eigenvalues

$$\lambda_k = \frac{k^2}{1 + k^2}, \qquad k \in \mathbb{Z},$$

this, by Proposition 7.1, shows that the bound (7.12) on γ, rendering renormalization possible, does not hold.

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