University of Trento

Ph.D. Thesis

Geometry of moduli spaces of higher spin curves

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Introduction

A moduli space is, roughly speaking, a parameter space for equivalence classes of geometric objects of fixed topological type and with prescribed characteristics. The moduli space $\mathcal{M}_g$ of smooth genus g algebraic curves (or compact Riemann surfaces with g "holes") was firstly introduced by Riemann in 1857 and some of the most important aspects of modern algebraic geometry were developed around it, such as geometric invariant theory and the theory of stacks. The points of this space represent isomorphism classes of complex manifold structures that can be put on a fixed compact genus g Riemann surface. When $g = 0$ there is only one Riemann sphere (corresponding to the projective line $\mathbb{P}^1$) and hence the corresponding moduli space is a single point. When $g = 1$ we are dealing with the so-called *elliptic curves* that, for instance, play a very important role in number theory and cryptography. Each elliptic curve is in one-to-one correspondence with a complex number called *j-invariant* and so the moduli space of curves of genus $g = 1$ is just the complex line $\mathbb{C}$. For higher values of g, Riemann calculated that the *number of moduli*, that is the dimension of the space, is equal to $3g - 3$ but the precise nature of $\mathcal{M}_g$ remained an open problem for over a century, at least until the first rigorous construction carried out by Mumford in 1965 in the book [MFK65].

Over time moduli spaces of curves with additional structures were considered. Theta-characteristics appeared for the first time in the context of characteristic theory of odd and even theta functions in the paper of Göpel [Göp47] and Rosenhain [Ros50] while the modern study of the families of curves endowed with a theta-characteristic begun with the works of Atiyah [Ati71] and Mumford [Mum71]. The moduli space $\mathcal{S}_g$ of the so-called *spin curves* parametrizes isomorphism classes of pairs (C, L) where C is a smooth curve of genus g and $L \in \mathrm{Pic}(C)$ is a theta-characteristic, that is a line bundle whose second tensor power is isomorphic to the canonical bundle ω_C. This space is the disjoint union of two irreducible components $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$ corresponding respectively to even and odd theta-characteristics, where the parity refers to the parity of $h^0(L)$. The moduli space of higher spin curves, the main object of this thesis, is a natural generalization of this space. Roughly speaking, it corresponds to equivalence classes of pairs (C, L) where $L \in \mathrm{Pic}(C)$ is a line bundle whose r-th tensor power is isomorphic to the canonical bundle of the curve.

A common characteristic of all these spaces is that they are not compact. This lackness can be intuitevely seen for instance if one thinks of a Riemann surface of genus $g \geq 2$ pinched between two of the holes in such a way that the resulting surface looks like made of two components with only a point in common. This new surface is clearly the limit of a sequence of elements of $\mathcal{M}_g$ but it is not itself a point of the moduli space, because the corresponding curve has a node. In order to compactify the space it is necessary to construct a boundary that contains any accumulation points of $\mathcal{M}_g$. The problem of finding a suitable compactification for $\mathcal{M}_g$ or $\mathcal{M}_{g,n}$, the moduli space of genus g curves with n marked points, has a long and rich history and it has been solved with the constructions of several natural compactifications. The one we will deal with is the Deligne-Mumford [DM69] compactification $\overline{\mathcal{M}}_g := \overline{\mathcal{M}}_g^{DM}$. Here the additional points of the boundary correspond to stable nodal curves, still required to be connected and compact. The points of a nodal curve are either smooth or analytically isomorphic to a neighborhood of the origin of $\{xy = 0\} \subset \mathbb{C}^2$ and the stability condition corresponds to the requirement of a finite automorphism group. Similarly, the boundary of $\overline{\mathcal{M}}_{g,n}$ consists of stable nodal n-pointed curve of genus g. The compactification of the moduli space $\mathcal{S}_g$ of spin curves was carried out by Cornalba in [Cor89], who constructed a proper Deligne-Mumford moduli space $\overline{\mathcal{S}}_g$ of stable spin curves. Our first paper of this thesis focuses on the problem of constructing a good compactification for the moduli space $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}$ of r-spin genus g curves of type $\mathbf{m}$, which has been addressed in particular by [Jar01] and [CCC07]. Here we provide a new set-theoretical description of the space and we investigate its rational Picard group.

Another topic of great interest in algebraic geometry since its remote origins is the so-called rationality problem. We recall that a rational variety is an algebraic variety birationally equivalent to a projective space of some dimension while a unirational variety is an algebraic variety only dominated by a rational variety. In general, the second condition is strictly weaker than the first. Rational varieties are those that appear most frequently in applications of mathematics also because they are for the most part tangible objects adequate for a concrete study, though they are only a small fraction of the class of all varieties. In the context of moduli spaces of curves, the starting point in this study was a conjecture of Severi [Sev15]: he suggested the unirationality of $\mathcal{M}_g$ for all genera and he proved it for $g \leq 10$. After almost 60 years, with major surprise, [HM82], [Har84] and [EH87] proved it wrong showing that for $g \geq 24$ the moduli space $\mathcal{M}_g$ is very far from being rational or unirational since it is of general type. After that, there has been a great deal of work trying to describe the geometry in the intermediate cases $11 \leq g \leq 23$. For an exhaustive survey on the subject one can see [Far08] or [Ver13]. Summarizing, now it is known that moduli spaces of curves are characterized by the fact of being rational or unirational in the simplest and manageable cases, that is for small values of g, and of general type for large g. In particular, the transition of $\mathcal{M}_g$ from the uniruledness case, that is $g \leq 16$, to the case where $\mathcal{M}_g$ is of general type is still open because the Kodaira dimension for $g = 17, 18, 19, 20$ and 21 is still unknown. In the case of moduli of spin curves, instead, the situation has been recently completely settled in [Far10], [FV10] and [FV12]. On the other hand, the global geometry of the moduli spaces of higher spin curves and their birational classification is largely unknown for $g \geq 2$ and $r > 2$. In our second paper, joint with Alessandro Verra, we almost complete the picture of the known results in genus $g \leq 4$ using quite related geometric constructions. In particular, we show the rationality of the moduli spaces of even and odd 4-spin curves of genus 3, of odd spin curves of genus 4 and of 3-spin curves of genus 4.

The last chapter deals with the study of weighted Gaussian maps on curves. The theory of standard Gaussian maps was developed by Jonathan Wahl in the late 80's [Wah87], [Wah90] and attracted considerable interest especially in the context of deformation theory. A great amount of work was done by studying the surjectivity of these maps (see for instance [CHM88] and [Par95]) while some results on the injectivity can be found in [Far05]. In particular, in this last work Farkas obtained also some results on the codimension of a particular locus $\mathcal{S}_g^r$ inside the moduli space of spin curves, that is the closure of the locus of spin curve (C, L) with $h^0(L) = r + 1$ for some integer $r \geq 0$, integrating in that way the results of [Har82] at least for small values of the genera. This has been possible thanks to the connection between Gaussian maps and theta-characteristics given by a tangent space computation [Nag90]. Our third paper, joint with Edoardo Ballico, studies a generalization of these Gaussian maps, the so-called *weighted Gaussian maps* introduced in [BF06a] and provides some criteria for their surjectivity. Similarly to the previous case, these maps present a connection with the moduli space of even higher spin curves given by an analogous tangent space computation [Fon02]. Thanks to this, we obtain a bound on the codimension of the analogous of the locus $\mathcal{S}_g^r$ for even higher spin curves, which turns out to be exact in a particular case.

Let us see more in details the contents of each chapter of this thesis.

Moduli of higher spin curves and line bundles

In [Jar01], the moduli space of higher spin curves with the additional data of *coherent net of roots* is described using torsion-free sheaves. The aim of this chapter is to refine the notion of root given in the line bundles setting by [CCC07] with such additional data and investigate the geometry of the corresponding moduli space. This space is less singular than the one described in [CCC07] and it is thus possible to study its Picard group.

A stable r-spin curve of type $\mathbf{m}$ is an n-pointed curve of genus g with only ordinary nodes as singularities together with the data of a coherent net of r-th roots of type $\mathbf{m}$ on it. The set $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ of stable r-spin curves up to isomorphisms is in one-to-one correspondence with the moduli space $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ of higher spin curves described by [Jar01], hence we can transport the scheme structure of the latter to the former and consider the two spaces as isomorphic schemes.

Proposition 0.0.1. *There is a scheme with underlying set $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ (which we will still call $\overline{S}_{g,n}^{1/r,\mathbf{m}}$) isomorphic to the scheme $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ introduced by Jarvis in [Jar01]. In particular, $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ is normal and projective.*

In order to describe the rational Picard group of this moduli space, we focus on the case without marked points and we restrict to one irreducible component whenever $S_g^{1/r} := S_{g,0}^{1/r,\mathbf{0}}$ is not irreducible. The low dimensional homology and cohomology of the open part $S_g^{1/r}$ has been studied in [RW12], together with the relations between boundary divisors and other classes, such as the classes λ and μ. It turns out that, when the genus g is greater or equal to 9, the first holomology group over $\mathbb{Q}$ is zero, while the second cohomology group has rank 1. This means that, for instance, the Hodge class generates the Picard group of $S_g^{1/r}$ over $\mathbb{Q}$.

By [Jar01], the classes λ, $\{\alpha_i^{(a,b)}\}$ and $\{\gamma_{j,\eta}^{(a,b)}\}$, where the $\alpha_i^{(a,b)}$'s and $\gamma_{j,\eta}^{(a,b)}$'s denote suitable boundary divisors (to be defined in §1.3), are independent in $\mathrm{Pic}(\overline{S}_g^{1/r})$. By [Jar00] and [Jar01] the space $\overline{S}_g^{1/r}$ is normal and with quotient singularities, so that, as in [BF06b], its rational Picard group is isomorphic to the correspondent Chow group over $\mathbb{Q}$. This implies that the whole Picard group $\mathrm{Pic}(\overline{S}_g^{1/r})$ is generated over $\mathbb{Q}$ by the generators of the Chow group of the open part $S_g^{1/r}$ together with the set of boundary classes of $\overline{S}_g^{1/r}$. In the end we obtain a complete description of the generators of the rational Picard group.

Theorem 0.0.2. *Assume $g \geq 9$. Then $\mathrm{Pic}(\overline{S}_g^{1/r})$ is freely generated over $\mathbb{Q}$ by the classes λ, $\{\alpha_i^{(a,b)}\}$ and $\{\gamma_{j,\eta}^{(a,b)}\}$, where λ is the Hodge class and $\{\alpha_i^{(a,b)}\}$ and $\{\gamma_{j,\eta}^{(a,b)}\}$ are the boundary divisors.*

On the rationality of the moduli spaces of higher spin curves in low genus

In this chapter we study the rationality of the stack $\mathcal{S}_g^{1/r}$ of the moduli space of higher spin curves for some low genus cases, that is $r = 4$ and $g = 3$ and $r = 2, 3$ and $g = 4$.

The Kodaira dimension of the moduli of ordinary spin curves is completely known, as well as several facts about rationality or unirationality in low genus. The picture is as follows for even spin curves:

- $\mathcal{S}_g^+$ *is uniruled for* $g \leq 7$,
- $\mathcal{S}_8^+$ *has Kodaira dimension zero,*
- $\mathcal{S}_g^+$ *is of general type for* $g \geq 9$.

For odd spin curves one has

- $\mathcal{S}_g^-$ *is uniruled for* $g \leq 11$,
- $\mathcal{S}_g^-$ *is of general type for* $g \geq 12$.

In particular it is natural asking whether a suitable minimal model of $\mathcal{S}_8^+$ is Calabi-Yau. On the other hand $\mathcal{S}_g^-$ and $\mathcal{S}_g^+$ are known to be unirational respectively for $g \leq 8$ and $g \leq 6$. Concerning the rationality problem, $\mathcal{S}_g^{\pm}$ is classically known to be rational for $g \leq 3$, while the rationality of $\mathcal{S}_4^+$ has been recently proved. For more details on this picture see [Far10], [FV10], [FV12],[TZ09], [Ver13].

Higher spin curves generalize spin curves. With the exception of the case of genus 1, the global geometry of the spaces $\mathcal{S}_g^{1/r}$ and their birational classification is largely unknown for $g \geq 2$ and $r > 2$.

On the other hand a natural, though elementary, remark is that for every curve C the canonical sheaf ω_C admits square roots, the theta characteristics on C, but also $2g-2$ and $g-1$ roots.

Restricting to $g-1$ roots, they form sets of $(g-1)^{2g}$ line bundles of degree two on C which are worth of being studied. In particular the forgetful map

$$f : \mathcal{S}^{1/(g-1)} \to \mathcal{M}_g$$

has degree $(g-1)^{2g}$. Since it grows up very fast, it is perhaps natural to expect that $\mathcal{S}_g^{1/(g-1)}$ becomes of general type after few values of g. With respect to this, assume that g is odd so that $\eta^{\otimes(g-1)/2}$ is a theta characteristic. Then it follows that, if g is odd, every irreducible component of $\mathcal{S}_g^{1/(g-1)}$ dominates $\mathcal{S}_g^+$ or $\mathcal{S}_g^-$. In particular, in view of the mentioned results, there exist irreducible components of $\mathcal{S}_g^{1/(g-1)}$ having non negative Kodaira dimension as soon as $g \geq 8$. In this frame the first unknown case to be considered is the genus 4 case. Somehow surprisingly this is still an exception. Indeed we prove that

Theorem 0.0.3. *The moduli space of 3-spin curves of genus 4 is rational.*

Let (C, η) be a general spin curve of genus 4 and order 3. The starting point for proving the theorem is the remark that giving (C, η) is equivalent to give the unique effective divisor $t \in |\eta^{\otimes 2}|$. Furthermore, let C be canonically embedded in $\mathbf{P}^3$, then $3t$ is the complete intersection of two quadrics and a cubic surface. We show that the GIT-quotient of the family of these complete intersections is rational and that it admits a natural birational map to the moduli space $\mathcal{S}_4^{1/3}$.

Adding up the previous picture and the above theorem we have a list of cases of genus $g \leq 4$ where the rationality of $\mathcal{S}_g^{1/r}$ is confirmed. The complementary list of cases is as follows:

- *Moduli of* 4*-spin curves of genus* 3.
- *Moduli of odd spin curves of genus* 4.
- *Moduli of* 6*-spin curves of genus* 4.

In particular it seems that the case of odd spin curves of genus 4 was not considered in the literature. Notice also that $\mathcal{S}_g^{1/(2g-2)}$ splits into the union of two components: the moduli of pairs (C, η) such that $\eta^{\otimes g-1}$ is an even theta characteristic and the complementary component. We will denote them respectively by

$$\mathcal{S}_g^{1/(2g-2)+}\ ,\ \mathcal{S}_g^{1/(2g-2)-}.$$

We will say that (C, η) is an even (odd) r-spin curve if $\eta^{\otimes r}$ is an even (odd) theta characteristic. In the final part of this chapter we almost complete the picture of the known results in genus $g \leq 4$. Building on quite related geometric constructions and methods, we prove the following theorems.

Theorem 0.0.4. *The moduli space of odd spin curves of genus 4 is rational.*

Theorem 0.0.5. *The moduli spaces of 4-spin curves of genus 3 are rational.*

In spite of several natural approaches, we have not found any evidence of the uniruledness of $\mathcal{S}_g^{1/r}$ for the only two missing cases in genus $g \leq 4$, namely for $\mathcal{S}_4^{1/6+}$ and $\mathcal{S}_4^{1/6-}$. The same lack of evidence appears for further low values of g, say for $g = 5, 6, 7$ and $r > 2$. Already for all these cases, it appears therefore interesting to apply some recent results on the structure of the Picard group of the Deligne-Mumford compactification of $\mathcal{S}_g^{1/r}$ for checking whether the canonical class is effective (cfr. for instance [Per13] and [RW12]).

Weighted Gaussian maps and loci inside $\mathcal{S}_g^{1/r}$

The theory of Gaussian maps on curves was developed by Jonathan Wahl in [Wah87] and [Wah90]. We recall that they are defined as

$$\Phi_L : R(L) \to H^0(\Omega^1_X \otimes L^{\otimes 2}), \qquad \sigma \otimes \tau \mapsto \tau \mathrm{d}\sigma - \sigma \mathrm{d}\tau,$$

where X is a smooth projective variety over $\mathbb{C}$, $L \in \operatorname{Pic}(X)$ a line bundle of positive degree and $R(L)$ is the kernel of the multiplication map $H^0(L) \otimes H^0(L) \to H^0(L^{\otimes 2})$. Furthermore, as for instance in [Far05], it makes sense to define the Gaussian map also as the restriction $\psi_L := \Phi_{L|\Lambda^2 H^0(L)}$ since the first map always vanishes on symmetric tensors. The interpretation of the name "Gaussian" can be found for instance in [Wah90]: if $X = C$ is a curve embedded in a projective space $\mathbb{P}^n$ and $L = \mathcal{O}_C(1)$, one can consider the Gauss mapping $C \to \operatorname{Gr}(1, n)$ sending each point of the curve to its tangent line in $\mathbb{P}^n$. The composition of this map with the Plücker embedding $\operatorname{Gr}(1, n) \hookrightarrow \mathbb{P}^N$ gives rise to the "associated curve" $\phi : C \to \mathbb{P}^N$ and the restriction of the hyperplane section ϕ^* of ϕ corresponds the so-called Gaussian map ψ_L.

The original interest in these maps came from the study of Φ_{K_C}, where K_C is the canonical bundle on a smooth curve (see for instance [CHM88] and [CLM00]) and in general they have been explored in particular when X is a curve, in relation with the deformation theory of the vertex of the cone over X (see [Wah87]). More recently, Edoardo Ballico and Claudio Fontanari provided a generalization of these maps in [BF06a]. They defined the so-called "weighted Gaussian maps" as

$$\begin{array}{rccc} \gamma_{a,b}(X, L): & H^0(X, L^{\otimes a}) \otimes H^0(X, L^{\otimes b}) & \to & H^0(\Omega^1_X \otimes L^{\otimes a} \otimes L^{\otimes b}) \\ & \sigma \otimes \tau & \mapsto & b\tau \mathrm{d}\sigma - a\sigma \mathrm{d}\tau. \end{array} \tag{1}$$

where X is always a smooth projective variety, $L \in \operatorname{Pic}(X)$ and $a, b > 0$ are two positive integers. When $a = b = 1$ we recover the standard Gaussian map.

In the first section, following the approach of [Wah90], we investigate the surjectivity of the map $\gamma_{a,b}(X, L)$ by studying first the weighted Gaussian maps for $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(e)$, with $e \in \mathbb{N}$ a generic positive integer, and then the restriction map given by $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(t)) \to H^0(X, \Omega^1_X(t))$. In particular, we prove a general result (Theorem 3.2.5) from which we deduce the following.

Corollary 0.0.6. *Let C be a smooth curve of genus g and $L \in \mathrm{Pic}(C)$ a line bundle of degree $\deg(L) \geq 2g+2$. Denote by $X \subset \mathbb{P}^n$ the linearly normal embedding of C induced by $|L|$. Then the map $\gamma_{a,b}(X, \mathcal{O}_X(1))$ is surjective for all positive integers a, b such that $a+b \geq 3$.*

Corollary 0.0.7. *Let C be a smooth curve of genus $g \geq 3$. Assume that C is neither hyperelliptic nor trigonal nor isomorphic to a plane quintic and let $X \subset \mathbb{P}^{g-1}$ be its canonical model. Then the map $\gamma_{a,b}(X, \omega_X)$ is surjective for all $a, b > 0$ such that $a+b \geq 4$.*

In the second section we relate the weighted Gaussian maps $\gamma_{1,h-1}(C,L)$, $h \geq 2$, to the locus $\mathfrak{Th}^r_{g,h} \subset \mathcal{M}_g$ defined, for $h \geq 2$ an arbitrary integer, by

$$\mathfrak{Th}^r_{g,h} := \{[C] \in \mathcal{M}_g \ | \ \exists L \in \mathrm{Pic}(C) \text{ s.t. } h^0(C,L) \geq r+1, \ hL = K_C\}. \quad (2)$$

It is a generalization for h-spin curves of the locus studied by J. Harris in [Har82] for theta-characteristics. The case in which h is even was considered in [Fon02], but the proof of the odd case is identical (see Theorem 3.3.2). In particular, J. Harris proved that each component of the locus

$$\mathcal{S}^r_g := \{[C,L] \in \mathcal{S}_g \mid h^0(L) \geq r+1, \quad h^0(L) \equiv r+1 \mod 2\}$$

has codimension at most $\binom{r+1}{2}$ in $\mathcal{S}_g$. We recall that $\mathcal{S}_g$ is the moduli space of pairs $[C,L]$ where C is a genus g curve and $L \in \mathrm{Pic}(C)$ is a theta-characteristic on C, that is, a square root of its canonical bundle. In [Far05] G. Farkas showed that for $r = 1, 2, \ldots, 9$ and 11 there exists an explicit integer $g(r)$ such that for all $g \geq g(r)$ the moduli space $\mathcal{S}^r_g$ has at least one component of codimension exactly $\binom{r+1}{2}$ and he made a conjecture (recently proved by L. Benzo in [Ben13]) on the existence of a component attaining the maximum codimension for any $r \geq 1$ and $g \geq \binom{r+2}{2}$. His proof is based on the connection between Gaussian maps and spin curves provided by a tangent space computation done by Nagaraj in [Nag90], whose main ingredient is the identification $T_{[C,L]}\mathcal{S}^r_g \cong \mathrm{Im}(\psi_L)^\perp$. In this work we use the analogous relation between the weighted Gaussian map and the locus $\mathfrak{Th}^r_{g,h}$ proved in [Har82, Theorem 3] for h even: there is an identification between the tangent space $T_C\mathfrak{Th}^r_{g,h}$ and the dual of the cokernel $\mathrm{Coker}(\gamma_{1,h-1}(C,L))$. As claimed before, the same relation holds identically also when h is odd. Using these facts we prove the following result.

Theorem 0.0.8. *For every $g, h \geq 2$ and every $[C] \in \mathfrak{Th}^0_{g,h}$ with an h-theta L satisfying $h^0(L) = 1$, the Zariski tangent space at $[C]$ has codimension $(g-1)(h-2)/h$ in the tangent space $H^0(C, K_C^{\otimes 2})^\vee$ of the local deformation space of C.*

We conclude our work with a focus on complete intersection curves. In Theorem 3.3.5 we prove that, for general m and r, if $\mathfrak{Th}^r_{g,h}$ does contain a complete intersection, then $\mathfrak{Th}^r_{g,h}$ has a component whose general element is a complete intersection.

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Chapter 1

On the rational Picard group of the moduli space of higher spin curves

ABSTRACT: We refine the notion of higher spin curves defined in terms of line bundles by adding the additional structure of coherent nets of roots firstly introduced by Jarvis in terms of torsion-free sheaves and we describe the boundary part of their moduli space in the case without marked points. We provide then a presentation of the rational Picard group of this space.

1.1 Introduction

In [Jar01], the moduli space of higher spin curves with the additional data of *coherent net of roots* is described using torsion-free sheaves. The aim of this note is to refine the notion of root given in the line bundles setting by [CCC07] with such additional data and investigate the geometry of the corresponding moduli space. This space is less singular than the one described in [CCC07] and it is thus possible to study its Picard group.

A stable r-spin curve of type $\mathbf{m}$ is an n-pointed curve of genus g with only ordinary nodes as singularities together with the data of a coherent net of r-th roots of type $\mathbf{m}$ on it. The set $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ of stable r-spin curves up to isomorphisms is in one-to-one correspondence with the moduli space $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ of higher spin curves described by [Jar01], hence we can transport the scheme structure of the latter to the former and consider the two spaces as isomorphic schemes.

Proposition 1.1.1. *There is a scheme with underlying set $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ (which we will still call $\overline{S}_{g,n}^{1/r,\mathbf{m}}$) isomorphic to the scheme $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ introduced by Jarvis in [Jar01]. In particular, $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ is normal and projective.*

In order to describe the rational Picard group of this moduli space, we focus on the case without marked points and we restrict to one irreducible component whenever $S_g^{1/r} := S_{g,0}^{1/r,\mathbf{0}}$ is not irreducible. The low dimensional homology and cohomology of the open part $S_g^{1/r}$ has been studied in [RW12], together with the relations between boundary divisors and other classes, such as the classes λ and μ. It turns out that, when the genus g is greater or equal to 9, the first holomology group over $\mathbb{Q}$ is zero, while the second cohomology group has rank 1. This means that, for instance, the Hodge class generates the Picard group of $S_g^{1/r}$ over $\mathbb{Q}$.

By [Jar01], the classes λ, $\{\alpha_i^{(a,b)}\}$ and $\{\gamma_{j,\eta}^{(a,b)}\}$, where the $\alpha_i^{(a,b)}$'s and $\gamma_{j,\eta}^{(a,b)}$'s denote suitable boundary divisors (to be defined in §1.3), are independent in $\mathrm{Pic}(\overline{S}_g^{1/r})$. By [Jar00] and [Jar01] the space $\overline{S}_g^{1/r}$ is normal and with quotient singularities, so that, as in [BF06], its rational Picard group is isomorphic to the correspondent Chow group over $\mathbb{Q}$. This implies that the whole Picard group $\mathrm{Pic}(\overline{S}_g^{1/r})$ is generated over $\mathbb{Q}$ by the generators of the Chow group of the open part $S_g^{1/r}$ together with the set of boundary classes of $\overline{S}_g^{1/r}$. In the end we obtain a complete description of the generators of the rational Picard group.

Theorem 1.1.2. *Assume $g \geq 9$. Then $\mathrm{Pic}(\overline{S}_g^{1/r})$ is freely generated over $\mathbb{Q}$ by the classes λ, $\{\alpha_i^{(a,b)}\}$ and $\{\gamma_{j,\eta}^{(a,b)}\}$, where λ is the Hodge class and $\{\alpha_i^{(a,b)}\}$ and $\{\gamma_{j,\eta}^{(a,b)}\}$ are the boundary divisors.*

We work over the complex field $\mathbb{C}$.

1.2 The moduli space

1.2.1 Coherent nets

Le C be a stable curve of genus g. Let us start by recalling the two basic definitions of r-th root of a line bundle and of a torsion-free sheaf.

Definition 1.2.1 ([Jar01], r-th root of a torsion-free sheaf)**.** *Given a semistable curve C and a rank-one torsion-free sheaf $\mathcal{K}$ on C, an* r-th root *of $\mathcal{K}$ is a pair $(\mathcal{E}, b)$ of a rank-one torsion-free sheaf $\mathcal{E}$ and an $\mathcal{O}_C$-module homomorphism $b : \mathcal{E}^{\otimes r} \to \mathcal{K}$ such that*

1. $r \cdot \deg \mathcal{E} = \deg \mathcal{K}$,
2. *b is an isomorphism on the locus of C where $\mathcal{E}$ is not singular,*
3. *for every $p \in C$ where $\mathcal{E}$ is singular, one has $\mathrm{length}_p(\mathrm{coker}(b)) = r - 1$.*

Definition 1.2.2 ([CCC07], limit r-th root of a line bundle)**.** *Given a stable curve C, a line bundle $K \in \mathrm{Pic}(C)$ and an integer r dividing $\deg(K)$, a* limit r-th root *of (C, K) is a triple (X, L, α) where $\pi : X \to C$ is a blow-up of C at a set of nodes $\Delta = \{p_i\}$, $L \in \mathrm{Pic}(X)$ and α is a homomorphism $L^{\otimes r} \to \pi^* K$ satisfying*

1. *the restriction of L to every exceptional component has degree one,*

2. *the map α is an isomorphism outside the exceptional components,*

3. *for every exceptional components E_i of X, the orders of vanishing of α at $p_i^+, p_i^- \in E_i$ add up to r, where $\{p_i^+, p_i^-\} = \pi^{-1}(p_i)$.*

Since the moduli space of stable curves with r-th roots of a fixed sheaf is not smooth when r is not prime, Jarvis rigidified the space through the additional structure of *coherent net of roots* (see [Jar01]). We aim to translate this construction in terms of line bundles on semi-stable curves.

Definition 1.2.3 ([Jar01], coherent net of roots)**.** *Given a semistable curve C and a rank-one torsion-free sheaf $\mathcal{K}$ on C, a* coherent net of roots *for $\mathcal{K}$ is a collection $\{\mathcal{E}_d, c_{d,d'}\}$ consisting of a rank-one torsion-free sheaf $\mathcal{E}_d$ for every d dividing r and a homomorphism $c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \to \mathcal{E}_{d'}$ for each d' dividing d such that*

- *$\mathcal{E}_1 = \mathcal{K}$ and $c_{d,d} = \mathrm{Id}$ for each d dividing r,*
- *for every $d'|d|r$ the pair $(\mathcal{E}_d, c_{d,d'})$ is a d/d'-th root of $\mathcal{E}_{d'}$ in such a way that all the maps are compatible. In other words, one has $c_{d',d''} \circ c_{d,d'}^{\otimes d'/d''} = c_{d,d''}$ for every $d''|d'|d|r$.*

If r is prime, then this costruction reduces to a simple r-th root of $\mathcal{K}$. Moreover, if for some d the sheaf $\mathcal{E}_d$ is locally free, then it determines up to isomorphism all pairs $(\mathcal{E}_{d'}, c_{d,d'})$ such that $d'|d$ by the relations $\mathcal{E}_{d'} = \mathcal{E}_d^{\otimes d/d'}$, $c_{d',1} = c_{d,1}$ and $c_{d,d'} = \mathrm{Id}$. This means that the whole net is not already encoded in the "top" root $(\mathcal{E}_r, c_{r,1})$ only when, for some $d < r$, at a node p the sheaf $\mathcal{E}_d$ is locally free but $\mathcal{E}_r$ is not. In this case the net corresponds to the choice of a locally free d-th root $\mathcal{E}_d$ of $\mathcal{K}$ and a non-locally-free (r/d)-th root of $\mathcal{E}_d$ ([Jar00, §2.3.2]).

Let us go more into details. Consider a stable curve C and a coherent net of roots $\{\mathcal{E}_d, c_{d,d'}\}$ of ω_C. Let $(\mathcal{E}_r, c_{r,1})$ be the top root and let Δ be the set of points of C where $\mathcal{E}_r$ is singular. Let us call $\nu : Y \to C$ the partial normalization of C at Δ. For simplicity suppose $\Delta = \{p\}$ and let $\{u, v\}$ be the order of the r-th root $(\mathcal{E}_r, c_{r,1})$ at p. This means (see [Jar01, §2.2.2]) that

$$\nu^\natural \mathcal{E}_r^{\otimes r} \overset{\tilde{c}_{r,1}}{\cong} \nu^* \omega_C(-up^+ - vp^-) = \omega_Y(-(u-1)p^+ - (v-1)p^-),$$

where $\nu^\natural$ is defined as $\nu^\natural \mathcal{F} := \nu^* \mathcal{F}/tors$ for every sheaf $\mathcal{F}$, the map $\widetilde{c}_{r,1}$ is the isomorphism induced by the map $c_{r,1}$ at the level of the partial normalization Y and the points $\{p^+, p^-\}$ are the preimages of the point p under the map ν. Moreover, for every $d|r$, the order of the d-th root $(\mathcal{E}_d, c_{d,1})$ at p is given by $\{u_d, v_d\}$, where u_d and v_d are the least non-negative integers congruent respectively to u and v modulo d (see [Jar01]). From this it is clear that $\mathcal{E}_d$ is smooth at p if and only if d divides u (and v). We can thus distinguish between the case in which u and v are relatively prime and the case in which they are not. Let us define

$$\nu^\natural \mathcal{E}_d := \nu^\natural \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}_Y \left(\frac{u-u_d}{d} p^+ + \frac{v-v_d}{d} p^- \right).$$

If $\gcd(u,v)=1$ then no $c_{d,1}$ is an isomorphism, all $\mathcal{E}_d$'s are singular at p and furthermore they are completely determined by the top root by the relations $\mathcal{E}_d = \nu_* \nu^\natural \mathcal{E}_d$ and $c_{d,1} = c_{r,1}$ (see [Jar01, §2.2.2]).

If instead $\gcd(u,v) = l > 1$ then the sheaf $\mathcal{E}_l$ is smooth at p and it is necessary an additional gluing datum to construct $\mathcal{E}_l$ from $\nu^\natural \mathcal{E}_l$. In particular, the sheaf $\mathcal{E}'_l := \nu^\natural \mathcal{E}_l$ is a smooth (that is, of order $\{0,0\}$) l-th root of $\omega_Y(p^+ + p^-)$ and thus the l-th root $\mathcal{E}_l$ is recovered once we choose an isomorphism $\phi : \mathcal{E}'_{l\,|p^+} \to \mathcal{E}'_{l\,|p^+}$ compatible with the maps

$$\mathcal{E}'^{\otimes l}_{l\ \ |p^+} \xrightarrow{\sim} \omega_Y(p^+ + p^-) \xleftarrow{\sim} \mathcal{E}'^{\otimes l}_{l\ \ |p^-}.$$

In other words, the gluing datum corresponds to a (non-canonically determined) l-th root of unity and hence there are exactly l distinct possibilities for the gluing. We will see later that in some cases different gluings correspond to automorphisms of the underlying curve while in other cases different gluings generate different spin curves.

1.2.2 Torsion-free sheaves and line bundles

We restate here the main theorem linking torsion-free sheaves on stable curves and line bundles on semi-stable curves.

Proposition 1.2.4 ([CCC07], Proposition 4.2.2.)**.** *Let B an integral scheme and $f : \mathcal{C} \to B$ and $f : \mathcal{C} \to B$ a family of nodal curves.*

1. *Let $\pi : \mathcal{X} \to \mathcal{C}$ be a family of blow-ups of $\mathcal{C}$ and let $\mathcal{L} \in \mathrm{Pic}\mathcal{X}$ be a line bundle having degree one on every exceptional component. Then $\pi_*(\mathcal{L})$ is a relatively torsion-free sheaf of rank one, flat over B.*

2. *Conversely, suppose that $\mathcal{F}$ is a relatively torsion-free sheaf of rank one on $\mathcal{C}$, flat over B. Then there exist a family $\pi : \mathcal{X} \to \mathcal{C}$ of blow-ups of $\mathcal{C}$ an a line bundle $\mathcal{L} \in \mathrm{Pic}\mathcal{X}$ having degree one on all exceptional components, such that $\mathcal{F} \cong \pi_*(\mathcal{L})$.*

3. *Let $\pi : \mathcal{X} \to \mathcal{C}$ and $\pi' : \mathcal{X}' \to \mathcal{C}$ be families of blow-ups of $\mathcal{C}$ and $\mathcal{L} \in \mathrm{Pic}\mathcal{X}$, $\mathcal{L}' \in \mathrm{Pic}\mathcal{X}'$ line bundles having degree one on every exceptional component. Then $\pi_*(\mathcal{L}) \cong \pi'_*(\mathcal{L}') \iff \exists \sigma : \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ isomorphism over $\mathcal{C}$ s.t. $\mathcal{L} \cong \sigma^*(\mathcal{L}')$.*

Consider a coherent net of roots $\{\mathcal{E}_d, c_{d,d'}\}$ for (C, ω_C). By Proposition 1.2.4, we can associate to every sheaf $\mathcal{E}_d$ a blow-up $\pi_d : X_d \to C$ of C at the points where $\mathcal{E}_d$ is singular and a line bundle $L_d \in \mathrm{Pic}(X_d)$ having degree one on all exceptional components and such that $\mathcal{E}_d \cong \pi_{d*}(L_d)$. Furthermore, for every d dividing r, the map $c_{d,1} : \mathcal{E}_d^{\otimes d} \to \omega_C$ corresponds to an homomorphism

$\alpha_d : L_d^{\otimes d} \to \pi_d^* \omega_C$ that makes (X_d, L_d, α_d) a limit d-th root of (C, ω_C) in the sense of [CCC07]. In particular, if $\{u_d, v_d\}$ is the order of $(\mathcal{E}_d, c_{d,1})$ at a point $p \in C$ where $\mathcal{E}_d$ is singular, then it is also the order of vanishing of α_d at the points $\{p^+, p^-\} = \pi_d^{-1}(p)$ of X_d.

Suppose that there exists two integers d and d' such that at a point $p \in C$ the sheaf $\mathcal{E}_d$ is singular while the sheaf $\mathcal{E}_{d'}$ is smooth.

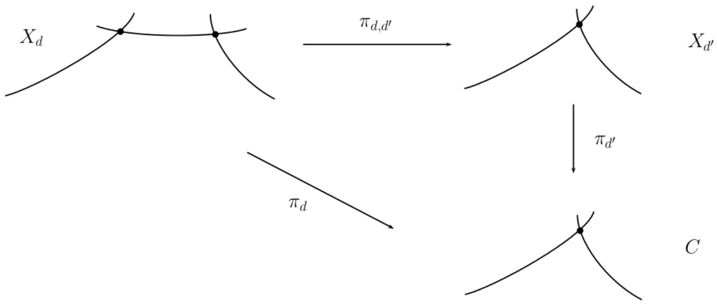

Figure 1.1: u and v not relatively prime.

As illustrated for instance in Figure 1.1, in this case $X_d \neq X_{d'}$ and we have a sequence of blow-ups

$$X_d \xrightarrow{\pi_{d,d'}} X_{d'} \xrightarrow{\pi_{d'}} C, \qquad \pi_d : X_d \to C$$

and two line bundles $L_d \in \mathrm{Pic}(X_d)$ and $L_{d'} \in \mathrm{Pic}(X_{d'})$ on two different bases. But if $\{p^+, p^-\}$ are the two points in the preimage $\pi_{d,d'}^{-1}(p) = \pi_d^{-1}(p)$, we know that the line bundle $L_{d'}$ is completely determined by the line bundle $\widetilde{N}_{d'} := \pi_{d,d'}^*(L_d)$ plus a gluing datum $\eta_{d'}^p : \widetilde{N}_{d'|p^+} \xrightarrow{\sim} \widetilde{N}_{d'|p^-}$. In other words, giving a line bundle $N \in \mathrm{Pic}(X_{d'})$ is the same (modulo isomorphisms) as giving a line bundle $\widetilde{N} \in \mathrm{Pic}(X_d)$ together with an isomorphism $\eta^p : \widetilde{N}_{|p^+} \xrightarrow{\sim} \widetilde{N}_{|p^-}$ such that $\widetilde{N}_{X_d \setminus E} = N_{X_{d'} \setminus \{p\}}$ and $\widetilde{N}_{|E} \cong \mathcal{O}_E(1)$, where $E = \pi_{d,d'}^{-1}(p)$ is the exceptional component coming from the blow-up.

Consider again the coherent net of roots $\{\mathcal{E}_d, c_{d,d'}\}$ and let r be the order of the top root. Define $X := X_r$ and $\pi := \pi_r$ and let $\Delta := \{p_1, \ldots, p_k\} \subset C$ be the set of points where $\mathcal{E}_r$ is singular. As we have seen above, to the coherent net of roots corresponds the collection of limit roots $\{(X_d, L_d, \alpha_d)\}$ for every d dividing r. Since we prefer to have a fixed base space X for all line bundles, we replace the collection $\{(X_d, L_d, \alpha_d)\}$ with the collection $(X, \{N_d, \beta_d, \eta_d^{p_j}\})$ where $N_d := \pi_{r,d}^* L_d \in \mathrm{Pic}(X)$, the map β_d is the homomorphism $N_d^{\otimes d} \to \pi^* \omega_C$ turning N_d into a limit d-th root of (C, ω_C) and $\eta_d^{p_j}$ is an isomorphism $N_{d|p_j^+} \xrightarrow{\sim} N_{d|p_j^+}$ every time that β_d has order zero at $\{p_j^+, p_j^-\}$.

We may note that, if $N := N_r$ is the top root and $\nu : Y \to C$ is the partial normalization at Δ, then the line bundle N is exactly the gluing of a degree one line bundle on each exceptional component with a line bundle $\widetilde{N} \in \mathrm{Pic}(Y)$

satisfying

$$\widetilde{N}^{\otimes r} = \nu^*\omega_C\left(-\sum_j (u^j p_j^+ + v^j p_j^-)\right), \tag{1.1}$$

where $\{u^j, v^j\}$ are the order of vanishing of $\beta := \beta_r$ at the points p_j^+ and p_j^- (see [CCC07, §2.2]). Furthermore the map β is defined to agree with the inclusion $\nu^*\omega_C\left(-\sum_j (u^j p_j^+ + v^j p_j^-)\right) \hookrightarrow \nu^*\omega_C$ on Y and to be zero on the exceptional components. Any other line bundle of the collection is then obtained by gluing the line bundle

$$\widetilde{N}_d := \widetilde{N}^{\otimes r/d} \otimes \mathcal{O}_Y\left(\sum_j \frac{u^j - u_d^j}{d}p^+ + \frac{v^j - v_d^j}{d}p^-\right) \tag{1.2}$$

on Y with the degree one line bundle on the exceptional components plus the datum of a gluing $\eta_d^{p_j}$ every time is needed, that is every time that d divides u (or v).

We are now ready to state the main definition.

Definition 1.2.5 (Coherent net of roots)**.** *Given a stable curve C and an integer r dividing $2g-2$, a* coherent net of roots *of order r for (C, ω_C) is the datum $(X, \{N_d, \beta_{d,d'}, \eta_d^{p_j}\}_{d|r})$ where*

- *$\pi : X \to C$ is a blow-up of C at a set of nodes $\Delta = \{p_1, \dots, p_k\}$,*
- *$N_1 = \pi^*\omega_C$ and $\beta_{d,d} = \mathrm{Id}$ for every d,*
- *for every $d|r$, the triple $(X, N_d, \beta_{d,1})$ is a limit d-th root of (C, ω_C) and the map $\beta_{d,1}$ has order $\{u_d^j, v_d^j\}$ at the points p_j^+ and p_j^- for every $j \in \{1, \dots, k\}$,*
- *for every $d'|d$, the map $\beta_{d,d'} : N_d^{\otimes d/d'} \to N_d$ is an isomorphism outside the exceptional components and has order $\{(u_d^j - u_{d'}^j)/d', (v_d^j - v_{d'}^j)/d'\}$ at the points p_j^+ and p_j^- for every $j \in \{1, \dots, k\}$,*
- *for every $j \in \{1, \dots, k\}$, if d divides u then $\eta_d^{p_j} : N_{d|p_j^+} \xrightarrow{\sim} N_{d|p_j^-}$ is a gluing datum, otherwise it is the null map.*
- *all the maps $\beta_{d,d'}$ are compatible.*

As in [Jar01] we can define what an isomorphism of coherent net of r-th roots is.

Definition 1.2.6 (Isomorphism of coherent nets)**.** *Given a stable curve C, an* isomorphism of coherent nets of r-th roots *of (C, ω_C) from $(X, \{N_d, \beta_{d,d'}, \eta_d^{p_j}\}_{d|r})$ to $(X', \{N'_d, \beta'_{d,d'}, \eta_d'^{p_j}\}_{d|r})$ is an isomorphism of curves $\tau : X \to X'$ together with a system of isomorphisms $\{\gamma_d : \tau^*N'_d \to N_d\}$ compatible with all the maps $\tau^*\beta'_{d,d'}$, $\beta_{d,d'}$, $\tau^*\eta_d'^{p_j}$ and $\eta_d^{p_j}$ and such that γ_1 is the canonical isomorphism $\beta_1 : \tau^*\pi'^*\omega_C \to \pi^*\omega_C$.*

Define the set

$$\overline{S}_g^{1/r} := \left\{ \left[C, (X, \{N_d, \beta_{d,d'}, \eta_d^{p_j}\}_{d|r}) \right] \;\middle|\; \begin{array}{c} C \text{ is a stable curve of genus } g, \\ (X, \{N_d, \beta_{d,d'}, \eta_d^{p_j}\}_{d|r}) \text{ is a coherent net of} \\ r\text{-th root in the sense of Def. 1.2.5} \end{array} \right\},$$

where $[\cdot,\cdot]$ denotes the class of r-spin curves modulo isomorphisms.

Proposition 1.2.7. *There is a scheme with underlying set $\overline{S}_g^{1/r}$ (which we will still call $\overline{S}_g^{1/r}$) isomorphic to the scheme $\overline{\mathcal{S}}_g^{1/r}$ introduced by Jarvis in [Jar01]. In particular, $\overline{S}_g^{1/r}$ is normal and projective.*

Proof. By construction, there is a set-theoretical bijection between $\overline{S}_g^{1/r}$ and the coarse moduli space $\overline{\mathcal{S}}_g^{1/r}$ of $\overline{\mathfrak{S}}_g^{1/r}$. Thanks to this correspondence, we can transport the scheme structure of the latter to the former and consider the two spaces as isomorphic schemes. Since the latter is normal and projective (see [Jar01, Proposition 3.1.1]), the proof is over. □

1.2.3 Marked points

The whole construction can be repeated considering also curves with marked points. We state here the analogous definitions. Let C be an n-pointed curve of genus g with only ordinary nodes as singularities, $\mathbf{m} = (m^1, \ldots, m^n)$ an n-tuple of integers and $r \in \mathbb{N}$ a positive integer dividing $2g - 2 - \sum m^i$. Following [CCC07], we define an *r-th-root of type* $\mathbf{m}$ as follows.

Definition 1.2.8 (Limit r-th root of type $\mathbf{m}$)**.** *Let $(C, (q_1, \ldots, q_n))$ be an n-pointed curve with only ordinary nodes as singularities. We say that the triple $((X, (q_1, \ldots, q_n)), L, \alpha)$ is an r-th root of $((C, (q_1, \ldots, q_n)), \omega_C)$ of type* $\mathbf{m}$ *if $(X, (q_1, \ldots, q_n))$ is a blow-up of $(C, (q_1, \ldots, q_n))$ at a set of nodes $\Delta = \{p_1, \ldots, p_k\}$, L is a line bundle on X and α is a homomorphism $L^{\otimes r} \to \pi^* \omega_C(-\sum m^i q_i)$ satisfying*

1. *the restriction of L to every exceptional component has degree one,*
2. *the map α is an isomorphism outside the exceptional components*
3. *for every exceptional component E_j, the order of vanishing u^j and v^j of α at $\{p_j^+, p_j^-\} = \pi^{-1}(p_j)$ add up to r,*

The arguments of §1.2.1 and §1.2.2 still hold with little modifications. In particular, equation (1.1) becomes

$$\widetilde{N}^{\otimes r} = \nu^* \omega_C \left(-\sum_j (u^j p_j^+ + v^j p_j^-) - \sum_i m^i q_i \right), \tag{1.3}$$

while equation (1.2) becomes

$$\widetilde{N}_d := \widetilde{N}^{\otimes r/d} \otimes \mathcal{O}_Y \left(\sum_j \frac{u^j - u_d^j}{d} p^+ + \sum_j \frac{v^j - v_d^j}{d} p^- + \sum_i \frac{m^i - m_d^i}{d} q_i \right). \tag{1.4}$$

The definition of coherent net of roots is completely analogous to the one without marked points. The only additional requirement is that each N_d is now a d-th root of (C, ω_C) of type $\mathbf{m}_d := (m_d^1, \ldots, m_d^n)$ and that the order of the maps $\beta_{d,d'}$ is consequently modified. Since the results of [Jar01] used in Proposition 1.2.7 hold for $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ and $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$, we can state the generalized results. The new notations have the obvious meaning.

Proposition 1.2.9. *There is a scheme with underlying set $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ (which we will still call $\overline{S}_{g,n}^{1/r,\mathbf{m}}$) isomorphic to the scheme $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ introduced by Jarvis in [Jar01]. In particular, $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ is normal and projective.*

1.3 Boundary divisors

In order to describe the boundary divisors we restrict to the case without marked points. Since the two schemes $\overline{S}_g^{1/r}$ and $\overline{\mathcal{S}}_g^{1/r}$ are isomorphic, the description of the boundary in $\overline{S}_g^{1/r}$ comes straightforward from the one of $\overline{\mathcal{S}}_g^{1/r}$ made by Jarvis.

1.3.1 Two irreducible components and one node

Let C be a stable curve with two irreducible components C_1 and C_2 of genus respectively i and $g-i$ meeting in a double point p. Let $\pi : X \to C$ be its blow-up at p and $\nu : Y \to C$ the normalization. Given a coherent net of roots for (C, ω_C), let us call the top root (X, N_r, β_r) and let $\{u, v\}$ be the order of $\beta_{r,1}$ at $\{p^+, p^-\}$. We have already seen that outside the exceptional component one has

$$\widetilde{N}_r^{\otimes r} = \nu^*\omega_C(-up^+ - vp^-) = \omega_Y(-(u-1)p^+ - (v-1)p^-).$$

We must have the degree of $\nu^*\omega_C(-up^+ - vp^-)$ divisible by r and this implies there exists a unique choice possible for u (and $v = r - u$), given by

$$u \equiv \deg \omega_{C_1} = 2i - 1 \mod r, \qquad v \equiv \deg \omega_{C_2} = 2g - 2i - 1 \mod r.$$

See for instance [CCC07, p. 29].

If $u \equiv 0 \mod r$, then $X = C$ and $N_r \in \mathrm{Pic}(C)$ corresponds to a line bundle $N_{1,r} \in \mathrm{Pic}(C_1)$ satisfying $N_{1,r}^{\otimes r} \cong \omega_{C_1}(p^+)$ and a line bundle $N_{2,r} \in \mathrm{Pic}(C_2)$ satisfying $N_{2,r}^{\otimes r} \cong \omega_{C_2}(p^-)$ plus the datum of a gluing datum $\eta_r^p : N_{1,r|p^+} \xrightarrow{\sim} N_{2,r|p^-}$. Any other line bundle of the net is then determined by $N_d := N_r^{\otimes r/d}$, $\beta_{d,d'} = \mathrm{Id}$ and $\eta_d^p := \eta_d^{p\otimes r/d}$. However, different choices for the gluing datum η_r^p correspond to automorphisms of the r-spin structure of C_1 or C_2 and hence induces different but isomorphic r-spin structure on C. In other words, different gluings correspond to the same point in $\overline{S}_g^{1/r}$. See for instance [JKV01, §1.7.1].

If $u \not\equiv 0 \mod r$, we may distinguish two cases. If $\gcd(u, v) = 1$, then there is no need of gluing datum and, as said before, all the remaining of the net is completely determined by the top root by equation (1.2). If $\gcd(u, v) = l > 1$

then there is the need to specify a gluing datum $\eta_p^d : N_{d|p^+} \xrightarrow{\sim} N_{d|p^-}$ but, as in the case $u \equiv 0$, all d gluing data will yield non-canonically isomorphic N_d and hence isomorphic nets of roots.

In conclusion, a spin curve with two irreducible components and one node corresponds to an element of $\mathfrak{S}_{i,1}^{1/r,u-1} \times \mathfrak{S}_{g-i,1}^{1/r,v-1}$ for u and v adding up to r. (see [Jar01, Example 1])

1.3.2 One irreducible component and one node

Let C be a stable curve with one node p, let X be its blow-up in p and Y its normalization. Given a coherent net or r-th root as before, this time there are r possible choices for u and v that allow a spin structure on X: or $u = v = 0$, in which case $X = C$ and the map $\beta_{r,1}$ is an isomorphism, or $u \in \{1, \dots, r-1\}$ and $v = r - u$, in which case the map $\beta_{r,1}$ has order exactly $\{u, v\}$ at $\{p^+, p^-\}$.

If $\gcd(u, v) = 1$, then the spin structure on C is completely determined by the r-th root $(X, N_r, \beta_{r,1})$ which in turn is induced by an r-th root of $\omega_Y(-(u-1)p^+ - (v-1)p^-)$, where Y is this time connected and irreducible. In particular, this kind of spin curve corresponds to an element of $\mathfrak{S}_{g-1,2}^{1/r,(u-1,v-1)}$ and, furthermore, there is a morphism $\mathfrak{S}_{g-1,2}^{1/r,(u-1,v-1)} \to \overline{\mathfrak{S}}_g^{1/r}$ whose image contains both points with $\widetilde{N}_r^{\otimes r} = \omega_Y(-(u-1)p^+ - (v-1)p^-)$ and $\widetilde{N}_r^{\otimes r} = \omega_Y(-(v-1)p^+ - (u-1)p^-)$.

If $\gcd(u, v) = l > 1$ or $u = v = 0$, then the coherent net requires the additional datum of a gluing isomorphism η_l^p (or η_r^p in the second case). This time, an automorphism of the r-spin structure on Y induces the same automorphism on both sides of the isomorphism η_l^p and hence it preserves the gluing. Consequently, we have l distinct gluing morphism (or r distinct gluing morphims in the second case) corresponding to $l/2$ (rounding up) different points in $\overline{S}_g^{1/r}$ belonging to $l/2$ distinct irreducible boundary divisors (see [Jar01, Example 2 and §3.2.2.]), since the two points in the normalization are not distinguishable.

Recall that when the genus g satisfies $g > 1$, then the moduli spaces $\overline{S}_{g,n}^{1/r,\mathbf{m}}$ are irreducible if $\gcd(r, m^1, \dots, m^n)$ is odd and they are the disjoint union of two irreducible component if $\gcd(r, m^1, \dots, m^n)$ is even. The two components corresponds respectively to r-spin curves with $h^0(N_2)$ even and $h^0(N_2)$ odd. When $g = 1$, the moduli space $\overline{S}_{1,n}^{1/r,\mathbf{m}}$ is the disjoint union of ρ irreducible components, where ρ is the number of divisors of $\gcd(r, m^1, \dots, m^n)$ (see [JKV01, §1.3] and [Jar01, Theorem 2.7]).

We can now summerize the description of the boundary divisors of $\overline{S}_g^{1/r}$ keeping the same notation of [Jar01]. We have seen that, over a stable curve in $\delta_i \subset \overline{M}_g$, given the order r of the root, there is a unique possibility for the order of the top root. This means that over δ_i there are the irreducible divisors $\alpha_i^{(a,b)}$ corresponding to the locus of spin curves with a spin structure of index a on the genus i component and of index b on the genus $g - i$ component. Here the index a parametrizes the components of $S_{i,1}^{1/r,u-1}$ while the index b parametrizes the components of $S^{1/r,v-1}_{g-i,1}$ (see [Jar01, §3.2.2.]).

Over δ_0 there are different types of spin structures. For any choice of the order $\{u,v\}$, when u and v are not relatively prime the spin structure is determined except for the glue. Also in this situation, for a particular choice of order $\{u,v\}$, of index a of the component and of the gluing η, the corresponding divisor of spin curves of the given order, index and glue is irreducible (see [Jar01, §3.2.2]). We denote these divisors by $\gamma^{(a)}_{j,\eta}$, where $j := \min(u,v)$, η is the gluing datum and a is the index of the correspondent component of $S^{1/r,(u-1,v-1)}_{g-1,2}$. Since the two points in the normalization are not distinguishable, there is the additional relation $\gamma^{(a)}_{j,\eta} = \gamma^{(a)}_{r-j,\eta^{-1}}$.

Finally, we recall the definition of the Hodge class. For the moduli space $\mathcal{M}_g$ of smooth curves of genus g the Hodge class is defined as the first Chern class of the Hodge bundle, which is the direct image of the dualizing sheaf ω_π of the universal curve $\pi : \mathcal{C}_g \to \mathcal{M}_g$. In a similar way, we can define the Hodge class $\lambda \in \mathrm{Pic}(\mathcal{S}^{1/r}_g)$ as the first Chern class of the Hodge bundle $\pi^{1/r}_* \omega_{\pi^{1/r}}$ where this time $\omega_{\pi^{1/r}}$ is the dualizing sheaf of the universal curve $\pi^{1/r} : \mathcal{C}^{1/r}_g \to \mathcal{S}^{1/r}_g$.

Theorem 1.3.1. *Assume $g \geq 9$. Then $\mathrm{Pic}(\overline{S}^{1/r}_g)$ is freely generated over $\mathbb{Q}$ by the classes λ, $\{\alpha^{(a,b)}_i\}$ and $\{\gamma^{(a,b)}_{j,\eta}\}$ just described.*

Proof. By [Jar00, Theorem 2.3], the space $\overline{\mathfrak{S}}^{1/r}_g$ is a smooth proper Deligne-Mumford stack over $\mathbb{Z}[1/r]$ and by [Jar01, Theorem 2.7] its coarse moduli space is normal and projective. By [Vis89, Proposition 2.8], if a scheme of finite type over a field of characteristic zero is the moduli space of some smooth stack, then its normalization has quotient singularities. Since the coarse moduli space of $\overline{\mathfrak{S}}^{1/r}_g$ is itself normal, we can conclude that it has quotient singularities and so also $\overline{S}^{1/r}_g$ has quotient singularities. This implies that every Weil divisor is a $\mathbb{Q}$-Cartier divisor and hence there is an isomorphism

$$\mathrm{Pic}(\overline{S}^{1/r}_g) \otimes \mathbb{Q} \cong A_{3g-4}(\overline{S}^{1/r}_g) \otimes \mathbb{Q}.$$

In particular, as in [BF06], thanks to the exact sequence

$$A_{3g-4}(\overline{S}^{1/r}_g \setminus S^{1/r}_g) \to A_{3g-4}(\overline{S}^{1/r}_g) \to A_{3g-4}(S^{1/r}_g) \to 0$$

we know that $\mathrm{Pic}(\overline{S}^{1/r}_g) \otimes \mathbb{Q}$ is generated by the generators of $A_{3g-4}(S^{1/r}_g)$ together with the set of boundary classes of $\overline{S}^{1/r}_g$.

Consider first the open part and restrict to one irreducible component whenever $S^{1/r}_g$ is not irreducible. To do this, denote by $S^{1/r}_g[\epsilon]$ the whole $S^{1/r}_g \otimes_{\mathbb{Z}[1/r]} \mathbb{C}$ if r is odd or the component of Arf invariant ϵ if r is even. Theorem 1.4 of [RW12] shows that for $g \geq 6$ one has

$$H_1(S^{1/r}_g[\epsilon]; \mathbb{Q}) = 0$$

and for $g \geq 9$ the second cohomology group $H^2(S^{1/r}_g[\epsilon]; \mathbb{Q})$ has rank 1 and it is thus generated by only one class. Since the Hodge class λ is a non trivial class in this group, we can conclude that the open part $\mathrm{Pic}(S_g^{1/r}[\epsilon]) \otimes \mathbb{Q}$ is generated, for instance, by λ.

Now we take care of the contributions coming from the boundary part. By [Jar01] the $\{\alpha_i^{(a,b)}\}$ and the $\{\gamma_{j,n}^{(a,b)}\}$ are generators for the boundary divisors of the space $\overline{S}_g^{1/r}$. Moreover, by Proposition 3.4 of [Jar01], the classes λ, $\alpha_i^{(a,b)}$ and $\gamma_{j,n}^{(a,b)}$ are independent in $\mathrm{Pic}(\overline{\mathcal{S}}_g^{1/r})$ for $g > 1$, hence they are independent also in $\mathrm{Pic}(\overline{S}_g^{1/r})$ and the claim follows. □

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Chapter 2

On the rationality of the moduli of higher spin curves in low genus

joint work with Alessandro Verra.

ABSTRACT: The global geometry of the moduli spaces of higher spin curves and their birational classification is largely unknown for $g \geq 2$ and $r > 2$. Using quite related geometric constructions, we almost complete the picture of the known results in genus $g \leq 4$ showing the rationality of the moduli spaces of even and odd 4-spin curves of genus 3, of odd spin curves of genus 4 and of 3-spin curves of genus 4.

2.1 Introduction

Let C be a smooth, irreducible complex projective curve of genus g, a theta characteristic on C is a square root η of the canonical sheaf ω_C. By definition a pair (C, η) is a spin curve, which is said to be even or odd according to the parity of $h^0(\eta)$.

Starting from Cornalba's paper [Cor89], the moduli space $\mathcal{S}_g$ of spin curves of genus g and its compactifications are object of systematic investigations. As is well known $\mathcal{S}_g$ is split in two irreducible connected components $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$. They respectively correspond to moduli of even and odd spin curves. The Kodaira dimension of $\mathcal{S}_g^{\pm}$ is completely known, as well as several facts about rationality or unirationality in low genus. The picture is as follows for even spin curves:

- ◦ *$\mathcal{S}_g^+$ is uniruled for $g \leq 7$,*
- ◦ *$\mathcal{S}_8^+$ has Kodaira dimension zero,*
- ◦ *$\mathcal{S}_g^+$ is of general type for $g \geq 9$.*

For odd spin curves one has

- ◦ $\mathcal{S}_g^-$ *is uniruled for* $g \leq 11$,
- ◦ $\mathcal{S}_g^-$ *is of general type for* $g \geq 12$.

In particular it is natural asking whether a suitable minimal model of $\mathcal{S}_8^+$ is Calabi-Yau. On the other hand $\mathcal{S}_g^-$ and $\mathcal{S}_g^+$ are known to be unirational respectively for $g \leq 8$ and $g \leq 6$. Concerning the rationality problem, $\mathcal{S}_g^{\pm}$ is classically known to be rational for $g \leq 3$, while the rationality of $\mathcal{S}_4^+$ has been recently proved. For more details on this picture see [Far10], [FV10], [FV12],[TZ09], [Ver13].

Higher spin curves generalize spin curves. By definition a higher spin curve of genus g and order r is a pair (C, η) such that $\eta^{\otimes r} \cong \omega_C$. The moduli spaces of these pairs are denoted by $\mathcal{S}_g^{1/r}$. They were constructed by Jarvis in [Jar98], [Jar00], [Jar01] and then studied by several authors. With the exception of the case of genus 1, the global geometry of these spaces and their birational classification is largely unknown for $g \geq 2$ and $r > 2$.

On the other hand a natural, though elementary, remark is that for every curve C the canonical sheaf ω_C admits square roots, the theta characteristics on C, but also $2g-2$ and $g-1$ roots.

Restricting to $g-1$ roots, they form sets of $(g-1)^{2g}$ line bundles of degree two on C which are worth of being studied. In particular the forgetful map

$$f : \mathcal{S}^{1/(g-1)} \to \mathcal{M}_g$$

has degree $(g-1)^{2g}$. Since it grows up very fast, it is perhaps natural to expect that $\mathcal{S}_g^{1/(g-1)}$ becomes of general type after few values of g. With respect to this, assume that g is odd so that $\eta^{\otimes(g-1)/2}$ is a theta characteristic. Then it follows that, if g is odd, every irreducible component of $\mathcal{S}_g^{1/(g-1)}$ dominates $\mathcal{S}_g^+$ or $\mathcal{S}_g^-$. In particular, in view of the mentioned results, there exist irreducible components of $\mathcal{S}_g^{1/(g-1)}$ having non negative Kodaira dimension as soon as $g \geq 8$. In this frame the first unknown case to be considered is the genus 4 case. Somehow surprisingly this is still an exception. Indeed we prove that

Theorem 2.1.1. *The moduli space of 3-spin curves of genus 4 is rational.*

Let (C, η) be a general spin curve of genus 4 and order 3. The starting point for proving the theorem is the remark that giving (C, η) is equivalent to give the unique effective divisor $t \in |\eta^{\otimes 2}|$. Furthermore, let C be canonically embedded in $\mathbf{P}^3$, then $3t$ is the complete intersection of two quadrics and a cubic surface. We show that the GIT-quotient of the family of these complete intersections is rational and that it admits a natural birational map to the moduli space $\mathcal{S}_4^{1/3}$.

Adding up the previous picture and the above theorem we have a list of cases of genus $g \leq 4$ where the rationality of $\mathcal{S}_g^{1/r}$ is confirmed. The complementary list of cases is as follows:

- ◦ *Moduli of* 4*-spin curves of genus* 3.

- *Moduli of odd spin curves of genus* 4.
- *Moduli of* 6-*spin curves of genus* 4.

In particular it seems that the case of odd spin curves of genus 4 was not considered in the literature. Notice also that $\mathcal{S}_g^{1/(2g-2)}$ splits into the union of two components: the moduli of pairs (C,η) such that $\eta^{\otimes g-1}$ is an even theta characteristic and the complementary component. We will denote them respectively by

$$\mathcal{S}_g^{1/(2g-2)+}\ ,\ \mathcal{S}_g^{1/(2g-2)-}.$$

We will say that (C,η) is an even (odd) r-spin curve if $\eta^{\otimes r}$ is an even (odd) theta characteristic. In the final part of this paper we almost complete the picture of the known results in genus $g \leq 4$. Building on quite related geometric constructions and methods, we prove the following theorems.

Theorem 2.1.2. *The moduli space of odd spin curves of genus 4 is rational.*

Theorem 2.1.3. *The moduli spaces of* 4*-spin curves of genus 3 are rational.*

In spite of several natural approaches, we have not found any evidence of the uniruledness of $\mathcal{S}_g^{1/r}$ for the only two missing cases in genus $g \leq 4$, namely for $\mathcal{S}_4^{1/6+}$ and $\mathcal{S}_4^{1/6-}$. The same lack of evidence appears for further low values of g, say for $g = 5,6,7$ and $r > 2$. Already for all these cases, it appears therefore interesting to apply some recent results on the structure of the Picard group of the Deligne-Mumford compactification of $\mathcal{S}_g^{1/r}$ for checking whether the canonical class is effective (cfr. for instance [Per13] and [RW12]).

2.2 Third roots on genus 4 curves

Let (C,η) be a spin curve of genus g and order r. We will assume that C is canonically embedded in $\mathbf{P}^{g-1}$.

Putting $k = [\frac{g-1}{r}]+1$, we have $h^0(\eta^{\otimes k}) \geq 1$ by Riemann-Roch. This implies that each effective divisor $t \in |\eta^{\otimes k}|$ satisfies the condition $rt = Q \cdot F$, where F is a hypersurface of degree k. If g is even then $\deg t = g$ and we expect that t is isolated, which is equivalent to have $h^1(\eta^{\otimes k}) = 0$.

Let us focus on the case $g = 4$ and $r = 3$. In this situation $C \subset \mathbf{P}^3$ is a genus 4 curve of degree 6 and t is a divisor in the linear system $\left|\eta^{\otimes 2}\right|$. Then $3t$ is a bicanonical divisor and there exists a quadric surface S such that

$$3t = C \cdot S.$$

Lemma 2.2.1. *If C be a general curve of genus* 4*, then $h^0(\eta) = 0$ for every spin curve (C,η) of order 3 .*

Proof. We can assume that $C = Q \cap F$, where Q is a fixed, smooth quadric and F a cubic surface. Now assume $h^0(\eta) = 1$ for some cubic root η of ω_C. Then there exist points $x, y \in C$ such that $x + y \in |\eta|$ and $3x + 3y = C \cdot H$, where $H \in |\mathcal{O}_Q(1)|$. Let $\mathcal{F}$ be the family of complete intersections $3x' + 3y' = C' \cdot H'$, where $H' \in |\mathcal{O}_Q(1)|$ and $C' \in |\mathcal{O}_Q(3)|$ is smooth. Notice that the action of Aut$Q$ on $\mathcal{F}$ has finitely many orbits. On the other hand, since $3x + 3y$ is a complete intersection, one computes $\dim |\mathcal{I}_{3x+3y}(C)| = 8$, where $\mathcal{I}_{3x+3y}$ is the ideal sheaf of $3x + 3y$. But then, since the moduli space of C is 9-dimensional, C is not general. This implies the statement. □

From now on we will assume that our spin curve (C, η) is sufficiently general. In particular we assume that

- $C = Q \cap F \subset \mathbf{P}^3$ *is a complete intersection of a smooth quadric* Q *and a cubic* F,
- *for each* $x \in C$ *one has* $h^0(\mathcal{O}_C(3x)) = 1$,
- $h^0(\eta) = 0$ *so that* $h^0(\eta^{\otimes 2}) = 1$.

The second condition is just equivalent to say that the two g^1_3's on C have simple ramification. The third one is satisfied iff the unique effective divisor $t \in |\eta^{\otimes 2}|$ is not contained in any plane. It is clear from the previous remarks that the scheme $3t$ is a complete intersection, namely

$$3t = F \cdot Q \cdot S,$$

where S is a second quadric. We will set

$$E := Q \cdot S.$$

The curve E is a quartic curve of arithmetic genus one. Let us point out that E is uniquely defined by (C, η).

Lemma 2.2.2. *Under the previous assumptions* E *is reduced.*

Proof. If E is not reduced then either $E = 2H$, with $H \in |\mathcal{O}_Q(1)|$, or $E = 2R + R' + R''$, where R, R', R'' are distinct lines and $R', R'' \in |H - R|$. If $E = 2H$, then $3t = 2(H \cdot C) = 6x + 6y$ for some $x, y \in C$. But then we have $t = 2x + 2y \subset H$, a contradiction. Let then $E = 2R + R' + R''$ and notice that under our assumptions the divisor $C \cdot R$ has one simple point, say x. Since $C \cdot E$ has multiplicity $3m$ at x, it follows that $x \in R' + R''$. Since $R' \cap R'' = \emptyset$, we can assume that $x \in R'$. Notice also that $C \cdot R' := x + y' + z' \neq 3x$. But then the multiplicity of $C \cdot E$ is not divisible by 3 at y' or z': a contradiction. □

We will denote by $\mathcal{I}_{at}$ the ideal sheaf in Q of the divisor $at \subset C$. Let $o \in t$ be a closed point, we can fix local parameters x, y at o so that y is a local equation of C and x restricts to a local parameter in $\mathcal{O}_{C,o}$. Then $\mathcal{I}_{at}$ is generated at o by x^{am} and y, where m is the multiplicity of t at o.

We observe that $3t$ is a 0-dimensional scheme of length 12, embedded in the smooth curve C. Since $3t$ is the complete intersection of C and E one can compute $h^0(\mathcal{I}_{3t}(3)) = 5$. Then, from the standard exact sequence

$$0 \to \mathcal{I}_{3t}(3) \to \mathcal{O}_Q(3) \to \mathcal{O}_{3t}(3) \to 0,$$

it follows $h^1(\mathcal{I}_{3t}(3)) = 1$. If $s \in H^0(\mathcal{I}_{2t/E}(3))$ we denote as Z_s the subscheme of E defined by s.

Lemma 2.2.3. *Assume s is general, then we have:*

1) $h^0(\mathcal{I}_{2t/E}(3)) = 4$ *and* $h^1(\mathcal{I}_{2t/E}(3)) = 0$,

2) *the base scheme of* $|\mathcal{I}_{2t/E}(3)|$ *is* t,

3) $Z_s = t + t'$ *where* t' *is smooth and* $t' \cap \operatorname{Sing} E = \emptyset$,

4) $3t' = E \cdot C'$ *where* $C' \in |\mathcal{O}_Q(3)|$.

Proof. 1) Consider the standard exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{E/Q}(3) \to \mathcal{I}_{2t}(3) \to \mathcal{I}_{2t/E}(3) \to 0.$$

Since $\mathcal{I}_{E/Q}(3) \cong \mathcal{O}_Q(1)$ we have $h^0(\mathcal{I}_{/Q}(3)) = 4$ and $h^1(\mathcal{I}_{E/Q}(1)) = 0$. To compute $h^i(\mathcal{I}_{2t}(3))$ consider the other standard exact sequence

$$0 \to \mathcal{I}_{C/Q}(3) \to \mathcal{I}_{2t}(3) \to \mathcal{I}_{2t/C}(3) \to 0.$$

This yelds $h^0(\mathcal{I}_{2t}(3)) = 8$ and $h^i(\mathcal{I}_{2t}(3)) = 0$ for $i > 0$. Hence 1) follows.

2) The previous exact sequence implies the surjectivity of the restriction map $r : H^0(\mathcal{I}_{2t}(3)) \to H^0(\mathcal{I}_{2t/C}(3))$. We have $\mathcal{I}_{2t/C}(3) \cong \omega_C(t)$ and moreover $\omega_C(t)$ is globally generated by [GL86]. Then there exists C' in $|\mathcal{I}_{2t}(3)|$ such that $C' \cdot C = t + d$ with $d \cap t = \emptyset$. If b denotes the base scheme of $|\mathcal{I}_{2t}(3)|$, we have $2t \subseteq b \subset 3t \subset C$ so that $b \cap d = \emptyset$. This implies $C \cdot C' = b + d$ and $2t = b$.

3) just follows from 2) and Bertini theorem.

4) To prove 4) observe that $2C \cdot E = 6t$ and $3C_1 \cdot E = 6t + 3t'$, hence $3t' \in |\mathcal{O}_E(3)|$. □

Notice that (C, η) uniquely defines on E a non trivial third root of $\mathcal{O}_E$. Indeed, let h be a plane section of E and define

$$\epsilon := \mathcal{O}_E(t' - h).$$

It is clear that $\epsilon^{\otimes 3} \cong \mathcal{O}_E$. Finally we deduce the following useful result.

Theorem 2.2.4. *On a dense open set $U \subset \mathcal{S}_4^{1/3}$ every point is the moduli point of a spin curve (C, η) such that:*

- (C, η) *satisfies the previous assumptions of generality,*
- E *is a smooth quartic elliptic curve,*
- t *is a smooth divisor of* E,
- $t \in |\epsilon(1)|$, *where ϵ is a non trivial third root of* $\mathcal{O}_E$.

Proof. Let $\mathcal{D}$ be the family of pairs (C, η, t) such that E is singular. We consider the morphism $\phi : \mathcal{D} \to |\mathcal{O}_Q(3)|$ such that $\phi(C, \eta, t) := E$. Fixing E, a triple (C', η', t') is in the fibre at E iff: (i) $t' \in |\epsilon(1)| \cong |\mathcal{I}_{2t}(3)|$, (ii) $3t'$ is a complete intersection, (iii) $\dim |\mathcal{I}_{3t'}(3)| = 4$, (iv) $\dim |\epsilon(1)| = 3$. Then the fibre of ϕ at any point $E \in \phi(\mathcal{D})$ is 7-dimensional. Indeed we have finitely many non trivial third roots of $\mathcal{O}_E$ and each of them defines an irreducible 7-dimensional component of $\phi^{-1}(E)$, as above. Since the singular E's form a 7-dimensional family, it follows $\dim \mathcal{D} = 14$. Let now $m : \mathcal{D} \to \mathcal{S}_4^{1/3}$ be the natural map. We know that $\mathcal{S}_4^{1/3}$ has pure dimension 9 and, moreover, Aut$Q$ is 6-dimensional and acts on $\mathcal{D}$ with finite stabilizer at each point. Then $\mathcal{S}_4^{1/3} - m(\mathcal{D})$ contains a dense open set and this implies the statement. We omit further details. □

2.3 Projective bundles related to $\mathcal{S}_4^{1/3}$

Let (C, η) be a general spin curve of order 3 and genus 4. We keep the previous conventions, so that C is canonically embedded in $\mathbf{P}^3$ as $Q \cap F$.

It follows from the above theorem that the moduli point $[C, \eta]$ uniquely defines, up to isomorphisms, a triple (E, ϵ, t) such that E is a smooth quartic elliptic curve in $\mathbf{P}^3$ and ϵ is a non trivial third root of $\mathcal{O}_E$.

Moreover, t is a smooth element of $|\epsilon(1)|$ and it is the complete intersection

$$3t = C \cdot E = F \cdot Q \cdot S \subset \mathbf{P}^3,$$

where S is a quadric. Finally, as a divisor in C, t is also the unique element of $|\eta^{\otimes 2}|$.

In order to prove the rationality of $\mathcal{S}_4^{1/3}$ our strategy is as follows. We consider the moduli space of level 3 elliptic curves

$$\mathcal{R}_{1,3} := \{[E, \epsilon] \mid g(E) = 1, \quad \epsilon \neq \mathcal{O}_E, \quad \epsilon^{\otimes 3} \cong \mathcal{O}_E\}.$$

Over it we have the moduli space $\mathcal{P}_{1,4}$ of triples (E, ϵ, H) such that $H \in \mathrm{Pic}^4 E$. This can be also defined via the Cartesian square

$$\begin{array}{ccc} \mathcal{P}_{1,4} & \longrightarrow & \mathcal{P}ic_{4,1} \\ \downarrow & & \downarrow \\ \mathcal{R}_{1,3} & \longrightarrow & \mathcal{M}_1. \end{array}$$

As usual, $\mathcal{P}ic_{4,1}$ denotes the universal Picard variety, that is, the moduli space of pairs (H, E) such that E is an elliptic curve and $H \in \mathrm{Pic}^4\ E$.

The space $\mathcal{P}_{4,1}$ is a rational surface. Proving its unirationality, so that the rationality follows, is an easy exercise. Starting from $\mathcal{P}_{4,1}$ we construct a suitable "tower"

$$\mathbb{P}_c \xrightarrow{c} \mathbb{P}_b \xrightarrow{b} \mathbb{P}_a \xrightarrow{a} \mathcal{P}_{4,1}$$

of projective bundles a, b, c. Clearly, as a "tower" of projective bundles over a rational base, $\mathbb{P}$ is rational. Let $\phi : \mathcal{S}_4^{1/3} \to \mathcal{P}_{4,1}$ be the rational map defined as $\phi([C, \eta]) := [E, \epsilon]$. Then we will show that ϕ factors through a natural birational map between $\mathcal{S}_4^{1/3}$ and $\mathbb{P}_c$, so proving that $\mathcal{S}_4^{1/3}$ is rational. In the next subsections we produce the projective bundles which are needed.

2.3.1 The ambient bundle $\mathbb{P}$

Let us start with the universal curve over $\mathcal{M}_1$ and its pull-back $\mathcal{E} \to \mathcal{R}_{1,3}$. Consider the open set $U \subset \mathcal{R}_{1,3}$ of points $[E, \epsilon]$ such that $\mathrm{AutPic}^0 E$ is trivial. On $\mathcal{P}_{4,1} \times_U \mathcal{E}$ there exists a Poincaré bundle $\mathcal{P}$. Then the restriction of $\mathcal{P}$ to the fibre at $[E, \epsilon, H]$ of the projection map

$$\alpha : \mathcal{P}_{4,1} \times_U \times \mathcal{E} \to \mathcal{P}_{4,1}$$

is given by $\mathcal{P} \otimes \mathcal{O}_{\{[E,\epsilon,H]\} \times E} \cong H$. Note that $(\alpha_* \mathcal{P})_{[E,\epsilon,H]} = H^0(H)$ has constant dimension 4. Defining $\mathcal{H} := \alpha_* \mathcal{P}$ we have, by Grauert's theorem, that $\mathcal{H}$ is a vector bundle of rank 4 over $\mathcal{P}_{4,1}$. We define the the ambient bundle $\mathbb{P}$ as follows:

$$\mathbb{P} := \mathbf{P}\mathcal{H}^*.$$

It is a $\mathbf{P}^3$-bundle over $\mathcal{P}_{4,1}$ and its structure map will be denoted by $p : \mathbb{P} \to \mathcal{P}_{4,1}$. In particular, the tautological bundle $\mathcal{O}_{\mathbb{P}}(1)$ defines an embedding

$$\mathcal{P}_{4,1} \times_U \mathcal{E} \subset \mathbb{P}.$$

At $x := [E, \epsilon, H]$ this is the embedding $E \subset \mathbb{P}_x = \mathbf{P}H^0(H)^*$ defined by H.

2.3.2 The bundle of quadrics $a : \mathbb{P}_a \to \mathcal{P}_{4,1}$

Let us consider the map

$$\mu : \mathrm{Sym}^2 \mathcal{H} \to \alpha_*(\mathcal{P}^{\otimes 2})$$

of vector bundles on $\mathcal{P}_{4,1}$. At $x := [E, \epsilon, H]$ we have $\alpha_*(\mathcal{P}^{\otimes 2})_x = H^0(H^{\otimes 2})$ and

$$\mu_x : \mathrm{Sym}^2 H^0(H) \to H^0(H^{\otimes 2})$$

is the multiplication map. Putting $\mathcal{Q} := \ker \mu$ and $\mathbb{P}_a := \mathbf{P}\mathcal{Q}$, we denote as

$$a : \mathbb{P}_a \to \mathcal{P}_{4,1}$$

the structure map. The bundle a is a $\mathbf{P}^1$-bundle and the fibre $\mathbb{P}_x$ parametrizes the quadrics containing the tautological embedding $E \subset \mathbb{P}_x$ defined by H.

2.3.3 The $\mathbf{P}^3$-bundle $b : \mathbb{P}_b \to \mathbb{P}_a$

At first we define the $\mathbf{P}^3$-bundle

$$e : \mathbb{P}_e \to \mathcal{P}_{4,1}.$$

Its fibre $\mathbb{P}_{e,x}$ will be $|\epsilon \otimes H|$ at $x := [E, \epsilon, H]$. On $\mathcal{P}_{4,1} \times_U \mathcal{E}$ we fix a vector bundle $\mathcal{N}$ whose restriction to the fibre of $\alpha : \mathcal{P}_{4,1} \times_U \mathcal{E} \to \mathcal{P}_{4,1}$ at x is

$$\mathcal{N} \otimes \mathcal{O}_{\alpha^* x} \cong \epsilon.$$

In order to define $\mathcal{N}$, let $\beta : \mathcal{P}_{4,1} \times_U \mathcal{E} \to \mathcal{R}_{1,3} \times_U \mathcal{E}$ be the natural map. Then we define $\mathcal{N} := \beta^* \mathcal{L}$, where $\mathcal{L}$ is a Poincaré bundle on $\mathcal{R}_{1,3} \times_U \mathcal{E}$. In particular the restriction of $\mathcal{L}$ to the fibre at $[E, \epsilon]$ of the projection $\gamma : \mathcal{R}_{1,3} \times_U \mathcal{E} \to \mathcal{R}_{1,3}$ is the line bundle ϵ. We consider the tensor product $\mathcal{H} \otimes \mathcal{N}$ and finally $\alpha_*(\mathcal{H} \otimes \mathcal{N})$. This is a rank 4 vector bundle with fibre $H^0(H \otimes \epsilon)$ at x. Let us define

$$\mathbb{P}_b := a^* \mathbf{P} \alpha_*(\mathcal{H} \otimes \epsilon).$$

The bundle $\mathbb{P}_b$ is a $\mathbf{P}^3$-bundle over $\mathbb{P}_a$. The fibre of $a \circ b : \mathbb{P}_b \to \mathcal{P}_{4,1}$ at x is the Segre product $|\epsilon \otimes H| \times |\mathcal{I}_E(2)|$, where $\mathcal{I}_E$ is the ideal sheaf of $E \subset \mathbb{P}_x$.

2.3.4 The $\mathbf{P}^3$-bundle $c : \mathbb{P}_c \to \mathbb{P}_b$

In the fibre product $\mathbb{P}_b \times_{\mathcal{P}_{4,1}} \mathbb{P}$ we define as follows the subvarieties

$$\mathbf{t} \subset \mathbf{E} \subset \mathbf{Q} \subset \mathbb{P}_{\mathbf{b}} \times_{\mathcal{P}_{\mathbf{4,1}}} \mathbb{P}.$$

A point $o \in \mathbb{P}_b \times_U \mathbb{P}$ defines a pair (v, z) where $z \in \mathbb{P}_x$ and $x := a \circ b(o)$ $= [E, \epsilon, H]$. Moreover, the point o is an element $t \in |\epsilon \otimes H|$ of the fibre of $\mathbb{P}_b$ at $b(o)$. Finally $b(o)$ is an element $Q \in |\mathcal{I}_E(2)|$, where $\mathcal{I}_{\mathbf{E}}$ is the ideal sheaf of the tautological embedding $\mathbf{E} \subset \mathbb{P}_{\mathbf{x}}$. Clearly we have $\mathbf{t} \subset \mathbf{E} \subset \mathbf{Q}$.

The conditions $z \in \mathbf{t}$, $z \in \mathbf{E}$, $z \in \mathbf{Q}$ respectively define the closed sets $\mathbf{t}$, $\mathbf{E}$, $\mathbf{Q}$ as well as the previous inclusions. In particular $\mathbf{E}$ is an embedding of $\mathcal{P}_{1,4} \times_U \mathcal{E}$ and $\mathbf{t}$ is a Weil divisor in $\mathbf{E}$. Let us consider the standard exact sequence

$$0 \to \mathcal{I}_{3\mathbf{t}} \to \mathcal{O}_{\mathbf{Q}} \to \mathcal{O}_{3\mathbf{t}} \to 0$$

where $\mathcal{I}_{3\mathbf{t}}$ is the ideal sheaf of $\mathbf{t}$ in $\mathbf{Q}$. We pull-back the line bundle $\mathcal{O}_{\mathbb{P}}(3)$ to the fibre product $\mathbb{P}_b \times_{\mathcal{P}_{4,1}} \mathbb{P}$ and tensor the above exact sequence by it. The resulting exact sequence is denoted in the following way:

$$0 \to \mathcal{I}_{3\mathbf{t}}(3) \to \mathcal{O}_{\mathbf{Q}}(3) \to \mathcal{O}_{3\mathbf{t}}(3) \to 0.$$

Let $\beta : \mathbb{P}_b \otimes \mathbb{P} \to \mathbb{P}_b$ be the projection onto $\mathbb{P}_b$. Then we apply the push-down functor β_* to this new exact sequence. We obtain the exact sequence

$$0 \to \beta_* \mathcal{I}_{3\mathbf{t}}(3) \to \beta_* \mathcal{O}_{\mathbf{Q}}(3) \to \beta_{*|\mathbf{Q}} \mathcal{O}_{3\mathbf{t}}(3) \to R^1 \beta_* \mathcal{I}_{3\mathbf{t}}(3) = 0.$$

Here the sheaf $R^1 \beta_* \mathcal{I}_{3\mathbf{t}}(3)$ is zero because at any point $p = (t, Q, [E, \epsilon, H]) \in \mathbb{P}_b$ its fibre is $H^1(\mathcal{I}_{3t/Q}(3)) = 0$. Notice also that the sheaf $\mathcal{F} := \beta_* \mathcal{I}_{3\mathbf{t}}(3)$ is a rank 5 vector bundle with fibre $H^0(\mathcal{I}_{3t/Q}(3)) = 0$ at the same point p. Finally we define

$$\mathbb{P}_c := \mathbf{P}\mathcal{F}.$$

We denote the structure map of this $\mathbf{P}^4$-bundle as $c : \mathbb{P}_c \to \mathbb{P}_b$.

The fibre of c at p is the linear system of cubic sections C of Q containing the scheme $3t \subset E$. So a smooth C is a canonical curve of genus 4 endowed with the order 3 spin structure

$$\eta := \omega_C(-t).$$

2.4 The rationality of $\mathcal{S}_4^{1/3}$

Let $\mathcal{I}_{2t/\mathbf{P}^3}$ be the ideal sheaf of $2t \subset C \subset \mathbf{P}^3$. Notice also that

Lemma 2.4.1. *$|\mathcal{I}_{2t/\mathbf{P}^3}(2)|$ is a pencil of quadrics with base locus E.*

Proof. Observe that $\omega_C^{\otimes 2}(-2t) \cong \eta^{\otimes 2}$. Moreover, this is also the sheaf $\mathcal{I}_{2t/C}(2)$. Consider the standard exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{C/\mathbf{P}^3}(2) \to \mathcal{I}_{2t/\mathbf{P}^3}(2) \to \eta^{\otimes 2} \to 0.$$

Since we have $h^0(\mathcal{I}_{C/\mathbf{P}^3}(2)) = h^0(\eta^{\otimes 2}) = 1$, the statement follows. □

Due to the latter construction there exists a natural moduli map

$$\phi : \mathbb{P}_c \to \mathcal{S}_4^{1/3}$$

which sends a point $z = (C, t, Q, [E, \epsilon, H]) \in \mathbb{P}_c$ to the point

$$\phi(z) := (C, \eta),$$

with $\eta = \omega_C(-t)$. Clearly ϕ is defined at z iff C is smooth.

Proposition 2.4.2. *$\mathcal{S}_4^{1/3}$ is irreducible.*

Proof. The domain $\mathbb{P}_c$ is irreducible and $\mathcal{S}_4^{1/3}$ has pure dimension 9. Hence it suffices to show that $V := \mathcal{S}_4^{1/3} - \phi(\mathbb{P}_c)$ has dimension ≤ 8. Given $[C, \eta] \in V$, we can assume that C is not hyperelliptic and that its canonical model $C \subset \mathbf{P}^3$ is contained in a smooth quadric Q. Due to the results of §2.2, we can also assume that $h^0(\eta^{\otimes 2}) = 1$ and that E is smooth, where E is the base locus of the pencil of quadrics $|\mathcal{I}_{2t/\mathbf{P}^3}(2)|$ and $t \in |\eta^{\otimes 2}|$. Let $\epsilon = H(-t)$ with $H = |\mathcal{O}_E(1)|$. If $\mathrm{AutPic}^0 E$ is trivial, we have $[C, \eta] = \phi([C, t, Q, [E, \epsilon, H])$ with $H = \mathcal{O}_E(1)$ and hence $[E, \epsilon]$ is in a finite set of $\mathcal{R}_{1,3}$. Fixing E it is easy to see that the family of 4-tuples $(C, t, Q, [E, \epsilon, H])$ is 8-dimensional modulo $\cong$. This implies the statement. □

Since $\mathbb{P}_c$ is rational we can finally deduce the rationality of $\mathcal{S}_4^{1/3}$, stated in the Introduction. We show that

Theorem 2.4.3. *The map $\phi : \mathbb{P}_c \to \mathcal{S}_4^{1/3}$ is birational, so that $\mathcal{S}_4^{1/3}$ is rational.*

Proof. At first we show that the map ϕ is dominant. Starting with a general point $[C, \eta] \in \mathcal{S}_4^{1/3}$ it is possible to reconstruct a point $z = (C, t, Q, [E, \epsilon, H]) \in \mathbb{P}_c$ such that $\phi(z) = [C, \eta]$. Indeed t is the unique element of $|\eta^{\otimes 2}|$. Then, from the canonical embedding $C \subset \mathbf{P}^3$, we reconstruct E as the smooth base locus of the pencil of quadrics $|\mathcal{I}_{2t}(2)|$, considered above. Then we have $H := \mathcal{O}_E(1)$ and $\epsilon := H(-t)$. The quadric Q is the unique quadric of $|\mathcal{I}_{2t/\mathbf{P}^3}(2)|$ containing C. It is clear that $[C, \eta] = \phi(z)$, with $z = (C, t, Q, [E, \epsilon, H])$. Conversely the inverse map of ϕ is well-defined too. Starting from a general $[C, \eta]$ the point z is indeed uniquely reconstructed as above. Hence ϕ^{-1} is well defined and ϕ is birational. □

In the next sections we prove the other rationality results announced in the Introduction.

2.5 The rationality of $\mathcal{S}_4^-$

We start from an odd spin curve (C, η) of genus 4. As in the previous sections, C will be sufficiently general. Thus, passing to its canonical model, we have

$$C \subset Q \subset \mathbf{P}^3,$$

where $Q = \mathbf{P}^1 \times \mathbf{P}^1$ is a smooth quadric and C has bidegree $(3,3)$ in it. Since η is odd, there exists a unique $d \in |\eta|$ and we have

$$2d = L \cdot C,$$

where L is a plane section of Q and a conic tritangent to C. We omit the proof of the next, very easy, lemma:

Lemma 2.5.1. *For a general C both the divisor d and the plane section L are smooth.*

Let o_1, o_2, o_3 be the three points of d. They are not collinear because $h^0(\eta) = 1$. Hence we can fix projective coordinates $(x_0 : x_1) \times (y_0 : y_1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ so that

$$o_1 = (1:0) \times (1:0)\ ,\ o_2 = (0:1) \times (0:1)\ ,\ o_3 = (1:1) \times (1:1).$$

In particular we can assume that these points are in the diagonal

$$L := \{x_0 y_1 - x_1 y_0 = 0\}$$

of $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\mathcal{I}_{2d}$ be the ideal sheaf of $2d$ in $\mathbf{P}^1 \times \mathbf{P}^1$ and let

$$I := H^0(\mathcal{I}_{2d}(3,3)).$$

We consider the 9-dimensional linear system $\mathbf{P}I$. This is endowed with the map

$$m : \mathbf{P}I \to \mathcal{S}_4^-$$

defined as follows. Let $C \in \mathbf{P}I$ be smooth, then $m(C) := [C, \eta]$, where $\eta := \mathcal{O}_C(o_1 + o_2 + o_3)$. It is clear from the construction that m is dominant. Let

$$\mathbb{G} \subset \mathrm{Aut}\mathbf{P}^1 \times \mathbf{P}^1 \tag{2.1}$$

be the stabilizer of the set $\{o_1, o_2, o_3\}$. We have:

Lemma 2.5.2. *Assume $C_1, C_2 \in \mathbf{P}I$ are smooth. Then $m(C_1) = m(C_2)$ if and only if $C_2 = \alpha(C_1)$ for some $\alpha \in \mathbb{G}$.*

Proof. Let $m(C_i) = [C_i, \eta_i]$, $i = 1, 2$. If $m(C_1) = m(C_2)$ there exists a biregular map $a : C_2 \to C_1$. Since $\mathcal{O}_{C_i}(1,1) \cong \omega_{C_i}$, it follows that a induces an isomorphism $a^* : H^0(\mathcal{O}_{C_1}(1,1)) \to H^0(\mathcal{O}_{C_2}(1,1))$. This implies that a is induced by some $\alpha \in \mathrm{Aut}\mathbf{P}^1 \times \mathbf{P}^1$. Furthermore, the condition $m(C_1) = m(C_2)$ also implies that $a^* \mathcal{O}_{C_2}(o_1 + o_2 + o_3) \cong \mathcal{O}_{C_1}(o_1 + o_2 + o_3)$. Hence $\alpha \in \mathbb{G}$. The converse is obvious. □

Now observe that $\mathbb{G}$ acts on $\mathbf{P}I$ in the natural way and $m : \mathbf{P}I \to \mathcal{S}_4^-$ is dominant. Then, as an immediate consequence of the previous lemma, we have

Corollary 2.5.3. *$\mathcal{S}_4^-$ is birational to the quotient $\mathbf{P}I/\mathbb{G}$.*

Thus the rationality of $\mathcal{S}_4^-$ follows if $\mathbf{P}I/\mathbb{G}$ is rational. In order to prove this, we preliminarily describe the group $\mathbb{G}$ and its action on $\mathbf{P}I$. We recall that the natural inclusion $\mathrm{Aut}\mathbf{P}^1 \times \mathrm{Aut}\mathbf{P}^1 \subset \mathrm{Aut}\mathbf{P}^1 \times \mathbf{P}^1$ induces the exact sequence

$$0 \to \mathrm{Aut}\mathbf{P}^1 \times \mathrm{Aut}\mathbf{P}^1 \to \mathrm{Aut}\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbb{Z}_2 \to 0,$$

where $\mathbb{Z}_2$ is generated by the class of the projective involution

$$\iota : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$$

exchanging the factors. From the above exact sequence we have the exact sequence

$$0 \to \mathbb{G}_3 \to \mathbb{G} \to \mathbb{Z}_2 \to 0.$$

Here $\mathbb{G}_3$ denotes the stabilizer of the set $\{o_1, o_2, o_3\}$ in $\mathrm{Aut}\mathbf{P}^1 \times \mathrm{Aut}\mathbf{P}^1$. Since this set of three points is fixed by $\mathbb{G}_3$, it follows that the diagonal L is also fixed. Hence $\mathbb{G}_3$ is the diagonal embedding in $\mathrm{Aut}\mathbf{P}^1 \times \mathrm{Aut}\mathbf{P}^1$ of the stabilizer of $\{o_1, o_2, o_3\}$ in $\mathrm{Aut}\mathbf{P}^1$. As is very well known, this is a copy of the symmetric group S_3.

Now we aim to describe explicitely the $\mathbb{G}$-invariant subspaces of $\mathbf{P}I$. We fix the notation $l := x_0y_1 - x_1y_0$ for the equation of the diagonal L. Let

$$R = \oplus_{a,b\in\mathbb{Z}} R_{a,b}$$

be the coordinate ring of $\mathbf{P}^1 \times \mathbf{P}^1$, where $R_{a,b}$ is the vector space of forms of bidegree a, b. We can assume that $\iota^* : R \to R$ is the involution such that $\iota^* x_i = y_i$, $i = 0, 1$. On the other hand let

$$h_1 := x_0(y_1-y_0)+y_0(x_1-x_0)\ ,\ h_2 := x_1(y_0-y_1)+y_1(x_0-x_1)\ ,\ h_3 := x_0y_1+x_1y_0,$$

so that $\{l, h_1, h_2, h_3\}$ is a basis of $R_{1,1}$. We can also assume that, for each $\sigma \in \mathbb{G}_3$, the map $\sigma^* : R \to R$ is such that $\sigma^* l = l$ and σ^* permutes the elements of the set $\{h_1, h_2, h_3\}$. Then we observe that the eigenspaces of $\iota^* : R_{1,1} \to R_{1,1}$ are

$$R_{1,1}^- =< l >\ ,\ R_{1,1}^+ =< h_1, h_2, h_3 > .$$

This implies that

$$R_{1,1} =< l > \oplus < h_1 + h_2 + h_3 > \oplus < h_1 - h_3, h_2 - h_3 >$$

where all the summands are $\mathbb{G}$-invariant. Considering the multiplication map

$$\mu : \mathrm{Sym}^2 R_{1,1} \to R_{2,2}$$

one can check that

$$\mathrm{Ker}\,\mu =< h_3^2 - l^2 - (h_1 - h_3)(h_2 - h_3) > .$$

Then, putting $h := h_1 + h_2 + h_3$ and $h_{ij} := h_i - h_j$, it is easy to deduce that the eigenspaces of $\iota^* : R_{2,2} \to R_{2,2}$ decompose as follows:

$$R_{2,2}^+ =< h^2 > \oplus < hh_{13} \ , \ hh_{23} > \oplus < h_{13}^2 \ , \ h_{23}^2 > \oplus < h_{13}h_{23} >$$

and

$$R_{2,2}^- =< lh > \oplus < lh_{13} \ , \ lh_{23} >,$$

where each summand appearing above is $\mathbb{G}$-invariant. Finally, we consider the vector space I and observe that, taking the multiplication by l, we have an injection

$$< l > \otimes R_{2,2} \hookrightarrow I.$$

Its image $lR_{2,2} \subset I$ is a subspace codimension one. Moreover we have

$$lR_{2,2}^+ \subseteq I^- \ , \ lR_{2,2}^- \subseteq I^+,$$

where I^+, I^- are the eigenspaces of $\iota^* : I \to I$. Let us consider

$$c = x_0x_1(x_0 - x_1) + y_0y_1(y_0 - y_1).$$

Notice that $c \in I$ and that $\div(c)$ is $\mathbb{G}$-invariant. Indeed, $\div(c)$ is the union of the six lines in the quadric $Q = \mathbf{P}^1 \times \mathbf{P}^1$ passing through the points o_1, o_2, o_3. Notice also that c is not in $lR_{2,2}$, in particular $I =< c > \oplus lR_{2,2}$. Notice also that $\iota^* c = c$.

Summing all the previous remarks up, we can finally describe the eigenspaces of $\iota^* : I \to I$ and their decompositions as a direct sum of $\mathbb{G}$-invariant summands. Actually, each summand is an irreducible representation of $\mathbb{G}$.

Lemma 2.5.4. *Let I^+, I^- be the eigenspaces of $\iota^* : I \to I$, then we have*

- $I^+ =< c > \oplus < l^2h > \oplus < l^2h_{13} \ , \ l^2h_{23} >$.
- $I^- =< lh^2 > \oplus < lhh_{13} \ , \ lhh_{23} > \oplus < lh_{13}^2 \ , \ lh_{23}^2 > \oplus < lh_{13}h_{23} >$.

Now let us consider

$$B := \mathbf{P}I^+ \times \mathbf{P}I^-$$

and then the variety

$$\mathbb{P} := \{(x, p) \in \mathbf{P}I \times B \ | \ x \in \mathbb{P}_p\} \subset \mathbf{P}I \times B,$$

where $p := (p^+, p^-) \in \mathbf{P}I^+ \times \mathbf{P}I^-$ and $\mathbb{P}_p$ denotes the line joining p^+ and p^-. The variety $\mathbb{P}$ is endowed with its two natural projections

$$\mathbf{P}I \xleftarrow{\beta} \mathbb{P} \xrightarrow{\alpha} B.$$

It is clear that $\beta : \mathbb{P} \to \mathbf{P}I$ is birational, since there exists a unique line $\mathbb{P}_p$ passing through a point in $\mathbf{P}I - (\mathbf{P}I^+ \cup \mathbf{P}I^-)$. Moreover $\alpha : \mathbb{P} \to B$ is a $\mathbf{P}^1$-bundle structure with fibre $\mathbb{P}_p$ at $p = (p^+, p^-)$. It is also clear that the action of $\mathbb{G}$ on $\mathbf{P}I$ induces an action of $\mathbb{G}$ on $\mathbb{P}$ and that $\mathbf{P}I/\mathbb{G} \cong \mathbb{P}/\mathbb{G}$.

More precisely, the map ι^* acts as the identity on B, since its two factors are projectivized eigenspaces of ι^*. Moreover each fibre $\mathbb{P}_p$ is ι^*-invariant: $\iota^*/\mathbb{P}_p$ is a projective involution with fixed points p^+, p^-.

On the other hand, the action of $\mathbb{G}_3$ is generically free since the 2-dimensional summands of $I^\pm$ are the standard representations of S_3. We are now in position to deduce the rationality of $\mathbf{P}I/\mathbb{G}$. Keeping the previous notations, we show that:

Theorem 2.5.5. *The quotient $\mathbb{P}/\mathbb{G}$ is rational.*

Proof. The group $\mathbb{G}_3$ acts linearly on the fibres of the $\mathbf{P}^1$-bundle $\alpha : \mathbb{P} \to B$. Indeed, for each $p = (p^+, p^-) \in B$, each $\phi \in \mathbb{G}_3$ induces a projective isomorphism $\phi/\mathbb{P}_p : \mathbb{P}_p \to \mathbb{P}_{\phi(p)}$, where the latter is the line spanned by the points $\phi(p^+)$ and $\phi(p^-)$. Since the action of $\mathbb{G}_3$ is generically free, it follows that $\mathbb{P}$ descends to a $\mathbf{P}^1$-bundle $\overline{\alpha} : \overline{\mathbb{P}} \to U$ over a dense open set $U \subset B/\mathbb{G}_3$. Moreover, the projective involution ι induces a projective involution $\overline{\iota} : \overline{\mathbb{P}} \to \overline{\mathbb{P}}$ fixing each fibre of $\overline{\alpha}$. Let $\overline{p} \in U$ be the orbit of $p = (p^+, p^-) \in B$. Then the fixed points of $\overline{\iota} : \overline{\mathbb{P}}_{\overline{p}} \to \overline{\mathbb{P}}_{\overline{p}}$ are the orbits $\overline{p}^+, \overline{p}^-$ of p^+, p^-. Passing to the quotient of $\overline{\mathbb{P}}$ by $\overline{\iota}$, these two fixed points define on it two sections $s^\pm : U \to \overline{\mathbb{P}}/ < \overline{\iota} >$ defined by the relation $s^\pm(\overline{p}) = \overline{p}^\pm$. Since every fibration with fibre $\mathbf{P}^1$ and a section is locally trivial in the Zariski topology, it follows that

$$\mathbb{P}/\mathbb{G} \cong U \times \mathbf{P}^1.$$

To complete the proof it remains to show the rationality of $U \cong (\mathbf{P}I^+ \times \mathbf{P}I^-)/\mathbb{G}_3$. Since each projectivized $\mathbb{G}$-invariant summand of I^+ and I^- has a rational quotient under the action of $\mathbb{G}$, the rationality of any such quotient is standard. $\square$

We have already seen that $\mathbb{P}/\mathbb{G}$ is birational to the moduli space of odd spin curves of genus four. Hence it follows:

Corollary 2.5.6. *The moduli space $\mathcal{S}_4^-$ is rational.*

2.6 The rationality of $\mathcal{S}_3^{1/4-}$

As we will see, the rationality result to be proven in this section naturally relies on the geometry of odd spin curves of genus four we considered above. To see this relation let us fix from now on a general curve C of genus three and two distinct points $n_1, n_2 \in C$. As is well known, the line bundle $\omega_C(n_1+n_2)$ defines a morphism

$$\phi : C \to C_n \subset Q := \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$$

which is generically injective and such that C_n is a curve of bidegree $(3,3)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ with exactly one node $n := \phi(n_1) = \phi(n_2)$, see [GL86]. The condition that $\mathcal{O}_C(n_1 + n_2)$ is a theta characteristic is reflected by the projective model C_n as follows:

Lemma 2.6.1. *Let R_1 and R_2 be the two lines of Q passing through the node n. Then the following conditions are equivalent:*

∘ R_1, R_2 *are tangent to the branches of* n*, that is* $\phi^* R_i = 2n_i + n_j$ *for* $i \neq j$.

∘ $\mathcal{O}_C(n_1 + n_2)$ *is a theta characteristic.*

Assume now that $[C, \eta]$ is a general point of $\mathcal{S}_3^{1/4-}$ so that $\eta^{\otimes 2}$ is an odd theta characteristic on C. This is equivalent to say that there exist two distinct points $n_1, n_2 \in C$ such that

$$\mathcal{O}_C(n_1 + n_2) \cong \eta^{\otimes 2} \text{ and } \eta^{\otimes 6} \cong \omega_C(n_1 + n_2) \cong \mathcal{O}_C(3n_1 + 3n_2).$$

Considering the morphism ϕ defined by $\eta^{\otimes 6}$, we have as above that its image

$$C_n \subset Q = \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$$

is a curve with exactly one node $n = \phi(n_1) = \phi(n_2)$ and no other singular point. Now we observe that the linear system $|\eta^{\otimes 6}|$ contains the two distinct elements:

∘ $3n_1 + 3n_2$, where $n_1 + n_2 \in |\eta^{\otimes 2}|$,

∘ $2o_1 + 2o_2 + 2o_3$, where $o_1 + o_2 + o_3 \in |\eta^{\otimes 3}|$.

Lemma 2.6.2. *One has* $h^0(\eta) = 0$*, so that* $h^0(\mathcal{O}_C(o_1 + o_2 + o_3)) = 1$.

Proof. If $h^0(\eta) \geq 1$ then $\eta \cong \mathcal{O}_C(p)$ for some point $p \in C$. But then $4p \in |\omega_C|$, which is impossible on a general curve C of genus three. Since $h^0(\eta) = 0$ and $\omega_C(-o_1 - o_2 - o_3) \cong \eta$, we have from Riemann-Roch and Serre duality that $h^0(\mathcal{O}_C(o_1 + o_2 + o_3)) = 1 + h^0(\eta) = 1$. □

Lemma 2.6.3. *For a general* C*, the points* o_1, o_2, o_3 *are distinct and furthermore* $\{o_1\ o_2\ o_3\} \cap \{n_1\ n_2\} = \emptyset$*. Moreover* $2o_1 + 2o_2 + 2o_3 = L \cdot C$ *where* $L \in |\mathcal{O}_Q(1)|$ *is smooth.*

It is clear from the previous lemmas and remarks that η uniquely defines a pair of divisors $(o_1 + o_2 + o_3, n_1 + n_2)$ such that $\eta \cong \mathcal{O}_C(o_1 + o_2 + o_3 - n_1 - n_2)$ and moreover $\mathcal{O}_C(n_1 + n_2) \cong \eta^{\otimes 2}$ and $\mathcal{O}_C(o_1 + o_2 + o_3) \cong \eta^{\otimes 3}$.

We proceed as in the previous section. We fix coordinates $(x_0 : x_1) \times (y_0 : y_1)$ on Q so that $o_1 = (1 : 0, 1 : 0), o_2 = (0 : 1, 0 : 1)$ and $o_3 = (1 : 1, 1 : 1)$. Then the diagonal $L = \{x_0 y_1 - x_1 y_0\}$ is tritangent to C_n at o_1, o_2, o_3 and $n \in Q - L$.

Then we observe that C_n belongs to the 4-dimensional linear system $\mathbb{F}_n$ of all curves D of bidegree $(3, 3)$ such that:

(i) $2o_1 + 2o_2 + 2o_3 \subset L \cdot D$,

(ii) D has multiplicity ≥ 2 at n,

(iii) $R_i \cdot D = 3n$ for $i = 1, 2$, where R_1 and R_2 are the lines of Q passing through the point n.

Putting $U := Q - L$, we have on U a natural $\mathbf{P}^4$-bundle

$$\mathbb{F} := \{(D, n) \in |\mathcal{O}_Q(3)| \times U \mid D \in \mathbb{F}_n\}.$$

Indeed the projection $\pi : \mathbb{F} \to U$ realizes $\mathbb{F}$ as a $\mathbf{P}^4$-bundle. Furthermore we define

$$m : \mathbb{F} \to \mathcal{S}_3^{1/4-}$$

as follows. Let $C_n \in \mathbb{F}_n$ be nodal and such that its normalization $\nu : C \to C_n$ is a smooth genus three curve. Then

$$m(C_n) := [C, \eta],$$

where $\eta := \nu^*\mathcal{O}_C(o_1 + o_2 + o_3 - \nu^* n)$. We have omitted some standard details.

Then we observe that the group $\mathbb{G}$ defined analogously to (2.1) acts on $\mathbb{F}$ in the natural way. The action of $\alpha \in \mathbb{G}$ on $\mathbb{F}$ is the function $f_\alpha : \mathbb{F} \to \mathbb{F}$ sending (D, n) to $(\alpha(D), \alpha(n))$, for each $(D, n) \in |\mathcal{O}_Q(3)| \times U$. The proof of the next lemma is completely analogous to the proof of Lemma 2.5.2 and hence we omit it. The corollary is immediate.

Lemma 2.6.4. *Let $D_1, D_2 \in \mathbb{F}$. Then $m(D_1) = m(D_2)$ iff there exists $\alpha \in \mathbb{G}$ such that $\alpha(D_1) = \alpha(D_2)$.*

Corollary 2.6.5. *The quotient $\mathbb{F}/\mathbb{G}$ is birational to $\mathcal{S}_3^{1/4-}$.*

Now we can easily deduce that

Theorem 2.6.6. *$\mathcal{S}_3^{1/4-}$ is rational.*

Proof. Observe that $\mathbb{G}$ acts linearly between the fibres of $\pi : \mathbb{F} \to U$ and that the induced action on U is generically free. Hence the projective bundle $\pi : \mathbb{F} \to U$ descends to a projective bundle $\overline{\mathbb{F}} \to U/\mathbb{G}$ which is birational to $\mathbb{F}/\mathbb{G}$. On the other hand $\overline{\mathbb{F}}$ is birational to $\mathbf{P}^4 \times U/\mathbb{G}$. Since U is a rational surface, the quotient $U/\mathbb{G}$ is rational. Hence $\overline{\mathbb{F}}$ is rational and, by Corollary 2.6.5, the space $\mathcal{S}_3^{1/4-}$ is rational too. □

2.7 The rationality of $\mathcal{S}_3^{1/4+}$

Let us recall that, for any smooth curve C and any divisor e of degree two on it, the line bundle $\omega_C(e)$ is very ample iff $h^0(\mathcal{O}_C(e)) = 0$. Let C be a general curve of genus 3 and let η be any 4-th root of ω_C. Then $\eta^{\otimes 2}$ is a theta characteristic. We have considered the case where $\eta^{\otimes 2}$ is odd, that is $[C, \eta] \in \mathcal{S}_3^{1/4-}$, in the previous section. From now on we assume that $[C, \eta]$ is in $\mathcal{S}^{1/4+}$, so that $h^0(\eta^{\otimes 2}) = 0$. Then the line bundle $\omega_C \otimes \eta^{\otimes 2}$ is very ample and moreover it defines an embedding of C in $\mathbf{P}^3$ as a projectively normal curve whose ideal is generated by cubics, see [Dol12, §6.3]. Obviously no quadric contains C and we cannot argue as in the previous section. Though the beautiful geometry of cubic surfaces through C can be used, it is simpler to consider the canonical model of C. Hence we will assume that C is embedded in $\mathbf{P}^2$ as a general plane quartic.

Lemma 2.7.1.

1. *One has $h^0(\eta^{\otimes 3}) = 1$. Moreover, the unique divisor of $|\eta^{\otimes 3}|$ is supported on three distinct points o_1, o_2, o_3.*

2. *There exists exactly one cubic E such that $C \cdot E = 4(o_1 + o_2 + o_3)$ and E is smooth.*

Proof. We have $h^0(\eta^{\otimes 3}) \geq 2$ iff $h^0(\omega_C \otimes \eta^{-\otimes 3}) = 1$. This implies that C has a Weierstrass point p such that $4p \in |\omega_C|$. But then C is not a general curve. To complete the proof of (1) and to prove (2) it suffices to construct a pair (C, η) with the required properties. Starting from a smooth cubic E this construction is standard. $\square$

Furthermore, let $H := \mathcal{O}_E(1)$ and, as above, $4(o_1 + o_2 + o_3) = E \cdot C$. Let

$$\epsilon := H(-o_1 - o_2 - o_3).$$

Clearly ϵ is a 4-th root of $\mathcal{O}_E$. Moreover:

Lemma 2.7.2. *The line bundle $\epsilon^{\otimes 2}$ is not trivial.*

Proof. If $\epsilon^{\otimes 2}$ were trivial then $2o_1 + 2o_2 + 2o_3 = B \cdot E$, where B is a conic. This would imply that $h^0(\eta^{\otimes 2}) = h^0(\mathcal{O}_C(B - 2o_1 - 2o_2 - 2o_3)) = 1$, which is against our assumption that $\eta^{\otimes 2}$ is an even theta. $\square$

Moving $o_1 + o_2 + o_3$ in $|H \otimes \epsilon^{-1}|$, we can see that a general $d \in |H \otimes \epsilon^{-1}|$ defines a linear system of genus 3 spin curves (D, η_D) of order 4, such that $\eta_D^{\otimes 2}$ is an even theta characteristic on D. Indeed, let $\mathcal{I}_{4d}$ be the ideal sheaf of $4d \subset E$. Then

$$|\mathcal{I}_{4d}(4)|$$

is a 3-dimensional linear system of plane quartics D such that the line bundle $\eta_D := \omega_D(-d)$ satisfies the previous requirements.

Since the previous curve C was general in moduli, the construction implies that a dense open set of $\mathcal{S}_3^{1/4+}$ is filled up by points $[D, \eta_D]$ realized as above. We now use the previous remarks to prove that $\mathcal{S}_3^{1/4+}$ is birational to a suitable tower of projective bundles over the rational modular curve $\mathcal{S}_1^{1/4+}$.

Let $\mathcal{T}$ be the moduli space of abelian curves, polarized by a $(1,3)$ polarization and morepver endowed with a 4-torsion point whose square is not trivial. We can think of a general point of $\mathcal{T}$ as of the isomorphism class of a triple (E, H, t) such that E is a smooth genus one curve where the zero element o of the natural sum is fixed, H is a polarization of degree three and moreover t is a point of E satisfying $4t \sim 4o$ and $2t \not\sim 2o$. The space $\mathcal{T}$ is a finite cover of the moduli space $\mathcal{A}_1(3)$ of abelian curves endowed with a $(1,3)$ polarization. In the following proposition we identify E and $\mathrm{Pic}^0 E$ because, at the level of moduli, the zero of the abelian curves is fixed.

Proposition 2.7.3. *The moduli space $\mathcal{T}$ is rational.*

Proof. Observe that, on a smooth plane cubic E, a line bundle t such that $t^{\otimes 4} \cong \mathcal{O}_E$ is of the form $t = \mathcal{O}_E(p - o)$, where: (i) $3o \in |\mathcal{O}_E(1)|$, (ii) $4p + q + r \in |\mathcal{O}_E(2)|$, (iii) $q + r \in |2o|$. Indeed these conditions are just equivalent to say that $4p \sim 4o$. Notice also that they are fulfilled iff there exists a conic B such that $B \cdot E = 4p + q + r$ and $q + r \sim 2o$. Furthermore, it is easy to see that either $t^{\otimes 2}$ is not trivial and B is smooth or B is a double line and $B \cdot E = 2(2p + o)$. Assuming the former case we consider the plane cubic $A + B$, where A is the flex tangent to E at o. Let P be the pencil of cubics generated by E and $A+B$, then its base locus is the 0-dimensional scheme $4p+q+r+3o \subset E$. Let $F \in P$ be a smooth plane cubic, then F is endowed with the line bundles $t_F := \mathcal{O}_F(p-o)$ and $H_F := \mathcal{O}_F(1)$. Hence there exists a rational map $\tau : P \to \mathcal{T}$ defined as follows: $\tau(F) = [F, H_F, t_F]$. This map clearly dominates $\mathcal{T}$ and hence $\mathcal{T}$ is a rational curve. □

Let $f : \mathcal{T} \to \mathcal{A}_1(3)$ be the above mentioned natural finite covering and let $\mathcal{E}$ be the universal family of abelian curves over $\mathcal{A}_1(3)$. Over a non empty open set of $\mathcal{A}_1(3) \times_{\mathcal{M}_1} \mathcal{E}$ we have a Poincaré line bundle $\mathcal{P}$ whose restriction to the curve $[E, H] \times E$ is the line bundle H. In order to use $\mathcal{P}$ we consider the morphism

$$f \times id_{\mathcal{E}} : \mathcal{T} \times_{\mathcal{M}_1} \mathcal{E} \to \mathcal{A}_1(3) \times_{\mathcal{M}_1} \mathcal{E}$$

and the pull-back $\tilde{\mathcal{P}} := (f \times id_{\mathcal{E}})^* \mathcal{P}$ of $\mathcal{P}$. On the other hand, let $S := \mathcal{T} \times_{\mathcal{M}_1} \mathcal{E}$ and let $u : S \to \mathcal{T}$ be the projection onto $\mathcal{T}$ of the surface S. We have two natural sections of u, namely $s_1, s_2 : \mathcal{T} \to S$, defined as follows: $s_1([E, H, t]) = t \in E$ and $s_0([E, H, t]) = o \in E$. Let D_i be the image of s_i in S. Then we have on S the line bundle $\mathcal{O}_S(D_1 - D_0)$. Finally, we can define the following $\mathbf{P}^2$-bundles over (a suitable non empty open set of) $\mathcal{T}$:

- $\mathbb{T} := \mathbf{P} u_*(\tilde{\mathcal{P}}(D_1 - D_0))$,
- $\mathbb{P} := \mathbf{P} u_* \tilde{\mathcal{P}}^*$.

The fibre of $\mathbb{T}$ at the point $[E, H, t]$ is the linear system $|H(t-o)|$, while the fibre of $\mathbb{P}$ at the same point is $\mathbf{P}H^0(H)^*$. Now we consider the tautological embedding

$$\tilde{\mathcal{E}} \subset \mathbb{P}^*.$$

At the point $e := [E, H, t]$ the fibre of $\mathcal{E}$ is E and the tautological embedding restricts to the embedding $E \subset \mathbb{P}_e = \mathbf{P}H^0(H)^*$, defined by H.

Finally we consider the fibre product

$$\mathbb{F} := \mathbb{T} \times_{\mathcal{T}} \mathbb{P}$$

and the incidence correspondence

$$\mathcal{Z} \subset \mathbb{F}$$

parametrizing the points $[E, H, t; d, x] \in \mathbb{T} \times_{\mathcal{T}} \mathbb{P}$ such that

- $x \in d \subset E \subset \mathbf{P}H^0(H)^*$,
- $d \in |H(t-o)|$.

Let $\pi_1 : \mathbb{F} \to \mathbb{T}$ and $\pi_2 : \mathbb{F} \to \mathbb{P}$ be the projection maps, we have the embeddings

$$\mathcal{Z} \subset \ \pi_1^* \tilde{\mathcal{E}} \subset \mathbb{F}.$$

Notice that $\mathcal{Z}$ is a divisor in $\pi_1^* \tilde{\mathcal{E}}$ and that the latter, up to shrinking its base, is a smooth family of elliptic curves. In such a family we can therefore consider the divisor $4\mathcal{Z}$ and this is a subscheme of $\mathbb{F}$. Let $\mathcal{J}$ be its ideal sheaf, then our construction yields the projective bundle

$$\mathbb{Q} := \mathbf{P}\pi_{1*}(\mathcal{J} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}}(4))$$

over $\mathbb{T}$. If $\tau = [E, H, t, d]$ is a general point of $\mathbb{T}$, then the fibre of $\mathbb{Q}$ at τ is

$$Q_\tau = |\mathcal{I}_{4d}(4)|,$$

where $\mathcal{I}_{4d}$ is the ideal sheaf of $4d$ in $E \subset \mathbf{P}H^0(H)^*$. In particular, by Grauert's theorem, $\mathbb{Q}$ is a $\mathbf{P}^3$-bundle over $\mathbb{T}$. Furthermore, the bundle $\mathbb{T}$ is rational, since it is a projective bundle over the rational curve $\mathcal{T}$, and hence also $\mathbb{Q}$ is rational.

The conclusion is now quite clear: we can construct a birational map between $\mathbb{Q}$ and $\mathcal{S}_3^{1/4+}$. Indeed let

$$m : \mathbb{Q} \to \mathcal{S}_3^{1/4+}$$

be the rational map which is defined as follows. If τ is a general point as above, to a general element $D \in \mathbb{Q}_\tau = |\mathcal{I}_{4d}(4)|$ the map m associates the point $[D, \omega_D(-d)]$. Arguing as in the other cases considered in this paper, one can show that m is dominant. In particular it follows $\mathcal{S}_3^{1/4+}$ is irreducible.

Furthermore, m is generically invertible: indeed its inverse is just the rational map sending a general $[C, \eta]$ to the point $[\hat{E}, \hat{H}, \hat{t}, C]$, where $E \cdot C = 4d$, $\hat{E} := \mathrm{Pic}^0 E$, $\hat{t} := H(-d)$ as a 4-torsion point of $\mathrm{Pic}^0 E$ and $\hat{H} := \mathcal{O}_{\hat{E}}(3\hat{o})$ is the $(1,3)$ polarization defined by the zero point $\hat{o} := \mathcal{O}_E$ of $\mathrm{Pic}^0 E$. We have shown that

Theorem 2.7.4. *$\mathcal{S}_3^{1/4+}$ is rational.*

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Chapter 3

On the surjectivity of weighted Gaussian maps

joint work with Edoardo Ballico.

ABSTRACT: We study the surjectivity of suitable weighted Gaussian maps $\gamma_{a,b}(X, L)$ which provide a natural generalization of the standard Gaussian maps and encode the local geometry of the locus $\mathfrak{Th}^r_{g,h} \subset \mathcal{M}_g$ of curves endowed with an h-th root L of the canonical bundle satisfying $h^0(L) \geq r+1$. In particular, we get a bound on the dimension of its Zariski tangent space, which turns out to be sharp in the special case $r = 0$. Finally, we describe this locus in the case of complete intersection curves.

3.1 Introduction

The theory of Gaussian maps on curves was developed by Jonathan Wahl in [Wah87] and [Wah90]. We recall that they are defined as

$$\Phi_L : R(L) \to H^0(\Omega^1_X \otimes L^{\otimes 2}), \qquad \sigma \otimes \tau \mapsto \tau \mathrm{d}\sigma - \sigma \mathrm{d}\tau,$$

where X is a smooth projective variety over $\mathbb{C}$, $L \in \mathrm{Pic}(X)$ a line bundle of positive degree and $R(L)$ is the kernel of the multiplication map $H^0(L) \otimes H^0(L) \to H^0(L^{\otimes 2})$. Furthermore, as for instance in [Far05], it makes sense to define the Gaussian map also as the restriction $\psi_L := \Phi_{L|\Lambda^2 H^0(L)}$ since the first map always vanishes on symmetric tensors. The interpretation of the name "Gaussian" can be found for instance in [Wah90]: if $X = C$ is a curve embedded in a projective space $\mathbb{P}^n$ and $L = \mathcal{O}_C(1)$, one can consider the Gauss mapping $C \to \mathrm{Gr}(1, n)$ sending each point of the curve to its tangent line in $\mathbb{P}^n$. The composition of this map with the Plücker embedding $\mathrm{Gr}(1, n) \hookrightarrow \mathbb{P}^N$ gives rise to the "associated curve" $\phi : C \to \mathbb{P}^N$ and the restriction of the hyperplane section ϕ^* of ϕ corresponds the so-called Gaussian map ψ_L.

The original interest in these maps came from the study of Φ_{K_C}, where K_C is the canonical bundle on a smooth curve (see for instance [CHM88] and

[CLM00]) and in general they have been explored in particular when X is a curve, in relation with the deformation theory of the vertex of the cone over X (see [Wah87]). More recently, Edoardo Ballico and Claudio Fontanari provided a generalization of these maps in [BF06]. They defined the so-called "weighted Gaussian maps" as

$$\begin{array}{rccc} \gamma_{a,b}(X,L): & H^0(X,L^{\otimes a})\otimes H^0(X,L^{\otimes b}) & \to & H^0(\Omega^1_X\otimes L^{\otimes a}\otimes L^{\otimes b}) \\ & \sigma\otimes\tau & \mapsto & b\tau\mathrm{d}\sigma - a\sigma\mathrm{d}\tau. \end{array} \tag{3.1}$$

where X is always a smooth projective variety, $L \in \mathrm{Pic}(X)$ and $a, b > 0$ are two positive integers. When $a = b = 1$ we recover the standard Gaussian map.

In the first section, following the approach of [Wah90], we investigate the surjectivity of the map $\gamma_{a,b}(X, L)$ by studying first the weighted Gaussian maps for $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(e)$, with $e \in \mathbb{N}$ a generic positive integer, and then the restriction map given by $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(t)) \to H^0(X, \Omega^1_X(t))$. In particular, we prove a general result (Theorem 3.2.5) from which we deduce the following corollaries.

Corollary 3.1.1. *Let C be a smooth curve of genus g and $L \in \mathrm{Pic}(C)$ a line bundle of degree $\deg(L) \geq 2g + 2$. Denote by $X \subset \mathbb{P}^n$ the linearly normal embedding of C induced by $|L|$. Then the map $\gamma_{a,b}(X, \mathcal{O}_X(1))$ is surjective for all positive integers a, b such that $a + b \geq 3$.*

Corollary 3.1.2. *Let C be a smooth curve of genus $g \geq 3$. Assume that C is neither hyperelliptic nor trigonal nor isomorphic to a plane quintic and let $X \subset \mathbb{P}^{g-1}$ be its canonical model. Then the map $\gamma_{a,b}(X, \omega_X)$ is surjective for all $a, b > 0$ such that $a + b \geq 4$.*

In the second section we relate the weighted Gaussian maps $\gamma_{1,h-1}(C, L)$, $h \geq 2$, to the locus $\mathfrak{Th}^r_{g,h} \subset \mathcal{M}_g$ defined, for $h \geq 2$ an arbitrary integer, by

$$\mathfrak{Th}^r_{g,h} := \{[C] \in \mathcal{M}_g \mid \exists L \in \mathrm{Pic}(C) \text{ s.t. } h^0(C, L) \geq r + 1,\ hL = K_C\}. \tag{3.2}$$

It is a generalization for h-spin curves of the locus studied by J. Harris in [Har82] for theta-characteristics. The case in which h is even was considered in [Fon02], but the proof of the odd case is identical (see Theorem 3.3.2). In particular, J. Harris proved that each component of the locus

$$\mathcal{S}^r_g := \{[C, L] \in \mathcal{S}_g \mid h^0(L) \geq r + 1, \quad h^0(L) \equiv r + 1 \mod 2\}$$

has codimension at most $\binom{r+1}{2}$ in $\mathcal{S}_g$. We recall that $\mathcal{S}_g$ is the moduli space of pairs $[C, L]$ where C is a genus g curve and $L \in \mathrm{Pic}(C)$ is a theta-characteristic on C, that is, a square root of its canonical bundle. In [Far05] G. Farkas showed that for $r = 1, 2, \ldots, 9$ and 11 there exists an explicit integer $g(r)$ such that for all $g \geq g(r)$ the moduli space $\mathcal{S}^r_g$ has at least one component of codimension exactly $\binom{r+1}{2}$ and he made a conjecture (recently proved by L. Benzo in [Ben13]) on the existence of a component attaining the maximum codimension for any $r \geq 1$ and $g \geq \binom{r+2}{2}$. His proof is based on the connection between Gaussian maps and spin curves provided by a tangent space computation done by Nagaraj in [Nag90], whose main ingredient is the identification $T_{[C,L]}\mathcal{S}_g^r \cong \mathrm{Im}(\psi_L)^\perp$. In this work we use the analogous relation between the weighted Gaussian map and the locus $\mathfrak{Th}_{g,h}^r$ proved in [Fon02, Theorem 3] for h even: there is an identification between the tangent space $T_C\mathfrak{Th}_{g,h}^r$ and the dual of the cokernel $\mathrm{Coker}(\gamma_{1,h-1}(C,L))$. As claimed before, the same relation holds identically also when h is odd. Using these facts we prove the following result.

Theorem 3.1.3. *For every $g,h \geq 2$ and every $[C] \in \mathfrak{Th}_{g,h}^0$ with an h-theta L satisfying $h^0(L) = 1$, the Zariski tangent space at $[C]$ has codimension $(g-1)(h-2)/h$ in the tangent space $H^0(C, K_C^{\otimes 2})^\vee$ of the local deformation space of C.*

We conclude our work with a focus on complete intersection curves. In Theorem 3.3.5 we prove that, for general m and r, if $\mathfrak{Th}_{g,h}^r$ does contain a complete intersection, then $\mathfrak{Th}_{g,h}^r$ has a component whose general element is a complete intersection. We work over the complex field $\mathbb{C}$.

3.2 Surjectivity

Let X be a smooth projective variety and $L \in \mathrm{Pic}(X)$. Let $b > a > 0$ be two integers. Let us fix a very ample line bundle $\mathcal{O}_X(1)$ on X and set $n = h^0(\mathcal{O}_X(1)) - 1$. Then we use $|\mathcal{O}_X(1)|$ to embed X inside $\mathbb{P}^n$. Following [Wah87], for any $e \in \mathbb{N}_{>0}$ and for any integers $b > a > 0$ we have the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(ae)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(be)) & \xrightarrow{\gamma_{a,b}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e))} & H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(ae+be)) \\ \downarrow & & \downarrow \alpha_{ae+be,X} \\ H^0(X, \mathcal{O}_X(ae)) \otimes H^0(X, \mathcal{O}_X(be)) & \xrightarrow{\gamma_{a,b}(X, \mathcal{O}_X(e))} & H^0(X, \Omega^1_X(ae+be)). \end{array} \tag{3.3}$$

As in [Wah87], we show that the top map is always surjective.

Lemma 3.2.1. *For any $e \in \mathbb{N}_{>0}$ and for any integers $b > a > 0$ the map*

$$\gamma_{a,b}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e)) : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(ae)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(be)) \to H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(ae+be))$$

is surjective. In particular, the map $\gamma_{a,b}(X, \mathcal{O}_X(e))$ is surjective if the map $\alpha_{ae+be,X}$ is surjective.

Proof. Let us call for the moment $\gamma := \gamma_{a,b}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e))$. First we show that the map γ is not identically zero. Fix homogeneous coordinates $X_0, \ldots, X_n$ for $\mathbb{P}^n$ and consider, for instance, the mononomials X_0^{ae} and X_1^{be}. Then

$$\gamma(X_0^{ae} \otimes X_1^{be}) = abeX_0^{ae}X_1^{be}\left(\frac{\mathrm{d}X_0}{X_0} - \frac{\mathrm{d}X_1}{X_1}\right)$$

which is a non trivial function. Notice also that γ is a $\mathrm{GL}(n+1)$-equivariant map. For every $t \geq 2$ the space $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(t))$ corresponds to the kernel of the multiplication map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t-1)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)),$$

and hence $\Omega^1_{\mathbb{P}^n}(t)$ is a stable homogeneous bundle. The space $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(t))$ is thus an irreducible $\mathrm{GL}(n+1)$-representation for every integer $t \geq 2$. Indeed it corresponds to the Young diagram $(t-1, 1)$. This means that the codomain $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(t))$ has no proper $GL(n+1)$-invariant subspaces and hence the map γ is surjective, since it is equivariant and non-zero. □

We study the surjectivity of the right-vertical map $\alpha_{t,X}$. For every $t > 0$ it factors via two maps:

$$H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(t)) \xrightarrow{\rho_{t,X}} H^0(X, \Omega^1_{\mathbb{P}^n|X}(t)) \xrightarrow{\beta_{t,X}} H^0(X, \Omega^1_X(t)) \tag{3.4}$$

where $\rho_{t,X}$ is the restriction map and $\beta_{t,X}$ comes from the cohomology of the conormal sequence. Considering indeed the conormal sequence $0 \to N^*_X(t) \to \Omega^1_{\mathbb{P}^n|X}(t) \to \Omega^1_X(t) \to 0$, we have that the second one is surjective if $h^1(N^*_X(t)) = 0$. When X is a complete intersection we have the following results on the first cohomology of the twisted conormal bundle.

Lemma 3.2.2. *Assume that X is a complete intersection of $n-k$ hypersurfaces of degree $d_1 \geq \cdots \geq d_{n-k} \geq 2$. Then*

$$h^1(N^*_X(t)) = 0 \iff k \geq 2 \quad or \quad \begin{cases} k = 1, \\ t > 2d_1 + d_2 + \cdots + d_{n-1} - n - 1. \end{cases}$$

Proof. Recall that $N^*_X(t) = \bigoplus_{i=1}^{n-k} \mathcal{O}_X(t - d_i)$. If $k \geq 2$ then we have that the first cohomology group $H^1(X, \mathcal{O}_X(c))$ vanishes for all $c \in \mathbb{Z}$ since X is a complete intersection. If $k = 1$ then $\omega_X = \mathcal{O}_X(\sum_i d_i - n - 1)$ and hence $N^*_X(t)$ does not have special terms if and only if $t > 2d_1 + d_2 + \cdots + d_{n-1} - n - 1$. □

In this situation we can prove the surjectivity of the weighted Gaussian maps in some ranges. In order to do so, we need one more lemma.

Lemma 3.2.3. *Let $X \subset \mathbb{P}^n$ be a smooth variety. Assume $h^1(\mathbb{P}^n, \mathcal{I}_X(t-1)) = 0$ and that the homogeneous ideal of X contains no minimal generator of degree t. Then the map $\rho_{t,X} : H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(t)) \to H^0(X, \Omega^1_{\mathbb{P}^n|X}(t))$ is surjective.*

The last hypothesis is equivalent to asking for the surjectivity of the map

$$V \otimes H^0(\mathbb{P}^n, \mathcal{I}_X(t-1)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{I}_X(t)),$$

where V is the $(n+1)$-dimensional vector space $V := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. From the short exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

tensored by $\Omega^1_{\mathbb{P}^n}(t)$, it follows that the map $\rho_{t,X}$ is surjective if the first cohomology group $H^1(\mathcal{I}_X \otimes \Omega^1_{\mathbb{P}^n}(t))$ vanishes. But the latter follows from our assumptions by considering the Euler sequence for $\mathbb{P}^n \simeq \mathbb{P}(V)$ twisted by the ideal sheaf $\mathcal{I}_X(t)$

$$0 \to \mathcal{I}_X \otimes \Omega^1_{\mathbb{P}^n}(t) \to \mathcal{I}_X(t-1) \otimes V \to \mathcal{I}_X(t) \to 0. \tag{3.5}$$

and hence the proof is over. □

Proposition 3.2.4. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of $n-k$ hypersurfaces of degree $d_1 \geq \cdots \geq d_{n-k} \geq 2$. Then the weighted Gaussian map $\gamma_{a,b}(X, \mathcal{O}_X(1))$ is surjective if $a+b \neq d_i$ and*

$$k \geq 2 \quad \text{or} \quad \begin{cases} k = 1, \\ t > 2d_1 + d_2 + \cdots + d_{n-1} - n - 1. \end{cases}$$

Proof. Defining $t = a + b$ we know by Lemma 3.2.1 that $\gamma_{a,b}(X, \mathcal{O}_X(1))$ is surjective if the map $\alpha_{t,X}$ is surjective. By Lemma 3.2.2 we know that the map $\beta_{t,X}$ is surjective if and only if $k \geq 2$ or $k = 1$ and $t > 2d_1+d_2+\cdots+d_{n-1}-n-1$. Since X is projectively normal one has $h^1(X, \mathcal{I}_X(t-1)) = 0$. Furthermore, we have just seen in the previous proof that the map $\rho_{t,X}$ is surjective if $h^1(\mathcal{I}_X \otimes \Omega^1_{\mathbb{P}^n}(t)) = 0$. In this situation the ideal sheaf $\mathcal{I}_X$ is a quotient of $\bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-d_i)$ because it is minimally generated by the $n-k$ forms of degree $d_1, \ldots, d_{n-k}$ defining X. The map $\alpha_{t,X}$ is thus surjective if $h^1(\Omega^1_{\mathbb{P}^n}(t-d_i)) = 0$ for all i, that is $t \neq d_i$ for all i. □

Let us move to the general situation where C is a smooth curve and L a very ample line bundle on it.

Theorem 3.2.5. *Let C be a smooth curve and $L \in \mathrm{Pic}(C)$ be a very ample line bundle on C. Denote by $X \subset \mathbb{P}^n$ the linearly normal embedding of C induced by $|L|$. Let $t_0 \in \mathbb{N}$ be an integer such that $t_0 \geq 3$. Assume that*

1. *$h^1(\mathbb{P}^n, \mathcal{I}_X(t-1)) = 0$ for every $t \geq t_0$,*
2. *the map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \times H^0(\mathbb{P}^n, \mathcal{I}_X(t-1)) \to H^0(\mathbb{P}^n, \mathcal{I}_X(t))$ is surjective for every $t \geq t_0$,*
3. *$h^1(\mathcal{O}_X(1)) = 0$.*

Then, for every $t \geq t_0$ and for every $a, b > 0$ such that $a + b = t$, the weighted Gaussian map $\gamma_{a,b}(X, \mathcal{O}_X(1))$ is surjective.

Proof. Notice that hypothesis (2) implies that the homogeneous ideal of X is generated by forms of degree $\leq t_0 - 1$ and hence X is scheme-theoretically cut out by forms of degree $t_0 - 1$. By Lemma 3.2.1 and the factorization of $\alpha_{t,X}$ given in (3.4), it is enough to prove that $\rho_{t,X}$ and $\beta_{t,X}$ are surjective for all $t \geq t_0$. The first map is surjective by Lemma 3.2.3. Recall that in order to prove the surjectivity of the second map it is enough to show that $h^1(N^*_X(t)) = 0$ for every $t \geq t_0$. Since X is scheme-theoretically cut out by forms of degree $t_0 - 1$, the ideal $\mathcal{I}_X(t_0-1)$ is spanned by its global section. Therefore its quotient $N_X^*(t_0-1) = \mathcal{I}_X/\mathcal{I}_X^2(t_0-1)$ is spanned by global sections. Hence for some $N \in \mathbb{N}$ we have an exact sequence of $\mathcal{O}_X$-sheaves

$$0 \to E \to \mathcal{O}_X(1)^{\oplus N} \to N_X^*(t_0) \to 0.$$

Since X is a curve, we have $h^2(X, E) = 0$. Furthermore, from $h^1(\mathcal{O}_X(1)) = 0$ we obtain $h^1(N_X^*(t_0)) = 0$ and the same argument works for $t \geq t_0$. □

From this we can deduce some special cases as the following ones.

Corollary 3.2.6. *Let C be a smooth curve of genus g and $L \in \mathrm{Pic}(C)$ a line bundle of degree $\deg(L) \geq 2g+2$. Denote by $X \subset \mathbb{P}^n$ the linearly normal embedding of C induced by $|L|$. Then the map $\gamma_{a,b}(X, \mathcal{O}_X(1))$ is surjective for all positive integers a, b such that $a+b \geq 3$.*

Proof. Since $\deg(L) \geq 2g+2$, one has $h^1(L) = 0$ and L is very ample. In particular, X is projectively normal ([ACGH85, p. 140]) and one has $h^1(\mathcal{I}_X(2)) = 0$. Since $\deg(L) \geq 2g+2$, the homogeneous ideal of X is generated by quadrics ([Gre84] and [GL88, p. 302]). Hence, for $t \geq 3$, the map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \times H^0(\mathbb{P}^n, \mathcal{I}_X(t-1)) \to H^0(\mathbb{P}^n, \mathcal{I}_X(t))$ is surjective. We can then apply Theorem 3.2.5 with $t_0 = 3$ and this completes the proof. □

Corollary 3.2.7. *Let C be a smooth curve of genus $g \geq 3$. Assume that C is neither hyperelliptic nor trigonal nor isomorphic to a plane quintic and let $X \subset \mathbb{P}^{g-1}$ be its canonical model. Then the map $\gamma_{a,b}(X, \omega_X)$ is surjective for all $a, b > 0$ such that $a+b \geq 4$.*

Proof. Every canonically embedded smooth curve is projectively normal by a theorem of Max Noether [ACGH85, p.117] and hence $h^1(\mathcal{I}_X(2)) = 0$. Since C is neither trigonal nor a plane quintic, a theorem of K. Petri says that the homogeneous ideal of X is generated by quadrics [ACGH85, p. 131]. Hence condition (2) of Theorem 3.2.5 is satisfied for $t_0 = 3$. The same argument used in the proof of Theorem 3.2.5 shows that $N_X^*(a+b-2)$ is spanned by global section and we have thus an analogous exact sequence

$$0 \to E \to \mathcal{O}_X(2)^{\oplus N} \to N_X^*(a+b) \to 0.$$

Since $h^1(\mathcal{O}_X(2)) = h^1(\omega_X^{\otimes 2}) = 0$ we get $h^1(N_X^*(a+b)) = 0$ and hence we conclude that the map $\beta_{a+b,X}$ is surjective. Since the homogeneous ideal of X is generated by forms of degree $\leq a+b-1$ (actually, of degree $\leq a+b-2$), from Lemma 3.2.3 we deduce that the map $\rho_{a+b,X}$ is surjective. Hence $\gamma_{a,b}(X, \omega_X)$ is surjective. □

3.3 Gaussian maps and h-theta-characteristics

In this section we identify some objects in the kernel of the Gaussian maps, in particular the ones of the kind $v^{\otimes a} \otimes v^{\otimes b}$ for $v \in H^0(L)$. We get a lower bound on the dimension of the kernel, which in general is far from being sharp, but that it is achieved in the case of interest for us. Consider the multiplication map

$$\eta_{a+b} : \mathrm{Sym}^{a+b}(H^0(L)) \to H^0(L^{\otimes a+b}).$$

Lemma 3.3.1. *For every $a, b > 0$, the kernel of $\gamma_{a,b}(X, L)$ has dimension $\geq \mathrm{rank}(\eta_{a+b})$.*

Proof. Let v be an element in $H^0(L)$. Define $\sigma := v^{\otimes a} \in H^0(L^{\otimes a})$ and $\tau := \sigma^{\otimes b} \in H^0(L^{\otimes b})$. We have

$$\gamma_{a,b}(X, L)(\sigma \otimes \tau) = \frac{\tau^{a+1}}{\sigma^{b-1}} d\left(\frac{\sigma^b}{\tau^a}\right) = 0$$

where the first equality follows from the proof of [BF06, Lemma 1] and the second follows since $\sigma^b = \tau^a$ by definition. This means that the kernel of $\gamma_{a,b}(X, L)$ contains the linear span of all elements of the kind $v^a \otimes v^b$ for $v \in H^0(L)$. In order to prove $\dim(\ker \gamma_{a,b}(X, L)) \geq \mathrm{rank}(\eta_{a+b})$ it is enough to show that, given the multiplication map $\nu_{a,b} : H^0(L^{\otimes a}) \otimes H^0(L^{\otimes b}) \to H^0(L^{\otimes a} \otimes L^{\otimes b})$, one has

$$\dim(\nu_{a,b}(\ker \gamma_{a,b}(X, L))) \geq \mathrm{rank}(\eta_{a+b}).$$

We have just shown that $\ker \gamma_{a,b}(X, L)$ contains all elements of the kind $v^{\otimes a+b}$ for $v \in H^0(L)$. Since we work over algebraically closed base field with characteristic zero, any symmetric polynomial of degree $a + b$ is a linear combination of $(a + b)$-th powers of linear forms, that is

$$\mathrm{Im}(\eta_{a+b}) \subset \mathrm{lin\_span}\{v^{\otimes a+b} \in H^0(L^{\otimes a+b}) \ \mid \ v \in H^0(L)\} \subset \nu_{a,b}(\ker \gamma_{a,b}(X, L)).$$

The thesis follows. □

Notice that, when $a = 1$, $b = h - 1$ and h is even, we are in the situation of Theorem 3 of [Har82] and the map $\gamma_{a,b}(C, L)$ corresponds to the map

$$\begin{array}{rrcl} \mu_h := \gamma_{1,h-1}(C, L) : & H^0(C, L) \otimes H^0(C, K_C - L) & \to & H^0(C, 2K_C) \\ & \sigma \otimes \tau & \mapsto & (h-1)\tau \mathrm{d}\sigma - \sigma \mathrm{d}\tau \end{array}$$

Notice the misprint in the statement of [Har82, Theorem 3] and [BF06, Theorem 1]. Since in Theorem 3 of [Har82] only the case h even is considered, we restate here the theorem in its full generality.

Theorem 3.3.2. *Let $\mathfrak{Th}^r_{g,h}$ be the space defined in (3.2) for $g, h \geq 2$ and define the map*

$$\begin{array}{rrcl} \mu_h := & H^0(C, L) \otimes H^0(C, K_C - L) & \to & H^0(C, 2K_C) \\ & \sigma \otimes \tau & \mapsto & (h-1)\tau \mathrm{d}\sigma - \sigma \mathrm{d}\tau. \end{array}$$

Then $T_C\left(\mathfrak{Th}^r_{g,h}\right) = (\mathrm{coker}\mu_h)^*$.

Proof. The proof of [Har82, Theorem 3] works verbatim when $h \geq 3$ is an odd integer. Indeed, just use $\frac{1}{h}$ instead of $\frac{1}{2m}$ in the proof of [Har82, Lemma 3] and use h instead of $2m$ in [Har82, §3.3]. □

In particular, when C is a smooth and connected projective curve, Proposition 1 of [BF06] gives us the bound

$$\operatorname{rank}(\gamma_{a,b}(C,L)) \geq h^0(L^{\otimes a}) + h^0(L^{\otimes b}) - 3, \tag{3.6}$$

where the inequality is strict if $bs - at \neq 0$, with $s = h^0(L^{\otimes a}) - 1$ and $t = h^0(L^{\otimes b}) - 1$. Recalling the definition (3.2) of $\mathfrak{Th}^r_{g,h}$ given in the Introduction, Lemma 3.3.1 provides a bound on the dimension of the Zariski tangent space $\operatorname{Zar}_{[C]}\mathfrak{Th}^r_{g,h}$ of $\mathfrak{Th}^r_{g,h}$ at $[C] \in \mathfrak{Th}^r_{g,h}$ inside the Zariski tangent space $H^0(C, K_C^{\otimes 2})^\vee$ of the deformation space of C. In particular, one has

$$\operatorname{rank}(\mu_h) = \operatorname{codimZar}_{(C,L)}\mathfrak{Th}^r_{g,h} \leq (r+1)\left(r+1+(g-1)\frac{h-2}{h}\right) - \operatorname{rank}(\eta_h). \tag{3.7}$$

In his work G. Farkas studied the classical locus $\mathcal{S}^r_g := \{[C, L] \in \mathcal{S}_g \mid h^0(L) \geq r+1, h^0(L) \equiv_2 r+1\}$ corresponding to the choice $h = 2$ and he showed that, for small values of r, there is a specific integer $g(r)$ such that for every $g \geq g(r)$ there exists a component of $\mathcal{S}^r_g$ realizing the bound. He also made a conjecture (see [Far05, Conjecture 3.4]), recently proved by [Ben13], on the existence of such a component for every $r \geq 3$ and for every $g \geq \binom{r+2}{2}$.

When h is general and $r = 0$, we know the exact dimension of the tangent space of $\mathfrak{Th}^0_{g,h}$ at any $[C]$ with $h^0(L) = 1$, by seeing it as a subspace of the tangent space $H^0(K_C^{\otimes 2})^\vee$ of the local deformation space of C.

Theorem 3.3.3. *For every $g, h \geq 2$ and every $[C] \in \mathfrak{Th}^0_{g,h}$ with an h-theta L satisfying $h^0(L) = 1$, the Zariski tangent space at $[C]$ has codimension $(g-1)(h-2)/h$ in the tangent space $H^0(C, K_C^{\otimes 2})^\vee$ of the local deformation space of C.*

Proof. Let C be a smooth curve and $L \in \operatorname{Pic}(X)$ an h-th canonical root such that $h^0(L) = 1$. Notice that, when $r = 0$, the rank of the map η_s is always equal to 1 for every $s \in \mathbb{N}$. From (3.6) we know that

$$\operatorname{rank}(\mu_h) \geq h^0(L) + h^0(K_C - L) - 3 = h^0(K_C - L) - 2 \tag{3.8}$$

and from Lemma 3.3.1 we have

$$\operatorname{rank}(\mu_h) \leq h^0(L) \cdot h^0(K_C - L) - 1 = h^0(K_C - L) - 1 \tag{3.9}$$

When $g \geq 2$ and $h > 2$, the inequality in (3.8) is strict because $s = h^0(L) - 1 = 0$ and $t = h^0(K_C - L) - 1 = g - \frac{2g-2}{h} \neq 0$ ([BF06, Proposition 1]). Furthermore, if $h = 2$ then $h^0(K_C - L) = 1$ and thus, from (3.9), we have $\operatorname{rank}(\mu_2) = h^0(K_C - L) - 1 = 0$ even in this case. We can thus conclude that, for all $g, h \geq 2$, the map μ_h has always rank $h^0(K_C - L) - 1 = (g-1)(h-2)/h$. $\square$

For any g, r and h we cannot expect a complete intersection C to be an isolated point of $\mathfrak{Th}^r_{g,h}$, since curves of this kind move in families. However, we show that the dimension of a component of $\mathfrak{Th}^r_{g,h}$ containing a complete intersection is the smallest possible one. In other words, we show that if a component of $\mathfrak{Th}^r_{g,h}$ contains a complete intersection curve, then its general element is again a complete intersection. Let $C \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of degree $d_1, \dots, d_{n-1}$ with

$$d_1 \geq \cdots \geq d_{n-1} \geq 2, \qquad n \geq 2.$$

If $n = 3$ assume $d_1 \geq 3$ and if $n = 2$ assume $d_1 \geq 4$. Let us call $\xi := \sum_i d_i - n - 1$ and recall that $\omega_C = \mathcal{O}_C(\xi)$. We need a preliminary lemma.

Lemma 3.3.4. *The cokernel of the cotangent map*

$$\alpha : H^0(C, \Omega^1_{\mathbb{P}^n|C}(\xi)) \to H^0(C, \omega_C^{\otimes 2})$$

has dimension $h^0(N_C) - (n+1)^2 + 1$*.*

Proof. We use the conormal exact sequence

$$0 \to N_C^*(\xi) \to \Omega^1_{\mathbb{P}^n|C}(\xi) \to \omega_C(\xi) \to 0. \tag{3.10}$$

From its cohomology and Serre duality we find

$$\dim \operatorname{coker}(\alpha) = h^0(N_C) - h^1(\Omega^1_{\mathbb{P}^n|C}(\xi)).$$

In order to compute $h^1(\Omega^1_{\mathbb{P}^n|C}(\xi))$, define the vector space $V := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ $= H^0(C, \mathcal{O}_C(1)) \cong \mathbb{C}^{n+1}$ and consider the restriction to C of the Euler sequence of $\Omega^1_{\mathbb{P}^n}$:

$$0 \to \Omega_{\mathbb{P}^n|C}(\xi) \to V \otimes \mathcal{O}_C(\xi - 1) \xrightarrow{\mu} \mathcal{O}_C(\xi) \to 0. \tag{3.11}$$

Since $\xi - 1 \geq 0$ and C is projectively normal, the map μ is a surjective multiplication map on global sections. From the long exact sequence of cohomology of (3.11) and Serre duality, it thus follows that $h^1(\Omega^1_{\mathbb{P}^n|C}(\xi)) = (n+1)^2 - 1$, proving the lemma. □

Now define $g := \frac{\xi \prod_i d_i}{2} + 1$, $\zeta := \frac{\xi}{h}$, $r := h^0(C, \mathcal{O}_C(\zeta)) - 1$ and consider the diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\zeta)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((h-1)\zeta)) & \longrightarrow & H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\xi)) \\ \downarrow & & \downarrow \alpha \\ H^0(C, \mathcal{O}_C(\zeta)) \otimes H^0(C, \mathcal{O}_C((h-1)\zeta)) & \xrightarrow{\gamma_{1,h-1}} & H^0(C, \Omega^1_C(\xi)). \end{array} \tag{3.12}$$

We can state the following Theorem.

Theorem 3.3.5. *For every $h \geq 2$ and g, r as above, the curve $[C]$ is a smooth point of an irreducible component of $\mathfrak{Th}^r_{g,h}$ whose general member is a complete intersection curve.*

Proof. Let $\mathbb{D}$ be the subset of the Hilbert scheme $\operatorname{Hilb}(\mathbb{P}^n)$ of $\mathbb{P}^n$ parametrizing the smooth complete intersection of type $d_1, \dots, d_{n-1}$ of $\mathbb{P}^n$. The set $\mathbb{D}$ is an open subset of $\operatorname{Hilb}(\mathbb{P}^n)$ ([Ser06, p. 236]). Hence for each $C \in \mathbb{D}$ the tangent space of $\mathbb{D}$ at C has dimension $h^0(N_C)$. Since $\mathbb{D}$ is smooth ([Ser06, p. 236]), we have $\dim(\mathbb{D}) = h^0(N_C)$. The set $\mathbb{D}$ is irreducible, because it is parametrized by an open subset of the vector space $\prod_{i=1}^{n-1} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))$. Since C has genus ≥ 2, only finitely many automorphisms of $\mathbb{P}^n$ sends C into itself. The adjunction formula gives $\omega_C \cong \mathcal{O}_C(d_1 + \cdots + d_{n-1} - n - 1)$. Set $e := d_1 + \cdots + d_{n-1} - n - 1$. Since C has only finitely many line bundles L such that $L^{\otimes e} \cong \omega_C$, up to projective isomorphism C has only finitely many embeddings into $\mathbb{P}^n$ as a complete intersection of type $d_1, \ldots, d_{n-1}$. Hence the locus $\Gamma \subset \mathcal{M}_g$ of all complete intersection curves is an irreducible variety of dimension $h^0(N_C) - \dim \mathrm{Aut}(\mathbb{P}^n) = h^0(N_C) - (n+1)^2 + 1$ and it is contained in an irreducible component $\mathcal{Z}$ of $\mathfrak{Th}^r_{g,h}$. From diagram (3.12), Lemma 3.2.1 and the fact that C is projectively normal, we see that the map $\gamma_{1,h-1}(C, \mathcal{O}_C(\zeta))$ has cokernel equal to the cokernel of the map α and hence of dimension exactly $h^0(N_C) - (n+1)^2 + 1$. Then, Theorem 3.3.2 implies $\dim(T_C\mathcal{Z}) = h^0(N_C) - (n+1)^2 + 1$ and we can conclude that the curve $[C]$ is a smooth point of the component. In particular, $\Gamma = \mathcal{Z}$ and hence the claim follows. $\square$

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