STABLE DISCRETIZATIONS OF ELASTIC FLOW IN RIEMANNIAN MANIFOLDS*

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Abstract. The elastic flow, which is the L^2-gradient flow of the elastic energy, has several applications in geometry and elasticity theory. We present stable discretizations for the elastic flow in two-dimensional Riemannian manifolds that are conformally flat, i.e., conformally equivalent to the Euclidean space. Examples include the hyperbolic plane, the hyperbolic disk, and the elliptic plane, as well as any conformal parameterization of a two-dimensional manifold in $\mathbb{R}^d$, $d \geq 3$. Numerical results show the robustness of the method, as well as quadratic convergence with respect to the space discretization.

Key words. elastic flow, hyperbolic plane, hyperbolic disk, elliptic plane, Riemannian manifolds, geodesic elastic flow, finite element approximation, stability, equidistribution

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1. Introduction. Elastic flow of curves in a two-dimensional Riemannian manifold $(\mathcal{M}, g)$ is given as the L^2-gradient flow of the elastic energy $\frac{1}{2} \int \varkappa_g^2$, where $\varkappa_g$ is the geodesic curvature. It has been shown (see [8] for the general case and [12] for the hyperbolic plane) that the gradient flow of the elastic energy is given as

$$(1.1) \qquad \mathcal{V}_g = -(\varkappa_g)_{s_g s_g} - \frac{1}{2}\,\varkappa_g^3 - S_0\,\varkappa_g\,,$$

where $\mathcal{V}_g$ is the normal velocity of the curve with respect to the metric g, $\partial_{s_g} = g^{-\frac{1}{2}}\,\partial_s$, s denoting arclength, and S_0 is the sectional curvature of g. The evolution law (1.1) decreases the curvature energy $\frac{1}{2} \int \varkappa_g^2$, and long term limits are expected to be critical points of this energy. These critical points are called free elasticae and are of interest in geometry [17] and mechanics [22, 2]. In particular, let us mention that a curve is an absolute minimizer if and only if it is a geodesic. Recently the flow (1.1) was studied in [12, 13] for the case of the hyperbolic plane, relying on earlier results in [15] for a flat background metric. The hyperbolic plane is a particular case of a manifold with nonpositive sectional curvature, which is of particular interest as the set of free elasticae is much richer; see [17].

In this paper, we allow for a general conformally flat metric. Examples include the hyperbolic plane, the hyperbolic disk, and the elliptic plane, as well as any conformal parameterization of a two-dimensional manifold in $\mathbb{R}^d$, $d \geq 3$. For parameterized hypersurfaces in $\mathbb{R}^3$, earlier authors (see, e.g., [10, 18, 19, 3, 5]) used the surrounding space in their numerical approximations, which leads to errors in directions normal to the hypersurface. This will be avoided by the intrinsic approach used in this paper. In

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particular, our numerical method leads to approximate solutions which remain on the hypersurface after application of the parameterization map. In addition, in this paper we will present a first numerical analysis for elastic flow in manifolds not embedded in $\mathbb{R}^3$. This in particular makes it possible to compute elastic flow of curves in the hyperbolic plane in a stable way.

For finite element approximations of (1.1) introduced in [8] it does not appear possible to prove a stability result. It is the aim of this paper to introduce novel approximations for (1.1) that can be shown to be stable. In particular, we will show that the semidiscrete continuous-in-time approximations admit a gradient flow structure. For relevant literature on conformal metrics we refer to [20, 16]. Curvature driven flows in hyperbolic spaces have been studied by [11, 1, 12, 13, 8], and related numerical approximations of elastic flow of curves can be found in [15, 14, 6, 9] for the Euclidean case and in [10, 18, 19, 3, 5] for the case of curves on hypersurfaces in $\mathbb{R}^3$.

The outline of this paper is as follows. After formulating the problem in detail in the next section, we will derive in section 3 weak formulations which will be the basis for our finite element approximation. In section 4 we introduce continuous-in-time, discrete-in-space discretizations which are based on the weak formulations. For these semidiscrete formulations a stability result will be shown, which is the main contribution of this work. In section 5 we then formulate fully discrete variants for which we show existence and uniqueness. In section 6 we present several numerical computations which show convergence rates as well as the robustness of the approach. Finally, in the appendix we show the consistency of the weak formulations presented in section 3.

2. Mathematical formulations. Let $I = \mathbb{R}/\mathbb{Z}$ be the periodic interval $[0,1]$. Let $\vec{x} : I \to \mathbb{R}^2$ be a parameterization of a closed curve $\Gamma \subset \mathbb{R}^2$. On assuming that $|\vec{x}_\rho| > 0$ on I, we introduce the arclength s of the curve, i.e., $\partial_s = |\vec{x}_\rho|^{-1} \partial_\rho$, and set

$$(2.1) \qquad \vec{\tau} = \vec{x}_s \qquad \text{and} \qquad \vec{\nu} = -\vec{\tau}^\perp ,$$

where $\cdot^\perp$ denotes a clockwise rotation by $\frac{\pi}{2}$. For the curvature $\varkappa$ of Γ it holds that

$$(2.2) \qquad \varkappa \vec{\nu} = \vec{\varkappa} = \vec{\tau}_s = \vec{x}_{ss} = \frac{1}{|\vec{x}_\rho|} \left[\frac{\vec{x}_\rho}{|\vec{x}_\rho|} \right]_\rho .$$

Let $H \subset \mathbb{R}^2$ be an open set with metric tensor

$$(2.3) \qquad [(\vec{v}, \vec{w})_g](\vec{z}) = g(\vec{z}) \, \vec{v} \cdot \vec{w} \quad \forall \, \vec{v}, \vec{w} \in \mathbb{R}^2 \qquad \text{for } \vec{z} \in H ,$$

where $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$ is the standard Euclidean inner product, and where $g : H \to \mathbb{R}_{>0}$ is a smooth positive weight function. The length induced by (2.3) is defined as

$$(2.4) \qquad [|\vec{v}|_g](\vec{z}) = ([(\vec{v}, \vec{v})_g](\vec{z}))^{\frac{1}{2}} = g^{\frac{1}{2}}(\vec{z}) \, |\vec{v}| \quad \forall \, \vec{v} \in \mathbb{R}^2 \qquad \text{for } \vec{z} \in H .$$

For $\lambda \in \mathbb{R}$, we define the generalized elastic energy as

$$(2.5) \qquad W_{g,\lambda}(\vec{x}) = \frac{1}{2} \int_I (\varkappa_g^2 + 2\,\lambda) \, |\vec{x}_\rho|_g \, \mathrm{d}\rho ,$$

where

$$(2.6) \qquad \varkappa_g = g^{-\frac{1}{2}}(\vec{x}) \left[\varkappa - \frac{1}{2} \vec{\nu} \cdot \nabla \ln g(\vec{x}) \right]$$

is the curvature of the curve with respect to the metric g; see [8] for details. Generalized elastic flow is defined as the L^2-gradient flow of (2.5), and it was established in [8] that a strong formulation is given by

$$(2.7) \qquad \mathcal{V}_g = g^{\frac{1}{2}}(\vec{x})\, \vec{x}_t \,.\, \vec{\nu} = -(\varkappa_g)_{s_g s_g} - \frac{1}{2}\, \varkappa_g^3 - S_0\, \varkappa_g + \lambda\, \varkappa_g \,,$$

where $\partial_{s_g} = g^{-\frac{1}{2}}(\vec{x})\, \partial_s$ and

$$(2.8) \qquad S_0 = -\frac{\Delta \ln g}{2\, g}$$

is the sectional curvature of g. We refer to [8] for further details.

The two weak formulations of (2.7), for $\lambda = 0$, introduced in [8] are based on the equivalent equation

$$(2.9) \qquad g(\vec{x})\, \vec{x}_t \,.\, \vec{\nu} = -\frac{1}{|\vec{x}_\rho|}\left(\frac{[\varkappa_g]_\rho}{g^{\frac{1}{2}}(\vec{x})\, |\vec{x}_\rho|}\right)_\rho - \frac{1}{2}\, g^{\frac{1}{2}}(\vec{x})\, \varkappa_g^3 - g^{\frac{1}{2}}(\vec{x})\, S_0(\vec{x})\, \varkappa_g \,.$$

The first uses $\varkappa$ as a variable, while the second uses $\varkappa_g$ as a variable.

($\mathcal{U}$): Let $\vec{x}(0) \in [H^1(I)]^2$. For $t \in (0, T]$ find $\vec{x}(t) \in [H^1(I)]^2$ and $\varkappa(t) \in H^1(I)$ such that

(2.10a)

$$\int_I g(\vec{x})\, \vec{x}_t \,.\, \vec{\nu}\, \chi\, |\vec{x}_\rho|\, \mathrm{d}\rho = \int_I g^{-\frac{1}{2}}(\vec{x})\left(g^{-\frac{1}{2}}(\vec{x})\left[\varkappa - \frac{1}{2}\, \vec{\nu} \,.\, \nabla \ln g(\vec{x})\right]\right)_\rho \chi_\rho\, |\vec{x}_\rho|^{-1}\, \mathrm{d}\rho$$

$$- \frac{1}{2}\int_I g^{-1}(\vec{x})\left[\varkappa - \frac{1}{2}\, \vec{\nu} \,.\, \nabla \ln g(\vec{x})\right]^3 \chi\, |\vec{x}_\rho|\, \mathrm{d}\rho$$

$$- \int_I S_0(\vec{x})\left[\varkappa - \frac{1}{2}\, \vec{\nu} \,.\, \nabla \ln g(\vec{x})\right]\chi\, |\vec{x}_\rho|\, \mathrm{d}\rho \quad \forall\, \chi \in H^1(I)\,,$$

(2.10b)

$$\int_I \varkappa\, \vec{\nu} \,.\, \vec{\eta}\, |\vec{x}_\rho|\, \mathrm{d}\rho + \int_I (\vec{x}_\rho \,.\, \vec{\eta}_\rho)\, |\vec{x}_\rho|^{-1}\, \mathrm{d}\rho = 0 \quad \forall\, \vec{\eta} \in [H^1(I)]^2\,.$$

($\mathcal{W}$): Let $\vec{x}(0) \in [H^1(I)]^2$. For $t \in (0, T]$ find $\vec{x}(t) \in [H^1(I)]^2$ and $\varkappa_g(t) \in H^1(I)$ such that

(2.11a)

$$\int_I g(\vec{x})\, \vec{x}_t \,.\, \vec{\nu}\, \chi\, |\vec{x}_\rho|\, \mathrm{d}\rho = \int_I g^{-\frac{1}{2}}(\vec{x})\, [\varkappa_g]_\rho\, \chi_\rho\, |\vec{x}_\rho|^{-1}\, \mathrm{d}\rho - \frac{1}{2}\int_I g^{\frac{1}{2}}(\vec{x})\, \varkappa_g^3\, \chi\, |\vec{x}_\rho|\, \mathrm{d}\rho$$

$$- \int_I S_0(\vec{x})\, g^{\frac{1}{2}}(\vec{x})\, \varkappa_g\, \chi\, |\vec{x}_\rho|\, \mathrm{d}\rho \quad \forall\, \chi \in H^1(I)\,,$$

(2.11b)

$$\int_I g(\vec{x})\, \varkappa_g\, \vec{\nu} \,.\, \vec{\eta}\, |\vec{x}_\rho|\, \mathrm{d}\rho + \int_I \left[\nabla g^{\frac{1}{2}}(\vec{x}) \,.\, \vec{\eta} + g^{\frac{1}{2}}(\vec{x})\, \frac{\vec{x}_\rho \,.\, \vec{\eta}_\rho}{|\vec{x}_\rho|^2}\right]|\vec{x}_\rho|\, \mathrm{d}\rho = 0 \quad \forall\, \vec{\eta} \in [H^1(I)]^2\,.$$

For the numerical approximations based on ($\mathcal{U}$) and ($\mathcal{W}$) it does not appear possible to prove stability results that show that discrete analogues of (2.5), for $\lambda = 0$, decrease monotonically in time. The reason is that these formulations are directly based on the divergence form in (2.9), which does not immediately capture the variational structure of the gradient flow for (2.5). For the same reason, the first stable finite

element approximations for Euclidean elastic flow were given in [14], whereas the earlier schemes in [15] do not appear to admit a stability proof. In fact, based on the ideas in [14], and utilizing the techniques in [6], it is possible to introduce alternative weak formulations of (2.7), for which semidiscrete continuous-in-time finite element approximations admit such a stability result. Novel aspects compared to our previous work [6] include the highly nonlinear nature of the energy (2.5) and the curvature definition (2.6). Their variations give rise to new nontrivial contributions; see, e.g., (3.3) below. Moreover, exploiting the variational structure of the problem by treating $\varkappa_g$ as an independent variable is, of course, new to this work compared to [6].

We end this section with some example metrics that are of particular interest in differential geometry. Two families of metrics are given by

$$(2.12a) \qquad g(\vec{z}) = (\vec{z}.\vec{e}_2)^{-2\mu}, \ \mu \in \mathbb{R}, \quad \text{with} \quad H = \mathbb{H}^2 = \{\vec{z} \in \mathbb{R}^2 : \vec{z}.\vec{e}_2 > 0\},$$

and

$$(2.12b) \quad g(\vec{z}) = \frac{4}{(1 - \alpha\,|\vec{z}|^2)^2} \quad \text{with} \quad H = \begin{cases} \mathbb{D}_\alpha = \{\vec{z} \in \mathbb{R}^2 : |\vec{z}| < \alpha^{-\frac{1}{2}}\}, & \alpha > 0, \\ \mathbb{R}^2, & \alpha \le 0. \end{cases}$$

The case (2.12a) with $\mu = 1$ models the hyperbolic plane, while $\mu = 0$ corresponds to the Euclidean case. The case (2.12b) with $\alpha = 1$ gives a model for the hyperbolic disk, while $\alpha = -1$ models the geometry of the elliptic plane. Of course, $\alpha = 0$ collapses to the Euclidean case.

Further metrics of interest are induced by conformal parameterizations $\vec{\Phi} : H \to \mathbb{R}^d$, $d \ge 3$, of the two-dimensional Riemannian manifold $\mathcal{M} \subset \mathbb{R}^d$, i.e., $\mathcal{M} = \vec{\Phi}(H)$ and $|\partial_{\vec{e}_1}\vec{\Phi}(\vec{z})|^2 = |\partial_{\vec{e}_2}\vec{\Phi}(\vec{z})|^2$ and $\partial_{\vec{e}_1}\vec{\Phi}(\vec{z}).\partial_{\vec{e}_2}\vec{\Phi}(\vec{z}) = 0$ for all $\vec{z} \in H$. Here examples include the stereographic projection of the unit sphere without the north pole, $\vec{\Phi}(\vec{z}) = (1 + |\vec{z}|^2)^{-1}\,(2\,\vec{z}.\vec{e}_1, 2\,\vec{z}.\vec{e}_2, |\vec{z}|^2 - 1)^T$, so that $g(\vec{z}) = 4\,(1 + |\vec{z}|^2)^{-2}$ and $H = \mathbb{R}^2$, which yields a geometric interpretation to (2.12b) with $\alpha = -1$. Further examples are the Mercator projection of the unit sphere without the north and the south pole, $\vec{\Phi}(\vec{z}) = \cosh^{-1}(\vec{z}.\vec{e}_1)\,(\cos(\vec{z}.\vec{e}_2), \sin(\vec{z}.\vec{e}_2), \sinh(\vec{z}.\vec{e}_1))^T$, so that

$$(2.12c) \qquad\qquad g(\vec{z}) = \cosh^{-2}(\vec{z}.\vec{e}_1) \quad \text{with} \quad H = \mathbb{R}^2,$$

as well as the catenoid parameterization $\vec{\Phi}(\vec{z}) = (\cosh(\vec{z}.\vec{e}_1)\cos(\vec{z}.\vec{e}_2), \cosh(\vec{z}.\vec{e}_1)\sin(\vec{z}.\vec{e}_2), \vec{z}.\vec{e}_1)^T$, so that

$$(2.12d) \qquad\qquad g(\vec{z}) = \cosh^2(\vec{z}.\vec{e}_1) \quad \text{with} \quad H = \mathbb{R}^2.$$

Based on [21, p. 593] we also recall the following conformal parameterization of a torus with large radius $R > 1$ and small radius $r = 1$ from [8]. In particular, we let $\mathfrak{s} = [R^2 - 1]^{\frac{1}{2}}$ and define $\vec{\Phi}(\vec{z}) = \mathfrak{s}\,([\mathfrak{s}^2 + 1]^{\frac{1}{2}} - \cos(\vec{z}.\vec{e}_2))^{-1}\,(\mathfrak{s}\cos\frac{\vec{z}.\vec{e}_1}{\mathfrak{s}}, \mathfrak{s}\sin\frac{\vec{z}.\vec{e}_1}{\mathfrak{s}}, \sin(\vec{z}.\vec{e}_2))^T$, so that

$$(2.12e) \qquad g(\vec{z}) = \mathfrak{s}^2\,([\mathfrak{s}^2 + 1]^{\frac{1}{2}} - \cos(\vec{z}.\vec{e}_2))^{-2} \quad \text{with} \quad H = \mathbb{R}^2.$$

3. Weak formulations. We define the first variation of a quantity depending in a differentiable way on $\vec{x}$, in the direction $\vec{\chi}$ as

$$(3.1) \qquad\qquad \left[\frac{\delta}{\delta\vec{x}}\,A(\vec{x})\right](\vec{\chi}) = \lim_{\varepsilon \to 0} \frac{A(\vec{x} + \varepsilon\,\vec{\chi}) - A(\vec{x})}{\varepsilon},$$

and observe that, for $\vec{x}$ sufficiently smooth,

$$(3.2) \qquad\qquad \left[\frac{\delta}{\delta\vec{x}}\,A(\vec{x})\right](\vec{x}_t) = \frac{\mathrm{d}}{\mathrm{d}t}\,A(\vec{x}).$$

For later use, on noting (3.1), (2.1), and (2.4), we observe that

$$(3.3a) \qquad \left[\frac{\delta}{\delta\vec{x}}g^\beta(\vec{x})\right](\vec{\chi}) = \beta\, g^{\beta-1}(\vec{x})\, \vec{\chi}\,.\,\nabla g(\vec{x}) = \beta\, g^\beta(\vec{x})\, \vec{\chi}\,.\,\nabla \ln g(\vec{x}) \quad \forall\, \beta \in \mathbb{R}\,,$$

$$(3.3b) \quad \left[\frac{\delta}{\delta\vec{x}}\nabla \ln g(\vec{x})\right](\vec{\chi}) = (D^2 \ln g(\vec{x}))\,\vec{\chi}\,,$$

$$(3.3c) \qquad \left[\frac{\delta}{\delta\vec{x}}|\vec{x}_\rho|\right](\vec{\chi}) = \frac{\vec{x}_\rho\,.\,\vec{\chi}_\rho}{|\vec{x}_\rho|} = \vec{\tau}\,.\,\vec{\chi}_\rho = \vec{\tau}\,.\,\vec{\chi}_s\,|\vec{x}_\rho|\,,$$

$$(3.3d) \qquad \left[\frac{\delta}{\delta\vec{x}}|\vec{x}_\rho|_g\right](\vec{\chi}) = \left(\vec{\tau}\,.\,\vec{\chi}_s + \frac{1}{2}\vec{\chi}\,.\,\nabla \ln g(\vec{x})\right)|\vec{x}_\rho|_g\,,$$

$$\left[\frac{\delta}{\delta\vec{x}}\vec{\tau}\right](\vec{\chi}) = \left[\frac{\delta}{\delta\vec{x}}\frac{\vec{x}_\rho}{|\vec{x}_\rho|}\right](\vec{\chi}) = \frac{\vec{\chi}_\rho}{|\vec{x}_\rho|} - \frac{\vec{x}_\rho}{|\vec{x}_\rho|^2}\frac{\vec{x}_\rho\,.\,\vec{\chi}_\rho}{|\vec{x}_\rho|} = \vec{\chi}_s - \vec{\tau}\,(\vec{\chi}_s\,.\,\vec{\tau})$$

$$(3.3e) \qquad\qquad\qquad = (\vec{\chi}_s\,.\,\vec{\nu})\,\vec{\nu}\,,$$

$$(3.3f) \qquad \left[\frac{\delta}{\delta\vec{x}}\vec{\nu}\right](\vec{\chi}) = -\left[\frac{\delta}{\delta\vec{x}}\vec{\tau}^\perp\right](\vec{\chi}) = -(\vec{\chi}_s\,.\,\vec{\nu})\,\vec{\nu}^\perp = -(\vec{\chi}_s\,.\,\vec{\nu})\,\vec{\tau}\,,$$

$$(3.3g) \qquad \left[\frac{\delta}{\delta\vec{x}}\vec{\nu}\,|\vec{x}_\rho|\right](\vec{\chi}) = -\left[\frac{\delta}{\delta\vec{x}}\vec{x}_\rho^\perp\right](\vec{\chi}) = -\vec{\chi}_\rho^\perp = -\vec{\chi}_s^\perp\,|\vec{x}_\rho|\,,$$

where we always assume that $\vec{\chi}$ is sufficiently smooth so that all the quantities are defined almost everywhere. For example, $\vec{\chi} \in [L^\infty(I)]^2$ for (3.3a), (3.3b), and $\vec{\chi} \in [W^{1,\infty}(I)]^2$ for (3.3c)–(3.3g). In addition, on recalling (2.1), we have for all $\vec{a},\,\vec{b} \in \mathbb{R}^2$ that

$$(3.4a) \quad \vec{a}\,.\,\vec{b}^\perp = -\vec{a}^\perp\,.\,\vec{b}\,,$$

$$(3.4b) \quad \vec{a}^\perp = (\vec{a}^\perp\,.\,\vec{\tau})\,\vec{\tau} + (\vec{a}^\perp\,.\,\vec{\nu})\,\vec{\nu} = (\vec{a}^\perp\,.\,\vec{\nu}^\perp)\,\vec{\tau} - (\vec{a}^\perp\,.\,\vec{\tau}^\perp)\,\vec{\nu} = (\vec{a}\,.\,\vec{\nu})\,\vec{\tau} - (\vec{a}\,.\,\vec{\tau})\,\vec{\nu}\,.$$

Let $(\cdot,\cdot)$ denote the L^2-inner product on I. In the following we will discuss the L^2-gradient flow of the energy
(3.5)

$$W_{g,\lambda}(\vec{x}) = \left(\frac{1}{2}\varkappa_g^2 + \lambda, |\vec{x}_\rho|_g\right) = \left(\frac{1}{2}g^{-\frac{1}{2}}(\vec{x})\left(\varkappa - \frac{1}{2}\vec{\nu}\,.\,\nabla \ln g(\vec{x})\right)^2 + \lambda\, g^{\frac{1}{2}}(\vec{x}), |\vec{x}_\rho|\right),$$

treating either $\varkappa$ or $\varkappa_g$ formally as an independent variable that has to satisfy the side constraint (2.10b) or (2.11b), respectively. The necessary techniques are obtained from the formal calculus of PDE constrained optimization and were used by the authors for the first time in [6] in the present context. For the weak formulations of the L^2-gradient flow obtained in this way, it can be shown that they are consistent with the strong formulation (2.7); see the appendix. Moreover, we will formally establish that solutions to these weak formulations are indeed solutions to the L^2-gradient flow of (3.5). Mimicking these stability proofs on the discrete level, which in essence reduces to the seminal idea introduced in [14, Rem. 2.1], will yield the main theoretical results of this paper.

3.1. Based on $\varkappa$. We define the Lagrangian

$$\mathcal{L}(\vec{x}, \varkappa^\star, \vec{y}) = \frac{1}{2}\left(g^{-\frac{1}{2}}(\vec{x})\left(\varkappa^\star - \frac{1}{2}\vec{\nu}\,.\,\nabla \ln g(\vec{x})\right)^2 + 2\,\lambda\, g^{\frac{1}{2}}(\vec{x}), |\vec{x}_\rho|\right)$$

$$(3.6) \qquad\qquad - (\varkappa^\star\,\vec{\nu}, \vec{y}\,|\vec{x}_\rho|) - (\vec{x}_s, \vec{y}_s\,|\vec{x}_\rho|)\,,$$

which is obtained on combining (3.5) and the side constraint

$$(3.7) \qquad (\varkappa^\star \, \vec{\nu}, \vec{\eta} \, |\vec{x}_\rho|) + (\vec{x}_s, \vec{\eta}_s \, |\vec{x}_\rho|) = 0 \quad \forall \, \vec{\eta} \in [H^1(I)]^2 \, ;$$

recall (2.10b) and (2.1). Taking variations $\vec{\eta} \in [H^1(I)]^2$ in $\vec{y}$ and setting $[\frac{\delta}{\delta \vec{y}} \mathcal{L}](\vec{\eta}) = 0$ we obtain (3.7). Combining (3.7) and (2.10b) yields, on recalling (2.1), that $\varkappa^\star = \varkappa$, and we are going to use this identity from now. Taking variations $\chi \in L^2(I)$ in $\varkappa^\star$ and setting $[\frac{\delta}{\delta \varkappa^\star} \mathcal{L}](\chi) = 0$ leads to

$$(3.8) \qquad \left(g^{-\frac{1}{2}}(\vec{x}) \left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right) - \vec{y} . \vec{\nu}, \chi \, |\vec{x}_\rho| \right) = 0 \qquad \forall \, \chi \in L^2(I) \,,$$

which implies that
(3.9)

$$\vec{y} . \vec{\nu} = g^{-\frac{1}{2}}(\vec{x}) \left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right) \quad \Longleftrightarrow \quad \varkappa = g^{\frac{1}{2}}(\vec{x}) \, \vec{y} . \vec{\nu} + \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \,.$$

Taking variations $\vec{\chi} \in [H^1(I)]^2$ in $\vec{x}$, and then setting $(\mathcal{V}_g, g^{\frac{1}{2}}(\vec{x}) \, \vec{\chi} . \vec{\nu} \, |\vec{x}_\rho|_g) = (g^{\frac{3}{2}}(\vec{x}) \, \vec{x}_t . \vec{\nu}, \vec{\chi} . \vec{\nu} \, |\vec{x}_\rho|) = -[\frac{\delta}{\delta \vec{x}} \mathcal{L}](\vec{\chi})$, where we have noted (2.7) and (2.4), yields, on recalling (2.1), that

(3.10)

$$\left(g^{\frac{3}{2}}(\vec{x}) \, \vec{x}_t . \vec{\nu}, \vec{\chi} . \vec{\nu} \, |\vec{x}_\rho| \right) = -\frac{1}{2} \left(\left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right)^2, \left[\frac{\delta}{\delta \vec{x}} \, g^{-\frac{1}{2}}(\vec{x}) \, |\vec{x}_\rho| \right] (\vec{\chi}) \right)$$

$$+ \frac{1}{2} \left(g^{-\frac{1}{2}}(\vec{x}) \left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right), \left[\frac{\delta}{\delta \vec{x}} \vec{\nu} . \nabla \ln g(\vec{x}) \right] (\vec{\chi}) \, |\vec{x}_\rho| \right)$$

$$+ \left(\varkappa \vec{y}, \left[\frac{\delta}{\delta \vec{x}} \vec{\nu} \, |\vec{x}_\rho| \right] (\vec{\chi}) \right) + \left(\vec{y}_\rho, \left[\frac{\delta}{\delta \vec{x}} \vec{\tau} \right] (\vec{\chi}) \right) - \lambda \left(1, \left[\frac{\delta}{\delta \vec{x}} \, g^{\frac{1}{2}}(\vec{x}) \, |\vec{x}_\rho| \right] (\vec{\chi}) \right)$$

for all $\vec{\chi} \in [H^1(I)]^2$. On choosing $\vec{\chi} = \vec{x}_t$ in (3.10) we obtain, on noting (3.2), that

$$\left(g^{\frac{3}{2}}(\vec{x}) \, (\vec{x}_t . \vec{\nu})^2, |\vec{x}_\rho| \right) = -\frac{1}{2} \left(\left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right)^2, \left[g^{-\frac{1}{2}}(\vec{x}) \, |\vec{x}_\rho| \right]_t \right)$$

$$+ \frac{1}{2} \left(g^{-\frac{1}{2}}(\vec{x}) \left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right), [\vec{\nu} . \nabla \ln g(\vec{x})]_t \, |\vec{x}_\rho| \right)$$

$$(3.11) \qquad + \left(\varkappa \vec{y}, [\vec{\nu} \, |\vec{x}_\rho|]_t \right) + (\vec{y}_\rho, \vec{\tau}_t) - \lambda \left(1, \left[g^{\frac{1}{2}}(\vec{x}) \, |\vec{x}_\rho| \right]_t \right).$$

Differentiating (3.7) with respect to time, and then choosing $\vec{\eta} = \vec{y}$, yields, on recalling that $\varkappa^\star = \varkappa$, that

$$(3.12) \qquad (\varkappa_t, \vec{y} . \vec{\nu} \, |\vec{x}_\rho|) + (\varkappa \vec{y}, (\vec{\nu} \, |\vec{x}_\rho|)_t) + (\vec{\tau}_t, \vec{y}_\rho) = 0 \,.$$

Combining (3.11), (3.12), and (3.9) gives, on noting (3.5), that

$$\left(g^{\frac{3}{2}}(\vec{x}) \, (\vec{x}_t . \vec{\nu})^2, |\vec{x}_\rho| \right) = -\frac{1}{2} \left(\left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right)^2, \left[g^{-\frac{1}{2}}(\vec{x}) \, |\vec{x}_\rho| \right]_t \right)$$

$$+ \frac{1}{2} \left(g^{-\frac{1}{2}}(\vec{x}) \left(\varkappa - \frac{1}{2} \vec{\nu} . \nabla \ln g(\vec{x}) \right), [\vec{\nu} . \nabla \ln g(\vec{x})]_t \, |\vec{x}_\rho| \right)$$

$$
-\left(\varkappa_t, g^{-\frac{1}{2}}(\vec{x})\left(\varkappa - \frac{1}{2}\vec{\nu}\cdot\nabla\ln g(\vec{x})\right)|\vec{x}_\rho|\right) - \lambda\left(1,\left[g^{\frac{1}{2}}(\vec{x})\,|\vec{x}_\rho|\right]_t\right)
$$

$$
(3.13)\qquad = -\frac{\mathrm{d}}{\mathrm{d}t}\,W_{g,\lambda}(\vec{x})\,.
$$

The above yields the gradient flow property of the new weak formulation, on noting from (2.7) and (2.4) that the left-hand side of (3.13) can be equivalently written as $(\mathcal{V}_g^2,|\vec{x}_\rho|_g)$.

In order to derive a suitable weak formulation, we now return to (3.10). Combining (3.10), (3.3), and (3.4a) yields that

(3.14)

$$
\left(g^{\frac{3}{2}}(\vec{x})\,\vec{x}_t\cdot\vec{\nu},\vec{\chi}\cdot\vec{\nu}\,|\vec{x}_\rho|\right) = -\frac{1}{2}\left(g^{-\frac{1}{2}}(\vec{x})\left(\varkappa - \frac{1}{2}\vec{\nu}\cdot\nabla\ln g(\vec{x})\right)^2 + 2\,\lambda\,g^{\frac{1}{2}}(\vec{x}),\vec{\chi}_s\cdot\vec{\tau}\,|\vec{x}_\rho|\right)
$$

$$
+\frac{1}{4}\left(g^{-\frac{1}{2}}(\vec{x})\left(\varkappa - \frac{1}{2}\vec{\nu}\cdot\nabla\ln g(\vec{x})\right)^2 - 2\,\lambda\,g^{\frac{1}{2}}(\vec{x}),\vec{\chi}\cdot(\nabla\ln g(\vec{x}))\,|\vec{x}_\rho|\right)
$$

$$
+\frac{1}{2}\left(g^{-\frac{1}{2}}(\vec{x})\left(\varkappa - \frac{1}{2}\vec{\nu}\cdot\nabla\ln g(\vec{x})\right)\vec{\nu},(D^2\ln g(\vec{x}))\,\vec{\chi}\,|\vec{x}_\rho|\right)
$$

$$
-\frac{1}{2}\left(g^{-\frac{1}{2}}(\vec{x})\left(\varkappa - \frac{1}{2}\vec{\nu}\cdot\nabla\ln g(\vec{x})\right)[\ln g(\vec{x})]_s,\vec{\nu}\cdot\vec{\chi}_s\,|\vec{x}_\rho|\right) + (\vec{y}_s\cdot\vec{\nu},\vec{\chi}_s\cdot\vec{\nu}\,|\vec{x}_\rho|)
$$

$$
+\left(\varkappa\vec{y}^\perp,\vec{\chi}_s\,|\vec{x}_\rho|\right)\qquad \forall\,\vec{\chi}\in[H^1(I)]^2\,.
$$

Overall we obtain the following weak formulation.

($\mathcal{P}$): Let $\vec{x}(0)\in[H^1(I)]^2$. For $t\in(0,T]$ find $\vec{x}(t),\vec{y}(t)\in[H^1(I)]^2$ and $\varkappa\in L^2(I)$ such that (3.14), (3.8), and

$$
(3.15)\qquad\qquad (\varkappa\vec{\nu},\vec{\eta}\,|\vec{x}_\rho|) + (\vec{x}_s,\vec{\eta}_s\,|\vec{x}_\rho|) = 0\quad \forall\,\vec{\eta}\in[H^1(I)]^2
$$

hold. We remark that in the Euclidean case (3.8) collapses to $\varkappa = \vec{y}\cdot\vec{\nu}$, and so on eliminating $\varkappa$ from (3.14) and (3.15), and on noting (3.4b), we obtain that the formulation ($\mathcal{P}$) collapses to [6, (2.4)] for the Euclidean elastic flow.

3.2. Based on $\varkappa_g$. We recall that ($\mathcal{P}$) was inspired by the formulation ($\mathcal{U}$), which is based on $\varkappa$ acting as a variable. In order to derive an alternative formulation, we now start from ($\mathcal{W}$), where the curvature $\varkappa_g$ is a variable.

We begin by equivalently rewriting the side constraint (2.11b) as

(3.16)

$$
\left(g^{\frac{1}{2}}(\vec{x})\,\varkappa_g\,\vec{\nu},\vec{\eta}\,|\vec{x}_\rho|_g\right) + (\vec{x}_s,\vec{\eta}_s\,|\vec{x}_\rho|_g) + \frac{1}{2}\left(\nabla\ln g(\vec{x}),\vec{\eta}\,|\vec{x}_\rho|_g\right) = 0\quad \forall\,\vec{\eta}\in[H^1(I)]^2\,,
$$

where we have noted (2.1), (2.4), and $\frac{1}{2}\nabla\ln g(\vec{x}) = g^{-\frac{1}{2}}(\vec{x})\,\nabla g^{\frac{1}{2}}(\vec{x})$. Combining (2.5) and (3.16) leads to the Lagrangian

$$
\mathcal{L}_g(\vec{x},\varkappa_g^\star,\vec{y}_g) = \frac{1}{2}\left((\varkappa_g^\star)^2 + 2\,\lambda,|\vec{x}_\rho|_g\right) - \left(g^{\frac{1}{2}}(\vec{x})\,\varkappa_g^\star\,\vec{\nu},\vec{y}_g\,|\vec{x}_\rho|_g\right) - (\vec{x}_s,(\vec{y}_g)_s\,|\vec{x}_\rho|_g)
$$

$$
(3.17)\qquad\qquad -\frac{1}{2}\left(\nabla\ln g(\vec{x}),\vec{y}_g\,|\vec{x}_\rho|_g\right).
$$

Taking variations $\vec{\eta}\in[H^1(I)]^2$ in $\vec{y}_g$, and setting $[\frac{\delta}{\delta\vec{y}_g}\,\mathcal{L}_g](\vec{\eta}) = 0$ we obtain

(3.18)

$$
\left(g^{\frac{1}{2}}(\vec{x})\,\varkappa_g^\star\,\vec{\nu},\vec{\eta}\,|\vec{x}_\rho|_g\right) + (\vec{x}_s,\vec{\eta}_s\,|\vec{x}_\rho|_g) + \frac{1}{2}\left(\nabla\ln g(\vec{x}),\vec{\eta}\,|\vec{x}_\rho|_g\right) = 0\quad \forall\,\vec{\eta}\in[H^1(I)]^2\,.
$$

Combining (3.18) and (3.16) yields that $\varkappa_g^\star = \varkappa_g$, and we are going to use this identity from now. Taking variations $\chi \in L^2(I)$ in $\varkappa_g^\star$ and setting $[\frac{\delta}{\delta \varkappa_g^\star} \mathcal{L}_g](\chi) = 0$ yields that

$$(3.19) \qquad \left(\varkappa_g - g^{\frac{1}{2}}(\vec{x})\, \vec{y}_g \,.\, \vec{\nu}, \chi\, |\vec{x}_\rho|_g\right) = 0 \qquad \forall\, \chi \in L^2(I)\,,$$

which implies that

$$(3.20) \qquad \varkappa_g = g^{\frac{1}{2}}(\vec{x})\, \vec{y}_g \,.\, \vec{\nu}\,.$$

Taking variations $\vec{\chi} \in [H^1(I)]^2$ in $\vec{x}$, and then setting $(\mathcal{V}_g, g^{\frac{1}{2}} \vec{\chi} \,.\, \vec{\nu}\, |\vec{x}_\rho|_g) = (g(\vec{x})\, \vec{x}_t \,.\, \vec{\nu}, \vec{\chi} \,.\, \vec{\nu}\, |\vec{x}_\rho|_g) = -[\frac{\delta}{\delta \vec{x}} \mathcal{L}_g](\vec{\chi})$, where we have noted (2.7), yields, on recalling (2.1) and (2.4), that

$$(g(\vec{x})\, \vec{x}_t \,.\, \vec{\nu}, \vec{\chi} \,.\, \vec{\nu}\, |\vec{x}_\rho|_g) = -\frac{1}{2}\left(\varkappa_g^2 + 2\,\lambda, \left[\frac{\delta}{\delta \vec{x}} |\vec{x}_\rho|_g\right](\vec{\chi})\right)$$

$$+ \left(\varkappa_g\, \vec{y}_g, \left[\frac{\delta}{\delta \vec{x}} g^{\frac{1}{2}}(\vec{x})\, \vec{\nu}\, |\vec{x}_\rho|_g\right](\vec{\chi})\right) + \left((\vec{y}_g)_\rho, \left[\frac{\delta}{\delta \vec{x}} g^{\frac{1}{2}}(\vec{x})\, \vec{\tau}\right](\vec{\chi})\right)$$

$$(3.21) \qquad + \frac{1}{2}\left(\vec{y}_g, \left[\frac{\delta}{\delta \vec{x}} (\nabla \ln g(\vec{x}))\, |\vec{x}_\rho|_g\right](\vec{\chi})\right) \qquad \forall\, \vec{\chi} \in [H^1(I)]^2\,.$$

Choosing $\vec{\chi} = \vec{x}_t$ in (3.21), and noting (3.2), yields that

$$(g(\vec{x})\, (\vec{x}_t \,.\, \vec{\nu})^2, |\vec{x}_\rho|_g) = -\frac{1}{2}\left((\varkappa_g)^2 + 2\,\lambda, (|\vec{x}_\rho|_g)_t\right) + \left(\varkappa_g\, \vec{y}_g, (g^{\frac{1}{2}}(\vec{x})\, \vec{\nu}\, |\vec{x}_\rho|_g)_t\right)$$

$$(3.22) \qquad + \left((\vec{y}_g)_\rho, (g^{\frac{1}{2}}(\vec{x})\, \vec{\tau})_t\right) + \frac{1}{2}\left(\vec{y}_g, ((\nabla \ln g(\vec{x}))\, |\vec{x}_\rho|_g)_t\right)\,.$$

On differentiating (3.16) with respect to time, and then choosing $\vec{\eta} = \vec{y}_g$, we obtain, on recalling (2.1) and (2.4), that

$$\left((\varkappa_g)_t\, \vec{y}_g, g^{\frac{1}{2}}(\vec{x})\, \vec{\nu}\, |\vec{x}_\rho|_g\right) + \left(\varkappa_g\, \vec{y}_g, (g^{\frac{1}{2}}(\vec{x})\, \vec{\nu}\, |\vec{x}_\rho|_g)_t\right)$$

$$(3.23) \qquad + \left((\vec{y}_g)_\rho, (g^{\frac{1}{2}}(\vec{x})\, \vec{\tau})_t\right) + \frac{1}{2}\left(\vec{y}_g, ((\nabla \ln g(\vec{x}))\, |\vec{x}_\rho|_g)_t\right) = 0\,.$$

Choosing $\chi = (\varkappa_g)_t$ in (3.19), and combining with (3.22) and (3.23), yields, on recalling (3.5), that

$$(3.24) \qquad \left(g(\vec{x})\, (\vec{x}_t \,.\, \vec{\nu})^2, |\vec{x}_\rho|_g\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\, W_{g,\lambda}(\vec{x})\,,$$

which once again reveals the gradient flow structure, on noting from (2.7) that the left-hand side of (3.24) can be equivalently written as $(\mathcal{V}_g^2, |\vec{x}_\rho|_g)$.

In order to derive a suitable weak formulation, we now return to (3.21). Substituting (3.3) into (3.21) yields, on noting (2.4), that

(3.25)

$$(g(\vec{x})\, \vec{x}_t \,.\, \vec{\nu}, \vec{\chi} \,.\, \vec{\nu}\, |\vec{x}_\rho|_g)$$

$$= -\frac{1}{2}\left(\varkappa_g^2 + 2\,\lambda - \vec{y}_g \,.\, \nabla \ln g(\vec{x}), \left[\frac{\delta}{\delta \vec{x}} |\vec{x}_\rho|_g\right](\vec{\chi})\right)$$

$$+ \frac{1}{2}\left(\vec{y}_g, \left[\frac{\delta}{\delta \vec{x}} (\nabla \ln g(\vec{x}))\right](\vec{\chi})\, |\vec{x}_\rho|_g\right) + \left(\varkappa_g\, \vec{y}_g \,.\, \vec{\nu}, \left[\frac{\delta}{\delta \vec{x}} g(\vec{x})\right](\vec{\chi})\, |\vec{x}_\rho|\right)$$

$$+ \left(g(\vec{x})\, \varkappa_g\, \vec{y}_g, \left[\frac{\delta}{\delta \vec{x}} \vec{\nu}\, |\vec{x}_\rho|\right](\vec{\chi})\right) + \left((\vec{y}_g)_\rho \,.\, \vec{\tau}, \left[\frac{\delta}{\delta \vec{x}} g^{\frac{1}{2}}(\vec{x})\right](\vec{\chi})\right)$$

$$+ \left(g^{\frac{1}{2}}(\vec{x})\, (\vec{y}_g)_\rho, \left[\frac{\delta}{\delta \vec{x}} \vec{\tau}\right](\vec{\chi})\right)$$

$$= -\frac{1}{2}\left(\varkappa_g^2 + 2\lambda - \vec{y}_g \cdot \nabla \ln g(\vec{x}), \left[\vec{\tau} \cdot \vec{\chi}_s + \frac{1}{2}\vec{\chi} \cdot \nabla \ln g(\vec{x})\right] |\vec{x}_\rho|_g\right)$$

$$+ \frac{1}{2}\left((D^2 \ln g(\vec{x}))\vec{y}_g, \vec{\chi} |\vec{x}_\rho|_g\right) + \left(g^{\frac{1}{2}}(\vec{x})\varkappa_g \vec{y}_g \cdot \vec{\nu} + \frac{1}{2}(\vec{y}_g)_s \cdot \vec{\tau}, (\nabla \ln g(\vec{x})) \cdot \vec{\chi} |\vec{x}_\rho|_g\right)$$

$$- \left(g^{\frac{1}{2}}(\vec{x})\varkappa_g \vec{y}_g, \vec{\chi}_s^\perp |\vec{x}_\rho|_g\right) + ((\vec{y}_g)_s \cdot \vec{\nu}, \vec{\chi}_s \cdot \vec{\nu} |\vec{x}_\rho|_g) \qquad \forall \vec{\chi} \in [H^1(I)]^2.$$

Then, on recalling (3.4a), we obtain the following weak formulation.

($\mathcal{Q}$): Let $\vec{x}(0) \in [H^1(I)]^2$. For $t \in (0,T]$ find $\vec{x}(t), \vec{y}_g(t) \in [H^1(I)]^2$ and $\varkappa_g(t) \in L^2(I)$ such that

(3.26)

$$(g(\vec{x})\vec{x}_t \cdot \vec{\nu}, \vec{\chi} \cdot \vec{\nu} |\vec{x}_\rho|_g)$$

$$= -\frac{1}{2}\left(\varkappa_g^2 + 2\lambda - \vec{y}_g \cdot \nabla \ln g(\vec{x}), \left[\vec{\chi}_s \cdot \vec{\tau} + \frac{1}{2}\vec{\chi} \cdot \nabla \ln g(\vec{x})\right] |\vec{x}_\rho|_g\right)$$

$$+ \frac{1}{2}\left((D^2 \ln g(\vec{x}))\vec{y}_g, \vec{\chi} |\vec{x}_\rho|_g\right) + \left(g^{\frac{1}{2}}(\vec{x})\varkappa_g \vec{y}_g \cdot \vec{\nu} + \frac{1}{2}(\vec{y}_g)_s \cdot \vec{\tau}, \vec{\chi} \cdot (\nabla \ln g(\vec{x})) |\vec{x}_\rho|_g\right)$$

$$+ \left(g^{\frac{1}{2}}\varkappa_g, \vec{\chi}_s \cdot \vec{y}_g^\perp |\vec{x}_\rho|_g\right) + ((\vec{y}_g)_s \cdot \vec{\nu}, \vec{\chi}_s \cdot \vec{\nu} |\vec{x}_\rho|_g) \qquad \forall \vec{\chi} \in [H^1(I)]^2,$$

(3.19), and (3.16) hold. We remark that in the Euclidean case (3.19) collapses to $\varkappa_g = \vec{y}_g \cdot \vec{\nu}$, and so on eliminating $\varkappa_g$ from (3.26) and (3.16), and on noting (3.4b), we obtain that the formulation ($\mathcal{Q}$) collapses to [6, (2.4)] for the Euclidean elastic flow.

4. Semidiscrete finite element approximations. Let $[0,1] = \cup_{j=1}^J I_j$, $J \geq 3$, be a decomposition of $[0,1]$ into intervals given by the nodes q_j, $I_j = [q_{j-1}, q_j]$. For simplicity, and without loss of generality, we assume that the subintervals form an equipartitioning of $[0,1]$, i.e., that

(4.1) $$q_j = j\,h \quad \text{with} \quad h = J^{-1}, \qquad j = 0,\ldots,J.$$

Clearly, as $I = \mathbb{R}/\mathbb{Z}$ we identify $0 = q_0 = q_J = 1$.

The necessary finite element spaces are defined as follows:

$$V^h = \{\chi \in C(I) : \chi|_{I_j} \text{ is linear } \forall j = 1 \to J\} \quad \text{and} \quad \underline{V}^h = [V^h]^2.$$

Let $\{\chi_j\}_{j=1}^J$ denote the standard basis of V^h, and let $\pi^h : C(I) \to V^h$ be the standard interpolation operator at the nodes $\{q_j\}_{j=1}^J$. We require also the local interpolation operator $\pi_j^h \equiv \pi^h|_{I_j}$, $j = 1,\ldots,J$.

We define the mass lumped L^2–inner product $(u,v)^h$, for two piecewise continuous functions, with possible jumps at the nodes $\{q_j\}_{j=1}^J$, via

(4.2) $$(u,v)^h = \sum_{j=1}^J \int_{I_j} \pi_j^h[u\,v]\,\mathrm{d}\rho = \frac{1}{2}h\sum_{j=1}^J \left[(u\,v)(q_j^-) + (u\,v)(q_{j-1}^+)\right],$$

where we define $u(q_j^\pm) = \lim_{\delta \searrow 0} u(q_j \pm \delta)$. The interpolation operators π^h, π_j^h and the definition (4.2) naturally extend to vector valued functions.

Let $(\vec{X}^h(t))_{t\in[0,T]}$, with $\vec{X}^h(t) \in \underline{V}^h$, be an approximation to $(\vec{x}(t))_{t\in[0,T]}$. Then, similarly to (2.1), we set

$$(4.3) \qquad \vec{\tau}^h = \vec{X}^h_s = \frac{\vec{X}^h_\rho}{\left|\vec{X}^h_\rho\right|} \qquad \text{and} \qquad \vec{\nu}^h = -(\vec{\tau}^h)^\perp .$$

For later use, we let $\vec{\omega}^h \in \underline{V}^h$ be the mass-lumped L^2-projection of $\vec{\nu}^h$ onto $\underline{V}^h$, i.e.,

$$(4.4) \qquad \left(\vec{\omega}^h, \vec{\varphi}\left|\vec{X}^h_\rho\right|\right)^h = \left(\vec{\nu}^h, \vec{\varphi}\left|\vec{X}^h_\rho\right|\right) = \left(\vec{\nu}^h, \vec{\varphi}\left|\vec{X}^h_\rho\right|\right)^h \qquad \forall\, \vec{\varphi} \in \underline{V}^h .$$

On noting (3.1), (4.3), and (2.4), we have the following discrete analogues of (3.3) for all $\vec{\chi} \in \underline{V}^h$ and for $j = 1, \ldots, J$:

$$\left[\frac{\delta}{\delta \vec{X}^h}\, g^\beta\left(\vec{X}^h\right)\right](\vec{\chi}) = \beta\, g^{\beta-1}\left(\vec{X}^h\right)\, \vec{\chi}\cdot\nabla g\left(\vec{X}^h\right)$$

$$(4.5\text{a}) \qquad\qquad = \beta\, g^\beta\left(\vec{X}^h\right)\, \vec{\chi}\cdot\nabla \ln g\left(\vec{X}^h\right) \quad \text{on } I_j \quad \forall\, \beta \in \mathbb{R},$$

$$(4.5\text{b})$$
$$\left[\frac{\delta}{\delta \vec{X}^h}\, \nabla \ln g\left(\vec{X}^h\right)\right](\vec{\chi}) = \left(D^2 \ln g\left(\vec{X}^h\right)\right)\vec{\chi} \quad \text{on } I_j,$$

$$(4.5\text{c}) \qquad \left[\frac{\delta}{\delta \vec{X}^h}\left|\vec{X}^h_\rho\right|\right](\vec{\chi}) = \frac{\vec{X}^h_\rho \cdot \vec{\chi}_\rho}{\left|\vec{X}^h_\rho\right|} = \vec{\tau}^h\cdot\vec{\chi}_\rho = \vec{\tau}^h\cdot\vec{\chi}_s \left|\vec{X}^h_\rho\right| \quad \text{on } I_j,$$

$$(4.5\text{d}) \qquad \left[\frac{\delta}{\delta \vec{X}^h}\left|\vec{X}^h_\rho\right|_g\right](\vec{\chi}) = \left(\vec{\tau}^h\cdot\vec{\chi}_s + \frac{1}{2}\,\vec{\chi}\cdot\nabla \ln g\left(\vec{X}^h\right)\right)\left|\vec{X}^h_\rho\right|_g \quad \text{on } I_j,$$

$$\left[\frac{\delta}{\delta \vec{X}^h}\, \vec{\tau}^h\right](\vec{\chi}) = \left[\frac{\delta}{\delta \vec{X}^h}\, \frac{\vec{X}^h_\rho}{\left|\vec{X}^h_\rho\right|}\right](\vec{\chi}) = \frac{\vec{\chi}_\rho}{\left|\vec{X}^h_\rho\right|} - \frac{\vec{X}^h_\rho}{\left|\vec{X}^h_\rho\right|^2}\,\frac{\vec{X}^h_\rho\cdot\vec{\chi}_\rho}{\left|\vec{X}^h_\rho\right|}$$

$$(4.5\text{e}) \qquad\qquad = \vec{\chi}_s - \vec{\tau}^h\left(\vec{\chi}_s\cdot\vec{\tau}^h\right) = \left(\vec{\chi}_s\cdot\vec{\nu}^h\right)\vec{\nu}^h \quad \text{on } I_j,$$

$$(4.5\text{f}) \qquad \left[\frac{\delta}{\delta \vec{X}^h}\, \vec{\nu}^h\right](\vec{\chi}) = -\left[\frac{\delta}{\delta \vec{X}^h}\,(\vec{\tau}^h)^\perp\right](\vec{\chi}) = -\left(\vec{\chi}_s\cdot\vec{\nu}^h\right)\vec{\tau}^h \quad \text{on } I_j,$$

$$(4.5\text{g}) \qquad \left[\frac{\delta}{\delta \vec{X}^h}\, \vec{\nu}^h\left|\vec{X}^h_\rho\right|\right](\vec{\chi}) = -\left[\frac{\delta}{\delta \vec{X}^h}\left(\vec{X}^h_\rho\right)^\perp\right](\vec{\chi}) = -\vec{\chi}_\rho^\perp = -\vec{\chi}_s^\perp\left|\vec{X}^h_\rho\right| \quad \text{on } I_j.$$

4.1. Based on κ^h. In the following we will discuss the L^2-gradient flow of the energy
$$(4.6)$$
$$W^h_{g,\lambda}\left(\vec{X}^h, \kappa^h\right) = \frac{1}{2}\left(g^{-\frac{1}{2}}\left(\vec{X}^h\right)\left(\kappa^h - \frac{1}{2}\,\frac{\vec{\omega}^h}{|\vec{\omega}^h|}\cdot\nabla \ln g\left(\vec{X}^h\right)\right)^2 + 2\lambda\, g^{\frac{1}{2}}\left(\vec{X}^h\right), \left|\vec{X}^h_\rho\right|\right)^h,$$

subject to the side constraint

$$(4.7) \qquad \left(\kappa^h\,\vec{\nu}^h, \vec{\eta}\left|\vec{X}^h_\rho\right|\right)^h + \left(\vec{X}^h_s, \vec{\eta}_s\left|\vec{X}^h_\rho\right|\right) = 0 \quad \forall\, \vec{\eta} \in \underline{V}^h .$$

On recalling (4.4), we see that (4.6) and (4.7) are discrete analogues of (3.5) and (2.10b), respectively. We define the Lagrangian

$$\mathcal{L}^h\left(\vec{X}^h,\kappa^h,\vec{Y}^h\right) = \frac{1}{2}\left(g^{-\frac{1}{2}}\left(\vec{X}^h\right)\left(\kappa^h - \frac{1}{2}\frac{\vec{\omega}^h}{|\vec{\omega}^h|}.\nabla\ln g\left(\vec{X}^h\right)\right)^2 + 2\lambda g^{\frac{1}{2}}\left(\vec{X}^h\right),\left|\vec{X}^h_\rho\right|\right)^h$$

$$(4.8) \qquad - \left(\kappa^h\,\vec{\nu}^h,\vec{Y}^h\left|\vec{X}^h_\rho\right|\right)^h - \left(\vec{X}^h_s,\vec{Y}^h_s\left|\vec{X}^h_\rho\right|\right),$$

which is the corresponding discrete analogue of (3.6).

In addition to (4.5), we will require $[\frac{\delta}{\delta\vec{X}^h}\pi^h[\frac{\vec{\omega}^h}{|\vec{\omega}^h|}]](\vec{\chi})$ in order to compute variations of (4.8). We establish this along the lines of [6, (3.2)–(3.7)]. To this end, we introduce the following operators. On recalling (4.2) and (4.3), let $D_s, \widehat{D}_s : V^h \to V^h$ be such that for any $t \in [0,T]$

$$(D_s\eta)(q_j) = \frac{|\vec{X}^h(q_j,t) - \vec{X}^h(q_{j-1},t)|\,\eta_s(q_j^-) + |\vec{X}^h(q_{j+1},t) - \vec{X}^h(q_j,t)|\,\eta_s(q_j^+)}{|\vec{X}^h(q_j,t) - \vec{X}^h(q_{j-1},t)| + |\vec{X}^h(q_{j+1},t) - \vec{X}^h(q_j,t)|}$$

$$(4.9\text{a}) \qquad = \frac{\eta(q_{j+1}) - \eta(q_{j-1})}{|\vec{X}^h(q_j,t) - \vec{X}^h(q_{j-1},t)| + |\vec{X}^h(q_{j+1},t) - \vec{X}^h(q_j,t)|}, \qquad j = 1,\dots,J,$$

(4.9b)

$$(\widehat{D}_s\eta)(q_j) = \frac{(D_s\eta)(q_j)}{|(D_s\vec{X}^h(t))(q_j)|} = \frac{\eta(q_{j+1}) - \eta(q_{j-1})}{|\vec{X}^h(q_{j+1},t) - \vec{X}^h(q_{j-1},t)|}, \qquad j = 1,\dots,J,$$

where $q_{J+1} = q_1$. Here, we make the following natural assumption:

$$(\mathcal{C}^h) \qquad \vec{X}^h(q_j,t) \neq \vec{X}^h(q_{j+1},t) \qquad \text{and} \qquad \vec{X}^h(q_{j-1},t) \neq \vec{X}^h(q_{j+1},t),$$
$$j = 1,\dots,J, \qquad \forall\, t \in [0,T].$$

Hence (4.9) is well-defined. As usual, $D_s, \widehat{D}_s : \underline{V}^h \to \underline{V}^h$ are defined componentwise.

It follows from (4.4), (4.3), and (4.9a) that, for all $\vec{\varphi} \in \underline{V}^h$,

(4.10)

$$\left(\vec{\omega}^h,\vec{\varphi}\left|\vec{X}^h_\rho\right|\right)^h = -\left((\vec{\tau}^h)^\perp,\vec{\varphi}\left|\vec{X}^h_\rho\right|\right)^h = -\left((\vec{X}^h_\rho)^\perp,\vec{\varphi}\right)^h = -\left((D_s\vec{X}^h)^\perp,\vec{\varphi}\left|\vec{X}^h_\rho\right|\right)^h.$$

Therefore, we have from (4.10), $(\mathcal{C}^h)$, and (4.9b) that

$$(4.11) \qquad \vec{\omega}^h = -(D_s\vec{X}^h)^\perp \qquad \text{and} \qquad \pi^h\left[\frac{\vec{\omega}^h}{|\vec{\omega}^h|}\right] = -(\widehat{D}_s\vec{X}^h)^\perp.$$

Then it is a simple matter to compute, for any $\vec{\chi} \in \underline{V}^h$,

$$\left[\frac{\delta}{\delta\vec{X}^h}\widehat{D}_s\vec{X}^h\right](\vec{\chi}) = \vec{\pi}^h\left[\left[\underline{\underline{\mathrm{Id}}} - (\widehat{D}_s\vec{X}^h)\otimes(\widehat{D}_s\vec{X}^h)\right]\left(\widehat{D}_s\vec{\chi}\right)\right]$$

$$= \vec{\pi}^h\left[|\vec{\omega}^h|^{-2}\left((\widehat{D}_s\vec{\chi}).\vec{\omega}^h\right)\vec{\omega}^h\right],$$

so that
(4.12)

$$\left[\frac{\delta}{\delta\vec{X}^h}\vec{\pi}^h\frac{\vec{\omega}^h}{|\vec{\omega}^h|}\right](\vec{\chi}) = -\left(\left[\frac{\delta}{\delta\vec{X}^h}\widehat{D}_s\vec{X}^h\right](\vec{\chi})\right)^\perp = -\vec{\pi}^h\left[|\vec{\omega}^h|^{-2}\left((\widehat{D}_s\vec{\chi}).\vec{\omega}^h\right)(\vec{\omega}^h)^\perp\right].$$

Similarly to (4.10), we have for any $\vec{\eta} \in V^h$ that

$$(4.13) \qquad \left(\vec{\eta}_s,\vec{\varphi}\left|\vec{X}^h_\rho\right|\right)^h = \left(D_s\vec{\eta},\vec{\varphi}\left|\vec{X}^h_\rho\right|\right)^h \qquad \forall\,\vec{\varphi} \in \underline{V}^h.$$

Hence, it follows from (4.13), (4.9b), and (4.11) that

$$(4.14) \qquad \left(|\vec{\omega}^h|^{-1} \, \vec{\eta}_s, \vec{\varphi} \left| \vec{X}_\rho^h \right| \right)^h = \left(\widehat{D}_s \, \vec{\eta}, \vec{\varphi} \left| \vec{X}_\rho^h \right| \right)^h \qquad \forall \, \vec{\eta}, \, \vec{\varphi} \in \underline{V}^h \, .$$

Therefore, combining (4.12) and (4.14) yields for any $\vec{\varphi}, \vec{\chi} \in \underline{V}^h$ that

(4.15)

$$\left(\vec{\varphi}, \left[\frac{\delta}{\delta \vec{X}^h} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \right] (\vec{\chi}) \left| \vec{X}_\rho^h \right| \right)^h = - \left(|\vec{\omega}^h|^{-2} \, \vec{\varphi}, \left((\widehat{D}_s \, \vec{\chi}) \cdot \vec{\omega}^h \right) (\vec{\omega}^h)^\perp \left| \vec{X}_\rho^h \right| \right)^h$$

$$= - \left(|\vec{\omega}^h|^{-3} \, (\vec{\varphi} \cdot (\vec{\omega}^h)^\perp) \, \vec{\omega}^h, \vec{\chi}_s \left| \vec{X}_\rho^h \right| \right)^h = - \left(|\vec{\omega}^h|^{-1} \, \vec{\varphi} \cdot \left(\frac{\vec{\omega}^h}{|\vec{\omega}^h|} \right)^\perp, \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \vec{\chi}_s \left| \vec{X}_\rho^h \right| \right)^h .$$

Taking variations $\chi \in V^h$ in κ^h and setting $[\frac{\delta}{\delta \kappa^h} \, \mathcal{L}^h](\chi) = 0$ leads to

(4.16)

$$\left(g^{-\frac{1}{2}} \left(\vec{X}^h \right) \left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \nabla \ln g \left(\vec{X}^h \right) \right) - \vec{Y}^h \cdot \vec{\nu}^h, \chi \left| \vec{X}_\rho^h \right| \right)^h = 0 \quad \forall \, \chi \in V^h \, ,$$

which, on recalling (4.4), implies the discrete analogue of (3.9)

$$\pi^h \left[\vec{Y}^h \cdot \vec{\omega}^h \right] = \pi^h \left[g^{-\frac{1}{2}} \left(\vec{X}^h \right) \left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \nabla \ln g \left(\vec{X}^h \right) \right) \right]$$

$$(4.17) \qquad \Longleftrightarrow \qquad \kappa^h = \pi^h \left[g^{\frac{1}{2}} \left(\vec{X}^h \right) \vec{Y}^h \cdot \vec{\omega}^h + \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \nabla \ln g \left(\vec{X}^h \right) \right] .$$

Taking variations $\vec{\eta} \in \underline{V}^h$ in $\vec{Y}^h$, and setting $[\frac{\delta}{\delta \vec{Y}^h} \, \mathcal{L}^h](\vec{\eta}) = 0$, we obtain (4.7). Setting $(g^{\frac{3}{2}}(\vec{X}^h) \, \vec{X}_t^h \cdot \vec{\omega}^h, \vec{\chi} \cdot \vec{\omega}^h \, |\vec{X}_\rho^h|)^h = -[\frac{\delta}{\delta \vec{X}^h} \, \mathcal{L}^h](\vec{\chi})$, for variations $\vec{\chi} \in \underline{V}^h$ in $\vec{X}^h$ yields, as a discrete analogue to (3.10),

(4.18)

$$\left(g^{\frac{3}{2}} \left(\vec{X}^h \right) \, \vec{X}_t^h \cdot \vec{\omega}^h, \vec{\chi} \cdot \vec{\omega}^h \left| \vec{X}_\rho^h \right| \right)^h$$

$$= -\frac{1}{2} \left(\left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \nabla \ln g \left(\vec{X}^h \right) \right)^2, \left[\frac{\delta}{\delta \vec{X}^h} g^{-\frac{1}{2}} \left(\vec{X}^h \right) \left| \vec{X}_\rho^h \right| \right] (\vec{\chi}) \right)^h$$

$$- \lambda \left(1, \left[\frac{\delta}{\delta \vec{X}^h} g^{\frac{1}{2}} \left(\vec{X}^h \right) \left| \vec{X}_\rho^h \right| \right] (\vec{\chi}) \right)^h$$

$$+ \frac{1}{2} \left(g^{-\frac{1}{2}} \left(\vec{X}^h \right) \left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \nabla \ln g \left(\vec{X}^h \right) \right), \left[\frac{\delta}{\delta \vec{X}^h} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \nabla \ln g \left(\vec{X}^h \right) \right] (\vec{\chi}) \left| \vec{X}_\rho^h \right| \right)^h$$

$$+ \left(\kappa^h \, \vec{Y}^h, \left[\frac{\delta}{\delta \vec{X}^h} \vec{\nu}^h \left| \vec{X}_\rho^h \right| \right] (\vec{\chi}) \right)^h + \left(\vec{Y}_\rho^h, \left[\frac{\delta}{\delta \vec{X}^h} \vec{\tau}^h \right] (\vec{\chi}) \right) .$$

Choosing $\vec{\chi} = \vec{X}_t^h$ in (4.18), where we observe a discrete variant of (3.2), yields that

(4.19)

$$\left(g^{\frac{3}{2}} \left(\vec{X}^h \right) (\vec{X}_t^h \cdot \vec{\omega}^h)^2, \left| \vec{X}_\rho^h \right| \right)^h$$

$$= -\frac{1}{2} \left(\left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} \cdot \nabla \ln g \left(\vec{X}^h \right) \right)^2, \left[g^{-\frac{1}{2}} \left(\vec{X}^h \right) \left| \vec{X}_\rho^h \right| \right]_t \right)^h$$

$$- \lambda \left(1, \left[g^{\frac{1}{2}} \left(\vec{X}^h\right) \left|\vec{X}^h_\rho\right|\right]_t\right)^h$$

$$+ \frac{1}{2} \left(g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} . \nabla \ln g \left(\vec{X}^h\right)\right), \left[\frac{\vec{\omega}^h}{|\vec{\omega}^h|} . \nabla \ln g \left(\vec{X}^h\right)\right]_t \left|\vec{X}^h_\rho\right|\right)^h$$

$$+ \left(\kappa^h \vec{Y}^h, \left[\vec{\nu}^h \left|\vec{X}^h_\rho\right|\right]_t\right)^h + \left(\vec{Y}^h_\rho, \vec{\tau}^h_t\right).$$

Differentiating (4.7) with respect to time and then choosing $\vec{\eta} = \vec{Y}^h$ yields that

$$(4.20) \qquad \left(\kappa^h_t, \vec{Y}^h . \vec{\nu}^h \left|\vec{X}^h_\rho\right|\right)^h + \left(\kappa^h \vec{Y}^h, (\vec{\nu}^h \left|\vec{X}^h_\rho\right|)_t\right)^h + \left(\vec{\tau}^h_t, \vec{Y}^h_\rho\right) = 0.$$

Combining (4.19), (4.20), and (4.16) with $\chi = \kappa^h_t$ gives, on noting (4.6), that

(4.21)

$$\left(g^{\frac{3}{2}} \left(\vec{X}^h\right) (\vec{X}^h_t . \vec{\omega})^2, \left|\vec{X}^h_\rho\right|\right)^h = -\frac{1}{2} \left(\left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}}{|\vec{\omega}|} . \nabla \ln g \left(\vec{X}^h\right)\right)^2, \left[g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left|\vec{X}^h_\rho\right|\right]_t\right)^h$$

$$+ \frac{1}{2} \left(g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}}{|\vec{\omega}|} . \nabla \ln g \left(\vec{X}^h\right)\right), \left[\frac{\vec{\omega}}{|\vec{\omega}|} . \nabla \ln g \left(\vec{X}^h\right)\right]_t \left|\vec{X}^h_\rho\right|\right)^h$$

$$- \left(\kappa^h_t, g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left(\kappa^h - \frac{1}{2} \frac{\vec{\omega}}{|\vec{\omega}|} . \nabla \ln g \left(\vec{X}^h\right)\right) \left|\vec{X}^h_\rho\right|\right)^h - \lambda \left(1, \left[g^{\frac{1}{2}} \left(\vec{X}^h\right) \left|\vec{X}^h_\rho\right|\right]_t\right)^h$$

$$= -\frac{d}{dt} W^h_{g,\lambda}(\vec{X}^h, \vec{\kappa}^h).$$

In order to derive a suitable approximation of $(\mathcal{P})$, we now return to (4.18). Combining (4.18), (4.5), and (4.15), on noting (3.4a), yields

(4.22)

$$\left(g^{\frac{3}{2}} \left(\vec{X}^h\right) \vec{X}^h_t . \vec{\omega}^h, \vec{\chi} . \vec{\omega}^h \left|\vec{X}^h_\rho\right|\right)^h = \left(\vec{Y}^h_s . \vec{\nu}^h, \vec{\chi}_s . \vec{\nu}^h \left|\vec{X}^h_\rho\right|\right)$$

$$- \frac{1}{2} \left(g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left[\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} . \nabla \ln g \left(\vec{X}^h\right)\right]^2 + 2 \lambda g^{\frac{1}{2}} \left(\vec{X}^h\right), \vec{\chi}_s . \vec{\tau}^h \left|\vec{X}^h_\rho\right|\right)^h$$

$$+ \frac{1}{4} \left(g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left[\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} . \nabla \ln g \left(\vec{X}^h\right)\right]^2 - 2 \lambda g^{\frac{1}{2}} \left(\vec{X}^h\right), \vec{\chi} . (\nabla \ln g \left(\vec{X}^h\right)) \left|\vec{X}^h_\rho\right|\right)^h$$

$$+ \frac{1}{2} \left(g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left[\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} . \nabla \ln g \left(\vec{X}^h\right)\right] \frac{\vec{\omega}^h}{|\vec{\omega}^h|}, (D^2 \ln g \left(\vec{X}^h\right)) \vec{\chi} \left|\vec{X}^h_\rho\right|\right)^h$$

$$- \frac{1}{2} \left(g^{-\frac{1}{2}} \left(\vec{X}^h\right) \left[\kappa^h - \frac{1}{2} \frac{\vec{\omega}^h}{|\vec{\omega}^h|} . \nabla \ln g \left(\vec{X}^h\right)\right] \frac{\nabla \ln g \left(\vec{X}^h\right)}{|\vec{\omega}^h|} . \left(\frac{\vec{\omega}^h}{|\vec{\omega}^h|}\right)^\perp, \frac{\vec{\omega}^h}{|\vec{\omega}^h|} . \vec{\chi}_s \left|\vec{X}^h_\rho\right|\right)^h$$

$$+ \left(\kappa^h (\vec{Y}^h)^\perp, \vec{\chi}_s \left|\vec{X}^h_\rho\right|\right)^h \quad \forall \vec{\chi} \in \underline{V}^h,$$

which is the discrete analogue of (3.14), on noting that $\vec{\nu}^\perp = \vec{\tau}$.

Hence we obtain the following approximation of $(\mathcal{P})$.

$(\mathcal{P}_h)^h$: Let $\vec{X}^h(0) \in \underline{V}^h$. For $t \in (0, T]$ find $(\vec{X}^h(t), \kappa^h(t), \vec{Y}^h(t)) \in \underline{V}^h \times V^h \times \underline{V}^h$ such that (4.22), (4.16), and (4.7) hold.

We note that in the Euclidean case it follows from (4.17) that $\kappa^h = \pi^h [\vec{Y}^h . \vec{\omega}^h]$, and so on eliminating κ^h, and on noting (4.4), the approximation $(\mathcal{P}_h)^h$ collapses to the isotropic closed curve version of (3.36), with $\beta = 0$, in [6].

THEOREM 4.1. *Let the assumption $(\mathcal{C}^h)$ be satisfied and let $(\vec{X}^h(t), \vec{Y}^h(t)) \in \underline{V}^h \times \underline{V}^h$, for $t \in (0, T]$, be a solution to $(\mathcal{P}_h)^h$. Then the solution satisfies the stability bound (4.21).*

Proof. The proof is given in (4.19), (4.20), and (4.21). □

Remark 4.2. We note why we choose $\frac{\vec{\omega}^h}{|\vec{\omega}^h|}$ in (4.6) as opposed to $\vec{\nu}^h$ or $\vec{\omega}^h$. In the case of $\vec{\nu}^h$, (4.17) and (4.18) still hold with $\frac{\vec{\omega}^h}{|\vec{\omega}^h|}$ replaced by $\vec{\omega}^h$ and $\vec{\nu}^h$, respectively. However, then the elimination of κ^h from the modified (4.18) via the modified (4.17) now leads to a far more complicated version of (4.22). In the case of $\vec{\omega}^h$, one needs to compute $[\frac{\delta}{\delta \vec{X}^h} \vec{\omega}^h]$ as opposed to $[\frac{\delta}{\delta \vec{X}^h} \frac{\vec{\omega}^h}{|\vec{\omega}^h|}]$. However, on noting (4.11) and (4.9), it is easier to compute the latter. Hence, the choice of $\frac{\vec{\omega}^h}{|\vec{\omega}^h|}$ in (4.6).

Remark 4.3. Due to (4.7), the approximation $(\mathcal{P}_h)^h$ satisfies an equidistribution property, i.e., any two neighboring elements are either parallel or of the same length, at every $t > 0$. For this property to hold, it is crucial to employ mass lumping in (4.7). We refer to [4, Rem. 2.4] for more details.

4.2. Based on κ_g^h. Let $(\cdot, \cdot)^\diamond$ denote a discrete L^2–inner product based on some numerical quadrature rule. In particular, for two piecewise continuous functions, with possible jumps at the nodes $\{q_j\}_{j=1}^J$, we let $(u, v)^\diamond = I^\diamond(u\, v)$, where

(4.23)
$$I^\diamond(f) = \sum_{j=1}^J h_j \sum_{k=1}^K w_k\, f(\alpha_k\, q_{j-1} + (1-\alpha_k)\, q_j)\,, \quad w_k > 0\,, \ \alpha_k \in [0, 1]\,, \quad k = 1, \dots, K\,,$$

with $K \geq 2$, $\sum_{k=1}^K w_k = 1$, and with distinct α_k, $k = 1, \dots, K$. A special case is $(\cdot, \cdot)^\diamond = (\cdot, \cdot)^h$ (recall (4.2)), but we also allow for more accurate quadrature rules.

We define the Lagrangian

$$\mathcal{L}_g^h(\vec{X}^h, \kappa_g^h, \vec{Y}_g^h) = \frac{1}{2} \left((\kappa_g^h)^2 + 2\,\lambda, \left|\vec{X}_\rho^h\right|_g \right)^\diamond - \left(g^{\frac{1}{2}} \left(\vec{X}^h\right) \kappa_g^h\, \vec{\nu}^h, \vec{Y}_g^h \left|\vec{X}_\rho^h\right|_g \right)^\diamond$$

(4.24)
$$- \left(\vec{X}_s^h, \left(\vec{Y}_g^h\right)_s \left|\vec{X}_\rho^h\right|_g \right)^\diamond - \frac{1}{2} \left(\nabla \ln g \left(\vec{X}^h\right), \vec{Y}_g^h \left|\vec{X}_\rho^h\right|_g \right)^\diamond\,,$$

which is the corresponding discrete analogue of (3.17). Taking variations $\chi \in V^h$ in κ_g^h and setting $[\frac{\delta}{\delta \kappa_g^h} \mathcal{L}_g^h](\chi) = 0$ yields that

(4.25)
$$\left(\kappa_g^h - g^{\frac{1}{2}} \left(\vec{X}^h\right) \vec{Y}_g^h \cdot \vec{\nu}^h, \chi \left|\vec{X}_\rho^h\right|_g \right)^\diamond = 0 \qquad \forall\, \chi \in V^h\,.$$

Taking variations $\vec{\eta} \in \underline{V}^h$ in $\vec{Y}_g^h$ and setting $[\frac{\delta}{\delta \vec{Y}_g^h} \mathcal{L}_g^h](\vec{\eta}) = 0$ we obtain

(4.26)
$$\left(g^{\frac{1}{2}} \left(\vec{X}^h\right) \kappa_g^h\, \vec{\nu}^h, \vec{\eta} \left|\vec{X}_\rho^h\right|_g \right)^\diamond + \left(\vec{X}_s^h, \vec{\eta}_s \left|\vec{X}_\rho^h\right|_g \right)^\diamond + \frac{1}{2} \left(\nabla \ln g \left(\vec{X}^h\right), \vec{\eta} \left|\vec{X}_\rho^h\right|_g \right)^\diamond = 0\,,$$

for all $\vec{\eta} \in \underline{V}^h$, as a discrete analogue of (3.16). Taking variations $\vec{\chi} \in \underline{V}^h$ in $\vec{X}^h$, and then setting $(g(\vec{X}^h)\, \vec{X}_t^h \cdot \vec{\omega}^h, \vec{\chi} \cdot \vec{\omega}^h\, |\vec{X}_\rho^h|_g)^\diamond = -[\frac{\delta}{\delta \vec{X}^h} \mathcal{L}_g^h](\vec{\chi})$, we obtain

(4.27)

$$
\left(g\left(\vec{X}^h\right) \vec{X}_t^h \cdot \vec{\omega}^h, \vec{\chi} \cdot \vec{\omega}^h \left|\vec{X}_\rho^h\right|_g \right)^\diamond
$$

$$
= -\frac{1}{2} \left((\kappa_g^h)^2 + 2\lambda, \left[\frac{\delta}{\delta \vec{X}^h} \left|\vec{X}_\rho^h\right|_g \right] (\vec{\chi}) \right)^\diamond + \left(\kappa_g^h \vec{Y}_g^h, \left[\frac{\delta}{\delta \vec{X}^h} g\left(\vec{X}^h\right) \vec{\nu}^h \left|\vec{X}_\rho^h\right| \right] (\vec{\chi}) \right)^\diamond
$$

$$
+ \left(\left(\vec{Y}_g^h\right)_\rho, \left[\frac{\delta}{\delta \vec{X}^h} g^{\frac{1}{2}}\left(\vec{X}^h\right) \vec{\tau}^h \right] (\vec{\chi}) \right)^\diamond + \frac{1}{2} \left(\vec{Y}_g^h, \left[\frac{\delta}{\delta \vec{X}^h} \left(\nabla \ln g\left(\vec{X}^h\right) \right) \left|\vec{X}_\rho^h\right|_g \right] (\vec{\chi}) \right)^\diamond
$$

for all $\vec{\chi} \in \underline{V}^h$. Choosing $\vec{\chi} = \vec{X}_t^h$ in (4.27), and noting a discrete variant of (3.2), as well as (2.4), yields that

$$
\left(g\left(\vec{X}^h\right) (\vec{X}_t^h \cdot \vec{\omega}^h)^2, \left|\vec{X}_\rho^h\right|_g \right)^\diamond
$$

$$
= -\frac{1}{2} \left((\kappa_g^h)^2 + 2\lambda, \left(\left|\vec{X}_\rho^h\right|_g \right)_t \right)^\diamond + \left(\kappa_g^h \vec{Y}_g^h, \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \vec{\nu}^h \left|\vec{X}_\rho^h\right|_g \right)_t \right)^\diamond
$$

(4.28) $\qquad + \left(\left(\vec{Y}_g^h\right)_\rho, \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \vec{\tau}^h \right)_t \right)^\diamond + \frac{1}{2} \left(\vec{Y}_g^h, \left(\left(\nabla \ln g\left(\vec{X}^h\right) \right) \left|\vec{X}_\rho^h\right|_g \right)_t \right)^\diamond .$

On differentiating (4.26) with respect to time, and then choosing $\vec{\eta} = \vec{Y}_g^h$, we obtain, on recalling (4.3) and (2.4), that

(4.29)

$$
\left((\kappa_g^h)_t \vec{Y}_g^h, g^{\frac{1}{2}}\left(\vec{X}^h\right) \vec{\nu}^h \left|\vec{X}_\rho^h\right|_g \right)^\diamond + \left(\kappa_g^h \vec{Y}_g^h, \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \vec{\nu}^h \left|\vec{X}_\rho^h\right|_g \right)_t \right)^\diamond
$$

$$
+ \left(\left(\vec{Y}_g^h\right)_\rho, \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \vec{\tau}^h \right)_t \right)^\diamond + \frac{1}{2} \left(\vec{Y}_g^h, \left(\left(\nabla \ln g\left(\vec{X}^h\right) \right) \left|\vec{X}_\rho^h\right|_g \right)_t \right)^\diamond = 0.
$$

Choosing $\chi = (\kappa_g^h)_t$ in (4.25), and combining with (4.28) and (4.29), yields that

(4.30) $\qquad \left(g\left(\vec{X}^h\right) (\vec{X}_t^h \cdot \vec{\omega}^h)^2, \left|\vec{X}_\rho^h\right|_g \right)^\diamond + \frac{1}{2} \frac{d}{dt} \left((\kappa_g^h)^2 + 2\lambda, \left|\vec{X}_\rho^h\right|_g \right)^\diamond = 0,$

which reveals the discrete gradient flow structure. Also note that (4.28)–(4.30) are the discrete analogues of (3.22)–(3.24).

In order to derive a suitable finite element approximation, we now return to (4.27). Substituting (4.5) into (4.27) yields, on noting (4.3) and (2.4), that

(4.31)

$$
\left(g\left(\vec{X}^h\right) \vec{X}_t^h \cdot \vec{\omega}^h, \vec{\chi} \cdot \vec{\omega}^h \left|\vec{X}_\rho^h\right|_g \right)^\diamond = \frac{1}{2} \left(\vec{Y}_g^h, \left[\frac{\delta}{\delta \vec{X}^h} \left(\nabla \ln g\left(\vec{X}^h\right) \right) \right] (\vec{\chi}) \left|\vec{X}_\rho^h\right|_g \right)^\diamond
$$

$$
- \frac{1}{2} \left((\kappa_g^h)^2 + 2\lambda - \vec{Y}_g^h \cdot \nabla \ln g\left(\vec{X}^h\right), \left[\frac{\delta}{\delta \vec{X}^h} \left|\vec{X}_\rho^h\right|_g \right] (\vec{\chi}) \right)^\diamond
$$

$$
+ \left(\kappa_g^h \vec{Y}_g^h \cdot \vec{\nu}^h \left[\frac{\delta}{\delta \vec{X}^h} g\left(\vec{X}^h\right) \right] (\vec{\chi}) \left|\vec{X}_\rho^h\right| \right)^\diamond + \left(g\left(\vec{X}^h\right) \kappa_g^h \vec{Y}_g^h, \left[\frac{\delta}{\delta \vec{X}^h} \vec{\nu}^h \left|\vec{X}_\rho^h\right| \right] (\vec{\chi}) \right)^\diamond
$$

$$
+ \left(\left(\vec{Y}_g^h\right)_\rho \cdot \vec{\tau}^h, \left[\frac{\delta}{\delta \vec{X}^h} g^{\frac{1}{2}}\left(\vec{X}^h\right) \right] (\vec{\chi}) \right)^\diamond + \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \left(\vec{Y}_g^h\right)_\rho, \left[\frac{\delta}{\delta \vec{X}^h} \vec{\tau}^h \right] (\vec{\chi}) \right)^\diamond
$$

$$
= -\frac{1}{2} \left((\kappa_g^h)^2 + 2\lambda - \vec{Y}_g^h \cdot \nabla \ln g\left(\vec{X}^h\right), \left[\vec{\tau}^h \cdot \vec{\chi}_s + \frac{1}{2} \vec{\chi} \cdot \nabla \ln g\left(\vec{X}^h\right) \right] \left|\vec{X}_\rho^h\right|_g \right)^\diamond
$$

$$+ \frac{1}{2} \left((D^2 \ln g\left(\vec{X}^h\right)) \vec{Y}^h_g , \vec{\chi} \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond$$

$$+ \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \kappa^h_g \vec{Y}^h_g . \vec{\nu}^h + \frac{1}{2} \left(\vec{Y}^h_g\right)_s . \vec{\tau}^h , \left(\nabla \ln g\left(\vec{X}^h\right)\right) . \vec{\chi} \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond$$

$$- \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \kappa^h_g \vec{Y}^h_g , \vec{\chi}^\perp_s \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond + \left(\left(\vec{Y}^h_g\right)_s . \vec{\nu}^h , \vec{\chi}_s . \vec{\nu}^h \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond \qquad \forall \, \vec{\chi} \in \underline{V}^h .$$

Then (4.31), (4.25), and (4.26), on recalling (3.4a), give rise to the following approximation of $(\mathcal{Q})$.

$(\mathcal{Q}_h)^\diamond$: Let $\vec{X}^h(0) \in \underline{V}^h$. For $t \in (0, T]$ find $(\vec{X}^h(t), \kappa^h_g(t), \vec{Y}^h_g(t)) \in \underline{V}^h \times V^h \times \underline{V}^h$ such that

(4.32)

$$\left(g\left(\vec{X}^h\right) \vec{X}^h_t . \vec{\omega}^h , \vec{\chi} . \vec{\omega}^h \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond - \left(\left(\vec{Y}^h_g\right)_s . \vec{\nu}^h , \vec{\chi}_s . \vec{\nu}^h \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond$$

$$= -\frac{1}{2} \left((\kappa^h_g)^2 + 2\,\lambda - \vec{Y}^h_g . \nabla \ln g\left(\vec{X}^h\right) , \left[\vec{\chi}_s . \vec{\tau}^h + \frac{1}{2} \vec{\chi} . \nabla \ln g\left(\vec{X}^h\right) \right] \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond$$

$$+ \frac{1}{2} \left(\left(D^2 \ln g\left(\vec{X}^h\right)\right) \vec{Y}^h_g , \vec{\chi} \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond + \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \kappa^h_g , \vec{\chi}_s . \left(\vec{Y}^h_g\right)^\perp \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond$$

$$+ \left(g^{\frac{1}{2}}\left(\vec{X}^h\right) \kappa^h_g \vec{Y}^h_g . \vec{\nu}^h + \frac{1}{2} \left(\vec{Y}^h_g\right)_s . \vec{\tau}^h , \vec{\chi} . \left(\nabla \ln g\left(\vec{X}^h\right)\right) \left. \left| \vec{X}^h_\rho \right|\right._g \right)^\diamond \quad \forall \, \vec{\chi} \in \underline{V}^h ,$$

(4.25), and (4.26) hold.

THEOREM 4.4. *Let $|\vec{X}^h_\rho| > 0$ almost everywhere in $I \times (0, T)$. Let $(\vec{X}^h(t), \kappa^h_g(t), \vec{Y}^h_g(t)) \in \underline{V}^h \times V^h \times \underline{V}^h$, for $t \in (0, T]$, be a solution to $(\mathcal{Q}_h)^\diamond$. Then the solution satisfies the stability bound* (4.30).

Proof. We have already shown that a solution to $(\mathcal{Q}_h)^\diamond$ satisfies (4.28) and (4.29). Hence choosing $\chi = (\kappa^h_g)_t$ in (4.25), and combining with (4.28) and (4.29), yields (4.30) as before. □

Remark 4.5. We stress that unlike for $(\mathcal{P}_h)^h$ (recall Remark 4.3), it is not possible to prove an equidistribution property for $(\mathcal{Q}_h)^\diamond$, even if we employ mass lumping in (4.26). It is for this reason that we also consider higher order quadrature rules. The motivation behind considering $(\mathcal{Q}_h)^\diamond$ as an alternative to $(\mathcal{P}_h)^h$ is twofold. First, from a variational point of view, it is more natural to work with $\varkappa_g$ as a variable, since (2.5) is naturally defined in terms of $\varkappa_g$. Second, the techniques introduced for $(\mathcal{Q}_h)^\diamond$ will be exploited in [7] for stable approximations of Willmore flow for axisymmetric hypersurfaces in $\mathbb{R}^3$.

5. Fully discrete finite element approximations. Let $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$ be a partitioning of $[0, T]$ into possibly variable time steps $\Delta t_m = t_{m+1} - t_m$, $m = 0 \to M - 1$. We set $\Delta t = \max_{m=0 \to M-1} \Delta t_m$. For a given $\vec{X}^m \in \underline{V}^h$ we set $\vec{\nu}^m = -\frac{[\vec{X}^m_\rho]^\perp}{|\vec{X}^m_\rho|}$, as the discrete analogue to (2.1). We also let $\vec{\omega}^m \in \underline{V}^h$ be the natural fully discrete analogue of $\vec{\omega}^h \in \underline{V}^h$; recall (4.4). Given $\vec{X}^m \in \underline{V}^h$, the fully discrete approximations we propose in this section will always seek a parameterization $\vec{X}^{m+1} \in \underline{V}^h$ at the new time level, together with a suitable approximation of curvature.

For the metrics we consider in this paper, we summarize in Table 1 the quantities that are necessary in order to implement the numerical schemes presented below.

Expressions for terms that are relevant for the implementation of the presented finite element approximations.

g	$\nabla \ln g(\vec{x})$	$D^2 \ln g(\vec{x})$
(2.12a)	$-2\,\mu\,(\vec{x}\,.\,\vec{e}_2)^{-1}\,\vec{e}_2$	$2\,\mu\,(\vec{x}\,.\,\vec{e}_2)^{-2}\,\vec{e}_2 \otimes \vec{e}_2$
(2.12b)	$\frac{4\,\alpha}{1-\alpha\,|\vec{x}|^2}\,\vec{x}$	$\frac{4\,\alpha}{1-\alpha\,|\vec{x}|^2}\,\underline{\mathrm{Id}} + \frac{8\,\alpha^2}{(1-\alpha\,|\vec{x}|^2)^2}\,\vec{x}\otimes\vec{x}$
(2.12c)	$-2\,\tanh(\vec{x}\,.\,\vec{e}_1)\,\vec{e}_1$	$-2\,\cosh^{-2}(\vec{x}\,.\,\vec{e}_1)\,\vec{e}_1 \otimes \vec{e}_1$
(2.12d)	$2\,\tanh(\vec{x}\,.\,\vec{e}_1)\,\vec{e}_1$	$2\,\cosh^{-2}(\vec{x}\,.\,\vec{e}_1)\,\vec{e}_1 \otimes \vec{e}_1$
(2.12e)	$-2\,\dfrac{\sin(\vec{x}\,.\,\vec{e}_2)}{[\mathsf{s}^2+1]^{\frac{1}{2}}-\cos(\vec{x}\,.\,\vec{e}_2)}\,\vec{e}_2$	$2\,\dfrac{1-[\mathsf{s}^2+1]^{\frac{1}{2}}\,\cos(\vec{x}\,.\,\vec{e}_2)}{([\mathsf{s}^2+1]^{\frac{1}{2}}-\cos(\vec{x}\,.\,\vec{e}_2))^2}\,\vec{e}_2 \otimes \vec{e}_2$

5.1. Based on κ^{m+1}. We propose the following fully discrete approximation of $(\mathcal{P}_h)^h$.

$(\mathcal{P}_m)^h$: Let $(\vec{X}^0, \kappa^0, \vec{Y}^0) \in \underline{V}^h \times V^h \times \underline{V}^h$. For $m = 0, \ldots, M-1$, we define $\kappa_g^m = \pi^h[g^{-\frac{1}{2}}(\vec{X}^m)\,[\kappa^m - \frac{1}{2}\,\frac{\vec{\omega}^m}{|\vec{\omega}^m|}\,.\,\nabla\ln g(\vec{X}^m)]]$, and then find $(\vec{X}^{m+1}, \kappa^{m+1}, \vec{Y}^{m+1}) \in \underline{V}^h \times V^h \times \underline{V}^h$ such that

$$\left(g^{\frac{3}{2}}\left(\vec{X}^m\right)\frac{\vec{X}^{m+1}-\vec{X}^m}{\Delta t_m}\,.\,\vec{\omega}^m, \vec{\chi}\,.\,\vec{\omega}^m\,|\vec{X}_\rho^m| \right)^h - \left(\vec{Y}_s^{m+1}, \vec{\chi}_s\,|\vec{X}_\rho^m| \right)$$

$$+ \left(\vec{Y}_s^m\,.\,\vec{\tau}^m, \vec{\chi}_s\,.\,\vec{\tau}^m\,|\vec{X}_\rho^m| \right) = -\frac{1}{2}\left(g^{\frac{1}{2}}\left(\vec{X}^m\right)[(\kappa_g^m)^2 + 2\,\lambda], \vec{\chi}_s\,.\,\vec{\tau}^m\,|\vec{X}_\rho^m| \right)^h$$

$$+ \frac{1}{4}\left(g^{\frac{1}{2}}\left(\vec{X}^m\right)[(\kappa_g^m)^2 - 2\,\lambda], \vec{\chi}\,.\,\left(\nabla\ln g\left(\vec{X}^m\right)\right)|\vec{X}_\rho^m| \right)^h$$

$$+ \frac{1}{2}\left(\kappa_g^m\,\frac{\vec{\omega}^m}{|\vec{\omega}^m|}, (D^2\ln g\left(\vec{X}^m\right))\,\vec{\chi}\,|\vec{X}_\rho^m| \right)^h + \left(\kappa^m\,(\vec{Y}^m)^\perp, \vec{\chi}_s\,|\vec{X}_\rho^m| \right)^h$$

$$- \frac{1}{2}\left(\frac{\kappa_g^m}{|\vec{\omega}^m|}\,\nabla\ln g\left(\vec{X}^m\right)\,.\,\left(\frac{\vec{\omega}^m}{|\vec{\omega}^m|}\right)^\perp, \vec{\chi}_s\,.\,\frac{\vec{\omega}^m}{|\vec{\omega}^m|}\,|\vec{X}_\rho^m| \right)^h \qquad \forall\,\vec{\chi}\in\underline{V}^h,$$

(5.1a)

$$\left(g^{\frac{1}{2}}\left(\vec{X}^m\right)\vec{Y}^{m+1}\,.\,\vec{\omega}^m, \vec{\eta}\,.\,\vec{\omega}^m\,|\vec{X}_\rho^m| \right)^h + \frac{1}{2}\left(\frac{\vec{\omega}^m}{|\vec{\omega}^m|}\,.\,\nabla\ln g\left(\vec{X}^m\right), \vec{\eta}\,.\,\vec{\omega}^m\,|\vec{X}_\rho^m| \right)^h$$

(5.1b)

$$+ \left(\vec{X}_s^{m+1}, \vec{\eta}_s\,|\vec{X}_\rho^m| \right) = 0 \qquad \forall\,\vec{\eta}\in\underline{V}^h,$$

and

(5.2) $$\kappa^{m+1} = \pi^h\left[g^{\frac{1}{2}}\left(\vec{X}^m\right)\vec{Y}^{m+1}\,.\,\vec{\omega}^m + \frac{1}{2}\,\frac{\vec{\omega}^m}{|\vec{\omega}^m|}\,.\,\nabla\ln g\left(\vec{X}^m\right) \right].$$

Notice that (5.1b) was obtained on combining (5.2) with a fully discrete variant of (4.7), and noting (4.4), in order to obtain a lower dimensional linear system to solve for the unknowns $\vec{X}^{m+1}$ and $\vec{Y}^{m+1}$ that is decoupled from (5.2). Moreover, (5.1a) is a fully discrete approximation of (4.22), on noting the definition of κ_g^m.

We make the following mild assumption.

$(\mathfrak{A})^h$ Let $|\vec{X}_\rho^m| > 0$ for almost all $\rho \in I$, let $\dim \mathrm{span}\{\vec{\omega}^m(q_j) : j = 1, \ldots, J\} = 2$, and let $\vec{\omega}^m(q_j) \neq \vec{0}$, $j = 1, \ldots, J$.

The above assumption can be violated only if all the vertex normals $\vec{\omega}^m(q_j)$ of Γ^m are collinear, or if two neighboring edges of Γ^m overlap. Clearly, this almost never happens in practice, and it certainly cannot happen if Γ^m has no self-intersections. See also [4, Remark 2.2] for more details.

LEMMA 5.1. *Let the assumption* $(\mathfrak{A})^h$ *hold. Then there exists a unique solution* $(\vec{X}^{m+1}, \kappa^{m+1}, \vec{Y}^{m+1}) \in \underline{V}^h \times V^h \times \underline{V}^h$ *to* $(\mathcal{P}_m)^h$.

Proof. As (5.1) is linear, existence follows from uniqueness. To investigate the latter, we consider the following system: Find $(\vec{X}, \vec{Y}) \in \underline{V}^h \times \underline{V}^h$ such that

$$(5.3a) \quad \left(g^{\frac{3}{2}}\left(\vec{X}^m\right) \vec{X} . \vec{\omega}^m, \vec{\chi} . \vec{\omega}^m |\vec{X}_\rho^m|\right)^h - \Delta t_m \left(\vec{Y}_s, \vec{\chi}_s |\vec{X}_\rho^m|\right) = 0 \qquad \forall \, \vec{\chi} \in \underline{V}^h,$$

$$(5.3b) \quad \left(g^{\frac{1}{2}}\left(\vec{X}^m\right) \vec{Y} . \vec{\omega}^m, \vec{\eta} . \vec{\omega}^m |\vec{X}_\rho^m|\right)^h + \left(\vec{X}_s, \vec{\eta}_s |\vec{X}_\rho^m|\right) = 0 \qquad \forall \, \vec{\eta} \in \underline{V}^h.$$

Choosing $\vec{\chi} = \vec{X}$ in (5.3a) and $\vec{\eta} = \vec{Y}$ in (5.3b), and combining, yields that

$$(5.4) \qquad \pi^h [\vec{X} . \vec{\omega}^m] = \pi^h [\vec{Y} . \vec{\omega}^m] = 0 \in V^h.$$

As a consequence, it follows from choosing $\vec{\chi} = \vec{Y}$ in (5.3a) and $\vec{\eta} = \vec{X}$ in (5.3b) that $\vec{X}$ and $\vec{Y}$ are constant vectors. Combining (5.4) and the assumption $(\mathfrak{A})^h$ then yields that $\vec{X} = \vec{Y} = \vec{0} \in \underline{V}^h$.

Hence we have shown the existence of a unique $(\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{V}^h \times \underline{V}^h$ solving (5.1), which via (5.2) yields existence and uniqueness of $\kappa^{m+1} \in V^h$. □

5.2. Based on κ_g^{m+1}. We propose the following fully discrete approximation of $(\mathcal{Q}_h)^\diamond$.

$(\mathcal{Q}_m)^\diamond$: Let $(\vec{X}^0, \kappa_g^0, \vec{Y}_g^0) \in \underline{V}^h \times V^h \times \underline{V}^h$. For $m = 0, \ldots, M-1$, find $(\vec{X}^{m+1}, \kappa_g^{m+1}, \vec{Y}_g^{m+1}) \in \underline{V}^h \times V^h \times \underline{V}^h$ such that

(5.5a)

$$\left(g\left(\vec{X}^m\right) \frac{\vec{X}^{m+1} - \vec{X}^m}{\Delta t_m} . \vec{\omega}^m, \vec{\chi} . \vec{\omega}^m |\vec{X}_\rho^m|_g\right)^\diamond - \left(\left(\vec{Y}_g^{m+1}\right)_s, \vec{\chi}_s |\vec{X}_\rho^m|_g\right)^\diamond$$

$$+ \left(\left(\vec{Y}_g^m\right)_s . \vec{\tau}^m, \vec{\chi}_s . \vec{\tau}^m |\vec{X}_\rho^m|_g\right)^\diamond$$

$$= -\frac{1}{2}\left((\kappa_g^m)^2 + 2\lambda - \vec{Y}_g^m . \nabla \ln g\left(\vec{X}^m\right), \left[\vec{\chi}_s . \vec{\tau}^m + \frac{1}{2} \vec{\chi} . \nabla \ln g\left(\vec{X}^m\right)\right] |\vec{X}_\rho^m|_g\right)^\diamond$$

$$+ \frac{1}{2}\left((D^2 \ln g\left(\vec{X}^m\right)) \vec{Y}_g^m, \vec{\chi} |\vec{X}_\rho^m|_g\right)^\diamond$$

$$+ \left(g^{\frac{1}{2}}\left(\vec{X}^m\right) \kappa_g^m \vec{Y}_g^m . \vec{\nu}^m + \frac{1}{2} (\vec{Y}_g^m)_s . \vec{\tau}^m, \vec{\chi} . \left(\nabla \ln g\left(\vec{X}^m\right)\right) |\vec{X}_\rho^m|_g\right)^\diamond$$

$$+ \left(g^{\frac{1}{2}}\left(\vec{X}^m\right) \kappa_g^m, \vec{\chi}_s . (\vec{Y}_g^m)^\perp |\vec{X}_\rho^m|_g\right)^\diamond \qquad \forall \, \vec{\chi} \in \underline{V}^h,$$

(5.5b)
$$\left(\kappa_g^{m+1} - g^{\frac{1}{2}} \left(\vec{X}^m \right) \vec{Y}_g^{m+1} . \vec{\nu}^m , \chi \, |\vec{X}_\rho^m|_g \right)^\diamond = 0 \quad \forall \, \chi \in V^h \, ,$$

$$\left(g^{\frac{1}{2}} \left(\vec{X}^m \right) \kappa_g^{m+1} \vec{\nu}^m , \vec{\eta} \, |\vec{X}_\rho^m|_g \right)^\diamond + \left(\vec{X}_s^{m+1} , \vec{\eta}_s \, |\vec{X}_\rho^m|_g \right)^\diamond + \frac{1}{2} \left(\nabla \ln g \left(\vec{X}^m \right) , \vec{\eta} \, |\vec{X}_\rho^m|_g \right)^\diamond$$

(5.5c)
$$= 0 \quad \forall \, \vec{\eta} \in \underline{V}^h \, .$$

Of course, in the case $(\cdot, \cdot)^\diamond = (\cdot, \cdot)^h$, (5.5b) gives rise to $\kappa_g^{m+1} = \pi^h \, [g^{\frac{1}{2}}(\vec{X}^m)$
$\vec{Y}_g^{m+1} . \vec{\omega}^m]$, on noting (4.4), and so κ_g^{m+1} can be eliminated from (5.5a) to give rise
to a coupled linear system involving only $\vec{X}^{m+1}$ and $\vec{Y}_g^{m+1}$, similarly to (5.1).

We make the following mild assumption.

$(\mathfrak{B})^\diamond$ Let $|\vec{X}_\rho^m| > 0$ for almost all $\rho \in I$, and let $\dim \operatorname{span} \mathcal{Z}^\diamond = 2$, where

$$\mathcal{Z}^\diamond = \left\{ \left(g^{\frac{1}{2}}(\vec{X}^m) \, \vec{\nu}^m , \chi \, |\vec{X}_\rho^m|_g \right)^\diamond : \chi \in V^h \right\} \subset \mathbb{R}^2 .$$

In the case $(\cdot, \cdot)^\diamond = (\cdot, \cdot)^h$ the above assumption collapses to the first part of $(\mathfrak{A})^h$,
as was demonstrated below (3.11) in [8]. For more general quadrature rules, it is
violated only if some g-weighted vertex normals of Γ^m are all collinear, which means
that $(\mathfrak{B})^\diamond$ is clearly a very mild constraint.

LEMMA 5.2. *Let the assumptions $(\mathfrak{A})^h$ and $(\mathfrak{B})^\diamond$ hold. Then there exists a unique
solution $(\vec{X}^{m+1}, \kappa_g^{m+1}, \vec{Y}_g^{m+1}) \in \underline{V}^h \times V^h \times \underline{V}^h$ to $(\mathcal{Q}_m)^\diamond$.*

Proof. As (5.5) is linear, existence follows from uniqueness. To investigate the
latter, we consider the following system: Find $(\vec{X}, \kappa_g, \vec{Y}_g) \in \underline{V}^h \times V^h \times \underline{V}^h$ such that

(5.6a) $\left(g \left(\vec{X}^m \right) \vec{X} . \vec{\omega}^m , \vec{\chi} . \vec{\omega}^m \, |\vec{X}_\rho^m|_g \right)^\diamond - \Delta t_m \left((\vec{Y}_g)_s , \vec{\chi}_s \, |\vec{X}_\rho^m|_g \right)^\diamond = 0 \quad \forall \, \vec{\chi} \in \underline{V}^h \, ,$

(5.6b) $\left(\kappa_g - g^{\frac{1}{2}} \left(\vec{X}^m \right) \vec{Y}_g . \vec{\nu}^m , \chi \, |\vec{X}_\rho^m|_g \right)^\diamond = 0 \quad \forall \, \chi \in V^h \, ,$

(5.6c) $\left(g^{\frac{1}{2}} \left(\vec{X}^m \right) \kappa_g \vec{\nu}^m , \vec{\eta} \, |\vec{X}_\rho^m|_g \right)^\diamond + \left(\vec{X}_s , \vec{\eta}_s \, |\vec{X}_\rho^m|_g \right)^\diamond = 0 \quad \forall \, \vec{\eta} \in \underline{V}^h \, .$

Choosing $\vec{\chi} = \vec{X}$ in (5.6a), $\chi = \kappa_g$ in (5.6b), and $\vec{\eta} = \vec{Y}_g$ in (5.6c) yields that

$$\left(g \left(\vec{X}^m \right) (\vec{X} . \vec{\omega}^m)^2 , |\vec{X}_\rho^m|_g \right)^\diamond + \Delta t_m \left((\kappa_g)^2 , |\vec{X}_\rho^m|_g \right)^\diamond = 0 \, ,$$

and so it follows from (4.23) (recall $K \geq 2$) and the positivities of $g(\vec{X}^m)$ and $|\vec{X}_\rho^m|$
that

(5.7) $\kappa_g = 0 \in V^h \quad \text{and} \quad \left(g \left(\vec{X}^m \right) \vec{X} . \vec{\omega}^m , \eta \, |\vec{X}_\rho^m|_g \right)^\diamond = 0 \qquad \forall \, \eta \in C(\overline{I}) \, .$

As a consequence, it follows from choosing $\vec{\chi} = \vec{Y}_g$ in (5.6a) and $\vec{\eta} = \vec{X}$ in (5.6c)
that $\vec{X}$ and $\vec{Y}_g$ are constant vectors. Combining (5.6b), $\kappa_g = 0$, and the assumption
$(\mathfrak{B})^\diamond$ then yields that $\vec{Y}_g = \vec{0} \in \underline{V}^h$. Moreover, it follows from (5.7), (4.23) (recall
$K \geq 2$), and $\vec{X}$ being a constant that $\vec{X} . \vec{\omega}^m = 0 \in V^h$. Combining this with the
assumption $(\mathfrak{A})^h$ yields that $\vec{X} = \vec{0} \in \underline{V}^h$. Hence there exists a unique solution
$(\vec{X}^{m+1}, \kappa_g^{m+1}, \vec{Y}_g^{m+1}) \in \underline{V}^h \times V^h \times \underline{V}^h$ to $(\mathcal{Q}_m)^\diamond$. $\qquad \square$

6. Numerical results. Unless otherwise stated, in all our computations we set $\lambda = 0$. For the scheme $(\mathcal{Q}_m)^\diamond$ we consider either $(\mathcal{Q}_m)^h$ (recall (4.2)) or $(\mathcal{Q}_m)^\star$, where $(\cdot, \cdot)^\star$ denotes a quadrature that is exact for polynomials of degree up to five.

On recalling (4.6) and (4.17), for solutions of the scheme $(\mathcal{P}_m)^h$ we define $W_{g,\lambda}^{m+1} = \frac{1}{2}((\vec{Y}^{m+1} \cdot \vec{\omega}^m)^2 + 2\lambda, g^{\frac{1}{2}}(\vec{X}^m)|\vec{X}_\rho^m|)^h$ as the natural discrete analogue of (3.5), while for solutions of $(\mathcal{Q}_m)^\diamond$ we let $\widetilde{W}_{g,\lambda}^{m+1} = \frac{1}{2}((\kappa_g^{m+1})^2 + 2\lambda, |\vec{X}_\rho^m|_g)^\diamond$.

We also consider the ratio

$$(6.1) \qquad \mathfrak{r}^m = \frac{\max_{j=1 \to J} |\vec{X}^m(q_j) - \vec{X}^m(q_{j-1})|}{\min_{j=1 \to J} |\vec{X}^m(q_j) - \vec{X}^m(q_{j-1})|}$$

between the longest and shortest elements of Γ^m, and we are often interested in the evolution of this ratio over time.

In order to define the initial data for the schemes $(\mathcal{P}_m)^h$ and $(\mathcal{Q}_m)^\diamond$ we define, given $\Gamma^0 = \vec{X}^0(\overline{I})$, the discrete curvature vector $\vec{\kappa}^0 \in \underline{V}^h$ such that

$$\left(\vec{\kappa}^0, \vec{\eta} \, |\vec{X}_\rho^0|\right)^h + \left(\vec{X}_s^0, \vec{\eta}_s \, |\vec{X}_\rho^0|\right) = 0 \quad \forall \, \vec{\eta} \in \underline{V}^h \, ;$$

recall (2.2). Then we set $\kappa^0 = \pi^h[\frac{\vec{\kappa}^0 \cdot \vec{\omega}^0}{|\vec{\omega}^0|}]$ and, as a discrete analogue to (2.6), we let $\kappa_g^0 = \pi^h[g^{-\frac{1}{2}}(\vec{X}^0)[\kappa^0 - \frac{1}{2}\frac{\vec{\omega}^0}{|\vec{\omega}^0|} \cdot \nabla \ln g(\vec{X}^0)]]$. Finally, on recalling (4.17) and (4.25), we set $\vec{Y}^0 = \vec{\pi}^h[|\vec{\omega}^0|^{-2} \kappa_g^0 \vec{\omega}^0]$ and $\vec{Y}_g^0 = \vec{\pi}^h[g^{-\frac{1}{2}}(\vec{X}^0)|\vec{\omega}^0|^{-2} \kappa_g^0 \vec{\omega}^0]$.

6.1. Elliptic plane: (2.12b) with $\alpha = -1$. For the elliptic plane, we recall the true solution

$$(6.2) \qquad \vec{x}(\rho, t) = a(t)\,\vec{e}_2 + r(t)\,[\cos 2\pi\rho\,\vec{e}_1 + \sin 2\pi\rho\,\vec{e}_2], \qquad \rho \in I,$$

with

$$(6.3) \qquad a(t) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} r^4(t) = \frac{1}{8}(1 - \alpha^2 r^4(t))(1 - 6\alpha r^2(t) + \alpha^2 r^4(t)),$$

for $\alpha = -1$, from Appendix A.2 in [8]. An explicit formula for $r(t)$ is stated in [8, (A.17)]. We use this true solution for a convergence test. To this end, we start with the initial data

$$(6.4) \qquad \vec{X}^0(q_j) = a(0)\,\vec{e}_2 + r(0)\begin{pmatrix} \cos[2\pi q_j + 0.1\sin(2\pi q_j)] \\ \sin[2\pi q_j + 0.1\sin(2\pi q_j)] \end{pmatrix}, \qquad j = 1, \ldots, J$$

(recall (4.1)), with $r(0) = 1.5$ and $a(0) = 0$, for $J \in \{32, 64, 128, 256, 512\}$. We compute the error $\|\Gamma - \Gamma^h\|_{L^\infty} = \max_{m=1,\ldots,M} \max_{j=1,\ldots,J} \|\vec{X}^h(q_j) - a(t_m)\,\vec{e}_2| - r(t_m)|$ over the time interval $[0, 1]$ between the true solution (6.2) and the discrete solutions for the schemes $(\mathcal{P}_m)^h$, $(\mathcal{Q}_m)^h$, and $(\mathcal{Q}_m)^\star$. We note that the circle is shrinking and reaches a radius $r(T) = 1.148$ at time $T = 1$. Here, and in the convergence experiments that follow, we use the time step size $\Delta t = 0.1\,h_{\Gamma^0}^2$, where h_{Γ^0} is the maximal edge length of Γ^0. The computed errors are reported in Table 2.

6.2. Hyperbolic disk: (2.12b) with $\alpha = 1$. For the hyperbolic disk, we recall the true solution (6.2), (6.3) for $\alpha = 1$, from Appendix A.2 in [8]. A nonlinear equation satisfied by $r(t)$ is stated in [8, (A.19)], which we solve in practice with the help of a Newton iteration. Similarly to Table 2 we start with the initial data (6.4) with $r(0) = 0.1$ and $a(0) = 0$. We compute the error $\|\Gamma - \Gamma^h\|_{L^\infty}$ over the

TABLE 2
Errors for the convergence test for (6.2) with (6.3) for $\alpha = -1$, with $r(0) = 1.5$, over the time interval $[0,1]$. The ratios (6.1) at time $t = 1$ for the last row are 1.14, 1.49, and 1.14, respectively.

	$(\mathcal{P}_m)^h$		$(\mathcal{Q}_m)^h$		$(\mathcal{Q}_m)^\star$	
h_{Γ^0}	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC
2.1544e-01	7.1380e-03	—	1.4510e-02	—	1.2582e-02	—
1.0792e-01	1.7446e-03	2.04	3.5351e-03	2.04	3.0547e-03	2.05
5.3988e-02	4.3377e-04	2.01	8.7838e-04	2.01	7.5835e-04	2.01
2.6997e-02	1.0829e-04	2.00	2.1926e-04	2.00	1.8926e-04	2.00
1.3499e-02	2.7064e-05	2.00	5.4795e-05	2.00	4.7295e-05	2.00

TABLE 3
Errors for the convergence test for (6.2) with (6.3) for $\alpha = 1$, with $r(0) = 0.1$, over the time interval $[0,1]$. The ratios (6.1) at time $t = 1$ for the last row are equal to 1.00 for all three schemes.

	$(\mathcal{P}_m)^h$		$(\mathcal{Q}_m)^h$		$(\mathcal{Q}_m)^\star$	
h_{Γ^0}	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC
2.1544e-01	1.8356e-03	—	1.8655e-03	—	2.3602e-03	—
1.0792e-01	4.5233e-04	2.03	4.5938e-04	2.03	5.8378e-04	2.02
5.3988e-02	1.1270e-04	2.01	1.1444e-04	2.01	1.4583e-04	2.00
2.6997e-02	2.8151e-05	2.00	2.8590e-05	2.00	3.6450e-05	2.00
1.3499e-02	7.0364e-06	2.00	7.1460e-06	2.00	9.1121e-06	2.00

time interval $[0,1]$ between the true solution (6.2) and the discrete solutions for the schemes $(\mathcal{P}_m)^h$, $(\mathcal{Q}_m)^h$, and $(\mathcal{Q}_m)^\star$. We note that the circle is expanding and reaches a radius $r(T) = 0.404$ at time $T = 1$. The computed errors are reported in Table 3. Surprisingly, and in contrast to the other tables, the ratio (6.1) reaches the value 1 for all three schemes, and for all presented values of J, at the final time. It is not clear why the tangential motion implicit in $(\mathcal{Q}_h)^\diamond$ appears to lead to equidistribution in this example, but it may have to do with the fact that we compute an expanding circle solution in $\mathbb{D}_1$, the hyperbolic disk; recall (2.12b). For more general evolutions we do not observe equidistribution for $(\mathcal{Q}_m)^h$ or $(\mathcal{Q}_m)^\star$ in practice.

6.3. Hyperbolic plane: (2.12a) with $\mu = 1$. For the hyperbolic plane, we recall the true solution (6.2) with

$$(6.5) \qquad a(t) = a(0) \exp\left(-t + \frac{1}{2} \int_0^t \sigma^2(u) \, \mathrm{d}u\right) \quad \text{and} \quad r(t) = \frac{a(t)}{\sigma(t)},$$

where σ satisfies the ODE $\sigma'(t) = \sigma(t)\left(1 - \frac{1}{2}\sigma^2(t)\right)(\sigma^2(t) - 1)$, from Appendix A.1 in [8]. A nonlinear equation satisfied by $\sigma(t)$ is stated in [8, Appendix A.1], which we solve in practice with the help of a Newton iteration. Moreover, $a(t)$ can be obtained from (6.5) via numerical integration using, e.g., Romberg's method. As initial data we use (6.4) with $r(0) = 1$ and $a(0) = 2$. We recall from Appendix A.1 in [8] that the circle will raise and expand. In fact, at time $T = 1$ it holds that $r(T) = 1.677$ and $a(T) = 2.411$. The computed errors are reported in Table 4, and they can be compared with the corresponding numbers in [8, Table 7]. In particular, $(\mathcal{P}_m)^h$ exhibits smaller errors than $(\mathcal{U}_m)^h$ in [8], while the errors of $(\mathcal{W}_m)^h$ in [8] are very close to $(\mathcal{Q}_m)^\star$. We repeat the convergence test with the initial data $r(0) = 1$ and $a(0) = 1.1$, so that the circle will now sink and shrink. In fact, at time $T = 1$ it holds that $r(T) = 0.645$ and $a(T) = 0.792$. The computed errors are reported in Table 5, and they can be compared with the corresponding numbers in [8, Table 6]. In particular, $(\mathcal{P}_m)^h$ exhibits significantly smaller errors than $(\mathcal{U}_m)^h$ in [8], and similarly $(\mathcal{Q}_m)^h$ and $(\mathcal{Q}_m)^\star$ show significantly smaller errors than $(\mathcal{W}_m)^h$ in [8]. We observe that the approximation $(\mathcal{Q}_m)^h$ exhibits nonoptimal convergence rates for this

TABLE 4

Errors for the convergence test for (6.2) *with* (6.5), *with* $r(0) = 1$, $a(0) = 2$, *over the time interval* $[0,1]$. *The ratios* (6.1) *at time* $t = 1$ *for the last row are* 1.07, 2.59, *and* 1.65, *respectively.*

| | $(\mathcal{P}_m)^h$ | | $(\mathcal{Q}_m)^h$ | | $(\mathcal{Q}_m)^\star$ | |
h_{Γ^0}	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC
2.1544e-01	1.2690e-01	—	7.5442e-02	—	4.3265e-02	—
1.0792e-01	3.1923e-02	2.00	1.9548e-02	1.95	1.0719e-02	2.02
5.3988e-02	7.9911e-03	2.00	4.9076e-03	2.00	2.6764e-03	2.00
2.6997e-02	1.9984e-03	2.00	1.2291e-03	2.00	6.6898e-04	2.00
1.3499e-02	4.9966e-04	2.00	3.0741e-04	2.00	1.6723e-04	2.00

TABLE 5

Errors for the convergence test for (6.2) *with* (6.5), *with* $r(0) = 1$, $a(0) = 1.1$, *over the time interval* $[0,1]$. *The ratios* (6.1) *at time* $t = 1$ *for the last row are* 1.07, 2.76, *and* 1.20, *respectively.*

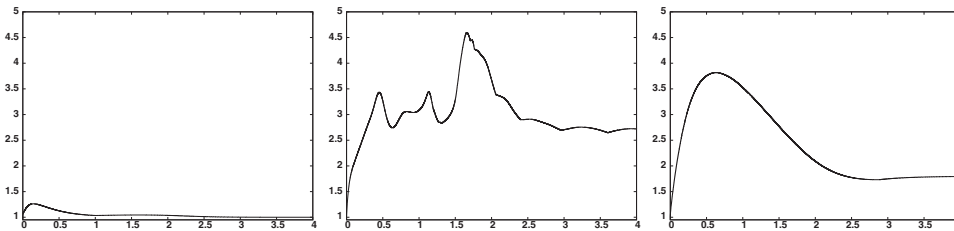| | $(\mathcal{P}_m)^h$ | | $(\mathcal{Q}_m)^h$ | | $(\mathcal{Q}_m)^\star$ | |
h_{Γ^0}	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC	$\|\Gamma - \Gamma^h\|_{L^\infty}$	EOC
2.1544e-01	2.9884e-03	—	5.3699e-02	—	1.1530e-02	—
1.0792e-01	9.7352e-04	1.62	1.6346e-02	1.72	2.9345e-03	1.98
5.3988e-02	2.6531e-04	1.88	5.3475e-03	1.61	7.3673e-04	2.00
2.6997e-02	6.7844e-05	1.97	2.5787e-03	1.05	1.8436e-04	2.00
1.3499e-02	1.7057e-05	1.99	5.8915e-04	2.13	4.6102e-05	2.00

FIG. 1. *A plot of the ratio* (6.1) *for the schemes* $(\mathcal{P}_m)^h$, $(\mathcal{Q}_m)^h$, *and* $(\mathcal{Q}_m)^\star$.

experiment, which appears to have two causes. First, the induced tangential motion of $(\mathcal{Q}_m)^h$ leads to a large ratio $\mathfrak{r}^m$, with comparatively large elements at the bottom of the evolving circle. And second, compared to the experiments in Table 4, the evolving circle is now closer to the $\vec{e}_1$-axis, and hence the associated singularity of g has a stronger effect. All the other experiments, and all the other schemes, always show the expected quadratic convergence rate.

We recall that in Figures 10, 11, and 13 of [8], the authors show some curve evolutions for elastic flow in the hyperbolic plane. Repeating these simulations, for the same discretization parameters, for the newly introduced schemes $(\mathcal{P}_m)^h$, $(\mathcal{Q}_m)^h$, and $(\mathcal{Q}_m)^\star$, yields very similar curve evolutions. As expected, the main difference is in the evolution of the ratio (6.1); recall Remarks 4.3 and 4.5. As an example, we show the evolution of (6.1) for the experiment in [8, Fig. 10] in Figure 1.

6.4. Geodesic elastic flow. We begin with two computations for geodesic elastic flow on a Clifford torus. To this end, we employ the metric induced by (2.12e) with $\mathfrak{s} = 1$, so that the torus has radii $r = 1$ and $R = 2^{\frac{1}{2}}$. As initial data we choose a circle in H with radius 3 and center $(0, 2)^T$. For the simulation in Figure 2 we use the scheme $(\mathcal{P}_m)^h$ with the discretization parameters $J = 256$ and $\Delta t = 10^{-3}$. The scheme $(\mathcal{Q}_m)^h$ was not able to compute this evolution, due to a blow-up in the tangential part of $\vec{Y}^{m+1}$. Hence we only present a comparison with $(\mathcal{Q}_m)^\star$, which gives nearly identical results to $(\mathcal{P}_m)^h$. However, the ratio (6.1) at time $t = 50$ is
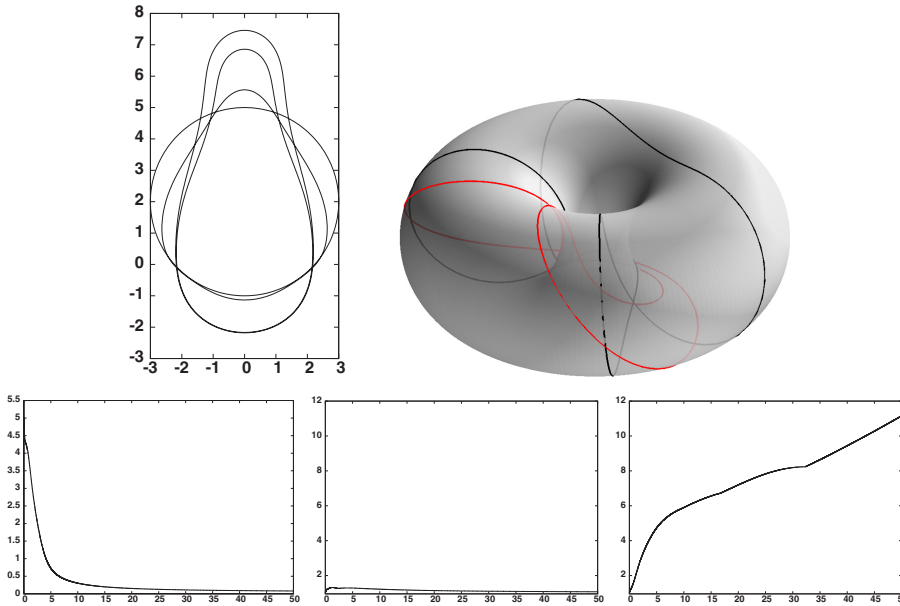
FIG. 2. $(\mathcal{P}_m)^h$ Geodesic elastic flow on a Clifford torus. The solutions $\vec{X}^m$ at times $t = 0, 1, 10, 50$. On the right we visualize $\vec{\Phi}(\vec{X}^m)$ at times $t = 0$ (red) and $t = 50$ (black) for (2.12e) with $\mathfrak{s} = 1$. Below is a plot of the discrete energy $W_{g,\lambda}^{m+1}$, as well as of the ratio (6.1) for $(\mathcal{P}_m)^h$ and $(\mathcal{Q}_m)^\star$.

11.2 for $(\mathcal{Q}_m)^\star$, while it is only 1.1 for $(\mathcal{P}_m)^h$; recall the equidistribution property from Remark 4.3. Repeating the experiment with $\lambda = 1$ gives the evolution shown in Figure 3. In the case $\lambda = 0$, the flow reduces the elastic energy and the absolute minimizer is given by geodesics which have geodesic curvature zero. However, in Figure 3 the elastic energy does not settle down to zero, and the curves instead seem to converge to a nontrivial critical point of the elastic energy. This is in accordance with the analysis in [17], which showed that in cases of hypersurfaces for which the Gaussian curvature is not nonnegative at all points, the set of free elasticae, i.e., the set of critical points, is much richer.

We end this section with some computations of geodesic elastic flow on the unit sphere, inspired, for example, by the numerical results presented in [10, Fig. 1–8] and [5, Fig. 36, 37]. To this end, we employ the metric induced by (2.12c), which means that $\vec{\Phi}(H)$, the surface on which we compute geodesic elastic flow, is the unit sphere without the north and the south pole. In particular, geodesic elastic flow evolutions on the unit sphere that pass through these poles cannot be computed with our formulation. We demonstrate this problem with a first simulation for (2.12c). The initial data is chosen such that elastic flow on the unit sphere would lead to a simple covering of a great circle. But as the curve would need to pass through the north pole, this represents a blow-up in H in finite time, and so our approximation cannot compute the evolution beyond the crossing of the pole. As initial data we choose a unit circle in H centered at $2\,\vec{e}_1$. For the simulation in Figure 4 we use the scheme $(\mathcal{P}_m)^h$ with the discretization parameters $J = 256$ and $\Delta t = 10^{-6}$. We observe that Γ^m expands into an ellipse-like shape in H, leading to a blow-up to infinity some time after $t = 0.005$. We remark that an alternative to the Mercator projection is given
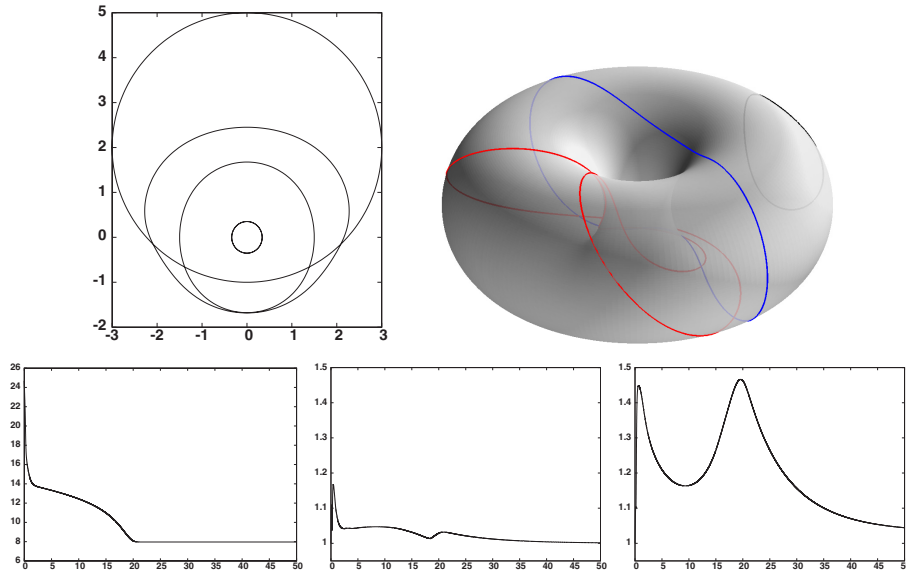
Fig. 3. $(\mathcal{P}_m)^h$ *Generalized geodesic elastic flow, with $\lambda = 1$, on a Clifford torus. The solutions $\vec{X}^m$ at times $t = 0, 1, 10, 30, 50$. On the right we visualize $\vec{\Phi}(\vec{X}^m)$ at times $t = 0$ (red), $t = 10$ (blue), and $t = 50$ (black) for (2.12e) with $\mathfrak{s} = 1$. Below is a plot of the discrete energy $W_{g,\lambda}^{m+1}$, as well as of the ratio (6.1) for $(\mathcal{P}_m)^h$ and $(\mathcal{Q}_m)^\star$.*
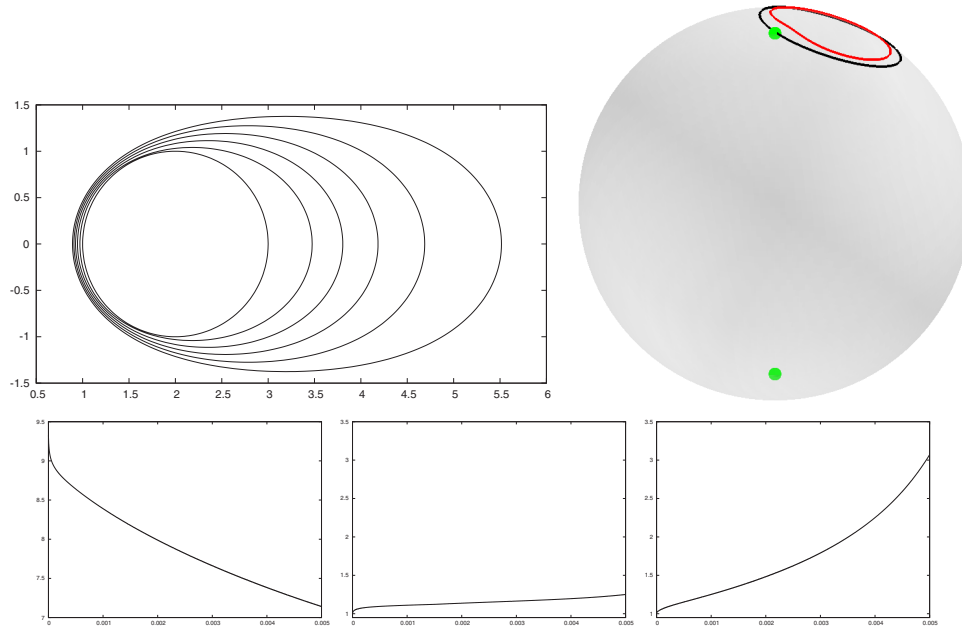

Fig. 4. $(\mathcal{P}_m)^h$ *Geodesic elastic flow on the unit sphere. The solutions $\vec{X}^m$ at times $t = 0, 0.001, \ldots, 0.005$. On the right we visualize $\vec{\Phi}(\vec{X}^m)$ at times $t = 0$ (red) and $t = 0.005$ (black) for (2.12c), with the two poles, $\pm\vec{e}_3$, represented by green dots. Below is a plot of the discrete energy $W_{g,\lambda}^{m+1}$, as well as of the ratio (6.1) for $(\mathcal{P}_m)^h$ and $(\mathcal{Q}_m)^\star$.*
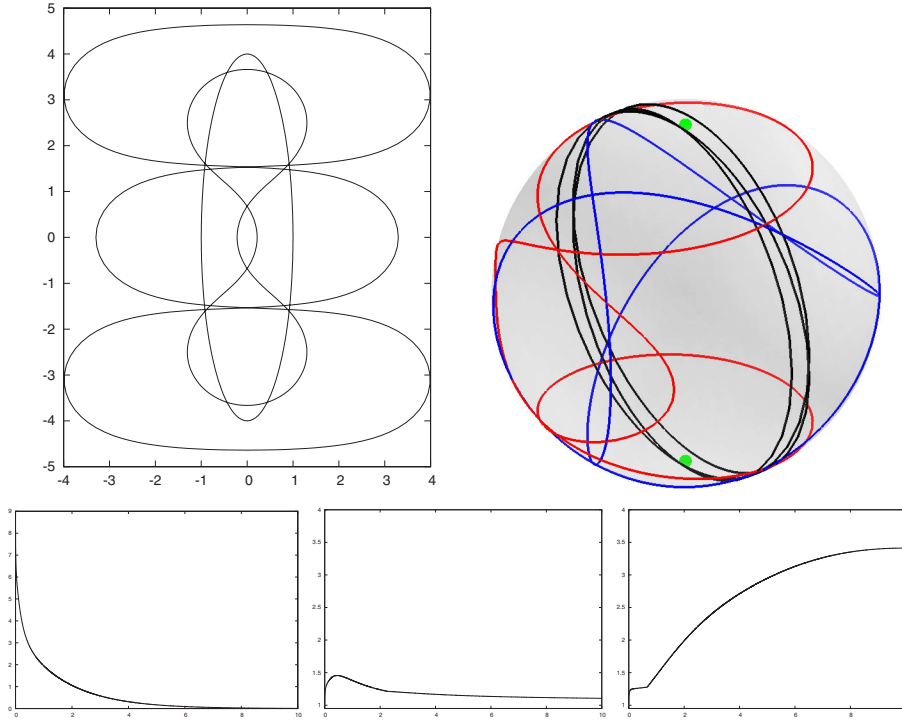
FIG. 5. $(\mathcal{P}_m)^h$ Geodesic elastic flow on the unit sphere. The solutions $\vec{X}^m$ at times $t = 0, 1, 10$. On the right we visualize $\vec{\Phi}(\vec{X}^m)$ at times $t = 0$ (red), $t = 1$ (blue), and $t = 10$ (black) for (2.12c), with the two poles, $\pm\vec{e}_3$, represented by green dots. Below is a plot of the discrete energy $W_{g,\lambda}^{m+1}$, as well as of the ratio (6.1) for $(\mathcal{P}_m)^h$ and $(\mathcal{Q}_m)^\star$.

by the stereographic projection of the unit sphere, so that only one of the two poles is missing, rather than two. This allows for the computation of a slightly larger class of evolutions on the unit sphere. Recall that the appropriate g is defined by (2.12b) with $\alpha = -1$.

A more involved simulation is shown in Figure 5, where we choose as initial data a 2×8 ellipse in H centered at the origin and use the scheme $(\mathcal{P}_m)^h$ with the discretization parameters $J = 256$ and $\Delta t = 10^{-3}$. We observe the evolution of the initial curve toward a triple covering of a great circle on the sphere. Note that eventually the solution would like to settle on the two poles, which would represent a singularity for the flow in H. In Figure 6 we show the same evolution for $\lambda = 0.4$. We note that the final shape is not a steady state. Of course, by considering a length-preserving variant, where the parameter λ depends on time, steady state solutions as shown in, e.g., [5, Fig. 36, 37] could also be produced by the numerical schemes presented here. However, as this goes beyond the scope of the paper, we omit such details and the corresponding evolutions here.

Conclusions. We have derived and analyzed two finite element schemes for the numerical approximation of elastic flow in two-dimensional Riemannian manifolds. The Riemannian manifolds that can be considered in our framework include the hyperbolic plane, the hyperbolic disk, and the elliptic plane. More generally, any metric conformal to the two-dimensional Euclidean metric can be considered. An example of
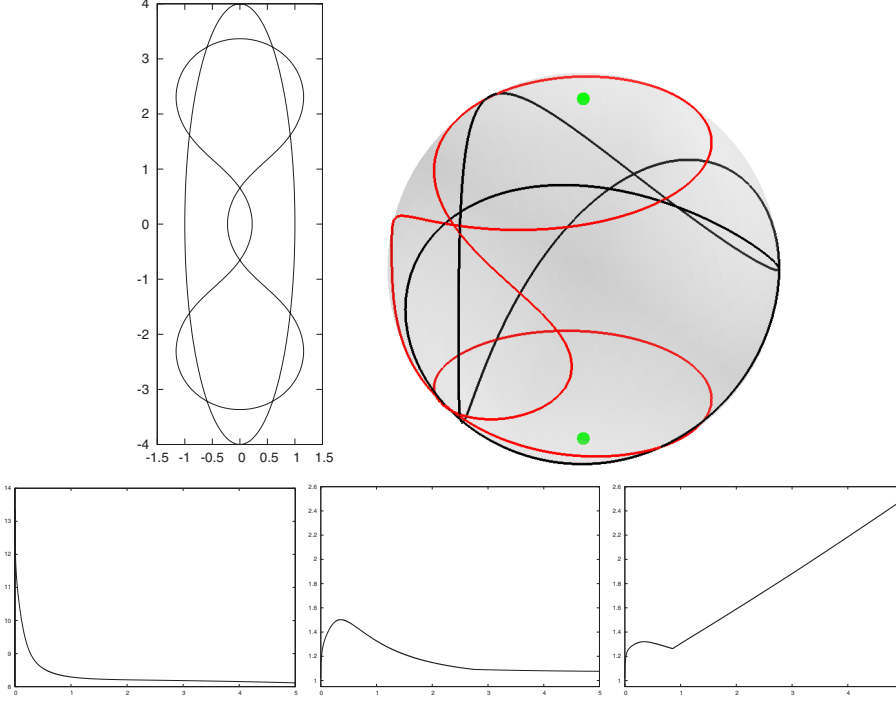
FIG. 6. $(\mathcal{P}_m)^h$ Generalized geodesic elastic flow, with $\lambda = 0.4$, on the unit sphere. The solutions $\vec{X}^m$ at times $t = 0, 5$. On the right we visualize $\vec{\Phi}(\vec{X}^m)$ at times $t = 0$ (red) and $t = 5$ (black) for (2.12c), with the two poles, $\pm \vec{e}_3$, represented by green dots. Below is a plot of the discrete energy $W_{g,\lambda}^{m+1}$, as well as of the ratio (6.1) for $(\mathcal{P}_m)^h$ and $(\mathcal{Q}_m)^\star$.

this is two-dimensional manifolds in $\mathbb{R}^d$, $d \geq 3$, which are conformally parameterized.

Our numerical simulations are based on the fully discrete schemes $(\mathcal{P}_m)^h$, $(\mathcal{Q}_m)^h$, and $(\mathcal{Q}_m)^\star$. Due to the nonoptimal convergence rates exhibited by $(\mathcal{Q}_m)^h$ in some numerical experiments, as well as numerical breakdown in others, we would advocate to use either $(\mathcal{P}_m)^h$ or $(\mathcal{Q}_m)^\star$ in practice. Here the former has the advantage that the vertices will be nearly equidistributed in practice and that the assembly of the linear systems is easier due the use of a mass-lumping quadrature.

Appendix A. Consistency of weak formulations. In this appendix we prove that solutions to $(\mathcal{P})$ and $(\mathcal{Q})$ indeed satisfy the strong form (2.7). Throughout this appendix we suppress the dependence of g on $\vec{x}$.

For later use we note, on recalling (2.1), (2.2), and (2.8), that

(A.1a) $$\vec{\nu}_s = -\varkappa \vec{\tau},$$

(A.1b) $$g^{-\frac{1}{2}} (g^{\frac{1}{2}})_s = -g^{\frac{1}{2}} (g^{-\frac{1}{2}})_s = \frac{1}{2} (\ln g)_s,$$

(A.1c) $$(\ln g)_s = \vec{\tau} \cdot \nabla \ln g,$$

(A.1d) $$(\ln g)_{ss} = \vec{\tau} \cdot (\nabla \ln g)_s + \vec{\tau}_s \cdot \nabla \ln g = \vec{\tau} \cdot (D^2 \ln g) \vec{\tau} + \varkappa \vec{\nu} \cdot \nabla \ln g,$$

(A.1e) $$(\vec{\nu} \cdot \nabla \ln g)_s = \vec{\nu} \cdot (\nabla \ln g)_s + \vec{\nu}_s \cdot \nabla \ln g = \vec{\nu} \cdot (D^2 \ln g) \vec{\tau} - \varkappa (\ln g)_s,$$

$$g^{-\frac{1}{2}} (\varkappa_g)_{ss} = g^{-\frac{1}{2}} (g^{\frac{1}{2}} (\varkappa_g)_{s_g})_s = g^{\frac{1}{2}} (\varkappa_g)_{s_g s_g} + g^{-\frac{1}{2}} (g^{\frac{1}{2}})_s (\varkappa_g)_{s_g}$$

(A.1f) $$= g^{\frac{1}{2}} (\varkappa_g)_{s_g s_g} + \frac{1}{2} (\ln g)_s (\varkappa_g)_{s_g} ,$$

(A.1g) $$-2\, g\, S_0(\vec{x}) = \Delta \ln g = \vec{\nu} \,.\, (D^2 \ln g)\, \vec{\nu} + \vec{\tau} \,.\, (D^2 \ln g)\, \vec{\tau} .$$

A.1. ($\mathcal{P}$). We note from (3.9), (2.6), and (A.1a) that

(A.2) $$\vec{y} \,.\, \vec{\nu} = \varkappa_g = g^{-\frac{1}{2}}(\vec{x}) \left[\varkappa - \frac{1}{2} \vec{\nu} \,.\, \nabla \ln g(\vec{x}) \right] \quad \text{and} \quad \vec{y}_s \,.\, \vec{\nu} = (\varkappa_g)_s + \varkappa \vec{y} \,.\, \vec{\tau} ,$$

and so it follows from (3.14), $\mathcal{V}_g = g^{\frac{1}{2}} \vec{x}_t \,.\, \vec{\nu}$, (2.4), (2.1), and (3.4b) that

(A.3)

$$
\left(g^{\frac{1}{2}} \mathcal{V}_g, \vec{\chi} \,.\, \vec{\nu} \,|\vec{x}_\rho|_g \right) = -\frac{1}{2} \left(g^{\frac{1}{2}} [\varkappa_g^2 + 2\,\lambda], \vec{\tau} \,.\, \vec{\chi}_\rho \right) + \frac{1}{4} \left(g^{\frac{1}{2}} [\varkappa_g^2 - 2\,\lambda], (\nabla \ln g) \,.\, \vec{\chi}\, |\vec{x}_\rho| \right)
$$

$$
+ \frac{1}{2} \left(\varkappa_g \, (D^2 \ln g)\, \vec{\nu}, \vec{\chi}\, |\vec{x}_\rho| \right) - \frac{1}{2} \left(\varkappa_g\, [\ln g]_s, \vec{\nu} \,.\, \vec{\chi}_\rho \right)
$$

$$
+ ([(\varkappa_g)_s + \varkappa \vec{y} \,.\, \vec{\tau}]\, \vec{\nu}, \vec{\chi}_\rho) + \left(\left[g^{\frac{1}{2}} \varkappa_g + \frac{1}{2} \vec{\nu} \,.\, \nabla \ln g \right] \vec{y}^\perp, \vec{\chi}_\rho \right)
$$

$$
= \frac{1}{2} \left(\left[g^{\frac{1}{2}} [\varkappa_g^2 - 2\,\lambda] + \varkappa_g \vec{\nu} \,.\, \nabla \ln g \right] \vec{\tau}, \vec{\chi}_\rho \right)
$$

$$
+ \left(\left[(\varkappa_g)_s + \varkappa \vec{y} \,.\, \vec{\tau} - \frac{1}{2} \varkappa_g \, (\ln g)_s - (\vec{y} \,.\, \vec{\tau})\, (g^{\frac{1}{2}} \varkappa_g + \frac{1}{2} \vec{\nu} \,.\, \nabla \ln g) \right] \vec{\nu}, \vec{\chi}_\rho \right)
$$

$$
+ \frac{1}{4} \left(g^{\frac{1}{2}} [\varkappa_g^2 - 2\,\lambda], (\nabla \ln g) \,.\, \vec{\chi}\, |\vec{x}_\rho| \right) + \frac{1}{2} \left(\varkappa_g\, (D^2 \ln g)\, \vec{\nu}, \vec{\chi}\, |\vec{x}_\rho| \right)
$$

$$
= \sum_{i=1}^{4} S_i(\vec{\chi}) \qquad \forall\, \vec{\chi} \in [H^1(I)]^2 .
$$

Combining (A.3), (2.6), integration by parts, (2.2), and (2.4) yields that

$$
S_1(\vec{\chi}) = -\frac{1}{2} \left(g^{\frac{1}{2}} \left[\varkappa_g^2 + 2\,\lambda - 2\, g^{-\frac{1}{2}} \varkappa \varkappa_g \right] \vec{\tau}, \vec{\chi}_\rho \right)
$$

$$
= \frac{1}{2} \left(\varkappa \left[\varkappa_g^2 + 2\,\lambda - 2\, g^{-\frac{1}{2}} \varkappa \varkappa_g \right] \vec{\nu}, \vec{\chi}\, |\vec{x}_\rho|_g \right)
$$

(A.4)
$$
+ \frac{1}{2} \left(g^{-\frac{1}{2}} \left[g^{\frac{1}{2}} \left[\varkappa_g^2 + 2\,\lambda - 2\, g^{-\frac{1}{2}} \varkappa \varkappa_g \right] \right]_s \vec{\tau}, \vec{\chi}\, |\vec{x}_\rho|_g \right) .
$$

Combining (A.3) and (A.2), on noting (2.6), (A.1a), (A.1f), and (2.4), yields that

$$
S_2(\vec{\chi}) = \left(\left[(\varkappa_g)_s - \frac{1}{2} \varkappa_g\, (\ln g)_s \right] \vec{\nu}, \vec{\chi}_\rho \right)
$$

$$
= -\left(g^{-\frac{1}{2}} \left[(\varkappa_g)_{ss} - \frac{1}{2} ((\ln g)_s \varkappa_g)_s \right] \vec{\nu}, \vec{\chi}\, |\vec{x}_\rho|_g \right)
$$

$$
+ \left(g^{-\frac{1}{2}} \varkappa \left[(\varkappa_g)_s - \frac{1}{2} (\ln g)_s \varkappa_g \right] \vec{\tau}, \vec{\chi}\, |\vec{x}_\rho|_g \right)
$$

$$
= -\left(g^{\frac{1}{2}} (\varkappa_g)_{s_g s_g} - \frac{1}{2} g^{-\frac{1}{2}} (\ln g)_{ss} \varkappa_g, \vec{\chi} \,.\, \vec{\nu}\, |\vec{x}_\rho|_g \right)
$$

(A.5)
$$
+ \left(\varkappa \left[(\varkappa_g)_{s_g} - \frac{1}{2} g^{-\frac{1}{2}} (\ln g)_s \varkappa_g \right], \vec{\chi} \,.\, \vec{\tau}\, |\vec{x}_\rho|_g \right) .
$$

Combining (A.3) and (2.6), on noting (2.4) and (A.1c), yields that

$$S_3(\vec{\chi}) = \frac{1}{4}\left(\varkappa_g^2 - 2\lambda, (\nabla \ln g).\vec{\chi}\,|\vec{x}_\rho|_g\right)$$

(A.6)
$$= \frac{1}{4}\left(\varkappa_g^2 - 2\lambda, \left[2\,(\varkappa - g^{\frac{1}{2}}\,\varkappa_g)\,\vec{\chi}.\vec{\nu} + (\ln g)_s\,\vec{\chi}.\vec{\tau}\right]|\vec{x}_\rho|_g\right).$$

It follows from (A.3) and (2.4) that

(A.7)
$$S_4(\vec{\chi}) = \frac{1}{2}\left(g^{-\frac{1}{2}}\,\varkappa_g\,(D^2\ln g)\,\vec{\nu}, [(\vec{\chi}.\vec{\nu})\,\vec{\nu} + (\vec{\chi}.\vec{\tau})\,\vec{\tau}]\,|\vec{x}_\rho|_g\right).$$

Choosing $\vec{\chi} = \chi\,\vec{\tau}$, for $\chi \in H^1(I)$, in (A.3), and noting (A.4), (A.5), (A.6), and (A.7), we obtain for the right-hand side of (A.3) the value

(A.8)

$$\sum_{i=1}^{4} S_i(\chi\,\vec{\tau}) = \frac{1}{2}\left(g^{-\frac{1}{2}}\left[g^{\frac{1}{2}}\left[\varkappa_g^2 + 2\lambda - 2\,g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g\right]\right]_s, \chi\,|\vec{x}_\rho|_g\right)$$

$$+ \left(\varkappa\left[(\varkappa_g)_{s_g} - \frac{1}{2}\,g^{-\frac{1}{2}}\,(\ln g)_s\,\varkappa_g\right], \chi\,|\vec{x}_\rho|_g\right) + \frac{1}{4}\left(\varkappa_g^2 - 2\lambda, (\ln g)_s\,\chi\,|\vec{x}_\rho|_g\right)$$

$$+ \frac{1}{2}\left(g^{-\frac{1}{2}}\,\varkappa_g\,\vec{\tau}.(D^2\ln g)\,\vec{\nu}, \chi\,|\vec{x}_\rho|_g\right)$$

$$= \frac{1}{2}\left(g^{-\frac{1}{2}}\,(g^{\frac{1}{2}})_s\left[\varkappa_g^2 + 2\lambda - 2\,g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g\right] + \left[\varkappa_g^2 - 2\,g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g\right]_s, \chi\,|\vec{x}_\rho|_g\right)$$

$$+ \left(\varkappa\left[(\varkappa_g)_{s_g} - \frac{1}{2}\,g^{-\frac{1}{2}}\,(\ln g)_s\,\varkappa_g\right] + \left(\frac{1}{4}\,\varkappa_g^2 - \frac{1}{2}\,\lambda\right)(\ln g)_s, \chi\,|\vec{x}_\rho|_g\right)$$

$$+ \frac{1}{2}\left(g^{-\frac{1}{2}}\,\varkappa_g\left[(\vec{\nu}.\nabla\ln g)_s + \varkappa\,(\ln g)_s\right], \chi\,|\vec{x}_\rho|_g\right)$$

$$= \frac{1}{2}\left((\ln g)_s\left[\varkappa_g^2 - g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g\right] + \left[\varkappa_g^2 - 2\,g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g\right]_s + 2\,g^{-\frac{1}{2}}\,\varkappa\,(\varkappa_g)_s, \chi\,|\vec{x}_\rho|_g\right)$$

$$+ \left(g^{-\frac{1}{2}}\,\varkappa_g\,(\varkappa - g^{\frac{1}{2}}\,\varkappa_g)_s, \chi\,|\vec{x}_\rho|_g\right)$$

$$= \frac{1}{2}\left((\ln g)_s\left[\varkappa_g^2 - g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g + g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g - \varkappa_g^2\right], \chi\,|\vec{x}_\rho|_g\right)$$

$$+ \left(\varkappa_g\,(\varkappa_g)_s + g^{-\frac{1}{2}}\left[-(\varkappa\,\varkappa_g)_s + \varkappa\,(\varkappa_g)_s + \varkappa_g\,\varkappa_s\right] - \varkappa_g\,(\varkappa_g)_s, \chi\,|\vec{x}_\rho|_g\right)$$

$$= 0\,,$$

as required, where we have recalled (A.1b) and (2.6).

Choosing $\vec{\chi} = \chi\,\vec{\nu}$, for $\chi \in H^1(I)$, in (A.3), and noting (A.4), (A.5), (A.6), and (A.7), we obtain

(A.9)

$$\left(g^{\frac{1}{2}}\,\mathcal{V}_g, \chi\,|\vec{x}_\rho|_g\right) = \sum_{i=1}^{4} S_i(\chi\,\vec{\nu}) = \frac{1}{2}\left(\varkappa\left[\varkappa_g^2 + 2\lambda - 2\,g^{-\frac{1}{2}}\,\varkappa\,\varkappa_g\right], \chi\,|\vec{x}_\rho|_g\right)$$

$$- \left(g^{\frac{1}{2}}\,(\varkappa_g)_{s_g s_g} - \frac{1}{2}\,g^{-\frac{1}{2}}\,(\ln g)_{ss}\,\varkappa_g, \chi\,|\vec{x}_\rho|_g\right)$$

$$+ \frac{1}{2}\left(\varkappa_g^2 - 2\lambda, (\varkappa - g^{\frac{1}{2}}\,\varkappa_g)\,\chi\,|\vec{x}_\rho|_g\right) + \frac{1}{2}\left(g^{-\frac{1}{2}}\,\varkappa_g\,\vec{\nu}.(D^2\ln g)\,\vec{\nu}, \chi\,|\vec{x}_\rho|_g\right)$$

$$= \left(-g^{\frac{1}{2}}\,(\varkappa_g)_{s_g s_g} + g^{\frac{1}{2}}\,\lambda\,\varkappa_g + \frac{1}{2}\,g^{-\frac{1}{2}}\left[\vec{\tau}.(D^2\ln g)\,\vec{\tau} + \vec{\nu}.(D^2\ln g)\,\vec{\nu}\right]\varkappa_g, \chi\,|\vec{x}_\rho|_g\right)$$

$$+ \frac{1}{2} \left(\varkappa \varkappa_g^2 - 2 g^{-\frac{1}{2}} \varkappa^2 \varkappa_g + g^{-\frac{1}{2}} \varkappa_g \varkappa \vec{\nu} \cdot \nabla \ln g + \varkappa_g^2 (\varkappa - g^{\frac{1}{2}} \varkappa_g), \chi |\vec{x}_\rho|_g \right)$$

$$= - \left(g^{\frac{1}{2}} \left[(\varkappa_g)_{s_g s_g} + \frac{1}{2} \varkappa_g^3 + (S_0(\vec{x}) - \lambda) \varkappa_g \right], \chi |\vec{x}_\rho|_g \right)$$

$$+ \left(g^{-\frac{1}{2}} \varkappa \varkappa_g \left[g^{\frac{1}{2}} \varkappa_g - \varkappa + \frac{1}{2} \vec{\nu} \cdot \nabla \ln g \right], \chi |\vec{x}_\rho|_g \right)$$

$$= - \left(g^{\frac{1}{2}} \left[(\varkappa_g)_{s_g s_g} + \frac{1}{2} \varkappa_g^3 + (S_0(\vec{x}) - \lambda) \varkappa_g \right], \chi |\vec{x}_\rho|_g \right) \qquad \forall \chi \in H^1(I),$$

where we have recalled (A.1d), (A.1g), and (2.6). Clearly, it follows from (A.9) that (2.7) holds.

A.2. ($\mathcal{Q}$). It follows from (3.26), $\mathcal{V}_g = g^{\frac{1}{2}} \vec{x}_t \cdot \vec{\nu}$, (3.20), (2.1), (2.4), and (3.4b) that

(A.10)

$$\left(g^{\frac{1}{2}} \mathcal{V}_g, \vec{\chi} \cdot \vec{\nu} |\vec{x}_\rho|_g \right) = - \frac{1}{2} \left(\varkappa_g^2 + 2 \lambda - \vec{y}_g \cdot \nabla \ln g, \left[\vec{\tau} \cdot \vec{\chi}_s + \frac{1}{2} \vec{\chi} \cdot \nabla \ln g \right] |\vec{x}_\rho|_g \right)$$

$$+ \frac{1}{2} \left((D^2 \ln g) \vec{y}_g, \vec{\chi} |\vec{x}_\rho|_g \right) + \left(\varkappa_g^2 + \frac{1}{2} (\vec{y}_g)_s \cdot \vec{\tau}, (\nabla \ln g) \cdot \vec{\chi} |\vec{x}_\rho|_g \right) + \left(g \varkappa_g \vec{y}_g^\perp, \vec{\chi}_\rho \right)$$

$$+ \left(g^{\frac{1}{2}} (\vec{y}_g)_s \cdot \vec{\nu}, \vec{\chi}_\rho \cdot \vec{\nu} \right)$$

$$= - \frac{1}{2} \left(g^{\frac{1}{2}} \left[\varkappa_g^2 + 2 \lambda - \vec{y}_g \cdot \nabla \ln g \right] \vec{\tau}, \vec{\chi}_\rho \right) + \frac{1}{2} \left((D^2 \ln g) \vec{y}_g, \vec{\chi} |\vec{x}_\rho|_g \right)$$

$$+ \left(\frac{1}{4} \left[3 \varkappa_g^2 - 2 \lambda + \vec{y}_g \cdot \nabla \ln g \right] + \frac{1}{2} (\vec{y}_g)_s \cdot \vec{\tau}, (\nabla \ln g) \cdot \vec{\chi} |\vec{x}_\rho|_g \right)$$

$$+ \left(g \varkappa_g \vec{y}_g^\perp, \vec{\chi}_\rho \right) + \left(g^{\frac{1}{2}} (\vec{y}_g)_s \cdot \vec{\nu}, \vec{\chi}_\rho \cdot \vec{\nu} \right)$$

$$= \frac{1}{2} \left(g^{\frac{1}{2}} \left[\varkappa_g^2 - 2 \lambda + \vec{y}_g \cdot \nabla \ln g \right] \vec{\tau}, \vec{\chi}_\rho \right) + \left(g^{\frac{1}{2}} [(\vec{y}_g)_s \cdot \vec{\nu}] \vec{\nu} - g \varkappa_g (\vec{y}_g \cdot \vec{\tau}) \vec{\nu}, \vec{\chi}_\rho \right)$$

$$+ \left(\frac{1}{4} \left[3 \varkappa_g^2 - 2 \lambda + \vec{y}_g \cdot \nabla \ln g + 2 (\vec{y}_g)_s \cdot \vec{\tau} \right], (\nabla \ln g) \cdot \vec{\chi} |\vec{x}_\rho|_g \right)$$

$$+ \frac{1}{2} \left((D^2 \ln g) \vec{y}_g, \vec{\chi} |\vec{x}_\rho|_g \right) = \sum_{i=1}^{4} T_i(\vec{\chi}) \qquad \forall \vec{\chi} \in [H^1(I)]^2 .$$

It follows from (2.6), (3.20), and (A.1c) that

$$\vec{y}_g \cdot \nabla \ln g = (\vec{y}_g \cdot \vec{\nu}) \vec{\nu} \cdot \nabla \ln g + (\vec{y}_g \cdot \vec{\tau}) \vec{\tau} \cdot \nabla \ln g$$

(A.11) $$= g^{-\frac{1}{2}} \varkappa_g 2 (\varkappa - g^{\frac{1}{2}} \varkappa_g) + \vec{y}_g \cdot \vec{\tau} (\ln g)_s = 2 g^{-\frac{1}{2}} \varkappa \varkappa_g - 2 \varkappa_g^2 + \vec{y}_g \cdot \vec{\tau} (\ln g)_s .$$

Combining (A.10), (A.11), (A.4), (2.2), and (2.4) yields that

(A.12)

$$T_1(\vec{\chi}) = - \frac{1}{2} \left(g^{\frac{1}{2}} \left[\varkappa_g^2 + 2 \lambda - 2 g^{-\frac{1}{2}} \varkappa \varkappa_g - \vec{y}_g \cdot \vec{\tau} (\ln g)_s \right] \vec{\tau}, \vec{\chi}_\rho \right)$$

$$= S_1(\vec{\chi}) + \frac{1}{2} \left(g^{\frac{1}{2}} \vec{y}_g \cdot \vec{\tau} (\ln g)_s \vec{\tau}, \vec{\chi}_\rho \right)$$

$$= S_1(\vec{\chi}) - \frac{1}{2} \left(\varkappa [\vec{y}_g \cdot \vec{\tau} (\ln g)_s] \vec{\nu}, \vec{\chi} |\vec{x}_\rho|_g \right) - \frac{1}{2} \left(g^{-\frac{1}{2}} \left[g^{\frac{1}{2}} [\vec{y}_g \cdot \vec{\tau} (\ln g)_s] \right]_s \vec{\tau}, \vec{\chi} |\vec{x}_\rho|_g \right) .$$

It follows from (3.20), (A.1a), and (A.1c) that

$$g^{\frac{1}{2}}(\vec{y}_g)_s \cdot \vec{\nu} = \left[g^{\frac{1}{2}}\vec{y}_g \cdot \vec{\nu}\right]_s - \vec{y}_g \cdot \left[g^{\frac{1}{2}}\vec{\nu}\right]_s = (\varkappa_g)_s - (\vec{y}_g \cdot \vec{\nu})(g^{\frac{1}{2}})_s + g^{\frac{1}{2}}\varkappa\,\vec{y}_g \cdot \vec{\tau}$$

$$(A.13) \qquad = (\varkappa_g)_s - \varkappa_g\,g^{-\frac{1}{2}}(g^{\frac{1}{2}})_s + g^{\frac{1}{2}}\varkappa\,\vec{y}_g \cdot \vec{\tau} = (\varkappa_g)_s - \frac{1}{2}(\ln g)_s\,\varkappa_g + g^{\frac{1}{2}}\varkappa\,\vec{y}_g \cdot \vec{\tau}.$$

Combining (A.10) and (A.13), on noting (A.1a), (A.1f), and (2.4), yields that

(A.14)

$$T_2(\vec{\chi}) = -\left(g^{-\frac{1}{2}}\left[g^{\frac{1}{2}}(\vec{y}_g)_s \cdot \vec{\nu} - g\,\varkappa_g\,(\vec{y}_g \cdot \vec{\tau})\right]_s \vec{\nu}, \vec{\chi}\,|\vec{x}_\rho|_g\right)$$

$$+ \left(\varkappa\left[(\vec{y}_g)_s \cdot \vec{\nu} - g^{\frac{1}{2}}\varkappa_g\,(\vec{y}_g \cdot \vec{\tau})\right]\vec{\tau}, \vec{\chi}\,|\vec{x}_\rho|_g\right)$$

$$= -\left(g^{-\frac{1}{2}}\left[(\varkappa_g)_{ss} - \frac{1}{2}((\ln g)_s\,\varkappa_g)_s + \left[\vec{y}_g \cdot \vec{\tau}\,(g^{\frac{1}{2}}\varkappa - g\,\varkappa_g)\right]_s\right]\vec{\nu}, \vec{\chi}\,|\vec{x}_\rho|_g\right)$$

$$+ \left(\varkappa\left[(\vec{y}_g)_s \cdot \vec{\nu} - g^{\frac{1}{2}}\varkappa_g\,(\vec{y}_g \cdot \vec{\tau})\right]\vec{\tau}, \vec{\chi}\,|\vec{x}_\rho|_g\right)$$

$$= -\left(g^{\frac{1}{2}}(\varkappa_g)_{s_g s_g} - \frac{1}{2}g^{-\frac{1}{2}}(\ln g)_{ss}\,\varkappa_g + g^{-\frac{1}{2}}\left[\vec{y}_g \cdot \vec{\tau}\,(g^{\frac{1}{2}}\varkappa - g\,\varkappa_g)\right]_s, \vec{\chi} \cdot \vec{\nu}\,|\vec{x}_\rho|_g\right)$$

$$+ \left(\varkappa\left[(\varkappa_g)_{s_g} - \frac{1}{2}g^{-\frac{1}{2}}(\ln g)_s\,\varkappa_g + \vec{y}_g \cdot \vec{\tau}\,(\varkappa - g^{\frac{1}{2}}\varkappa_g)\right], \vec{\chi} \cdot \vec{\tau}\,|\vec{x}_\rho|_g\right)$$

$$= S_2(\vec{\chi}) - \left(g^{-\frac{1}{2}}\left[\vec{y}_g \cdot \vec{\tau}\,(g^{\frac{1}{2}}\varkappa - g\,\varkappa_g)\right]_s, \vec{\chi} \cdot \vec{\nu}\,|\vec{x}_\rho|_g\right)$$

$$+ \left(\varkappa\,\vec{y}_g \cdot \vec{\tau}\,(\varkappa - g^{\frac{1}{2}}\varkappa_g), \vec{\chi} \cdot \vec{\tau}\,|\vec{x}_\rho|_g\right).$$

It follows from (2.2) and (3.20) that

$$(A.15) \qquad (\vec{y}_g)_s \cdot \vec{\tau} = (\vec{y}_g \cdot \vec{\tau})_s - \vec{y}_g \cdot \vec{\tau}_s = (\vec{y}_g \cdot \vec{\tau})_s - \varkappa\,\vec{y}_g \cdot \vec{\nu} = (\vec{y}_g \cdot \vec{\tau})_s - g^{-\frac{1}{2}}\varkappa\,\varkappa_g.$$

Combining (A.10), (A.11), (A.15), and (2.6) yields that

(A.16)

$$T_3(\vec{\chi}) = \frac{1}{4}\left(3\,\varkappa_g^2 - 2\,\lambda + \vec{y}_g \cdot \nabla\ln g + 2\,(\vec{y}_g)_s \cdot \vec{\tau}, (\nabla\ln g)\cdot\vec{\chi}\,|\vec{x}_\rho|_g\right)$$

$$= \frac{1}{4}\left(\varkappa_g^2 - 2\,\lambda + \vec{y}_g \cdot \vec{\tau}\,(\ln g)_s + 2\,(\vec{y}_g \cdot \vec{\tau})_s, (\nabla\ln g)\cdot\vec{\chi}\,|\vec{x}_\rho|_g\right)$$

$$= \frac{1}{4}\left(\varkappa_g^2 - 2\,\lambda + \vec{y}_g \cdot \vec{\tau}\,(\ln g)_s + 2\,(\vec{y}_g \cdot \vec{\tau})_s, [(\vec{\nu}\cdot\nabla\ln g)\,\vec{\chi}\cdot\vec{\nu} + (\vec{\tau}\cdot\nabla\ln g)\,\vec{\chi}\cdot\vec{\tau}]\,|\vec{x}_\rho|_g\right)$$

$$= \frac{1}{4}\left(\varkappa_g^2 - 2\,\lambda + \vec{y}_g \cdot \vec{\tau}\,(\ln g)_s + 2\,(\vec{y}_g \cdot \vec{\tau})_s, \left[2\,(\varkappa - g^{\frac{1}{2}}\varkappa_g)\,\vec{\chi}\cdot\vec{\nu} + (\ln g)_s\,\vec{\chi}\cdot\vec{\tau}\right]|\vec{x}_\rho|_g\right)$$

$$= S_3(\vec{\chi}) + \frac{1}{4}\left(\vec{y}_g \cdot \vec{\tau}\,(\ln g)_s + 2\,(\vec{y}_g \cdot \vec{\tau})_s, \left[2\,(\varkappa - g^{\frac{1}{2}}\varkappa_g)\,\vec{\chi}\cdot\vec{\nu} + (\ln g)_s\,\vec{\chi}\cdot\vec{\tau}\right]|\vec{x}_\rho|_g\right).$$

It follows from (3.20) that

$$(D^2\ln g)\,\vec{y}_g = \vec{y}_g \cdot \vec{\nu}\,(D^2\ln g)\,\vec{\nu} + \vec{y}_g \cdot \vec{\tau}\,(D^2\ln g)\,\vec{\tau}$$

$$(A.17) \qquad\qquad = g^{-\frac{1}{2}}\varkappa_g\,(D^2\ln g)\,\vec{\nu} + \vec{y}_g \cdot \vec{\tau}\,(D^2\ln g)\,\vec{\tau}.$$

Combining (A.10) and (A.17) yields that

$$T_4(\vec{\chi}) = \frac{1}{2} \left(g^{-\frac{1}{2}} \varkappa_g \, (D^2 \ln g) \, \vec{\nu} + \vec{y}_g \, . \, \vec{\tau} \, (D^2 \ln g) \, \vec{\tau}, \vec{\chi} \, |\vec{x}_\rho|_g \right)$$

$$= \frac{1}{2} \left(g^{-\frac{1}{2}} \varkappa_g \, (D^2 \ln g) \, \vec{\nu} + \vec{y}_g \, . \, \vec{\tau} \, (D^2 \ln g) \, \vec{\tau}, [(\vec{\chi} \, . \, \vec{\nu}) \, \vec{\nu} + (\vec{\chi} \, . \, \vec{\tau}) \, \vec{\tau}] \, |\vec{x}_\rho|_g \right)$$

$$\text{(A.18)} \qquad = S_4(\vec{\chi}) + \frac{1}{2} \left(\vec{y}_g \, . \, \vec{\tau} \, (D^2 \ln g) \, \vec{\tau}, [(\vec{\chi} \, . \, \vec{\nu}) \, \vec{\nu} + (\vec{\chi} \, . \, \vec{\tau}) \, \vec{\tau}] \, |\vec{x}_\rho|_g \right).$$

Choosing $\vec{\chi} = \chi \, \vec{\tau}$, for $\chi \in H^1(I)$, in (A.10), and noting (A.12), (A.14), (A.16), (A.18), and (A.8), we obtain for the right-hand side of (A.10) the value

$$\sum_{i=1}^{4} T_i(\chi \, \vec{\tau}) = \sum_{i=1}^{4} S_i(\chi \, \vec{\tau}) + \frac{1}{2} \left(-g^{-\frac{1}{2}} \, (g^{\frac{1}{2}})_s \, (\ln g)_s - (\ln g)_{ss}, (\vec{y}_g \, . \, \vec{\tau}) \, \chi \, |\vec{x}_\rho|_g \right)$$

$$+ \frac{1}{2} \left(2 \, \varkappa \, (\varkappa - g^{\frac{1}{2}} \, \varkappa_g) + \frac{1}{2} \, [(\ln g)_s]^2 + \vec{\tau} \, . \, (D^2 \ln g) \, \vec{\tau}, (\vec{y}_g \, . \, \vec{\tau}) \, \chi \, |\vec{x}_\rho|_g \right)$$

$$= \left(\varkappa \left[(\varkappa - g^{\frac{1}{2}} \, \varkappa_g) - \frac{1}{2} \, \vec{\nu} \, . \, \nabla \ln g \right], (\vec{y}_g \, . \, \vec{\tau}) \, \chi \, |\vec{x}_\rho|_g \right) = 0,$$

as required, where we have recalled (A.1b), (A.1d), and (2.6).

Choosing $\vec{\chi} = \chi \, \vec{\nu}$, for $\chi \in H^1(I)$, in (A.10), and noting (A.12), (A.14), (A.16), (A.18), and (A.9), we obtain

(A.19)

$$\left(g^{\frac{1}{2}} \, \mathcal{V}_g, \chi \, |\vec{x}_\rho|_g \right) = \sum_{i=1}^{4} T_i(\chi \, \vec{\nu})$$

$$= \sum_{i=1}^{4} S_i(\chi \, \vec{\nu}) + \frac{1}{2} \left(-\varkappa \, (\ln g)_s - 2 \, g^{-\frac{1}{2}} \left[g^{\frac{1}{2}} \, (\varkappa - g^{\frac{1}{2}} \, \varkappa_g) \right]_s, (\vec{y}_g \, . \, \vec{\tau}) \, \chi \, |\vec{x}_\rho|_g \right)$$

$$+ \frac{1}{2} \left((\ln g)_s \, (\varkappa - g^{\frac{1}{2}} \, \varkappa_g) + (\vec{\nu} \, . \, \nabla \ln g)_s + \varkappa \, (\ln g)_s, (\vec{y}_g \, . \, \vec{\tau}) \, \chi \, |\vec{x}_\rho|_g \right)$$

$$= - \left(g^{\frac{1}{2}} \left[(\varkappa_g)_{s_g s_g} + \frac{1}{2} \, \varkappa_g^3 + (S_0(\vec{x}) - \lambda) \, \varkappa_g \right], \chi \, |\vec{x}_\rho|_g \right) \qquad \forall \, \chi \in H^1(I),$$

where we have recalled (A.1b) and (2.6). Clearly, it follows from (A.19) that (2.7) holds.

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