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Combinatorial properties of multidimensional continued fractions

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ABSTRACT

The study of combinatorial properties of mathematical objects is a very important research field and continued fractions have been deeply studied in this sense. However, multidimensional continued fractions, which are a generalization arising from an algorithm due to Jacobi, have been poorly investigated in this sense, up to now. In this paper, we propose a combinatorial interpretation of the convergents of multidimensional continued fractions in terms of counting some particular tilings, generalizing some results that hold for classical continued fractions.

1. Introduction

Multidimensional continued fractions were introduced by Jacobi [15] (and then generalized by Perron [20]) in the attempt to answer a problem posed by Hermite [13] who asked for an algorithm that provides periodic representations for algebraic irrationals of any degree, in the same way as continued fractions are periodic if and only if they represent quadratic irrationals. Unfortunately, the Jacobi–Perron algorithm does not solve the problem, which is still a beautiful open problem in number theory, but opened a new and rich research field. Indeed, there are many studies about multidimensional continued fractions and their modifications, aiming to generalize the results and properties of classical continued fractions.

Continued fractions have been widely studied from different points of view. Several works explore the combinatorial properties of continued fractions giving many interesting interpretations. In the book of Benjamin and Quinn [3], one chapter is devoted to show that numerators and denominators of convergents of continued fractions count some particular tilings, reporting also some results proved in [4]. In [10], the author extends these results studying tilings where the height conditions for the tiles can also be negative and the number of these tilings is computed by means of convergents of negative continued fractions. In [2], the authors provide further results regarding the properties of continued fractions in terms of counting tilings and give also a combinatorial interpretation to the expansion of e. Different works on combinatorial interpretations of continued fractions can be found in [7–9,11,12,16,18,19,21,22].

Regarding multidimensional continued fractions, there are just few works about their combinatorial properties. In [14], the Jacobi–Perron algorithm is used for giving a generating method of the so-called stepped surfaces. In [5], the authors used multidimensional continued fractions to obtain a method for the generation of discrete segments in the three-dimensional space. Finally, in [1], multidimensional continued fractions have been exploited for obtaining results about tilings, discrete approximations of lines and planes, and Markov partitions for toral automorphism.

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In this paper, we generalize the results of [4] and [10] giving a combinatorial interpretation for multidimensional continued fractions in terms of counting the number of tilings of a board using tiles of length one, two or three, where we can also stack such tiles. We also give an interpretation to negative conditions for the height of the stacks. In particular, Section 2 is devoted to the preliminary definitions and properties of multidimensional continued fractions, where we also introduce them from a formal point of view. Section 3 presents the main results.

2. Preliminaries

Definition 2.1. Let $(a_i)_{i\geq 0}, (b_i)_{i\geq 0}, (c_i)_{i\geq 0}$ be three sequences of integers, we define A_j^n, B_j^n, C_j^n, for any $j, n \in \mathbb{N}$ with $n \geq j$, by the following product of matrices:

$$\begin{pmatrix} A_j^n & A_j^{n-1} & A_j^{n-2} \\ B_j^n & B_j^{n-1} & B_j^{n-2} \\ C_j^n & C_j^{n-1} & C_j^{n-2} \end{pmatrix} := \prod_{i=j}^{n} \begin{pmatrix} a_i & 1 & 0 \\ b_i & 0 & 1 \\ c_i & 0 & 0 \end{pmatrix}. \tag{1}$$

The sequence (A_j^n) satisfies some recurrence properties that will be exploited later. In particular, by (1), we get

$$A_0^n = a_n A_0^{n-1} + b_n A_0^{n-2} + c_n A_0^{n-3} \tag{2}$$

for all $n \geq 3$, since

$$\begin{pmatrix} A_0^n & A_0^{n-1} & A_0^{n-2} \\ B_0^n & B_0^{n-1} & B_0^{n-2} \\ C_0^n & C_0^{n-1} & C_0^{n-2} \end{pmatrix} = \begin{pmatrix} A_0^{n-1} & A_0^{n-2} & A_0^{n-3} \\ B_0^{n-1} & B_0^{n-2} & B_0^{n-3} \\ C_0^{n-1} & C_0^{n-2} & C_0^{n-3} \end{pmatrix} \begin{pmatrix} a_n & 1 & 0 \\ b_n & 0 & 1 \\ c_c & 0 & 0 \end{pmatrix}.$$

Moreover, from

$$\begin{pmatrix} A_0^n & A_0^{n-1} & A_0^{n-2} \\ B_0^n & B_0^{n-1} & B_0^{n-2} \\ C_0^n & C_0^{n-1} & C_0^{n-2} \end{pmatrix} = \begin{pmatrix} a_0 & 1 & 0 \\ b_0 & 0 & 1 \\ c_0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1^n & A_1^{n-1} & A_1^{n-2} \\ B_1^n & B_1^{n-1} & B_1^{n-2} \\ C_1^n & C_1^{n-1} & C_1^{n-2} \end{pmatrix}$$

we obtain

$$C_0^n = c_0 A_1^n, \quad B_0^n = b_0 A_1^n + C_1^n, \quad A_0^n = a_0 A_1^n + B_1^n \tag{3}$$

and by substitutions we get

$$A_0^n = a_0 A_1^n + b_1 A_2^n + c_2 A_3^n. \tag{4}$$

When $c_i = 1$, for all $i \geq 0$, the sequences $(A_0^n)_{n\geq 0}, (B_0^n)_{n\geq 0}, (C_0^n)_{n\geq 0}$ are numerators and denominators of convergents of a multidimensional continued fraction (of dimension two). Given a pair of real numbers (α_0, β_0) the Jacobi algorithm evaluates a pair of sequences of integers $(a_i)_{i\geq 0}, (b_i)_{i\geq 0}$ as follows:

$$\begin{cases} a_i = \lfloor \alpha_i \rfloor \\ b_i = \lfloor \beta_i \rfloor \\ \alpha_{i+1} = \dfrac{1}{\beta_i - b_i} & \qquad i = 0, 1, 2, \ldots \\ \beta_{i+1} = \dfrac{\alpha_i - a_i}{\beta_i - b_i} \end{cases}$$

The elements of the sequences of real numbers $(\alpha_i)_{i\geq 0}, (\beta_i)_{i\geq 0}$ satisfy the following relations:

$$\alpha_i = a_i + \frac{\beta_{i+1}}{\alpha_{i+1}}, \quad \beta_i = b_i + \frac{1}{\alpha_{i+1}} \tag{5}$$

for all $i \geq 0$, from which

$$\alpha_0 = a_0 + \frac{\beta_1}{\alpha_1} = a_0 + \frac{b_1 + \dfrac{1}{\alpha_2}}{a_1 + \dfrac{\beta_2}{\alpha_2}} = \ldots$$

$$\beta_0 = b_0 + \frac{\beta_1}{\alpha_1} = b_0 + \cfrac{1}{a_1 + \cfrac{\beta_2}{\alpha_2}} = \dots$$

Then, it follows that

$$\alpha_0 = \frac{\alpha_i A_0^{i-1} + \beta_i A_0^{i-2} + A_0^{i-3}}{\alpha_i C_0^{i-1} + \beta_i C_0^{i-2} + C_0^{i-3}}, \quad \beta_0 = \frac{\alpha_i B_0^{i-1} + \beta_i B_0^{i-2} + B_0^{i-3}}{\alpha_i C_0^{i-1} + \beta_i C_0^{i-2} + C_0^{i-3}},$$

for all $i \geq 0$, where the A_0^i, B_0^i, C_0^i are as in Definition 2.1 (with $c_i = 1$ for all $i \geq 0$) and we observe that $A_0^{-1} = 1, A_0^{-2} = 0, B_0^{-1} = 0, B_0^{-2} = 1, C_0^{-1} = 0, C_0^{-2} = 0$ by (1) when $j = n = 0$. Moreover, equation (5) determines the multidimensional continued fraction that has the following expansion

$$\alpha_0 = a_0 + \cfrac{b_1 + \cfrac{1}{a_2 + \cfrac{b_3 + \cfrac{1}{\ddots}}{a_3 + \cfrac{\ddots}{\ddots}}}}{a_1 + \cfrac{b_2 + \cfrac{1}{a_3 + \cfrac{\ddots}{\ddots}}}{a_2 + \cfrac{b_3 + \cfrac{1}{\ddots}}{a_3 + \cfrac{\ddots}{\ddots}}}}, \quad \beta_0 = b_0 + \cfrac{1}{a_1 + \cfrac{b_2 + \cfrac{1}{a_3 + \cfrac{\ddots}{\ddots}}}{a_2 + \cfrac{b_3 + \cfrac{1}{\ddots}}{a_3 + \cfrac{\ddots}{\ddots}}}}$$

which is usually written shortly as $[(a_0, a_1, \dots), (b_0, b_1, \dots)]$ and

$$[(a_0, \dots, a_i), (b_0, \dots, b_i)] = \left(\frac{A_0^i}{C_0^i}, \frac{B_0^i}{C_0^i} \right),$$

for all $i \geq 0$. For more details about multidimensional continued fractions and the Jacobi–Perron algorithm (which is the generalization of the Jacobi algorithm to higher dimensions) we refer the reader to [6].

Multidimensional continued fractions can be introduced also in a formal way (see, e.g., [17, Section 2]), where the partial quotients are not in general obtained by an algorithm and the numerators are not necessarily equal to 1, as well as in the classical case. In fact, given two sequences $(a_i)_{i\geq 0}$, $(b_i)_{i\geq 0}$, one can introduce and study the continued fraction

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots}}.$$

In our case, we deal with three sequences of integers $(a_i)_{i\geq 0}, (b_i)_{i\geq 0}, (c_i)_{i\geq 0}$ and three sequences of real numbers $(\alpha_i)_{i\geq 0}, (\beta_i)_{i\geq 0}, (\gamma_i)_{i\geq 0}$ subjected to the relations

$$\alpha_i = a_i + \frac{\beta_i}{\alpha_i}, \quad \beta_i = b_i + \frac{\gamma_i}{\alpha_i}, \quad \gamma_i = c_i$$

for any $i \geq 0$. These relations determine the multidimensional continued fraction that has the following expansion

$$\alpha_0 = a_0 + \cfrac{b_1 + \cfrac{c_2}{a_2 + \cfrac{b_3 + \cfrac{c_4}{\ddots}}{a_3 + \cfrac{\ddots}{\ddots}}}}{a_1 + \cfrac{b_2 + \cfrac{c_3}{a_3 + \cfrac{\ddots}{\ddots}}}{a_2 + \cfrac{b_3 + \cfrac{c_4}{\ddots}}{a_3 + \cfrac{\ddots}{\ddots}}}}, \quad \beta_0 = b_0 + \cfrac{c_1}{a_1 + \cfrac{b_2 + \cfrac{c_3}{a_3 + \cfrac{\ddots}{\ddots}}}{a_2 + \cfrac{b_3 + \cfrac{c_4}{\ddots}}{a_3 + \cfrac{\ddots}{\ddots}}}}, \tag{6}$$

which can be shortly written as $[(a_0, a_1, \ldots), (b_0, b_1, \ldots), (c_0, c_1, \ldots)]$. Then, we have

$$[(a_0, \ldots, a_i), (b_0, \ldots, b_i), (c_0, \ldots, c_i)] = \left(\frac{A_0^i}{C_0^i}, \frac{B_0^i}{C_0^i} \right),$$

where the A_0^i, B_0^i, C_0^i are as in Definition 2.1, for all $i \geq 0$. Since $\frac{A_0^0}{C_0^0} = a_0$ and $\frac{B_0^0}{C_0^0} = b_0$, it is a natural choice to set $c_0 = 1$. Moreover, we would like to highlight that A_0^n does not depend on b_0, c_0, c_1 and B_0^n does not depend on a_0, b_1, c_0, c_2.

3. Counting the number of tilings using multidimensional continued fractions

In this section, we give a combinatorial interpretation to the convergents of a multidimensional continued fraction in terms of tilings of some boards, extending the approach of Benjamin and Quinn [3,4] for the classical continued fractions.

In the following, a $(n+1)$–board is a $1 \times (n+1)$ chessboard (composed by $n+1$ cells), a *square* is a 1×1 tile, a *domino* is a 1×2 tile and a *bar* is a 1×3 tile. The $n+1$ cells of the $(n+1)$–board are labeled from 0 to n (i.e., we refer to the cell of position 0 for the first cell and so on). Thus, a square can cover a single cell of the chessboard, a domino can cover two consecutive cells of the chessboard and a bar three consecutive cells of the chessboard. A *tiling* of a n–board is a full covering of the chessboard using squares, dominoes and bars that can also be stacked. The *height conditions* for stacking the tiles are given by the finite sequences $(a_0, \ldots, a_n)$, $(b_0, \ldots, b_n)$ and $(c_0, \ldots, c_n)$, where

- a_i denotes the number of squares that may be stacked in the i–th position of the $(n+1)$–board (e.g., a_0 is the number of stackable squares in the cell of position 0), for any $i \geq 0$;
- b_i denotes the number of dominoes that may be stacked to cover the positions $i-1$ and i of the $(n+1)$–board (e.g., b_1 is the number of stackable dominoes covering the positions 0 and 1), for any $i \geq 1$;
- c_i denotes the number of bars that may be stacked to cover the positions $i-2$, $i-1$ and i of the $(n+1)$–board (e.g., c_2 is the number of stackable dominoes covering the positions 0, 1 and 2), for any $i \geq 2$.

In this setting the elements b_0, c_0, c_1 do not give height conditions, however their role will be important later when discussing different types of tilings. We will denote by M_0^n the number of possible tilings of a $(n+1)$–board with height conditions $(a_0, \ldots, a_n)$, $(b_0, \ldots, b_n)$, $(c_0, \ldots, c_n)$, for any $n \geq 0$.

When we allow the first and the last tile of a chessboard to be bordering, i.e., position n next to 0, we call it a *circular* $(n+1)$*-board* and the tiling covering it a *circular tiling*.

Example 3.1. Consider $n = 5$ with the following height conditions

$$(1, 2, 3, 2, 3, 2), \quad (b_0, 6, 5, 4, 3, 2), \quad (c_0, c_1, 1, 2, 3, 1),$$

where we do not explicit the values of b_0, c_0, c_1 since, in this case, they are not relevant for the tilings. Examples of valid tilings are represented in Fig. 1, while in Fig. 2 is represented a non-valid tiling for these height conditions: in this case there are too many bars covering the last three cells. Given the above sequences of partial quotients, the sequence of the

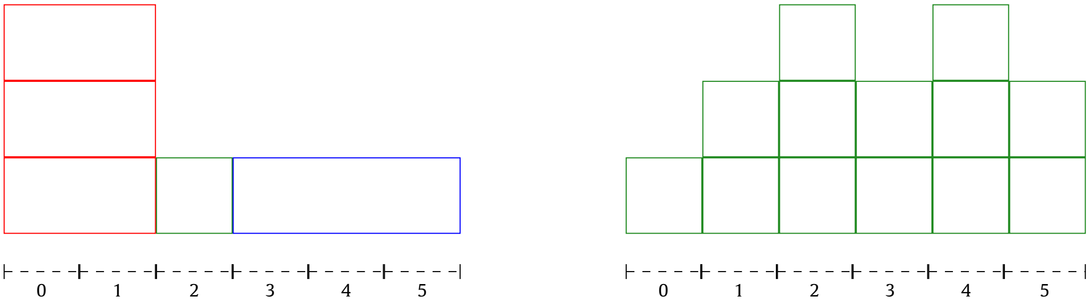


Fig. 1. Examples of valid tilings.

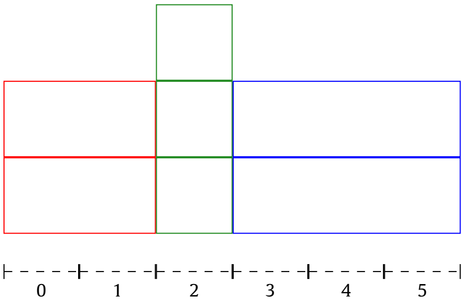


Fig. 2. Example of a non-valid tiling.

convergents $\left(\frac{A_0^n}{C_0^n}\right)$ is the following:

n	0	1	2	3	4	5
Convergents	1	4	$\frac{30}{11}$	$\frac{47}{16}$	$\frac{44}{15}$	$\frac{202}{69}$

In this case, we have not specified the value of b_0, since it does not provide any height condition, and consequently we can not write explicitly the sequence of convergents $\left(\frac{B_0^n}{C_0^n}\right)$.

Theorem 3.1. *Given three finite sequences of integers* $(a_0, \dots, a_n)$, $(b_0, \dots, b_n)$, $(c_0, \dots, c_n)$ *and* A_0^n, B_0^n, C_0^n *as in Definition 2.1, then*

i) A_0^n *counts the number of possible tilings of a* $(n+1)$*–board with height conditions* $(a_0, \dots, a_n)$, $(b_0, \dots, b_n)$, $(c_0, \dots, c_n)$.
ii) B_0^n *counts the number of possible tilings of a* $(n+2)$*–board, where the first cell is labeled with -1 (i.e., we add a cell on the left to a* $(n+1)$*–board), with height conditions* $(0, a_0, \dots, a_n)$, $(b_{-1}, b_0, \dots, b_n)$, $(c_{-1}, c_0, \dots, c_n)$, *i.e., such that the first tile of the tiling is a stack of dominoes or bars.*
iii) C_0^n *counts the number of possible tilings of a n–board with height conditions* $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$, $(c_1, \dots, c_n)$, *where the first cell is labeled with 1, i.e. removing the first cell from a* $n+1$*–board.*

Proof. i) We want to show that the sequences $(M_0^i)_{i\geq 0}$ and $(A_0^i)_{i\geq 0}$ have the same initial values and satisfy the same recurrence. Clearly, for a 1–board we have

$$M_0^0 = a_0 = A_0^0,$$

and for a 2–board

$$M_0^1 = a_0 a_1 + b_1 = A_0^1.$$

Then for a 3–board we can have tilings with:
- three stacks of squares,
- a stack of squares in the first cell and a stack of dominoes covering the second and third positions,
- a stack of dominoes covering the first and second cells and a stack of squares in the third position,
- a stack of bars.

Thus,

$$M_0^2 = a_0a_1a_2 + a_0b_2 + a_2b_1 + c_2 = A_0^2.$$

For a $(n+1)$–board, with $n > 2$, we can observe that the number of tilings satisfies the following recursive formula:

$$M_0^n = a_0M_1^n + b_1M_2^n + c_2M_3^n,$$

since we can count the tilings dividing them in three sets: tilings that start with a stack of squares, tilings that start with a stack of dominoes and tilings that start with a stack of bars. Thus, the number of tilings of a $(n+1)$–board starting with a stack of squares is $a_0M_1^n$ and similarly for the other two situations. Finally using (4), we obtain that $M_0^n = A_0^n$.

ii) In this case, we want to show that the sequences $(M_{-1}^n)_{n\geq -1}$ and $(B_0^n)_{n\geq -1}$ have the same initial conditions and satisfy the same recurrence. We can observe that if the board has only one cell (i.e., the -1 cell) there are no possible tilings (since $a_{-1} = 0$), then $M_{-1}^{-1} = 0$ and this is consistent with $B_0^{-1} = 0$ (see Section 2). Moreover, $M_{-1}^0 = b_0 = B_0^0$, since we can tile the 2–board only with a domino. Similarly, $M_{-1}^1 = b_0a_1 + c_1 = B_0^1$, because we only have two possibilities:
- a tiling composed by one domino (in cells -1 and 0) for which we have b_0 possibilities and one square (in the cell 1) for which we have a_1 possibilities;
- a tiling composed by one bar (in the cells $-1, 0, 1$) for which we have c_1 possibilities.

Finally, we can observe that the number of tilings for a $(n+2)$–board, with $n > 1$, satisfies the recurrence

$$M_{-1}^n = b_0M_1^n + c_1M_2^n = b_0A_1^n + c_1A_2^n = b_0A_1^n + C_1^n = B_0^n,$$

since in this case we may start the tiling either with a stack of dominoes in positions $-1, 0$, which are exactly $b_0A_1^n$, or a stack of bars in positions $-1, 0, 1$, giving a contribution of $c_1A_2^n$. The result follows from the previous point and (3).

iii) From (3), considering $c_0 = 1$, we have $C_0^n = A_1^n$, for all $n \geq 1$. Moreover, from i), we have that A_1^n counts the number of possible tilings of a n–board with height conditions $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$, $(c_1, \dots, c_n)$. □

With the same notation as Theorem 3.1 we have the following corollary for a $(n+1)$–circular board.

Corollary 3.2. *The number of circular tilings of a $(n+1)$–circular board with height conditions $(a_0, a_1, \dots, a_n)$, $(b_0, b_1 \dots, b_n)$, $(0, c_1 \dots, c_n)$ (i.e. we forbid bars covering the cells $0, n, n-1$) is $A_0^n + B_0^{n-1}$.*

Proof. The previous theorem ensures that A_0^n counts the number of all the possible tilings where the cells 0 and n are not covered by the same stack of dominoes or bars. The tilings that are missing are the ones where a stack of dominoes covers 0 and n or a stack of bars covers $1, 0, n$, (the only other possible case, where a stack of bars covers $0, n, n-1$ is impossible since $c_0 = 0$). In particular we may suppose that in both cases the stack begins in the n–th cell. By the previous theorem B_0^{n-1} counts the number of tilings of a $n+1$ board starting from cell -1, which does not start with a stack of squares. We can notice that this is the same as considering a $(n-1)$–board where the cell -1 is replaced with the cell n. □

3.1. *Negative dominoes and bars*

Now, we want to generalize Theorem 3.1 in order to allow negative values for b_i, c_i, following the ideas of [10].

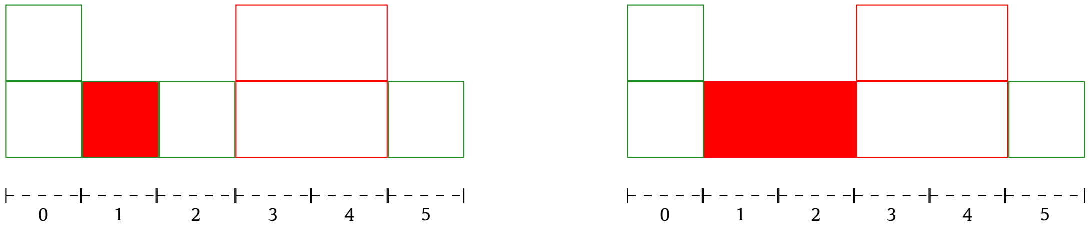
We notice that a positive b_i adds b_i number of ways to tile cells $i-1, i$. So a natural way to explain negative coefficient is to impose some restrictions such that a negative b_i give us $|b_i|$ less way to cover the cells $i-1, i$. More specifically, the idea is to discard those tilings which have a full stack of squares on the cell $i-1$ (i.e. exactly a_{i-1} squares) and up to $|b_i|$ squares in the cell k. An analogous argument can be done for c_i.

Definition 3.1 *(Mixed Tiling).* Let $(a_i)_{i\geq 0}$ be a sequence of positive integers and $(b_i)_{i\geq 0}$, $(c_i)_{i\geq 0}$ be sequences of integers such that

- if $b_i < 0$ and $c_i > 0$, then $a_i > |b_i|$;
- if $b_i > 0$ and $c_i < 0$, then either $a_i > |c_i|$ or $b_i > |c_i|$;
- if $b_i < 0$ and $c_i < 0$, then $a_i > |b_i| + |c_i|$.

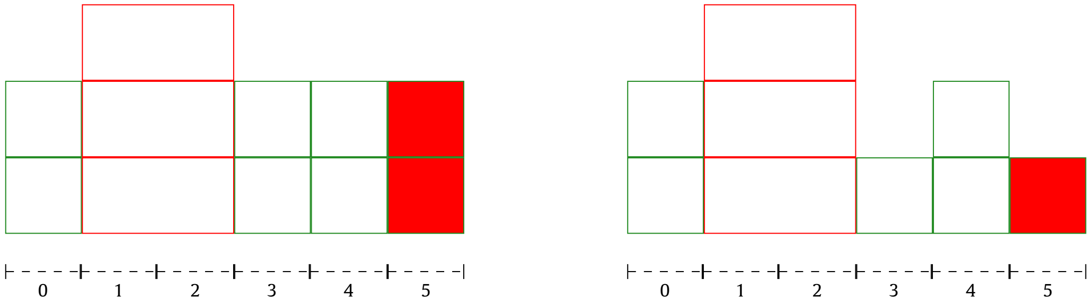
We define a *mixed tiling* of an $(n+1)$—board with height condition respectively given by $(a_0, \dots a_n)$, $(b_0, \dots b_n)$ and $(c_0, \dots c_n)$, as follows: for any $k \in \mathbb{N}$,

1. if $b_k \geq 0$ and $c_k \geq 0$, we fall back in the same case defined at the beginning of Section 3;
2. if $b_k < 0$ and $c_k > 0$, when there is a full stack of exactly a_{k-1} squares in the cell $k-1$, we discard all the tilings having up to $|b_k|$ squares in the cell k and we refer to them as *negligible* tilings;
3. if $c_k < 0$ and $b_k > 0$, we have two cases:



(a) Negligible tiling (case 2).

(b) Negligible tiling (case 3b).



(c) Negligible tiling (case 4a).

(d) Negligible tiling (case 4b).

Fig. 3. Some examples of negligible tilings.

  (a) if $a_k > |c_k|$, when there is a full stack of a_{k-2} squares in the cell $k-2$ and a full stack of a_{k-1} squares in the cell $k-1$, then we consider as negligible all the tilings having up to $|c_k|$ squares in the cell k;
  (b) otherwise, necessarily $b_k > |c_k|$. In this case when there is a full stack of a_{k-2} squares in the cell $k-2$, the negligible tilings are those with up to $|c_k|$ dominoes covering the cells $k-1, k$;

4. if $c_k < 0$ and $b_k < 0$, we have two cases:
   (a) when at the same time there is a full stack of a_{k-2} squares in the cell $k-2$ and a full stack of a_{k-1} squares in the cell $k-1$, we discard all the tilings having up to $|c_k| + |b_k|$ squares in the cell k;
   (b) when there is a full stack of a_{k-1} squares in the cells $k-1$, the negligible tilings have up to $|b_k|$ squares in the cell k.

Remark 1. Notice that the condition 4b applies when there are less than a_{k-2} squares in the cell $k-2$ to compensate the negative b_k in the case 4a.

Example 3.2. Consider the height conditions given by

$$(2, 3, 1, 2, 2, 3), \quad (b_0, -1, 3, 3, 2, -1), \quad (c_0, c_1, -2, 2, 1, -1).$$

In this case there are several restrictions given by these choice of conditions:

- Since $b_1 = -1$, then when we have $a_0 = 2$ squares in position 0, we need to exclude all the tilings having one square in the cell in position 1 (see Fig. 3a).
- Since $c_2 = -2$ and $a_2 = 1 < |c_2|$, then we are in the case 3b and we need to exclude the tilings having two squares in position 0 and 1 or two dominoes in the positions 1 and 2 (see Fig. 3b).
- Finally, $b_5 = c_5 = -1$ so we are in the fourth case. Therefore the negligible tilings are those having $a_3 = 2$ squares in position 3, $a_4 = 2$ squares in position 4 and one or two squares in position 5 (see Fig. 3c). Moreover we also need to discard the tilings having $a_4 = 2$ squares in position 4 and $|b_5| = 1$ square in position 5 (see Fig. 3d).

Theorem 3.3. *Consider the height conditions given by* $(a_0, \ldots, a_n)$, $(b_0, \ldots, b_n)$, $(c_0, \ldots, c_n)$ *such that the conditions in Definition 3.1 hold. Then* A_0^n *is the number of mixed tilings with these height conditions.*

Proof. In the following proof we will exclude the case of b_n and c_n being both not negative, since it follows easily by Theorem 3.1.

First we want to show that the number of mixed tiling M_0^n satisfies the same initial condition and recurrence relations of A_0^n.

- If $n=0$ we trivially have $M_0^0 = a_0 = A_0^0$.
- If $n=1$ we have $M_0^1 = a_0a_1 + b_1 = A_0^1$, since $b_1 < 0$ we may cover using only squares, that are a_0a_1, but we need to subtract $|b_1|$ negligible tilings, when we have a_0 squares in the cell in position 0 and less than $|b_1|+1$ squares in the cell in position 1.
- If $n=2$ we have $M_0^2 = a_0a_1a_2 + a_0b_2 + a_2b_1 + c_2 = A_0^2$, indeed $a_0a_1a_2$ is the total number of tilings consisting in only squares, a_0b_2 and b_1a_2 are the number of tilings involving a stack of squares and a stack of dominoes that we need to add (when $b_i \geq 0$) or subtract (when $b_i < 0$). Finally c_2 is the number of tiling using only bars we need to add (when $c_2 \geq 0$)) or the number of tilings we need to subtract ($c_2 < 0$).

We now need to prove that M_0^n satisfies the same recurrence of A_0^n.

- If $b_n < 0$ and $c_n > 0$, then every tiling must finish either with a stack of squares or a stack of bars. By induction there are $c_nM_0^{n-3}$ tilings that end with a stack of bars and $a_nM_0^{n-1}$ tilings that end with a stack of squares, ignoring the condition stated in Definition 3.1. Among these, we need to subtract $|b_n|M_0^{n-2}$ negligible tiling, namely those having a full stack of a_{n-1} squares in the cell $n-1$ and less than $|b_n|+1$ squares in the cell n. Therefore the number of all tilings is

$$M_0^n = a_nM_0^{n-1} + b_nM_0^{n-2} + c_nM_0^{n-3},$$

which is the same as (2).
- If $b_n > 0$ and $c_n < 0$ then every tiling must finish either with a stack of squares or a stack of dominoes. By induction these are respectively $a_nM_0^{n-1}$ and $b_nM_0^{n-2}$. Now we need to distinguish two possible cases:
  - If $a_n > |c_n|$, then we need to subtract $|c_n|M_0^{n-3}$ negligible tilings, i.e. those having a full stack of a_{n-1} squares in cell $n-1$, a full stack of a_{n-2} squares in the cell $n-2$ and less than $|c_n|+1$ squares in cell n.
  - If $a_n \leq |c_n|$, then $b_n > |c_n|$ by hypothesis and so we need to subtract $|c_n|M_0^{n-3}$ negligible tiling, which in this case are those having a full stack of a_{n-2} squares in the cell $n-2$ and less than $|c_n|+1$ dominoes covering the cells in positions $n-1$, n.
- Finally, if $c_n < 0$ and $b_n < 0$ then every tiling must finish with a stack of squares. By induction there are $a_nM_0^{n-1}$ tilings that end with a stack of squares. From this number we need to subtract $|b_n|M_0^{n-2}$ negligible tilings, those when there is a full stack of a_{n-1} squares in the cell $n-1$ and less than $|b_n|+1$ squares in the cell n. Moreover we also need to subtract $(|b_n|+|c_n|)M_0^{n-3}$ negligible tilings, i.e. when there are full stacks of a_{n-1} and a_{n-2} squares in the cells $n-1$ and $n-2$ respectively and less than $|c_n|+|b_n|+1$ squares in the cell n. However in this way we have counted twice the tilings having full stacks of a_{n-1} and a_{n-2} squares in the cells $n-1$ and $n-2$, and less than $|b_n|+1$ squares in the last cell, so we have to add up this coverings again. These are a total of $|b_n|M_0^{n-3}$ tiling, obtaining the result stated by the thesis. □

3.2. *Generalization to higher dimensions*

The Jacobi algorithm has been generalized to higher dimensions by Perron [20] as follows:

$$\begin{cases} a_n^{(i)} = \lfloor \alpha_n^{(i)} \rfloor \\ \alpha_{n+1}^{(1)} = \dfrac{1}{\alpha_n^{(m)} - a_n^{(m)}} \\ \alpha_{n+1}^{(i)} = \dfrac{\alpha_n^{(i-1)} - a_n^{(i-1)}}{\alpha_n^{(m)} - a_n^{(m)}} \end{cases} \qquad n = 0, 1, 2, \ldots$$

starting from m real numbers $\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}$ and providing a multidimensional continued fraction

$$[(a_0^{(1)}, a_1^{(1)}, \ldots), \ldots, (a_0^{(m)}, a_1^{(m)}, \ldots)]$$

of degree m which is defined by the following relation

$$\begin{cases} \alpha_n^{(i-1)} = a_n^{(i-1)} + \dfrac{\alpha_{n+1}^{i}}{\alpha_{n+1}^{(1)}}, \quad i = 2, \ldots, m \\ \alpha_n^{(m)} = a_n^{(m)} + \dfrac{1}{\alpha_{n+1}^{(1)}} \end{cases} \qquad n = 0, 1, 2, \ldots$$

Our results about the multidimensional continued fraction of degree 2 easily extend to a multidimensional continued fraction of degree m by considering $m+1$ different tiles of length $1, 2, \ldots, m+1$. In this paper we dealt with the case of degree 2 for simplicity of notation.

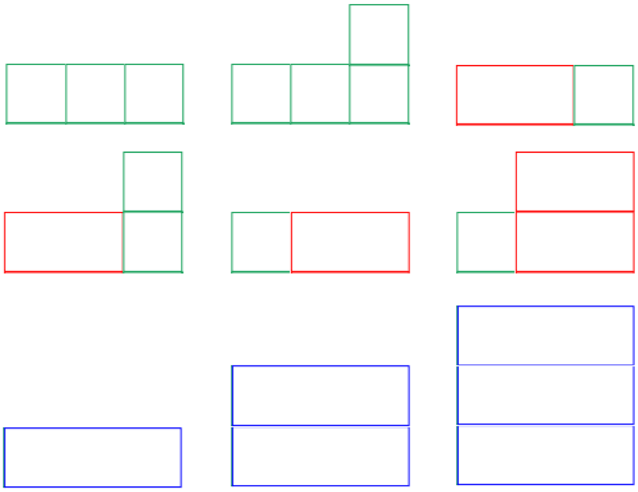


Fig. 4. Possible tiling of a 3–board with height conditions $(1, 1, 2, 3, 5)$, $(1, 1, 2, 5, 14)$, $(2, 1, 3, 4, 7)$.

Example 3.3. We conclude with an example of application of Theorem 3.1 and how it can be generalized to higher dimensions, specifically when we also have a tile of length four.

First, let us consider a board that we want to tile using squares, dominoes and bars with the following height conditions:

$$(a_n) = (1, 1, 2, 3, 5), \quad (b_n) = (1, 1, 2, 5, 14), \quad (c_n) = (2, 1, 3, 4, 7).$$

Thus, the corresponding sequences of numerators and denominators of the multidimensional continued fractions with the above partial quotients are

$$(A_0^n) = (1, 2, 9, 41, 345), \quad (B_0^n) = (1, 2, 6, 32, 258), \quad (C_0^n) = (2, 2, 8, 42, 336)$$

These sequences can be evaluated by the sequences of partial quotients exploiting (1). If we have a board of length one, we have only one tiling, since $a_0 = 1$. Given a board of length two, the number of possible tilings is $A_0^1 = 2$, indeed we only have the tiling composed by two squares and the tiling composed by one domino. Considering a 3–board, $A_0^2 = 9$ is the number of possible tilings, which are all displayed in Fig. 4.

Then, by Theorem 3.1, we know that 41 and 345 are the number of possible tilings for a 4–board and a 5–board, respectively.

If we want to tile a chessboard using also tiles of length four, then we need a fourth sequence for the height conditions of this kind of tiles. In order to count the number of possible tilings in this case we need the convergents of a multidimensional continued fraction of degree 3. Let us consider the following height conditions for the tiles, which correspond to the partial quotients of our 3-dimensional continued fraction:

$$\begin{aligned}&(a_n) = (1, 1, 2, 3, 5), \quad (b_n) = (1, 1, 2, 5, 14),\\ &(c_n) = (2, 1, 3, 4, 7), \quad (d_n) = (1, 2, 5, 12, 29).\end{aligned}$$

In this case, we have four sequences $(A_0^n), (B_0^n), (C_0^n), (D_0^n)$ for the numerators and denominators of convergents and they can be evaluated by

$$\begin{pmatrix} A_0^n & A_0^{n-1} & A_0^{n-2} & A_0^{n-3} \\ B_0^n & B_0^{n-1} & B_0^{n-2} & B_0^{n-3} \\ C_0^n & C_0^{n-1} & C_0^{n-2} & C_0^{n-3} \\ D_0^n & D_0^{n-1} & D_0^{n-2} & D_0^{n-3} \end{pmatrix} := \prod_{i=0}^{n} \begin{pmatrix} a_i & 1 & 0 & 0 \\ b_i & 0 & 1 & 0 \\ c_i & 0 & 0 & 1 \\ d_i & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we obtain

$$\begin{aligned}&(A_0^n) = (1, 2, 9, 53, 434), \quad (B_0^n) = (1, 2, 11, 47, 432),\\ &(C_0^n) = (2, 4, 12, 64, 574), \quad (D_0^n) = (1, 1, 4, 21, 197).\end{aligned}$$

Note that the sequence A_0^n has the first three elements equal to the previous case, since the tile of length four clearly does not affect the tiling of boards with length less than four. Now, we can observe that an analogous of Theorem 3.1 still works. The number of possible tilings of a 4–board can be evaluated as

$$a_3 A_0^2 + b_3 A_0^1 + c_3 A_0^0 + d_3 A_0^{-1} = 3 \cdot 9 + 5 \cdot 2 + 4 \cdot 1 + 12 \cdot 1 = 53 = A_0^3,$$

where, as usual, we consider $A_0^{-1} = 1$ and we have exploited the analogous of recurrence (2). Indeed, we know that A_0^2, A_0^1, A_0^0 are the number of possible tilings of a 3–board, 2–board, 1–board, respectively. Thus the sum in the equation above gives the number of possible tilings of a 4–board remembering that a_3 is the height condition for squares in the cell of position 3, b_3 is the height condition for dominoes in the cells in positions 2 and 3, c_3 is the height condition for bars in the cells in positions 1, 2, 3, d_3 is the height condition for the tile of length four in the cells in positions $0, 1, 2, 3$.

Declaration of competing interest

There is no conflict of interest.

Data availability

No data was used for the research described in the article.

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