ON REGULARITY FOR INTEGRAL CURRENTS

The smooth approximation of cycles

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Abstract

This thesis is devoted to the study of regularity properties of generalized surfaces in the Plateau problem. The first part of the thesis focuses on the regularity theory for Federer-Fleming integral currents, proving that every integral cycle can be approximated by a smooth submanifold up to a singular set of codimension 5. This is based on a joint study with William Browder and Camillo De Lellis, completing the program started by the former author with Frederick J. Almgren. The second part of the thesis extends the previous result to unoriented domains, showing that integral cycles mod 2 can be approximated by smooth submanifolds up to a singular set of codimension 3; in addition, this estimate on the singular set can be refined depending on the codimension of the cycle. Each estimate on the singular set is optimal, and the submanifolds can be taken free of singularities if the homology class admits a smooth embedded representative.

Contents

Introduction

Motivations

Over the past century the subject of geometric measure theory has matured from a collection of isolated results into a unified body of fundamental knowledge, characterized by both a rich internal structure and strong connections to a multitude of different areas of mathematics. These developments have improved our comprehension of the analytic and topological foundations of geometry, marking new directions in the calculus of variations. In particular, geometric variational problems – such as finding energy minimizing representatives in homology or homotopy classes of maps, minimizers with prescribed topological singularities or within classes of diffeomorphisms – have attracted growing interest and have been increasingly well understood in recent years.

The archetypal geometric variational problem is the so-called *Plateau problem*, named after the Belgian physicist Joseph Plateau, who extensively studied the structure of soap bubbles and soap films, *cfr.* [69]. Its embryonic formulations can be traced back to the second half of the 18th century when Lagrange, in Appendix I of his article [55], posed (in fact, solved) the first version of what can essentially now be stated as the following problem:[1]

"Find the surface of smallest area spanning a given contour".

Arguably, what is most fascinating about the Plateau problem is that, depending on the mathematical rigorous meaning that one attaches to the words *surface*, *area* and *spanning*, it is possible to develop different formulations of it, each providing beautiful mathematics; it is therefore fair to say that the Plateau problem is not just a single problem, but rather a collection of them, *cfr.* [5].

The theory of *integral currents*, which developed in the late 1950s after the early-stage contributions of De Giorgi in the setting of finite perimeter sets and of Federer & Fleming (*cfr.* [18, 19, 20] and [44]), is one of the most fruitful mathematical theories allowing one to solve the Plateau problem in higher dimensions and without topological restrictions. It partakes of the smoothness of differentiable manifolds and of the combinatorial structure of polyhedral chains with integer coefficients; in addition, the class of integral currents enjoys suitable compactness properties that makes it a useful tool

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[1] "Pour ne donner là-dessus qu'un exemple très-simple, supposons qu'il faille trouver la surface qui est la moindre de toutes celles qui ont un même périmètre donné".

for solving geometric variational problems, with particular attention to those that may exhibit singularities. These qualities make them one of the most satisfactory analytic and topological formulations of the concept of generalized surfaces to solve the oriented Plateau problem: the problem of finding an oriented generalized surface of smallest area spanning a given boundary or representing a given integral homology class.

After establishing the existence of solutions in this class of objects, the natural subsequent question is that of their regularity. The most studied approach of regularity theory focuses on how regular integral currents solving the Plateau problem turn out to be *a posteriori*. In particular, integral currents are studied under the strong variational assumption of being minimizers: this allows powerful regularity theorems to be proven and, in the last years, also a refined analysis of the structure of their singularities both in the interior and at the boundary; it is indeed well-known that solutions to the Plateau problem may be singular, as shown by examples of complex algebraic varieties like the complex cusp

$$\{(z, w) \in \mathbb{C}^2 : z^2 = w^3\},$$

admitting a branch point singularity at the origin. A full classification of singularities of minimizing integral currents is still missing and many questions remain unsolved, both when the current has codimension one and when it has higher codimension in the ambient space.

The presence of branching singularities when the codimension is higher than one required the development of an entirely new regularity theory and more sophisticated machinery, which was initially developed in a 1728-page, 10 cm-thick typewritten preprint by Frederick J. Almgren, *cfr.* [3, 6]. This result, establishing that the interior singularities of m-dimensional minimizing integral currents have Hausdorff codimension at least two, has only recently been understood, simplified and improved in a series of papers by Camillo De Lellis and Emanuele Spadaro, *cfr.* [32, 33, 34, 35, 36]; building upon their theory, a series of important achievements have been obtained in the last years, above all the $m-2$-rectifiable structure of the singular set due to De Lellis-Minter-Skorobogatova and, independently, Krummel-Wickramasekera, *cfr.* [29, 30, 31] and [52, 53, 54]. The subsequent important question is whether it is possible to obtain quantitative measure bounds on singularities or, more ambitiously, some generic tame structure for them. In this direction, together with Anna Skorobogatova, we have been able to prove that a specific meaningful subset of the interior singular set, that is the set of flat singular points of highest density, has locally finite $m-2$-dimensional Hausdorff measure, *cfr.* [17].

However, there is another way to interpret in what sense an integral current is more regular than in its original definition, and it corresponds to the question – without assuming any variational hypothesis – of whether it can be approximated by smoother objects, *cfr.* [41]. This approach to regularity theory goes back to the late 1950s and to the origin of integral currents with the seminal article of Federer and Fleming, see [41, 44]. In particular, in [44, page 458, lines 30-31], the authors write: “Integral currents are actually much smoother than one might expect from the preceding definition”, introducing the well-known deformation theorem of integral currents and the strong approximation theorem by means of polyhedral chains with integer coefficients, see [44, Theorems 5.5 and 8.22]. The deformation theorem represents a cornerstone in the theory, showing that the space of integer coefficients polyhedral chains is dense in the space of integral currents with respect to the flat topology and convergence of the masses.

The natural next question in the theory of integral currents is thus the following.

Question 1. *"How closely can one approximate an integral current T representing a given homology class τ by a smooth submanifold?"*

It may happen, in full generality, that integral currents are singular due to topological obstructions: in [86], Thom provides an example of a homology class of dimension 7 in a manifold of dimension 14 which is not realizable by means of a submanifold, *cfr.* Example 2.4.1. Moreover, it turns out that for each dimension greater than 7 there exist (in some manifold of arbitrarily large dimension) nonrealizable integral homology classes, see [86, Théorème III.9]; therefore, any integral current representing such a class must have singularities, see also [44, page 518, lines 3-7]. Nevertheless, these obstructions motivate the following basic question.

Question 2. *"Suppose that a given homology class τ is realizable by a smooth submanifold, is it always possible to approximate any integral current T representing τ (and hence,* a fortiori*, any T which is area-minimizing) by smooth submanifolds?"*

In the 1980s Frederick J. Almgren and William Browder started a research program to address this type of problems. In particular, in 1988 at the conference for the 60th birthday of Manfredo P. do Carmo, the former author posed these basic questions formally, announcing a solution to Questions 1 and 2 a few years later in the following form, see [7, page 20, line 9] and [5, page 44, line 21].

Theorem 1 (Optimal smooth approximation)**.** *Let $\mathcal{M}$ be a connected smooth closed oriented Riemannian manifold of dimension $m+n$. Let $\varepsilon > 0$, τ be an m-dimensional integral homology class in $H_m(\mathcal{M},\mathbb{Z})$, and T be an integral cycle representing τ. Then, there is a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and an oriented m-dimensional smooth submanifold Σ of $\mathcal{M} \setminus \mathcal{K}^{m-5}$ (where $\mathcal{K}^{m-5}$ denotes the $m-5$-skeleton of $\mathcal{K}$) with the following properties.*

1. *The m-dimensional volume of Σ does not exceed the mass of T by more than ε, that is $\mathcal{H}^m(\Sigma) \leq \mathbb{M}(T) + \varepsilon$.*
2. *The current $[\![\Sigma]\!]$ is an integral cycle homologous to T and there is an integral $m+1$-dimensional current S in $\mathcal{M}$ such that $\partial S = [\![\Sigma]\!] - T$ and $\mathbb{M}(S) < \varepsilon$.*
3. *If τ admits a smooth representative, then Σ can be chosen to be a smooth submanifold of $\mathcal{M}$.*

Nevertheless, the program was never completed and no proof of Theorem 1 ever appeared. In [7], among other things, the authors hint at the strategy of using Thom's criterion in the context of homotopy classes of mappings from $\mathcal{M}$ (less a skeleton) to the Thom complex $T(\widetilde{\gamma}^n)$.

Building upon this *homotopy with holes* technique, the first part of this thesis provides a complete proof of Theorem 1.

The second part of the thesis deals, instead, with regularity theory for the unoriented Plateau problem. In fact, Federer and Fleming's theory of integral currents necessarily requires the domain of integration to be oriented and homology classes to have integer coefficients; in order to generalize Federer and Fleming's theory to unoriented domains, it is possible to replace $\mathbb{Z}$ as coefficient group by the cyclic group $\mathbb{Z}_2$ of order 2: this has been done at the beginning of the sixties by Ziemer, later improved and generalized to any finite coefficient group by Fleming, see [95, 45]. Many regularity properties have been later derived for integral currents mod 2, both in terms of *a posteriori* regularity for minimizers and in terms of approximation theorems.

About the former aspect of regularity, it is a corollary of Federer's dimension reduction argument, *cfr.* [42], that mod 2 minimizing integral currents are induced by smooth submanifolds, up to an interior singular set of Hausdorff codimension at least 2, which turns out to be discrete for two-dimensional mod 2 currents *cfr.* [2]. In codimension higher than one, it is a theorem of Simon the $m-2$-rectifiable structure of interior singularities, *cfr.* [78, Corollary 1], which is also now known to be of locally finite $m-2$-Hausdorff measure by the methods of Naber and Valtorta in [68].

About the latter aspect of regularity, the only *a priori* approximation-type result for mod 2 integral currents was the classical mod 2 deformation theorem, *cfr.* [95, Theorem 4.2], and as a corollary the mod 2 strong polyhedral approximation theorem, *cfr.* [43, $(4.2.20)^\nu$, $(4.2.21)^\nu$], both developed as simple adaptations of the ones for integral currents. A natural question is thus to ask whether it is possible to approximate any intergral mod 2 current representing a $\mathbb{Z}_2$ homology class of a (possibly non-orientable) smooth closed Riemannian manifold by a smooth embedded submanifold. In general the answer is negative, because of the existence of mod 2 homology classes not admitting any smooth embedded representative, *cfr.* [88, Section 3]. In analogy with Theorem 1, the second part of this thesis provides an affirmative answer to the previous question every time the mod 2 homology class admits such a representative and, moreover, it provides sharp estimates on the singular sets of the approximating sequence.

Theorem 2 (Optimal unoriented smooth approximation)**.** *Let $\mathcal{M}$ be a connected smooth closed (not necessarily orientable) Riemannian manifold of dimension $m+n$. Let $\varepsilon > 0$, τ an m-dimensional homology class in $H_m(\mathcal{M}, \mathbb{Z}_2)$, and T an integral cycle mod 2 representing τ. Then there is a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and an m-dimensional smooth submanifold Σ of $\mathcal{M} \setminus \mathcal{K}^{m-n-1}$ (where $\mathcal{K}^{m-n-1}$ denotes the $m-n-1$-skeleton of $\mathcal{K}$) with the following properties.*

1. *The m-dimensional volume of Σ does not exceed the mass of T by more than ε, that is $\mathcal{H}^m(\Sigma) \leq \mathbb{M}(T) + \varepsilon$.*
2. *The integral mod 2 cycle $[\![\Sigma]\!]_{\mathbb{Z}_2}$ is homologous to T and there is an $m+1$-dimensional integral mod 2 current S in $\mathcal{M}$ such that $\partial S = [\![\Sigma]\!]_{\mathbb{Z}_2} - T$ and $\mathbb{M}(S) < \varepsilon$.*
3. *If τ admits a smooth embedded representative, then Σ can be chosen to be a smooth submanifold of $\mathcal{M}$.*

The main differences with respect to integral homology are mostly of a topological nature and allow for the development of a finer argument than that used in the oriented setting. This yields a significantly better estimate for the singular set, which further improves with higher codimensions; this is due to the particularly well-behaved homotopy type of the Thom space of the (unoriented) n-plane bundle, which in turn is due to the tame structure of the cohomology of the (unoriented) Grassmannians, *cfr.* Section 1.4.3.

A corollary of Theorems 1 and 2 is the absence of the so-called *Lavrentiev gap phenomenon* for the homological Plateau problem in the absence of topological obstructions, see Theorems 2.0.4 and 3.0.7. The Lavrentiev gap phenomenon refers to the surprising property of some functionals in the calculus of variations to admit different infima depending on whether the infimum is taken over the whole class of admissible objects or over some smaller class of more regular ones. The first of such examples was discovered by Lavrentiev in 1927, see [56], and the absence of such a phenomenon is a fundamental and desirable property in a variational theory, *cfr.* [14].

Finally, Theorems 1 and 2 are both optimal in terms of dimensional estimates for the singularities of the approximating sequence of submanifolds; each singular set turns out to represent meaningful topological obstructions, whose nature depends on the specific problem.

In the oriented setting, the codimension 5 singular set represents singularities which are of a worse nature compared to those in the unoriented setting. In particular, optimality of the codimension 5 construction in Theorem 1 is proved by a refined study of the singularities of cycles representing the innately singular homology class discovered by Thom, and then by exploiting a geometric resolution of singularity technique developed by Dennis Sullivan, see Theorem 2.4.3.

In the unoriented setting, instead, proving that the codimension $n+1$ estimate on the singular set in Theorem 2 for mod 2 homology is optimal is much subtler than in the integral case, since singularities that appear in this context all arise from the impossibility of finding embeddings in low codimensions, and not – as in integral classes – by innate singularities also obstructing Steenrod representability, *i.e.* representability by means of a continuous map, *cfr.* Section 3.4.

The proofs of Theorems 1 and 2 are based on a fruitful interplay between geometric measure theory with tools and techniques from differential topology and homotopy theory. Many additional questions can now be explored following the development of this new setup, which are currently under investigation and will be the object of study in the upcoming years.

Guide to the thesis

We briefly summarize the content of each chapter below.

Chapter 1. Preliminaries

The aim of Chapter 1 is to present the main definitions, notation and classical results needed throughout the thesis. In Section 1.1 we recall some preliminaries on the theory of integral currents. Particular attention is devoted to the celebrated Federer-Fleming decomposition theorem for integral currents and the strong polyhedral approximation theorem, *cfr.* [44, Theorems 5.5 and 8.22]. In Section 1.2 we recall some elementary facts (and their modifications) about smooth triangulations and piecewise-linear topology needed in the subsequent chapters. In Section 1.3 we recall some tools from differential topology and homotopy theory; particular attention is devoted to the celebrated Thom criterion, characterizing which homology classes can be represented by submanifolds, *cfr.* [86, Théorème II.1]. Finally, in Section 1.4 we recall some theory on cohomology operations, characteristic classes and on the cohomology of Grassmannians.

Chapter 2. Smooth approximation for integral cycles

The aim of Chapter 2 is to present the smooth approximation theorem for integral cycles. We give a full proof of Theorem 1, showing that every integral cycle can be approximated by a smooth embedded submanifold up to a singular set of codimension 5. In particular, if the integral cycle belongs to a homology class admitting a smooth embedded representative, then the submanifold can be taken free of singularitites. The chapter concludes by recalling Sullivan's geometric resolution of singularities and applies it to show the optimality of the codimension 5 construction. This chapter is based on [8], which is a joint study with William Browder and Camillo De Lellis, based on some previous preliminary work of the former author with Frederick J. Almgren.

Chapter 3. Smooth approximation for integral cycles mod 2

The aim of Chapter 3 is to present the smooth approximation theorem for integral cycles mod 2, proving Theorem 2. This extends the previous result to unoriented domains, showing that integral cycles mod 2 can be approximated by smooth submanifolds up to a singular set of codimension 3; in addition, this estimate on the singular set can be refined depending on the codimension of the cycle and the submanifolds can be taken free of singularities if the homology class admits a smooth embedded representative. Finally, after recalling a few results in the theory of stable mappings and their singularitites, we prove the optimality of the estimate on the singular set. This chapter is based on [15].

Chapter 4. Further results on the smooth approximation theorem

The aim of Chapter 4 is to present further partial results related to the smooth approximation theorem. In particular, a $\mathbb{Z}_k$-manifold approximation theorem is stated for integral cycles mod k, with k being an odd integer greater than 2. In addition, an example is provided of an area-minimizing cycle whose support cannot be locally approximated by a smooth submanifold.

Additional research

In this third part of the introduction we provide a short summary of the additional research carried out during the doctoral studies. In particular, we briefly describe the results obtained and the techniques involved in the proofs, referring to the articles for a complete treatment of each subject.

A.1 Hausdorff measure bounds for density-Q flat singularities of minimizing integral currents

As mentioned above, after the achievement of the $m-2$-rectifiable structure of the interior singular set of area-minimizing integral currents due to De Lellis-Skorobogatova-Minter and, independently, Krummel-Wickramasekera, *cfr.* [29, 30, 31] and [52, 53, 54], the subsequent important question is whether it is possible to obtain quantitative measure bounds on the singular set or, more modestly, on some parts of it. In this direction, with Anna Skorobogatova, we proved that a specific meaningful subset of the interior singular set, that is the set of flat singular points of highest density, has locally finite $m-2$-dimensional Hausdorff measure, *cfr.* [17].

0.1. Assumption. Let $\kappa_0 \in (0,1]$, $m, l \in \mathbb{N}_{\geq 1}$ and $n \geq \overline{n} \geq 2$ be integers. Let Σ be an $m+\overline{n}$-dimensional embedded complete submanifold of $\mathbb{R}^{m+n} = \mathbb{R}^{m+\overline{n}+l}$ of class C^{3,κ_0}, and let T be an m-dimensional integral current in $\Sigma \cap \mathbf{B}_{7\sqrt{m}}$ with $\partial T \llcorner \mathbf{B}_{7\sqrt{m}} = 0$. We assume T is *area-minimizing* in $\Sigma \cap \mathbf{B}_{7\sqrt{m}}$, that is $\operatorname{spt}(T) \subset \Sigma \cap \mathbf{B}_{7\sqrt{m}}$ and

$$\mathbb{M}(T + \partial S) \geq \mathbb{M}(T), \quad \text{for every } m+1\text{-integral current } S \text{ with } \operatorname{spt}(S) \subset \Sigma \cap \mathbf{B}_{7\sqrt{m}}.$$

Moreover, for every $p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}$ we assume that $\Sigma \cap \mathbf{B}_{7\sqrt{m}}$ is the graph of a C^{3,κ_0} function $\Psi_p : \mathrm{T}_p\Sigma \cap \mathbf{B}_{7\sqrt{m}} \to \mathrm{T}_p\Sigma^{\perp}$. We denote $\boldsymbol{c}(\Sigma) := \sup_{p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}} \|D\Psi_p\|_{C^{2,\kappa_0}}$ and we assume $\boldsymbol{c}(\Sigma) \leq \overline{\varepsilon} \leq 1$, where $\overline{\varepsilon}$ is a small positive constant whose choice will be specified in each statement.

For T and Σ as in Assumption 0.1, we recall that a point $p \in \operatorname{spt}(T)$ is called a *regular point* if there is a positive radius $r > 0$, a C^{3,κ_0}-regular embedded m-dimensional oriented submanifold $\mathcal{M} \subset \Sigma$ and a positive integer Q such that $T \llcorner \mathbf{B}_r(p) = Q[\![\mathcal{M}]\!]$. The set of regular points of T, which is relatively open in $\operatorname{spt}(T)$, is denoted by $\operatorname{Reg}(T)$. Its complement, *i.e.* $\operatorname{spt}(T) \setminus \operatorname{Reg}(T)$, is denoted by $\operatorname{Sing}(T)$ and is called the *singular set* of T. For $Q \in \mathbb{N}$, we denote by $\mathrm{D}_Q(T)$ the points of density Q of the current T, and set $\operatorname{Sing}_Q(T) := \operatorname{Sing}(T) \cap \mathrm{D}_Q(T)$.

For any $r > 0$ and $p \in \mathbb{R}^{m+n}$, $\iota_{p,r} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is the map $y \mapsto \frac{y-p}{r}$ and we denote $T_{p,r} := (\iota_{p,r})_\sharp T$, *i.e.* the pushforward of T by the map $\iota_{p,r}$. We also denote by $\Sigma_{p,r}$ the rescaled ambient manifold $\iota_{p,r}(\Sigma)$. The classical monotonicity formula of mass ratios (see [77, Theorem 17.6] and [33, Lemma A.1]) implies that, for every $r_k \downarrow 0$ and $p \in \operatorname{spt}(T)$, there is a subsequence (not relabelled) for which T_{p,r_k} converges to an integral cycle S which is a cone (*i.e.*, $S_{0,r} = S$ for all $r > 0$ and $\partial S = 0$) and which is area-minimizing in $\mathbb{R}^{m+n}$. Such a cone is referred to as a *tangent cone* to T at p.

Recall that a tangent cone supported in an m-dimensional plane is called *flat*, and a point $p \in \mathrm{Sing}(T)$ with at least one flat tangent cone is called *a flat singular point.* We denote by $\mathfrak{F}(T)$ the set of flat singular points. We remark that from the constancy lemma, see [77, Theorem 26.27], a flat tangent cone at a singular point q must be an oriented m-dimensional plane with positive integer density $\Theta(T,p)$; we remark that by Allard's Regularity Theorem, see [1], $\Theta(T,q) > 1$. Hence, we can write the following subdivision

$$\mathfrak{F}(T) = \bigcup_{Q=2}^{\infty} \mathfrak{F}_Q(T),$$

where $\mathfrak{F}_Q(T) := \{p \in \mathfrak{F}(T) : \Theta(T,p) = Q\}$ is the set of *flat singular points of density* Q. The typical examples of flat singular points are branching singularities of area-minimizers induced by complex subvarieties of $\mathbb{C}^n$. We introduce a further parameter which is a real number belonging to $[1,\infty)$, which is called the *singularity degree of* T *at* p and that we denote $\mathrm{I}(T,p)$. For a fixed $z \in [1,\infty)$, we will denote by $\mathfrak{F}_{Q,\geqslant z}(T)$ the set of *flat singular points of* T *with density* Q *and singularity degree* $\geq z$, that is

$$\mathfrak{F}_{Q,\geqslant z}(T) := \mathfrak{F}_Q(T) \cap \{\mathrm{I}(T,p) \geq z\}.$$

0.2. Assumption. Suppose that T, Σ are as in Assumption 0.1. Moreover, suppose that 0 is a flat singular point of T and $Q \in \mathbb{N} \setminus \{0,1\}$ is the density of T at 0. Moreover, assume that there exists an m-dimensional plane $\pi_0 \in T_0\Sigma$ such that $(\mathbf{p}_{\pi_0})_\sharp T \llcorner \mathbf{B}_{6\sqrt{m}} = Q[\![B_{6\sqrt{m}}(0,\pi_0)]\!]$, where $\mathbf{p}_{\pi_0}$ is the orthogonal projection onto the plane π_0.

Let us state the main result of [17] on the fine structural properties of $\mathfrak{F}_Q(T)$.

Theorem 0.3. *Suppose that* T *and* Σ *satisfy Assumption 0.2. Then* $\mathfrak{F}_Q(T)$ *has finite* $m-2$*-dimensional Hausdorff measure, namely*

$$\mathcal{H}^{m-2}(\mathfrak{F}_Q(T)) < \infty.$$

A.2 Generic uniqueness for the Plateau problem

Given a complete $C^{h,\beta}$-smooth Riemannian manifold $\mathcal{M}$ of dimension $m+n$ and an oriented closed submanifold $\Gamma \subset \mathcal{M}$ of dimension $m-1$ which is a boundary in integral homology, in collaboration with Andrea Marchese, Andrea Merlo and Simone Steinbrüchel, we construct in [16] a complete metric space $\mathcal{B}$ of $C^{h,\alpha}$-perturbations of Γ inside $\mathcal{M}$ $(\alpha < \beta)$ enjoying the following property.

Theorem 0.4. *For the generic element* $b \in \mathcal{B}$*, in the sense of Baire categories, there exists a unique* m*-dimensional area-minimizing integral current* T *in* $\mathcal{M}$ *with* $\partial T = b$ *and it has multiplicity one* $\|T\|$*-almost everywhere.*

The proof can be seen as a consequence of the important boundary regularity theory developed by De Lellis, De Philippis, Hirsch and Massaccesi in [22], coupled with the observation that *one-sided* regular boundary points are generic compared to *two-sided* ones; from this, previous conditional results by Morgan on generic uniqueness can be applied to achieve our conclusions, *cfr.* [62, 63, 64].

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My last thought goes to Bill Browder: now it's time for me to make my own mistake. This thesis is dedicated to him.

List of symbols

Here we list some common notation used throughout the thesis:

$\mathcal{H}^k(A)$	k-dimensional Hausdorff measure of the set A;
A°	interior of the set A;
L^j	j-skeleton of a cubical decomposition of $\mathbb{R}^d$;
$\mathcal{K}^j$	j-skeleton of the triangulation $\mathcal{K}$;
$b(S)$	barycenter of the simplex S;
$B_\delta(A)$	δ-neighborhood of A, *i.e.* $\{x : \mathrm{dist}\{x, A\} < \delta\}$;
$V_\delta(\mathcal{K}^j)$	neighborhood of $\mathcal{K}^j$ defined in Subsection 2.1.1;
$U_\delta(\mathcal{K}^j)$	neighborhood of $\mathcal{K}^j$ with smooth boundary defined in Lemma 2.1.2;
$\mathrm{spt}(T)$	support of the current T;
$\mathbb{M}(T)$	mass of T;
$\mathbb{F}(T)$	integral flat norm of T;
$f_\sharp T$	push-forward of T by the map f;
$\mathcal{D}^k(U)$	space of smooth compactly supported k-forms on U;
$\mathcal{D}_k(U)$	space of (de Rham) k-currents in U;
$\mathbf{N}_k(U)$	space of normal k-currents in U;
$\mathbf{R}_k(U)$	space of integer rectifiable k-currents in U;
$\mathbf{I}_k(U)$	space of integral k-currents in U;
$\mathbf{P}_k(U)$	space of polyhedral k-currents in U;
$\mathbf{F}_k(U)$	space of integral flat k-chains in U;
$\mathcal{Z}_k(\mathcal{M},\mathrm{G})$	space of k-cycles with coefficients in G and support in $\mathcal{M}$;
$\mathcal{B}_k(\mathcal{M},\mathrm{G})$	space of k-boundaries with coefficients in G and support in $\mathcal{M}$;
$H_k(\mathcal{M},\mathrm{G})$	k-homology group with coefficients in G and support in $\mathcal{M}$;
$\mathcal{Z}_{k,Lip}(\mathcal{M},\mathrm{G})$	space of k-Lipschitz cycles with coefficients in G and support in $\mathcal{M}$;
$BSO(n)$	classifying space for oriented n-plane bundles;
$BO(n)$	classifying space for n-plane bundles;
$\widetilde{\gamma}^n$	universal oriented n-plane bundle over $BSO(n)$;
γ^n	universal n-plane bundle over $BO(n)$;
$T(\widetilde{\gamma}^n)$	Thom space of the universal oriented n-plane bundle;
$T(\gamma^n)$	Thom space of the universal n-plane bundle;
$K(\pi, n)$	Eilenberg-MacLane space of type (π, n);
Sq^i	Steenrod squares;
$\mathcal{P}^i_p$	Steenrod reduced p^{th} power for odd prime p;
β_p	Bockstein homomorphism for odd prime p;
$w_i(\xi)$	i^{th} Stiefel-Whitney class of the n-plane bundle ξ;
$p_i(\xi)$	i^{th} Pontrjagin class of the n-plane bundle ξ;
$e(\xi)$	Euler class of the n-plane bundle ξ;
$St_p^{2r(p-1)+1}$	Thom's Steenrod powers defined after Remark 2.0.5;
$\widetilde{\Omega}_r$	r-dimensional oriented cobordism group.

Chapter 1

Preliminary results

The aim of Chapter 1 is to present the main definitions, notation and classical results needed throughout the thesis. In Section 1.1 we recall some preliminaries on the theory of integral currents and we define a generalized version of the oriented Plateau problem in this category. Particular attention is devoted to the celebrated Federer-Fleming decomposition theorem for integral currents and the strong polyhedral approximation theorem, *cfr.* [44, Theorems 5.5 and 8.22]; in addition, in Section 1.2 we describe a generalization of a few results in Whitehead's theory of smooth triangulations. In Section 1.3 we recall some tools from Thom's cobordism theory and homotopy theory. Particular attention is devoted to the celebrated Thom criterion, characterizing which homology classes can be represented by smooth embedded submanifolds, *cfr.* [86, Théorème II.1]; in addition, in Section 1.4 we recall some notions concerning cohomology operations, characteristic classes and the cohomology of Grassmannians.

We do not aim to be exhaustive and most of the proofs will be omitted: we will focus on definitions and results that will be relevant for the subsequent chapters. For a complete treatment of these subjects, we refer the reader to [43, 44, 46, 77] for the Federer-Fleming theory of integral currents and to [58, 74] for an introduction to piecewise-linear topology; to [39, 60, 80, 84, 86] for the main notions of cobordism and homotopy theory and, in particular, to [39, 66, 81] for an introduction to cohomology operations and to [60] for characteristic classes.

1.1 The Federer-Fleming theory of integral currents

The notion of current[1] first appeared, albeit in a less general and less precise form, in the 1930s in the work by Georges de Rham [37, 38]. It was only after Schwartz's introduction of the concept of distribution in 1945, see [75], that de Rham reframed its definition from the one dealing with homologies on forms to the cleaner one that we are going to define now.

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[1] The choice of the term "current" is motivated by the fact that in a 3-dimensional space "1-dimensional currents" can be interpreted as electrical currents and indeed, in [37] and [38], de Rham thought of them as cables carrying an electrical current of unit intensity.

1.1.1 Currents and generalized Plateau problems

Let $U \subset \mathbb{R}^d$ is an open set and $0 \leq k \leq d$. We denote by $\mathcal{D}^k(U)$ the space of smooth compactly supported k-forms on U with the colimit topology[2] (*cfr.* Remark 1.1.3), where the notion of convergence is characterized by the following definition.

Definition 1.1.1. A sequence $(\omega^n)_{n\in\mathbb{N}} \in \mathcal{D}^k(U)$, that in local coordinates takes the form

$$\omega^n(x) = \sum_{I\in I(k,d)} \omega^n_I(x)dx_I,$$

converges to $\omega \in \mathcal{D}^k(U)$ as $n \to \infty$ if there exists a compact set $K \subset U$ such that

1. $\mathrm{spt}(\omega^n_I) \subset K$ for any $I \in I(k,d)$ and for any $n \in \mathbb{N}$,

2. for every choice of the multi-index α we have $D^\alpha\omega^n_I \to D^\alpha\omega_I$ uniformly in K for every $I \in I(k,d)$.

Remark 1.1.2. Note that in local coordinates we are just equipping $\mathcal{D}^k(U)$ with the topology on the predual of the space of distributions $\mathcal{D}^0(U)$: there is indeed the following identification

$$\mathcal{D}^k(U) \simeq \left(\mathcal{D}^0(U)\right)^{|I(k,d)|}.$$

Remark 1.1.3. The topology described above turns $\mathcal{D}^k(U)$ into a locally convex separable topological vector space. We remark that $\mathcal{D}^k(U)$ is not a Fréchet space[3] since its topology is not induced by a countable family of seminorms (and hence it is not metrizable), but each space $\mathcal{D}^k_K(U)$ of smooth k-forms supported in the compact set K is. Note that $\mathcal{D}^k(U)$ is the colimit of the spaces $\mathcal{D}^k_K(U)$, with inclusions $\mathcal{D}^k_K \subset \mathcal{D}^k_{K'}$, where $K \subset K'$. Using an exhaustion of U by compact sets, *i.e.* a sequence $(K_n)_n$ of compacts with $U = \cup_n K_n$, with $K_n \subset K^\circ_{n+1}$, it is possible to prove that a sequence in $\mathcal{D}^k(U)$ converges in the colimit topology if and only if it converges as in Definition 1.1.1. Accordingly, we can define the following.

Definition 1.1.4. A *k-dimensional current* in U is a continuous linear functional on $\mathcal{D}^k(U)$. The space of k-dimensional currents in U is denoted by $\mathcal{D}_k(U)$.

Hence, $\mathcal{D}_k(U)$ is the inverse limit of the dual spaces of $\mathcal{D}^k_K(U)$ in the category of locally convex topological vector spaces.

Definition 1.1.5. Given $T \in \mathcal{D}_k(U)$, we define the *boundary* of T as the $k-1$-current defined as

$$\partial T(\omega) := T(d\omega), \ \text{ for all } \omega \in \mathcal{D}^{k-1}(U).$$

---

[2] Sometimes called also direct limit or final topology, or even inductive limit topology.

[3] It is, however, a Montel space, *i.e.* a topological vector space that is barrelled and such that every closed bounded set is compact; this shows that $\mathcal{D}^k(U)$ enjoys many desirable topological properties, in particular on the notions of convergence on its dual space, *cfr.* [89] for more on this.

Remark 1.1.6. The functional ∂T is well-defined, linear and continuous. Note that ∂ on k-currents is just the adjoint operator of d on smooth, compactly supported k-forms. The counterpart of the fact that $d \circ d = 0$ is that $\partial(\partial T) = 0$ for all $T \in \mathcal{D}_k(U)$.

Definition 1.1.7. Given $T \in \mathcal{D}_k(U)$, we define the *mass* of T as

$$\mathbb{M}(T) := \sup \left\{ T(\omega) \,|\, \omega \in \mathcal{D}^k(U), \|\omega(x)\|^* \leq 1 \ \ \forall x \in U \right\},$$

where $\|\omega(x)\|^*$ is the comass norm of $\omega(x)$.

Remark 1.1.8. Definitions 1.1.4, 1.1.5 and 1.1.7 can be considered as generalized concepts for the notions of manifold, boundary of a manifold and volume of a manifold respectively. Indeed, let Σ be a smooth, oriented, k-dimensional submanifold in $\mathbb{R}^d$. In the spirit of Poincaré duality, we define the following linear functional $T_\Sigma(\cdot)$ on $\mathcal{D}^k(U)$:

$$\omega \mapsto \int_\Sigma \omega.$$

Such a current is often denoted by $[\![\Sigma]\!]$, mirroring the notation for the fundamental class of Σ. Note that T_Σ is uniquely determined by Σ in the sense that $\Sigma \neq \Sigma'$ (as oriented submanifolds) implies $T_\Sigma \neq T_{\Sigma'}$. We can rewrite Stokes' theorem in the following way

$$\partial T_\Sigma = T_{\partial \Sigma}$$

and it is possible to prove that

$$\mathcal{H}^k(\Sigma) = \sup \left\{ T_\Sigma(\omega) \,|\, \omega \in \mathcal{D}^k\left(U\right), \|\omega(x)\|^* \leq 1 \ \ \forall x \in U \right\},$$

justifying the terminology of "generalized surfaces".

Since we are interested in a variational problem, as a dual space, $\mathcal{D}_k(U)$ is naturally equipped with the weak* topology[4].

Definition 1.1.9. Given a sequence of k-currents $(T_n)_n$ and a k-current T, we say that $(T_n)_n$ *converges to T in the sense of currents* if the sequence converges to T with respect to the weak* topology, that is

$$\lim_{n \to \infty} T_n(\omega) = T(\omega) \quad \text{for all } \ \omega \in \mathcal{D}^k(U).$$

Remark 1.1.10. Note that the boundary operator is continuous and the mass is a lower semicontinuous functional with respect to the weak* convergence of currents.

As for distributions, we have a notion of support of a current.

Definition 1.1.11. The *support* of a k-current T in $\mathcal{D}_k(U)$ is defined as

$$\operatorname{spt}(T) := \mathbb{R}^d \setminus \bigcup \left\{ V \subset U, V \text{ open} : \, \omega \in \mathcal{D}^k(U), \, \operatorname{spt}(\omega) \subset V \Rightarrow T(\omega) = 0 \right\}.$$

---

[4] Even if, since $\mathcal{D}^k(U)$ is a Montel space, a sequence in $\mathcal{D}_k(U)$ is converging in the weak* topology if and only if it is strongly converging.

Remark 1.1.12. The notion of current is so general that it is possible to define for every $k = 1, 2, \dots, d$ a k-current $T \in \mathcal{D}_k(\mathbb{R}^d)$ whose support is a singleton: calling them *k-dimensional* objects does not encode the existence of a surrounding k-dimensional geometry.

Important subclasses of currents need to be defined to obtain a reasonable geometric object in the solution of the generalized Plateau problem.

Definition 1.1.13. We say that a k-current T has *finite mass* if $\mathbb{M}(T) < \infty$.

Remark 1.1.14. Note that 0-currents with finite mass can be identified with signed measures. The definition of current with finite mass is not trivial. In fact the colimit topology on $\mathcal{D}^k(\mathbb{R}^d)$ to which currents are dual induces a notion of convergence (that of Definition 1.1.1) which is clearly finer than that induced by the sup-norm

$$\|\omega(x)\|_\infty := \sup\left\{\|\omega(x)\|^* \colon x \in \mathbb{R}^d\right\}.$$

Standard examples of currents with infinite mass are derivatives of the Dirac delta at a point $x_0 \in U$: fix $x_0 \in U \subset \mathbb{R}$ and take the 0-current T on $\mathbb{R}$ such that $T(\varphi) = \varphi'(x_0)$, for all $\varphi \in \mathcal{C}_c^\infty(U)$.

Given $T \in \mathcal{D}_k(\mathbb{R}^d)$ with finite mass we obtain

$$|T(\omega)| \leq \mathbb{M}(T)\|\omega\|_\infty \quad \text{for all } \omega \in \mathcal{D}^k(\mathbb{R}^d).$$

Hence, T can be extended by density to a linear continuous functional defined on the space of continuous and infinitesimal at infinity k-forms $\omega \in \mathcal{C}_0\left(\mathbb{R}^d, \Lambda^k(\mathbb{R}^d)\right)$, where $\Lambda^k(\mathbb{R}^d)$ is the space of k-covectors on $\mathbb{R}^d$. By the Riesz representation theorem, T can be represented by integration with respect to a measure with values in $\Lambda^k(\mathbb{R}^d)^*$, that we denote by $\Lambda_k(\mathbb{R}^d)$. Thus, there exists a positive and finite measure μ on $\mathbb{R}^d$ and a vector field $\tau : \mathbb{R}^d \to \Lambda_k(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d, \mu)$, unitary (in mass norm) μ-a.e. such that

$$T(\omega) = \int_{\mathbb{R}^d} \langle \omega(x), \tau(x)\rangle d\mu(x) \qquad \text{for all } \omega \in \mathcal{D}^k(\mathbb{R}^d). \tag{1.1}$$

We denote by $T = \tau \cdot \mu$ a k-current whose action on a form is as in (1.1). It is customary to call μ the *total variation measure* associated with T and denote it by $\|T\|$. It can be checked that $\mathbb{M}(T) = \mu(\mathbb{R}^d)$, that is the mass of T equals the mass of the measure $\|T\|$. In particular, if $T \in \mathcal{D}_k(U)$ with finite mass, if $B \subset U$ is any Borel set we can define the *restriction* of T on B, denoted with $T \llcorner B$, by

$$T \llcorner B(\omega) := \int_B \langle \omega(x), \tau(x)\rangle d\mu(x).$$

Definition 1.1.15. A k-current $T \in \mathcal{D}_k(U)$ is called *normal* if both T and ∂T have finite mass, *i.e.* $\mathbb{M}(T) < \infty$ and $\mathbb{M}(\partial T) < \infty$. The space of normal k-currents is denoted by $\mathbf{N}_k(U)$.

Normal currents have good compactness properties:

Proposition 1.1.16. *Let* $(T_n)_n$ *be a sequence of normal* k*-currents s.t.*

$$\sup_n \left(\mathbb{M}(T_n) + \mathbb{M}(\partial T_n)\right) < \infty.$$

Then, up to subsequences, $(T_n)_n$ *converges in the sense of currents to a* k*-current* T*. Moreover, we have that*

$$\mathbb{M}(T) \leq \liminf_{n\to\infty} \mathbb{M}(T_n) \quad \textit{and} \quad \mathbb{M}(\partial T) \leq \liminf_{n\to\infty} \mathbb{M}(\partial T_n).$$

In particular, T *is a normal* k*-current.*

If the ambient space is a compact oriented smooth manifold, it is possible to define its real homology in terms of currents. More formally, let $\mathcal{M}^{m+n}$ be an oriented closed smooth submanifold of dimension $m+n$ smoothly embedded by Whitney's theorem in $\mathbb{R}^d$ for some large d. For $k = 0, 1, \ldots$, we define the space of normal cycles and boundaries respectively as follows:

$$\begin{aligned} Z_k(\mathcal{M}, \mathbb{R}) &:= \{T \in \mathbf{N}_k(U) \mid \ \mathrm{spt}\,(T) \subset \mathcal{M}, \partial T = 0\} \\ B_k(\mathcal{M}, \mathbb{R}) &:= \{\partial S \mid S \in \mathbf{N}_{k+1}(U), \ \mathrm{spt}\,(S) \subset \mathcal{M}\}, \end{aligned}$$

where U is a tubular neighborhood of $\mathcal{M}$. The k-homology group $H_k(\mathcal{M}, \mathbb{R})$ of a $\mathcal{M}$ turns out to be well-defined as the following quotient space

$$H_k(\mathcal{M}, \mathbb{R}) := Z_k(\mathcal{M}, \mathbb{R})/B_k(\mathcal{M}, \mathbb{R}).$$

Two sets of pairings are naturally associated to homology and de Rham cohomology, denoted by $H^*_{\mathrm{dR}}(\cdot)$. One is the set of de Rham pairings between currents and forms which factors to homology and cohomology

$$\langle \, , \rangle \, : H_k(\mathcal{M}, \mathbb{R}) \times H^k_{\mathrm{dR}}(\mathcal{M}) \to \mathbb{R}$$

defined by $\langle [T], [\omega] \rangle := T(\omega)$. The others are the pairings

$$(\omega, \eta) \in \mathcal{D}^k(\mathcal{M}) \times \mathcal{D}^{n-k}(\mathcal{M}) \to \int_{\mathcal{M}} \omega \wedge \eta$$

which, by Stokes theorem, induce Poincaré pairings in cohomology

$$\mathrm{Poinc}\langle \, , \rangle \, : H^k_{\mathrm{dR}}(\mathcal{M}) \times H^{n-k}_{\mathrm{dR}}(\mathcal{M}) \to \mathbb{R}$$

given by $\mathrm{Poinc}\langle [\omega], [\eta] \rangle := \int \omega \wedge \eta$. Classical theorems, respectively the de Rham and the Poincaré duality theorems, state in fact that de Rham and Poincaré pairings are non degenerate; hence there are the following *Poincaré duality isomorphisms*

$$P : H^{n-k}_{\mathrm{dR}}(\mathcal{M}) \to H_k(\mathcal{M}, \mathbb{R}).$$

At this point one could formulate a very general version of the oriented Plateau problem, both for fixed boundaries and in real homology classes.

The Plateau problem for normal currents. *Given* $S \in \mathbf{N}_{k-1}(U)$ *such that* $S = \partial T$ *for some* $T \in \mathbf{N}_k(U)$*, does there exist* $T' \in \mathbf{N}_k(U)$ *such that* $\partial T' = S$ *and* $\mathbb{M}(T')$ *is minimized? Alternatively, given a nontrivial* $\tau \in H_k(X, \mathbb{R})$*, does there exist a normal cycle* $Z \in \tau$ *such that* $\mathbb{M}(Z) = \inf\{\mathbb{M}(T) \,|\, T \in \tau\}$*?*

Remark 1.1.17. By Proposition 1.1.16 it is possible to apply the direct methods in the calculus of variations: the $\mathbb{M}(\cdot)$ functional is lower semicontinuous by Remark 1.1.10 with respect to convergence in the sense of currents and by sequential compactness in Proposition 1.1.16 we solve the very general Plateau problem. For the homological version, this comes as a corollary of, again, lowersemicontinuity of the mass and the fact that $B_k(\mathcal{M}, \mathbb{R})$ is weak* closed (we do not comment further on this now, since to formalize this problem and solve it we need to introduce the Federer-Fleming deformation theorem, see Section 1.1.2).

Nevertheless, the Plateau problem formulation for normal currents is not satisfactory since the space $\mathbf{N}_k(U)$ still contains elements that do not have the geometric meaning of being genuinely "k-dimensional". Indeed, we can construct the following example, *cfr.* [21, Example 2.8].

Example 1.1.18. Consider the south and north poles S and N in the sphere $S^2 \subset \mathbb{R}^3$ and let Z be the 0-dimensional current $[\![N]\!] - [\![S]\!]$. For any meridian γ joining S to N the corresponding current $[\![\gamma]\!]$ is a minimizer of the mass among all currents T with $\partial T = Z$ and $\operatorname{spt}(T) \subset S^2$. However the same holds if we parametrize the meridians as a one-parameter family $(\gamma_t)_{t \in S^1}$ where t is the intersection of γ_t with the equator $\{x_3 = 0\} \cap S^2$. If μ is a probability measure on S^1, then the current

$$T_1(\omega) := \int_{S^1} [\![\gamma_t]\!](\omega)\, d\mu(t)$$

is also mass-minimizing among all currents T with $\operatorname{spt}(T) \subset S^2$ and $\partial T = Z$.

One could argue that Example 1.1.18 is not particularly troublesome, since among all minimizers it is still possible to find genuine 1-dimensional minimizers. In fact, deeper issues arise as the following theorem shows (see [91] for a short proof).

Theorem 1.1.19 (Lavrentiev gap phenomenon)**.** *Given a smooth closed embedded curve* γ *in* $\mathbb{R}^4$ *define the following quantities:*

$$\begin{aligned} M(\gamma) &:= \inf\left\{\mathcal{H}^2(\Sigma) : \Sigma \textit{ is immersed, oriented and } \partial\Sigma = \gamma\right\}, \\ m(\gamma) &:= \min\{\mathbb{M}(T) : T \in \mathbf{N}_2(\mathbb{R}^4),\, \partial T = [\![\gamma]\!]\}. \end{aligned}$$

Then there are γ*'s for which* $m(\gamma) < M(\gamma)$*.*

As a result, we need to add more structure to enclose some reasonable geometric meaning in the notion of current. In particular, we need to focus on a particular subclass partaking of the smoothness of differentiable manifolds and of the combinatorial structure of polyhedral chains with integer coefficients: *integral currents.*

Existence of solutions for the generalized Plateau problem

We recall that the notion of rectifiable set turns out to be the appropriate measure-theoretic generalization of k-dimensional C^1-submanifolds. Roughly, k-rectifiable sets are characterized by the equivalent properties of being countable unions of measurable pieces of k-dimensional C^1-submanifolds, or of admitting a measure-theoretic notion of tangent space $\mathcal{H}^k$-almost everywhere.

Definition 1.1.20. A set $E \subset \mathbb{R}^d$ is said to be *countably k-rectifiable* (or simply *k-rectifiable*) if it is $\mathcal{H}^k$-measurable and there exist k-dimensional embedded submanifolds $\mathcal{N}_1, \mathcal{N}_2, \ldots$ and $E_0 \subset \mathbb{R}^d$ with $\mathcal{H}^k(E_0) = 0$ such that

$$E = E_0 \cup \bigcup_{i=1}^{\infty} E_i,$$

where E_i are pairwise disjoint Borel subsets of $\mathcal{N}_i$.

Given a k-rectifiable set E in $\mathbb{R}^d$, one can define an *orientation* of E as a Borel function $\tau_E : E \to \Lambda_k(\mathbb{R}^d)$ such that $\tau_E(x)$ is a simple unit k-vector spanning $T_x E$ for $\mathcal{H}^k$-a.e. $x \in E$.

Definition 1.1.21. Let U be an open set in $\mathbb{R}^d$. We call a k-current T *rectifiable* if T admits the following integral representation

$$T(\omega) = \int_E \langle \omega(x), \tau_E(x) \rangle \, \theta(x) d\mathcal{H}^k(x),$$

where E is a k-rectifiable set, τ_E is an orientation of E, and θ is a real-valued function $\theta \in L^1(U, \mathcal{H}^k \llcorner E)$. The function θ is often called *multiplicity* of the current T. We will often use the notation $T = [\![E, \tau_E, \theta]\!]$.

Remark 1.1.22. Given $T = [\![E, \tau_E, \theta]\!]$ a k-rectifiable current, then E, τ_E, θ in the representation of T are not uniquely determined: one could write equivalently $[\![E, -\tau_E, -\theta]\!]$ instead of $[\![E, \tau_E, \theta]\!]$. If we require in addition that $\theta > 0$ for $\mathcal{H}^k$-a.e. $x \in E$, then E, τ_E, m are uniquely determined (up to $\mathcal{H}^k$-null sets).
In particular we have that

$$\mathbb{M}(T) = \int_E |\theta(x)| d\mathcal{H}^k(x) = \|\theta\|_{L^1(U, \mathcal{H}^k \llcorner E)}.$$

Definition 1.1.23. A rectifiable current $T = [\![E, \tau_E, \theta]\!]$ whose multiplicity θ takes values in $\mathbb{Z}$ is called an *integer (multiplicity) rectifiable current.* The set of integer rectifiable k-currents in U is denoted by $\mathbf{R}_k(U)$.

Remark 1.1.24. Any $T \in \mathbf{R}_0(U)$ can be written as finite sum of weighted Dirac masses. More formally, let $x_i \in \mathbb{R}^d$, $\theta_i \in \mathbb{Z}$ and δ_{x_i} the Dirac mass at point x_i for $i = 1, \ldots, N$. Then we can write

$$T = \sum_{i=1}^{N} \theta_i \delta_{x_i}.$$

Indeed, a function $\theta \in L^1(U, \mathcal{H}^0 \llcorner E)$ with values in $\mathbb{Z}$ is a function that attains a finite number of values in a finite number of points, vanishing elsewhere.

Remark 1.1.25. Unlike the space of rectifiable currents, the set of integer rectifiable currents is clearly not a real vector space anymore, since in general $\lambda T \in \mathbf{R}_k(U)$ only if $T \in \mathbf{R}_k(U)$ and λ is an integer. As a result, there is no hope to invoke any simple functional-analytic principle to obtain good compactness properties in $\mathbf{R}_k(U)$. Still, $\mathbf{R}_k(U)$ maintains the algebraic structure of abelian group with respect to the sum.

Definition 1.1.26. If both T and ∂T are integer rectifiable currents, then T is called an *integral current.* The corresponding space is denoted by $\mathbf{I}_k(U)$.

The following theorem by Federer and Fleming is one of the cornerstones in the theory of integral currents, see [44, Section 8].

Theorem 1.1.27 (Closure theorem)**.** *Let $(T_n)_n$ be a sequence of integral k-currents in $U \subset \mathbb{R}^d$ such that*

$$\sup_n (\mathbb{M}(T_n) + \mathbb{M}(\partial T_n)) < \infty.$$

Then there exist $T \in \mathbf{I}_k(U)$ and a subsequence $(T_{n_j})_j$ such that T_{n_j} converges in the sense of currents to T.

It is worth noticing that, for a uniformly bounded sequence of integral currents $(T_n)_n$, the existence of a converging subsequence and a limit current T follows from Proposition 1.1.16. The fact that the limit current T is not merely normal but integral relies crucially on the geometry of integral currents: hence the name *closure* theorem, asserting that bounded sets (with respect to the mass and the mass of the boundary) in the space $\mathbf{I}_k(U)$ are (sequentially) weak* closed in $\mathbf{N}_k(U)$.

The closure theorem was initially proved for codimension one currents, i.e. $d-1$-dimensional currents in $\mathbb{R}^d$ by De Giorgi, see [18] and [19], in the language of *sets of finite perimeter* few years before the appearence of Federer and Fleming foundational article [44]. It is worth mentioning this since the ideas introduced by De Giorgi in [18] and [19] still play an important role in the general case.

Federer and Fleming's proof of closure theorem is based on the celebrated *Besicovitch-Federer structure theorem*, which is a deep rectifiability criterion for sets of finite Hausdorff measure, the proof of which is rather demanding. Several years later, two different proofs of the closure theorem were proposed by Solomon [79] and by Almgren [4], without employing the machinery of the structure theorem: their proofs relied on various facts about multivalued functions. Moreover, another different proof of the closure theorem without the structure theorem nor multivalued functions and that develops in the same spirit as De Giorgi's codimension 1 proof is due to White [92], and is based on a rectifiability criterion for measures and a key lower density lemma, *cfr.* also [46, Section 2.7].

Another important theorem in the theory of integral currents and very much related to the closure theorem is the *boundary rectifiability theorem*, telling us that

$$\mathbf{R}_k(U) \cap \mathbf{N}_k(U) = \mathbf{I}_k(U).$$

Theorem 1.1.28 (Boundary rectifiability theorem). *Let T be an integer multiplicity rectifiable current with $\mathbb{M}(\partial T) < \infty$. Then T is an integral current.*

As a corollary of Federer and Fleming closure theorem we can prove existence of a solution of the generalized oriented Plateau problem in the class of integral currents.

Theorem 1.1.29 (Generalized oriented Plateau solution). *Given $S \in \mathbf{I}_k(\mathbb{R}^d)$ with $\partial S = 0$, there exists a $k+1$-dimensional integral current T' such that $\partial T' = S$ and $\mathbb{M}(T')$ is minimized among all integral currents $T \in \mathbf{I}_{k+1}(\mathbb{R}^d)$ satisfying $\partial T = S$. We call such a solution an* area-minimizing integral current.

Proof. Let $m := \inf\{\mathbb{M}(T) \,|\, T \in \mathbf{I}_{k+1}(\mathbb{R}^d) : \partial T = S\}$ and let $(T_n)_n$ be a minimizing sequence. Note that m is finite, since it is always possible to construct the cone over S. Since $\mathbb{M}(T_n)$ is bounded and $\mathbb{M}(\partial T_n)$ is constant, we can apply Theorem 1.1.27 to the sequence $(T_n)_n$ getting a subsequence converging in the sense of currents to $T' \in \mathbf{I}_{k+1}(\mathbb{R}^d)$. By Remark 1.1.10 we get $\partial T' = S$ and $\mathbb{M}(T') \leq m$. □

If the ambient space is a compact oriented smooth manifold, it is possible to define its integral homology in terms of currents. In particular, it is possible to show that homology classes are closed with respect to convergence of currents; in fact, integral homology classes are connected components of integral cycles. As a consequence of this, it is possible to prove that also the generalized oriented *homological* Plateau problem always admits an area-minimizing representative; we postpone the formal details after presenting the deformation theorem.

1.1.2 Deformation theorem

After introducing a few other preliminary results – such as the notion of slicing, the homotopy formula, the constancy theorem and a representation formula for normal currents – we state and prove in this section one of the cornerstones in the theory of normal and integral currents: the celebrated Federer-Fleming *deformation theorem*. Then, we conclude mentioning some of its most important consequences as, above all, the isoperimetric inequality, the strong polyhedral approximation theorem and the definition of homology with currents.

Further preliminary results in the theory of currents

If M is a compact local Lipschitz neighborhood retract, we recall that the *integral flat norm* $\mathbb{F}(T)$ of $T \in \mathbf{I}_k(M)$ is defined by:

$$\mathbb{F}(T) := \min\{\mathbb{M}(R) + \mathbb{M}(S) \mid T = R + \partial S,\ R \in \mathbf{I}_k(M),\ S \in \mathbf{I}_{k+1}(M)\}.$$

Remark 1.1.30. In full generality, it is still an open problem if convergence in the integral flat topology is equivalent to the weak* convergence of integral currents; what is known and easy to show is that convergence in the integral flat topology implies weak* convergence.

Given a smooth, proper map $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ and a k-current T in $\mathbb{R}^d$, the *push-forward* of T according to the map f is the k-current $f_\sharp T$ in $\mathbb{R}^{d'}$ defined by

$$f_\sharp T(\omega) := T(f^*\omega), \quad \text{for every } \omega \in \mathcal{D}^k(\mathbb{R}^{d'}),$$

where $f^*\omega$ denotes the pullback of ω through f. If T is such that $\mathbb{M}(T), \mathbb{M}(\partial T) < \infty$ and $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ a Lipschitz map such that $f_{|\mathrm{spt}(T)}$ is proper, then the pushforward of T via f can be defined as follows. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a standard mollifier, denote $\varphi_\tau(x) := \tau^{-n}\varphi(\tau^{-1}x)$, for $\tau > 0$, and let $f_\tau := f * \varphi_\tau$ be the smoothing of f. The pushforward of T via f is defined as

$$f_\sharp T(\omega) := \lim_{\tau \to 0} f_{\tau\sharp} T(\omega), \quad \text{for every } \omega \in \mathcal{D}^k(\mathbb{R}^{d'}).$$

Definition 1.1.31. A k-dimensional *polyhedral current* (or *k-polyhedral chain*) is a k-current P of the form

$$P := \sum_{i=1}^N \theta_i [\![\sigma_i]\!],$$

where $\theta_i \in \mathbb{R} \setminus \{0\}$, σ_i are nontrivial k-dimensional simplexes in $\mathbb{R}^d$ with disjoint relative interiors and oriented by constant k-vectors τ_i such that $[\![\sigma_i]\!] = [\![\sigma_i, \tau_i, 1]\!]$ is the multiplicity-one rectifiable current naturally associated to the simplex σ_i. A polyhedral current with integer coefficients θ_i is called *integer polyhedral* and we denote the space of integer polyhedral currents as $\mathbf{P}_k(U)$.

Definition 1.1.32. For every $k = 0, \dots, m+n$, we denote by $\mathcal{Z}_{k,Lip}(\mathcal{M})$ the set of k-dimensional integer Lipschitz cycles with support in $\mathcal{M}$, that is the set of cycles of the form $f_\sharp(P)$ where $f : \mathbb{R}^d \to \mathcal{M}$ is a Lipschitz map and P is an integer polyhedral cycle in $\mathbb{R}^d$.

In general there is no natural definition for the notion of intersection of two currents; in fact, even in the intersection theory for smooth manifolds some “safety” conditions are required. However, it is possible to define the intersection of a normal k-current T and a level set $f^{-1}(y)$ of a smooth map $f : \mathbb{R}^d \to \mathbb{R}^m$ (with $k \le m \le d$) for almost every y, resulting in a normal current T_y with the expected dimension $m - k$. This operation is called *slicing* and to define it properly we need to recall two important results, see also [46, Section 2.2.5].

Theorem 1.1.33 (Sard theorem)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be of class C^k for some $k \ge \max\{d-m, 1\}$. Denote by*

$$\mathcal{S}_f := \big\{x \in \mathbb{R}^d : \mathrm{rank}(Df(x)) < m\big\}$$

the set of critical points of f. Then $\mathscr{L}^m(f(\mathcal{S}_f)) = 0$, that is the set of critical values of f is of null $\mathscr{L}^m$-measure.

Corollary 1.1.34. *Let $0 < m \le k \le d$. Let M be a smooth k-surface in $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}^m$ be smooth. Denote $M_y := M \cap f^{-1}(y)$. Then for $\mathscr{L}^m$-a.e. y, M_y is a smooth surface of dimension $k - m$ (or it is empty).*

In order to extend Corollary 1.1.34 where M is replaced by a rectifiable set E and f is Lipschitz we need to recall the following version of the coarea formula.

Theorem 1.1.35 (Coarea formula)**.** *Let $E \subset \mathbb{R}^d$ be k-rectifiable and $f : \mathbb{R}^d \to \mathbb{R}^m$ a Lipschitz function. For $y \in \mathbb{R}^m$, denote $E_y := E \cap f^{-1}(y)$. For $\mathcal{H}^k$-a.e. $x \in E$ denote $D_\tau f(x)$ the tangential gradient of f at x and denote the tangential Jacobian as*

$$J_\tau f(x) := |D_\tau f_1(x) \wedge \cdots \wedge D_\tau f_m(x)|$$

Then, for every Borel function $g : E \to [0, +\infty]$, we have

$$\int_{\mathbb{R}^m} \left(\int_{E_y} g(x)\, d\mathcal{H}^{k-m}(x) \right) d\mathscr{L}^m(y) = \int_E g(x)\, J_\tau f(x)\, d\mathcal{H}^k(x). \tag{1.2}$$

By Sard theorem and coarea formula we have the following proposition, allowing us to define the notion of slice of a rectifiable current.

Proposition 1.1.36 (Slicing of rectifiable currents)**.** *Let $T = [\![E, \tau, \theta]\!]$ be a rectifiable k-current in $\mathbb{R}^d$, with $\mathbb{M}(T) < \infty$. Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz function, with $0 < m \leq k \leq d$. Denote*

$$\widetilde{E} := \{x \in E : D_\tau f(x) \textit{ is defined and has rank } m\}.$$

For $y \in \mathbb{R}^m$, denote $E_y := E \cap f^{-1}(y)$. Then the following hold true:

1. *$\mathcal{H}^{k-m}(E_y \setminus \widetilde{E}) = 0$, for $\mathscr{L}^m$-a.e. y;*
2. *E_y is $k-m$-rectifiable for $\mathscr{L}^m$-a.e. y;*
3. *Denoting $\eta(x) := D_\tau f_1(x) \wedge \cdots \wedge D_\tau f_m(x)$, we have that*

$$T_x E = T_x E_y \oplus \mathrm{Span}\{\eta(x)\},$$

for $\mathscr{L}^m$-a.e. y and for $\mathcal{H}^{k-m}$-a.e. $x \in E_y$. Hence, for $\mathscr{L}^m$-a.e. y, we can define the orientation on E_y as the $k-m$-vector $\widetilde{\tau}$ such that

$$\frac{\eta(x)}{|\eta(x)|} \wedge \widetilde{\tau}(x) = \tau(x), \qquad \textit{for } \mathcal{H}^{k-m}\textit{-a.e. } y \in E_y.$$

Definition 1.1.37. Under the assumptions of Proposition 1.1.36, the $k-m$-rectifiable current $T_y := [\![E_y, \widetilde{\tau}, \theta \llcorner E_y]\!]$ is well-defined for $\mathscr{L}^m$-a.e. y and it is defined as the *slice* of T at y according to f. We will use the notation $\langle T, f, y\rangle$ to highlight f.

By the coarea formula with $g = |\theta|$ we get:

Corollary 1.1.38. *Under the assumptions of Proposition 1.1.36, let $T_y := [\![E_y, \widetilde{\tau}, \theta \llcorner E_y]\!]$. Then*

$$\int_{\mathbb{R}^m} \mathbb{M}(T_y) d\mathscr{L}^m(y) = \int_E |\theta(x)| J_\tau f(x) d\mathcal{H}^k(x) \leq (\mathrm{Lip}(f))^m \mathbb{M}(T).$$

Proposition 1.1.39. *Let $T \in \mathbf{N}_k(\mathbb{R}^d)$ and rectifiable. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz and C^1 function. Let T_y be as in Definition 1.1.37, then for $\mathscr{L}^1$-a.e. $y \in \mathbb{R}$ we have that*

$$T_y = \partial(T \llcorner \{f \le y\}) - \partial T \llcorner \{f \le y\}.$$

Motivated by the above characterization for codimension one slices of rectifiable and normal currents, it is possible to define the slice of a normal current.

Definition 1.1.40. Let $T \in \mathbf{N}_k(\mathbb{R}^d)$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz and C^1 function. For every $y \in \mathbb{R}$ we define the *slice of a normal current* as

$$T_y := \partial(T \llcorner \{f \le y\}) - (\partial T) \llcorner \{f \le y\}.$$

It is a bit more complicated to deal with normal currents slices of codimension $m > 1$.

Definition 1.1.41. Let $T \in \mathbf{N}_k(\mathbb{R}^d)$ and $f : \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz and C^1 function. Denote $f_1, \dots, f_m$ the components of f. For every $y \in \mathbb{R}^m$ such that $y = (y_1, \dots, y_m)$ we define recursively

$$\begin{aligned} T_{y_1} &:= \partial(T \llcorner \{f_1 \le y_1\}) - (\partial T) \llcorner \{f_1 \le y_1\}, \\ T_{y_1,y_2} &:= \partial(T_{y_1} \llcorner \{f_2 \le y_2\}) - (\partial T_{y_1}) \llcorner \{f_2 \le y_2\}, \\ &\dots \\ \langle T, f, y\rangle = T_y &:= \partial(T_{y_1,\dots,y_{m-1}} \llcorner \{f_m \le y_m\}) - (\partial T_{y_1,\dots,y_{m-1}}) \llcorner \{f_m \le y_m\}. \end{aligned}$$

Remark 1.1.42. When $m > 1$, for T_y to be well-defined we have to ensure that the slices are still normal currents after each iteration.

We finally recall the following useful property of slicing for normal currents.

Proposition 1.1.43. *Let $T \in \mathbf{N}_k(\mathbb{R}^d)$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz and C^1 function. Then, for every $y \in \mathbb{R}$ we have that*

$$\partial(T_y) = -(\partial T)_y$$

and

$$\int_{\mathbb{R}} \mathbb{M}(T_y) dy \le \mathrm{Lip}(f)\, \mathbb{M}(T).$$

We recall now the important homotopy formula for currents. Let U, V be open sets in $\mathbb{R}^d$, consider two smooth maps $f, g : U \to V$ and a smooth homotopy

$$h : [0,1] \times U \to V, \quad h(0,\cdot) = f, \quad h(1,\cdot) = g.$$

Suppose h is proper in $[0,1] \times \mathrm{spt}(T)$. Then $h_\sharp([\![(0,1)]\!] \times T)$ is well defined and we have

$$\begin{aligned} \partial h_\sharp([\![(0,1)]\!] \times T) &= h_\sharp \partial([\![(0,1)]\!] \times T) \\ &= h_\sharp(\delta_1 \times T - \delta_0 \times T - [\![(0,1)]\!] \times \partial T) \\ &= g_\sharp T - f_\sharp T - h_\sharp([\![(0,1)]\!] \times \partial T). \end{aligned}$$

Hence, we have the following formula, *cfr.* [46, Proposition 4, Section 2.2.3].

Proposition 1.1.44 (Homotopy formula). *Let U, V be open sets in $\mathbb{R}^d$, $f, g : U \to V$ be smooth maps, $h : [0,1] \times U \to V$ be a smooth homotopy between f and g and $T \in \mathcal{D}_k(U)$. Assume finally that $h^{-1}(K) \cap \operatorname{spt}(T)$ is compact in $[0,1] \times U$ for any compact set $K \subset V$. Then we have that*

$$g_\sharp T - f_\sharp T = \partial h_\sharp([\![(0,1)]\!] \times T) + h_\sharp([\![(0,1)]\!] \times \partial T),$$

where $h_\sharp([\![(0,1)]\!] \times \partial T) := 0$ if $k = 0$.

We recall the following characterization of d-currents in $U \subset \mathbb{R}^d$ without boundary in an open subset $V \subset U$, *cfr.* [46, Theorem 1, Section 4.3.1].

Theorem 1.1.45 (Constancy theorem). *Let $T \in \mathcal{D}_d(U)$ be a d-dimensional current in an open set U of $\mathbb{R}^d$, and let V be an open set in U. Suppose that*

$$\partial T \llcorner V = 0$$

or, equivalently, $\operatorname{spt}(\partial T) \subset \overline{U} \backslash V$. Then, for each connected component V_i of V there exists a constant $c_i \in \mathbb{R}$ such that

$$T = \sum_i c_i [\![V_i]\!] \quad \text{in } V.$$

In other words, for any $f \in C^\infty_c(V)$ smooth function with compact support in V we have

$$T(f(x)dx) = \int f(x)c(x)dx,$$

where $c(x)$ is piecewise constant, that is $c(x) = c_i$ on V_i.

In fact, it is possible to prove the following more general result, which is a representation formula for normal currents, *cfr.* [46, Theorem 2, Section 4.3.1].

Theorem 1.1.46. *Let $T \in \mathcal{D}_d(U)$ be a d-dimensional current in a bounded open set $U \subset \mathbb{R}^d$. Suppose that $\mathbb{M}(\partial T) < \infty$, then there exists a function $u \in BV_{loc}(U)$ such that*

$$T(f(x)dx) = \int_U f(x)u(x)dx \quad \text{for all } f \in C^\infty_c(U).$$

Moreover, $\|\partial T\|(V) = |Du|(V)$ for any open set $V \subset U$.

An immediate corollary of Theorem 1.1.46 is the following characterization of normal d-dimensional currents in $U \subset \mathbb{R}^d$.

Corollary 1.1.47 (Representation of top dimensional normal currents). *A d-dimensional current $T \in \mathcal{D}_d(U)$ belongs to $\mathbf{N}_d(U)$ if and only if it is representable as*

$$T(f(x)dx) = \int f(x)u(x)dx, \quad \text{for some } u \in BV(U).$$

In this case we also have that

$$\mathbb{M}(T) = \int_U |u(x)|dx, \quad \mathbb{M}(\partial T) = \int_U |Du|.$$

In particular, we can improve Theorem 1.1.45 for integer rectifiable currents.

Theorem 1.1.48 (Constancy theorem for integral currents)**.** *Let T be an integer rectifiable d-current in an open set U of $\mathbb{R}^d$. If U is connected and $\partial T \llcorner U = 0$, then*

$$T = c\,[\![U]\!], \quad \textit{with } c \in \mathbb{Z}.$$

The Federer-Fleming deformation theorem

We now prove the celebrated Federer-Fleming deformation theorem, and see its relevance in studying the homology of manifolds. In particular, we also prove the isoperimetric inequality and the strong approximation theorem; we will mostly follow [46, Chapter 5] and [44].

First we fix some notation. We decompose $\mathbb{R}^{m+n}$ into cubes with edges parallel to the axes and of unitary length. More precisely we denote by $\mathcal{L}^{m+n}$ the family of cubes

$$\mathcal{L}^{m+n} := \left\{ z \in [0,1]^{m+n} \mid z \in \mathbb{Z}^{m+n} \right\}$$

so that

$$\mathbb{R}^{m+n} = \bigcup \left\{ F \mid F \in \mathcal{L}^{m+n} \right\},$$

and for any j, $0 \leq j < m+n$, we denote by $\mathcal{L}^j$ the collection of all j-faces of the cubes in $\mathcal{L}^{m+n}$

$$\mathcal{L}^j := \left\{ F \mid F \text{ is a } j\text{-face of } Q \in \mathcal{L}^{m+n} \right\}.$$

We instead denote by L_j the j-skeleton of the subdivision

$$L^j := \bigcup \left\{ F \mid F \in \mathcal{L}^j \right\}$$

and, as we also need the translate of L^j, we also set

$$L^j(a) := a + L^j, \quad a \in \mathbb{R}^{m+n}.$$

Finally, for any $\alpha \in I(j, m+n)$ we denote by $\mathbb{R}_\alpha$ the coordinate j-plane generated by $e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_\alpha}$, and let $\pi_\alpha : \mathbb{R}^{m+n} \to \mathbb{R}_\alpha$ be the orthogonal projection onto $\mathbb{R}_\alpha$.

Lemma 1.1.49. *We have*

(*i*) *For* $0 \leq j \leq m+n$

$$L^j(a) = \bigcup_{\alpha \in I(j,m+n)} \bigcup_{z \in \mathbb{Z}^{m+n}} \left(z + \pi_{\bar{\alpha}}^{-1}(a) \right)$$

(*ii*) $L^m(a) \cap L^{n-1} = \emptyset$ *if* $a \in (0,1)^{m+n}$.

Proof. The claim (*i*) is trivial since each j-face $F \in L^j(a)$ is parallel to an α plane $|\alpha| = j$, hence $F \subset \pi_{\bar{\alpha}}^{-1}\pi_{\bar{\alpha}}(a)$. To prove (*ii*) we assume by contradiction that $x \in L^m(a) \cap L^{n-1}$. As $x \in L^{n-1}(a)$, at least $m+n-(n-1) = m+1$ coordinates of x are integers, hence at least $m+1$ coordinates of $x-a$ are not integer. As $x \in L^m(a)$, the same argument yields that at least $m+n-m = n$ coordinates of $x-a$ are integers: a contradiction. $\square$

The geometric idea in the proof the deformation theorem is that of projecting or retracting m-currents into the m-skeleton L^m of the standard subdivision, and doing that by a suitably chosen center of projection. Next lemma contains the construction of a suitable class of retraction maps.

Let $q := (1/2, \dots, 1/2)$ denote the center of the cube $[0,1]^{m+n}$. By reasoning similarly to Lemma 1 *(ii)* we see that

$$\operatorname{dist}\left(L^{m-1}(a), L^n\right) \geq \frac{1}{4} \quad \text{if } a \in B(q, 1/4).$$

For any ρ, $0 < \rho < 1/4$ we denote by

$$L^j(a,\rho) := \left\{x \in \mathbb{R}^{m+n} \mid \operatorname{dist}\left(x, L^j(a)\right) < \rho\right\}$$

the tubular neighbourhood of L^j of radius ρ.

Lemma 1.1.50. *For any $a \in B(q,1/4)$ there is a locally Lipschitz map $\psi : \mathbb{R}^{m+n}\backslash L^{n-1}(a) \to \mathbb{R}^{m+n}\backslash L^{n-1}(a)$ such that*

(i) $\psi\left([0,1]^{m+n}\backslash L^{n-1}(a)\right) = [0,1]^{m+n} \cap L^m$;

(ii) $\psi\left([0,1]^{m+n} \cap L^m\right) = \mathrm{id}_{[0,1]^{m+n}\cap L^m}$;

(iii) $\psi(z+x) = \psi(x) + z$ *for all* $x \in \mathbb{R}^{m+n}$, $z \in \mathbb{Z}^{m+n}$;

(iv) $|D\psi(x)| \leq c/\rho$ *for a.e.* $x \in \mathbb{R}^{m+n}\backslash L^{n-1}(a,\rho)$, $0 < \rho < 1/4$.

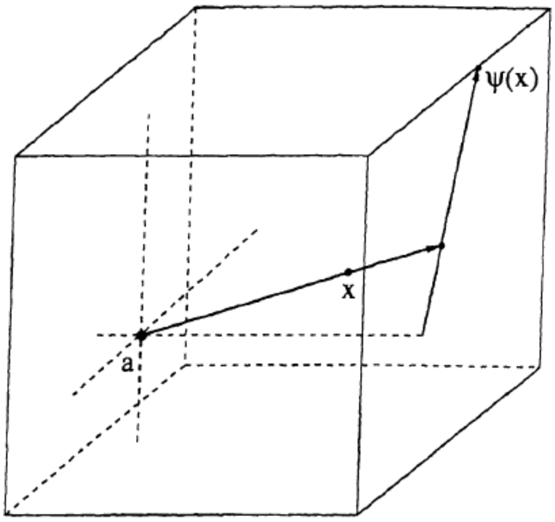

Figure 1.1: The map ψ.

Proof. We split the proof into three steps.

Step 1. Let F be a j-face in L^j, $j \geq m+1$. Denote by a_F the orthogonal projection of a into F and let $\psi_F : F\backslash\{a_F\} \to \partial F$ be the retraction map which takes $x \in F\backslash\{a_F\}$ to the point $y \in \partial F$ such that for some $\lambda \in (0,1]$

$$x = (1-\lambda)a_F + \lambda y. \tag{1.3}$$

Of course $\psi_{F|\partial F} = \mathrm{id}_{|\partial F}$. Also for the segment $\overline{aa_F}$ we have

$$\overline{aa_F} \subset \bigcap_{\alpha \in I(j,m+n)} \pi_{\bar{\alpha}}^{-1}\pi_{\bar{\alpha}}(a) \subset \bigcap_{\alpha \in I(m+1,m+n)} \pi_{\bar{\alpha}}^{-1}\pi_{\bar{\alpha}}(a), \tag{1.4}$$

since a_F is the orthogonal projection of a into F. Therefore, Lemma 1.1.49 (i) yields that $\overline{aa_F}$ lies in $L^{n-1}(a)$. This shows that, by (1.3), ψ_F maps $F\backslash L^{n-1}(a)$ into $\partial F\backslash L^{n-1}(a)$,

$$\psi_F : F\backslash L^{n-1}(a) \longrightarrow F\backslash L^{n-1}(a)$$

is onto, ψ_F is the identity on ∂F and it is locally Lipschitz.

Step 2. Gluing the ψ_F for each j-face in $[0,1]^{m+n}$, we define a map

$$\psi^{(j)} : [0,1]^{m+n} \cap L^j\backslash L^{n-1}(a) \longrightarrow [0,1]^{m+n} \cap L^{j-1}\backslash L^{n-1}(a),$$

$\psi^{(j)}(x) = \psi_F(x)$ if $x \in F$, which is onto and locally Lipschitz. The composition map

$$\psi_0 := \psi^{(m+1)} \circ \psi^{(m+2)} \circ \cdots \circ \psi^{(m+n)}$$

is well-defined, maps $[0,1]^{m+n}\backslash L^{n-1}(a)$ onto $[0,1]^{m+n} \cap L^m\backslash L^{n-1}(a)$ which is equal to $[0,1]^{m+n} \cap L^m$ by Lemma 1.1.49 (ii), and is the identity on $[0,1]^{m+n} \cap L^m$. Furthermore, if $x, x+z \in [0,1]^{m+n}$, $z \in \mathbb{Z}$, then

$$\psi_0(x+z) = z + \psi_0(x). \tag{1.5}$$

In fact if $x, x+z \in [0,1]^{m+n}$, either x and $x+z$ belong to L^m where ψ_0 is the identity, or x and $x+z$ belong to the interior of two j-faces $F_1, F_2, j \geq m+1$ which are parallel and $F_2 = z + F_1$. Hence $a_{F_2} = z + a_{F_1}$ and (1.5) follows from (1.3). By (1.4) we can extend by periodicity the map ψ_0 to a map

$$\psi : \mathbb{R}^{m+n}\backslash L^{n-1}(a) \longrightarrow \mathbb{R}^{m+n}\backslash L^{n-1}(a)$$

which satisfies the claims $(i), (ii), (iii)$.

Step 3. Finally, let us prove (iv); in fact, we will prove slightly more: for all $x \in [0,1]^m\backslash L^{n-1}(a)$ we have

$$|\bar{D}\psi(x)| \leq \frac{c}{\operatorname{dist}(x, L^{n-1}(a))} \tag{1.6}$$

where $c = c(m,n)$ and

$$|\bar{D}\psi(x)| := \limsup_{x' \to x} \frac{|\psi(x') - \psi(x)|}{|x'-x|}.$$

By induction on n, if $n = 1$, (1.6) is trivial since $y = \psi(x)$ is given by

$$y = a + \lambda(x-a), \qquad \text{where } \lambda = \frac{|y-a|}{|x-a|}.$$

Assume (1.6) holds in case $n-1$ replaces n and let $\widetilde{\psi} = \psi^{(m+1)} \circ \cdots \circ \psi^{(m+n-1)}$ so that $\psi = \widetilde{\psi} \circ \psi^{(m+n)}$. If $x \in [0,1]^m \backslash L^{n-1}(a)$ and $y = \psi^{m+n}(x)$, as in the case $n = 1$, we immediately infer

$$\left|D\psi^{(m+n)}(x)\right| \leq \sqrt{m+n}\,\frac{|y-a|}{|x-a|}. \tag{1.7}$$

If $y \in L^m(a)$ then $\widetilde{\psi}(y) = y$ and (1.6) is trivial. Assume therefore that this is not the case, and let F be the $m+n-1$-face of $\mathcal{L}^{m+n-1}$ such that $y \in F$. We can assume that $F \subset \mathbb{R}^{m+n-1} \times \{0\}$. Let π_F be the orthogonal projection of $\mathbb{R}^{m+n}$ into $\mathbb{R}^{m+n-1} \times \{0\}$ and let $a_F \in \operatorname{int} F$ be the projection of a on F. It is easily seen that

$$\widetilde{L}^{n-2}(a_F) = L^{n-1}(a) \cap \left(\mathbb{R}^{m+n-1} \times \{0\}\right)$$

is the singular set of $\widetilde{\psi}_{|\mathbb{R}^{m+n-1}\times\{0\}}$, and therefore by the inductive assumption

$$|\bar{D}\widetilde{\psi}(y)| \leq \frac{c}{\operatorname{dist}(y, \widetilde{L}^{n-2}(a_F))}. \tag{1.8}$$

By similarity

$$\frac{|y-a|}{|x-a|} = \frac{\operatorname{dist}(y, \widetilde{L}^{n-2}(a_F))}{\operatorname{dist}(x, \pi_F^{-1}\widetilde{L}^{n-2}(a_F))},$$

while

$$\pi_F^{-1}\widetilde{L}^{n-2}(a_F) \subset L^{n-1}(a),$$

hence

$$\operatorname{dist}(x, \pi_F^{-1}\widetilde{L}^{n-2}(a_F)) \geq \operatorname{dist}\left(x, L^{n-1}(a)\right).$$

We then infer that

$$\operatorname{dist}(y, \widetilde{L}^{n-2}(a_F)) \geq \frac{|y-a|}{|x-a|}\operatorname{dist}\left(x, L^{N-1}(a)\right)$$

and

$$|\bar{D}\widetilde{\psi}(y)| \leq c\,\frac{|y-a|}{|x-a|} = \frac{1}{\operatorname{dist}\left(x, L^{n-1}(a)\right)}, \tag{1.9}$$

on account of (1.8).

Finally, (1.9) and (1.7) yield the conclusion. □

Theorem 1.1.51 (Deformation theorem, unscaled version). *Let T be a normal current, $T \in \mathbf{N}_m(\mathbb{R}^{m+n})$. Then* (*i*) *We can decompose T as*

$$T = P + \partial R + S$$

where P is a polyhedral chain made of faces of the m-skeleton L^m of the classical subdivision of $\mathbb{R}^{m+n}$

$$P = \sum_i \beta_i [\![F_i]\!], \quad \beta_i \in \mathbb{R}.$$

Moreover,

$$\begin{aligned} &\operatorname{spt} P \subset L^m, \quad \operatorname{spt} \partial P \subset L^{m-1}, \\ &\operatorname{spt} R \subset \cup \left\{ \bar{Q} \mid Q \in \mathcal{L}^{m+n}, \bar{Q} \cap \operatorname{spt} T \neq \emptyset \right\}, \\ &\operatorname{spt} S \subset \cup \left\{ \bar{Q} \mid Q \in \mathcal{L}^{m+n}, \bar{Q} \cap \operatorname{spt} \partial T \neq \emptyset \right\} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{M}(P) \leq c\,\mathbb{M}(T), \quad \mathbb{M}(\partial P) \leq c\,\mathbb{M}(\partial T), \quad \mathbb{M}(R) \leq c\,\mathbb{M}(T), \\ &\mathbb{M}(S) \leq c\,\mathbb{M}(\partial T), \end{aligned}$$

where $c = c(m, n)$. Moreover,

(*ii*) *If T is integer rectifiable, we can choose P, R integer rectifiable and $\beta \in \mathbb{Z}$, and, if also ∂T is integer rectifiable, we can choose S integer rectifiable too.*

(*iii*) *If T is a Lipschitz chain, then P, S, R are Lipschitz chains.*

(*iv*) *If ∂T is a Lipschitz chain, then S is a Lipschitz chain.*

(*v*) *If ∂T is supported in L^{m-1}, then $S = 0$.*

Proof. We would like to project the current T into the m-skeleton L^m according to Lemma 1.1.50 but, in order to do that efficiently, we must choose a in such a way that not much of the mass of T is concentrated close to $L^{n-1}(a)$.

Step 1. Let T be a normal current. We claim that there is a constant $c_1 = c(m, n)$ and a point $a \in B(q, 1/4)$ such that for any ρ, $0 < \rho < 1/4$

$$\begin{aligned} &\mathbb{M}\left(T \llcorner L^{n-1}(a, \rho)\right) \leq c_1 \rho^{m+1} \mathbb{M}(T) \\ &\mathbb{M}\left(\partial T \llcorner L^{n-1}(a, \rho)\right) \leq c_1 \rho^{m+1} \mathbb{M}(\partial T). \end{aligned} \tag{1.10}$$

For any $\alpha \in I(m+1, m+n)$, we consider the $m+1$-face of $[0,1]^{m+n}$, $F_\alpha := \mathbb{R}_\alpha \cap [0,1]^{m+n}$ and let q_α be its center. In $F_\alpha \cap B(q_\alpha, 1/4)$ we define a *good set* G_α by $g \in G_\alpha$ if and only if

$$\mathbb{M}\left(T \llcorner \bigcup_{z \in \mathbb{Z}^{m+n} \cap \mathbb{R}_\alpha} \pi_\alpha^{-1}(B(g+z, \rho))\right) \leq \beta \rho^{m+1} \mathbb{M}(T), \quad \text{for all } \rho \in (0, 1/4), \tag{1.11}$$

and β a constant to be chosen later.

We now claim that the *bad set* $B_\alpha := F_\alpha \cap [0,1]^{m+n} \backslash G_\alpha$ is small; more precisely, that

$$\mathcal{H}^{m+1}(B_\alpha) \leq c_2 \frac{1}{\beta},$$

which clearly is small if β is large. Indeed, for any $b \in B_\alpha$ there is a $\rho_b \in (0,1/4)$ such that

$$\mathbb{M}\left(T \llcorner \bigcup_{z\in\mathbb{Z}^{m+n}\cap\mathbb{R}_\alpha} \pi_\alpha^{-1}\left(B\left(g+z,\rho_b\right)\right)\right) \geq \beta\rho_b^{m+1}\mathbb{M}(T).$$

By Besicovitch covering theorem there is a pairwise disjoint and countable subcollection such that

$$B_\alpha \subset \bigcup_k B\left(b_k, 5\rho_k\right),$$

but then, setting $b = b_k, \rho_b = \rho_k$ in (1.11), we get $\beta\sum_k \rho_k^{m+1}\mathbb{M}(T) \leq \mathbb{M}(T)$, *i.e.*

$$\sum_k \rho_k^{m+1} \leq \frac{1}{\beta},$$

and hence

$$\mathcal{H}^{m+1}\left(B_\alpha\right) \leq \frac{c_4(m)}{\beta} \quad \text{and} \quad \mathcal{H}^{m+n}\left(\pi_\alpha^{-1}\left(B_\alpha\right)\cap B(q,1/4)\right) \leq \frac{c_5(m,n)}{\beta}.$$

Finally, summing on $\alpha \in I(m+1,m+n)$, we get

$$\mathcal{H}^{m+n}\left(\bigcup_\alpha \pi_\alpha^{-1}\left(B_\alpha\right)\cap B(q,1/4)\right) \leq \frac{c_1(m,n)}{\beta},$$

and hence $\cap_\alpha\pi_\alpha^{-1}\left(G_\alpha\right)\cap B(q,1/4)$ is most of $B(q,1/4)$ for β large. Repeating the same argument for ∂T, and denoting by G'_α, B'_α the good and bad sets for ∂T we also find that for β sufficiently large the set $\cap_\alpha\pi_\alpha^{-1}\left(G'_\alpha\right)\cap B(q,1/4)$ is most of $B(q,1/4)$. Therefore, we can find a in the good set

$$a \in \bigcap_\alpha \pi_\alpha^{-1}\left(G_\alpha\right) \cap \bigcap \pi_\alpha^{-1}\left(G'_\alpha\right)\cap B(q,1/4)$$

for which (1.11) holds. Since

$$L^{n-1}(a,\rho) = \bigcup_\alpha \bigcup_{z\in\mathbb{Z}^{m+n}\cap\mathbb{R}_\alpha} \pi_\alpha^{-1}\left(B\left(a_\alpha+z,\rho\right)\right), \quad \text{with } a_\alpha = \pi_\alpha(a),$$

the claim is proved.

Step 2. We now project T into the m-skeleton L^m by means of the projection map in Lemma 1.1.50 associated to a point $a \in B(q,1/4)$ for which (1.10) of Step 1 holds. For $0<\rho<1/4$ we set

$$T_\rho := T \llcorner L^{n-1}(a,\rho), \quad (\partial T)_\rho := \partial T \llcorner L^{n-1}(a,\rho)$$

so that by (1.10)

$$\mathbb{M}\left(T_\rho\right) \leq c\rho^{m+1}\mathbb{M}(T), \quad \mathbb{M}\left((\partial T)_\rho\right) \leq c\rho^{m+1}\mathbb{M}(\partial T).$$

Then, by (iv) in Lemma 1.1.50 we have

$$\begin{aligned} \mathbb{M}\left(\psi_\sharp(T_\rho - T_{\rho/2})\right) &\le \frac{c}{\rho^m}\rho^{m+1}\mathbb{M}(T) \le c\rho\mathbb{M}(T), \\ \mathbb{M}\left(\psi_\sharp(\partial T)_\rho - \psi_\sharp(\partial T)_{\rho/2})\right) &\le \frac{\rho}{\rho^{m-1}}\rho^{m+1}\mathbb{M}(T) \le c\rho\mathbb{M}(\partial T). \end{aligned} \tag{1.12}$$

Similarly, if $h(t,x) := (1-t)x + t\psi(x)$, by the homotopy formula we get

$$\begin{aligned} &\mathbb{M}\left(h_\sharp([\![(0,1)]\!] \times \left(T_\rho - T_{\rho/2}\right)\right) \le c\rho\mathbb{M}(T)\ , \\ &\mathbb{M}\left(h_\sharp([\![(0,1)]\!] \times \left((\partial T)_\rho - (\partial T)_{\rho/2}\right)\right) \le c\rho\mathbb{M}(\partial T)\ . \end{aligned} \tag{1.13}$$

Finally, by slicing we find $\rho^* \in (\rho/2, \rho)$ such that

$$\begin{aligned} &\mathbb{M}\left(\psi_\sharp\langle T, d, \rho^*\rangle\right) \le \frac{c}{\rho^{n-1}}\rho^{n+1}\mathbb{M}(T_\rho - T_{\rho/2}) \le c\rho\mathbb{M}(T), \\ &\mathbb{M}\left(h_\sharp([\![(0,1)]\!] \times \langle T, d, \rho^*\rangle)\right) \le c\rho\mathbb{M}(\partial T). \end{aligned} \tag{1.14}$$

Notice that if T is integer rectifiable we can choose ρ^* so that also $\langle T, d, \rho^*\rangle$ is integer rectifiable.

Now we select $\rho_\nu = 2^{-\nu}\rho$ and $\rho_\nu^* \in (2^{-\nu-1}\rho, 2^{-\nu}\rho)$ in such a way that (1.14) holds, and we let $T_\nu := T_{\rho_\nu^*}$. From (1.12), (1.13) and (1.14) we easily get that the sequence of currents

$$\begin{array}{ll} \psi_\sharp(T - T_\nu), & h_\sharp([\![(0,1)]\!] \times (T - T_\nu)), \\ \psi_\sharp(\partial T - (\partial T)_\nu), & h_\sharp([\![(0,1)]\!] \times \partial(T - T_\nu)) \end{array}$$

are Cauchy sequences with respect to the mass convergence, and moreover

$$\mathbb{M}(\langle T, d, \rho_\nu^*\rangle) + \mathbb{M}(\psi_\sharp\langle T, d, \rho_\nu^*\rangle) \to 0.$$

Therefore there are currents $P_1, S_1 \in \mathcal{D}_m(\mathbb{R}^{m+n})$ and $R_1 \in \mathcal{D}_{m+1}(\mathbb{R}^{m+n})$ such that

$$\begin{aligned} &\mathbb{M}(P_1 - \psi_\sharp(T - T_\nu)) \to 0, \\ &\mathbb{M}(R_1 - h_\sharp([\![(0,1)]\!] \times (T - T_\nu))) \to 0, \\ &\mathbb{M}(S_1 - h_\sharp([\![(0,1)]\!] \times \partial(T - T_\nu))) \to 0; \end{aligned} \tag{1.15}$$

in particular

$$\mathbb{M}(P_1) \le c\,\mathbb{M}(T), \quad \mathbb{M}(R_1) \le c\,\mathbb{M}(T), \quad \mathbb{M}(S_1) \le c\,\mathbb{M}(\partial T), \tag{1.16}$$

and by semicontinuity

$$\mathbb{M}(\partial P_1) \le c\,\mathbb{M}(\partial T). \tag{1.17}$$

Furthermore, by the homotopy formula we obtain

$$T - T_\nu - \psi_\sharp(T - T_\nu) = \partial h_\sharp([\![(0,1)]\!] \times (T - T_\nu)) - h_\sharp([\![(0,1)]\!] \times \partial(T - T_\nu))$$

and, as $\partial T_\nu = (\partial T)_\nu - \langle T, d, \rho_\nu^*\rangle$, we infer that

$$T - P_1 = \partial R_1 + S_1. \tag{1.18}$$

Observe that we also have that

$$\begin{aligned}&\text{if } T \text{ is integer rectifiable, then } P_1, R_1 \text{ are integer rectifiable;}\\&\text{if } T \text{ is integer rectifiable, then } S_1 \text{ is integer rectifiable;}\\&\text{if } T \text{ is a Lipschitz chain, then } P_1, R_1 \text{ are locally Lipschitz chains;}\\&\text{if } \partial T \text{ is a Lipschitz chain, then } S_1 \text{ is a Lipschitz chain.}\end{aligned} \tag{1.19}$$

Finally, since ψ retracts $\mathbb{R}^{m+n}\backslash L^{n-1}(a)$ onto L^m (*cfr.* Lemma 1.2.7) and ψ is the identity on L^m we have that

$$\begin{aligned}&\operatorname{spt} P_1 \subset L^m,\\&\operatorname{spt} R_1 \subset \cup\left\{\overline{Q} \mid Q \in \mathcal{L}^{m+n},\, \overline{Q} \cap \operatorname{spt} T \neq \emptyset\right\},\\&\operatorname{spt} S_1 \subset \cup\left\{\overline{Q} \mid Q \in \mathcal{L}^{m+n},\, \overline{Q} \cap \operatorname{spt} \partial T \neq \emptyset\right\}\end{aligned} \tag{1.20}$$

and

$$S_1 = 0, \quad \text{if } \operatorname{spt} \partial T \subset L^m. \tag{1.21}$$

In fact from $L^m \cap L^{n-1}(a, \rho_\nu) = \emptyset$ we infer $(\partial T)_\nu = 0$, and hence

$$\partial(T - T_\nu) = \partial T - \langle T, d, \rho_\nu^* \rangle \rightharpoonup^* \partial T.$$

Consequently,

$$R_1 = \lim_{\nu \to \infty} h_\sharp([\![(0,1)]\!] \times \partial(T - T_\nu)) = h_\sharp([\![(0,1)]\!] \times \partial T) = 0,$$

being ψ the identity on L^m.

Step 3. Assume now that spt $\partial T \subset L^{m-1}$ (for instance, when T is a cycle). In this case (1.19) and (1.19) yield

$$T = P_1 + \partial R_1, \qquad \text{spt } P_1 \subset L^m,$$

and hence $\operatorname{spt}(\partial T - \partial R_1) = \emptyset$. We can then apply the constancy theorem and conclude that P_1 is a polyhedral chain

$$P_1 = \sum_i \beta_i [\![F_i]\!], \quad \beta_i \in \mathbb{R},$$

and easily conclude the proof of the theorem.

In the general case we know that $P_1 \in \mathbf{N}_m(\mathbb{R}^{m+n})$ and is supported in the m-skeleton L^m. By the representation formula for normal currents in Corollary 1.1.47, we can represent P_1 over any face $F \in \mathcal{L}^m$ as

$$P_1 \llcorner F^\circ(\omega) = \int \langle \omega(x), \vec{F}(x) \rangle \theta_F(x) d\mathcal{H}^m(x),$$

where $\vec{F}(x)$ orients F, F° denotes the interior of F and $\theta_F \in BV_{\text{loc}}(\mathbb{R}^m)$. Moreover

$$\mathbb{M}(P_1 \llcorner F^\circ) = \int_{F^\circ} |\theta_F| d\mathcal{H}^m, \quad \mathbb{M}(\partial(P_1 \llcorner F^\circ)) = \int_{F^\circ} |D\theta_F|.$$

Furthermore, if $\beta \in \mathbb{R}$,

$$\begin{aligned}\mathbb{M}(P_1 \llcorner F^\circ - \beta[\![F]\!]) &= \int_{F^\circ} |\theta_F - \beta| d\mathcal{H}^m,\\ \mathbb{M}(\partial(P_1 \llcorner F^\circ - \beta[\![F]\!])) &= \int_{\mathbb{R}^m} |D(\chi_F(\theta_F - \beta))|.\end{aligned}$$

We now select β_F so that $\min\{\mathscr{L}^m(\{\theta_F \geq \beta\}), \mathscr{L}^m(\{\theta_F < 1/2\})\} \geq 1/2\,\mathscr{L}^m(F)$ (and notice that θ_F can be chosen integer if θ_F is integer-valued, *i.e.*, if P_1 is integer rectifiable) and get

$$\begin{aligned}|\beta_F| &\leq 2\int |\theta_F| d\mathcal{H}^m \llcorner F^\circ,\\ \int |\theta_F - \beta_F| d\mathcal{H}^m \llcorner F^\circ \ &\leq\ c\,|D\theta_F| \llcorner F^\circ.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{M}(P_1 \llcorner F^\circ - \beta_F[\![F]\!]) &\leq c\,\mathbb{M}(\partial(P_1 \llcorner F^\circ)),\\ \mathbb{M}(\partial(P_1 \llcorner F^\circ - \beta_F[\![F]\!])) &\leq c\,\mathbb{M}(\partial(P_1 \llcorner F^\circ))\end{aligned}$$

and $P_1 \llcorner \partial F = 0$, since P_1 is normal. Summing over the faces $F \in L^m$ we set

$$P := \sum_{F \in \mathcal{L}^m} \beta_F[\![F]\!].$$

Hence, we have

$$\begin{aligned}&\mathbb{M}(P - P_1) \leq c\,\mathbb{M}(\partial P_1), \quad \mathbb{M}(\partial(P - P_1)) \leq c\,\mathbb{M}(\partial P_1),\\ &\mathbb{M}(P) \leq \sum_\beta |\beta_F| \leq 2\,\mathbb{M}(P_1)\end{aligned}$$

and therefore, if $S = S_1 - P_1 - P$, we obtain

$$\begin{aligned}&\mathbb{M}(S) \leq \mathbb{M}(S_1) + \mathbb{M}(P_1 - P) \leq c\,\mathbb{M}(\partial T),\\ &\mathbb{M}(\partial P) \leq c\,\mathbb{M}(\partial P_1) \leq c\,\mathbb{M}(T)\end{aligned}$$

and, for $T = P + \partial R_1 + S$, one easily checks that all claims in the theorem hold true. □

The rescaled version of the deformation theorem follows now easily by first changing scale $x \mapsto \varepsilon^{-1}x$, then applying Theorem 1.1.51, and then changing scale back by $x \mapsto \varepsilon x$.

Theorem 1.1.52 (Deformation theorem, rescaled version)**.** *Let T be a normal current, $T \in \mathbf{N}_m(\mathbb{R}^{m+n})$. Then we can decompose T as*

$$T = P + \partial R + S$$

where P is a polyhedral chain made of faces of the m-skeleton $L^{m,\varepsilon}$ of the subdivision of $\mathbb{R}^{m+n}$ in cubes of side ε

$$P = \sum_i \beta_i[\![F_i]\!], \quad \beta_i \in \mathbb{R}.$$

Moreover

$$\begin{aligned}&\operatorname{spt} P \subset L^{m,\varepsilon}, \quad \operatorname{spt} \partial P \subset L^{m-1,\varepsilon},\\ &\operatorname{spt} R \subset \cup \left\{ \overline{Q} \mid Q \in \mathcal{L}^{m+n,\varepsilon}, \, \overline{Q} \cap \operatorname{spt} T \neq \emptyset \right\},\\ &\operatorname{spt} S \subset \cup \left\{ \overline{Q} \mid Q \in \mathcal{L}^{m+n,\varepsilon}, \, \bar{Q} \cap \operatorname{spt} \partial T \neq \emptyset \right\}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{M}(P) \le c\,\mathbb{M}(T), \quad \mathbb{M}(\partial P) \le c\,\mathbb{M}(\partial T), \quad \mathbb{M}(R) \le c\,\varepsilon\,\mathbb{M}(T),\\ &\mathbb{M}(S) \le c\,\varepsilon\,\mathbb{M}(\partial T),\end{aligned}$$

where $c = c(m,n)$. *Moreover,*

- (i) *if* T *is integer rectifiable, then* P, R *can be chosen integer rectifiable and* $\beta \in \mathbb{Z}$*; if also* ∂T *is integer rectifiable, then we can choose* S *to be integer rectifiable;*
- (ii) *if* T *is a Lipschitz chain, then* P, S, R *are Lipschitz chains;*
- (iii) *if* ∂T *is a Lipschitz chain, then* S *is a Lipschitz chain;*
- (iv) *if* ∂T *is supported in* $L^{m-1,\varepsilon}$*, then* $S = 0$.

Summarizing, such a theorem is obtained by deforming T into $\mathcal{L}^{m,\varepsilon}$, so that the result is automatically a polyhedral chain. The main error term is ∂R, where R is the surface through which T is deformed; S is a secondary error term due to moving ∂T into the skeleton $L^{m,\varepsilon}$. It is not difficult to see that there is no real problem to perform such a procedure by projecting from points inside the $m+1$-cubes defined by $\mathcal{L}^{m,\varepsilon}$, if T does not wind about the center projection and has no autointersections. In the general case, the key point is to suitably choose the center of projection, by observing that in the average the distortion is still controlled.

We conclude by stating and proving the Federer-Fleming polyhedral strong approximation theorem.

Theorem 1.1.53. *Let* $T \in \mathbf{R}_m(\mathbb{R}^{m+n})$ *be an integral current in* $\mathbb{R}^{m+n}$ *with compact support. Let* $\varepsilon > 0$*, then there exist an integer polyhedral chain* P *in* $\mathbb{R}^{m+n}$ *and a* C^1*-diffeomorphism of* $\mathbb{R}^{m+n}$ *into* $\mathbb{R}^{m+n}$ *such that*

$$\mathbb{M}(P - f_\sharp T) + \mathbb{M}(\partial P - \partial f_\sharp T) \le \varepsilon. \tag{1.22}$$

Moreover

$$\begin{aligned}&\operatorname{spt} P \subset \{x \mid \operatorname{dist}(x, \operatorname{spt} T) \le \varepsilon\},\\ &\operatorname{Lip} f \le 1+\varepsilon, \quad \operatorname{Lip} f^{-1} \le 1+\varepsilon,\\ &|f(x) - x| \le \varepsilon \quad \text{for all } x \in \mathbb{R}^{m+n},\\ &f(x) = x \quad \text{for } x \in \{x \mid \operatorname{dist}(x, \operatorname{spt} T) \ge \varepsilon\}.\end{aligned} \tag{1.23}$$

Note that the inequality (1.22) means that P and $f_\sharp T$, as well as ∂P and $f_\sharp \partial T$, coincide up to a piece of small mass.

Let us start first recalling the following Proposition, *cfr.* [43, 3.1.23] for a proof.

Proposition 1.1.54. *Let N be an m-dimensional submanifold of class $k \geq 1$ of $\mathbb{R}^{m+n}$, $z \in N$, $0 < t < 1$. Denote by π the m-plane through z which is tangent to N. Then for every sufficiently small positive number r there exists a diffeomorphism $f : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ of class k such that the Lipschitz constants of f and f^{-1} are less than $1/t$, f is the identity outside $B(z, r)$, and $B(z, tr) \cap N = B(z, tr) \cap f^{-1}(\pi)$.*

Proof of Theorem 1.1.53. First we also assume that ∂T is polyhedral. We know that $T = [\![E, \tau_E, \theta]\!]$, where E is n-rectifiable and θ integer-valued; also E is contained in a countable union $\cup E_i$ of C^1 submanifolds of $\mathbb{R}^{m+n}$. Furthermore, at almost every $x \in E$ the density of E and $\cup E_i$ is 1 and there is a single E_i such that E and E_i coincide at x except for a set of density 0. By a covering argument, we can now find open balls $B_i \subset \mathbb{R}^{m+n} \backslash \operatorname{spt} \partial T$ and C^1 submanifolds N_i of B_i such that $\cup N_i$ coincides with E except for a set of $\|T\|$-measure small. Covering each N_i with small balls centered at N_i of sufficiently small radii in such a way that the conclusion of Proposition 1.1.54 holds (with t very close to 1), we obtain a diffeomorphism f of $\mathbb{R}^{m+n}$ which flattens most of E into m-planes and for which $f_\sharp T$ differs from a polyhedral chain P_1 by a small quantity. However the error $f_\sharp T - P_1$, although small in mass, may have huge boundary. Now we use the deformation theorem to decompose the error

$$f_\sharp T - P_1 = P_2 + \partial R + S.$$

As f leaves $\operatorname{spt} \partial T$ fixed, S is actually zero and R has small mass if the grid size is small; also, as $f_\sharp T - P_1$ has small mass, so does P_2 and hence so does the remaining term ∂R. If we set

$$P := P_1 + P_2,$$

we then see that $f_\sharp T - P$ is equal to ∂R, hence has small mass and no boundary. In the general case in which ∂T is not polyhedral, first we approximate ∂T as previously

$$f_{1\sharp} \partial T = P_1 + \partial R_1$$

with $\mathbb{M}(R_1)$ and $\mathbb{M}(\partial R_1)$ small. Now, $f_{1\sharp} T - R_1$ has a polyhedral boundary and can be approximated by an integral polyhedral chain P_2

$$f_{2\sharp}(f_{1\sharp} T - R_1) = P_2 + \partial R_2$$

with small $\mathbb{M}(\partial R_2)$. Therefore,

$$(f_2 \circ f_1)_\sharp T = P_2 + f_{2\sharp} R_1 + \partial R_2$$

and the error term $f_{2\sharp} R_1 + \partial R_2$ and its boundary have small mass, as desired. □

Finally, we obtain the following result as a simple corollary of the deformation theorem, Theorem 1.1.52, and Theorem 1.1.53.

Corollary 1.1.55 (Strong polyhedral approximation theorem). *Let K be a compact subset of $\mathbb{R}^{m+n}$. For each integral current T such that $\operatorname{spt} T \subset K^\circ$, then there exists a sequence of polyhedral chains P_i with integer coefficients such that* (*i*) $\lim_{i\to\infty} \mathbb{F}(P_i - T) = 0$;

(*ii*) $\lim_{i\to\infty} \mathbb{M}(P_i) = \mathbb{M}(T)$, $\lim_{i\to\infty} \mathbb{M}(\partial P_i) = \mathbb{M}(\partial T)$.

Another consequence of the deformation theorem is the following.

Theorem 1.1.56 (Isoperimetric inequality). *Let T be an $m-1$-dimensional integer rectifiable current in $\mathbb{R}^{m+n}$ with compact support and $\partial T = 0$. Then there exists an integer rectifiable m-current R with compact support such that $\partial R = T$ and*

$$\mathbb{M}(R) \le c\,\mathbb{M}(T)^{m/(m-1)}$$

where $c = c(m,n)$.

Integral homology

If the ambient space is a compact oriented smooth manifold $\mathcal{M}$, it is possible to define its integral homology in terms of currents.

Let $\mathcal{M}^{m+n}$ be an oriented closed smooth manifold of dimension $m+n$, smoothly embedded by Whitney's theorem in $\mathbb{R}^d$ for some large d, and denote by U its smooth tubular neighnourhood. For $k = 0, 1, \ldots, m+n$, we define the space of integral cycles and boundaries respectively as follows:

$$\begin{aligned}\mathcal{Z}_k(\mathcal{M},\mathbb{Z}) &:= \{T \in \mathbf{I}_k(U),\ \partial T = 0,\ \operatorname{spt} T \subset \mathcal{M}\}\,,\\ \mathcal{B}_k(\mathcal{M},\mathbb{Z}) &:= \{\partial S,\ S \in \mathbf{I}_{k+1}(U),\ \operatorname{spt} S \subset \mathcal{M}\}\,.\end{aligned}$$

Note that $\mathcal{B}(\mathcal{M},\mathbb{Z})$ is a normal subgroup of $\mathcal{Z}(\mathcal{M},\mathbb{Z})$; then the integral k-homology group of $\mathcal{M}$ is defined by

$$H_k(\mathcal{M},\mathbb{Z}) := \mathcal{Z}_k(\mathcal{M},\mathbb{Z})/\mathcal{B}_k(\mathcal{M},\mathbb{Z}),$$

and it is possible to prove, as a consequence of the deformation theorem, that the homology theory so defined satisfies all seven Eilenberg-Steenrod Axioms, and hence it is equivalent to the singular homology of $\mathcal{M}$ with coefficients in $\mathbb{Z}$.

A relevant advantage in representing the singular homology in terms of homology groups of currents is that the cosets in $\mathcal{Z}_k(\mathcal{M},\mathbb{Z})$ turn out to be closed with respect to the convergence of currents. In fact, in the case of integral homology, the cosets are connected components of $\mathcal{Z}_k(\mathcal{M},\mathbb{Z})$. The key step in proving it is an isoperimetric inequality, which again comes as a corollary of the deformation theorem.

Theorem 1.1.57 (Isoperimetric inequality on manifolds). *For $k = 0, \ldots, m+n$, there is a constant $\varepsilon = \varepsilon(k, m, n, \varepsilon_A)$ such that if $T \in \mathcal{Z}_k(\mathcal{M},\mathbb{Z})$ with $\mathbb{M}(T) \le \varepsilon$, then $T \in \mathcal{B}_k(\mathcal{M},\mathbb{Z})$; more precisely, $T = \partial R$, $R \in \mathbf{I}_{k+1}(U)$, spt $R \subset \mathcal{M}$ and*

$$\mathbb{M}(R) \le c\,\mathbb{M}(T)^{1+1/k}$$

where $c = c(k,m,n)$.

The following proposition yields the key decomposition lemma. The proof comes again from the deformation theorem, combined with a suitable diagonal argument.

Proposition 1.1.58. *If* $T_i, T \in \mathbf{I}_m(\mathcal{M})$ *and* T_i *converges to* T *with equibounded masses and masses of the boundaries, then we can find integral currents* $R_i \in \mathbf{I}_{k+1}(\mathcal{M})$ *and* $S_i \in \mathcal{Z}_k(M,\mathbb{Z})$ *such that*

$$\mathbb{M}(R_i) + \mathbb{M}(S_i) \to 0, \quad T_i - T = S_i + \partial R_i.$$

From Proposition 1.1.58 and Theorem 1.1.57 we deduce the following important conclusion.

Theorem 1.1.59. *Let* $T_i, T \in \mathcal{Z}_k(\mathcal{M},\mathbb{Z})$ *be integral k-cycles,* $T_i \to T$*. Then, for i large enough,*

$$T_i - T = \partial R_i,$$

with $R_i \in \mathbf{I}_{k+1}(\mathcal{M})$*. Moreover* $\mathbb{M}\left(R_i\right) \to 0$*.*

Proof. By Proposition 1.1.58 we can write

$$T_i - T = S_i + \partial R_i',$$

$S_i \in \mathcal{Z}_k(\mathcal{M},\mathbb{Z}), R_i' \in \mathbf{I}_{k+1}(\mathcal{M})$ and $\mathbb{M}(R_i') + \mathbb{M}(S_i) \to 0$. The isoperimetric inequality of Theorem 1.1.57 implies that $S_i = \partial\Sigma_i$, with $\Sigma_i \in \mathbf{I}_{k+1}(\mathcal{M})$ and $\mathbb{M}(\Sigma_i) \leq c\,\mathbb{M}(S_i)^{1+1/k}$ for large i. Thus,

$$T_i - T = \partial(R_i' + \Sigma_i),$$

and the claim follows. □

The following corollary collects some important consequences of Theorem 1.1.59; in particular, it is possible to prove that also the generalized oriented homological Plateau problem always admits an area-minimizing representative.

The generalized oriented homological Plateau problem. *Let* $\mathcal{M}^{m+n}$ *be a smooth closed oriented Riemannian manifold and* $\tau \in H_m(\mathcal{M},\mathbb{Z})$ *an integral homology class. Find* $T \in \mathcal{Z}_m(\mathcal{M},\mathbb{Z})$ *such that* $T \in \tau$ *and* $\mathbb{M}(T) = \inf\{\mathbb{M}(S) \mid S \in \tau\}$*.*

Corollary 1.1.60 (Generalized oriented homological Plateau solution)**.** *With the same assumptions as above, we have that*

- *(i)* $\mathcal{B}_k(\mathcal{M},\mathbb{Z})$ *is closed in the weak* topology.*
- *(ii)* $\mathcal{Z}_k(\mathcal{M},\mathbb{Z})\backslash\mathcal{B}_k(\mathcal{M},\mathbb{Z})$ *is closed in the weak* topology.*
- *(iii) In each integral homology class* τ *there is a mass minimizing integral cycle, that is, there is* $T \in \mathcal{Z}_m(\mathcal{M},\mathbb{Z})$, $T \in \tau$, *such that* $\mathbb{M}(T) := \inf\{\mathbb{M}(S) \mid S \in \tau\}$.
- *(iv) Given* $c > 0$*, there are at most a finite number of integral homology classes* τ *such that* $\inf\{\mathbb{M}(S) \mid S \in \tau\} \leq c$*.*
- *(v) For a nontrivial integral homology class* $\inf\{\mathbb{M}(S) \mid S \in \tau\} > 0$*.*

1.1.3 Integral currents mod 2

To set up the main terminology, we briefly recall the theory of integral mod 2 currents as developed by Ziemer and completed (in fact, generalized) by Fleming in [95, 45] respectively. As a standard reference we also refer to [43, 4.2.26].

We denote by $\mathbf{F}_k(\mathbb{R}^d)$ the space of k-dimensional integral flat chains in $\mathbb{R}^d$, that is

$$\mathbf{F}_k(\mathbb{R}^d) := \{T + \partial S : T \in \mathbf{R}_k(\mathbb{R}^d),\ S \in \mathbf{R}_{k+1}(\mathbb{R}^d)\}.$$

As usual, if $M \subset \mathbb{R}^d$ is a compact (local) Lipschitz neighborhood retract, then $\mathbf{R}_k(M)$ will denote the space of $T \in \mathbf{R}_k(\mathbb{R}^d)$ with support $\operatorname{spt}(T) \subset M$; analogously for $\mathbf{F}_k(M)$ and $\mathbf{I}_k(M)$. For $T \in \mathbf{F}_k(M)$, its mod 2 flat norm can be defined as

$$\mathbb{F}^2(T) := \inf\{\mathbb{F}(T + 2Q) : Q \in \mathbf{F}_k(M)\},$$

its mod 2 mass as

$$\mathbb{M}^2(T) := \inf\{\mathbb{M}(T + 2Q) : Q \in \mathbf{F}_k(M)\}$$

and its mod 2 support as

$$\operatorname{spt}^2(T) := \bigcap\{\operatorname{spt}(R) : R \in \mathbf{F}_k(M), \mathbb{F}^2(T - R) = 0\}.$$

We will usually drop the index 2 when this is clear from the context.

The space of flat chains mod 2 in M is defined as the quotient group

$$\mathbf{F}_k(M, \mathbb{Z}_2) := \mathbf{F}_k(M)/2\mathbf{F}_k(M)$$

and if $T \in \mathbf{F}_k(M)$, we will denote with $[T]$ or T mod 2 the coset in $\mathbf{F}_k(M, \mathbb{Z}_2)$ containing T. Analogously, we can define the space of integer rectifiable currents mod 2 and of integral currents mod 2, respectively denoted by

$$\mathbf{R}_k(M, \mathbb{Z}_2) := \mathbf{R}_k(M)/2\mathbf{R}_k(M), \quad \mathbf{I}_k(M, \mathbb{Z}_2) := \mathbf{I}_k(M)/2\mathbf{I}_k(M).$$

If T is flat chain mod 2 with an integer coefficient polyhedral chain as a representative, then T will be called a polyhedral chain mod 2; analogously for Lipschitz chains mod 2.

In particular, we remark that an integer rectifiable current $T = [\![E, \tau_E, \theta]\!] \in \mathbf{R}_k(M)$ is called *representative mod 2* whenever $|\theta(x)| \leq 1$ for $\|T\|$-*a.e.* point. Hence, every integer rectifiable current can be written as

$$T + 2Q,$$

with $T, Q \in \mathbf{R}_k(M)$ and T is a representative mod 2. In particular, $\mathbb{M}^2([T]) = \mathbb{M}(T)$ and $\operatorname{spt}^2([T]) = \operatorname{spt}(T)$, whenever $T \in \mathbf{R}_k(M)$ and T is a representative mod 2. If $T \in \mathbf{R}^2_k(M)$ and $B \subset \mathbb{R}^d$ is a Borel subset, we define $T \llcorner B := [T \llcorner B]$, where T is a rectifiable representative of T mod 2.

If $T \in \mathbf{F}_k(M)$ is a flat chain, then the boundary operator ∂^2 on $\mathbf{F}^2_k(M)$ is defined by $\partial^2[T] = [\partial T]$, since if $T = S$ mod 2, then also $\partial T = \partial S$ mod 2. A flat chain $T \in \mathbf{F}_k(M)$

is a cycle mod 2 if $\partial T = 0 \bmod 2$, *i.e.* $\partial^2[T] = 0$ and it is a boundary mod 2 if there exists $S \in \mathbf{F}_{k+1}(M)$ such that $T = \partial S \bmod 2$, *i.e.* $[T] = \partial^2[S]$; the spaces of k-dimensional integral mod 2 cycles and k-dimensional integral mod 2 boundaries with $\mathrm{spt}^2(\cdot) \subset M$ are denoted by $\mathcal{Z}_k(M, \mathbb{Z}_2)$ and $\mathcal{B}_k(M, \mathbb{Z}_2)$, respectively; note that $\mathcal{B}(M, \mathbb{Z}_2)$ is a normal subgroup of $\mathcal{Z}(\mathcal{M}, \mathbb{Z}_2)$. If $\mathcal{M}$ is a closed smooth manifold, we define its mod 2 homology groups as

$$H_*(\mathcal{M}, \mathbb{Z}_2) = \mathcal{Z}_*(\mathcal{M}, \mathbb{Z}_2)/\mathcal{B}_*(M, \mathbb{Z}_2),$$

and it is possible to prove, as a consequence of the deformation theorem for mod 2 currents, that the homology theory so defined satisfies all seven Eilenberg-Steenrod Axioms, and hence it is equivalent to the singular homology of $\mathcal{M}$ with coefficients in $\mathbb{Z}_2$. In Chapter 3, we will slightly abuse notation between currents mod 2 and their representatives.

1.2 Piecewise-Linear topology

In this section we collect a few elementary facts about triangulating regions and some important adaptations needed in Chapters 2 and 3.

1.2.1 Algorithm to subdivide a convex polytope

We understand a convex polytope P of $\mathbb{R}^d$ as a closed convex set with a finite number of extremal points. Given a convex polytope $P \subset \mathbb{R}^d$ we now describe an algorithm to triangulate it. For any convex polytope we define its barycenter as the point which is given by the convex combination of the extremal points with all equal weights (if $V_1, \ldots, V_m$ are the extremal points of P, then the baricenter is $\frac{1}{m}\sum_i V_i$).

A touching hyperplane π of P is a hyperplane such that

- $\pi \cap P$ is nonempty;
- P is contained in one of the two closed half-spaces bounded by π.

If the dimension of P is strictly smaller than d, then we let the set $\mathcal{F}$ of faces of P be the collection of convex subsets of P of the form $P \cap \pi$, where π varies among all touching hyperplanes. If the dimension of P is d, we add to $\mathcal{F}$ the polytope P itself. We subdivide $\mathcal{F}$ as

$$\bigcup_{k=0}^{\dim(P)} \mathcal{F}_k\,,$$

where $\mathcal{F}_k = \{F \in \mathcal{F} : \dim(F) = k\}$. Clearly, $\mathcal{F}_0$ consists of points and it is the set of extremal points of P (in particular, it is a finite set) and any other element of $\mathcal{F}$ is necessarily the convex hull of some appropriate subset of $\mathcal{F}_0$.

In order to triangulate P, we first observe that all 1-dimensional faces and all 0-dimensional faces are (by definition) simplices of the corresponding dimension. We then list the 2-dimensional faces. For each face F which is not a triangle we consider the barycenter $b(F)$ and we decompose F into the triangles formed by $b(F)$ and the sides of F (namely, the 1-dimensional faces of F). Note that the collection of all such triangles (as F also varies among all 2-dimensional faces) has the following property: the intersection of any pair of such triangles is either empty, or a common vertex, or a common side. Next, fix a 3-dimensional face F which is not a simplex. Each of its 2-dimensional faces G is decomposed in triangles T's in the previous step. Decompose F in the 3-dimensional simplices constructed as convex hulls of any such T and the barycenter $b(F)$ of F (we can think of them as pyramids with basis T and vertex $b(F)$). We have decomposed all 3-dimensional faces into 3-dimensional simplices S. Any pair of such S (irrespectively of whether they belong to the decomposition of the same 3-dimensional face or to the decompositions of two distinct faces) has the property that their intersection is a common lower-dimensional face. We proceed inductively increasing the dimension of the faces at each step until we reach (and include) the one of highest dimension, namely P.

The following elementary lemmas will play an important role.

Lemma 1.2.1. *If $P \subset \mathbb{R}^d$ is a convex polytope and $A : \mathbb{R}^d \to \mathbb{R}^d$ an affine invertible map, then the triangulation $\mathcal{T}'$ for $A(P)$ obtained through the algorithm above coincides with the image through A of the triangulation $\mathcal{T}$ obtained for P through the algorithm, namely $\mathcal{T}' = \{A(T) : T \in \mathcal{T}\}$.*

Lemma 1.2.2. *Let $P, P' \subset \mathbb{R}^d$ be two convex polytopes whose intersection is a common face of both. Consider the triangulation $\mathcal{T}$ of P and the triangulation $\mathcal{T}'$ of P' obtained applying the algorithm above. Then the union of the two triangulations is a triangulation, namely: the intersection of an arbitrary element of $\mathcal{T}$ with an arbitrary element of $\mathcal{T}'$ is a common face of both simplices.*

1.2.2 Embedding convex polytopes in skeleta of refined triangulations

In this section we prove the following proposition.

Proposition 1.2.3. *Consider a finite family of convex m-dimensional polytopes $\{P_i\}$ in $\mathbb{R}^d$ and a triangulation $\mathcal{T}$ of some closed subset of $\mathbb{R}^d$. Then there exists a triangulation $\mathcal{T}_f$ finer than $\mathcal{T}$ with the property that each polytope P_i is union of elements of the m-skeleton of $\mathcal{T}_f$. Moreover, the refinement of $\mathcal{T}$ is local in the following sense: if we denote by $\mathcal{T}'$ the collection of those d-dimensional simplices of $\mathcal{T}$ which intersect at least one P_i, then any simplex of $\mathcal{T}$ which does not intersect an element of $\mathcal{T}'$ is not refined (namely, it is also an element of $\mathcal{T}_f$).*

We will give an algorithm to produce $\mathcal{T}_f$. First of all, take all possible intersections of the P_i's with d-dimensional simplices of $\mathcal{T}$. This gives a collection of new m-dimensional polytopes $\bar{P}_j$: each of them is contained in a d-dimensional simplex of $\mathcal{T}$, which we denote by T_j.

We start with $\bar{P}_1$. Because it is a convex polytope of dimension $< d$, we write it as the intersection of a finite number of hyperplanes π_i and a finite number of closed halfspaces H_j. Then we build a finite collection $\mathcal{H}$ of pairs of halfspaces H_i^-, H_i^+ by adding for each $H_j^+ := H_j$ the closure of its complement H_j^-, and for ever π_i the pair of closed halfspaces which have π_i as a boundary. We then subdivide T_1 inductively in smaller convex d-dimensional polytopes in the following way. In the first step we keep T_1 if it is contained in one of the two halfspaces $\{H_1^+, H_1^-\}$, otherwise we replace it with the pair $\{H_1^+ \cap T_1, H_1^- \cap T_1\}$. At step j we assume to have a finite collection of closed convex d-dimensional polytopes and each of them is kept if it is contained in one of the two halfspaces H_{j+1}^+, H_{j+1}^-, otherwise it is replaced by the two intersections with them.

The resulting collection is a partition of T_1 into convex polytopes with the property that any two faces of any two polytopes intersect in a common face. Moreover, the original P_1 is the m-dimensional face of some convex polytope of this partition. We apply to each of these d-dimensional polytopes the triangulating algorithm of Section 1.2.1 and, by Lemma 1.2.1, we obtain a triangulation of T_1. However, T_1 has faces in common with other simplices of $\mathcal{T}$ which are not yet partitioned. In order to remedy, we proceed inductively as follows. We first denote by $\mathcal{S}_k$ be the collection of k-dimensional faces of T_1; we start with the edges $\mathcal{S}_1$ and add to $\mathcal{S}_2$ every triangle of $\mathcal{T}$ which contains an edge $\sigma \in \mathcal{S}_1$ and add it to $\mathcal{S}_2$. The new triangles are triangulated compatibly with the elements of $\mathcal{S}_1$ by adding its barycenter and connecting it with edges to all its vertices and all the new points in the edges of $\mathcal{S}_1$ that it might contain. Observe that this procedure does not subdivide any edge of the initial triangulation $\mathcal{T}$ which is not in $\mathcal{S}_1$. Similarly, at step j we enlarge $\mathcal{S}_{j+1}$ with all the $j+1$-dimensional simplices which contain an element of $\mathcal{S}_j$ as a face. Each new simplex S added is triangulated by considering its barycenter $b(S)$ and subdividing S into the pyramids which:

- have vertex $b(S)$ and basis a j-dimensional face F of S, in case F does not belong to $\mathcal{S}_j$;
- have vertex $b(S)$ and basis a j-dimensional simplex of the subdivision of the face $F \in \mathcal{S}_j$ obtained so far inductively, in the other case;

We stop the procedure when we have subdivided the final new elements added to the collection $\mathcal{S}_d$. The final result is a triangulation.

Observe that, by construction, in the new triangulation $\mathcal{T}_1$ the polytope P_1 is the union of elements of the m-dimensional skeleton.

We now proceed inductively with the subsequent polytopes. However, observe that, at the $j+1$-th step, the simplex T_{j+1} might not be an element of the triangulation $\mathcal{T}_j$. However, if that is the case, by construction there is a collection $\mathcal{C}$ of d-dimensional simplices of $\mathcal{T}_j$ whose union is precisely T_{j+1}. The first subdivision algorithm in which we intersected T_1 with halfspaces is, in this case, applied simultaneously to all of the elements of $\mathcal{C}$. This then results into a subdivision $\mathcal{S}$ of T_{j+1} into convex polytopes which has the two properties of the subdivision obtained in the previous argument for T_1. This subdivision has however the additional feature that, for any element C of $\mathcal{C}$, there is an appropriate subcollection $\mathcal{S}'$ of $\mathcal{S}$ which is in fact a subdivision of it. The remaining part of the algorithm outlined above is then applied *verbatim*, and the result is the next triangulation $\mathcal{T}_{j+1}$.

1.2.3 Embedding polytopes in skeleta of refinements of polyhedra

In this section we extend the algorithm of the previous one to handle more general piecewise linear closed ambient manifolds. For simplicity, we assume that the latter are suitably embedded into some high-dimensional Euclidean space.

Definition 1.2.4. A finite polyhedron in $\mathbb{R}^d$ is the collection of finitely many simplices of $\mathbb{R}^d$.

A finite poyhedron K always admits a finite triangulation, namely a finite collection of simplices $\mathcal{T}$ with the following properties:

- Any face of an element of $\mathcal{T}$ belongs to $\mathcal{T}$;
- The intersection of any two elements of $\mathcal{T}$ is always either empty or a face of both;
- The union of the elements of $\mathcal{T}$ is K.

Although this is a classical fact, note that it is also a consequence of Proposition 1.2.3.

Definition 1.2.5. We will consider continuous maps f over finite polyhedra K taking values into a smooth manifold $\mathcal{M}$. Such maps f will be called *piecewise smooth* if there is a triangulation $\mathcal{T}$ of K with the property that the restriction of f to every simplex in $\mathcal{T}$ is smooth. The map will be called a *piecewise smooth homeomorphism* if in addition it is a homeomorphism with the image and if the triangulation can be chosen so that, for every simplex $\sigma \in \mathcal{T}$, the differential $D(f|_\sigma)$ of the restriction of f to σ has maximal rank at every point. When such a map exists between some polyhedron K and some smooth closed manifold $\mathcal{M}$, we say that K is an embedded piecewise linear closed submanifold of $\mathbb{R}^d$.

It is a classical result of Whitehead that if $f : K \to \mathcal{M}$ is a piecewise smooth homeomorphism, then every pair of triangulations of K admits a further triangulation which is a common refinement of both.

The generalization of Proposition 1.2.3 that we are looking for is then the following.

Proposition 1.2.6. *Consider a polyhedron K which is a piecewise linear submanifold of $\mathbb{R}^d$ of dimension $m+n$, let $\{P_i\}$ be a finite collection of m-dimensional convex polytopes all contained in K, and let $\mathcal{T}$ be a triangulation of K. Then there is a triangulation $\mathcal{T}_f$ of K which refines $\mathcal{T}$ and has the property that every P_i is the union of finitely many elements of the m-skeleton of $\mathcal{T}_f$.*

We quickly describe how to modify the algorithm explained in Section 1.2.2. As in there, we intersect the polytopes with the m-dimensional simplices of $\mathcal{T}$, reducing the proposition to the case in which each P_i is contained in an $m+n$-dimensional simplex of T_i. Moreover, as in there, we refine $\mathcal{T}_0 = \mathcal{T}$ into $\mathcal{T}_1$, $\mathcal{T}_2$, and so on, “embedding” one P_i at a time.

At the starting step the algorithm gives first a way to triangulate T_1 so that P_1 is the union of the m-skeleton of this local triangulation. In the argument, T_1 is supposed to be an $m+n$-dimensional simplex of $\mathbb{R}^{m+n}$, but this can be easily achieved identifying the $m+n$-dimensional affine plane π containing T_1 with $\mathbb{R}^{m+n}$. We then further triangulate all simplices of $\mathcal{T}$ which intersect T_1, proceeding inductively from the lower dimensional ones. Since at each stage of this second algorithm a single simplex is considered at a time, we can think of this as also taking place in some Euclidean space.

At the inductive step, when embedding P_{j+1} into a refinement of $\mathcal{T}_j$, the only difference is that the first subdivision is carried over all at once on all the $m+n$-dimensional simplices $\mathcal{C}_j$ of $\mathcal{T}_j$ which are contained in T_{j+1}. Again, the only important point is that, like above, T_{j+1} is an $m+n$-dimensional simplex. The second part, which refines the triangulation of $\mathcal{T}_j$ over all simplices intersecting at least one element of $\mathcal{C}_j$, is the same as in the initial step.

Lemma 1.2.7. *$\mathcal{M}$ is a connected smooth closed Riemannian manifold of dimension $m+n$, $\mathcal{K}$ a smooth triangulation of $\mathcal{M}$, $k \in \{0, \dots, m+n-1\}$ and $U_\delta(\mathcal{K}^k)$ a smooth neighborhood as in Lemma 2.1.2. Then the complement of $U_\delta(\mathcal{K}^k)$ is homotopy equivalent to a complex of dimension $m+n-k-1$.*

Proof. First we note that the complement of $U_\delta(\mathcal{K}^k)$ is a deformation retract of the complement of $\mathcal{K}^k$ by Lemma 2.1.1 and Lemma 2.1.2, and therefore $\mathcal{K} \setminus U_\delta(\mathcal{K}^k)$ is homotopy equivalent to $\mathcal{K} \setminus \mathcal{K}^k$; we then denote[5]

$$\mathcal{K}_c^k := \mathcal{K} \setminus \mathcal{K}^k.$$

Now we show that $\mathcal{K}_c^k$ is homotopy equivalent to a complex of dimension $m+n-k-1$, and in particular that

$$\mathcal{K}_c^k \sim \mathcal{K}_*^{m+n-k-1}, \tag{1.24}$$

where $\mathcal{K}_*$ is the dual cell complex of the triangulation $\mathcal{K}$, *cfr.* [67, §64]. To this aim, we first show the following:

$$\mathcal{K}_c^k \sim \mathrm{Bs}(\mathcal{K}) - \mathrm{Bs}(\mathcal{K}^k), \tag{1.25}$$

where $\mathrm{Bs}(\cdot)$ is the barycentric subdivision and the operation $\mathrm{Bs}(\mathcal{K}) - \mathrm{Bs}(\mathcal{K}^k)$ represents all simplexes of $\mathrm{Bs}(\mathcal{K})$ that are disjoint from $\mathrm{Bs}(\mathcal{K}^k)$. By definition, we have that

$$\mathcal{K}_c^k = \mathcal{K} \setminus \mathcal{K}^k = \mathrm{Bs}(\mathcal{K}) \setminus \mathrm{Bs}(\mathcal{K}^k).$$

Hence (1.25) follows just by noticing that, in terms of simplicial complexes, $\mathrm{Bs}(\mathcal{K}^k)$ is a full subcomplex of the complex $\mathrm{Bs}(\mathcal{K})$, *i.e.* every simplex of $\mathrm{Bs}(\mathcal{K})$ whose vertices are in $\mathrm{Bs}(\mathcal{K}^k)$ is itself in $\mathrm{Bs}(\mathcal{K}^k)$, and by applying [67, Lemma 70.1]. Since the complex $\mathrm{Bs}(\mathcal{K}) - \mathrm{Bs}(\mathcal{K}^k)$ corresponds exactly to the $m+n-k-1$-skeleton $\mathcal{K}_*^{m+n-k-1}$ of the dual cell structure of $\mathcal{K}$, (1.24) follows, and hence the final result. ☐

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[5] With an abuse of notation between the simplicial complex and the t-image of the geometric realization of the simplicial complex itself.

1.3 Cobordism and homotopy theory

We briefly recall the main topological notions that will be used later, we refer also to [80, 84, 86, 60].

The *mapping cylinder* M_f of a continuous map $f : X \to Y$ is the quotient space formed from the disjoint union $(X \times [0, 1]) \sqcup Y$ by identifying, for each $x \in X$, the point $(x, 1)$ with $f(x) \in Y$; it contains $X \times \{0\}$ as a subspace and has Y as a deformation retract.

For $n \geq 1$ and an abelian group π, the *Eilenberg-MacLane space* $K(\pi, n)$ is a space with the homotopy type of a CW-compex such that $\pi_i(K(\pi, n))$ vanishes for $i \neq n$ and $\pi_n(K(\pi, n)) \simeq \pi$, where $\pi_i(X)$ denotes the i-th homotopy group of the topological space X. Recall that the Hopf homotopy classification theorem states that for a (connected) CW-complex X, an abelian group π and for every $n \in \mathbb{N} \setminus \{0\}$ there is a natural isomorphism

$$T : [X, K(\pi, n)] \to H^n(X, \pi),$$

where $[X, K(\pi, n)]$ represents the set of (unbasedpointed) homotopy classes of continuous maps from X to $K(\pi, n)$ and $H^n(X, \pi)$ is the n-th cohomology group of X with coefficients in π. The isomorphism has the form $T([f]) = f^*(\iota)$, for a certain $\iota \in H^n(K(\pi, n), \pi)$ called the *fundamental class*; $K(\pi, n)$ is therefore the classifying space of n-dimensional cohomology with coefficients in π. This determines $K(\pi, n)$ up to homotopy equivalence: that is, the homotopy type of $K(\pi, n)$ is determined by π and n, and the identity map of π determines, up to homotopy, a canonical homotopy equivalence between any two copies of $K(\pi, n)$.

We recall that a continuous map $f : X \to Y$ between path-connected CW-complexes is called an *n-equivalence* for $n \geq 1$ if the induced homomorphism

$$f_* : \pi_i(X) \to \pi_i(Y)$$

is an isomorphism for $0 < i < n$ and an epimorphism for $i = n$.

We also recall that $f : X \to Y$ is an n-equivalence if and only if the inclusion $i : X \to M_f$ is an n-equivalence. From the long exact sequence of relative homotopy groups, it follows that i is an n-equivalence if and only if the relative homotopy group $\pi_i(M_f, X) = 0$ vanishes for all $i \leq n$. In order to keep our notation lighter, with a slight abuse we will sometimes write $\pi_i(Y, X) = 0$, meaning $\pi_i(M_f, X) = 0$ when the map f is clear from the context.

We recall the following characterization of n-equivalence and the subsequent corollary.

Proposition 1.3.1 ([80, Theorem 7.6.22])**.** *If $f : X \to Y$ is an n-equivalence, then for every relative CW-complex (K, L) with K of dimension at most n, and every map $a : L \to X$ and $b : K \to Y$ with $b_{|L} = f \circ a$, there exists a map $c : K \to X$ with $c_{|L} = a$ and $f \circ c$ homotopic to b relative to L.*

Corollary 1.3.2 ([80, Corollary 7.6.23]). *If K is a CW-complex and $f : X \to Y$ is an n-equivalence, then the induced homomorphism*

$$f_* : [K, X] \to [K, Y]$$

is a bijection if $\dim K < n$ *and a surjection if* $\dim K = n$.

We recall a classical result due to Whitehead, which allows to deduce homotopic properties of a space from its cohomological ones, *cfr.* also [86, Theorem II.6].

Theorem 1.3.3. *Let $f : X \to Y$ a map between two simply connected CW-complexes X, Y and p a prime number. If for any group coefficient $\mathbb{Z}_p$ the induced homomorphism*

$$f^* : H^i(Y, \mathbb{Z}_p) \to H^i(X, \mathbb{Z}_p)$$

is an isomorphism when $i < k$ and a monomorphism when $i = k$, then the relative homotopy groups $\pi_i(Y, X) = 0$, for $i \leq k$ and f is a k-equivalence.

Proof. Consider the following exact sequence in cohomology:

$$H^r(M_f) \xrightarrow{f^*} H^r(X) \to H^{r+1}(M_f, X) \to H^{r+1}(M_f) \to H^{r+1}(X).$$

The assumptions on f are equivalent to $H^i(M_f, X, \mathbb{Z}_p) = 0$ for every prime p and $i \leq k$. By duality on the group coefficients $\mathbb{Z}_p$, we can write $H_i(M_f, X, \mathbb{Z}_p) = 0$ for $i \leq k$ which is equivalent, by the universal coefficient formula, to $H_i(M_f, X, \mathbb{Z}) = 0$ for $i \leq k$. Since X and Y are simply connected, by the relative Hurewicz theorem, *cfr.* [80, Theorem 7.5.4], we conclude that $\pi_i(M_f, X) = 0$ for $i \leq k$, namely our claims (recall that by $\pi_i(Y, X)$ we actually mean $\pi_i(M_f, X)$). □

Let $\widetilde{G}_n(\mathbb{R}^{n+k})$ be the oriented Grassmannian manifold, that is the space of oriented n-dimensional subspaces in $\mathbb{R}^{n+k}$. The natural embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, induces an embedding $\widetilde{G}_n(\mathbb{R}^n) \hookrightarrow \widetilde{G}_n(\mathbb{R}^{n+1})$. Thus, forming the union over increasing dimensions we obtain an infinite CW-complex named the *infinite oriented Grassmannian*

$$\widetilde{G}_n := \widetilde{G}_n(\mathbb{R}^\infty) = \bigcup_k \widetilde{G}_n(\mathbb{R}^{n+k}),$$

as the set of all n-dimensional linear subspaces of $\mathbb{R}^\infty$, endowed with the direct limit topology, *i.e.* a subset of $\widetilde{G}_n$ is open if and only if its intersection with $\widetilde{G}_n(\mathbb{R}^{n+k})$ is open as a subset of $\widetilde{G}_n(\mathbb{R}^{n+k})$ for each k. We denote as $\widetilde{\gamma}^n$ the *universal oriented n-plane bundle*, that is the canonical vector bundle over the base space $\widetilde{G}_n$

$$\widetilde{E} \xrightarrow{\widetilde{\gamma}^n} \widetilde{G}_n$$

with total space $\widetilde{E}$ consisting of pairs $(\ell, v) \in \widetilde{G}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$ such that $v \in \ell$, topologized as subset of the cartesian product, and with projection $\pi : \widetilde{E} \to \widetilde{G}_n$ such that $\pi(\ell, v) = \ell$.

Recall that any oriented n-plane[6] bundle ξ over a paracompact base B admits a bundle map $\xi \to \widetilde{\gamma}^n$ and that any two bundle maps $f, g : \xi \to \widetilde{\gamma}^n$ from an oriented n-plane bundle ξ to $\widetilde{\gamma}^n$ are bundle homotopic, meaning that there exists a one-parameter family of bundle maps $h_t : \xi \to \widetilde{\gamma}^n$, with $t \in [0,1]$ and $h_0 = f$ and $h_1 = g$ such that h is continuous as a function of both variables, *cfr.* [60, Theorems 5.6, 5.7]. Hence, any oriented n-plane bundle ξ over a paracompact space B determines, up to orientation-preserving isomorphism, a unique homotopy class of maps $\overline{f}_\xi : B \to \widetilde{G}_n$. Since the classifying space $\widetilde{G}_n$ for oriented n-plane vector bundles is the classifying space associated to the rotations group $SO(n)$, we will denote it as usual by $BSO(n)$. In fact, in almost all our considerations we just need to consider $\widetilde{G}_n(\mathbb{R}^{n+k})$ for a sufficiently large k and at all effects treat $BSO(n)$ as some fixed compact manifold $\widetilde{G}_n(\mathbb{R}^{n+k})$ for a suitably large k.

Analogously, in the unoriented case we denote by $G_n(\mathbb{R}^{n+k})$ the Grassmannian manifold, that is the space of n-dimensional subspaces in $\mathbb{R}^{n+k}$. The natural embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, induces an embedding $G_n(\mathbb{R}^n) \hookrightarrow G_n(\mathbb{R}^{n+1})$ and thus, forming the union over increasing dimensions, we obtain an infinite CW-complex named the *infinite Grassmannian*

$$G_n := G_n(\mathbb{R}^\infty) = \bigcup_k G_n(\mathbb{R}^{n+k}),$$

as the set of all n-dimensional linear subspaces of $\mathbb{R}^\infty$, endowed with the direct limit topology, *i.e.* a subset of G_n is open if and only if its intersection with $G_n(\mathbb{R}^{n+k})$ is open as a subset of $G_n(\mathbb{R}^{n+k})$ for each k. We denote γ^n the *universal n-plane bundle*, that is the canonical vector bundle over the base space G_n

$$E \xrightarrow{\gamma^n} G_n$$

with total space E consisting of pairs $(\ell, v) \in G_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$ such that $v \in \ell$, topologized as a subset of the cartesian product, and with projection $\pi : E \to G_n$ such that $\pi(\ell, v) = \ell$. Since the classifying space G_n for n-plane vector bundles is the classifying space associated to the orthogonal group $O(n)$, we will denote it as usual by $BO(n)$. In fact, in almost all our considerations we just need to consider $G_n(\mathbb{R}^{n+k})$ for a sufficiently large k and at all effects treat $BO(n)$ as some fixed compact manifold $G_n(\mathbb{R}^{n+k})$ for a suitably large k.

We recall now the main notions of Thom spaces and Thom's characterization of representability of a homology class by smooth embedded submanifolds.

Let ξ be an n-plane bundle with a Euclidean metric and $A \subset E(\xi)$ be the subset of the total space consisting of all vectors v with $|v| \geq 1$. Then the identification space $E(\xi)/A$ is called the *Thom space* $T(\xi)$ of ξ. Note that $T(\xi)$ has a preferred base point, denoted by ∞, and the complement $T(\xi) \setminus \{\infty\}$ consists of all vectors $v \in E(\xi)$ with $|v| < 1$. We note that if the base space B of ξ is a (finite) CW-complex, then the Thom space $T(\xi)$ is an $n-1$-connected (finite) CW-complex.

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[6] That is, all fibers are oriented n-dimensional real vector spaces.

If ξ is a smooth oriented n-plane bundle, then the base space B of ξ can be smoothly embedded as the zero-cross section in the total space $E(\xi)$, and hence in the Thom space $T(\xi)$; moreover we note that while $T(\xi)$ is not a manifold in general, the complement of the base point $T(\xi) \setminus \{\infty\}$ has the structure of a smooth manifold.

Let R be a commutative ring with unity and ξ an n-plane bundle $E \xrightarrow{\pi} B$. For a point $b \in B$, let S^n_b be the one-point compactification of the fiber $\pi^{-1}(b)$; since S^n_b is the Thom space of $\xi_{|b}$, we have a canonical map

$$i_b : S^n_b \to T(\xi)\,.$$

An *R-orientation*, or a *Thom class*, of ξ is defined to be an element $u \in \widetilde{H}^n(T(\xi), R)$ (the reduced n-th cohomology group of $T(\xi)$) such that, for every point $b \in B$, $i^*_b(u)$ is a generator of (the free R-module) $\widetilde{H}^n(S^n_b)$. We recall now a fundamental theorem, *cfr.* [84, Theorem 15.51].

Theorem 1.3.4 (Thom isomorphism theorem)**.** *Let $u \in \widetilde{H}^n(T(\xi), R)$ be a Thom class for an n-plane bundle ξ of the form $E \xrightarrow{\pi} B$. Define*

$$\Phi : H^i(B, R) \to \widetilde{H}^{n+i}(T(\xi), R)$$

by the cup product $\Phi(x) = \pi^*(x) \smile u$. Then Φ is an isomorphism for every integer i.

We remark that for any oriented n-plane bundle the Thom class with $\mathbb{Z}$ coefficients exists and it is unique; analogously, for every n-plane bundle there exists a unique Thom class with $\mathbb{Z}_2$ coefficients, *cfr.* [60, Theorems 9.1, 8.1].

1.3.1 Thom's representability criterion

It goes back to Poincaré and to the birth of algebraic topology the question of whether manifolds are good models for homology, *cfr.* [70]. Around the late 1940s and aiming at computing the homotopy groups of spheres $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$, Pontrjagin introduced a new approach that can now be regarded as the germ of the much more general theory developed by René Thom a few years later *cfr.* [71, 72, 73]. In his seminal 1954 article, which is considered one of the cornerstones of algebraic and differential topology, Thom completely developed cobordism theory from (almost) nothing to the proof of one the best theorems in the field, all just in one single paper, see [86]. In particular, his most important contribution was that of classifying all smooth closed manifolds of general dimension up to the cobordism equivalence relation; but, what is even more remarkable is that this classification was merely the mean to provide an answer to a profound question raised by Norman Steenrod asking whether homology classes can always be represented by manifolds, *cfr.* [40, Problem 25][7].

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[7] In December 1946, as part of the Bicentennial Celebration of Princeton University, a conference was held on “The Problems of Mathematics”. The participants of the topology session of this conference submitted a large number of unsolved problems, among which there is what is now simply known as *Steenrod's problem*.

Thom's celebrated result about realizability of homology classes by means of smooth embedded submanifolds can be stated as follows, *cfr.* [86, Théorème II.1].

Definition 1.3.5 (Smooth representability)**.** Let $\mathcal{M}$ a connected smooth closed oriented $m+n$-dimensional manifold, τ is an element of the m-dimensional integral homology group $H_m(\mathcal{M},\mathbb{Z})$ and Σ a smooth oriented m-dimensional closed embedded submanifolds of $\mathcal{M}$. We say that τ is *representable by a smooth submanifold* (or that τ *admits a smooth representative*) if there exists a smooth embedding $f:\Sigma\to\mathcal{M}$ such that the fundamental class of Σ equals τ, that is $f_*[\Sigma]=\tau$ (or, equivalently, that $[\![\Sigma]\!]\in\tau$).

Theorem 1.3.6 (Thom's representability criterion)**.** *Given $\mathcal{M}$ a connected smooth closed oriented $m+n$-dimensional manifold. A homology class $\tau\in H_m(\mathcal{M},\mathbb{Z})$ is representable by a m-dimensional smooth submanifold $\Sigma\subset\mathcal{M}$ of codimension n if and only if there exists a map $f:\mathcal{M}\to T(\widetilde{\gamma}^n)$ which pulls back the Thom class*[8] *$u\in H^n(T(\widetilde{\gamma}^n),\mathbb{Z})$ to the Poincaré dual of τ.*

Thus, just as terminology, we will say that a cohomology class $x\in H^n(\mathcal{M},\mathbb{Z})$ is representable by a smooth embedded submanifold if there exists a map $f:\mathcal{M}\to T(\widetilde{\gamma}^n)$ which pulls back the Thom class $u\in H^n(T(\widetilde{\gamma}^n),\mathbb{Z})$ to x.

Proof. Suppose that there exists an embedded m-dimensional submanifold Σ in $\mathcal{M}$ representing τ. Let N be the closed normal tubular neighborhood of Σ, with boundary ∂N; by assumption, since Σ is oriented, the normal bundle $\nu:N\to\Sigma$ has $SO(n)$ as structure group. Note that N is induced from the mapping cylinder $M_{\widetilde{\gamma}^n}$ of the universal oriented n-plane bundle $\widetilde{\gamma}^n$ by a map $g:\Sigma\to BSO(n)$. Hence, there exists a bundle map

$$F:N\to M_{\widetilde{\gamma}^n}$$

making the following diagram commute:

$$\begin{tikzcd}
N \arrow[r, "F"] \arrow[d, "\nu"'] & M_{\widetilde{\gamma}^n} \arrow[d, "\widetilde{\gamma}^n"] \\
\Sigma \arrow[r, "g"'] & BSO(n)
\end{tikzcd}$$

Note that $F|_{\partial N}$ sends ∂N to $\partial M_{\widetilde{\gamma}^n}$. Denoting φ^* and $\varphi^*_{SO(n)}$ the Thom isomorphisms, the following diagram also commutes:

$$\begin{tikzcd}
H^n(N,\partial N) \arrow[r, "F^*"] & H^n(M_{\widetilde{\gamma}^n},\partial M_{\widetilde{\gamma}^n}) \\
H^0(\Sigma) \arrow[r, "g^*"'] \arrow[u, "\varphi^*"'] & H^0(BSO(n)) \arrow[u, "\varphi^*_{SO(n)}"]
\end{tikzcd} \tag{1.26}$$

---

[8] Recall that, since $R=\mathbb{Z}$ and $n\geq 1$, the reduced cohomology group coincides with the standard cohomology group.

Moreover, let $j_* : H^n(N, \partial N) \to H^n(\mathcal{M})$ the canonical homomorphism defined by the inclusion. One can see that in $N \setminus \partial N$, by Poincaré duality, the class $\varphi^*(\omega)$ is nothing other than the fundamental class of the basis Σ. Hence, the class $j_*\varphi^*(\omega) \in H^n(\mathcal{M})$ is simply the Poincaré dual of τ.

Denoting $h : M_{\widetilde{\gamma}^n} \to T(\widetilde{\gamma}^n)$ the map obtained by identifying $\partial M_{\widetilde{\gamma}^n}$ to a point $\{\infty\}$, the composite map $h \circ F$ sends ∂N to the collapsed point $\{\infty\}$; hence, the map $h \circ F$ can be extended from N to the whole $\mathcal{M}$ just by sending $\mathcal{M} \setminus N$ to $\{\infty\}$. As a result, we obtained a map f from $\mathcal{M}$ to $T(\widetilde{\gamma}^n)$ such that, by the commutative diagram (1.26),

$$f^*(u) = f^*\varphi^*_{SO(n)}(\omega_{SO(n)}) = j_*\varphi^*(\omega) = PD(\tau),$$

where $PD(\tau)$ refers to the Poincaré dual of τ.

Conversely, suppose that there exists a map $f : \mathcal{M} \to T(\widetilde{\gamma}^n)$ such that

$$f^*(u) = PD(\tau).$$

Note that $T(\widetilde{\gamma}^n) \setminus \{\infty\}$ is a smooth manifold, and hence f can be approximated on $\mathcal{M} \setminus f^{-1}(\{\infty\})$ by a smooth map f_1, close (and hence homotopic) to f; moreover, by transversality, we can approximate arbitrarily f_1 by a smooth map F such that $F^{-1}(BSO(n))$ (in fact, a sufficiently large finite dimensional approximation of $BSO(n)$) is an embedded smooth oriented m-dimensional submanifold Σ of $\mathcal{M}$.

Denoting φ^* the Thom isomorphism associated to a normal tubular neighborhood of Σ in $\mathcal{M}$, the class $PD(\tau) = f^*(u) = F^*(u)$ is nothing but $j_*\varphi^*(\omega)$, where ω is the unit class of Σ. Hence, as before, this simply means that $PD(\tau) = PD([\Sigma])$. □

By suitably modifying the proof of Theorem 1.3.6, one sees that the natural analog of Theorem 1.3.6 holds for compact manifolds with boundary.

Theorem 1.3.7 (Relative Thom's representability criterion)**.** *Let $\mathcal{M}$ be a connected smooth oriented compact $m+n$-dimensional manifold with boundary $\partial\mathcal{M}$ and τ a relative homology class $\tau \in H_m(\mathcal{M}, \partial\mathcal{M}, \mathbb{Z})$. Then τ is representable*[9] *by a smooth compact embedded submanifold manifold $\Sigma \subset \mathcal{M}$ with $\partial\Sigma = \Sigma \cap \partial\mathcal{M}$ if and only if there exists a map $f : \mathcal{M} \to T(\widetilde{\gamma}^n)$ which pulls back the Thom class $u \in H^n(T(\widetilde{\gamma}^n), \mathbb{Z})$ to the relative Poincaré dual of τ.*

Proof. Assume that $\tau \in H_m(\mathcal{M}, \partial\mathcal{M}, \mathbb{Z})$ is representable by a smooth compact embedded submanifold $\Sigma \subset \mathcal{M}$ with $\partial\Sigma = \Sigma \cap \partial\mathcal{M}$, that is there exists a smooth embedding $h : \Sigma \to \mathcal{M}$ such that the fundamental class $[\Sigma]$ determined by the orientation of Σ equals τ. Denote the relative Poincaré dual of τ by $x \in H^n(\mathcal{M})$. If we consider $D(\mathcal{M})$ the double manifold of $\mathcal{M}$, then $D(\mathcal{M})$ is a smooth closed oriented manifold. In $D(\mathcal{M})$, we consider the double manifold of Σ, which is a smooth closed oriented embedded submanifold of $D(\mathcal{M})$, whose fundamental class is an absolute homology class in $H_m(D(\mathcal{M}))$; denote

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[9] With the clear modifications in Definition 1.3.5 for $\mathcal{M}, \Sigma$ with boundary and τ a relative homology class.

by $y \in H^n(D(\mathcal{M}))$ its Poincaré dual. By applying Thom's construction of Theorem 1.3.6, we find a map

$$F : D(\mathcal{M}) \to T(\widetilde{\gamma}^n)$$

such that $F^*(u) = y$. Denoting by $i : \mathcal{M} \to D(\mathcal{M})$ the inclusion map, we consider the restriction of F to $\mathcal{M}$, so that we obtain a new map $f : \mathcal{M} \to T(\widetilde{\gamma}^n)$ such that

$$f^*(u) = (F \circ i)^*(u) = i^*(y) = x.$$

Conversely, assume that there exists a map $f : \mathcal{M} \to T(\widetilde{\gamma}^n)$ such that $f^*(u) = x$. Consider the restriction of f to $\partial\mathcal{M}$ and denote it by ∂f. The space $T(\widetilde{\gamma}^n) \setminus \{\infty\}$ is a smooth manifold: hence by [90, Proposition 2.3.4], we can approximate the map of ∂f by means of a new map g_0 agreeing with ∂f on $\partial f^{-1}(U(\infty))$ and which is of class C^∞ on $\partial\mathcal{M} \setminus \partial f^{-1}(U(\infty))$, where U is a small smooth neighborhood of $\{\infty\}$; if the approximation is close enough, then g_0 is homotopic to ∂f. By the standard density argument, *cfr.* [90, Theorem 4.5.6], we can approximate g_0 by a homotopic map g_1 which is smooth and transverse to the zero section[10] of $\widetilde{\gamma}^n$. Since $\partial\mathcal{M}$ has a collar neighborhood, by the homotopy extension property there is a map f_1 defined on $\mathcal{M}$, homotopic to f and such that $\partial f_1 = g_1$. By [90, Proposition 2.3.4 (ii)], we can assume that f_1 is smooth and coincides with g_1 on $\partial\mathcal{M}$. By [90, Proposition 4.5.7], we obtain a final map $\widetilde{f}$ arbitrarily close to f, agreeing with it on $f^{-1}(U(\infty))$, of class C^∞ on $\mathcal{M} \setminus f^{-1}(U(\infty))$ and such that both $\widetilde{f}$ and $\partial\widetilde{f}$ are transverse to the zero section, *cfr.* also [90, Proposition 4.5.10]. By applying the same proof as in [10, Theorem 8.2], the preimage $\widetilde{f}^{-1}(BSO(n))$ is a smooth compact embedded submanifold $\Sigma \subset \mathcal{M}$ of codimension n, with $\partial\Sigma = \Sigma \cap \partial\mathcal{M}$. Hence we can conclude that $x = f^*(u) = \widetilde{f}^*(u)$. □

Remark 1.3.8. A direct way to prove the first implication of Theorem 1.3.7 is the following[11]. Let $\mathcal{M}, \tau, \Sigma$ as above. Choose an extension of the normal bundle ν_Σ of Σ to some smooth n-plane bundle $\xi : E \to U$, where U is an open neighborhood of Σ: this induces an isomorphism $q : \nu_\Sigma \to \xi|_\Sigma$; choose a connection on ξ. Working locally in each chart and then using a partition of unity, it is possible to construct a smooth section s of ξ which vanishes on Σ and such that the covariant derivative of s along Σ induces the isomorphism q. Up to shrinking the open set U, one can assume that s vanishes exactly on Σ and that it is transverse to the zero section. Now it is possible to conclude in analogy with Thom's construction: choose a bundle map $\xi \to \widetilde{\gamma}^n$ and, after having rescaled s, note that it is possible to extend the classifying map with domain U and with values in the corresponding subspace of the total space of $\widetilde{\gamma}^n$ to a map $f : \mathcal{M} \to T(\widetilde{\gamma}^n)$, sending the complement of U to the 0-cell of $T(\widetilde{\gamma}^n)$. It follows that $x = f^*(u)$, where x is the Poincaré dual of τ and $u \in H^n(T(\widetilde{\gamma}^n), \mathbb{Z})$ the Thom's class. This argument shows that Thom's construction can be performed without any need of a tubular neighborhood theorem.

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[10] With the usual approximation to the restriction of $\widetilde{\gamma}^n$ to a sufficiently large compact manifold $\widetilde{G}_n(\mathbb{R}^{n+k})$.

[11] We thank Jacob Lurie for suggesting this argument.

1.4 Cohomology operations and characteristic classes

In this section we collect a few results about cohomology operations, characteristic classes and the cohomology of $BSO(n)$, see also [66, 80, 60, 39]. These results are needed in Chapter 2, in the proof of Lemma 2.3.1, and in Chapter 3, in the proof of Theorem 3.3.1 and Lemma 3.3.4.

1.4.1 Cohomology operations

A *cohomology operation* of type $(\pi, n; \rho, m)$ is a family of functions

$$\theta_X : H^n(X, \pi) \to H^m(X, \rho),$$

one for each space X, satisfying the *naturality* condition $f^*\theta_Y = \theta_X f^*$ for any map $f : X \to Y$. The set of cohomology operations of type $(\pi, n; \rho, m)$ can be denoted by $\mathcal{O}(\pi, n; \rho, m)$. A cohomology operation θ is said to be *additive* if θ_X is a homomorphism for every X. An important result on the classification of these operations in terms of the cohomology of Eilenberg-MacLane spaces is the following.

Theorem 1.4.1. *There is a one-to-one correspondence*

$$\mathcal{O}(\pi, n; \rho, m) \to H^m(K(\pi, n), \rho),$$

given by $\theta \to \theta(\iota_n)$*, where* ι_n *is the fundamental class of* $K(\pi, n)$*.*

The *Steenrod squares* Sq^i, $i \geq 0$, are additive cohomology operations of type $(\mathbb{Z}_2, n; \mathbb{Z}_2, n+i)$,

$$Sq^i : H^n(X, \mathbb{Z}_2) \to H^{n+i}(X, \mathbb{Z}_2),$$

defined for all n and such that

1. $Sq^0 = 1$, the identity;
2. if deg $u = n$, then $Sq^n u = u \smile u$;
3. if $i >$ deg u, then $Sq^i u = 0$;
4. if $u, v \in H^*(X, \mathbb{Z}_2)$, then

$$Sq^k(u \smile v) = \sum_{i+j=k} Sq^i u \smile Sq^j v.$$

This condition is usually called *Cartan formula.*

The above properties characterize the cohomology operations Sq^i. It is then possible to prove existence and uniqueness of such operations, *cfr.* [81]. From the above properties it is possible to derive the following.

5. Sq^1 is the Bockstein homomorphism β induced by the short exact coefficient sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0;$$

6. if $0 < a < 2b$, then

$$Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j,$$

   where the binomial coefficient is taken mod 2. These relations are usually called *Adem relations*;

7. $Sq^i(\sigma(u)) = \sigma(Sq^i u)$, where $\sigma : H^n(X, \mathbb{Z}_2) \to H^{n+1}(\Sigma X, \mathbb{Z}_2)$ is the suspension isomorphism given by reduced cross-product with a generator of $H^1(S^1, \mathbb{Z}_2)$ and ΣX the reduced suspension of X.

This last property says that the Steenrod squares are *stable* operations, *i.e.* they commute with the cohomology suspension operation.

There are analogous additive operations for odd primary coefficients: the *reduced Steenrod p^{th}-powers* $\mathcal{P}^i$ of type $\mathcal{O}(\mathbb{Z}_p, n; \mathbb{Z}_p, n + 2i(p-1))$ for p an odd prime[12] and written

$$\mathcal{P}^i : H^n(X, \mathbb{Z}_p) \to H^{n+2i(p-1)}(X, \mathbb{Z}_p).$$

They satisfy the following properties.

1. $\mathcal{P}^0 = 1$, the identity;
2. if $\deg u = 2i$, then $\mathcal{P}^i u = u \smile \cdots \smile u$, p times;
3. if $2i > \deg u$, then $\mathcal{P}^i u = 0$;
4. if $u, v \in H^*(X, \mathbb{Z}_p)$, then

$$\mathcal{P}^k(u \smile v) = \sum_{i+j=k} \mathcal{P}^i u \smile \mathcal{P}^j v.$$

   This is the *Cartan formula*;

In analogy with Steenrod squares, the above properties characterize the cohomology operations $\mathcal{P}^i$: it is then possible to prove existence and uniqueness of such operations, *cfr.* [81]. From the above properties it is possible to derive the following.

5. $\mathcal{P}^i(\sigma(u)) = \sigma(\mathcal{P}^i u)$, where $\sigma : H^n(X, \mathbb{Z}_p) \to H^{n+1}(\Sigma X, \mathbb{Z}_p)$ is the suspension isomorphism given by reduced cross-product with a generator of $H^1(S^1, \mathbb{Z}_p)$;
6. if $a < pb$, then

$$\mathcal{P}^a \mathcal{P}^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \mathcal{P}^{a+b-j} \mathcal{P}^j.$$

   These are the *Adem relations.*

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[12] Sometimes the notation $\mathcal{P}^i_p$ is used to highlight the coefficient group $\mathbb{Z}_p$.

We note that by the Adem relations, the operation Sq^{2i+1} is the same as the composition $Sq^1Sq^{2i} = \beta Sq^{2i}$, so that Sq^{2i} can be understood as $\mathcal{P}^i$ for $p = 2$.

Another additive cohomology operation for odd primary coefficients is the *Bockstein homomorphism* β_p of type $\mathcal{O}(\mathbb{Z}_p, n; \mathbb{Z}_p, n+1)$ for p an odd prime and written

$$\beta_p : H^n(X, \mathbb{Z}_p) \to H^{n+1}(X, \mathbb{Z}_p),$$

which is obtained from the short exact coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$.

Composition endows the set of stable cohomology operations with a natural ring structure: this ring is known as the *Steenrod algebra* and usually denoted by $\mathcal{A}_p$. The Steenrod algebra $\mathcal{A}_2$ is defined to be the the algebra over $\mathbb{Z}_2$ that is the quotient of the algebra of polynomials in the noncommuting variables Sq^i, $i \geq 1$, by the two-sided ideal generated by the Adem relations. Analogously, the Steenrod algebra $\mathcal{A}_p$ for odd p is defined to be the algebra over $\mathbb{Z}_p$ formed by polynomials in the noncommuting variables $\beta_p, \mathcal{P}^i$, $i \geq 1$, modulo the Adem relations and the relations $\beta_p^2 = 0$. Thus, for every space X, $H^*(X, \mathbb{Z}_p)$ is a module over $\mathcal{A}_p$ for all primes p; the Steenrod algebra is a graded algebra with the elements of degree i being those that map $H^n(X, \mathbb{Z}_p)$ to $H^{n+i}(X, \mathbb{Z}_p)$ for all n. It is possible to prove that $\mathcal{A}_2$ is generated as an algebra by the elements Sq^{2^k} and $\mathcal{A}_p$ for p odd prime is generated by β_p and the elements $\mathcal{P}^{p^k}$.

More generally, let R be a commutative ring with unit. We recall that on the category of free chain complexes C over R and short exact sequences of R modules

$$0 \to G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \to 0 \tag{1.27}$$

there is a functorial connecting homomorphism

$$\beta^* : H^*(C, G'') \to H^*(C, G')$$

of degree 1 and a functorial exact sequence

$$\ldots \to H^n(C, G') \xrightarrow{\varphi^*} H^n(C, G) \xrightarrow{\psi^*} H^n(C, G'') \xrightarrow{\beta^*} H^{n+1}(C, G') \to \ldots,$$

cfr. [80, Theorem 4.5.11]. The connecting homomorphism β^* (sometimes just denoted β when the coefficient group is clear from the context) is called the Bockstein cohomology homomorphism corresponding to the coefficient sequence (1.27).

1.4.2 Characteristic classes

We define *Stiefel-Whitney cohomology classes* of a vector bundle axiomatically. For a proof of existence and uniqueness of cohomology classes satisfying these 4 axioms we refer to [60].

1. To each real vector bundle ξ with base space $B(\xi)$ there corresponds a sequence of cohomology classes

$$w_i(\xi) \in H^i(B(\xi), \mathbb{Z}_2), \quad i = 0, 1, 2, \ldots,$$

called the *Stiefel-Whitney classes* of ξ. The class $w_0(\xi)$ is equal to the unit element

$$1 \in H^0(B(\xi), \mathbb{Z}_2)$$

and $w_i(\xi)$ equals zero for $i > n$ if ξ is an n-plane bundle.

2. (Naturality) If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then

$$w_i(\xi) = f^* w_i(\eta).$$

3. (Whitney Product Theorem) If ξ and η are vector bundles over the same base space, then the Stiefel-Whitney class of a direct sum is the cup product of the summands' classes

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta).$$

For example $w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta)$, $w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi)w_1(\eta) + w_2(\eta)$, and so on (omitting the cup product in the notation).

4. For the canonical line bundle[13] γ_1^1 over $\mathbb{RP}(1)$, the Stiefel-Whitney class $w_1(\gamma_1^1)$ is non-zero.

Proposition 1.4.2. *Stiefel-Whitney classes $w_i(\xi) \in H^i(B)$ can be characterized in terms of the Steenrod operations by showing the following equality:*

$$w_i(\xi) = \Phi^{-1} Sq^i \Phi(1) = \Phi^{-1} Sq^i u,$$

where Φ is the Thom isomorphism and $u \in \widetilde{H}^n(T(\xi), \mathbb{Z}_2)$ is the Thom class of ξ.

This shows that $w_i(\xi)$ is the unique cohomology class in $H^i(B)$ such that $\Phi(w_i(\xi)) = \pi^* w_i(\xi) \smile u$ is equal to $Sq^i\Phi(1) = Sq^i u$, *cfr.* also [85].

All the discussion about Stiefel-Whitney classes works analogously for complex vector bundles, except that for complex vector bundles all the cohomology classes belong to $\mathbb{Z}$ coefficient cohomology: they are called the *Chern classes.* One possible way to define Chern classes is the following.

There is a unique sequence of functions $c_0, c_1, c_2, \cdots$ assigning to each complex vector bundle ω with $E \xrightarrow{\pi} B$ a class $c_i(\omega) \in H^{2i}(B, \mathbb{Z})$, depending only on the isomorphism type of ω, such that:

1. $c_0(\omega)$ equals the unit element $1 \in H^0(B, \mathbb{Z})$ and $c_i(\omega) = 0$ for $i > n$ if ω a complex n-plane bundle;
2. $c_i(f^*(\omega)) = f^*(c_i(\omega))$, for a pull-back bundle $f^*(\omega)$;

---

[13] Let $E(\gamma_n^1)$ be the subset of $\mathbb{RP}(n) \times \mathbb{R}^{n+1}$ consisting of all pairs $(\{\pm x\}, v)$ such that the vector v is a multiple of x. Recall that the *canonical line bundle* over the real projective space $\mathbb{RP}(n)$ is the vector bundle $\pi : E(\gamma_n^1) \to \mathbb{RP}(n)$ defined by $\pi(\{\pm x\}, v) = \{\pm x\}$. Thus each fiber $\pi^{-1}(\{\pm x\})$ can be identified with the line through x and $-x$ in $\mathbb{R}^{n+1}$; each such line is to be given the usual vector space structure.

3. if ω_1 is a complex n-plane bundle and ω_2 a complex m-plane bundle, then
$$c_k(\omega_1 \oplus \omega_2) = \sum_{i=0}^{k} c_i(\omega_1) \smile c_{k-i}(\omega_2);$$
4. for the canonical line bundle ω with $E \xrightarrow{\pi} \mathbb{CP}(1)$, $c_1(E)$ is a generator of $H^2(\mathbb{CP}(1), \mathbb{Z})$ specified in advance.

Besides the evident formal similarity between Stiefel-Whitney and Chern classes there is also the following direct relation: regarding an n-dimensional complex vector bundle ω as a $2n$-dimensional real vector bundle, then $w_{2i+1}(\omega) = 0$ and $w_{2i}(\omega)$ is the image of $c_i(\omega)$ under the coefficient homomorphism $H^{2i}(B(\omega), \mathbb{Z}) \to H^{2i}(B(\omega), \mathbb{Z}_2)$.

We now define the *Pontrjagin classes* $p_i(\xi) \in H^{4i}(B, \mathbb{Z})$ associated to a n-plane bundle ξ in terms of Chern classes. For an n-plane bundle ξ with $E \to B$, its complexification is the complex n-plane bundle $\xi^{\mathbb{C}}$ with $E^{\mathbb{C}} \to B$ obtained from the real n-plane bundle $\xi \oplus \xi$ by defining scalar multiplication by the complex number i in each fiber $\mathbb{R}^n \oplus \mathbb{R}^n$ via the rule $i(x, y) = (-y, x)$. Thus, each fiber $\mathbb{R}^n$ of ξ becomes a fiber $\mathbb{C}^n$ of $\xi^{\mathbb{C}}$. The Pontrjagin class $p_i(\xi)$ is then defined to be
$$p_i(\xi) := (-1)^i c_{2i}(\xi^{\mathbb{C}}) \in H^{4i}(B, \mathbb{Z}).$$

Let ξ be an oriented (real) n-plane bundle $E \xrightarrow{\pi} B$ and consider the restriction homomorphism $\widetilde{H}^*(T(\xi), \mathbb{Z}) \to H^*(E, \mathbb{Z})$ induced by the inclusion and denoted as $y \mapsto y_{|E}$. In particular, applying this homomorphism to the Thom class $u \in \widetilde{H}^n(T(\xi), \mathbb{Z})$, we obtain a new cohomology class
$$u_{|E} \in H^n(E, \mathbb{Z}).$$
Recalling that $H^n(E, \mathbb{Z})$ is canonically isomorphic to $H^n(B, \mathbb{Z})$, we can define the *Euler class* of an n-plane bundle ξ as the cohomology class
$$e(\xi) \in H^n(B, \mathbb{Z})$$
corresponding to $u_{|E}$ under the isomorphism $\pi^* : H^n(B, \mathbb{Z}) \to H^n(E, \mathbb{Z})$.

1.4.3 Cohomology of $BSO(n)$ and $BO(n)$

We describe now the mod p cohomology of the classifying space for oriented n-plane bundles $BSO(n)$.

We recall that the mod 2 cohomology of $BSO(n)$ can be computed as follows, *cfr.* [60, Theorem 12.4].

Proposition 1.4.3. *The cohomology $H^*(BSO(n), \mathbb{Z}_2)$ is a polynomial algebra over $\mathbb{Z}_2$, freely generated by the Stiefel-Whitney classes $w_2(\widetilde{\gamma}^n), \ldots, w_n(\widetilde{\gamma}^n)$.*

The cohomology ring of $BSO(n)$ with coefficients in an odd prime p has the following structure, *cfr.* [60, Theorem 15.9].

Proposition 1.4.4. *If* Λ *is an integral domain containing* $1/2$*, then the cohomology ring*

$$H^*(BSO(2n+1),\Lambda)$$

is a polynomial ring over Λ *generated by the Pontrjagin classes* $p_1(\widetilde{\gamma}^{2n+1}),\dots,p_n(\widetilde{\gamma}^{2n+1})$*. Similarly,*

$$H^*(BSO(2n),\Lambda)$$

is a polynomial ring over Λ *generated by the Pontrjagin classes* $p_1(\widetilde{\gamma}^{2n}),\dots,p_{n-1}(\widetilde{\gamma}^{2n})$ *and the Euler class* $e(\widetilde{\gamma}^{2n})$*.*

That is, for every value of n, even or odd, the ring $H^*(BSO(n),\Lambda)$ is generated by the characteristic classes $p_1,\dots,p_{\lfloor n/2\rfloor}$, and e. These generators are subject only to the relations $e=0$ for n odd and $e^2=p_{n/2}$ for n even.

We turn now the description of the mod p cohomology of the classifying space for n-plane bundles $BO(n)$. In particular, we recall Thom's algebraic computations in the study of the homotopy type of $T(\gamma^n)$, *cfr.* [86, Chapitre II, Section 6], which in turn are based on fundamental contributions of Serre, *cfr.* [76].

Recall that the cohomology $H^*(T(\gamma^n),\mathbb{Z}_2)$ is isomorphic to the ideal J of the algebra $H^*(BO(n),\mathbb{Z}_2)$ generated by the Stiefel-Whitney classes w_n. On the other hand, introducing n variables t_i we obtain a basis of $H^h(BO(n),\mathbb{Z}_2)$ generated by symmetrised monomials

$$\sum(t_1)^{a_1}(t_2)^{a_2}\dots(t_r)^{a_r}, \tag{1.28}$$

where the sum of the integers a_i is equal to h, and the symmetrisation sign $\sum$ means the summation over all *essential permuations* for (1.28)[14]. Hence, one obtains a basis for the dimension $n+h$ of the ideal J by multiplying the elements of the basis in (1.28) by $w_n=t_1\cdot t_2\cdot\dots t_n$, obtaining all symmetrised monomials

$$\sum(t_1)^{a_1+1}(t_2)^{a_2+1}\dots(t_r)^{a_r+1}t_{r+1}\dots t_n. \tag{1.29}$$

Indeed, any essential permutation for the monomial (1.28) is essential for (1.29), and vice versa.

Let P be an arbitrary polynomial in the variables t_j. A variable t_j is a called *dyadic* for the polynomial P if the exponent of this variable in terms of the polynomial P is either zero or 2^i (with $i=0$ included). Any dyadic variable t_j for the polynomial P is dyadic for Sq^iP as well, where Sq^i denotes the Steenrod squares. By a *non-dyadic factor* of the monomial $(t_1)^{a_1}(t_2)^{a_2}\dots(t_r)^{a_r}$ we mean the submonomial consisting of all non-dyadic variables; denote the number of these variables by u, and by v their total degree. We define a preorder[15] $\succsim$ on the set of monomials in (t_j) as follows: given two monomials x,y we say that $x\succsim y$ if $u(x)>u(y)$ or if $u(x)=u(y)$ and $v(x)<v(y)$. For any number $h\le n$ consider the classes

$$x^h_\omega=\sum(t_1)^{a_1+1}(t_2)^{a_2+1}\dots(t_r)^{a_r+1}\cdot t_{r+1}\dots t_n, \tag{1.30}$$

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[14] *i.e.* over representatives of classes of the symmetric group of n variables modulo the subgroup of permutations fixing the monomial (1.28).

[15] A binary relation satisfying reflexivity and transitivity only.

where $\omega = \{a_1, a_2, \ldots, a_r\}$ is an arbitrary decomposition of h into summands, with no summand of type $2^i - 1$ (non-dyadic decomposition of h); we denote the number of such decompositions by $d(h)$.

For any dimension $i \leq h$, consider the classes

$$x^i_{\omega_i},\ Sq^1 x^{i-1}_{\omega_{i-1}},\ Sq^2 x^{i-2}_{\omega_{i-2}},\ \ldots,\ Sq^{I_h} x^h_{\omega_h},\ \ldots,\ Sq^I w_k, \tag{1.31}$$

where Sq^{I_h} is an admissible sequence of total degree $i - h$, and ω_h is a non-dyadic decomposition of h as in (1.30); all classes in (1.31) are in fact linearly independent. In particular, it is possible to prove that the cohomology classes in (1.31) form a basis of $H^{n+i}(T(\gamma^n))$. We associate to each class x^h_ω a map

$$H_\omega : T(\gamma^n) \to K(\mathbb{Z}_2, n+h)$$

such that $H^*_\omega(\iota) = x^h_\omega$, where ι is the fundamental class of $K(\mathbb{Z}_2, n+h)$. The maps H_ω define a map H from $T(\gamma^n)$ into the following product of Eilenberg-MacLane spaces

$$Y := K(\mathbb{Z}_2, n) \times K(\mathbb{Z}_2, n+2) \times \ldots \times (K(\mathbb{Z}_2, n+h))^{d(h)} \times \ldots \times (K(\mathbb{Z}_2, 2n))^{d(n)}.$$

Since the classes (1.31) form a basis of $H^{n+i}(T(\gamma^n))$, then the homomorphism H^* induced by H is an isomorphism from $H^{n+i}(Y, \mathbb{Z}_2)$ to $H^{n+i}(T(\gamma^n))$ for all $i \leq n$. In mod p coefficients, with $p > 2$, the cohomology of Y is trivial and the cohomology of $T(\gamma^n)$ is trivial in dimensions strictly less than $2n$. Hence, H^* is again an isomorphism in dimensions strictly less than $2n$ and a monomorphism in dimension $2n$. Thus, for $T(\gamma^n)$ and Y one can apply Theorem 1.3.3, obtaining that there exists an inverse map g from the $2n$-skeleton of Y to $T(\gamma^n)$ such that $g \circ H$ is homotopic to the identity on the $2n-1$-skeleton of $T(\gamma^n)$. In particular, one obtains the following, *cfr.* Theorem 3.3.1.

Theorem 1.4.5. *There exists a map $H : T(\gamma^n) \to Y$ which is a $2n$-equivalence, for all positive integers n.*

Chapter 2

Smooth approximation for integral cycles

The main goal of Chapter 2 is to present the smooth approximation theorem for integral cycle, that we recall here[1].

Theorem 2.0.1 (Optimal smooth approximation)**.** *Let $\mathcal{M}$ be a connected smooth closed oriented Riemannian manifold of dimension $m+n$. Let $\varepsilon > 0$, τ be a fixed element of the m-dimensional integral homology group $H_m(\mathcal{M}, \mathbb{Z})$, and T be an integral cycle representing τ. Then, there is a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and an oriented m-dimensional smooth submanifold Σ of $\mathcal{M} \backslash \mathcal{K}^{m-5}$ (where $\mathcal{K}^{m-5}$ denotes the $m-5$-skeleton of $\mathcal{K}$) with the following properties.*

1. *The m-dimensional volume of Σ does not exceed the mass of T by more than ε, that is $\mathcal{H}^m(\Sigma) \leq \mathbb{M}(T) + \varepsilon$.*
2. *The current $[\![\Sigma]\!]$ is an integral cycle homologous to T and there is an integral $m+1$-dimensional current S in $\mathcal{M}$ such that $\partial S = [\![\Sigma]\!] - T$ and $\mathbb{M}(S) < \varepsilon$.*
3. *If τ admits a smooth representative, then Σ can be chosen to be a smooth submanifold of $\mathcal{M}$.*

Remark 2.0.2. The codimension 5 construction in Theorem 2.0.1 is optimal, as shown by the *innately singular* homology class discovered by Thom, see Theorem 2.4.3.

We recall here the underlying assumptions for all the theorems of Chapter 2, unless otherwise stated.

2.0.3. Assumption. $n, m \in \mathbb{N} \backslash \{0\}$ are arbitrary positive integers, $\mathcal{M}$ is a connected smooth oriented closed Riemannian manifold of dimension $m+n$, τ is an element of the m-dimensional integral homology group $H_m(\mathcal{M}, \mathbb{Z})$ and T is an integral current (hence a cycle) representing τ.

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[1] This chapter is based on [8].

When dealing with smooth triangulations of $\mathcal{M}$ we will tacitly assume to have fixed some simplicial complex $\mathcal{K}$ together with a piecewise smooth map $t : |\mathcal{K}| \to \mathcal{M}$. In order to keep our notation simpler, with a slight abuse we will in fact mostly avoid referring to the map t and we will use directly $\mathcal{K}$ also for the smooth triangulation of $\mathcal{M}$; thus, a simplex of the triangulation $\mathcal{K}$ will mean the t-image of a simplex of $|\mathcal{K}|$. $\mathcal{K}^j$ will denote the j-dimensional skeleton of $\mathcal{K}$, i.e. the union of all j-dimensional simplices of $\mathcal{K}$. Moreover, the letter Σ will be reserved to denote either smooth oriented m-dimensional closed embedded submanifolds of $\mathcal{M}$ or smooth oriented m-dimensional embedded submanifolds of $\mathcal{M} \setminus \mathcal{K}^j$ (for some integer j) whose topological closure is contained in $\mathcal{K}^j$.

2.0.1 Consequences of the main theorem

The first consequence of Theorem 2.0.1 is the absence of the *Lavrentiev gap phenomenon* for the homological Plateau problem.

Theorem 2.0.4 (Absence of Lavrentiev gaps). *Let $\mathcal{M}, \tau, T$ and Σ as in Assumption 2.0.3, and define the following quantities:*

$\mathbb{M}_T := \min\{\mathbb{M}(T) : T \in \mathcal{Z}_m(\mathcal{M}) \cap \tau\}$,
$\mathbb{M}_P := \inf\{\mathbb{M}(P) : P \in \mathcal{Z}_{m,Lip}(\mathcal{M}) \cap \tau\}$,
$\mathbb{M}_\Sigma := \inf\{Vol^m(\Sigma) : [\![\Sigma]\!] \in \tau \text{ and } \Sigma \text{ is smooth in } \mathcal{M} \setminus \mathcal{K}^{m-5} \text{ for some triangulation } \mathcal{K}\}$,
$\mathbb{M}_{\mathrm{Reg}} := \inf\{Vol^m(\Sigma) : [\![\Sigma]\!] \in \tau \text{ and } \Sigma \text{ is smooth in } \mathcal{M}\}$.

Then, $\mathbb{M}_T = \mathbb{M}_P = \mathbb{M}_\Sigma$ and, moreover, $\mathbb{M}_T = \mathbb{M}_{\mathrm{Reg}}$ when τ is representable by a smooth submanifold.

Remark 2.0.5. Note that any class τ is representable by a smooth submanifold when $n \in \{1, 2\}$ or when $m \in \{1, 2, 3, 4, 5, 6\}$, see Lemma 2.3.1 and Remark 2.3.2. In these cases, Theorem 2.0.1 implies that singularities of integral cycles can be *weakly resolved*, in the sense that they can be approximated (with respect to the flat topology) by smooth embedded submanifolds.

A necessary condition for τ to be representable by a smooth submanifold can often be expressed in terms of the vanishing of some suitable *obstruction*, which is represented by a cohomology operation; in particular, denoting x the Poincaré dual of τ and by p an odd prime, a necessary (and, in some particular dimensions, also sufficient) condition for an integral homology class τ to be representable by a smooth submanifold is that all $St_p^{2r(p-1)+1}x$ are null, see [86, Théorème II.20]. We recall that, following Thom's notation for the Bockstein reduced p^{th} powers reduction mod p, $St_p^{2r(p-1)+1}$ represent (up to a sign) the following cohomology operations:

$$St_p^{2r(p-1)+1} = \beta^* \circ \mathcal{P}_p^r \circ \theta_p : H^*(X, \mathbb{Z}) \to H^{*+2r(p-1)+1}(X, \mathbb{Z}),$$

where $\mathcal{P}_p^r$ is the reduced Steenrod p^{th} power, $\beta^* : H^*(X, \mathbb{Z}_p) \to H^{*+1}(X, \mathbb{Z})$ is the Bockstein associated to the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$$

and θ_p the reduction mod p such that $\beta_p = \theta_p \circ \beta^*$, with $\beta_p : H^*(X, \mathbb{Z}_p) \to H^{*+1}(X, \mathbb{Z}_p)$ the Bockstein associated to $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$.

This necessary condition is a consequence of the fact that the cohomology for odd primes of the oriented Grassmannians is concentrated in dimensions which are multiples of 4.

As a simple corollary of Theorem 2.0.1 we further deduce the following approximation theorem with cycles of prescribed singularities.

Theorem 2.0.6 (Approximation by cycles with prescribed singular sets)**.** *Let $\mathcal{M}, \tau$ and T be as in Assumption 2.0.3. Then there is a sequence of smooth triangulations $\mathcal{K}_j$ of $\mathcal{M}$ and a sequence of smooth embedded oriented m-dimensional submanifolds $(\Sigma_j)_j$ in $\mathcal{M} \setminus \mathcal{K}_j^{m-5}$ such that*

(a) $[\![\Sigma_j]\!] \to T$ *in the sense of currents,*

(b) $\lim_{j\to\infty} \mathcal{H}^m(\Sigma_j) = \mathbb{M}(T)$,

(c) $\partial[\![\Sigma_j]\!] = 0$ *and* $[\![\Sigma_j]\!]$ *is in the same homology class as T.*

In fact conclusion (c) is a simple consequence of the Federer flatness criterion: since $\partial[\![\Sigma_j]\!]$ is a flat current supported in a set which has $\mathcal{H}^{m-1}$-zero measure, it must be 0, see [43, 4.1.20]; therefore it also follows from the convergence to T that $[\![\Sigma_j]\!]$ is in the same homology class as T for every j sufficiently large.

2.0.2 Overview of the proof

The main idea of our study is to combine Federer and Fleming's theory of integral currents with tools and techniques from cobordism and homotopy theory.

The proof of Theorem 2.0.1 can be *grosso modo* described as follows. Starting from an m-dimensional integral cycle T in a homology class τ of $\mathcal{M}$, we first develop a delicate approximation theorem by means of a cycle P' which is a smooth submanifold outside of a (small) δ-neighborhood B_δ of the $m-2$-skeleton of some triangulation of $\mathcal{M}$, *cfr.* Proposition 2.2.1. In particular, a subset of the smooth part of P' is a compact smooth submanifold with boundary embedded in a compact manifold with boundary, which we denote by Ω (ideally we would define Ω as $\mathcal{M} \setminus B_\delta$, but the latter does not have a smooth boundary: we will get around this technical obstruction by a standard regularization procedure, *cfr.* Section 3.1): this object represents a relative homology class in $H_m(\Omega, \partial\Omega, \mathbb{Z})$; this will induce, by Theorem 1.3.7 a map

$$F : \Omega \to T(\tilde{\gamma}^n)$$

with values in the Thom space of the universal oriented n-plane bundle and such that the pull-back of the Thom class equals the Poincaré dual of τ, when restricted to Ω. This is known as the (relative) *Thom construction.*

Then, denoting by Q the complement of a small neighborhood U_δ of the $m-5$-skeleton of the triangulation of $\mathcal{M}$, we note that Q has the homotopy type of an $n+4$-dimensional skeleton of $\mathcal{M}$, *cfr.* Lemma 1.2.7. Thus, we exploit the $n+4$-equivalence between $T(\widetilde{\gamma}^n)$ and the Eilenberg-MacLane space $K(\mathbb{Z}, n)$, *cfr.* Lemma 2.3.1, to prove that the restriction of the Poincaré dual of τ to Q admits a lift

$$f : Q \to T(\widetilde{\gamma}^n) \tag{2.1}$$

pulling back the Thom class to itself. Applying Theorem 1.3.7 and after some technicalities, this provides an integral cycle R homologous to τ which is a closed smooth embedded submanifold with singularities all contained in the $m-5$-dimensional skeleton $\mathcal{K}^{m-5}$ of $\mathcal{M}$.

Since, by Lemma 1.2.7, Ω has the homotopy type of an $n+1$-dimensional complex, we observe that homotopy classes of maps defined on Ω and with values in $T(\widetilde{\gamma}^n)$ are in one-to-one correspondence with those with values in $K(\mathbb{Z}, n)$, *cfr.* Corollary 1.3.2. This allows us to conclude that the smooth part of P' coincides, up to a homotopy, with the smooth part of R restricted to Ω.

The conclusion then follows from a technical geometric measure theory construction, *cfr.* Proposition 2.2.6, saying that if two m-dimensional integral cycles P' and R agree outside of a sufficiently small neighborhood of the $m-2$-skeleton of the triangulation of $\mathcal{M}$, then we can find a smooth deformation R' of R which is almost coinciding with R and with mass close to the mass of P'. This provides the desired approximation R' of Theorem 2.0.1, satisfying (1) and (2).

Finally, part (3) of Theorem 2.0.1 is proved following the same lines: under the additional assumption that τ is representable by a smooth submanifold we immediately obtain a map

$$g : \mathcal{M} \to T(\widetilde{\gamma}^n)$$

which pulls-back the Thom class to the Poincaré dual of τ; the analogous construction can thus be performed just by replacing the map f in (3.1) with g.

In Section 3.1 we collect some technical preliminary lemmas about neighborhoods of skeleta and maps associated to them. In Section 3.2 we will prove the two main technical propositions from geometric measure theory: Proposition 2.2.1 and Proposition 2.2.6; an appendix with some elementary facts about triangulations and simplicial decompositions is listed in Section 1.2. Section 2.3 is dedicated to the proof of Theorem 2.0.1 and Section 3.4 shows the optimality of the construction. We add at the end another brief appendix, see Section 1.4, recalling some introductory results about cohomology operations and characteristic classes, useful in the proof of Lemma 2.3.1.

2.1 Regular neighborhoods and maps

We also refer to [50] for the theory of smooth regular neighborhoods.

2.1.1 Neighborhoods of skeleta

Having fixed a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and a skeleton $\mathcal{K}^j$, we denote by $B_\delta(\mathcal{K}^j)$ the usual metric neighborhoods of the skeleton, *i.e.* the sets of points p with $\text{dist}(p, \mathcal{K}^j) < \delta$.

In many instances we will use these neighborhoods for our considerations. However, for some important considerations we will in fact need a suitable variant, which will be denoted by $V_\delta(\mathcal{K}^j)$ and are defined in the following way. We first fix a (sufficiently large) constant C_0 which will depend on the triangulation $\mathcal{K}$, subdivide the simplices forming $\mathcal{K}^j$ into $\mathcal{S}_0 \cup \ldots \cup \mathcal{S}_j$ according to their dimension ($\mathcal{S}_i$ being the collection of simplices of dimension i) and hence set

$$V_\delta(\mathcal{K}^j) := \bigcup_{i=0}^{j} \bigcup_{\sigma \in \mathcal{S}_i} B_{C_0^{-i}\delta}(\sigma) \tag{2.2}$$

where

$$B_{C_0^{-i}\delta}(\sigma) = \{p : \operatorname{dist}(p, \sigma) < C_0^{-i}\delta\}\,.$$

The following is an elementary consequence of our definition.

Lemma 2.1.1. *For every smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and every $j \leq m+n-1$ there is a choice of $\bar\delta > 0$ (sufficiently small) and of C_0 sufficiently large such that the following holds. First of all, $\mathcal{M} \setminus V_{\bar\delta}(\mathcal{K}^j)$ is a deformation retract of $\mathcal{M} \setminus \mathcal{K}^j$.*

Moreover, for every point $p \in \partial V_{\bar\delta}(\mathcal{K}^j)$ there is at most one σ in each $\mathcal{S}_i$ (with $0 \leq i \leq j$) such that $p \in \partial B_{C_0^{-i}\bar\delta}(\sigma)$. In particular, there is a neighborhood of U of p, an integer $\bar j \in \{1, \ldots, j\}$ and a diffeomorphism $\phi : U \to B_1 \subset \mathbb{R}^{m+n}$ (the unit ball in $\mathbb{R}^{m+n}$) such that

$$\phi(U \setminus V_{\bar\delta}(\mathcal{K}^j)) = \{(x_1, \ldots, x_{m+n}) : x_i > 0 \text{ for } 1 \leq i \leq \bar j\}\,.$$

Note in particular that the boundary of $V_{\bar\delta}(\mathcal{K}^j)$ is a Lipschitz submanifold. This is, however, not suitable for our purposes; we need an appropriate regularization of it which, given the explicit local description of Lemma 2.1.1, is a consequence of a standard regularization procedure.

Lemma 2.1.2. *Let $\mathcal{K}$ be a smooth triangulation of $\mathcal{M}$, let $\bar\delta$ and C_0 be given by Lemma 2.1.1, and fix any pair of positive numbers $\delta' < \delta < \bar\delta$. Then there is a neighborhood $U_\delta(\mathcal{K}^j)$ of $\mathcal{K}^j$ with the following properties:*

- *$V_\delta(\mathcal{K}^j) \supset U_\delta(\mathcal{K}^j) \supset V_{\delta'}(\mathcal{K}^j)$;*
- *The boundary of $U_\delta(\mathcal{K}^j)$ is smooth;*
- *$\mathcal{M} \setminus U_\delta(\mathcal{K}^j)$ is a deformation retract of $\mathcal{M} \setminus V_{\delta'}(\mathcal{K}^j)$;*
- *There is a smooth tubular neighborhood $\mathcal{C}$ of $\partial U_\delta(\mathcal{K}^j)$ in $\mathcal{M}$ containing $\partial V_{\delta'}(\mathcal{K}^j)$.*

2.1.2 Squeezing maps

We will now build some special maps related to the neighborhoods B_δ and V_δ.

Lemma 2.1.3. *Let $\mathcal{M}$ be as in Assumption 2.0.3 and $\mathcal{K}$ a smooth triangulation of $\mathcal{M}$. For every $\varepsilon_a > 0$ and $\eta_a > 0$ there is a positive number $\delta_a < \eta_a$ with the following property. If $\gamma \in]0,1]$ is an arbitrary number, then there is a diffeomorphism Φ such that:*

1. *Φ is isotopic to the identity and it coincides with the identity on $\mathcal{M} \setminus B_{\eta_a}(\mathcal{K}^k)$;*
2. *$\mathrm{Lip}(\Phi) \leq 1 + \varepsilon_a$;*
3. *For every point $p \in B_{\delta_a}(\mathcal{K}^k)$ there is an orthonormal frame $e_1, \ldots, e_{m+n}$ such that*

$$|d\Phi_p(e_i)| \leq 1 + \varepsilon_a \qquad \forall i \in \{1, \ldots, k\}, \tag{2.3}$$
$$|d\Phi_p(e_j)| \leq \gamma \qquad \forall j \in \{k+1, \ldots, m+n\}\,. \tag{2.4}$$

Before coming to the proof of the latter lemma, we remark that a simple modification of the arguments gives the following one, which is in fact much simpler.

Lemma 2.1.4. *Let $\mathcal{M}$ be as in Assumption 2.0.3, $\mathcal{K}$ a smooth triangulation, $j \in \{0, \ldots, m+n-1\}$. If C_0 and $\bar{\delta}^{-1}$ in Lemma 2.1.1 are chosen sufficiently large, then for every $\delta_b > \delta'_b > 0$ there is a Lipschitz map $\Phi : \mathcal{M} \to \mathcal{M}$ with the following properties:*

- *Φ maps $V_{\delta'_b}(\mathcal{K}^j)$ into $\mathcal{K}^j$;*
- *Φ is a smooth diffeomorphism between $\mathcal{M} \setminus \Phi^{-1}(\mathcal{K}^j)$ and $\mathcal{M} \setminus \mathcal{K}^j$;*
- *$\Phi(p) = p$ for every $p \notin V_{\delta_b}(\mathcal{K}^j)$.*

Proof of Lemma 3.1.4. The proof is by induction over k.

We start with the first step, where $k = 0$. We enumerate the 0-skeleton as the points $p_1, \ldots, p_N$, we let d be the minimum of $\mathrm{dist}(p_i, p_j)$ and r_0 a radius which is smaller than the minimum of the injectivity radii for the exponential maps centered at p_i and whose choice will be specified later. We then set $\eta := \min\{\eta_a, \frac{d}{4}, \frac{r_0}{2}\}$. We fix $\delta_a < \eta$ and $\mu \geq 0$ (whose choice will be specified later) and let $\varphi : [0, \infty[\to [0, \infty[$ be the following piecewise linear increasing function:

$$\varphi(t) = \begin{cases} \mu t & \text{if } 0 \leq t \leq 2\delta_a, \\ \frac{\eta - \delta_a - 2\mu\delta_a}{\eta - 3\delta_a}(t - 2\delta_a) + 2\mu\delta_a & \text{if } 2\delta_a \leq t \leq \eta - \delta_a, \\ t & \text{if } t \geq \eta - \delta_a. \end{cases}$$

Observe that $0 \leq \varphi' \equiv \mu$ on $[0, 2\delta_a]$ while $0 \leq \varphi' \leq \frac{\eta}{\eta - 3\delta_a}$ everywhere else, and the latter number can be made arbitrarily close to 1 depending only on the ratio $\frac{\delta_a}{\eta}$, but independently of μ. We then regularize φ by convolution with a standard smooth nonnegative kernel, hence getting a smooth diffeomorphism of the real half line, which we denote by ψ. Its derivative ψ' will enjoy the same global upper bound and, by choosing the kernel suitably, we can ensure $\psi(t) = \mu t$ on the interval $[0, \delta_a]$ and $\psi(t) = t$ on $[\eta - \frac{\delta_a}{2}, \infty[$. We next define the map $\Psi_\mu(x) := \psi(|x|)\frac{x}{|x|}$ from $\mathbb{R}^{m+n}$ onto itself. Notice that $\mathrm{Lip}(\Psi_\mu)$ can be made arbitrarily close to 1 choosing the ratio $\frac{\delta_a}{\eta}$ very small, while clearly $\Psi_\mu(x) = \mu x$ in the ball of radius B_{δ_a}. We are now ready to define the map Φ. $\Phi(p) := p$ for $p \notin \mathcal{M} \setminus B_\eta(\mathcal{K}^0)$. Next $B_\eta(\mathcal{K}^0)$ is the disjoint union of $B_\eta(p_i)$. On each such ball we consider the exponential map $\exp_{p_i}$ and Φ is defined to be

$$\Phi := \exp_{p_i} \circ \Psi_\mu \circ \exp_{p_i}^{-1} .$$

By choosing the radius r_0 sufficiently small we can get the Lipschitz constant of the exponential maps and of their inverses arbitrarily close to 1 on the domains of our interest. So, if we fix some constant $\tilde{\varepsilon}$, after choosing $\frac{\delta_a}{\eta}$ and r_0 sufficiently small, we can easily achieve

$$\mathrm{Lip}\,(\Phi) \leq (1+\tilde{\varepsilon})^3$$

and

$$\mathrm{Lip}\,(\Phi_{|B_{\delta_a}(\mathcal{K}^0)}) \leq (1+\tilde{\varepsilon})^2\mu$$

In particular we first choose $\tilde{\varepsilon}$ so that $(1+\tilde{\varepsilon})^3 \leq 1+\varepsilon_a$ and we then choose μ so that $(1+\tilde{\varepsilon})^2\mu \leq \gamma$. Note that the choice of δ_a is then independent of γ.

We next wish to tackle the induction step, so we assume that the statement of the proposition holds for $k-1$ in place of k. We then fix ε_a, η_a, and γ. We apply the proposition in case $k-1$ with the same η_a but with γ_0 and ε_0 in place of γ and ε_a and get the corresponding δ_a, which we denote by δ_0. The choices of ε_0 and γ_0 will be specified later, but we anticipate that ε_0 will only depend on ε_a among all these parameters. We therefore then have a corresponding map, which we denote by Φ_{k-1}, with the property that $\mathrm{Lip}(\Phi_{k-1}) \leq 1+\varepsilon_0$, which is the identity outside of $B_{\eta_a}(\mathcal{K}^{k-1})$ and which in turn satisfies all the requirements of the lemma for the other parameters.

Next we consider δ_a, η, and μ, whose choices will be specified later. For each k-dimensional face F in the k-dimensional skeleton, we consider

$$F' := F \setminus B_{\delta_0/16}(\mathcal{K}^{k-1})$$

and we choose η small enough so that the normal neighborhoods $N_\eta(F')$ are pairwise disjoint and all diffeomorphic to $F' \times B^{m+n-k}_\eta$, where B^{m+n-k}_η denotes the $m+n-k$ dimensional ball of radius η and centered at 0 in $\mathbb{R}^{m+n-k}$. We then parametrize $N_\eta(F')$ as (x,y), where $x \in F'$ and $y \in B^{m+n-k}_\eta$.

We next introduce a function of two variables defined in the following way. First of all we define $\bar{\mu} : [0,\infty[\to [0,\infty[$ as

$$\bar{\mu}(s) = \begin{cases} 1 & \text{if } s \leq \frac{\delta_0}{4}, \\ 1 - 2\frac{(1-\mu)}{\delta_0}(s - \frac{\delta_0}{4}) & \text{if } \frac{\delta_0}{4} \leq s \leq \frac{3\delta_0}{4}, \\ \mu & \text{if } s \geq \frac{3\delta_0}{4}. \end{cases}$$

Hence we set

$$\varphi(s,t) = \begin{cases} t\bar{\mu}(s) & \text{if } t \leq 2\delta_a, \\ \frac{\eta-\delta_a-2\bar{\mu}(s)\delta_a}{\eta-3\delta_a}(t-2\delta_a) + 2\bar{\mu}(s)\delta_a & \text{if } 2\delta_a \leq t \leq \eta-\delta_a, \\ t & \text{if } t \geq \eta-\delta_a. \end{cases}$$

Note that $\partial_t\varphi \equiv \mu$ if $t \leq 2\delta_a$ and $s \geq \frac{3\delta_0}{4}$ while $\partial_t\varphi \equiv 1$ if $t \geq \eta - \delta_a$ or for every t if $s \leq \frac{\delta_0}{4}$. On the other hand we have the upper bound

$$\left|\frac{\partial\varphi}{\partial t}\right| \leq \frac{\eta}{\eta - 3\delta_a}\,,$$

where the right hand side can be made arbitrarily close to 1 by choosing $\frac{\delta_a}{\eta}$ small. Likewise we have the upper bound

$$\left|\frac{\partial\varphi}{\partial s}\right| \leq \frac{4\eta}{\delta_0}$$

and the right hand side can be made arbitrarily small by choosing $\frac{\eta}{\delta_0}$ small.

These two requirements (namely $\frac{\eta}{\eta-3\delta_a}$ being sufficiently close to 1 and $\frac{4\eta}{\delta_0}$ being sufficiently close to 0) will only depend on ε_a, ε_0 and the geometry of the triangulation, while ε_0 will only depend on ε_a and the geometry of the triangulation, so that ultimately δ_a will depend in fact only on ε_a, η_a, $\mathcal{M}$ and $\mathcal{K}$.

With a similar regularization procedure as the one outlined above, we can smooth φ to a function ψ. Then we also suitably smooth the distance function $p \mapsto \operatorname{dist}(p, \mathcal{K}^{k-1})$ in $B_{\delta_0}(\mathcal{K}^{k-1}) \setminus B_{\frac{\delta_0}{8}}(\mathcal{K}^{k-1})$ to a function d. We then define a function $\Phi_k : N_\eta(F') \to N_\eta(F')$ by setting

$$\Phi_k(x, y) := \left(x, \psi(d(x,0), |y|)\tfrac{y}{|y|}\right)\,.$$

Being the $N_\eta(F')$ pairwise disjoint, this define a function on the union of them. Finally, since the function is the identity at the boundary of this domain, we can extend it to all of $\mathcal{M}$ by being the identity. We then claim that the function $\Phi := \Phi_{k-1} \circ \Phi_k$ in fact satisfies all the requirements upon choosing our parameters correctly. The bound on the Lipschitz constant simply follows by multiplying the bounds of the Lipschitz constants of the two maps and choosing the parameters correctly. The fact that the map is equal to the identity outside of $B_{\eta_a}(\mathcal{K}^k)$ follows from choosing $\eta \leq \eta_a$. It remains to check the third claim. We will check that the claim holds at every p such that $\operatorname{dist}(p, \mathcal{K}^k) \leq \delta_a$ but $\operatorname{dist}(p, \mathcal{K}^{k-1}) \geq 3\frac{\delta_0}{4}$ and at every p such that $\operatorname{dist}(p, \mathcal{K}^{k-1}) \leq 3\frac{\delta_0}{4}$. The union of the two sets clearly contains $B_{\delta_a}(\mathcal{K}^k)$, and this completes the proof. First of all observe that if p is in the first set, then by construction there are $m + n - k$ orthonormal vectors $e_{k+1}, \ldots, e_{m+n}$ with the property that

$$|d\Phi_k|_p(e_i)| \leq 2\mu\,.$$

On the other hand because of the Lipschitz bound on Φ_{k-1} we immediately conclude that

$$|d\Phi|_p(e_i)| \leq 2(1 + \varepsilon_0)\mu$$

and thus choosing μ appropriately we can guarantee $2(1+\varepsilon_0)\mu \leq \gamma$. Completing the e_i's to an orthonormal basis we get the desired estimate on the other vectors simply using the global Lipschitz estimate on Φ.

Consider now a $p \in B_{3\delta_0/4}(\mathcal{K}^{k-1})$. By construction $q = \Phi_{k-1}(p)$ belongs to $B_{\delta_0}(\mathcal{K}^{k-1})$. There are therefore $m+n+1-k$ orthonormal vectors $v_k, \ldots, v_{m+n}$ with the property that

$$|d\Phi_{k-1}|_q(v_i)| \leq \gamma_0 \,.$$

Consider the vector space V spanned by these vectors. Then we have the estimate

$$|d\Phi_{k-1}(v)| \leq \sqrt{m+n+1-k}\,\gamma_0 |v|$$

for every such v. Because Φ is a diffeomorphism, there is an $m+n-k$-dimensional subspace W of $T_p\mathcal{M}$ which $d\Phi|_p$ maps onto V. If we choose an orthonormal base $e_{k+1}, \ldots, e_{m+n}$ of the latter, we can then estimate

$$|d\Phi_p(e_i)| \leq \sqrt{m+n-k}\,\gamma_0 \operatorname{Lip}(\Phi_k) \,.$$

We then conclude by choosing $\gamma_0 \leq \frac{\gamma}{2\sqrt{m+n-k}}$, given that our constructions certainly implies $\operatorname{Lip}(\Phi_k) \leq 1 + \varepsilon_a \leq 2$. □

2.2 Geometric measure theory propositions

The proof of our main theorems will make use of two technical propositions from geometric measure theory. The first one is an upgraded version of the Federer and Fleming's strong polyhedral approximation theorem, where the approximands are smooth embedded submanifolds outsode of the $m-2$-skeleton $\mathcal{K}^{m-2}$ of a smooth triangulation of $\mathcal{M}$.

2.2.1 Codimension 2 smooth approximation

Proposition 2.2.1. *Let $\mathcal{M}$ be as in Assumption 2.0.3 and T be an integral m-dimensional cycle in $\mathcal{M}$. For every fixed $\varepsilon_c > 0$ there is an integral cycle P homologous to T and a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ with the following properties:*

(a_0) $\mathbb{M}(P) \leq (1+\varepsilon_c)\mathbb{M}(T)$;

(b_0) $\mathbb{F}(T-P) \leq \varepsilon_c$;

(c_0) $\operatorname{spt}(P) \subset \{x : \operatorname{dist}(x, \operatorname{spt}(T)) \leq \varepsilon_c\}$;

(d_0) $P = \sum_{F\in\mathcal{F}^m} \beta_F [\![F]\!]$, *where $\beta_F \in \mathbb{Z}$ and $\mathcal{F}^m$ is the collection of m-dimensional cells of $\mathcal{K}$ with an appropriately chosen orientation.*

Furthermore, for a sufficiently small $\delta_c' > 0$ and any $\delta_c < \delta_c'$ we can find a second integral cycle P' homologous to P with the following properties:

(a) $\mathbb{M}(P') \leq (1+3\varepsilon_c)\mathbb{M}(T)$ *and* $\mathbb{F}(T-P') \leq 3\varepsilon_c$;

(b) $\operatorname{spt}(P') \subset \{x : \operatorname{dist}(x, \operatorname{spt}(T)) \leq 3\varepsilon_c\}$;

(c) $\|P'\|(B_{\delta_c'}(\mathcal{K}^{m-2})) \leq 3\varepsilon_c$;

(d) $P' \llcorner \mathcal{M} \backslash B_{\delta_c}(\mathcal{K}^{m-2}) = [\![\Gamma]\!]$ *for some smooth oriented submanifold* Γ *of* $\mathcal{M} \backslash B_{\delta_c}(\mathcal{K}^{m-2})$ *without boundary in* $\mathcal{M} \setminus B_{\delta_c}(\mathcal{K}^{m-2})$.

Remark 2.2.2. A routine modification of the arguments used to prove Proposition 2.2.1 implies in fact that the cycle P' can be chosen so that its singularities are all *contained* in $\mathcal{K}^{m-2}$. While this is a much weaker result than the one achieved by our main theorem of constructing an integral cycle with singularities all contained in $\mathcal{K}^{m-5}$, its proof can however be completed without recurring to any sophisticated topological fact.

2.2.2 Proof of Proposition 2.2.1: first approximation

In this section we show the existence of the first approximating cycle P as in Proposition 2.2.1. It is quite possible that the existence of a P with the desired properties is already proved in the existing literature; nevertheless, we have not been able to find a precise reference for our purposes and therefore we provide a proof. Instrumental to our argument is to consider the ambient manifold $\mathcal{M}$ to be smoothly isometrically embedded in some Euclidean space $\mathbb{R}^N$ (the codimension is irrelevant), which we can always assume without loss of generality thanks to Nash's Theorem.

Consider now the integral cycle T in $\mathcal{M}$ as an integral cycle of $\mathbb{R}^N$. By [43, Lemma 4.2.19] for every $\kappa > 0$ there is a diffeomorphism g and an integral $m+1$-dimensional current S such that

- $\mathrm{Lip}\,(g) \leq 1+\kappa$ and $|g(x)-x| \leq \kappa$ for all x;
- $\mathbb{M}(S) + \mathbb{M}(\partial S) \leq \kappa$;
- $\mathrm{spt}\,(S) \subset \{x : \mathrm{dist}(x, \mathrm{spt}\,(T)) \leq \kappa\}$;
- $g_\sharp T + \partial S \in \mathbf{P}_m(\mathbb{R}^N)$.

Note therefore that, if we set $\bar{P} := g_\sharp T + \partial S$, then $\bar{P}$ is a cycle,

$$\mathbb{M}(\bar{P}) \leq (1+\kappa)\mathbb{M}(T) + \kappa$$

and

$$\mathrm{spt}\,(\bar{P}) \subset \{x : \mathrm{dist}(x, \mathrm{spt}\,(T)) \leq \kappa\} \subset B_\kappa(\mathcal{M})\,.$$

We next observe that we can, without loss of generality, regard $\bar{P}$ as

$$\bar{P} = \sum_i k_i [\![P_i]\!]$$

where each k_i is a positive integer, each P_i is an oriented *closed* simplex, and for every pair of distinct P_i and P_j, either $P_i \cap P_j = \emptyset$ or $P_i \cap P_j$ is a common lower-dimensional face. This can be seen as follows: fix a representation as $\bar{P} = \sum_j \bar{k}_j [\![\bar{P}_j]\!]$ with $\bar{k}_j$ positive integers and $\bar{P}_j$ oriented simplices. Fix a triangulation $\mathcal{T}$ of $\mathbb{R}^N$ and apply Proposition 1.2.3 to refine $\mathcal{T}$ to a new triangulation $\mathcal{T}'$ with the property that each $\bar{P}_j$ is the union of m-dimensional simplices in $\mathcal{T}'$ (elements of the m-skeleton). The desired conclusion is then immediate.

Assuming κ to be sufficiently small, we can further assume that the orthogonal projection $\mathbf{p} : B_\kappa(\mathcal{M}) \to \mathcal{M}$ is smooth, well-defined and with $\mathrm{Lip}\,(\mathbf{p})$ arbitrarily close to 1. More precisely, $\mathrm{Lip}\,(\mathbf{p}) = 1 + \bar{\kappa}$ where $\bar{\kappa} \downarrow 0$ as $\kappa \downarrow 0$.

We now consider the cycle $\mathbf{p}_\sharp \bar{P}$. Because of the usual homotopy formula, and because of the estimate above, we can ensure that

$$\begin{aligned}
\mathbb{M}(\mathbf{p}_\sharp \bar{P}) &\leq (1 + C\bar{\kappa})\mathbb{M}(\bar{P}) \leq (1 + C\bar{\kappa})(\mathbb{M}(T) + \kappa), \\
\mathbb{F}(T - \mathbf{p}_\sharp \bar{P}) &\leq \mathbb{F}(T - \bar{P}) + \mathbb{F}(\bar{P} - \mathbf{p}_\sharp \bar{P}) \leq \kappa + C\bar{\kappa}\mathbb{M}(\bar{P}) \\
&\leq \kappa + C\bar{\kappa}(\mathbb{M}(T) + \kappa), \\
\mathrm{spt}\,(\mathbf{p}_\sharp \bar{P}) &\subset B_\kappa(\mathrm{spt}(\bar{P})) \subset B_{2\kappa}(\mathrm{spt}\,(T))\,.
\end{aligned}$$

Consider now that that $K := \mathrm{spt}(\bar{P})$ is a polyhedron in the sense of Definition 1.2.4 and that $\mathbf{p} : K \to \mathcal{M}$ is a piecewise smooth map in the sense of Definition 1.2.5. We next fix a triangulation $\mathcal{T}$ of $\mathcal{M}$, which again we understand as a piecewise smooth homeomorphism φ of some PL-submanifold $L \subset \mathbb{R}^{\bar{N}}$ onto $\mathcal{M}$, according to Definition 1.2.5. We next recall the following proposition about uniqueness of smooth triangulations, which is due to Whitehead, *cfr.* [93], and corresponding to [58, Lect. 5, Theorem 1]:

Proposition 2.2.3. *Consider two piecewise smooth homeomorphisms $f : L \to \mathcal{M}$ and $g : M \to \mathcal{M}$ where $L \subset \mathbb{R}^{N_1}$ and $M \subset \mathbb{R}^{N_2}$ are two finite polyhedra. Then for every $\eta > 0$ there are two piecewise smooth homeomorphisms $f' : L \to \mathcal{M}$ and $g' : M \to \mathcal{M}$ which are η-close in the C^1-sense to f and g and such that $(f')^{-1} \circ g'$ and $(g')^{-1} \circ f'$ are piecewise linear.*

Recall that being f and f' piecewise smooth, for both there are triangulations $\mathcal{T}$ and $\mathcal{T}'$ of L with the property that the restriction of each of the simplices of the corresponding triangulation is a smooth function. Closeness in the C^1-sense means that $\|f_{|\Delta\cap\Delta'} - f'_{|\Delta\cap\Delta'}\|_{C^1} \leq \eta$ for every $\Delta \in \mathcal{T}$ and $\Delta' \in \mathcal{T}'$.

An inspection of the argument given in [58] shows that the invertibility of both maps is only used to prove the piecewise linearity of both $(f')^{-1} \circ g'$ and $(g')^{-1} \circ f'$. Adapted to our setting, the arguments lead to the following conclusion.

Proposition 2.2.4. *For every $\eta > 0$ there is a piecewise smooth homeomorphism $\psi : L \to \mathcal{M}$ and a piecewise smooth map $\mathbf{q} : K \to \mathcal{M}$ such that:*

- *(α) ψ and $\mathbf{q}$ are η-close in the C^1-sense to φ and $\mathbf{p}$;*
- *(β) $\Psi := \psi^{-1} \circ \mathbf{q} : K \to L$ is piecewise linear.*

Point (β) means that there is a triangulation $\mathcal{T}_1$ of K and a triangulation $\mathcal{T}_2$ of L with the property that every simplex Δ in the triangulation $\mathcal{T}_1$ is mapped by $\psi^{-1} \circ \mathbf{q}$ inside some simplex Δ' of $\mathcal{T}_2$ and that the restriction $\Psi|_\Delta$ is an affine map.

We are now ready to declare that our cycle P is in fact given by $\mathbf{q}_\sharp \bar{P}$. It is immediate to see that

$$\begin{aligned}
\mathbb{M}(P) &\leq (1 + C(\bar{\kappa} + \eta))\mathbb{M}(\bar{P}) \leq (1 + C(\bar{\kappa} + \eta))(\mathbb{M}(T) + \kappa), \\
\mathbb{F}(T - P) &\leq \mathbb{F}(T - \bar{P}) + \mathbb{F}(\bar{P} - P) \leq \kappa + C(\bar{\kappa} + \eta)(\mathbb{M}(\bar{P})) \\
&\leq \kappa + C(\bar{\kappa} + \eta)(\mathbb{M}(T) + \kappa), \\
\operatorname{spt}(P) &\subset B_{\kappa+\eta}(\operatorname{spt}(\bar{P})) \subset B_{2\kappa+\eta}(\operatorname{spt}(T)) .
\end{aligned}$$

In particular, choosing $\kappa, \bar{\kappa}$ and η appropriately, P satisfies the three desired estimates (a_0), (b_0), and (c_0) in Proposition 2.2.1.

Consider now the finite collection $\mathcal{P}$ of simplices Γ_i which are images through Ψ of some m-dimensional simplex Δ_i of the triangulation of K. Some of these might have dimension strictly smaller than m (which would mean that the affine map $\Psi|_\Delta$ does not have full rank). We then discard them from $\mathcal{P}$. Upon choosing an orientation for the Γ_i's, we clearly have that

$$P = \sum_i \ell_i \psi_\sharp [\![\Gamma_i]\!] ,$$

for an appropriate choice of the multiplicities ℓ_i.

We now can apply Proposition 1.2.6 and find a triangulation $\mathcal{T}_3$ of L which refines the triangulation $\mathcal{T}_2$ and with the property that each Γ_i is the union of some elements in the m-dimensional skeleton of $\mathcal{T}_3$. The image through ψ of $\mathcal{T}_3$ gives the desired triangulation $\mathcal{K}$ of $\mathcal{M}$ which satisfies the requirement (d_0) in Proposition 2.2.1.

Finally, observe that there is an integral current Z in $\mathbb{R}^N$ such that $T - \bar{P} = \partial Z$ and with $\operatorname{spt}(Z) \subset B_{2\kappa}(\mathcal{M})$. In particular $\mathbf{p}_\sharp Z$ provides an integral current in $\mathcal{M}$ such that $\partial \mathbf{p}_\sharp Z = T - \mathbf{p}_\sharp \bar{P}$. Given that $\mathbf{p}$ and $\mathbf{q}$ are close in the Lipschitz norm, there is a Lipschitz homotopy of the two maps which takes values in $B_{2\kappa}(\mathcal{M})$. Composing the latter homotopy with $\mathbf{p}$, we find a Lipschitz homotopy Φ between the two maps which takes values in $\mathcal{M}$: through the homotopy formula this map provides an integral current Z' in $\mathcal{M}$ such that $\partial Z' = \mathbf{p}_\sharp \bar{P} - \mathbf{q}_\sharp \bar{P}$. In particular we conclude that P is in the same homology class of $\mathbf{p}_\sharp \bar{P}$ and hence in the same homology class of T in $\mathcal{M}$.

2.2.3 Proof of Proposition 2.2.1: second approximation

Starting with the approximation P and the triangulation $\mathcal{K}$ of the first part of Proposition 2.2.1 we now construct the approximation P' of the second part. This is done in two steps:

Step 1. Regularization on $\mathcal{M} \setminus \mathcal{K}^{m-1}$. In this first step we modify P using the following algorithm.

We start by fixing, for the triangulation $\mathcal{K}$, a suitable polyhedral submanifold K (*cfr.* Definition 1.2.5) of some Euclidean space $\mathbb{R}^N$ and a piecewise smooth homeomorphism $\psi : K \to \mathcal{M}$ (*cfr.* again Definition 1.2.5) which realizes the triangulation $\mathcal{K}$ in the sense that, for some suitable triangulation $\mathcal{T}$ of the polyhedron K, the following holds: for every cell F of $\mathcal{K}$, its diffeomorphic preimage $\psi^{-1}(F)$ is a simplex of $\mathcal{T}$. The current P is then given

$$P = \sum_{F \in \mathcal{F}^m} \beta_F [\![F]\!] ,$$

where $\beta_F \in \mathbb{Z}$ and $\mathcal{F}^m$ is the collection of m-dimensional cells of $\mathcal{K}$. Without loss of generality we can assume that $\beta_F \geq 0$. For every cell F with $\beta_F > 0$ we consider $\Delta := \psi^{-1}(F)$ and we let Γ be an $m+1$-dimensional simplex of the triangulation $\mathcal{T}$ which contains Δ. We will replace $\beta_F [\![F]\!]$ with $\sum_{j=1}^{\beta_F} [\![\psi(\Delta_j)]\!]$, where the $\Delta_j \subset \Gamma$ are diffeomorphic images of $\Delta \subset \Gamma$ with $\partial \Delta_j = \partial \Delta$ and $\Delta_{j'} \cap \Delta_{j''} = \partial \Delta$ for every $j' \neq j''$. In order to define the Δ_j we will use the following elementary lemma.

Lemma 2.2.5. *Consider the m-dimensional simplex $\Omega \subset \mathbb{R}^m$ which is the convex hull of $\{e_0, e_1, \ldots, e_m\}$, where $e_0 = 0$ and $e_1, \ldots, e_m$ is the standard basis. For each $i \in \{0, 1, \ldots, m\}$ let F_i be the relative interior of the $m-1$-dimensional face of Ω spanned by $e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_m$. In other words, F_i consists of those points p which can be written as convex combinations $\sum_j \lambda_j e_j$ with $\lambda_i = 0$ and $\lambda_j > 0$ for every $j \neq i$.*

Then there is a Lipschitz function $f : \Omega \to \mathbb{R}$ and a neighborhood V of $\bigcup_i F_i$ such that

- *(a) f is positive and smooth in the interior of Ω;*
- *(b) $f(x) = \mathrm{dist}(x, \partial\Omega)$ for every $x \in V$;*
- *(c) $\mathrm{Lip}\,(f) \leq C$ for some constant $C = C(m)$.*

With Lemma 2.2.5 at hand, we are ready to define Δ_j. First of all let $\{v_0, v_1, \ldots, v_m\}$ be the extremal points of Δ, π the m-dimensional linear space spanned by $\{v_1 - v_0, \ldots, v_m - v_0\}$ and then let v_{m+1} be the only unit vector orthogonal to π with the property that $\frac{1}{2}\sum_i v_i + \gamma v_{m+1} \in \Gamma$ for every γ sufficiently small. Let $A : \Delta \to \Omega$ be the affine map defined by $A(v_i) = e_i$. Choose then positive numbers $0 < \delta_1 < \delta_2 < \ldots < \delta_{\beta_F}$ and define Δ_j as

$$\Delta_j = \{x + \delta_j\, f(A(x)) v_{m+1} : x \in \Delta\} .$$

Provided the δ_{β_F} is chosen sufficiently small, each Δ_j is contained in Γ. Note moreover that, by construction, $\psi(\Delta_{j'}) \cap \psi(\Delta_{j''}) \subset \mathcal{K}^{m-1}$, the $m-1$-dimensional skeleton of $\mathcal{K}$.

We perform the above construction for all F's with $\beta_F > 0$. Upon enumerating them as $\{F^i\}$, we denote by Δ^i_j the corresponding m-dimensional cells in the polyhedron K, by δ^i the number $\delta_{\beta_{F_i}}$ and by F^i_j their images through ψ. We note further that, choosing the $\delta_{\beta_{F_i}}$ sufficiently small, we can ensure that $F^i_j \cap F^{i'}_{j'} \subset \mathcal{K}^{m-1}$ for every choice of distinct pairs (i, j) and (i', j').

Consider now the integral current

$$\tilde{P} := \sum_i \sum_j [\![F^i_j]\!] .$$

Clearly $\partial(\sum_j [\![F^i_j]\!]) - \partial(\beta^i [\![F_i]\!]) = 0$ and so $\sum_j [\![F^i_j]\!] - \beta^i [\![F_i]\!]$ is a cycle T_i. Moreover we can use the homotopy formula to ensure that $\mathbb{M}(T_i)$ is as small as needed provided

δ^i is chosen comparably small. We can also ensure that $\mathcal{H}^m(F^i_j)$ is as close as needed to $\mathcal{H}^m(F)$ using the area formula. In particular we can conclude that, upon suitably choosing the δ^i's,

- $\tilde{P}$ is in the same homology class as P (in $\mathcal{M}$);
- $\mathrm{spt}(\tilde{P}) \subset B_{2\varepsilon_c}(\mathrm{spt}(T))$;
- $\mathbb{M}(\tilde{P}) \leq (1+2\varepsilon_c)\mathbb{M}(T)$;
- $\mathbb{F}(T-\tilde{P}) \leq 2\varepsilon_c$;
- $\mathrm{spt}(\tilde{P}) \setminus \mathcal{K}^{m-1}$ is smooth and is taken with multiplicity 1 by $\tilde{P}$, more precisely:
  (S) for every $p \in \mathrm{spt}(\tilde{P}) \setminus \mathcal{K}^{m-1}$ there is a neighborhood U of p and a smooth oriented m-dimensional submanifold Λ of $\mathcal{M}\cap U$ with boundary contained in $\mathcal{M}\cap\partial U$ and such that $\tilde{P}\llcorner U = [\![\Lambda]\!]$.

In the next (and final) step of the proof of Proposition 2.2.1 we will perturb $\tilde{P}$ by removing its $m-1$-dimensional singularities away from a small neighborhood of $\mathcal{K}^{m-2}$. But before coming to that, we provide the elementary proof of Lemma 2.2.5.

Proof of Lemma 2.2.5. We denote by Ω^{m-2} the $m-2$-dimensional skeleton of $\partial\Omega$, namely the set of points which are convex combinations $\sum_i \lambda_i e_i$ where at least two among the coefficients λ_i vanish, while we denote by π_j the affine $m-1$-dimensional space which is formed by linear combinations $\sum_i \lambda_i e_i$ with $\lambda_j = 0$. Moreover we denote by A_j the affine function which vanishes on π_j and coincides with $\mathrm{dist}(p, A_j)$ on the halfspace containing Ω.

We then observe that there is a (sufficiently small) positive constant ε_0 and a (sufficiently large) positive constant C_0, both depending on the dimension m, such that the following holds.

(L) On the open set $\{p \in \Omega : \mathrm{dist}(p,\pi_i) < \varepsilon_0 \text{ and } \mathrm{dist}(p,\pi_i) < C_0^{-1}\mathrm{dist}(p,\Omega^{m-2})\}$ the function $\mathrm{dist}(p,\partial\Omega)$ coincides with the affine function A_i.

We now let $V_k := \{p \in \Omega : 2^{-k-2} < \mathrm{dist}(p,\partial\Omega) < 2^{-k}\}$ for $k \geq 1$, while $V_0 := \{p \in \Omega : \mathrm{dist}(p,\partial\Omega) > \frac{1}{2}\}$ and we consider a partition of unity φ_k subordinate to it with the property that $\|\nabla\varphi_k\|_{C^0} \leq C2^k$. Finally, we let $\psi \in C^\infty_c(B_1)$ be a nonnegative mollifier with $\int \psi = 1$.

The function $f : \Omega \to \mathbb{R}$ is then defined as

$$f := \sum_k \varphi_k \mathrm{dist}(\cdot,\partial\Omega) * \psi_{c_0 2^{-k}}$$

for a sufficiently small constant c_0. Using (L) and the property that $A_i * \psi_\lambda = A_i$ for every choice of λ, we see immediately that f coincides with $\mathrm{dist}(\cdot,\partial\Omega)$ in a neighborhood of $\bigcup_i F_i$. The positivity and smoothness of f in Ω is in turn obvious. Finally, we can compute

$$\nabla f = \sum_k \nabla \varphi_k \mathrm{dist}(\cdot, \partial\Omega) * \psi_{c_0 2^{-k}} + \sum_k \varphi_k \nabla(\mathrm{dist}(\cdot, \partial\Omega) * \psi_{c_0 2^{-k}}).$$

The second summand is bounded by

$$\sum_k \varphi_k = 1$$

because the distance function is 1-Lipschitz. As for the first summand, given that $\sum_k \nabla\varphi_k = 0$, it equals

$$\sum_k \nabla\varphi_k (\mathrm{dist}(\cdot, \partial\Omega) * \psi_{c_0 2^{-k}} - \mathrm{dist}(\cdot, \partial\Omega))\,.$$

For every fixed $p \in \Omega$, there is a j such that p is not in the support of φ_k for $k > j+2$ and $k < j$. We can thus write

$$\begin{aligned}&\left|\sum_k \nabla\varphi_k(p)(\mathrm{dist}(p, \partial\Omega) * \psi_{c_0 2^{-k}} - \mathrm{dist}(p, \partial\Omega))\right| \\ \leq & C 2^{j+2} \sum_{k=j}^{j+2} |\mathrm{dist}(p, \partial\Omega) * \psi_{c_0 2^{-k}} - \mathrm{dist}(p, \partial\Omega))| \leq C\,,\end{aligned}$$

where we have used that $|\mathrm{dist}(p, \partial\Omega) * \psi_\lambda - \mathrm{dist}(p, \partial\Omega)| \leq \lambda$ for every p such that $\mathrm{dist}(p, \partial\Omega) > \lambda$. □

Step 2. Removing the $m-1$-dimensional singularities. Next consider an arbitrary face F^k as in the previous subsection and let σ_i be an arbitrary $m-1$-dimensional face of F^k. Fix a $\delta_c > 0$: the goal is to modify $\tilde{P}$ in a neighborhood of $\sigma \setminus B_{\delta_c}(\mathcal{K}^{m-2})$ to a new current P' in the same homology class, close to it in terms of mass, support and flat norm, and with the property that P' is smooth in that neighborhood. The neighborhood in which we will perturb $\tilde{P}$ is of the form $B_\lambda(\sigma) \setminus B_{\delta_c}(\mathcal{K}^{m-2})$. First of all, given the structure of $\tilde{P}$ obtained in the previous subsection, if λ is sufficiently small, there is an open subset $\Delta_i \subset \mathbb{R}^{m-1}$ and a smooth parametrization

$$\Phi : \Delta_i \times B^{n+1}_{\lambda_i} \to \mathcal{M}$$

of a normal neighborhood $\mathcal{N}_i$ of $\sigma \setminus B_{\delta_c}(\mathcal{K}^{m-2})$ with thickness λ_i and with the property that $\mathrm{spt}(\tilde{P}\setminus \llcorner \mathcal{N}_i)$ can be described in the following way. There are a finite number of distinct unit vectors $w_1, \ldots, w_L \in \partial B^{n+1}_1 \subset \mathbb{R}^{n+1}$ such that, if we let

$$\Lambda_\ell = \{(\sigma, s w_\ell) : \sigma \in \Delta_i, 0 < s < \lambda_i\}\,,$$

then $\tilde{P}\llcorner\mathcal{N}_i = \sum_{\ell=1}^{L} \epsilon_\ell \Phi_\sharp[\![\Lambda_\ell]\!]$, where ϵ_ℓ takes values in $\{-1,1\}$. Given that $\tilde{P}$ has no boundary in $\mathcal{N}_i$, we conclude that L must be an even number $2\bar{L}$ and that exactly $\bar{L}$ among the numbers ϵ_ℓ equal 1, while the remaining equal -1. We can thus write

$$\tilde{P}\llcorner\mathcal{N}_i = \sum_{\ell=1}^{\bar{L}} \Phi_\sharp[\![\Lambda_\ell]\!] - \sum_{\ell=\bar{L}+1}^{2\bar{L}} \Phi_\sharp[\![\Lambda_\ell]\!] \,.$$

Consider now the oriented halflines $H_\ell = \{\lambda w_\ell : \lambda > 0\}$ in $\mathbb{R}^{n+1}$. Upon reordering them, we can find disjoint smooth oriented curves γ_ℓ for $\ell \in \{1,\ldots,\bar{L}\}$ in $\mathbb{R}^{n+1}$ with the property that $[\![\gamma_\ell]\!]\llcorner\mathbb{R}^{n+1}\setminus B_1 = ([\![H_\ell]\!] - [\![H_{\bar{L}+\ell}]\!])\llcorner\mathbb{R}^{n+1}\setminus B_1$. Furthermore we let $\tau_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the homothety $y \mapsto ty$ and denote by $\gamma_{\ell,t}$ the curve $\tau_t(H_\ell\gamma_\ell)$. We are now ready to define a replacement for $\tilde{P}\llcorner\mathcal{N}_i$. We fix a smooth compactly supported function ψ_i in $\mathbb{R}^{n+1}$ which is positive on Δ_i and vanishes on $\partial\Delta_i$, a small positive number κ_i, and we define

$$\Sigma^i := \left\{(x,y) : x \in \Delta_i, y \in \bigcup_\ell \gamma_{\ell,\kappa_i\psi_i(x)}\right\} \cap \Delta_i \times B^{n+1}_{\lambda_i} \,.$$

Choosing κ_i sufficiently small we can ensure that the current

$$P^i := \Phi_\sharp\Sigma^i$$

satisfies $\partial P^i = \partial(\tilde{P}\llcorner\mathcal{N}_i)$. Moreover we can make $\mathbb{F}(P^i - \tilde{P}\llcorner\mathcal{N}_i)$ and $\mathbb{M}(P^i) - \mathbb{M}(\tilde{P}\llcorner\mathcal{N}_i)$ smaller than any desired threshold by choosing κ_i sufficiently small. Note finally that, clearly, Σ_i is smooth in $\mathcal{N}_i$.

We next enumerate all the $m-1$-dimensional faces σ_i of all the m-dimensional faces F^k as $\sigma_1, \sigma_2, \ldots, \sigma_N$. We choose our parameters in such a way that the sets $\mathcal{N}_i$ are pairwise disjoint. Our final current P' will then be defined to be

$$P' := \sum_i P^i + \tilde{P}\llcorner\mathcal{M}\setminus\bigcup_i \mathcal{N}_i \,.$$

$P'\llcorner\mathcal{M}\setminus B_{\delta_c}(\mathcal{K}^{m-2})$ is then smooth by construction and, choosing the parameters accordingly, P' is homologous to $\tilde{P}$ and we achieve the desired estimates since we can make $\mathrm{spt}(P')$ arbitrarily close to $\mathrm{spt}(\tilde{P})$, $\mathbb{M}(P')$ arbitrarily close to $\mathbb{M}(\tilde{P})$, and $\mathbb{F}(P' - \tilde{P})$ arbitrarily small.

Finally, coming to the estimate on $\|P'\|(B_{\delta'_c}(\mathcal{K}^{m-2}))$ observe that we know:

$$\begin{aligned}
&\|P'\|(\mathcal{M}) \le \|P\|(\mathcal{M}) + 2\varepsilon_c\\
&\|P'\|(\mathcal{M}\setminus B_{\delta'_c}(\mathcal{K}^{m-2})) \ge \|P\|(\mathcal{M}\setminus B_{\delta'_c}(\mathcal{K}^{m-2}))\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\|P'\|(B_{\delta'_c}(\mathcal{K}^{m-2})) &= \|P'\|(\mathcal{M}) - \|P'\|(\mathcal{M}\setminus B_{\delta'_c}(\mathcal{K}^{m-2}))\\
&\le \|P\|(\mathcal{M}) + 2\varepsilon_c - \|P\|(\mathcal{M}\setminus B_{\delta'_c}(\mathcal{K}^{m-2})) = \|P\|(B_{\delta'_c}(\mathcal{K}^{m-2})) + 2\varepsilon_c
\end{aligned}$$

for every $\delta'_c > \delta_c$. Hence δ'_c must be chosen small enough just to ensure that $\|P\|(B_{\delta'_c}(\mathcal{K}^{m-2})) \le \varepsilon_c$.

2.2.4 Squeezing lemma

In the second proposition we are given two m-dimensional integral cycles S and R which agree outside of a sufficiently small neighborhood of the $m-2$-dimensional skeleton $\mathcal{K}^{m-2}$. We will then show that:

- S and R represent the same homology class;
- There is a smooth deformation R' of R which is close, in terms of mass and in flat norm, to S;
- R' coincides with S outside a slightly larger neighborhood of $\mathcal{K}^{m-2}$.

Proposition 2.2.6. *Let m and $\mathcal{M}$ be as in Assumption 2.0.3 and let $\mathcal{K}$ be a smooth triangulation of $\mathcal{M}$. Then for every $\varepsilon_d > 0$ and every $\eta_d > 0$ there exists $\delta_d(\varepsilon_d, \eta_d, \mathcal{K}, \mathcal{M}) > 0$ with the following property. Suppose S and R are m-dimensional integral cycles in $\mathcal{M}$ and that*

$$S \llcorner \mathcal{M} \setminus B_{\delta_d}\left(\mathcal{K}^{m-2}\right) = R \llcorner \mathcal{M} \setminus B_{\delta_d}\left(\mathcal{K}^{m-2}\right) .$$

Then S and R are homologous and moreover there exist an integral cycle R' in their homology class and a diffeomorphism Φ of $\mathcal{M}$ with the following properties:

1. $\mathbb{M}(R') \leq (1+\varepsilon_d)\mathbb{M}(S)$;
2. $\mathbb{F}(R'-S) \leq C(\varepsilon_d \mathbb{M}(S) + 2\|S\|(B_{\eta_d}(\mathcal{K}^{m-2})))^{\frac{m+1}{m}}$, *with* $C = C(\mathcal{M})$;
3. $R' \llcorner \mathcal{M} \setminus B_{\eta_d}(\mathcal{K}^{m-2}) = S \llcorner \mathcal{M} \setminus B_{\eta_d}(\mathcal{K}^{m-2})$;
4. Φ *is in the isotopy class of the identity and* $R' = \Phi_\sharp R$.

Remark 2.2.7. Note that the parameter δ_d does not depend on S and R. Its dependence on the parameters ε_d and η_d can be computed through our arguments, but since such explicit dependence is irrelevant for our purposes, we will ignore the issue.

2.2.5 Proof of Proposition 2.2.6

First of all we observe the following consequence of the area formula.

Lemma 2.2.8. *Assume Φ, γ, and ε_a are as in Lemma 3.1.4. If Z is any integer rectifiable current of dimension $m > k$, then*

$$\mathbb{M}(\Phi_\sharp(Z \llcorner B_{\delta_d}(\mathcal{K}^k))) \leq C(1+\varepsilon_a)^k \gamma^{m-k} \|Z\|(B_{\delta_d}(\mathcal{K}^k)), \tag{2.5}$$

where C is a dimensional constant which depends only on m and n.

Proof. Using the area formula, we have

$$\mathbb{M}(\Phi_\sharp(Z \llcorner B_{\delta_d}(\mathcal{K}^k))) = \int_{B_{\delta_d}(\mathcal{K}^k)} |d\Phi_p(\vec{Z}(p))| \, d\|Z\|(p), \tag{2.6}$$

where $\vec{Z}(p)$ is a unit simple m-vector orienting Z at p. We can write $v = \pm v_1 \wedge \ldots \wedge v_m$ for any orthonormal base of the approximate tangent space V to Z at p and estimate

$$|d\Phi_p(\vec{Z}(p))| \leq \Pi_{i=1}^m |D\Phi_p(e_i)| \,.$$

Now consider the space W spanned by $e_1, \ldots, e_k$ and let $\mathbf{p}_V(W)$ be its orthogonal projection onto V. Clearly the dimension of $W' := \mathbf{p}_V(W)$ is at most k and hence its orthogonal complement W'' in V has dimension at least $m-k$. We can choose an orthonormal base of V by completing an orthonormal base of W''. On the other hand any element of W'' is in the span of $e_{k+1}, \ldots, e_m$. In particular, we conclude that $|D\Phi_p(w'')| \leq \sqrt{m-k}\,\gamma |w''|$ for any vector $w'' \in W''$. On the other hand the estimate $|D\Phi_p(v)| \leq \sqrt{m+n}\,(1+\varepsilon_a)|v|$ holds for any vector $v \in T_p\mathcal{M}$, thus completing the proof of the estimate. □

Proof of Proposition 2.2.6. We can assume, without loss of generality, that the homology class of S is nontrivial, so that $\mathbb{M}(S) > 0$. The conclusion that R and S are homologous follows from the fact that they coincide outside $B_{\delta_d}(\mathcal{K}^{m-2})$. In particular $\mathrm{spt}(S-R) \subset B_{\delta_d}(\mathcal{K}^{m-2})$: since for δ_d smaller than a constant $c(\mathcal{K})$ the latter deformation retracts onto $\mathcal{K}^{m-2}$, whose m-dimensional homology is trivial, it follows that $S-R$ is homologically trivial.

We now let ε_d and η_d be given as in the statements. We further fix $\bar{\varepsilon}_d$, whose choice will be specified later (but which will depend only on ε_d), and apply Lemma 3.1.4 with $\varepsilon_a = \bar{\varepsilon}_d$ and $\eta_a = \eta_d$. We therefore get the parameter $\delta_a =: \delta_d$ (which will be the one of the conclusion of the proposition) and, after fixing yet another γ (whose choice will now be dependent on R), we get the map Φ satisfying the requirements of Lemma 3.1.6. The requirements (3) and (4) of Proposition 2.2.6 are then satisfied by construction. Moreover estimate (2) follows from the isoperimetric inequality and from (1) and (3). Indeed there is an integral current T such that $\partial T = S - R'$ and

$$\mathbb{M}(T) \leq C\,(\mathbb{M}(S-R'))^{\frac{m+1}{m}}$$

with $C = C(\mathcal{M})$. Using (1) and (3) we then estimate

$$\begin{aligned}
\mathbb{M}(S-R') &= \|S-R'\|(B_{\eta_d}(\mathcal{K}^{m-2})) \leq \|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \|R'\|(B_{\eta_d}(\mathcal{K}^{m-2}))\\
&= \|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \mathbb{M}(R') - \|S\|(\mathcal{M} \setminus B_{\eta_d}(\mathcal{K}^{m-2}))\\
&= 2\|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \mathbb{M}(R') - \mathbb{M}(S)\\
&\leq 2\|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \varepsilon_d \mathbb{M}(S)\,. \qquad (2.7)
\end{aligned}$$

It remains to prove (1). Note that

$$\begin{aligned}
\mathbb{M}(R') &\leq \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + \mathbb{M}(\Phi_\sharp(R \llcorner \mathcal{M} \setminus B_{\delta_d}(\mathcal{K}^{m-2})))\\
&= \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + \mathbb{M}(\Phi_\sharp(S \llcorner \mathcal{M} \setminus B_{\delta_d}(\mathcal{K}^{m-2})))\\
&\leq \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + (\mathrm{Lip}\,\Phi)^m \mathbb{M}(S)\\
&\leq \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + (1+\bar{\varepsilon}_d)^m \mathbb{M}(S)\,. \qquad (2.8)
\end{aligned}$$

Hence we apply Lemma 3.1.6 and infer

$$\mathbb{M}(R') \leq C(1+\bar{\varepsilon}_d)^{m-2}\gamma^2\mathbb{M}(R) + (1+\bar{\varepsilon}_d)^m\mathbb{M}(S)\,.$$

Next we fix $\bar{\varepsilon}_d$ so that $(1+\bar{\varepsilon}_d)^m = 1+\frac{\varepsilon_d}{2}$, and then we choose γ sufficiently small so that

$$C(1+\bar{\varepsilon}_d)^{m-2}\gamma^2\mathbb{M}(R) \leq \frac{\varepsilon_d}{2}\mathbb{M}(S)\,.$$

Note that the choice of γ, unlike that of ε_d, will indeed depend on R and S. ☐

2.3 Proof of the main theorem

We start with the following crucial topological lemma.

Lemma 2.3.1. *Let $h : T(\widetilde{\gamma}^n) \to K(\mathbb{Z}, n)$ be a map representing the Thom class $u \in H^n(T(\widetilde{\gamma}^n), \mathbb{Z})$. Then h is an $n+4$-equivalence, for all positive integers n.*

Proof. The spaces are the same for $n \in \{1, 2\}$: $T(\widetilde{\gamma}^1)$ is homotopy equivalent to the circle S^1, which is a realization of $K(\mathbb{Z}, 1)$, while $T(\widetilde{\gamma}^2)$ is homotopy equivalent to the infinite complex projective space $\mathbb{CP}(\infty)$, of type $K(\mathbb{Z}, 2)$. Hence, we can assume $n \geq 3$. Towards an application of Theorem 1.3.3, we recall the computations of the cohomology rings of $K(\mathbb{Z}, n)$ and of the classifying space $BSO(n)$, for any group coefficient $\mathbb{Z}_p$.

By Serre's computations using spectral sequences of fibre spaces, the cohomology $H^{n+i}(K(\mathbb{Z}, n))$ with $\mathbb{Z}_2$ coefficients is generated by the Steenrod squares Sq^2, Sq^3 (and Sq^4 if $n \geq 4$) for $i \leq 4$, see [76, Théorème 3]. By calculations of Cartan with coefficients in $\mathbb{Z}_3$, the cohomology $H^{n+i}(K(\mathbb{Z}, n))$ is generated by $\mathcal{P}^1_3$ in dimensions less than or equal to $n + 4$, while for $\mathbb{Z}_p$ coefficients with prime $p > 3$ there are no generators between dimension n and dimension $n + 8$, see for example [86, Chapitre II, §8, §9] or [39, §10.5].

The cohomology ring of $BSO(n)$ with coefficients in $\mathbb{Z}_2$ is generated by the Stiefel-Whitney classes $w_2, \ldots, w_n$ of $\widetilde{\gamma}^n$, *cfr.* Proposition 1.4.3, that is

$$H^*(BSO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, w_4 \ldots w_n].$$

For odd primes and in dimensions $i \leq 5$, we have that, *cfr.* Proposition 1.4.4,

$$\begin{aligned} H^*(BSO(n), \mathbb{Z}_p) &= \mathbb{Z}_p[p_1] && \text{if } n \neq 4, \\ H^*(BSO(4), \mathbb{Z}_p) &= \mathbb{Z}_p[p_1, e], && \text{if } n = 4. \end{aligned}$$

For every p prime, let Φ_p denote the Thom isomorphism between $H^i(BSO(n), \mathbb{Z}_p)$ and $\widetilde{H}^{n+i}(T(\widetilde{\gamma}^n), \mathbb{Z}_p)$ and u_p the Thom class. Since by Proposition 1.4.2 and [60, Theorem 19.7] we have[2] that $\Phi_2(w_i) = Sq^i(u_2)$ and $\Phi_3(p_1) = \mathcal{P}^1_3(u_3)$, it follows that, for any group coefficient $\mathbb{Z}_p$, the induced map in cohomology

$$h^* : H^{n+i}(K(\mathbb{Z}, n), \mathbb{Z}_p) \to H^{n+i}(T(\widetilde{\gamma}^n), \mathbb{Z}_p)$$

---

[2] Denoting with a slight abuse p_1 for p_1 reduced mod 3.

is an isomorphism for dimensions less than or equal to $n+3$ and a monomorphism in dimension $n+4$. Since $K(\mathbb{Z},n)$ and $T(\widetilde{\gamma}^n)$ are simply connected, by Theorem 1.3.3, we conclude that

$$\pi_k(K(\mathbb{Z},n),T(\widetilde{\gamma}^n))=0 \quad \text{for } k\leq n+4,$$

ending the proof. □

Remark 2.3.2. Lemma 2.3.1 shows that in particular that, for dimension $m\in\{1,2,3,4\}$ and for any codimension $n\in\mathbb{N}\setminus\{0\}$, every homology class $\tau\in H_m(\mathcal{M},\mathbb{Z})$ is represented by an embedded smooth submanifold Σ in $\mathcal{M}$. It is important to remark that, by [86, p.56, footnote 9], we also know that every homology class of $H_m(\mathcal{M},\mathbb{Z})$ for $m\leq 6$ is representable by a smooth submanifolds, due to the vanishing of the obstruction of the corresponding Poincaré dual x, $St_3^5(x)$, where St_3^5 represents (up to a sign) the following cohomology operations, *cfr.* Remark 2.0.5,

$$St_3^5=\beta^*\circ\mathcal{P}_3^1\circ\theta_3 : H^*(\mathcal{M},\mathbb{Z})\to H^{*+5}(\mathcal{M},\mathbb{Z}).$$

Proof of Theorem 2.0.1. Fix $\varepsilon_c>0$, whose choice will be specified later, and an integral current T in a homology class $\tau\in H_m(\mathcal{M},\mathbb{Z})$. First of all apply Proposition 2.2.1 to find a sufficiently small $\delta_c'>0$, a suitable triangulation $\mathcal{K}$ of the manifold and a new integral current $P'=:S$ with the property that S is in the same homology class of T and the following facts hold:

- $\mathbb{M}(S)\leq\mathbb{M}(T)+3\varepsilon_c$ and $\mathbb{F}(S-T)<3\varepsilon_c$;
- $\|S\|(B_{\delta_c'}(\mathcal{K}^{m-2}))\leq 3\varepsilon_c$;
- $B_{\delta_c'}(\mathcal{K}^{m-2})$ is homotopy equivalent to $\mathcal{K}^{m-2}$;
- $S\llcorner\mathcal{M}\setminus B_{\delta_c}(\mathcal{K}^{m-2})=[\![\Gamma]\!]$ for a smooth submanifold Γ.

We observe the following important fact: if we first choose ε_c, then δ_c, δ_c' and $\frac{\delta_c}{\delta_c'}$ can all be made smaller than any desired constant, while the triangulation is instead kept fixed (because it depends only on ε_c).

We have now fixed a triangulation $\mathcal{K}$ and we can therefore fix constant C_0 and $\bar{\delta}$ so that Lemmas 2.1.1 and 2.1.2 apply. We now require that $V_{\delta_c/2}(\mathcal{K}^{m-2})\subset\subset B_{\delta_c'}(\mathcal{K}^{m-2})$ for some $\delta_c/2<<\delta_c'$. Hence we apply Lemma 2.1.2 (where $\delta'<\delta$ corresponds here to $\delta_c/2<\tilde{\delta}/2$) to find a $U_{\tilde{\delta}/2}(\mathcal{K}^{m-2})$ suitably close to $V_{\tilde{\delta}/2}(\mathcal{K}^{m-2})$. We will want that $B_{\delta_c}(\mathcal{K}^{m-2})\subset U_{\tilde{\delta}/2}(\mathcal{K}^{m-2})\subset V_{\tilde{\delta}/2}(\mathcal{K}^{m-2})\subset V_{\tilde{\delta}}(\mathcal{K}^{m-2})\subset B_{\delta_c'}(\mathcal{K}^{m-2})$. This step requires to take $\frac{\delta_c}{\delta_c'}$ sufficiently small and $\tilde{\delta}<\delta_c'$. Define now $\Omega:=\mathcal{M}\setminus U_{\tilde{\delta}/2}(\mathcal{K}^{m-2})$.

The current $[\![\Gamma]\!]$ obtained from Proposition 2.2.1 is (when restricted to Ω and not relabelled) a smooth compact oriented submanifold of Ω with $\partial\Gamma\subset\partial\Omega$, provided $\partial\Omega$ is transversal to Γ, which can be ensured via a small smooth perturbation. Denoting by $x\in H^n(\mathcal{M})$ the Poincaré dual of τ, note that its restriction $x_{|\Omega}\in H^n(\Omega)$ to Ω is the relative Poincaré dual of a relative homology class which is represented by the smooth compact embedded submanifold $\Gamma \subset \Omega$ with boundary $\partial\Gamma = \Gamma \cap \partial\Omega$. Hence, by Theorem 1.3.7, there exists a map

$$F : \Omega \to T(\widetilde{\gamma}^n)$$

such that $F^*(u) = x_{|\Omega}$; in addition, F is smooth and transverse on $BSO(n)$ (and such that $F_{|\partial\Omega}$ is also transverse), so that $F^{-1}(BSO(n)) = \Gamma$, which is the smooth part of S.

We then take δ sufficiently small so that $\Omega \subset \mathcal{M} \setminus U_\delta(\mathcal{K}^{m-5})$ for the U_δ given in Lemma 2.1.2. Then, by Lemma 1.2.7 we have that $\mathcal{M} \setminus U_\delta(\mathcal{K}^{m-5})$ is homotopy equivalent to a complex of dimension $n+4$. Denote

$$Q := \mathcal{M} \setminus U_\delta(\mathcal{K}^{m-5}).$$

Given the n-dimensional cohomology class $x \in H^n(\mathcal{M})$ which is the Poincaré dual of τ, we consider its restriction to Q, that is $x_{|Q} \in H^n(Q)$; note that $x_{|Q}$ can be represented by a continuous map

$$g : Q \to K(\mathbb{Z}, n)$$

(in a suitable homotopy class of continuous maps) pulling-back the fundamental class of $K(\mathbb{Z}, n)$ to itself, *i.e.* $g^*(\iota) = x_{|Q}$. By Lemma 2.3.1 and Proposition 1.3.1, there exists a map $f : Q \to T(\widetilde{\gamma}^n)$ such that the diagram commutes, *i.e.* f pulls-back the universal Thom class to $x_{|Q}$.

$$\begin{tikzcd} & T(\widetilde{\gamma}^n) \arrow[d, "h"] \\ Q \arrow[ur, "f"] \arrow[r, "g"'] & K(\mathbb{Z}, n) \end{tikzcd}$$

By the same construction of the second part of the proof of Theorem 1.3.7, we can assume without loss of generality that f is smooth throughout $Q \setminus f^{-1}(U(\infty))$ and transversal to (a sufficiently high dimensional approximation of) the zero cross-section $BSO(n) \subset T(\widetilde{\gamma}^n)$, with ∂f also transversal to it. Hence, $f^{-1}(BSO(n))$ is a compact smooth m-dimensional embedded submanifold, with boundary contained in ∂Q; denote it as

$$\mathcal{N} := f^{-1}(BSO(n)).$$

Moreover, $\mathcal{N}$ represents the relative Poincaré dual of $x_{|Q}$, which equals $j_*(\tau) \in H_m(Q, \partial Q)$, where $j_* : H_m(\mathcal{M}) \to H_m(Q, \partial Q)$.

We next wish to extend $[\![\mathcal{N}]\!]$ (which is an integral current in $\mathcal{M}$) to an integral current N with the property that $N \llcorner \mathcal{M} \setminus \mathcal{K}^{m-5}$ is a smooth submanifold with multiplicity 1 and $N \llcorner Q = [\![\mathcal{N}]\!]$. First of all, because $\mathcal{N}$ is transversal to ∂Q, we can extend it to a smooth submanifold over the union Q' of Q with any smooth collaring extension of ∂Q. We can then use Lemma 2.1.2 to find such an extension Q' (which consists of $Q \cup \mathcal{C}$, where $\mathcal{C}$ is the smooth tubular neighborhood in Lemma 2.1.2) containing $\mathcal{M} \setminus V_{\delta'}(\mathcal{K}^{m-5})$ for some $\delta' < \delta$ positive. Since $\mathcal{N}$ intersects ∂Q transversally, we can extend to a smooth submanifold of Q' with boundary in $\partial Q'$, meeting $\partial Q'$ transversally. With abuse of notation this extension is still denoted by $\mathcal{N}$. We can now use the map Φ of Lemma 3.1.5 and set

$$N := \Phi_\sharp [\![\mathcal{N}]\!].$$

The latter current is integer rectifiable because Φ is Lipschitz (and, in particular, N has finite mass). Given that Φ is a diffeomorphism over $\mathcal{M} \setminus \Phi^{-1}(\mathcal{K}^{m-5}) \subset \mathcal{M} \setminus V_{\delta'}(\mathcal{K}^{m-5})$, then $N \llcorner \mathcal{M} \setminus \mathcal{K}^{m-5} = [\![\Sigma]\!]$ for some smooth submanifold Σ. Moreover $\operatorname{spt}(\partial N) \subset \mathcal{K}^{m-5}$ and in particular, by Federer flatness theorem, $\partial N = 0$, namely N is a cycle.

Consider now the two maps $F : \Omega \to T(\widetilde{\gamma}^n)$ and $f : Q \to T(\widetilde{\gamma}^n)$ such that $F^{-1}(BSO(n)) = \Gamma$ and $f^{-1}(BSO(n)) = \mathcal{N} \cap Q$. If we consider the restriction of f to $\Omega \subset Q$, we obtain a new map $f|_\Omega \colon \Omega \to T(\widetilde{\gamma}^n)$ that pulls-back the universal Thom class to $x_{|\Omega}$. By Lemma 1.2.7 we observe that Ω has the homotopy type of an $n+1$-complex, so that by Corollary 1.3.2 we can conclude that F and $f_{|\Omega}$ are homotopic: the homotopy can be taken smooth by [86, Lemme IV.5]. In particular, we define the smooth homotopy $H : [0,1] \times \Omega$ such that $H(0,x) = f_{|\Omega}(x)$ and $H(1,x) = F(x)$. In a small collar neighborhood $\mathcal{C}$ of $\partial\Omega$ inside Ω, which we identify with $\partial\Omega \times (0,1]$, we then glue the maps f and F together. Using the notation $x = (y,s) \in \mathcal{C}$ and after defining a smooth function φ on $[0,1]$ which is identically equal to 0 in a neighborhood of 0 and identically equal to 1 in a neighborhood of 1, we set

$$\widehat{f}(x) := \begin{cases} F(x) & if\ x \in \Omega \setminus \mathcal{C}, \\ H\left(x, \varphi(s)\right) & if\ x \in \mathcal{C}, \\ f(x) & if\ x \in Q \setminus \Omega \end{cases} \tag{2.9}$$

Since $T(\widetilde{\gamma}^n) \setminus \{\infty\}$ is a smooth submanifold, it follows from [90, Proposition 2.3.4 (ii)] that we can find $\widehat{f} : Q \to T(\widetilde{\gamma}^n)$, not relabelled, which is smooth throughout $Q \setminus f^{-1}(U(\infty))$, coincides with $f(x)$ in a neighborhood of ∂Q and with F on $\mathcal{M} \setminus V_{\tilde{\delta}}(\mathcal{K}^{m-2})$. Analogously, by [90, Proposition 4.5.10], we can perturb $\widehat{f}$ so that it is transverse to $BSO(n)$ and coinciding with $f(x)$ in a neighborhood of ∂Q and with F on $\mathcal{M} \setminus V_{\tilde{\delta}}(\mathcal{K}^{m-2})$.

Consider now the submanifold Σ' of $\mathcal{M} \setminus \mathcal{K}^{m-5}$ which consists of:

- Σ in $V_{\delta'}(\mathcal{K}^{m-5}) \setminus \mathcal{K}^{m-5}$;
- $\mathcal{N}$ on $U_\delta(\mathcal{K}^{m-5}) \setminus V_{\delta'}(\mathcal{K}^{m-5})$;
- $\widehat{f}^{-1}(BSO(n))$ on Q.

This is a smooth submanifold because:

- f and $\widehat{f}$ coincide in a neighborhood of ∂Q and hence $\widehat{f}^{-1}(BSO(n))$ coincides with $\mathcal{N}$ in a neighborhood of ∂Q;
- $\Sigma = \Phi(\mathcal{N}) = \mathcal{N}$ in a neighborhood of $\partial V_{\delta'}(\mathcal{K}^{m-5})$.

Moreover, $R = [\![\Sigma']\!]$ is an integer rectifiable current with finite mass and such that $\operatorname{spt}(\partial R) \subset \mathcal{K}^{m-5}$; in particular it is a cycle by Federer's flatness theorem. Observe also that $R - S$ is supported, by construction, in $V_{\tilde{\delta}}(\mathcal{K}^{m-2})$, which is homotopy equivalent to $\mathcal{K}^{m-2}$, and thus has trivial m-homology. In particular $R - S$ is a boundary, namely R and S belong to the same homology class.

We now apply Proposition 2.2.6 to S and R, noticing that the ε_d in Proposition 2.2.6 is a parameter to be chosen in terms of the ε of the statement of Theorem 2.0.1, and the η_d in Proposition 2.2.6 is δ'_c here. This gives us a parameter δ_d, which depends on ε_d and δ'_c. In turn we impose that $\tilde{\delta} \leq \delta_d$ so that we can apply Proposition 2.2.6. Since ε_d will be specified only in terms of $\mathbb{M}(T)$ and of ε in the statement of Theorem 2.0.1, while δ'_c depends on ε_c, which will also be specified only in terms of $\mathbb{M}(T)$ and ε in the statement of Theorem 2.0.1, the parameter $\tilde{\delta}$ can be taken smaller than δ_d. We can then find a current $R' := \Phi_\sharp R$ for a smooth diffeomorphism Φ isotopic to the identity such that

$$\mathbb{M}(R') \leq (1 + \varepsilon_d)\,\mathbb{M}(S) \leq (1 + \varepsilon_d)(\mathbb{M}(T) + 3\varepsilon_c)\,.$$

We therefore conclude that R' is homologous to R, hence to S, and therefore to T. Moreover, if we choose

$$\varepsilon_d\,(3 + \mathbb{M}(T)) < \frac{\varepsilon}{2} \qquad \text{and} \qquad 3\varepsilon_c < \frac{\varepsilon}{2}\,,$$

then $\mathbb{M}(R') \leq \mathbb{M}(T) + \varepsilon$. Finally

$$\begin{aligned}\mathbb{F}(T - R') &\leq 3\varepsilon_c + \mathbb{F}(S - R') \leq 3\varepsilon_c + C(\varepsilon_d\,\mathbb{M}(S) + 2\|S\|(B_{\delta'_c}(\mathcal{K}^{m-2})))^{\frac{m+1}{m}}\\ &\leq 3\varepsilon_c + C(\varepsilon_d(\mathbb{M}(T) + \varepsilon_c) + 6\varepsilon_c)^{\frac{m+1}{m}}\,.\end{aligned}$$

Therefore it is clear that a suitable choice of ε_d and ε_c depending only on $\mathbb{M}(T)$ and ε suffices to show $\mathbb{F}(T - R') \leq \varepsilon$.

The proof of part (3) of Theorem 2.0.1 is analogous; by assumption we know that τ is represented by a smooth closed submanifold Σ and hence, by Theorem 1.3.6 there exists a map $g : \mathcal{M} \to T(\widetilde{\gamma}^n)$ which pulls-back the universal Thom class $u \in H^n(T(\widetilde{\gamma}^n), \mathbb{Z})$ to the Poincaré dual of τ. Substituting in the previous steps the map f with this new map g, defined over the whole ambient space $\mathcal{M}$, and defining a similar homotopy as that one in (3.7), the result follows by applying Proposition 2.2.6 to S and $[\![\Sigma]\!]$, where S is the integral cycle denoted P' in Proposition 2.2.1.

□

2.4 Optimality of the main theorem

The codimension 5 construction in Theorem 2.0.1 is the best possible result in full generality, as shown by Theorem 2.4.3.

We start by recalling Thom's example of an integral homology class of dimension 7 in an orientable smooth manifold of dimension 14 which is not realizable by means of a submanifold, *cfr.* [86, Théorème III.9][3].

Example 2.4.1 (Thom)**.** For $i = 1, 2$ consider the lens space $L_i := S^7/\mathbb{Z}_3$, which is the orbit space of the 7-sphere with the free action of $\mathbb{Z}_3$ generated by the rotation. Let v_i be generator of $H^1(L_i, \mathbb{Z}_3) \simeq \mathbb{Z}_3$ and call $u_i = \beta_3(v_i) \in H^2(L_i, \mathbb{Z}_3) \simeq \mathbb{Z}_3$. Consider the real analytic, closed and oriented 14-dimensional manifold $L := L_1 \times L_2$ and the following cohomology class, where the powers and $\cdot$ denote the cup product (seeing $H^*(L_i)$ as embedded in $H^*(L)$):

$$y = u_1 \cdot v_2 \cdot (u_2)^2 - v_1 \cdot (u_2)^3 \in H^7(L, \mathbb{Z}_3).$$

Note that y is actually the reduction mod 3 of an integral cohomology class, since $y = \beta_3(v_1 \cdot v_2 \cdot (u_2)^2)$ and hence $y = \theta_3(x)$, with $x \in H^7(L, \mathbb{Z})$ given by $x = \beta^*(v_1 \cdot v_2 \cdot (u_2)^2)$, *cfr.* Remark 2.0.5 for the notation. Denoting by $z \in H_7(L, \mathbb{Z})$ its Poincaré dual homology class, we see that z cannot be realized in L by a submanifold since

$$St_3^5(x) = \beta^* \circ \mathcal{P}_3^1(y) = \beta^*((u_1)^3 \cdot v_2 \cdot (u_2)^2) = (u_1 \cdot u_2)^3 \neq 0.$$

Remark 2.4.2. We remark that the obstruction to realizability comes from a cohomology operation mod 3 and since $y \in H^7(L, \mathbb{Z}_3)$, then the Poincaré dual of $3y$ can be realized as the image of the fundamental class of a manifold. In general, it is a theorem of Thom, *cfr.* [86, Théorème II.29], that for every integral homology class $z \in H_k(\mathcal{M}, \mathbb{Z})$ of a closed oriented manifold there exists a non-zero integer N such that the class Nz is realizable as the image of the fundamental class of a manifold.

Example 2.4.1 is the first example of *innately singular* homology classes: from a geometric point of view, it represents a codimension 5 non-removable singularity which is the geometric analogue of the algebraic obstruction given by the dual 3-torsion cohomology operation St_3^5. In particular, Thom's innately singular class can be represented by a 7-dimensional cycle T with a 2-dimensional stratum of singularities, *i.e.* a closed (equisingular) 2-dimensional manifold $\mathcal{S}_T$ whose neighborhood is homeomorphic to a product

$$\mathcal{S}_T \times C(\mathbb{CP}(2)),$$

where $C(\mathbb{CP}(2))$ denotes the cone over $\mathbb{CP}(2)$; the innate nature of these singularities turns out to be intrinsically linked to the well-known fact that $\mathbb{CP}(2)$ does not bound any compact oriented smooth 5-dimensional manifold, as observed in [82].

This geometric description is a consequence of another insightful work of Thom, *cfr.* [87], where he studied manifolds with singularities partitioning them into partially ordered *strata* of varying dimensions; each such stratum has a neighborhood which is a locally trivial bundle with fiber the cone on a compact manifold with singularities, whose partially ordered set of strata has smaller dimension. This gives rise to a recursive

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[3] We remark that dimension 14 of the ambient space is not crucial: this example can be easily adapted to the lowest possible dimension allowed, that is dimension 10; we also refer to [12] for an example of a 7 dimensional integral homology class which does not admit a smooth representative in a 10 dimensional manifold with torsion-free homology.

construction that enabled Thom to understand and provide a geometric description of singularities.

We will now exploit the geometric obstruction theory described in [82] for reducing the dimension of singularites of a cycle. In particular, suppose T is a triangulated space of dimension m, and $\mathcal{S}_T$ its *singularity locus* of dimension s. Then, for every s-dimensional simplex of $\mathcal{S}_T$ we can consider its *link*, which is a well-defined $m-s-1$-dimensional manifold; this link determines an element in a suitable cobordism group Ω and the sum of the singular simplices with these link coefficients forms a cycle which defines an obstruction, that is

$$\omega_T \in H_s(\mathcal{S}_T, \Omega).$$

If this obstruction vanishes, then it is possible to resolve the singularity by a blow-up technique and reduce their dimension, *cfr.* [82, Theorem D] and [13, Theorem 1.6]. Geometrically, this means that any singular cycle representing a homology class can be resolved by replacing each conic fiber of the top singularity stratum by compact manifolds bounding the links, provided each link bounds a compact submanifold; the recurrence stops as soon as a link of singularities which is not null-cobordant is met.

In particular, if T is an m-dimensional oriented geometric cycle, the natural obstructions lie in

$$H_s(T, \widetilde{\Omega}_r),$$

where $\widetilde{\Omega}_r$ denotes the r-dimensional oriented cobordism group and $r = m-s-1$, which coincides with the dimension of the link of each s-dimensional simplex of $\mathcal{S}_T$; we refer to [60, §17] for an introduction about the oriented cobordism graded ring $\widetilde{\Omega}_*$.

Theorem 2.4.3. *Let $z \in H_7(L, \mathbb{Z})$ be the Thom homology class of Example 2.4.1 and fix a smooth triangulation $\mathcal{K}$ of L. Then it is impossible to find a representative Σ for z which is a smooth embedded submanifold in the complement of the 1-dimensional skeleton $\mathcal{K}^1$ of $\mathcal{K}$.*

Proof. Denote by T the 7-dimensional cycle representing Thom's homology class, and consider its singularity locus $\mathcal{S}_T$. Towards a proof by contradiction, assume that the cycle is a substratified set of a Whitney stratification of L which only intersects the one-skeleton $\mathcal{K}^1$ of a triangulation compatible with the stratification[4].

For each 1-dimensional simplex in the cycle we consider its link, which is a 5-dimensional closed oriented manifold. Since the obstruction to the resolution of singularities is an element of

$$\omega_T \in H_1(\mathcal{S}_T, \widetilde{\Omega}_5)$$

and the 5-dimensional oriented cobordism group $\widetilde{\Omega}_5 \simeq \mathbb{Z}_2$, by [82, Theorem D] the cycle $2T$ can be resolved by blow-up to the lower dimensional stratum, *i.e.* the zero-skeleton $\mathcal{K}^0$.

Analogously, the link of each vertex is a 6-dimensional closed oriented manifold and the oriented cobordism group $\widetilde{\Omega}_6$ is trivial; thus, there is no obstruction to a full

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[4] We refer to [87] and [48] for the notions about stratification theory.

resolution of the singularities of $2T$, and hence of $2z$. This is clearly in contradiction with Thom's algebraic obstruction which is 3-torsion, and that cannot be resolved if we multiply Thom's homology class z by a factor 2. $\square$

Remark 2.4.4. We remark that in [7, page 20] the counterexample to the construction is not correct, since by [86, Corollaire II.28] every 5-dimensional integral homology class in an oriented closed smooth manifold is representable by a smooth embedded submanifold, and hence part (3) of Theorem 2.0.1 provides the desired smooth approximation.

Remark 2.4.5. As a byproduct of the proof of Lemma 3.1.4, it is also possible to show the following. Let $\mathcal{M}, \tau$ and T as in Assumption 2.0.3 and denote $\operatorname{Sing}(T)$ its singular set (in the sense of [36, Definition 0.2]). If $\mathcal{H}^k(\operatorname{Sing}(T)) = 0$ for any $k \in \{1, \dots, m\}$, then for every triangulation $\mathcal{K}$ of $\mathcal{M}$ there exists an integral current T' homologous to T such that $\operatorname{Sing}(T') \subset \mathcal{K}^{k-1}$ with T' smooth in $\mathcal{M} \setminus \mathcal{K}^{k-1}$. Theorem 2.4.3 implies that, in general, for an integral current T representing an integral homology class $\tau \in H_m(\mathcal{M}, \mathbb{Z})$ it is not possible to conclude that $\mathcal{H}^{m-5}(\operatorname{Sing}(T)) = 0$.

Chapter 3

Smooth approximation for integral cycles mod 2

The main goal of Chapter 3 is to present the smooth approximation theorem for integral cycles mod 2, that we recall here[1].

Theorem 3.0.1 (Optimal unoriented smooth approximation)**.** *Let* $\mathcal{M}$ *be a connected smooth closed (not necessarily orientable) Riemannian manifold of dimension* $m+n$*. Let* $\varepsilon > 0$*,* τ *an* m*-dimensional homology class in* $H_m(\mathcal{M}, \mathbb{Z}_2)$*, and* T *an integral cycle mod 2 representing* τ*. Then there is a smooth triangulation* $\mathcal{K}$ *of* $\mathcal{M}$ *and an* m*-dimensional smooth submanifold* Σ *of* $\mathcal{M} \setminus \mathcal{K}^{m-n-1}$ *(where* $\mathcal{K}^{m-n-1}$ *denotes the* $m-n-1$*-skeleton of* $\mathcal{K}$*) with the following properties.*

1. *The* m*-dimensional volume of* Σ *does not exceed the mass of* T *by more than* ε*, that is* $\mathcal{H}^m(\Sigma) \leq \mathbb{M}(T) + \varepsilon$*.*

2. *The integral mod 2 cycle* $[\![\Sigma]\!]_{\mathbb{Z}_2}$ *is homologous to* T *and there is an* $m+1$*-dimensional integral mod 2 current* S *in* $\mathcal{M}$ *such that* $\partial S = [\![\Sigma]\!]_{\mathbb{Z}_2} - T$ *and* $\mathbb{M}(S) < \varepsilon$*.*

3. *If* τ *admits a smooth embedded representative, then* Σ *can be chosen to be a smooth submanifold of* $\mathcal{M}$*.*

Remark 3.0.2. The codimension $n+1$ estimate on the singular set in Theorem 3.0.1 is sharp in full generality, see Theorem 3.4.2; in fact, proving that the construction in Theorem 3.0.1 for mod 2 homology is optimal is subtler than in the case of integral homology, *cfr.* Section 3.4.

Remark 3.0.3. By the foundational work of Thom [86], any mod 2 homology class $\tau \in H_m(\mathcal{M}, \mathbb{Z}_2)$ is representable by a smooth embedded submanifold when $m = 1, 2, 3$ or when $n = 1$; likewise, this is true every time $m \leq n$, *cfr.* [86, Théorème II.26]. The lowest dimensional example not covered by Thom's algebraic computations is the 4-dimensional $\mathbb{Z}_2$ homology group in a 6-dimensional closed smooth manifold $\mathcal{M}$; in fact, it is a corollary

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[1] This chapter is based on [15].

of a construction due to Teichner, see [88], the existence of a 6-dimensional manifold $\mathcal{M}$ with a 4-dimensional homology class $\tau \in H_4(\mathcal{M}, \mathbb{Z}_2)$ which cannot be represented by an embedded – in fact, not even immersed, *cfr.* [49] – smooth submanifold, see Section 3.4.

Since every mod 2 homology class of codimension $n = 1$ is representable by a smooth embedded submanifold, a straightforward application of the same techniques developed in Chapter 2 in the integral setting would provide the following result: any integral cycle mod 2 can be approximated in the sense of Theorem 3.0.1 with a smooth submanifold up to a singular set of codimension 3, regardless of the codimension $n \geq 2$. The key difference in the mod 2 setting is based on the better-behaved topological structure of the Thom space of the universal n-plane bundle over the infinite Grassmannian $BO(n)$, allowing for a refined result and improving the estimate on the singular set based on the codimension n of the cycle, see Theorem 3.3.1 and Remark 3.3.3.

Remark 3.0.4. We recall that every mod 2 homology class $\sigma \in H_m(\mathcal{M}, \mathbb{Z}_2)$ admits a *Steenrod representation*, *i.e.* there exists a smooth manifold Σ and a continuous map $f : \Sigma \to \mathcal{M}$ such that the fundamental class of Σ equals σ, *cfr.* [86, Théorème III.2]. This differs substantially from the integral setting, where there exist integral homology classes that cannot be represented in the sense of Steenrod, *cfr.* [86, Théorème III.9]. For this reason, the problem of finding smooth embedded representatives in mod 2 homology classes can be understood as a problem of whether a continuous map is homotopic to an embedding.

In Chapter 3 we will always rely on the following assumptions, unless otherwise stated.

3.0.5. Assumption. $\mathcal{M}$ is a connected smooth closed (not necessarily orientable) Riemannian manifold of dimension $m + n$, where $n, m \in \mathbb{N} \setminus \{0\}$ are arbitrary positive integers, τ is an element of the m-dimensional homology group $H_m(\mathcal{M}, \mathbb{Z}_2)$ and T is an integral mod 2 current (hence a cycle) representing τ. Σ will denote either smooth m-dimensional closed embedded (not necessarily orientable) submanifolds of $\mathcal{M}$ or smooth m-dimensional embedded submanifolds of $\mathcal{M} \setminus \mathcal{K}^j$ (for some integer j) whose topological closure is contained in $\mathcal{K}^j$; We will denote by $[\![\Sigma]\!]_{\mathbb{Z}_2}$ the integral cycle mod 2 induced by Σ.

Definition 3.0.6 (*Smooth representability in* $\mathbb{Z}_2$)**.** Let $\mathcal{M}, \tau$ and Σ be as in Assumption 3.0.5. We say that τ is *representable by a smooth submanifold* (or that τ *admits a smooth representative*) if there exists a smooth embedding $f : \Sigma \to \mathcal{M}$ such that $f_*[\Sigma] = \tau$, where $[\Sigma] \in H_m(\Sigma, \mathbb{Z}_2)$ is the fundamental class of Σ (or, equivalently, $[\![\Sigma]\!]_{\mathbb{Z}_2} \in \tau$). Analogously, denoting by $x \in H^n(\mathcal{M}, \mathbb{Z}_2)$ the Poincaré dual of τ, we say that x is realized by a smooth submanifold whenever τ is.

3.0.1 Consequences of the principal theorem

In analogy with Chapter 2, an immediate corollary of Theorem 3.0.1 is the absence of the *Lavrentiev gap phenomenon* for the homological unoriented Plateau problem in absence of topological ostructions to realizability of mod 2 homology classes.

Theorem 3.0.7 (Absence of Lavrentiev gaps). *Let $\mathcal{M}, \tau, T$ and Σ as in Assumption 3.0.5, and define the following quantities:*

$$\begin{aligned}
&\mathbb{M}_T := \min\{\mathbb{M}(T) : T \in \mathscr{Z}_m(\mathcal{M}, \mathbb{Z}_2) \cap \tau\},\\
&\mathbb{M}_P := \inf\{\mathbb{M}(P) : P \in \mathscr{Z}_{m,Lip}(\mathcal{M}, \mathbb{Z}_2) \cap \tau\},\\
&\mathbb{M}_\Sigma := \inf\{\, Vol^m(\Sigma) : [\![\Sigma]\!]_{\mathbb{Z}_2} \in \tau \text{ and } \Sigma \text{ is smooth in } \mathcal{M} \setminus \mathcal{K}^{m-n-1} \text{ for some triangulation } \mathcal{K}\},\\
&\mathbb{M}_{\mathrm{Reg}} := \inf\{\, Vol^m(\Sigma) : [\![\Sigma]\!]_{\mathbb{Z}_2} \in \tau \text{ and } \Sigma \text{ is smooth in } \mathcal{M}\}\,.
\end{aligned}$$

Then, $\mathbb{M}_T = \mathbb{M}_P = \mathbb{M}_\Sigma$ and, moreover, $\mathbb{M}_T = \mathbb{M}_{\mathrm{Reg}}$ when τ is representable by a smooth submanifold.

Remark 3.0.8. We remark that for an integral mod 2 cycle $T \in \mathscr{Z}_m(\mathcal{M}, \mathbb{Z}_2)$ its mod 2 mass $\mathbb{M}(T)$ coincides with its *size*, that is the m-dimensional Hausdorff measure $\mathcal{H}^m(R)$ of the corresponding rectifiable set R.

Another simple corollary of Theorem 3.0.1 is the following approximation theorem with integral mod 2 cycles of prescribed singularities.

Theorem 3.0.9 (Approximation by cycles with prescribed singular sets). *Let $\mathcal{M}, \tau$ and T be as in Assumption 3.0.5. Then, there is a sequence of smooth triangulations $\mathcal{K}_j$ of $\mathcal{M}$ and a sequence of smooth embedded m-dimensional submanifolds $(\Sigma_j)_j$ in $\mathcal{M} \setminus \mathcal{K}_j^{m-n-1}$ such that*

- *(a)* $[\![\Sigma_j]\!]_{\mathbb{Z}_2} \to T$ *in the mod 2 flat topology,*
- *(b)* $\lim_{j\to\infty} \mathcal{H}^m(\Sigma_j) = \mathbb{M}(T)$,
- *(c)* $\partial[\![\Sigma_j]\!]_{\mathbb{Z}_2} = 0$ *and* $[\![\Sigma_j]\!]_{\mathbb{Z}_2}$ *is in the same homology class as* T.

Remark 3.0.10. If every mod 2 homology class of $\mathcal{M}$ admits a smooth embedded representative, then it is customary to call such manifolds $\mathbb{Z}_2$ *totally realizable.* Examples of $\mathbb{Z}_2$ totally realizable manifolds are spheres S^k and products of spheres $S^{k_1} \times S^{k_2} \times \ldots \times S^{k_i}$, or projective spaces $\mathbb{RP}(n)$ and $\mathbb{CP}(n)$ of real and complex dimension n respectively, *cfr.* [83, Section 8]. Clearly, in all such manifolds every mod 2 integral cycles can be approximated by smooth submanifolds by Theorem 3.0.1.

3.0.2 Overview of the proof

Differences with integral homology and new constructions

We describe here the main ideas proof, whose construction closely follows the approach introduced in Chapter 2. In particular, we emphasize its main differences from integral homology, which are mostly of topological nature and allow for the development of a finer argument than the one used in Chapter 2. This yields a significantly better estimate for the singular set, which further improves with higher codimensions and which is due to the particularly well-behaved homotopy type of the Thom space of the (unoriented) n-plane bundle, see Theorem 3.3.1 and Corollary 3.3.2. The key geometric measure theory ingredients – like the *codimension 2* smooth approximation theorem [8, Proposition 4.1] and the *squeezing* lemma [8, Proposition 4.3] – are, instead, easily adapted from the integral setting and hold for integral mod 2 cycles with minor modifications, see Section 3.2. Moreover, it is worth mentioning that proving sharpness of the codimension $n+1$ singular set of Theorem 3.0.1 is even subtler in mod 2 homology than in integral homology, since singularities that appear in this context all arise from the impossibility of finding embeddings in low codimensions, and not – as for integral classes – by innate singularities obstructing Steenrod representability also; in particular, we cannot exploit Sullivan's geometric theory of resolution of singularities by blow-up in [82], as done in [8, Theorem 6.3]. Instead, we need to rely on the singularity theory of stable mappings, coupled with an elegant result due to Grant and Szűcs [49] about obstructions to realizability of mod 2 homology classes by immersions, see Section 3.4; we refer to [47] for an accessible overview on the theory of stable mappings.

Outline of the proof

Starting from an integral mod 2 cycle T representing a $\mathbb{Z}_2$ homology class τ of $\mathcal{M}$, we first approximate it with an integral mod 2 cycle P' which is a smooth submanifold out of a small δ-neighborhood B_δ of the $m-2$-skeleton of some smooth triangulation of $\mathcal{M}$, *cfr.* Proposition 3.2.1; in particular, a subset of the smooth part of P' is a compact smooth submanifold with boundary embedded in a compact manifold with boundary denoted Ω (as in Chapter 2, Ω can be thought as $\mathcal{M} \setminus B_\delta$, up to taking smooth neighborhoods of skeleta described in Section 3.1). This smooth part represents a relative homology class in $H_m(\Omega, \partial\Omega, \mathbb{Z}_2)$ and by the relative Pontrjagin-Thom construction, *cfr.* Theorem 1.3.7, it induces a map $F : \Omega \to T(\gamma^n)$ with values in the Thom space of the universal n-plane bundle such that the pull-back of the Thom class is the Poincaré dual of τ, once restricted to Ω.

At this point, had we used the strategy adopted in the integral setting, we would now have simply been able to exploit the $n+2$-equivalence between $T(\gamma^n)$ and the Eilenberg-MacLane space $K(\mathbb{Z}_2, n)$ to prove that the restriction of the Poincaré dual of τ to the complement of a small neighborhood U_δ of the $m-3$-skeleton of the smooth triangulation of $\mathcal{M}$ admits a lift to $T(\gamma^n)$ pulling back the Thom class to itself; this would have provided an integral mod 2 cycle in τ given by a closed smooth embedded submanifold with singularities all contained in the $m-3$-dimensional skeleton $\mathcal{K}^{m-3}$ of $\mathcal{M}$.

Instead, the homotopy type of $T(\gamma^n)$ behaves much better in the mod 2 case since, for every $n \geq 1$, $T(\gamma^n)$ is $2n$-equivalent to a complex Y which is a *product* of Eilenberg-Mac Lane spaces $K(\mathbb{Z}_2, n)$. This nicer structure allows us to exploit the $2n$-equivalence $H : T(\gamma^n) \to Y$ to obtain a map g from the $2n$-skeleton of $K(\mathbb{Z}_2, n)$ to $T(\gamma^n)$ pulling-back the Thom class to the fundamental class of $K(\mathbb{Z}_2, n)$, simply by taking the restriction to the first factor of the inverse map of H, *cfr.* Corollary 3.3.2. Nevertheless, we remark that it is still crucial for us to rely on the fact that the map $h : T(\gamma^n) \to K(\mathbb{Z}_2, n)$ itself is an $n+2$-equivalence for any $n \geq 1$, *cfr.* Lemma 3.3.4, since this allows us to perform the last part of the argument of the whole proof.

Hence, denoting by Q the complement of a small neighborhood U_δ of the $m-n-1$-skeleton of the smooth triangulation of $\mathcal{M}$, it is possible to note that Q has the homotopy type of a $2n$-dimensional skeleton of $\mathcal{M}$, *cfr.* Lemma 3.1.3. Thus, we exploit the $2n$-equivalence between $T(\widetilde{\gamma}^n)$ and the product Y of Eilenberg-MacLane space $K(\mathbb{Z}_2, n)$, *cfr.* Theorem 3.3.1, to obtain a map $g : K(\mathbb{Z}_2, n)^{2n} \to T(\gamma^n)$ pulling-back the Thom class to the fundamental class of $K(\mathbb{Z}_2, n)$ as described above; this allows us to prove that the restriction of the Poincaré dual of τ to Q admits a lift

$$f : Q \to T(\gamma^n) \tag{3.1}$$

pulling back the Thom class to itself. Applying Theorem 1.3.7 and after some technicalities, this provides an integral mod 2 cycle R homologous to τ which is a closed smooth embedded submanifold with singularities all contained in the $m-n-1$-dimensional skeleton $\mathcal{K}^{m-n-1}$ of $\mathcal{M}$.

Since, by Lemma 3.1.3, Ω has the homotopy type of an $n+1$-dimensional complex, the $n+2$-equivalence between $T(\gamma^n)$ and $K(\mathbb{Z}_2, n)$ of Lemma 3.3.4 allows us to conclude that homotopy classes of maps defined on Ω and with values in $T(\gamma^n)$ are in one-to-one correspondence with those with values in $K(\mathbb{Z}_2, n)$, *cfr.* Corollary 1.3.2. Hence, we can conclude that the smooth part of P' coincides, up to a homotopy, with the smooth part of R restricted to Ω.

The conclusion then follows in analogy with Chapter 2 from a technical geometric measure theory *squeezing* lemma, *cfr.* Proposition 3.2.3, saying that if two m-dimensional integral mod 2 cycles P' and R agree outside of a sufficiently small neighborhood of the $m-2$-skeleton of a smooth triangulation of $\mathcal{M}$, then we can find a smooth deformation R' of R which is almost coinciding with R and with mass close to the mass of P'. This provides the desired approximation R' of Theorem 3.0.1, satisfying (1) and (2).

Finally, part (3) of Theorem 3.0.1 is proved following the same lines: under the additional assumption that τ is representable by a smooth embedded submanifold we immediately obtain a map

$$\ell : \mathcal{M} \to T(\gamma^n)$$

which pulls-back the Thom class to the Poincaré dual of τ; the analogous construction can thus be performed just by replacing the map f in (3.1) with ℓ.

The rest of the chapter is organized as follows. In Section 3.1 we recall from Chapter 2 some technical preliminary lemmas about neighborhoods of skeleta and maps associated to them. In Section 3.2 we state the adapted mod 2 version of the two main technical propositions from geometric measure theory, *i.e.* Proposition 3.2.1 and Proposition 3.2.3, and we comment on their proofs. Finally, Section 3.3 is dedicated to the proof of Theorem 3.0.1 and Section 3.4 shows the optimality of the construction.

Recall Thom's fundamental result about realizability of cycles by means of submanifolds can be stated as follows, *cfr.* [86, Théorème II.1].

Theorem 3.0.11. *Given $\mathcal{M}$ and τ as in Assumption 3.0.5, a homology class $\tau \in H_m(\mathcal{M}, \mathbb{Z}_2)$ is representable by a m-dimensional smooth submanifold $\Sigma \subset \mathcal{M}$ of codimension n if and only if there exists a map $f : \mathcal{M} \to T(\gamma^n)$ which pulls back the Thom class $u \in H^n(T(\gamma^n), \mathbb{Z}_2)$ to the Poincaré dual of τ.*

By suitably modifying the proof of [86, Théorème II.1], it is possible to derive the analog of Theorem 3.0.11 for compact manifolds with boundary: for a proof we refer to [8, Theorem 2.6], which holds almost *verbatim*, just replacing integral homology with homology mod 2, and $T(\widetilde{\gamma}^n), BSO(n)$ with $T(\gamma^n), BO(n)$ respectively.

Theorem 3.0.12. *Let $\mathcal{M}$ be a connected smooth compact $m+n$-dimensional Riemannian manifold with boundary $\partial\mathcal{M}$ and τ a relative homology class $\tau \in H_m(\mathcal{M}, \partial\mathcal{M}, \mathbb{Z}_2)$. Then τ is representable[2] by a smooth compact embedded submanifold manifold $\Sigma \subset \mathcal{M}$ with $\partial\Sigma = \Sigma \cap \partial\mathcal{M}$ if and only if there exists a map $f : \mathcal{M} \to T(\gamma^n)$ which pulls back the Thom class $u \in H^n(T(\gamma^n), \mathbb{Z}_2)$ to the relative Poincaré dual of τ.*

3.1 Regular neighborhoods and maps

3.1.1 Neighborhoods of skeleta

We recall here a few technical tools introduced in Chapter 2 to deal with neighborhoods of skeleta of smooth triangulations and we recall the definition of some useful maps.

For a fixed smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and a skeleton $\mathcal{K}^j$, we denote by $B_\delta(\mathcal{K}^j)$ the metric neighborhoods of the skeleton, *i.e.* the sets of points p with $\operatorname{dist}(p, \mathcal{K}^j) < \delta$. Fix a (sufficiently large) constant C_0 which will depend on the triangulation $\mathcal{K}$, subdivide the simplices forming $\mathcal{K}^j$ into $\mathcal{S}_0 \cup \ldots \cup \mathcal{S}_j$ according to their dimension ($\mathcal{S}_i$ being the collection of simplices of dimension i) and set

$$V_\delta(\mathcal{K}^j) := \bigcup_{i=0}^{j} \bigcup_{\sigma \in \mathcal{S}_i} B_{C_0^{-i}\delta}(\sigma) \tag{3.2}$$

where

$$B_{C_0^{-i}\delta}(\sigma) = \{p : \operatorname{dist}(p, \sigma) < C_0^{-i}\delta\}\,.$$

As an elementary consequence of the definition, one obtains the following.

Lemma 3.1.1. *For every smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and every $j \leq m+n-1$ there is a choice of $\bar{\delta} > 0$ (sufficiently small) and of C_0 (sufficiently large) such that $\mathcal{M} \setminus V_{\bar{\delta}}(\mathcal{K}^j)$ is a deformation retract of $\mathcal{M} \setminus \mathcal{K}^j$ and such that for every point $p \in \partial V_{\bar{\delta}}(\mathcal{K}^j)$ there is at most one σ in each $\mathcal{S}_i$ (with $0 \leq i \leq j$) for which $p \in \partial B_{C_0^{-i}\bar{\delta}}(\sigma)$.*
In particular, there is a neighborhood of U of p, an integer $\bar{j} \in \{1, \ldots, j\}$ and a diffeomorphism $\phi : U \to B_1 \subset \mathbb{R}^{m+n}$ (the unit ball in $\mathbb{R}^{m+n}$) such that

$$\phi(U \setminus V_{\bar{\delta}}(\mathcal{K}^j)) = \{(x_1, \ldots, x_{m+n}) : x_i > 0 \text{ for } 1 \leq i \leq \bar{j}\}\,.$$

---

[2] With the clear modifications in Definition 3.0.6 for $\mathcal{M}, \Sigma$ with boundary and τ a relative mod 2 homology class.

We also recall the appropriate regularization we need, and a key lemma about the homotopy type of complements of skeleta.

Lemma 3.1.2. *Let $\mathcal{K}$ be a smooth triangulation of $\mathcal{M}$, let $\bar{\delta}$ and C_0 be given by Lemma 3.1.1 and fix any pair of positive numbers $\delta' < \delta < \bar{\delta}$. Then there is a neighborhood $U_\delta(\mathcal{K}^j)$ of $\mathcal{K}^j$ with the following properties:*

- $V_\delta(\mathcal{K}^j) \supset U_\delta(\mathcal{K}^j) \supset V_{\delta'}(\mathcal{K}^j)$;
- *The boundary of $U_\delta(\mathcal{K}^j)$ is smooth;*
- *$\mathcal{M} \setminus U_\delta(\mathcal{K}^j)$ is a deformation retract of $\mathcal{M} \setminus V_{\delta'}(\mathcal{K}^j)$;*
- *There is a smooth tubular neighborhood $\mathcal{C}$ of $\partial U_\delta(\mathcal{K}^j)$ in $\mathcal{M}$ containing $\partial V_{\delta'}(\mathcal{K}^j)$.*

Lemma 3.1.3. *Let $\mathcal{M}$ be as in Assumption 3.0.5, $\mathcal{K}$ a smooth triangulation of $\mathcal{M}$, $k \in \{0, \dots, m+n-1\}$ and $U_\delta(\mathcal{K}^k)$ as in Lemma 3.1.2. Then the complement of $U_\delta(\mathcal{K}^k)$ is homotopy equivalent to a complex of dimension $m+n-k-1$.*

Proof. See Lemma 1.2.7. □

3.1.2 Squeezing maps

We associate to the neighborhoods B_δ, V_δ some maps whose definition is recalled in the next lemma; we refer to Chapter 2 for a proof.

Lemma 3.1.4. *Let $\mathcal{M}$ be as in Assumption 3.0.5 and $\mathcal{K}$ a smooth triangulation of $\mathcal{M}$. For every $\varepsilon_a > 0$ and $\eta_a > 0$ there is a positive number $\delta_a < \eta_a$ with the following property. If $\gamma \in (0, 1]$ is an arbitrary number, then there is a diffeomorphism Φ such that:*

1. *Φ is isotopic to the identity and it coincides with the identity on $\mathcal{M} \setminus B_{\eta_a}(\mathcal{K}^k)$;*
2. $\mathrm{Lip}(\Phi) \le 1 + \varepsilon_a$;
3. *For every point $p \in B_{\delta_a}(\mathcal{K}^k)$ there is an orthonormal frame $e_1, \dots, e_{m+n}$ such that*

$$|d\Phi_p(e_i)| \le 1 + \varepsilon_a \qquad \forall i \in \{1, \dots, k\}, \tag{3.1}$$
$$|d\Phi_p(e_j)| \le \gamma \qquad \forall j \in \{k+1, \dots, m+n\}\,. \tag{3.2}$$

We also recall the following simpler lemma, obtained as a minor modification of the former.

Lemma 3.1.5. *Let $\mathcal{M}$ be as in Assumption 3.0.5, $\mathcal{K}$ a triangulation, $j \in \{0, \dots, m+n-1\}$. If C_0 and $\bar{\delta}^{-1}$ in Lemma 3.1.1 are chosen sufficiently large, then for every $\delta_b > \delta'_b > 0$ there is a Lipschitz map $\Phi : \mathcal{M} \to \mathcal{M}$ with the following properties:*

- *Φ maps $V_{\delta'_b}(\mathcal{K}^j)$ into $\mathcal{K}^j$;*

- *Φ is a smooth diffeomorphism between $\mathcal{M} \setminus \Phi^{-1}(\mathcal{K}^j)$ and $\mathcal{M} \setminus \mathcal{K}^j$;*
- *$\Phi(p) = p$ for every $p \notin V_{\delta_b}(\mathcal{K}^j)$.*

Finally we recall the following consequence of the area formula, that is easily adapted from the integral setting to integral currents mod 2, taking into account that if $[T]$ is an integral current mod 2, $\varepsilon > 0$ and U an open set containing $\mathrm{spt}[T]$, then there exists an integral current $S = T$ mod 2 such that $\mathrm{spt}(S) \subset U$ and $\mathbb{M}(S) < \mathbb{M}([T]) + \varepsilon$.

Lemma 3.1.6. *Assume Φ, γ, and ε_a are as in Lemma 3.1.4. If Z is any integer rectifiable current mod 2 of dimension $m > k$, then*

$$\mathbb{M}(\Phi_\sharp(Z \llcorner B_{\delta_d}(\mathcal{K}^k))) \leq C(1+\varepsilon_a)^k \gamma^{m-k} \|Z\|(B_{\delta_d}(\mathcal{K}^k)), \tag{3.3}$$

where C is a dimensional constant which depends only on m and n.

3.2 Geometric measure theory propositions

We state the mod 2 analogs of the main geometric measure theory tools needed in the proof of Theorem 3.0.1, which have independent interest.

3.2.1 Codimension 2 smooth approximation

The first statement – Proposition 3.2.1 – is the mod 2 version of the *codimension 2* smooth approximation theorem in [8, Section 4], that is an upgraded version of the Federer and Fleming's strong polyhedral approximation theorem in the spirit of [43, (4.2.21)$^\nu$][3], where the approximands are smooth submanifolds out from the $m-2$-skeleton $\mathcal{K}^{m-2}$ of a smooth triangulation of $\mathcal{M}$. The proof is an easy adaptation of the one in Chapter 2 for integral cycles and we remark that since we are dealing with integral cycles mod 2, the integral current representative has multiplicity one (and possibly boundary of multiplicity 2) by definition; hence, the proof of Proposition 3.2.1 is even simpler than in the integral setting since there is no need to perform Step 1 in the *second approximation* (*i.e.* clearing out the multiplicity and regularizing $\mathcal{M} \setminus \mathcal{K}^{m-1}$).

Proposition 3.2.1. *Let $\mathcal{M}$ be as in Assumption 3.0.5 and T be an m-dimensional integral cycle mod 2 in $\mathcal{M}$. For every fixed $\varepsilon_c > 0$ there is a mod 2 integral cycle P homologous to T and a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ with the following properties:*

(a_0) $\mathbb{M}(P) \leq (1+\varepsilon_c)\mathbb{M}(T)$;

(b_0) $\mathbb{F}(T-P) \leq \varepsilon_c$;

(c_0) $\mathrm{spt}(P) \subset \{x : \mathrm{dist}(x, \mathrm{spt}(T)) \leq \varepsilon_c\}$;

---

[3] See also [59, Theorem 3.4].

(d_0) *for every* $p \in \operatorname{spt}(P) \setminus \mathcal{K}^{m-1}$ *there is a neighborhood* U *of* p *and a smooth* m*-dimensional submanifold* Λ *of* $\mathcal{M} \cap U$ *with boundary contained in* $\mathcal{M} \cap \partial U$ *and such that* $P \llcorner U = [\![\Lambda]\!]$.

Furthermore, for a sufficiently small $\delta'_c > 0$ *and any* $\delta_c < \delta'_c$ *we can find another mod 2 integral cycle* P' *homologous to* P *with the following properties:*

(*a*) $\mathbb{M}(P') \leq (1 + 2\varepsilon_c)\mathbb{M}(T)$ *and* $\mathbb{F}(T - P') \leq 2\varepsilon_c$*;*

(*b*) $\operatorname{spt}(P') \subset \{x : \operatorname{dist}(x, \operatorname{spt}(T)) \leq 2\varepsilon_c\}$*;*

(*c*) $\|P'\|(B_{\delta'_c}(\mathcal{K}^{m-2})) \leq 2\varepsilon_c$*;*

(*d*) $P' \llcorner \mathcal{M} \setminus B_{\delta_c}(\mathcal{K}^{m-2}) = [\![\Gamma]\!]_{\mathbb{Z}_2}$ *for some smooth submanifold* Γ *of* $\mathcal{M} \setminus B_{\delta_c}(\mathcal{K}^{m-2})$ *without boundary in* $\mathcal{M} \setminus B_{\delta_c}(\mathcal{K}^{m-2})$.

Proof. Choose an integral mod 2 cycle T in $\mathcal{M}$ and consider it as a mod 2 integral cycle in $\mathbb{R}^d$. Apply now the *first approximation* of [8, Proposition 4.1], which can be readily adapted to mod 2 cycles, to obtain a mod 2 integral cycle P homologous to T with $\operatorname{spt}(P) \subset \mathcal{M}$ and such that

(a_0) $\mathbb{M}(P) \leq (1 + \varepsilon_c)\mathbb{M}(T)$;

(b_0) $\mathbb{F}(T - P) \leq \varepsilon_c$;

(c_0) $\operatorname{spt}(P) \subset \{x : \operatorname{dist}(x, \operatorname{spt}(T)) \leq \varepsilon_c\}$;

(d_0) $P = \sum_{F \in \mathcal{F}^m} [\![F]\!]$ mod 2, where $\mathcal{F}^m$ is the collection of m-dimensional oriented cells of $\mathcal{K}$, for a suitable choice of $\mathcal{K} = \mathcal{K}(T, \varepsilon)$.

Starting with P we now need to perform the *second approximation* of [8, Proposition 4.1], which was divided in Step 1 (regularization in the complement of $\mathcal{K}^{m-1}$) and Step 2 (removal of the $m-1$-dimensional singularities); note that since P has multiplicity 1 for $\|P\|$-a.e. point, P is already regular on $\mathcal{M} \setminus \mathcal{K}^{m-1}$, with no need to perform Step 1.

We now only need to perturb P by removing its $m-1$-singularities away from a small neighbourhood of $\mathcal{K}^{m-1}$, which follows closely the procedure done in Chapter 2; for the reader's convenience, we recall it below with the needed minor adjustments.

Removing the $m-1$-dimensional singularities.

Consider an arbitrary face F^k and let σ_i be an arbitrary $m-1$-dimensional face of F^k. Fixing a $\delta_c > 0$, we modify P in a neighborhood of $\sigma \setminus B_{\delta_c}(\mathcal{K}^{m-2})$ to a new current P' in the same homology class, close to it in terms of mass, support and flat norm, and with the property that P' is smooth in that neighborhood. The neighborhood in which we perturb P is of the form $B_\lambda(\sigma) \setminus B_{\delta_c}(\mathcal{K}^{m-2})$.

Given the structure of P just obtained, if λ is sufficiently small, there is an open subset $U_i \subset \mathbb{R}^{m-1}$ and a smooth parametrization

$$\Phi : U_i \times B^{n+1}_\lambda \to \mathcal{M}$$

of a normal neighborhood $\mathcal{N}_i$ of $\sigma \setminus B_{\delta_c}(\mathcal{K}^{m-2})$ with thickness λ_i and with the property that $\mathrm{spt}(P \llcorner \mathcal{N}_i)$ can be described in the following way. There are two distinct unit vectors $w_1, w_2 \in \partial B_1^{n+1} \subset \mathbb{R}^{n+1}$ such that, if we let

$$\Lambda_\ell = \{(\sigma, s w_\ell) : \sigma \in U_i, 0 < s < \lambda_i\},$$

then, since P has no mod 2 boundary in $\mathcal{N}_i$,

$$P \llcorner \mathcal{N}_i = \Phi_\sharp [\![\Lambda_1]\!] + \Phi_\sharp [\![\Lambda_2]\!] \bmod 2.$$

Consider now the halflines $H_\ell = \{\lambda w_\ell : \lambda > 0\}$ for $\ell \in \{1,2\}$ in $\mathbb{R}^{n+1}$. We can find a smooth curve γ in $\mathbb{R}^{n+1}$ such that $[\![\gamma]\!] \llcorner \mathbb{R}^{n+1} \setminus B_1 = ([\![H_1]\!] - [\![H_2]\!]) \llcorner \mathbb{R}^{n+1} \setminus B_1$ mod 2. Furthermore we let $\tau_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the homothety $y \mapsto ty$ and denote by γ_t the curve $\tau_t(\gamma)$. We are now ready to define a replacement for $P \llcorner \mathcal{N}_i$. We fix a smooth compactly supported function ψ_i in $\mathbb{R}^{n+1}$ which is positive on U_i and vanishes on ∂U_i, a small positive number κ_i, and we define

$$\Sigma^i := \left\{(x,y) : x \in U_i, y \in \gamma_{\kappa_i \psi_i(x)}\right\} \cap U_i \times B^{n+1}_{\lambda_i}.$$

Choosing κ_i sufficiently small we can ensure that the mod 2 current $P^i := \Phi_\sharp \Sigma^i$ satisfies $\partial P^i = \partial (P \llcorner \mathcal{N}_i)$. Moreover we can make $\mathbb{F}(P^i - P \llcorner \mathcal{N}_i)$ and $\mathbb{M}(P^i) - \mathbb{M}(P \llcorner \mathcal{N}_i)$ smaller than any desired threshold by choosing κ_i sufficiently small. Note finally that, clearly, Σ_i is smooth in $\mathcal{N}_i$.

We next enumerate all the $m-1$-dimensional faces σ_i of all the m-dimensional faces F^k as $\sigma_1, \sigma_2, \ldots, \sigma_N$. We choose our parameters in such a way that the sets $\mathcal{N}_i$ are pairwise disjoint. Our final mod 2 current P' will then be defined to be

$$P' := \sum_i P^i + P \llcorner \mathcal{M} \setminus \bigcup_i \mathcal{N}_i .$$

$P' \llcorner \mathcal{M} \setminus B_{\delta_c}(\mathcal{K}^{m-2})$ is then smooth by construction and, choosing the parameters accordingly, P' is homologous to P and we achieve the desired estimates since we can make $\mathrm{spt}(P')$ arbitrarily close to $\mathrm{spt}(P)$, $\mathbb{M}(P')$ arbitrarily close to $\mathbb{M}(P)$, and $\mathbb{F}(P' - P)$ arbitrarily small.

Finally, coming the estimate on $\|P'\|(B_{\delta_c'}(\mathcal{K}^{m-2}))$, the proof follows *verbatim* that of Chapter 2, with δ_c' chosen small enough just to ensure that $\|P\|(B_{\delta_c'}(\mathcal{K}^{m-2})) \leq \varepsilon_c$. $\square$

Remark 3.2.2. A routine modification of the arguments in the proof of Proposition 3.2.1 implies that the mod 2 cycle P' can be chosen so that its singularities are all *contained* in $\mathcal{K}^{m-2}$.

3.2.2 Squeezing lemma

In the second statement − Proposition 3.2.3, which is the mod 2 version of the *squeezing* lemma in Chapter 2 − we are given two m-dimensional integral mod 2 cycles S and R which agree outside of a sufficiently small neighborhood of the $m-2$-dimensional skeleton $\mathcal{K}^{m-2}$. We will then show that:

- S and R represent the same homology class;
- There is a smooth deformation R' of R which is close, in terms of mass and in flat norm, to S;
- R' coincides with S outside a slightly larger neighborhood of $\mathcal{K}^{m-2}$.

Proposition 3.2.3. *Let m and $\mathcal{M}$ be as in Assumption 3.0.5 and $\mathcal{K}$ a smooth triangulation of $\mathcal{M}$. Then for every $\varepsilon_d > 0$ and every $\eta_d > 0$ there exists $\delta_d(\varepsilon_d, \eta_d, \mathcal{K}, \mathcal{M}) > 0$ with the following property. Suppose S and R are m-dimensional integral cycles mod 2 in $\mathcal{M}$ and that*

$$S \llcorner \mathcal{M} \setminus B_{\delta_d}\left(\mathcal{K}^{m-2}\right) = R \llcorner \mathcal{M} \setminus B_{\delta_d}\left(\mathcal{K}^{m-2}\right) .$$

Then S and R are homologous and moreover there exist a mod 2 integral cycle R' in their homology class and a diffeomorphism Φ of $\mathcal{M}$ with the following properties:

1. $\mathbb{M}(R') \leq (1+\varepsilon_d)\mathbb{M}(S)$;
2. $\mathbb{F}(R'-S) \leq C(\varepsilon_d \mathbb{M}(S) + 2\|S\|(B_{\eta_d}(\mathcal{K}^{m-2})))^{\frac{m+1}{m}}$, *with* $C = C(\mathcal{M})$;
3. $R' \llcorner \mathcal{M} \setminus B_{\eta_d}(\mathcal{K}^{m-2}) = S \llcorner \mathcal{M} \setminus B_{\eta_d}(\mathcal{K}^{m-2})$;
4. Φ *is in the isotopy class of the identity and* $R' = \Phi_\sharp R$.

Proof. The proof follows *verbatim* that of Chapter 2; for the reader's convenience, we recall it below.

Without loss of generality we can assume that the $\mathbb{Z}_2$ homology class of S is nontrivial, so that $\mathbb{M}(S) > 0$. The conclusion that R and S are homologous follows from the fact that they coincide outside $B_{\delta_d}(\mathcal{K}^{m-2})$. In particular $\operatorname{spt}(S-R) \subset B_{\delta_d}(\mathcal{K}^{m-2})$: since for δ_d smaller than a constant $c(\mathcal{K})$ the latter has trivial m-dimensional homology, it follows that $S-R$ is homologically trivial.

We now let ε_d and η_d be given as in the statements. We further fix $\bar{\varepsilon}_d$, whose choice will be specified later (but which will depend only on ε_d), and apply Lemma 3.1.4 with $\varepsilon_a = \bar{\varepsilon}_d$ and $\eta_a = \eta_d$. We therefore get the parameter $\delta_a =: \delta_d$ (which will be the one of the conclusion of the proposition) and, after fixing yet another γ (whose choice will now depend on R), we get the map Φ satisfying the requirements of Lemma 3.1.6. The requirements (3) and (4) of Proposition 3.2.3 are then satisfied by construction. Moreover estimate (2) follows from the isoperimetric inequality for mod 2 integral currents and from (1) and (3). Indeed there is a mod 2 integral current T such that $\partial T = S - R'$ and

$$\mathbb{M}(T) \leq C\left(\mathbb{M}(S-R')\right)^{\frac{m+1}{m}}$$

with $C = C(\mathcal{M})$. Using (1) and (3) we then estimate

$$\begin{aligned}
\mathbb{M}(S-R') &= \|S-R'\|(B_{\eta_d}(\mathcal{K}^{m-2})) \leq \|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \|R'\|(B_{\eta_d}(\mathcal{K}^{m-2})) \\
&= \|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \mathbb{M}(R') - \|S\|(\mathcal{M} \setminus B_{\eta_d}(\mathcal{K}^{m-2})) \\
&= 2\|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \mathbb{M}(R') - \mathbb{M}(S) \\
&\leq 2\|S\|(B_{\eta_d}(\mathcal{K}^{m-2})) + \varepsilon_d \mathbb{M}(S) . \qquad (3.4)
\end{aligned}$$

It remains to prove (1). Note that

$$\begin{aligned}\mathbb{M}(R') &\leq \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + \mathbb{M}(\Phi_\sharp(R \llcorner \mathcal{M} \setminus B_{\delta_d}(\mathcal{K}^{m-2}))) \\ &= \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + \mathbb{M}(\Phi_\sharp(S \llcorner \mathcal{M} \setminus B_{\delta_d}(\mathcal{K}^{m-2}))) \\ &\leq \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + (\operatorname{Lip}\Phi)^m \mathbb{M}(S) \\ &\leq \mathbb{M}(\Phi_\sharp(R \llcorner B_{\delta_d}(\mathcal{K}^{m-2}))) + (1+\bar{\varepsilon}_d)^m \mathbb{M}(S)\,. \end{aligned} \tag{3.5}$$

Hence we apply Lemma 3.1.6 and infer

$$\mathbb{M}(R') \leq C(1+\bar{\varepsilon}_d)^{m-2}\gamma^2 \mathbb{M}(R) + (1+\bar{\varepsilon}_d)^m \mathbb{M}(S)\,.$$

Next we fix $\bar{\varepsilon}_d$ so that $(1+\bar{\varepsilon}_d)^m = 1 + \frac{\varepsilon_d}{2}$, and then we choose γ sufficiently small so that

$$C(1+\bar{\varepsilon}_d)^{m-2}\gamma^2 \mathbb{M}(R) \leq \frac{\varepsilon_d}{2}\mathbb{M}(S)\,.$$

Note that the choice of γ, unlike that of ε_d, will indeed depend on R and S. $\square$

3.3 Proof of the principal theorem

Before coming to the proof of Theorem 3.0.1, we state some preliminary topological results and refer to Section 1.4.3 for an extended discussion.

A refined study of the cohomology of the infinite Grassmannian $BO(n)$ allows to prove that the Thom space of the universal n-plane bundle $T(\gamma^n)$ is $2n$-equivalent to a product of mod 2 Eilenberg-MacLane spaces; we refer to Section 1.4.3 for the precise definition of the indices and for a proof of Theorem 3.3.1. In particular, denoting by Y the product of Eilenberg-MacLane spaces given below:

$$Y := K(\mathbb{Z}_2, n) \times K(\mathbb{Z}_2, n+2) \times \ldots \times (K(\mathbb{Z}_2, n+h))^{d(h)} \times \ldots \times (K(\mathbb{Z}_2, 2n))^{d(n)}\,,$$

we have the following theorem.

Theorem 3.3.1 ([86, Théorème II.10])**.** *There exists a map $H : T(\gamma^n) \to Y$ which is a $2n$-equivalence, for all positive integers n.*

Corollary 3.3.2. *There exists a map $g : K(\mathbb{Z}_2, n)^{2n} \to T(\gamma^n)$ such that $g^*(u) = \iota$, where $K(\mathbb{Z}_2, n)^{2n}$ denotes the $2n$-skeleton of $K(\mathbb{Z}_2, n)$ and u, ι the fundamental classes of $T(\gamma^n)$ and $K(\mathbb{Z}_2, n)$ respectively.*

Remark 3.3.3. We remark that, in the integral setting, the Thom space of the universal *oriented* n-plane bundle is much more complicated, since the equivalent complex is a nontrivial interated fibre bundle with different fibers of type $K(\mathbb{Z}_2, i), K(\mathbb{Z}, j)$ or possibly even $K(\mathbb{Z}_p, k)$; this prevents obtaining a map like g in Corollary 3.3.2 simply by restricting the one obtained from the equivalent complex.

In particular, it is possible to prove the following result, which can be seen as a corollary of Theorem 3.3.1 but, since it is fundamental for us in the conclusion of the proof of Theorem 3.0.1, we include an independent proof for completeness.

Lemma 3.3.4. *Let* $h : T(\gamma^n) \to K(\mathbb{Z}_2, n)$ *be a map representing the Thom class* $u \in H^n(T(\gamma^n), \mathbb{Z}_2)$*. Then* h *is an* $n+2$*-equivalence, for all positive integers* n*.*

Proof. We can assume without loss of generality that $n \geq 2$; indeed for $n = 1$ the two spaces are homotopy equivalent since $T(\gamma^1)$ is homotopy equivalent to the infinite real projective space $\mathbb{RP}^\infty$, which is a realization of $K(\mathbb{Z}_2, 1)$.

Using the Serre spectral sequences, one can show that the cohomology $H^{n+i}(K(\mathbb{Z}_2, n), \mathbb{Z}_2)$ is generated for $i \leq 2$ by the Steenrod squares Sq^1 and Sq^2 of the fundamental class, while for any prime $p > 2$ the cohomology $H^*(K(\mathbb{Z}_2, n), \mathbb{Z}_p)$ has no generators.

The cohomology ring of $BO(n)$ with coefficients in $\mathbb{Z}_2$ is generated by the Stiefel-Whitney classes $w_1, \dots, w_n$ of γ^n, *cfr.* [60, Theorem 7.1] that is

$$H^*(BO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_n],$$

while, for odd primes p, the cohomology $H^*(BO(n), \mathbb{Z}_p)$ is a polynomial ring generated by the (mod p) Pontrjagin classes $p_1, \dots, p_{\lfloor n/2 \rfloor}$ of γ^n, *i.e.*

$$H^*(BO(n), \mathbb{Z}_p) = \mathbb{Z}_p[p_1, \dots, p_{\lfloor n/2 \rfloor}]. \tag{3.6}$$

For every prime p we denote by Φ_p the Thom isomorphism between $H^i(BO(n), \mathbb{Z}_p)$ and $H^{n+i}(T(\gamma^n), \mathbb{Z}_p)$ and by u_p the Thom class. Since $Sq^i(u_2) = \Phi_2(w_i)$ and since it is known that from (3.6) one obtains that $\tilde{H}^{n+i}(T(\gamma^n), \mathbb{Z}_p)$ has no generators for $i < n$, it is possible to conclude that for any group coefficient $\mathbb{Z}_p$ the induced map in cohomology

$$h^* : H^{n+i}(K(\mathbb{Z}_2, n), \mathbb{Z}_p) \to H^{n+i}(T(\gamma^n), \mathbb{Z}_p)$$

is an isomorphism for dimensions smaller or equal than $n+1$ and a monomorphism in dimension $n+2$; in fact, for $i = 2$ then $H^{n+i}(T(\gamma^n), \mathbb{Z}_2)$ admits in general another generator given by $\Phi_2(w_1^2)$.

Since $K(\mathbb{Z}_2, n)$ and $T(\gamma^n)$ are simply connected, by Theorem 1.3.3 we conclude that

$$\pi_k(K(\mathbb{Z}_2, n), T(\gamma^n)) = 0 \quad \text{for } k \leq n+2,$$

and that h is an $n+2$-equivalence for all positive integers n, ending the proof. □

Remark 3.3.5. An obvious consequence of Lemma 3.3.4 is that for dimensions $m \in \{1, 2\}$ (and any codimension $n \geq 1$) every mod 2 homology class can be represented by a smooth embedded submanifold $\Sigma \subset \mathcal{M}$.

We can now provide a proof of Theorem 3.0.1, following the strategy developed in Chapter 2.

Proof of Theorem 3.0.1. Fix $\varepsilon_c > 0$, whose choice will be specified later, and a mod 2 integral cycle T in a homology class $\tau \in H_m(\mathcal{M}, \mathbb{Z}_2)$. We first apply Proposition 3.2.1 to find a sufficiently small $\delta_c' > 0$, a suitable smooth triangulation $\mathcal{K}$ of the manifold and a new mod 2 integral cycle $P' =: S$ with the property that S is in the same homology class of T and the following facts hold:

- $\mathbb{M}(S) \le \mathbb{M}(T) + 3\varepsilon_c$ and $\mathbb{F}(S - T) < 3\varepsilon_c$;
- $\|S\|(B_{\delta_c'}(\mathcal{K}^{m-2})) \le 3\varepsilon_c$;
- $B_{\delta_c'}(\mathcal{K}^{m-2})$ is homotopy equivalent to $\mathcal{K}^{m-2}$;
- $S \llcorner \mathcal{M} \setminus B_{\delta_c}(\mathcal{K}^{m-2}) = [\![\Gamma]\!]_{\mathbb{Z}_2}$ for a smooth submanifold Γ.

Note that if we first choose ε_c, then δ_c, δ_c' and $\frac{\delta_c}{\delta_c'}$ can all be made smaller than any desired constant, while the triangulation is instead kept fixed (because it depends only on ε_c).

We have now fixed a smooth triangulation $\mathcal{K}$ and we can therefore fix constant C_0 and $\bar{\delta}$ so that Lemmas 3.1.1 and 3.1.2 apply. We now require that $V_{\delta_c/2}(\mathcal{K}^{m-2}) \subset\subset B_{\delta_c'}(\mathcal{K}^{m-2})$ for some $\delta_c/2 << \delta_c'$. Hence we apply Lemma 3.1.2 (where $\delta' < \delta$ corresponds here to $\delta_c/2 < \tilde{\delta}/2$) to find a $U_{\tilde{\delta}/2}(\mathcal{K}^{m-2})$ suitably close to $V_{\tilde{\delta}/2}(\mathcal{K}^{m-2})$. We will want that $B_{\delta_c}(\mathcal{K}^{m-2}) \subset U_{\tilde{\delta}/2}(\mathcal{K}^{m-2}) \subset V_{\tilde{\delta}/2}(\mathcal{K}^{m-2}) \subset V_{\tilde{\delta}}(\mathcal{K}^{m-2}) \subset B_{\delta_c'}(\mathcal{K}^{m-2})$. This step requires to take $\frac{\delta_c}{\delta_c'}$ sufficiently small and $\tilde{\delta} < \delta_c'$. Define now $\Omega := \mathcal{M} \setminus U_{\tilde{\delta}/2}(\mathcal{K}^{m-2})$.

The current $[\![\Gamma]\!]_{\mathbb{Z}_2}$ obtained from Proposition 3.2.1 is (when restricted to Ω and not relabelled) a smooth compact submanifold of Ω with $\partial\Gamma \subset \partial\Omega$, provided $\partial\Omega$ is transversal to Γ, which can be ensured via a small smooth perturbation. Denoting by $x \in H^n(\mathcal{M})$ the Poincaré dual of τ, note that its restriction $x_{|\Omega} \in H^n(\Omega)$ to Ω is the relative Poincaré dual of a relative homology class which is represented by the smooth compact embedded submanifold $\Gamma \subset \Omega$ with boundary $\partial\Gamma = \Gamma \cap \partial\Omega$. Hence, by Theorem 3.0.12, there exists a map

$$F : \Omega \to T(\gamma^n)$$

such that $F^*(u) = x_{|\Omega}$; in addition, F is smooth and transverse on $BO(n)$ (and such that $F_{|\partial\Omega}$ is also transverse), so that $F^{-1}(BO(n)) = \Gamma$, which is the smooth part of S.

We then take δ sufficiently small so that $\Omega \subset \mathcal{M} \setminus U_\delta(\mathcal{K}^{m-n-1})$ for the U_δ given in Lemma 3.1.2. Then, by Lemma 3.1.3 we have that $\mathcal{M} \setminus U_\delta(\mathcal{K}^{m-n-1})$ is homotopy equivalent to a complex of dimension $2n$. Denote

$$Q := \mathcal{M} \setminus U_\delta(\mathcal{K}^{m-n-1}).$$

Given the n-dimensional cohomology class $x \in H^n(\mathcal{M})$ which is the Poincaré dual of τ, we consider its restriction to Q, that is $x_{|Q} \in H^n(Q)$; note that $x_{|Q}$ can be represented by a continuous map

$$c : Q \to K(\mathbb{Z}_2, n)$$

(in a suitable homotopy class of continuous maps) pulling-back the fundamental class of $K(\mathbb{Z}_2, n)$ to itself, *i.e.* $c^*(\iota) = x_{|Q}$. By Corollary 3.3.2 and cellular approximation, there exists a map $f : Q \to T(\gamma^n)$ such that the diagram commutes, *i.e.* f pulls-back the universal Thom class to $x_{|Q}$.

$$\begin{tikzcd}
& T(\gamma^n) \arrow[d, "h"] \\
Q \arrow[ur, "f"] \arrow[r, "c"'] & K(\mathbb{Z}_2, n) \arrow[u, dotted, bend right, "g"']
\end{tikzcd}$$

By the same construction of the second part of the proof of Theorem 3.0.12, we can assume without loss of generality that f is smooth throughout $Q \setminus f^{-1}(U(\infty))$ and transversal to (a sufficiently high dimensional approximation of) the zero cross-section $BO(n) \subset T(\gamma^n)$, with ∂f also transversal to it. Hence, $f^{-1}(BO(n))$ is a compact smooth m-dimensional embedded submanifold, with boundary contained in ∂Q; denote it as

$$\mathcal{N} := f^{-1}(BO(n)).$$

Moreover, $\mathcal{N}$ represents the relative Poincaré dual of $x_{|Q}$, which equals $j_*(\tau) \in H_m(Q, \partial Q)$, where $j_* : H_m(\mathcal{M}) \to H_m(Q, \partial Q)$.

We next extend $[\![\mathcal{N}]\!]_{\mathbb{Z}_2}$ (which is an integral current mod 2 in $\mathcal{M}$) to a mod 2 integral current N with the property that $N \llcorner \mathcal{M} \setminus \mathcal{K}^{m-n-1}$ is induced by a smooth submanifold and $N \llcorner Q = [\![\mathcal{N}]\!]_{\mathbb{Z}_2}$. First of all, because $\mathcal{N}$ is transversal to ∂Q, we can extend it to a smooth submanifold over the union Q' of Q with any smooth collaring extension of ∂Q. We can then use Lemma 3.1.2 to find such an extension Q' (which consists of $Q \cup \mathcal{C}$, where $\mathcal{C}$ is the smooth tubular neighborhood in Lemma 3.1.2) containing $\mathcal{M} \setminus V_{\delta'}(\mathcal{K}^{m-n-1})$ for some $\delta' < \delta$ positive. Since $\mathcal{N}$ intersects ∂Q transversally, we can extend it to a smooth submanifold of Q' with boundary in $\partial Q'$, meeting $\partial Q'$ transversally. With abuse of notation this extension is still denoted by $\mathcal{N}$. We can now use the map Φ of Lemma 3.1.5 and set

$$N := \Phi_\sharp [\![\mathcal{N}]\!]_{\mathbb{Z}_2}.$$

The latter current is mod 2 integer rectifiable because Φ is Lipschitz (and, in particular, N has finite mass). Given that Φ is a diffeomorphism over $\mathcal{M} \setminus \Phi^{-1}(\mathcal{K}^{m-n-1}) \subset \mathcal{M} \setminus V_{\delta'}(\mathcal{K}^{m-n-1})$, then $N \llcorner \mathcal{M} \setminus \mathcal{K}^{m-n-1} = [\![\Sigma]\!]_{\mathbb{Z}_2}$ for some smooth submanifold Σ. Moreover $\operatorname{spt}(\partial N) \subset \mathcal{K}^{m-n-1}$ and in particular, by Federer flatness theorem, $\partial N = 0$, namely N is a cycle.

Consider now the two maps $F : \Omega \to T(\gamma^n)$ and $f : Q \to T(\gamma^n)$ such that $F^{-1}(BO(n)) = \Gamma$ and $f^{-1}(BO(n)) = \mathcal{N} \cap Q$. If we consider the restriction of f to $\Omega \subset Q$, we obtain a new map $f|_\Omega \colon \Omega \to T(\gamma^n)$ that pulls-back the universal Thom class to $x_{|\Omega}$. By Lemma 3.1.3 we observe that Ω has the homotopy type of an $n+1$-complex, so that by Lemma 3.3.4 and Corollary 1.3.2 we can conclude that F and $f_{|\Omega}$ are homotopic: the homotopy can be taken smooth by [86, Lemme IV.5]. In particular, we define the smooth homotopy $H : [0, 1] \times \Omega$ such that $H(0, x) = f_{|\Omega}(x)$ and $H(1, x) = F(x)$. In a small collar neighborhood $\mathcal{C}$ of $\partial\Omega$ inside Ω, which we identify with $\partial\Omega \times (0, 1]$, we then glue the maps f and F together. Using the notation $x = (y, s) \in \mathcal{C}$ and after defining a smooth function φ on $[0,1]$ which is identically equal to 0 in a neighborhood of 0 and identically equal to 1 in a neighborhood of 1, we set

$$\widehat{f}(x) := \begin{cases} F(x) & if\ x \in \Omega \setminus \mathcal{C}, \\ H\left(x, \varphi(s)\right) & if\ x \in \mathcal{C}, \\ f(x) & if\ x \in Q \setminus \Omega \end{cases} \tag{3.7}$$

Since $T(\gamma^n) \setminus \{\infty\}$ is a smooth manifold, it follows from [90, Proposition 2.3.4 (ii)] that we can find $\widehat{f} : Q \to T(\gamma^n)$, not relabelled, which is smooth throughout $Q \setminus f^{-1}(U(\infty))$, coincides with $f(x)$ in a neighborhood of ∂Q and with F on $\mathcal{M} \setminus V_{\tilde{\delta}}(\mathcal{K}^{m-2})$; the approximation can be taken close enough so that $\widehat{f}$ is in the same homotopy class. Analogously, by [90, Proposition 4.5.10], we can perturb $\widehat{f}$ so that it is transverse to $BO(n)$ and coinciding with $f(x)$ in a neighborhood of ∂Q and with F on $\mathcal{M} \setminus V_{\tilde{\delta}}(\mathcal{K}^{m-2})$.

Consider now the submanifold Σ' of $\mathcal{M} \setminus \mathcal{K}^{m-n-1}$ which consists of:

- Σ in $V_{\delta'}(\mathcal{K}^{m-n-1}) \setminus \mathcal{K}^{m-n-1}$;
- $\mathcal{N}$ on $U_{\delta}(\mathcal{K}^{m-n-1}) \setminus V_{\delta'}(\mathcal{K}^{m-n-1})$;
- $\widehat{f}^{-1}(BO(n))$ on Q.

This is a smooth submanifold because:

- f and $\widehat{f}$ coincide in a neighborhood of ∂Q and hence $\widehat{f}^{-1}(BO(n))$ coincides with $\mathcal{N}$ in a neighborhood of ∂Q;
- $\Sigma = \Phi(\mathcal{N}) = \mathcal{N}$ in a neighborhood of $\partial V_{\delta'}(\mathcal{K}^{m-n-1})$.

Moreover, $R = [\![\Sigma']\!]_{\mathbb{Z}_2} \in \mathbf{R}_m(\mathcal{M}, \mathbb{Z}_2)$ and $\operatorname{spt}(\partial R) \subset \mathcal{K}^{m-n-1}$; in particular it is a cycle by Federer's flatness theorem. Observe also that $R - S$ is supported, by construction, in $V_{\tilde{\delta}}(\mathcal{K}^{m-2})$, which is homotopy equivalent to $\mathcal{K}^{m-2}$, and thus has trivial m-homology. In particular, R and S belong to the same homology class.

The conclusion now follows *verbatim* as in Chapter 2, but we recall the last passages here for completeness: we apply Proposition 3.2.3 to S and R, noticing that the ε_d in Proposition 3.2.3 is a parameter to be chosen in terms of the ε of the statement of Theorem 3.0.1, and the η_d in Proposition 3.2.3 is δ'_c here. This gives us a parameter δ_d, which depends on ε_d and δ'_c. In turn we impose that $\tilde{\delta} \leq \delta_d$ so that we can apply Proposition 3.2.3. Since ε_d will be specified only in terms of $\mathbb{M}(T)$ and of ε in the statement of Theorem 3.0.1, while δ'_c depends on ε_c, which will also be specified only in terms of $\mathbb{M}(T)$ and ε in the statement of Theorem 3.0.1, the parameter $\tilde{\delta}$ can be taken smaller than δ_d. We can then find a current $R' := \Phi_\sharp R$ for a smooth diffeomorphism Φ isotopic to the identity such that

$$\mathbb{M}(R') \leq (1 + \varepsilon_d)\,\mathbb{M}(S) \leq (1 + \varepsilon_d)(\mathbb{M}(T) + 3\varepsilon_c)\,.$$

We therefore conclude that R' is homologous to R, hence to S, and therefore to T. Moreover, if we choose

$$\varepsilon_d\,(3+\mathbb{M}(T)) < \frac{\varepsilon}{2} \qquad \text{and} \qquad 3\varepsilon_c < \frac{\varepsilon}{2}\,,$$

then $\mathbb{M}(R') \le \mathbb{M}(T) + \varepsilon$. Finally

$$\begin{aligned}\mathbb{F}(T-R') &\le 3\varepsilon_c + \mathbb{F}(S-R') \le 3\varepsilon_c + C(\varepsilon_d\,\mathbb{M}(S) + 2\|S\|(B_{\delta_c'}(\mathcal{K}^{m-2})))^{\frac{m+1}{m}}\\ &\le 3\varepsilon_c + C(\varepsilon_d(\mathbb{M}(T)+\varepsilon_c) + 6\varepsilon_c)^{\frac{m+1}{m}}\,.\end{aligned}$$

Therefore it is clear that a suitable choice of ε_d and ε_c depending only on $\mathbb{M}(T)$ and ε suffices to show $\mathbb{F}(T-R') \le \varepsilon$.

The proof of part (3) of Theorem 3.0.1 is analogous; by assumption we know that τ is represented by a smooth submanifold Σ and hence, by Theorem 3.0.11 there exists a map $\ell : \mathcal{M} \to T(\gamma^n)$ which pulls-back the universal Thom class $u \in H^n(T(\gamma^n), \mathbb{Z}_2)$ to the Poincaré dual of τ. Substituting in the previous steps the map f with this new map ℓ, defined over the whole ambient space $\mathcal{M}$, and defining a similar homotopy as that one in (3.7), the result follows by applying Proposition 3.2.3 to S and $[\![\Sigma]\!]_{\mathbb{Z}_2}$, where S is the integral cycle mod 2 denoted P' in Proposition 3.2.1. ☐

3.4 Optimality of the principal theorem

In this section we show that the construction of Theorem 3.0.1 is optimal, in the sense that there exists m-dimensional $\mathbb{Z}_2$ homology classes that cannot be represented by mod 2 cycles which are smooth embedded manifolds in the complement of an $m-n-2$-dimensional skeleton of any smooth triangulation of $\mathcal{M}$. In fact, proving sharpness of the construction becomes even subtler in mod 2 homology than in the integral setting since singularities that appear in this context all arise from the impossibility of finding embeddings in low codimensions, and not – as in integral classes – by innate singularities obstructing also Steenrod representability. For this reason, we cannot exploit Sullivan's geometric theory of resolution of singularities by blow-up in [82], as done in [8, Theorem 6.3] to derive a contradiction. Instead, we need to rely on singularity theory of stable mappings, coupled with an elegant result due to Grant and Szűcs [49] that characterizes their singular set; we refer to [47] for an accessible overview on the theory of stable mappings.

The mod 2 homology class of least dimension that cannot be represented by a smooth embedded submanifold is a 4-dimensional homology class in a 6-dimensional closed smooth manifold $\mathcal{M}$, and it comes as a corollary of Teichner's construction of S^2-bundles over m-dimensional base manifolds (for every $m \ge 4$) such that every 2-dimensional $\mathbb{Z}_2$ cohomology class restricting to the generator in the fibre cannot be written as the second Stiefel-Whitney class $w_2(E)$ of some real vector bundle E over the base manifold (recall that $w_2(E)^2 = p_1(E)$ (mod 2), where p_1 is the first Pontrjagin class), see [88, Section 3].

In fact, let h be the canonical map from $T(\gamma^2)$ to $K(\mathbb{Z}_2, 2)$; by the theory of Thom [86], there is an obstruction in extending the homotopy inverse g of h from the 4-skeleton to the 5-skeleton given by the Eilenberg-MacLane invariant associated to the second non-null homotopy group $\pi_4(T(\gamma^2))$, which is an element in $H^5(K(\mathbb{Z}_2, 2))$ with coefficients in $\pi_4(T(\gamma^2))$ generating the kernel of the homomorphism

$$h^* : H^5(K(\mathbb{Z}_2, 2), \mathbb{Z}) \to H^5(T(\gamma^2)), \mathbb{Z}).$$

Precisely, the kernel of h^* is generated by $(1/2)\beta\mathbf{p}(\iota)$, where $\mathbf{p}$ is the Pontrjagin square $\mathbf{p} : H^*(\mathcal{M}, \mathbb{Z}_2) \to H^{2*}(\mathcal{M}, \mathbb{Z}_4)$, *cfr.* [94], and β is the mod 2 Bockstein $\beta : H^*(\mathcal{M}, \mathbb{Z}_2) \to H^{*+1}(\mathcal{M}, \mathbb{Z})$ associated to the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

Hence, if a mod 2 homology class $\tau \in H_m(\mathcal{M}, \mathbb{Z}_2)$ admits a smooth embedded representative, then $\beta(x^2) = 0$, where $x \in H^n(\mathcal{M}, \mathbb{Z}_2)$ is the Poincaré dual of τ. In particular, Teichner's construction provides an orientable 6-dimensional closed manifold $\mathcal{M}$ with $\bar{x} \in H^2(\mathcal{M}, \mathbb{Z}_2)$ such that $\beta(\bar{x}^2) \neq 0$, see [88, Lemma 2]. It is a finer and very elegant result due to Grant and Szűcs in [49] that the cohomology operation $\beta(\bar{x}^2) \neq 0$ also obstructs realizability by immersions and, more precisely, they characterize the obstruction $\beta(\bar{x}^2) \in H^5(\mathcal{M}, \mathbb{Z})$ as the integral class whose Poincaré dual is realized by the singular set of any *stable* map realizing the Poincaré dual of $\bar{x}$.

In particular, recall that a smooth map $f : \Sigma \to \mathcal{M}$ of a smooth closed m-dimensional manifold Σ into a smooth $m + n$-dimensional manifold $\mathcal{M}$ is called *stable* if for any sufficiently close smooth map $f' : \Sigma \to \mathcal{M}$ there exist diffeomorphisms $g : \Sigma \to \Sigma$ and $h : \mathcal{M} \to \mathcal{M}$ such that $h^{-1} \circ f' \circ g = f$. In general, it is a remarkable result due to Thom and Levine [57] that stable maps are not always dense[4] in the space of smooth mappings $C^\infty(\Sigma, \mathcal{M})$ but, in codimension $n = 2$, stable maps are dense in $C^\infty(\Sigma, \mathcal{M})$ whenever $m < 21$. Hence, (the Poincaré dual of) Teichner's cohomology class $x \in H^2(\mathcal{M}, \mathbb{Z}_2)$ can be represented by a stable map $\bar{f} : \Sigma \to \mathcal{M}$, simply by approximating the continuous map given by Steenrod representability by a smooth map, and then by a stable map. The *singular set* of $\bar{f} : \Sigma \to \mathcal{M}$ is defined as

$$\mathcal{S}(\bar{f}) := \{x \in \Sigma \mid \operatorname{rank}(D\bar{f}(x)) < m\} \subseteq \Sigma,$$

where $D\bar{f}(x)$ is the differential of $\bar{f}$. The singular set $\mathcal{S}(\bar{f})$ is the closure in Σ of the top singularity stratum $\mathcal{S}^{1,0} \subseteq \Sigma$ of *cross-cap* points, singularities that persist under small perturbations, and it is an open dense subset of $\mathcal{S}(\bar{f})$. In addition, since $x \in H^2(\mathcal{M}, \mathbb{Z}_2)$, then $\mathcal{S}(f)$ has codimension 3 in Σ, and hence it is of dimension 1; in particular, there is no other stratum of singularities, since all other strata have codimension at least 6 in Σ, *cfr.* [11] or [47, Chapter VI]; moreover, $\mathcal{S} := \mathcal{S}(\bar{f})$ carries a fundamental class $[\mathcal{S}] \in H_1(\mathcal{S}, \mathbb{Z}_{\mathcal{S}^{1,0}})$ with twisted coefficients $\mathbb{Z}_{\mathcal{S}^{1,0}}$ according to the tangent bundle of $\mathcal{S}^{1,0}$.

In [49], Grant and Szűcs showed that $\beta(\bar{x}^2) \in H^5(\mathcal{M}, \mathbb{Z})$ can be interpreted in terms of the singular set $\mathcal{S}$ of any stable map realizing $\bar{x}$; more formally, denoting by $i : \mathcal{S} \to \Sigma$

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[4] In the usual C^∞ Whitney topology on $C^\infty(\Sigma, \mathcal{M})$.

the inclusion, and by $\tilde{f} = \bar{f} \circ i : \mathcal{S} \to \mathcal{M}$ the restriction of $\bar{f}$ to its singular set, they proved the following.

Proposition 3.4.1 ([49, Proposition 4.2])**.** *Let $x \in H^n(\mathcal{M}, \mathbb{Z}_2)$ be realized by a stable map $\bar{f} : \Sigma^m \to \mathcal{M}^{m+n}$ of closed smooth manifolds, where n is even. Then $\beta(x^2) \in H^{2n+1}(\mathcal{M}, \mathbb{Z})$ is the cohomology class in $\mathcal{M}$ realized by the singular set $\mathcal{S}$ of $\bar{f}$. In other words, $\beta(x^2)$ is the Poincaré dual of*

$$\tilde{f}_*[\mathcal{S}] \in H_{m-n-1}(\mathcal{M}, \mathbb{Z}_{\mathcal{M}}).$$

Hence, we obtain the following.

Theorem 3.4.2. *Let $z \in H_4(\mathcal{M}, \mathbb{Z}_2)$ be the codimension 2 homology class of Teichner's example in* [88, Lemma 2] *and fix a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$. Then it is impossible to find a representative Σ for z which is a smooth embedded submanifold in the complement of the* 0*-dimensional skeleton of $\mathcal{K}$.*

Proof. Recall that, by Steenrod representability, z can be represented by a continuous map $f : \Sigma \to \mathcal{M}$. We can now approximate f with a homotopic smooth map and, again, by a homotopic stable map, given that for $n = 2$ and $m = 4$ stable maps are dense (in the C^∞ Whitney topology) in the space of smooth maps $C^\infty(\Sigma, \mathcal{M})$. Hence, $\bar{x}$ is realized by a stable map $\bar{f} : \Sigma^m \to \mathcal{M}^{m+n}$ of closed smooth manifolds. In addition, by Teichner's construction in [88, Lemma 1] we know that $\beta(\bar{x}^2) \neq 0$, where $\bar{x}$ denotes the Poincaré dual of z.

Therefore, by [49, Proposition 4.2] we have that $\beta(\bar{x}^2)$ is realized by the singular set $\mathcal{S}$ of $\bar{f}$, which has dimension 1; it is therefore impossible to remove $\tilde{f}(\mathcal{S})$ by removing the 0-dimensional skeleton of $\mathcal{K}$, which is a finite collection of points. □

Remark 3.4.3. Note that the proof of Theorem 3.4.2 also shows that it is impossible to find a representative Σ for z which is a smooth *immersed* submanifold in the complement of the 0-dimensional skeleton $\mathcal{K}^0$ of $\mathcal{K}$.

Chapter 4

Further results on the smooth approximation theorem

The main goal of Chapter 4 is to present further partial results related to the smooth approximation theorem which are currently under investigation. In particular, a $\mathbb{Z}_k$-manifold approximation theorem is stated for integral cycles mod k, with k an odd integer greater than 2. In addition, an example is provided of area-minimizing cycle whose support cannot be locally approximated by a smooth submanifold.

4.1 $\mathbb{Z}_k$-manifold approximation theorem

It is possible to adapt techniques of the smooth approximations of Theorems 2.0.1 and 3.0.1 to deal with $\mathbb{Z}_k$ coefficients, for any integer $k > 2$.

Relying on the remarkable theory developed by Morgan and Sullivan in [65], the appropriate concept of smooth manifold in the $\mathbb{Z}_k$-coefficient setting is the geometric object that the authors called *$\mathbb{Z}_k$-manifold*, and which can be defined as an oriented manifold M with boundary identified with k disjoint copies of an oriented manifold δM, called the *Bockstein of M*. Gluing these k boundary components together gives a space which carries a fundamental $\mathbb{Z}_k$ homology class, and the image of the Bockstein operation of this class for the coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_k \to 0$$

is indeed the integral class $[\delta M]$; in short, a $\mathbb{Z}_k$-manifold can be just thought as a space which is nonsingular outside a codimension one submanifold where k-sheets join together.

A suitable variant of Thom's criterion applies also to the mod k realizability problem: a homology class $\tau \in H_m(\mathcal{M}, \mathbb{Z}_k)$ is realizable by a smooth $\mathbb{Z}_k$-manifold if and only if there exists a map $f : \mathcal{M} \to T(\widetilde{\gamma}) \wedge M(\mathbb{Z}_k)$ from the ambient manifold to the smash product of the Thom spectrum $T(\widetilde{\gamma})$ and the Moore spectrum $M(\mathbb{Z}_k)$ such that it pulls-back the $\mathbb{Z}_k$ Thom class to the Poincaré dual of τ. As one might expect, there exists mod

k homology classes which do not admit any $\mathbb{Z}_k$-manifold representative: for example, the generator of $H_8(K(\mathbb{Z},3),\mathbb{Z}_3)$ cannot be represented by a $\mathbb{Z}_3$-manifold. Coupling this information with the regularity theory for mod k minimizing hypercurrents by De Lellis, Hirsch, Marchese, Spolaor and Stuvard in [25, 26, 27] or, more generally, stable codimension one integral varifolds by Minter-Wickramasekera in [61], and for mod k minimizers in general codimension by De Lellis-Minter-Skorobogatova in [28] (see also [24, 23]), it becomes interesting to investigate the following.

Work in progress. *(Smooth approximation for mod k cycles)*

(i) Is it possible to develop a $\mathbb{Z}_k$*-manifold approximation theorem* – in the spirit of Theorems 2.0.1 and 3.0.1 – for any mod k integral cycle belonging to a mod k homology class admitting a $\mathbb{Z}_k$-manifold representative? In general, how much singularity does one need to allow in the $\mathbb{Z}_k$-manifold approximands?

(ii) Can a finer study of $\mathbb{Z}_k$-manifold representability obstructions provide insights about generic regularity properties of minimizing mod k cycles and their singular set?

Towards the development of a full solution to the $\mathbb{Z}_k$*-manifold approximation theorem*, it is possible to obtain the following theorem.

Theorem 4.1.1 (Optimal $\mathbb{Z}_k$-manifold approximation)**.** *Let $\mathcal{M}$ be a connected smooth closed oriented Riemannian manifold of dimension $m+n$. Let $\varepsilon > 0$, τ be a fixed element of the m-dimensional mod k homology group $H_m(\mathcal{M},\mathbb{Z}_k)$ with k odd natural number greater than 2, and T be an integral cycle mod k representing τ. Then, there is a smooth triangulation $\mathcal{K}$ of $\mathcal{M}$ and a $\mathbb{Z}_k$-submanifold Σ of $\mathcal{M}\setminus\mathcal{K}^{m-5}$ (where $\mathcal{K}^{m-5}$ denotes the $m-5$-skeleton of $\mathcal{K}$) with the following properties.*

1. *The m-dimensional volume of Σ does not exceed the mass of T by more than ε, that is $\mathcal{H}^m(\Sigma)\le\mathbb{M}(T)+\varepsilon$.*

2. *The integral mod k cycle $[\![\Sigma]\!]_{\mathbb{Z}_k}$ is homologous to T and there is an integral $m+1$-dimensional integral mod k current S in $\mathcal{M}$ such that $\partial S = [\![\Sigma]\!]_{\mathbb{Z}_k} - T$, $\mathbb{M}(S) < \varepsilon$.*

3. *Σ can be chosen to be a $\mathbb{Z}_k$-submanifold of $\mathcal{M}$, if τ admits a $\mathbb{Z}_k$-manifold representative,*

Remark 4.1.2. Examples show that the codimension 5 construction in Theorem 4.1.1 is optimal; one can either develop the analogue of Sullivan's theory of geometric resolution of singularities as in [82] for manifolds with boundary or rely on Kreck's theory of stratifolds, *cfr.* [51, 9].

4.2 Setwise approximation theorem

Another aspect in the smooth approximations of Theorems 2.0.1 and 3.0.1 that deserves further study is whether something can be said in terms of closedness of the approximating submanifolds to the support of the integral cycle T.

Suppose in Theorem 2.0.1 that T is area-minimizing and $\tau \neq 0$ is represented by some smooth embedded submanifold. It is not always possible to require that the approximating submanifold Σ be close to the support of T, or close to the support any area-minimizing current representing τ.

Example 4.2.1. Let P be a cycle in Thom's 7-dimensional integral homology class defined in Example 2.4.1, let

$$f : P \to S^7$$

be a mapping of degree 1, and let Σ_f denote the mapping cylinder of f (with $\Sigma_0 = P$ and $\Sigma_1 = S^7$) embedded in a high dimensional Euclidean space $\mathbb{R}^d$.

Take a smooth "tubular neighborhood" U of this embedded Σ_f and let the ambient manifold $\mathcal{M}$ be the double of U. As 7 dimensional integral homology class τ we take the one corresponding to Σ_0 (or equivalently to $-\Sigma_1$). This class admits a smooth representative, given by the submanifold Σ_1.

We however change the metric in $\mathcal{M}$ if necessary to be very small near Σ_0 and use the monotonicity formula for minimal surfaces to ensure that the support of any area-minimizing T must lie very close to Σ_0. Hence, it is not possible for a submanifold representing τ to be contained in a small neighborhood of Σ_0, since this would be in contradiction with the fact that the homology class of P is not Steenrod representable.

Nevetheless it is tempting to believe that, under additional assumptions on the smooth embedded representative, something more can be said in terms of *setwise approximation of integral cycles*; in particular, it is possible to investigate whether the following is true.

Work in progress. *(Setwise approximation of integral cycles)*

Let $\mathcal{M}$ be a smooth closed oriented Riemannian manifold and $[T]$ an integral homology class admitting a smooth representative Σ_0.

(*i*) If Σ_0 lies in a small ε-neighborhood of the support of T, then there exists a sequence of smooth approximands $(\Sigma_i)_i$ also lying in the same ε-neighborhood.

(*ii*) If moreover the ε_i-neighborhoods containing each Σ_i decrease, then the cobordism classes of the manifolds Σ_i need to change accordingly.

Bibliography

[1] W. Allard, *On the first variation of a varifold*, in “Annals of Mathematics”, **95** (3), 417–491, 1972.

Cited on p. XIV.

[2] F. J. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, in “Annals of Mathematics”, **84** (2), 277–292, 1966.

Cited on p. X.

[3] F. Almgren, *Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two*, in “Bulletin of the American Mathematical Society (N.S.)”, **8** (2), 327–328, 1983.

Cited on p. VIII.

[4] F. Almgren, *Deformations and multiple-valued functions*, in “Proceedings of Symposia in Pure Mathematics”, **44**, 29–130, 1986.

Cited on p. 8.

[5] F. J. Almgren, *Questions and Answers about Area-Minimizing Surfaces and Geometric Measure Theory*, proceedings of Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA July 8-28 1990) in “Proceedings of Symposia in Pure Mathematics”, **54** (1), 255–259, American Mathematical Society, Providence 1993.

Cited on pp. VII and IX.

[6] F. Almgren, *Almgren's Big Regularity Paper*, World Scientific Monograph Series in Mathematics, **1**, World Scientific Publishing Co. Inc., River Edge 2000.

Cited on p. VIII.

[7] F. Almgren, W. Browder, *Homotopy with holes and minimal surfaces*, proceedings of Differential Geometry: A Symposium in Honour of Manfredo Do Carmo (Rio de Janeiro, Brazil August 1988), in “Pitman monographs and surveys in pure and applied mathematics”, **52**, 15–24, Longman Scientific & Technical, Essex, 1991.

Cited on pp. IX and 72.

[8] F. Almgren, W. Browder, G. Caldini, C. De Lellis, *Optimal smooth approximation of integral cycles*, preprint arXiv:2411.17678.

Cited on pp. XII, 47, 76, 78, 80, 81, and 89.

[9] A. Angel, C. Segovia, A. F. Torres, *$\mathbb{Z}_k$-Stratifolds*, in “Algebraic and Geometric Topology”, **24** (4), 1863–1901, 2024.

Cited on p. 94.

[10] R. Benedetti, *Lectures on Differential Topology*, Graduate studies in mathematics, **218**, American Mathematical Society, Providence 2021.

Cited on p. 39.

[11] J. M. Boardman, *Singularties of differentiable maps*, in “Publications Mathématiques de l’Institut des Hautes Scientifiques”, **33**, 21–57, 1967.

Cited on p. 90.

[12] C. Bohr, B. Hanke, D. Kotschick, *Cycles, submanifolds, and structures on normal bundles*, in “manuscripta mathematica”, **108**, 483–494, 2002.

Cited on p. 70.

[13] S. Buoncristiano, M. Dedò, *On resolving singularities and relating bordism to homology*, in “Annali della Scuola Normale Superiore di Pisa - Classe di Scienze”, **7** (4), 605–624, 1980.

Cited on p. 71.

[14] G. Buttazzo, M. Belloni, *A Survey on Old and Recent Results about the Gap Phenomenon in the Calculus of Variations*, in “Mathematics and Its Applications”, **331**, 1–27, 1995.

Cited on p. XI.

[15] G. Caldini *On smooth approximation of integral cycles mod 2*, preprint arXiv:2511.10545.

Cited on pp. XII and 73.

[16] G. Caldini, A. Marchese, A. Merlo, S. Steinbrüchel, *Generic uniqueness for the Plateau problem,* in “Journal de Mathématiques Pures et Appliquées”, **181**, 1-21, 2024.

Cited on p. XIV.

[17] G. Caldini, A. Skorobogatova, *Hausdorff measure bounds for density-Q flat singularities of minimizing integral current,* preprint arXiv:2504.19234.

Cited on pp. VIII, XIII, and XIV.

[18] E. De Giorgi, *Su una teoria generale della misura* $(r-1)$*-dimensionale in uno spazio ad* r *dimensioni*, in “Annali di Matematica Pura ed Applicata”, **36** (4), 191–213, 1954.

Cited on pp. VII and 8.

[19] E. De Giorgi, *Nuovi teoremi relativi alle misure* $(r-1)$*-dimensionali in uno spazio ad* r *dimensioni*, in “Ricerche di Matematica”, **4**, 95–113, 1955.

Cited on pp. VII and 8.

[20] E. De Giorgi, *Frontiere orientate di misura minima*, in *Seminario di Matematica della Scuola Normale Superiore di Pisa A.A. 1960-1961*, Editrice Tecnico Scientifica, Pisa 1961, 1–56.

Cited on p. VII.

[21] C. De Lellis, *The size of the singular set of area-minimizing currents*, lectures given at Conference on Geometry and Topology (Cambridge, MA 13-14 September 2014) in “Surveys in Differential Geometry”, **21**, 1–83, 2016.

Cited on p. 6.

[22] C. De Lellis, G. De Philippis, J. Hirsch and A. Massaccesi, *On the boundary behavior of mass-minimizing integral currents*, in “Memoirs of the American Mathematical Society”, **291** (1446), 2023.

Cited on p. XIV.

[23] C. De Lellis, J. Hirsch, A. Marchese, S. Stuvard, *Regularity of area minimizing currents mod* p*,* in “Geometric and Functional Analysis”, **30** (5), 1224-1336, 2020.

Cited on p. 94.

[24] C. De Lellis, J. Hirsch, A. Marchese, S. Stuvard, *Area-Minimizing Currents mod* $2Q$*: Linear Regularity Theory,* in “Communications on Pure and Applied Mathematics”, **75** (1), 83-127, 2022.

Cited on p. 94.

[25] C. De Lellis, J. Hirsch, A. Marchese, L. Spolaor, S. Stuvard, *Area minimizing hypersurfaces modulo* p*: a geometric free-boundary problem,* in “Journal of Functional Analysis”, **290** (12), 111442, 2026.

Cited on p. 94.

[26] C. De Lellis, J. Hirsch, A. Marchese, L. Spolaor, S. Stuvard, *Fine structure of the singular set of area minimizing hypersurfaces modulo* p*,* to appear in “Acta Mathematica”.

Cited on p. 94.

[27] C. De Lellis, J. Hirsch, A. Marchese, L. Spolaor, S. Stuvard, *Excess decay for minimizing hypercurrents mod 2Q,* in "Nonlinear Analysis", **247**, 113606, 2024.

Cited on p. 94.

[28] C. De Lellis, P. Minter, A. Skorobogatova, *Fine structure of singularities in area-minimizing currents mod*(q), preprint arXiv:2403.15889.

Cited on p. 94.

[29] C. De Lellis, P. Minter, A. Skorobogatova, *The Fine Structure of the Singular Set of Area-Minimizing Integral Currents III: Frequency 1 Flat Singular Points and* $\mathcal{H}^{m-2}$*-a.e. uniqueness of tangent cones,* preprint arXiv:2304.11553.

Cited on pp. VIII and XIII.

[30] C. De Lellis, A. Skorobogatova, *The fine structure of the singular set of area-minimizing integral currents I: the singularity degree of flat singular points,* in "Ars Inveniendi Analytica", **3**, 2025.

Cited on pp. VIII and XIII.

[31] C. De Lellis, A. Skorobogatova, *The fine structure of the singular set of area-minimizing integral currents II: rectifiability of flat singular points with singularity degree larger than 1,* to appear in "Commentarii Mathematici Helvetici".

Cited on pp. VIII and XIII.

[32] C. De Lellis and E. Spadaro, *Q-valued functions revisited*, in "Memoirs of the American Mathematical Society", **211** (991), 2011.

Cited on p. VIII.

[33] C. De Lellis and E. Spadaro, *Regularity of area minimizing currents I: gradient* L^p *estimates*, in "Geometric and Functional Analysis", **24**, 1831–1884, 2014.

Cited on pp. VIII and XIII.

[34] C. De Lellis and E. Spadaro, *Multiple valued functions and integral currents*, in "Annali della Scuola Normale Superiore di Pisa - Classe di Scienze", **14** (5), 1239–1269, 2015.

Cited on p. VIII.

[35] C. De Lellis and E. Spadaro, *Regularity of area minimizing currents II: center manifold*, in "Annals of Mathematics", **183** (2), 499–575, 2016.

Cited on p. VIII.

[36] C. De Lellis and E. Spadaro, *Regularity of area minimizing currents III: blow-up*, in "Annals of Mathematics", **183** (2), 577–617, 2016.

Cited on pp. VIII and 72.

[37] G. de Rham, *Relations entre la topologie et la théorie des intégrales multiples*, in “L’Enseignement mathématique”, **35**, 213–228, 1936.

Cited on p. 1.

[38] G. de Rham, *Über mehlfache Integrale*, in “Abhandlungen aus dem mathematischen Seminar der Hansischen Universität”, **12**, 313–339, 1937.

Cited on p. 1.

[39] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern Geometry - Methods and Applications III*, Graduate Texts in Mathematics, **124**, Springer, Berlin-Heidelberg 1990.

Cited on pp. 1, 40, and 65.

[40] S. Eilenberg, *On the Problems of Topology*, in “Annals of Mathematics”, **50** (2), 247–260, 1949.

Cited on p. 36.

[41] H. Federer, *An approximation theorem concerning currents of finite mass*, proceedings of Sixty-fourth Summer Meeting and Thirty-eight Colloquium (Salt Lake City, Utah September 1-4 1959) in “Notices of the American Mathematical Society”, **6** (4), 326, American Mathematical Society, Providence 1959.

Cited on p. VIII.

[42] H. Federer, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, in “Bulletin of the American Mathematical Society”, **76** (4), 767–771, 1970.

Cited on p. X.

[43] H. Federer, *Geometric measure theory*, Classics in Mathematics, Springer, Berlin-Heidelberg 1996.

Cited on pp. X, 1, 23, 27, 49, 56, and 80.

[44] H. Federer and W. Fleming, *Normal and Integral Currents*, in “Annals of Mathematics”, **72** (3), 458–520, 1960.

Cited on pp. VII, VIII, IX, XII, 1, 8, and 14.

[45] W. Fleming, *Flat chains over a finite coefficient group*, in “Transaction of the American Mathematical Society”, **121**, 160–186, 1966.

Cited on pp. X and 27.

[46] M. Giaquinta, G. Modica and J. Souček, *Cartesian Currents in the Calculus of Variations I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **37**, Springer, Berlin-Heidelberg 1998.

Cited on pp. 1, 8, 10, 12, 13, and 14.

[47] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Graduate Texts in Mathematics, Springer, Berlin-Heidelberg 1973.

Cited on pp. 76, 89, and 90.

[48] M. Goresky , R. MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **14**, Springer, Berlin-Heidelberg 1988.

Cited on p. 71.

[49] M. Grant and A. Szűcs, *On realizing homology classes by maps of restricted complexity*, in "Bulletin of the London Mathematical Society", **45** (2), 329–340, 2013.

Cited on pp. 74, 76, 89, 90, and 91.

[50] M. W. Hirsch, *Smooth regular neighborhoods*, in "Annals of Mathematics", **76** (3), 524–530, 1962.

Cited on p. 50.

[51] M. Kreck, *Differential Algebraic Topology. From Stratifolds to Exotic Spheres*, Graduate studies in mathematics, **110**, American Mathematical Society, Providence 2010.

Cited on p. 94.

[52] B. Krummel and N. Wickramasekera, *Analysis of singularities of area minimizing currents: a uniform height bound, estimates away from branch points of rapid decay, and uniqueness of tangent cones,* preprint, arXiv:2304.10272.

Cited on pp. VIII and XIII.

[53] B. Krummel and N. Wickramasekera, *Analysis of singularities of area minimizing currents: planar frequency, branch points of rapid decay, and weak locally uniform approximation,* preprint, arXiv:2304.10653.

Cited on pp. VIII and XIII.

[54] B. Krummel and N. Wickramasekera, *Analysis of singularities of area minimising currents: higher order decay extimates at branch points and size and rectifiability of the singular set,* in preparation.

Cited on pp. VIII and XIII.

[55] J. L. Lagrange, *Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies,* in "Miscellanea Taurinensia", 173-195, 1762.

Cited on p. VII.

[56] M. Lavrentiev, *Sur quelques problèmes du calcul des variations,* in "Annali di Matematica Pura ed Applicata", **4** (1), 7–28, 1927.

Cited on p. XI.

[57] H. I. Levine, *Singularities of Differentiable Mappings*, proceedings of Liverpool Singularities Symposium I (Liverpool, UK September 1969 - August 1970) in "Lecture notes in mathematics", **192**, 1–89, Springer, Berlin-Heidelberg 1971.

Cited on p. 90.

[58] J. Lurie, *Topics in Geometric Topology (18.937)*, Lecture notes available at `https://www.math.ias.edu/~lurie/937.html`

Cited on pp. 1 and 57.

[59] A. Marchese and S. Stuvard, *On the structure of flat chains modulo p*, in "Advances in Calculus of Variations", **11** (3), 309–323, 2018.

Cited on p. 80.

[60] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, **76**, Princeton University Press, Princeton 1974.

Cited on pp. 1, 33, 35, 36, 40, 42, 44, 65, 71, and 85.

[61] P. Minter, N. Wickramasekera, *A Structure Theory for Stable Codimension 1 Integral Varifolds with Applications to Area Minimising Hypersurfaces mod p,* in "Journal of the American Mathematical Society", **37** (3), 861–927, 2024.

Cited on p. 94.

[62] F. Morgan, *Almost every curve in* $\mathbb{R}^3$ *bounds a unique area minimizing surface*, in "Inventiones Mathematicae", **45** (3), 253–297, 1978.

Cited on p. XIV.

[63] F. Morgan, *Generic Uniqueness for Hypersurfaces Minimizing the Integral of an Elliptic Integrand with Constant Coefficients*, in "Indiana University Mathematics Journal", **30** (1), 29–45, 1981.

Cited on p. XIV.

[64] F. Morgan, *Measures on Spaces of Surfaces*, in "Archive for Rational Mechanics and Analysis", **78** (4), 335–359, 1982.

Cited on p. XIV.

[65] J. W. Morgan, D. P. Sullivan, *The transversality characteristic class and linking cycles in surgery theory,* in "Annals of Mathematics", **99** (3), 463–544, 1974.

Cited on p. 93.

[66] R. E. Mosher and M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper's Series in Modern Mathematics, Harper & Row, New York 1968.

Cited on pp. 1 and 40.

[67] J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park 1984.

Cited on p. 32.

[68] A. Naber and D. Valtorta, *The singular structure and regularity of stationary varifolds*, in "Journal of the European Mathematical Society", **22** (10), 3305–3382, 2020.

Cited on p. X.

[69] J. Plateau, *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires*, Gauthier-Villars, Paris 1873.

Cited on p. VII.

[70] H. Poincaré, *Analysis situs*, in "Journal de l'École Polytechnique", **1**, 1–121, 1895.

Cited on p. 36.

[71] L. S. Pontrjagin, *A classification of continuous transformations of a complex into a sphere, I*, in "Doklady Akademii Nauk SSSR", **19**, 147–149, 1938.

Cited on p. 36.

[72] L. S. Pontrjagin, *Homotopy classification of the mappings of an* $(n+2)$*-dimensional sphere on an* n*-dimensional*, in "Doklady Akademii Nauk SSSR", **70**, 957–959, 1950.

Cited on p. 36.

[73] L. S. Pontrjagin, *Smooth manifolds and their applications in homotopy theory*, in "Topological Library. Part 1: Cobordisms and Their Applications", **39**, 1–130, World Scientific, Singapore 2007.

Cited on p. 36.

[74] C. P. Rourke, B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **69**, Springer, Berlin-Heidelberg 1972.

Cited on p. 1.

[75] L. Schwartz, *Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques*, in "Annales de l'Université de Grenoble", **21**, 57–74, 1945.

Cited on p. 1.

[76] J. P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, in "Commentarii Mathematici Helvetici", **27**, 198–232, 1953.

Cited on pp. 45 and 65.

[77] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, **3**, Australian National University, Canberra 1984.

Cited on pp. XIII, XIV, and 1.

[78] L. Simon, *Cylindrical tangent cones and the singular set of minimal submanifolds*, in "Journal of Differential Geometry", **38**, 585–652, 1993.

Cited on p. X.

[79] B. Solomon, *A New Proof of the Closure Theorem for Integral Currents*, in "Indiana University Mathematics Journal", **33** (3), 393–418, 1984.

Cited on p. 8.

[80] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York 1966.

Cited on pp. 1, 33, 34, 40, and 42.

[81] N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Annals of Mathematics Studies, **50**, Princeton University Press, Princeton 1962.

Cited on pp. 1, 40, and 41.

[82] D. Sullivan, *Singularities in spaces*, proceedings of Liverpool Singularities Symposium II (Liverpool, UK September 1969 - August 1970) in "Lecture notes in mathematics", **209**, 196–206, Springer, Berlin-Heidelberg 1971.

Cited on pp. 70, 71, 76, 89, and 94.

[83] H. Suzuki, *On the realization of Stiefel-Whitney characteristic classes by submanifolds*, in "Tohoku Mathematical Journal", **10** (1), 91–115, 1958.

Cited on p. 75.

[84] R. M. Switzer, *Algebraic Topology - Homology and Homotopy*, Classics in Mathematics, Springer, Berlin-Heidelberg 2002.

Cited on pp. 1, 33, and 36.

[85] R. Thom, *Espaces fibrés en sphères et carrés de Steenrod*, in "Annales scientifiques de l'École normale supérieure", **69**, 109–182, 1952.

Cited on p. 43.

[86] R. Thom, *Quelques propriétés globales des variétés différentiables*, in "Commentarii Mathematici Helvetici", **28**, 17–86, 1954.

Cited on pp. IX, XII, 1, 33, 34, 36, 37, 45, 48, 65, 66, 68, 70, 72, 73, 74, 77, 78, 84, 87, and 90.

[87] R. Thom, *Ensembles et morphismes stratifiés*, in "Bulletin of the American Mathematical Society", **75**, 240–284, 1969.

Cited on pp. 70 and 71.

[88] P. Teichner, *6-dimensional manifolds without totally algebraic homology*, in "Proceedings of the American Mathematical Society", **123** (9), 2909–2914, 1995.

Cited on pp. X, 74, 89, 90, and 91.

[89] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Pure and Applied Mathematics, **25**, Academic Press, New York-London 1967.

Cited on p. 2.

[90] C. T. C. Wall, *Differential Topology*, Cambridge studies in advanced mathematics, **156**, Cambridge University Press, Cambridge 2016.

Cited on pp. 39, 68, and 88.

[91] B. White, *The least area bounded by multiples of a curve*, in "Proceedings of the American Mathematical Society", **90** (2), 230–232, 1984.

Cited on p. 6.

[92] B. White, *A new proof of the compactness theorem for integral currents*, in "Commentarii Mathematici Helvetici", **64**, 207–220, 1989.

Cited on p. 8.

[93] J. H. C. Whitehead, *On C^1-complexes*, in "Annals of Mathematics", **41** (4), 809–824, 1940.

Cited on p. 57.

[94] J. H. C. Whitehead, *On Simply connected, 4-dimensional polyhedra*, in "Commentarii Mathematici Helvetici", **22**, 48–92, 1949.

Cited on p. 90.

[95] W. P. Ziemer, *Integral Currents Mod 2*, in "Transactions of the American Mathematical Society", **105** (3), 496–524, 1962.

Cited on pp. X and 27.