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Resource allocation and Uncertainties:

An application case study of portfolio decision analysis and a numerical analysis on evidence theory

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Abstract

The thesis is divided into two parts concerning different topics. The first is solving a multi-period portfolio decision problem, and the second, more theoretical, is a numerical comparison of uncertainty measures within evidence theory.
Nowadays, portfolio problems are very common and present in several fields of study. The problem is inspired by a real-world infrastructure management case in the energy distribution sector. The problem consists of the optimal selection of a set of activities and their scheduling over time. In scheduling, various constraints and limits that the company has to meet must be considered, and the selection must be based on prioritizing the activities with a higher priority value. The problem is addressed by Portfolio Decision Analysis: the priority value of activities is assigned using the Multi-Attribute Value Theory method, which is then integrated with a multi-period optimization problem with activities durations and constraints. Compared to other problems in the literature, in this case, the activities have different durations that must be taken into account for proper planning. The planning obtained is suitable for the user's requirements both in terms of speed in providing results and in terms of simplicity and comprehensibility.
In recent years, measures of uncertainty or entropy within evidence theory have again become a topic of interest in the literature. However, this has led to an increase in the already numerous measures of total uncertainty, that is, one that considers both conflict and nonspecificity measures. The research aims to find a unique measure, but none of those proposed so far can meet the required properties. The measures are often complex, and especially in the field of application, it is difficult to understand which is the best one to choose and to understand the numerical results obtained.

Therefore, a numerical approach that compares a wide range of measures in pairs is proposed alongside comparisons based on mathematical properties. Rank correlation, hierarchical clustering, and eigenvector centrality are used for comparison. The results obtained are discussed and commented on to gain a broader understanding of the behavior of the measures and the similarities and non-similarities between them.

Keywords
Portfolio decision analysis, investment planning, evidence theory, uncertainty measures

Contents

List of Publications

Gaia Gasparini, Matteo Brunelli, and Marius Dan Chiriac. Multi-period portfolio decision analysis: A case study in the infrastructure management sector. *Operations Research Perspectives*, 9:100213, 2022.

Michele Urbani, Gaia Gasparini, and Matteo Brunelli. A numerical comparative study of uncertainty measures in the dempster–shafer evidence theory. *Information Sciences*, 639:119027, 2023.

Chapter 1

Introduction

The organization and scheduling of projects are everyday problems for businesses, professionals, and technicians. It is a process that involves numerous stakeholders and burdens the so-called decision-makers, that is, those who have to finalize the planning of interventions, activities, economic investments, and the use of materials and resources. Recent decades have increased the difficulties of this type of operation. Enterprises have become increasingly large and complex, and often the activities they deal with are interdependent and involve several experts. For example, these situations are found in many different sectors such as energy, military, healthcare, infrastructure, and business. In general, the ultimate goal of project scheduling is optimization. The concept of optimization covers several areas, such as resource utilization, economic and financial aspects, and compliance with regulatory criteria or a company's mission. One of the sciences that have addressed these purposes over the past 70 years is Operations Research (OR). Through transposing problems into mathematical terms and solving algorithms, this subject can optimize the outcome by taking into account the factors involved. Since the end of World War II, during which OR was invented, numerous studies have been conducted. However, the increasing complexity of decision-making processes does not always yield satisfactory results. For this reason, the study of resource allocation has led to the use of various expertise. In particular, the development in the last decade of Portfolio Decision Analysis (PDA) has enabled the integration of Decision Analysis with Operations Research. The applications of PDA are many and are found in various fields, so the scope of study is very broad although still young, and allows problems with hundreds of activities to be addressed. However, since these are mainly applications to real situations, many variables may be encountered, e.g., different objectives to be optimized, constraints, and characteristics of activities. For this reason, there is still a wide scope of study to enable comprehensive support for practical applications.

One of the elements that can make resource management and allocation problems more complex is uncertainty. In the case of PDA, this is the uncertainty associated with the consequences of actions chosen from among the possible ones. In decision analysis, several methods are used to account for this element within problems, and there are several techniques to avoid it. However, uncertainty remains one of the literature's most studied aspects and is sometimes still little known. Uncertainty is often related to a lack of information, for example, the consequences of an investment depend on the performance of financial markets about which we do not have accurate predictions. One of the theories for studying different types of information-related uncertainty is evidence theory, a generalization of probability theory. Evidence theory or Dempster-Shafer theory (DST) was introduced by Dempster [25] and developed by Shafer [125] in 1976 and allows for the expression of concepts such as ignorance and the representation of both random uncertainty, which is intrinsic to an event, and epistemic uncertainty, which instead depends on the lack of information. While the former has been extensively studied in the literature and applied to real systems thanks to probability theory, the latter still requires scientific efforts. In addition, the ability to express total ignorance allows one to deal with special situations, an extreme example but one that can be declined to commonly used situations is to imagine having to answer the question "Is there life on a distant planet?" The answers can only be two, "yes" or "no", with the functions provided by DST, it is possible to answer that we have no information to choose from either. However, in other simpler situations it is possible to commune the level of knowledge and information possessed that allows us to understand how much we know about a phenomenon. In DST, it is possible to study different aspects of uncertainty, particularly nonspecificity, and conflict which take into account the amount of information and possible conflicts between them. In particular, the measurement of uncertainty is also called entropy. Attempts to quantify entropy have been going on for the past 40 years, and recently this study has again attracted the attention of many researchers. However, the result found in the literature is a multitude of measurements that are difficult to compare with each other. Therefore, given the wide application of DST even in the real world, there seems to be an implicit demand to compare measures to rationally and consciously choose which ones to employ and how to compare them.

1.1 Structure of the Thesis

The thesis is organized as follows. Section 1.2 lists the research questions that led to the two publications. Chapter 2 describes the first publication on the topic of Portfolio Decision Analysis. Specifically, the sections from 2.1 to 2.2.5 explain the theory behind the problem addressed. While Section 2.4 describes the content of Publication I and is divided into problem description, problem solution, and results. Section 3 describes Publication II on the topic of evidence theory. Specifically, from Section 3.1 to Section 3.5 the theoretical part is outlined, while Section 3.6 describes in detail the second publication divided into the explanation of the measures considered and their comparison and finally the results.

1.2 Research Questions

To adequately understand the objectives of the research, these are described here as research questions that subdivide the topics discussed.

- **Question 1.** How can Portfolio Decision Analysis be applied to a real-world multi-period problem? Several real-world applications of PDA exist in the literature, but some possible problem characteristics have not yet been addressed.

- **Question 2.** How to create a specific optimization model for a given request and make it user-friendly? Complex mathematical models are often used with the risk of excluding end users from understanding and managing the results. Seeking a simpler model while maintaining accuracy and correctness could lead to more interactions between the theoretical and real worlds with dual benefits.

- **Question 3.** How to choose and evaluate measures of uncertainty or entropy within evidence theory? There are a variety of measures for calculating total uncertainty in the field of evidence theory. So far, however, no measure has yet been found that satisfies all the characteristics and can therefore be classified as unique. To understand how to deal with this huge quantity, it is necessary to approach it numerically by affixing a new evaluation to the theoretical one on properties that have already been widely developed.

The first two questions are addressed by Publication I, 2.4, in which a model of a multi-period problem is developed. In particular, regarding *Question 2*, the section 2.4.3 highlights the need to make the model usable by experts in the company for which the model is intended and shows how it is possible to meet this requirement.
Question 3 is addressed by Publication II, 3.6, in which a variety of uncertainty measures within Dempster-Shafer theory are listed and compared with each other numerically. In particular, several methods of comparison are used that provide a broader overview of the results of uncertainty measures.

Chapter 2

Portfolio decision analysis: a case study

2.1 Introduction

Situations in which portfolio decisions need to be made can occur in a variety of fields. Companies are often faced with this type of problem: choosing a subset of alternatives, i.e. actions to be executed and investments, to achieve economic, strategic, and ethical goals. Each alternative uses different kinds of resources: economic, raw materials, or labor. In general, all resources are limited in usable quantity or simultaneous use. This type of problem can be complicated by the presence of multiple objectives, the need to plan actions over time, obligatory time for execution, and additional constraints specific to the different fields of interest.
Companies face portfolio decision problems involving multiple stakeholders, consequently, the responsibility of each decision-maker (DM) can be very high and requires increasing operational expertise. Over the years, the demand for support from decision-makers has increased, so research on this topic, which has its origins in the fields of Operations Research and Decision Analysis, has continued to advance and expand into different fields.

Publication I draws on the need for companies to have decision support for project planning and resource allocation. We speak of support because the tool to be developed must be able to assist and not replace the decision-maker. The role of DMs has become increasingly complicated in recent years because of the increasing size of companies and the multiplicity of resources available to them. In addition, the business objectives that need to be achieved tend to be increasingly varied and depend on the type of company, e.g., private, public, or nonprofit [68]. Although the situation is complex, a decision maker or an expert has a deep knowledge of the business reality, has experience in analyzing the consequences of planning alternatives, and understands the limits and potential of the business. But as is pointed out by Cognitive Psychology, decision-makers often make mistakes [39, pp. 36].

2.2 Background

Scheduling activities that optimize the use of resources and achieve the goals set or desired by a company is a complicated and delicate task that cannot simply rely on the intuition and experience of decision-makers. This problem falls into the category that we can classify as "resource allocation decisions" or "portfolio selection". In practice, it involves selecting a subset of alternatives from those available in the decision-making situation that is useful in achieving the firm's or DM's objective and that satisfies the constraints imposed. These are alternatives or actions that require the consumption of a portion of the almost always limited resources, which may be economic or physical. The choice of the subset of alternatives will lead to consequences that must lead to the achievement of the goal. The consequences of a choice are not always certain, in which case we speak of uncertainties.

The problem facing decision-makers can prove complicated because of the presence of single or multiple objectives, the number of alternatives and constraints, and the presence of uncertainties about the consequences of actions. Precisely because of this complexity, the focus of research in this field has grown increasingly on trying to provide support for a formal method through a mathematical model that could, based on human experience, be as objective as possible. This latter aspect has been the goal of a variety of studies that aim to eliminate potential psychological aspects that may

influence human judgment and have important repercussions on the success of the analysis.
The use of analytical tools is part of *prescriptive analytics*, which focuses on providing help to the decision-maker based on known data and using various mathematical models aimed at predicting the consequences of actions and thus making the best choices. These characteristics mean that the difference between prescriptive analytics and predictive analytics is very blurred, especially when the application moves from a theoretical situation to a practical decision support case. Both analyses cannot exclude the involvement of experts and therefore base the correction of any errors or biases on studies of descriptive analysis, which deals with describing what happens in a decision-making process and has enabled the development of debiasing methods [99]. The approach to understanding errors and biases in human judgments began in the late 1960s with a series of papers later collected in [134]. The findings of those years were revolutionary and spread to many areas to explain how a person can't develop judgments rationally and according to the normative rules of probability theory [46]. Specifically, in decision analysis and risk analysis, cognitive biases and motivational biases have been studied, i.e. systematic errors in human judgment [138] and errors due to self-interest in outcomes or specific desires for consequences and outcomes [99]. The analysis of biases also made it possible to study methods and arrangements to limit them and minimize their effect [99, 110].

The first approach to portfolio problems is found in the field of Operations Research (OR), which, with the *knapsack problem*, allows the optimization of the choice of a set of alternatives from a reference set. We will then proceed with a brief introduction of operations research to better understand the topic.

2.2.1 Operations Research

The foundation of Operations Research dates back to World War II [53, pp. 1], when several groups of scientists were involved by British and American military organizations [132]. The research groups involved different types of specialists, e.g. physicists, mathematicians, statisticians [124], engineers, and other figures as in the group called "Blackett's circus" which also included physiologists, astrophysicists, and a surveyor [43]. The use of scientists in military operations, initially, involved requesting support for the use and understanding of a new device: the radar technology [124]. But the capabilities of these teams proved to exceed expectations, and in the military's urgency to improve its strategies and make operations more precise, scientists were involved in increasing numbers in different fields. The discoveries made were quickly utilized giving the opportunity to analyze consequences and collect data in quantities and from real situations [132, 91]. The scientists' work often occurred in areas geographically close to the operations and in close coordination with military officers [124]. Some examples of scientific challenges were described by those who actively participated during the war years. For example, Professor A. Charlesby mentions that he studied the improvement of bombing accuracy and optimization of the airfield where planes refueled after bombing; the airfield was congested because of the speed with which planes were able to accomplish the action and return to base. Instead, T. E. Easterfield was concerned with calculating how many spare engines would be needed to ensure proper maintenance of the planes if the war moved to the Far East [124].

The impression gained from these new studies was that of an interdisciplinary subject following new and interesting directions [43], so much so that some of the scientists who had participated in military research were encouraged to use the material produced in future research in the civilian area. Cooperation with the armed forces continued in the following years, plus many companies in various industries (e.g., electrical equipment manufacturers and railways) saw it as an interesting opportunity to improve their production strategies. It also became a subject taught in universities [43]. The origin of the name Operations Research recalls its connection with the military world: the term "operations," coined by Sir R. Watt and A.P. Rowe, is intended to separate normal scientific research from that carried out in collaboration with the armed forces. As mentioned, in the early 1950s, the application of Operations Research methods spread to various sectors, both public and private, which due to their increasing size, had more and more difficulties in managing the available resources in the most functional way.

Under this thrust comes the application of OR to business management. These early years show the natural adaptability of the subject, which is not dedicated to a specific field but can be applied to fields as diverse as economics, urban planning, engineering, and medicine. The key point is that the mathematical model that is created by analysts reflects the real situation by summarizing through formulas the most important aspects. Solutions to the problem must be doubly in accordance with reality and mathematical analysis. To this end, it is very important to always provide specific analyses and experiments, i.e., sensitivity analyses, that evaluate the results and validate the model in both aspects. In short, the goal of Operations Research is to provide organizations with a decision-support tool that is concrete and easy to understand while at the same time being able to process a large amount of data and give objective answers that point to optimality. In this context, it becomes clear how setting precise objectives is the basis for solving these problems. For-profit organizations often have economic aims, measured, for example, by Net Present Value, but public or nonprofit organizations have different objectives that need to be measured differently, and it is often in the latter case where there are multiple objectives.

Objectives are modeled in the form of an *objective function* that quantitatively measures how close one gets to the final goal. The other classic elements of a mathematical model of Operations Research are *decision variables*, which represent the quantities that must be determined by the problem resolution in order for it to be optimal; *constraints*, which insert a maximum or minimum limit to the decision variables or their combinations and represent the number of resources available and/or limits imposed; and finally, *parameters*, which are the quantities used by the constraints and require precision in their determination.

This method of approaching decision-making and resource management problems has the advantage of leading to an objective solution by concisely describing all the fundamental aspects of the problem to be addressed. In addition, by applying certain strategies it is possible to avoid the biases typical of human decision-making. The experts and decision-makers who receive the results of this analysis must be able to understand them, so it is essential to maintain the connection with reality at all stages and the right balance between accuracy and complexity. This last point is important and supported by the fact that the absolute *optimum* is not always demanded by companies, which, on the contrary, are satisfied even with results sufficiently close to it, as emphasized by Herbert Simon with his concept of "satisficing" [49].

2.2.2 Knapsack problem

After this brief, general introduction to Operations Research, we can focus on one of the problems it addresses called the "Knapsack Problem" [96] which is part of integer and combinatorial optimization.
The basic problem involves choosing some items from a given set where each item is characterized by a cost or dimension, w_j, and a value, p_j. The goal is to maximize the value obtainable from the sum of the chosen items while remaining below a maximum budget or size, c constraint. The decision variables are constrained to be integer and binary. The problem is mathematically presented as follows:

$$\text{maximize} \quad \sum_{j=1}^{n} p_j x_j \tag{2.1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} w_j x_j \leq c \tag{2.2}$$

$$x_j \in \{0, 1\} \quad \forall j \tag{2.3}$$

where: $x_j (j = 1, \ldots, n)$ are the items.

The knapsack problem has been the subject of extensive studies since the 1950s as evidenced by the two monographs [96][67] entirely based on the problem, solutions, and variants, and the extensive recent work comprising two publications [13][14] in which it is demonstrated that the application and theoretical use of the problem has not stopped even in recent years. Some real-world applications can also be found, for example in budget allocation and production management problems [52]. The knapsack problem is also often employed as a subproblem of more complex questions [37]. The constant research has produced remarkable results in the discovery of a large number of exact algorithms for variations of the knapsack problem [14][37]. However, still, many solutions are based on heuristics [102][44] and meta-heuristics [41].

In addition, the real-world application involves the presence of several constraints such as time duration of activities [81], variation of the budget limit over time [90], and precedence relations between projects [102] that further complicate the resolution and lead to requiring high and onerous commitment from analysts [162]. Research on the knapsack problem has produced many variations from the basic problem, from classical ones (e.g. subset sum problem) to those involving the addition of different constraints (e.g. multiple choice knapsack problem, precedence constrained knapsack problem) to more complex problems (e.g. multiple knapsack problem, multidimensional knapsack problem, quadratic knapsack problem). In the areas of activity planning [162] and resource allocation [24], some research has been conducted, but as reported by [78] these issues require continuous updating with new information and consequently repetition of results. This process complicates solution methodologies, prompting experts to request simpler methods of tackling the problem.

The knapsack problem is NP-Complete [15], which means that there is no algorithm that gives the solution in polynomial time and can become very complex to solve, for example the Multiple-Choice Multi-Dimension Knapsack Problem, a variant of the basic problem, is an NP-Hard problem [5], but as reported by Dudziński and Walukiewicz [37] great progress has been made to be able to solve most problems efficiently. However, the more one tries to make the problem realistic by adding different constraints, the more complex the problem turns out to be. There is still a strong limitation in using the knapsack problem for problems with many alternatives and real constraints, for example, portfolio decision problems, where it is essential to involve experts and decision-makers, whose input into goal setting is crucial, especially in cases where they are not limited to purely financial purposes. Some attempts in the question field have been made, for example in the studies of Vaezi and Sadjadi [135], Kuchta et al. [79], Tavana et al.

[130]. Practical examples can be taken from the studies of Barbati et al. [9] and Nesticò et al. [103] who both proposed an application of portfolio decision-making using the knapsack problem in urban planning.

It can be said that in general, organizations aim to achieve the greatest possible benefit from carrying out a set of activities or investments with the least possible use of resources. Quantifying this benefit complicates goal setting and consequently the calculation of the objective function. There are several cases where Net Present Value [16, 78, 120] or project profit values [94] can be used in the objective function, but when this is not possible, the prioritization of alternatives becomes more challenging to quantify. The cause of this complexity is the influence of several objectives that have to be quantified and condensed into a single prioritization value. To sum up, several needs are placed before us, of calculating an appropriate prioritization value, involvement of experts, decision-makers, and if necessary different stakeholders, and the creation of a tool that is useful for decision support but at the same time is also inclusive for those who are involved in the process but are not familiar with mathematical algorithms. All these needs have recently led to the development of a branch of decision analysis, Portfolio Decision Analysis, which applies the formal methods of Decision Analysis to optimization problems. As defined by Salo et al. [122]:

> *PDA has well-established roots that go back to the very origins of operations research. It is based on sound approaches for problem structuring, preference elicitation, assessment of alternatives, characterization of uncertainties and engagement of stakeholders.*

Before looking at the most relevant aspects, it is worth explaining what decision analysis is and what tools are used in both DA and PDA.

2.2.3 Decision Analysis

Decision Analysis originated in the 1960s as a branch of Operations Research in response to the need to assist decision-makers in making decisions in complex situations involving many stakeholders with different, potentially conflicting goals and where the consequences of choices may be uncertain. The normative aspect of DA has its basis in Decision Theory of

which the analysis turns out to be the prescriptive application. DT provides rules and axioms to guide decision-maker's choices toward consistent decisions even when there is uncertainty about consequences. The methods proposed by DA make it possible to follow the rules of the theory leading to consistent and coherent choices.
It was Howard in 1965 [55] who first combined the theoretical and practical aspects and first named and defined DA specifying it later:

> *Decision Analysis results from combining the fields of systems analysis and statistical decision theory.*

Research work in this field was also later carried on by many other researchers who focused their attention on the development of various methods. Including, citing some of the most widely used methods, the Multi-Attribute Utility Theory (MAUT) and the Multi-Attribute Value Theory (MAVT) formulated by Keeney and Raiffa in 1976 [66], ELECTRE [115, 116], TOPSIS [156], Analytic Hierarchy Process (AHP) [119] and Analytic Network Process (ANP) [118]. Alongside these tools, there are also graphical approaches such as decision trees and influence diagrams.

The two "sister" theories MAUT and MAVT differ from each other in the consideration or not of risk and uncertainty and are part of the Multi-Criteria Decision Analysis (MCDA) methods, identifying the best alternative by utility or value function. MAVT will be explained in more detail in the next section.
Also in the context of MCDA is the family of methods called ELECTRE. They are mainly used when the purpose is to exclude alternatives unsuitable for the objectives. They are part of the *outranking* methods, in which the rules of intransitivity do not apply, i.e., the information obtained is accepted even if some of the pairs of alternatives compared are incomparable.
The TOPSIS method ranks the available alternatives based on a comparison with an ideal, unrealistic solution. This gives a distance of each alternative from the desired one. In the case of a multi-criteria problem, the analyst must consider the importance, the weight, of each criterion and calculate the geometric distance accordingly.

AHP has found considerable success in recent years; it can be used in both single-criteria and multi-criteria choices and has a simple structure that makes it less complex than other methods. It evaluates the relative measures of one alternative against another, but unlike the previous method, it is most useful when searching for the best alternative among those available. It is a useful tool when attributes are difficult to measure, as in the case of "intangible" attributes. The term "hierarchy" in the name indicates how this procedure creates a hierarchy of attributes and then evaluates the alternatives, compared in pairs for each attribute, through a weighted average. ANP is a generalization and, as the name reminds us, the decision problem is visualized as a network. Like the previously described method, the comparison is done in pairs until the weights are defined and the alternatives are finally ranked.
The influence diagram is intended to give a global view of the problem, this leads to sacrificing some details in favor of a broader view. The decision tree and the decision matrix, two very similar tools, so much so that one can easily switch between them, are generally used toward the end of the decision-making process. Graphical modeling can be useful in supporting decision-makers for more direct visualization of a complex system, e.g., the decision tree allows decision possibilities to be grouped together making it easy to understand.

In this context, we will only delve into the use of Multi-Attribute Value Theory, a methodology used in Publication I to which this introduction relates by referring to other applications for a more in-depth discussion of other techniques that can be used. The choice of this method stems from the observation that experts rely on different criteria to reach an outcome, and therefore a method belonging to MCDA was needed. In addition, MAVT has a general approach of involving experts in defining value functions and identifying attributes while allowing a simple understanding of the process and outcome.

2.2.4 Multi-Attribute Value Function

Multi-Attribute Value Function is subject to the assumption that the decision-making process occurs under conditions of certainty. It allows the decision maker's preferences to be represented by combining them for each attribute into a single function and returning a value that characterizes each alternative.
Before discussing value functions in more detail, it is essential to understand what the term attribute means. It is a value that characterizes each alternative by quantifying the level of achievement of the goal; it can be interpreted as an assessment of the consequences triggered by choosing one alternative over another. Based on this definition, it follows that the decision-maker's choice of one alternative over the other will depend heavily on the attributes, must therefore be chosen carefully, and, depending on the case, may be *natural* (e.g., "minimize the time it takes to transport goods." can be quantified with the attribute "hours taken to transport goods") or it must be constructed or researched in cases where no natural attribute exists or are challenging to use and attributes are defined that fall into the *artificial* or *proxy* categories [64], where the former is adopted when a natural attribute does not exist (e.g., measuring satisfaction or fear) and the construction of a scale is necessary, while the latter are attributes that do exist and are therefore more similar to natural attributes, but measure the level achieved indirectly (e.g., "minimize the number of outages due to breakdowns" can be quantified with "hours of machinery repair work" that does not directly count the number of breakdowns).
To properly define a set of attributes, a set of properties must be considered according to [66, pp. 50–53]:

- **completeness**: the attributes of the set are such that they consider the problem in its entirety;
- **operational**: the attributes should be simple to use in the set without requiring excessive effort from those involved;
- **decomposable**: the attributes are such that it is possible to decompose the problem to make it simpler;

- **nonredundant**: attributes evaluate consequences, therefore they must be chosen such that the impacts of the chosen alternatives are not calculated twice;

- **minimal**: attributes should be as few as possible and should be able to directly assess the level of achievement of a goal.

There are other desirable properties of attributes suggested by Keeney and Gregory [65] that are sufficient for good attributes and add some useful features; according to the authors, attributes should be direct, understandable, complete (of all possible consequences that may result from choosing an alternative) and unambiguous.

Returning to analyzing the value function, this can be seen as a mathematical function through which to represent the preferences expressed by decision-makers. Like any mathematical tool, it is subject to rules. It must be implemented through methods designed specifically to account for preferences about consequences. The value function v makes alternatives easily comparable by assigning a numerical value to each of them, i.e. let $A = \{a, b, c, \dots\}$ be a finite set of alternatives, when alternative a is preferable to alternative b, the value of alternative a will be greater than the value of alternative b:

$$v(a) > v(b) \Leftrightarrow a \succ b$$
$$a, b \in A \text{ analogous for } \prec \text{ and } \sim$$

In multi-attribute situations, it is necessary to determine a value function for each attribute, called *single attribute value function*. In this case, the value function assigns to each alternative a combination of values derived from the value functions of the individual attributes. Each alternative can be described by a vector $x_i = (x_i^1, ..., x_i^m) \in X_1 \times \cdots \times X_m$, whose jth component x_i^j is the jth attribute level of the ith alternative. The simplest form of the value function in multi-attribute conditions is the additive form. The additive model is very simple to use, which is why numerous applications of it can be found in the literature [133, 131, 42]:

$$v(v_1(x_1^i), \ldots, v_m(x_m^i)) = \sum_{j=1}^{m} w_j v_j(x_j^i) \tag{2.4}$$

where: $v_j(x_j^i)$ is the value of the single attribute value function when the level of the attribute j is equal to x_j^i for the alternative i, and w_j's are the *scaling constants* or weights, they are in the interval $[0, 1]$ and $\sum_{j=1}^{m} w_j = 1$. Scaling constants quantify the relative importance of attributes.

A measurable value function, expressed as in Eq. 2.4, must satisfy two conditions to correctly express the concept. The two conditions are called: mutual preferential independence and mutual differential independence. The first indicates that the decision maker's preference for a subset of attributes is independent of the levels of the complementary subset, a statement valid for any subset. The second indicates that if two alternatives differ in the level of only one attribute, the value required to switch from one to the other is always the same, regardless of the level of the other attributes, and this must hold for each attribute toward the complementary subset. If the decision-maker cannot define the value function of a single attribute without information on the level of one or more different attributes, the above conditions do not apply. A different form of value function should be used. It must be able to correctly represent correlations and preferences; if this is not possible, the objectives are analyzed again and defined differently.
Single attribute value function determination methods aim to create a mathematical tool that reliably accounts for the decision-maker's preferences. Each method works differently, but in general, the decision maker is asked to express his preference over several comparisons of alternatives, and his statements are used to determine value functions through punctual values. Considering a continuous function, points that are not directly elicited can be inferred using mathematical tools (e.g., a piecewise linear function, an exponential function). Some of the most widely used methods are the direct evaluation method, the standard difference sequence technique, and the bisection method [40, pp. 115–120]. The difference between

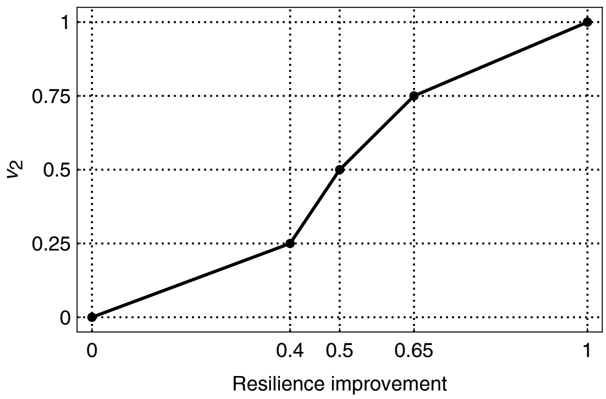


Figure 2.1: Example of value function. On the horizontal axis are shown the values from the least, x^-, to the greatest, x^+, that the attribute can assume. On the vertical axis are represented, normalized between $[0, 1]$, the values of the value function. In this case, the best conjunction of identification points with one of the methods of creating a value function is a piecewise linear function.

these methods concerns what is required of the decision-maker. A consistency check of the results is always recommended at the end of the process [144]. In addition, the functions obtained should be monotone, otherwise, they can be simplified by splitting them into two parts and adapting the demands made on the decision-maker. Figure 2.1 shows the characteristics of a generic single attribute value function. Scale constants can be determined by methods that interpret the decision-maker preferences through questions that seek trade-offs, i.e. values for which alternatives are equivalent. Other methods help the decision-maker rank the attributes until the relative values of the scale constants are obtained. Some examples of these methods are the trade-off method, the swing method, and the direct ratio method [40, pp. 135–141], just to cite a few of them.

As mentioned in Section 2.2 all decisions involving humans are affected by cognitive biases. In determining scaling constants, it is important to pay attention to, e.g., the range effect and the splitting effect [40, pp. 154–155]. The former concerns insensitivity to the range in which the relative attribute moves, but the larger the range, the higher the constant should be [137]. The second relates to the objectives, those described in more detail get a higher scaling constant value from the decision maker [145].

The direct involvement of decision-makers and experts leads other Multi-Attribute Value Theory procedures to be affected by biases as well. Among these, one of the most important is the anchoring bias, which is the tendency of humans to focus their attention on initial information and to make their judgments while keeping that datum as a basis [113].

2.2.5 Portfolio Decision Analysis

Now that some of the best-known methods of decision analysis have been described, we can focus on the analysis of portfolio decisions. PDA is a branch of Operations Research and Decision Analysis, in which it has its roots and methods; it can be interpreted as a combination of Decision Analysis methods with optimization problems and the use of mathematical algorithms.

As its name reminds us, PDA deals with decision problems in which a subset of alternatives, or portfolios, must be selected from a larger set to satisfy identified objectives. Specifically, PDA problems generally have some common characteristics: a large number of alternatives, multi-criteria, a large number of stakeholders, and, in addition to classic constraints, interdependencies between alternatives are often present. It is the latter aspect that makes the problems addressed by the PDA very complex because, without it, a ranking of alternatives could be made followed by selection based on the value obtained until the budget is exhausted. Examples of interdependencies between alternatives are situations where there is a precedence link between activities or a constraint that does not allow some activities to be performed if others are performed. In these contexts, an analytical and formal approach to portfolio selection such as PDA allows for a systematic method with greater transparency that enables easier understanding of the process to decision-makers and the possibility of reiteration. PDA has evolved mainly in the last few decades, but already after the first experiments with Decision Analysis in the late 1970s [61], one can find the use of methods that can be counted among those of PDA even though they were not yet recognized then. It is necessary to wait until the 1990s and 2000s for more illustrative procedures of applying Decision Analysis to portfolio and resource allocation problems to become established. The evolution of software and the increased ease of access to spreadsheets al-

lowed the expansion of these procedures into companies as well, and the application became especially popular in the pharmaceutical and energy industries [122, pp. 12]. With the beginning of the new millennium, the concept of PDA is being defined with a problem structure involving elicitation of preferences, ranking of alternatives set and collaboration with stakeholders, experts, and decision-makers, and the use of the resulting data in mathematical algorithms.
At present, the application of PDA has expanded in many areas because of the problem structure that can be encountered in many decision-making situations and the concrete possibility of support for decision-makers offered by the procedures. This last point is crucial because the complexity of portfolio decisions, unlike single-choice decisions, comes not only from the constraints but also from the number of possible choices. To better understand this statement, one can think of a situation in which one has to choose which activities to do or not to do in a known set of 30 projects; the number of possible portfolios is huge, amounting to 2^{30}. Typically, decision-makers find themselves working with hundreds of projects. Add to this the complication of possible interactions and the need to consider the preferences and goals of decision-makers and stakeholders, and it comes naturally to understand the importance of having a systematic decision support tool at one's disposal.
The fields in which PDA has been applied are many: in [123] the focus is on the selection and extraordinary planning of alternatives for urban development planning, in [80] the topics are those of the energy industry and environmental protection with different objectives on which to focus the optimization problem, in the field of environmental management, [82] provides guidance on the use of PDA, while [98, 97] and [95] on different fields, civil infrastructure, and nuclear industry respectively, address the problem from the perspective of safety and risk. The recent survey by Liesiö et al. [87] highlights the wide scope and numerous fields in which PDA has been applied in recent years.

The applications of PDA are numerous, but in a few cases the problem is addressed by considering the time dimension; in some cases (e.g., [10] and [123]) one of the purposes of the analysis is to select a portfolio of activities with an indication of when they should be performed, but their duration is not part of the optimization problem. For this reason, it is possible to deepen the study in this way.

2.3 Applications in civil engineering

The applications of PDA can also be extended to civil engineering, among the different areas one of the most interesting ones is related to infrastructure maintenance. In fact, this is a very sensitive issue in recent years due to the aging of infrastructures as for example in Europe, where in many countries there are infrastructures showing signs of deterioration caused by time, e.g. in France, UK and Germany [47] to the Italian situation highlighted by the tragic collapse of the Morandi Bridge [32]. This situation requires greater attention and planning of maintenance rehabilitation based on their current condition. To explore this topic further, we describe a practical application to the case of the Finnish Transport Agency (FTA). The study was performed by Mild et al. [98].

The problem

The problem addressed is the scheduling of bridge maintenance activities, and selecting an optimal portfolio of structures from those in need of work within 3 years. The method used is Robust Portfolio Modelling (RPM) from which the Core Index is derived. The problem is subject to constraints due to limited resources, has to consider numerous selection criteria, and the number of works managed is large, considering between 200 and 600 bridges. In addition, the choice to use RPM since the information may be incomplete. Finally, it is crucial to allow the company to reiterate the model several times by updating it with both the subjective data of the experts and the improvement changes made to the works, such as performing maintenance activities that change the current state of the work.

Robust Portfolio Modelling

Robust Portfolio Modelling [89] is a decision-making method for analyzing multi-criteria portfolio problems. It allows incomplete information on the relative importance of criteria to be handled. Specifically, a feasible set of weights is defined in which the relative weights of the criteria are described by linear constraints and the values of the alternatives by ranges large enough to contain the true value. The model also allows a comparison of portfolios with a *dominance* concept and consequently the selection of a set of non-dominated portfolios [88].
From this set, it is possible to deduce the Core Index, which allows the set of alternatives to be divided into three categories: (i) *core* projects that are found in all non-dominated portfolios, (ii) *exterior* projects that are not found in any non-dominated portfolios, and (iii) *borderline* projects that are found in only a few [89, 98].
This avoids the solution of indicating which projects to choose and which not and allows for greater flexibility. In addition, an advantage of this classification is the possibility of excluding exterior projects, which will retain their status even with more accurate information.

The solution

The process included a series of meetings with the company's experts for the appropriate development of the model. Holistic knowledge of the infrastructure and bridges enabled the technicians to evaluate the initial results of the model and make recommendations for improving it.

Four basic criteria are identified: minimize maintenance costs, minimize user costs, maximize safety, and maximize customer satisfaction. The value function used by the model is additive.
The authors also define an approximate approach because the size of the problem does not allow the use of the exact linear programming algorithm. Please refer to the publication [98] for further discussion. The result provided to the company is the list of bridges under analysis flanked by their respective Core Index values. Although these are not prioritization values, the experts reacted to the list by considering the Core Index in this

way. Although not formally correct, the authors believe that this use, as a business decision support, is functional and useful and supported by the Core Index definition if one considers all non-dominated portfolios as an optimal choice with equal probability.

Over the years the model has been used several times, demonstrating flexibility and adequate simplicity without excessive reduction in mathematical correctness. The flexibility also allows consideration of uncertainty about the actual costs of projects that may vary.

2.4 Publication I

2.4.1 The Problem

The published article is based on the creation of a support tool for the company SET s.p.a., which is the main operator of the electricity distribution network in the Province of Trento, Italy, with an extension of 12000 km of network. The company is responsible for supplying energy to 160 municipalities and 330000 both public and private users who use their services. The need for support in decision-making arises from the fact that it has to schedule a large number of different activities over a large time frame, i.e., 5 years. The planning has objectives that are not exclusively economic. The company's goals are to ensure high-quality service by decreasing the amount and duration of interruptions in energy supply. To achieve these objectives, the network must ensure adequate maintenance and should be suitable for both the customers and the area where the energy is supplied. In addition, there are environmental and regulatory requirements that cannot be overlooked. To respond adequately, the company divides its activities into three macro categories: *new connections*, *quality improvement*, and *load adjustment* interventions. The first category includes all those interventions that are carried out primarily at the request of users or energy suppliers; this characteristic means that the type and number of interventions in this category depend on the development of the area. The other two categories are strongly related to the improvement of the quality of the network or its maintenance. These are interventions that have as end

users inhabitants of rural or mountainous areas who, consequently, need specific interventions. Also included in these categories is the upgrading of some lines for user needs.
The description of activities highlights the nonprofit purpose of the company. The financial aspect appears in the planning problem only as an annual budget constraint. Before analyzing the planning problem mathematically, it is appropriate to specify the general situation to identify the different parts of the modeled problem. The decision situation consists of 368 activities to be carried out in the next 5 years; the company considers some activities more important than others according to a value defined as "priority". Optimal planning includes the greatest number of high-priority activities with the constraint of keeping costs below a predetermined annual budget. However, budget is not the only constraint decision makers have to worry about; they also have to consider the availability of work teams, which are not always sufficient to be able to work simultaneously on several activities and depend on the companies contracting the work; linked to this constraint is the need to consider the duration of each activity. Furthermore, the activities are interconnected, some cannot be performed unless preceded by other specific executions; added to this some work must be compulsorily performed from a certain date or the work must be finished by a deadline.

In short, the problem is characterized by a large set of activities from which to select the optimal portfolio, known costs and execution times, and possible constraints and interrelationships with other activities. The optimization problem must consider the constraints and is highly dependent on the priority assigned to each activity. The opportunity for dialogue with two experts, engineers from the company's Operations and Technological Innovation department, enabled us to understand how priority was a value assigned in a non-systematic way, but based on the experts' sensitivity and taking into account the relevant characteristics of the activities and the consequences of their execution. Based on the characteristics just described, we were convinced of the need to apply a Portfolio Decision Analysis that could accurately identify the value of the priorities of each activity on a solid basis and was able with an optimization model to provide an appropriate result for the company. In this way, the input data required by the analysis will be less and the decision support provided to the company will be more successful.

2.4.2 The Solution

Priority values

The calculation of the priority of alternatives is addressed through the Multi-Attribute Value Theory. This theory has already been used in infrastructure management [63, 161, 160]. Together with its sister theory, Multi-Attribute Utility Theory turns out to be the most widely used in decision-making models [87]. MAUT is used in the case of uncertainty. In the problem under consideration, uncertainty could be caused by changes in demands, the inclusion of new projects, or changes in some features. This type of uncertainty is overcome by running the optimization program at regular intervals with improved information. This method of reiteration is only possible by keeping the model simple enough that the company can also use it quickly and correctly.

The priority of activities is a fundamental value for the company's goals and consequently plays an important role in the objective function that will be made explicit later. The business idea is that the higher the priority of an activity, the greater should be the probability that it will be chosen before activities with lower priority.

To deal with a typical MAVT problem, it is necessary to define a finite set of n alternatives $A = \{A_1, A_2, \ldots, A_n\}$, provided in this case by the company, a finite set of m attributes $C = \{C_1, C_2, \ldots, C_m\}$, to be defined with the cooperation of the experts, together with the single attribute value functions whose combination will define the value function. The definition of these elements in the specific problem addressed in Publication I required 4 interviews with two experts. The approach followed in this context was inspired by Value Focused Thinking [64] by discussing with the decision-makers what the values were, i.e., what was interesting for them to achieve [62], to identify the fundamental objectives.

The interviews were conducted with both experts present at the same time. Initially, we focused on the company, the characteristics of the activities carried out, and, in general, how the company's planning is managed, to adequately identify the situation in which to carry out the analysis. We collected all the data useful for the analysis that could be easily provided by the company. In particular, the set of activities to be scheduled with their costs and durations. Having identified the problem, four fundamental objectives were identified: maximizing the number of users connected to the network, maximizing the quality of service, minimizing service interruptions, and minimizing delays in the execution of activities. Focusing on the objectives, the attributes that can quantify the level of achievement of each by the alternatives are:

Number of benefiting users: the higher the value of the investment, the greater the number of users who benefit from the execution of a specific activity.

Quality improvement: the company must comply with local and national regulations that require specific standards. The degree to which requirements are met assesses the quality of the service provided.

Resilience improvement: it is an attribute related to the tendency of the network to be subject to damage and service disruptions. Most of these situations are caused by extreme events, so the company has defined a formulation that considers the return time of events and allows for calculating the improvement in network resilience. The greater the resilience, the greater the ability to withstand extreme events (e.g., replacing overhead lines with underground lines).

Setup time: every activity requires a set-up time before it can be started. Activities with a shorter set-up time are those of greatest interest to the company. Set-up time is generally easy for the company to calculate, but it can vary due to external factors, so it may be subject to uncertainty. However, because it can be easily repeated with the additional information held by experts, it does not require specific attention in the model.

The attributes explicated above were analyzed to ensure that the properties necessary to obtain an appropriate set of criteria were met [65].

The information described is sufficient to move on to the next step: the definition of single attribute value functions. Experts were asked to identify the intervals $[\underline{x}_j, \overline{x}_j]$ where $\underline{x}_j$ and $\overline{x}_j$ are the worst and best values in terms of desirability, respectively. Then, using the method of the Mid-Value Splitting technique, we define which x_i corresponds to the 0.50, 0.25, and 0.75 values of the value function, i.e., the values in the middle of the function and the middle of the two consequent halves. Proceeding with this method for each attribute, four value functions are obtained to measure the value of the attribute for each level reached by each activity. The obtained functions have different shapes to best represent experts' preferences. It is interesting to observe the S-shaped trend of the value functions of the attributes "Resilience improvement" and "Quality improvement." This representation makes it clear that activities with an intermediate value are preferable for the company, i.e., activities that balance cost and improvement as opposed to activities that require a high use of resources to achieve high levels of improvement.
At this point, it is necessary to calculate the weights or scaling constants, w_j, of each attribute. The method chosen is the Trade-Off, which requires knowledge of the single attribute value functions and the assumption that the value function is additive. The first step in this process is to rank the four attributes in order of preference. The ranking makes it possible to identify between which pairs of attributes it is correct to make the direct comparison predicted by the chosen method, e.g., if experts find it complicated to compare and rank resilience and quality improvements, these two attributes will not be compared, but their weight value will be deduced by comparing them with the other attributes. The Trade-Off method involves comparing pairs of alternatives in which all but two attribute levels are equal. From this comparison, experts must indicate the attribute levels that make the two alternatives presented equally satisfactory. In this context, it is possible to require more comparisons than the minimum required, i.e., $m - 1$ equations, to have redundant information and decrease or highlight a possible inconsistency error [138, pp. 290] [66, pp. 22]. From the required tradeoffs, it is also possible to see any inconsistencies in the choice of additive form for the value function. For example, in the case

study, experts indicated that in the case where resilience or quality improvements were at the lowest level, no level of the other attributes would be sufficient to give a value greater than 0 to the activity. In the present case, however, we were able to maintain the additive form assumption of the value function, since in the real case activities with a zero improvement value are not included in the set of alternatives to be performed.
Once the results of the scaling constants are obtained, the value of each alternative is calculated with the equation 2.4.

Optimization problem

At this point, we have all the necessary information to develop the optimization problem whose objective is to plan activities over 5 years by favoring early execution of activities with a high priority value. The formulation of the *objective function* is inspired by a similar approach developed in [102] with maximization of the sum of the discounted values of the activities:

$$\text{maximize} \sum_{i \in A} \sum_{t \in T} \frac{\nu_i}{(1+r)^t} x_{i,t}. \tag{2.5}$$

where: $A = \{1, \dots, n\}$ is the set of activities; $T = \{1, \dots, m\}$ is the set of periods or time horizon; $x_{i,t}$ is a binary variable that when equal to 1 indicates that task i will be executed at time t; ν_i is the priority of task i and $r > 0$ is a discount factor.

The constraints set in the problem concern realistic situations made explicit by the company's experts. They will be described in detail in the following. The company created the set of alternatives considering that each of them can be executed only once in the given period T:

$$\sum_{t \in T} x_{i,t} \leq 1 \;\; \forall i \in A. \tag{C1}$$

SET s.p.a. is aware that the available resources and time will not be sufficient to carry out all the activities in the set A, so the constraint (C1) is in the form of inequality. In any case, all activities initiated must be completed:

$$l_i x_{i,t} \leq \sum_{k=t}^{\min\{t+l_i-1,|T|\}} z_{i,k} \ \ \forall i \in A,\ t \in T. \tag{C2}$$

where: $l_i \in \mathbb{N}_+$ is the execution time of activity i; $z_{i,k}$ is a binary variable that is equal to 1 when the ith activity is being executed in the kth period. The constraint C2 via the term $min\{t + l_i - 1, |T|\}$ requires the complete execution in l_i periods corresponding to the duration of the activity.

The resources available to the company are of two types, one financial, i.e. an annual budget for each contractor of which part $\tilde{b}_k$ can be allocated to specific types of activities, and another operational, i.e., labor teams that can work simultaneously. This last constraint depends on the availability of contractors performing certain activities. Consequently, we defined three constraints:

$$\sum_{i \in A} \sum_{t \in T_k} c_i x_{i,t} \leq b_k \ \ k \in \{1, \ldots, p\}. \tag{C3}$$

where: $b_k > 0\ \forall k = 1, \ldots, p$ is the budget associated at the kth period; c_i is the cost of activity i and in the formulation, it is considered to be paid at the beginning of the task. C3 is the budget constraint where the time horizon T has as many subperiods, $T_1, \ldots, T_p$, as there are years in which activities are to be planned.

$$\sum_{i \in A_k} \sum_{t \in T} c_i x_{i,t} \geq \tilde{b}_k \ \ k \in \{1, \ldots, r\}. \tag{C4}$$

where: $\tilde{b}_k > 0\ \forall k = 1, \ldots, r$ is the smallest fraction of the budget that must be invested in specific activities.
C4 is the constraint that allows a budget to be associated with specific subsets of A defined as $A_1, \ldots, A_r \subset A$.

$$\sum_{i \in A_j} z_{i,t} \leq u_j \ \ \forall j \in E,\ t \in T. \tag{C5}$$

where: $u_j \in \mathbb{N}_+$ are the teams that can work simultaneously for every jth contractor; $E = \{1, \ldots, q\}$ is the indexed set of contractors.

Some of the activities may have intercorrelations with each other, particularly in this case they are characterized by precedence constraints. We define $\mathcal{R} \subset A \times A$ the set of pairs (i, j) such that $(i, j) \in \mathcal{R}$ indicates that i cannot start if the execution of j is not completed.

$$x_{i,k} \leq \theta_{i,j,k} \sum_{t=1}^{\max\{k-l_j,1\}} x_{j,t} \;\; \forall (i,j) \in \mathcal{R}, \;\; k \in T. \tag{2.6}$$

In the constraint 2.6 the parameter $\theta_{i,j,k}$ is introduced for modeling the precedence correlation and defined as:

$$\theta_{i,j,k} = \begin{cases} 1, & \text{if } k > l_j \\ 0, & \text{if } k \leq l_j \end{cases} \quad \forall k \in T.$$

The constraint 2.6 can be explained by reasoning about the value assumed by $\theta_{i,j,k}$, which governs whether or not activity i subject to the precedence constraint can be scheduled:

- when $\theta_{i,j,k} = 0$, i.e., if $k \leq l_j$ and thus $x_{i,k} \leq 0$, activity i in fact has not yet elapsed the period required to complete activity j

- when $\theta_{i,j,k} = 1$, the planning of activity j turns out to be earlier than at least l_j periods before the start of i.

The last 3 constraints deal with "time" issues: some activities must be run within a deadline, some cannot be run during specific periods (e.g., winter or summer), and finally, some have imposed a start period. The latter constraint also takes into account all those activities that have already been scheduled the moment there is new information and the model is run again. These constraints can be modeled respectively:

$$\sum_{t=1}^{d_i-l_i} x_{i,t} = 1 \;\; \forall i \in A^d \tag{C7}$$

where $A^d \subset A$ is the set of activities with a deadline; $d_i \in T\ \forall i \in A^d$ are the deadlines.

$$z_{i,t} = 0 \ \ \forall (i,t) \in S \tag{C8}$$

where: $S \subset A \times T$ is the subset of pairs (i,t) in which t indicates the periods when activity i cannot be execute.

$$x_{i,t} = 1 \ \ \forall (i,t) \in \mathcal{S}' \tag{C9}$$

where: $\mathcal{S}' \subset P \times T$ is the of pairs (i,t) such that the ith project must start in period t. The constraint C9 forces execution in the indicated period by imposing value 1 on the variable $x_{i,t}$ in period t. Thus, the form of the optimization model is:

$$\begin{array}{ll}
\text{maximize} & \displaystyle\sum_{i\in A}\sum_{t\in T}\frac{\nu_i}{(1+r)^t}x_{i,t} \\
\text{subject to} & \displaystyle\sum_{t\in T} x_{i,t} \leq 1 \ \ \forall i \in A \\
& \displaystyle l_i x_{i,t} \leq \sum_{k=t}^{\min\{t+l_i-1,|T|\}} z_{i,k} \ \ \forall i \in A, t \in T \\
& \displaystyle\sum_{i\in A}\sum_{t\in T_k} c_i x_{i,t} \leq b_k \ \ k \in \{1,\dots,p\} \\
& \displaystyle\sum_{i\in A_k}\sum_{t\in T} c_i x_{i,t} \geq \tilde{b}_k \ \ k \in \{1,\dots,r\} \\
& \displaystyle\sum_{i\in A_j} z_{i,t} \leq u_j \ \ \forall j \in E,\ t \in T \\
& \displaystyle x_{i,k} \leq \theta_{i,j,k} \sum_{t=1}^{\max\{k-l_j,1\}} x_{j,t} \ \ \forall (i,j) \in \mathcal{R} \ \ k \in T \\
& \displaystyle\sum_{t=1}^{d_i-l_i} x_{i,t} = 1 \ \ \forall i \in A^d
\end{array}$$

$$z_{i,t} = 0 \;\; \forall (i,t) \in S$$
$$x_{i,t} = 1 \;\; \forall (i,t) \in S'$$
$$x_{i,t},\, z_{i,t} \in \{0,1\} \;\; \forall i \in A, t \in T$$

Sensitivity analysis

The sensitivity analysis of the problem focused on two elements: the value of the discount rate $r > 0$ in the objective function 2.5 and the convergence of the optimization problem.

The discount rate is a difficult value to estimate. Still, it plays a central role within the objective function because it pushes forward the execution of tasks with a higher priority. Since we could not perform discount rate studies, partly because of the need to be able to run the model multiple times, we directed the analysis to understand the impact that variation in the value of r has on the final result. As for the possible values to assign to r, these are subjective values that depend on how much you want the discount to weigh. In this case, they can be equated with risk-free because we are talking about investments in the public.
To prove that the variation of r values does not significantly inside the final result, in Table 2.1, are shown the values of the Jaccard similarity coefficient are given [1]. The convergence time of the optimization prob-

Table 2.1: Comparisons between solutions obtained with different discount rates. Values in the table are the Jaccard similarity coefficients of the two sets of activities selected by optimizing with two different discount rates: 1 indicates that the two solutions contain the same activities. Between parentheses, we give the average difference, in months, between the start of the same activity in models with different values of the discount rate.

r \ r	0.01	0.02	0.04	0.10
0.01	1	0.98 (1.6)	0.99 (1.3)	0.97 (3.3)
0.02		1	0.99 (1.4)	0.97 (2.7)
0.04			1	0.97 (3.1)
0.10				1

[1] The Jaccard similarity coefficients is the ratio of the intersection and union of two matrices $J(A,B) = \frac{|A \cap B|}{|A \cup B|}$. In this case, the matrices are those of the variable $z_{i,t}$.

lem turns out to be relevant for reasons such as offering the possibility to reiterate the problem as needed by changing the input data and providing a tool conveniently usable by users. Following the idea of *satisficing* solution of the aforementioned Herbert Simon [49], the goal is to operationalize the optimization process with an appropriate runtime by arriving at a near-optimal solution. To evaluate the convergence of the algorithm, the Lagrangian duality gap was used, i.e., the difference between the primal and dual problems [53, pp. 230], the distance of which decreases as the optimal solution approaches. In particular, such gap shall decrease as the algorithm approaches the global optimum. Figure 2.2 presents a graphical analysis of the relative Lagrangian duality gap as a function of the running time, after the first 20 minutes of computations. It can be seen that in around one hour the relative gap was reduced to the 0.1% (blue line), which guarantees that the incumbent solution is near optimal. The result

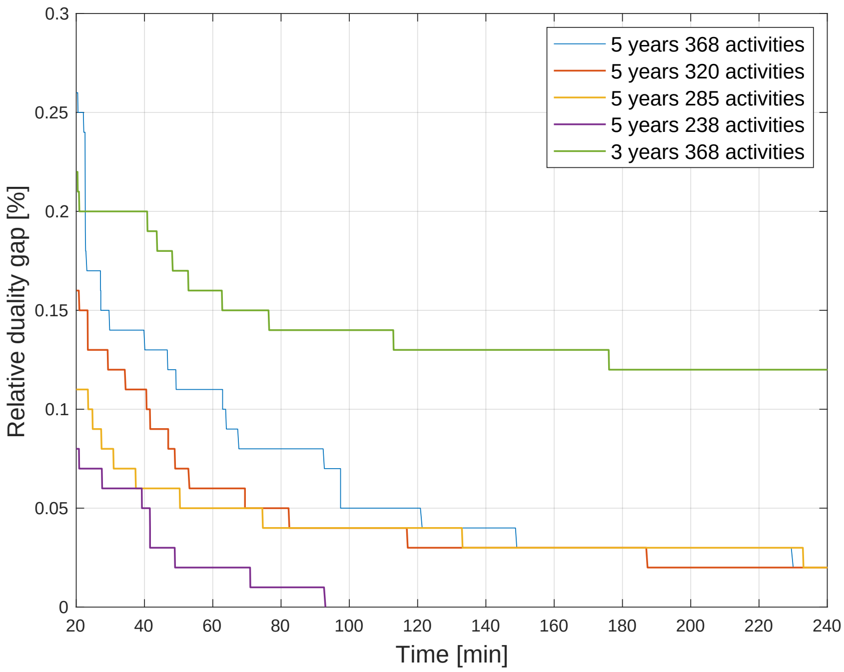


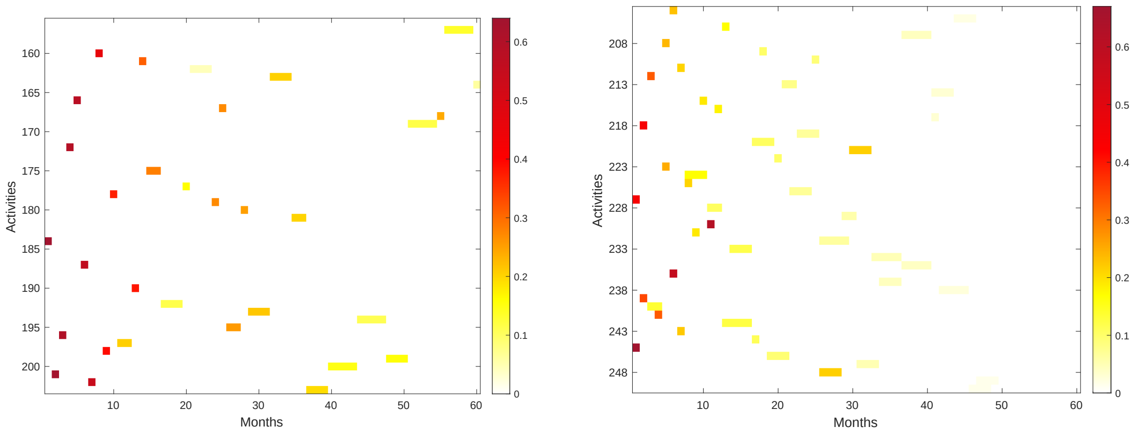
Figure 2.2: Evolution of the relative duality gap: the blue line represents the gap trend for the complete set of 368 activities while the other lines represent instances of subsets with fewer activities. The green line represents the gap trend for the set of 368 activities in 3 years.

of the analysis was also tested on larger sets of tasks and different time horizons showing in each case that a satisfactory solution could be reached after about an hour. It is considered in agreement with the experts that the time required is adequate.

2.4.3 Results

The results obtained were calculated based on the actual information provided by the company: 368 activities, 7 contractors, 9 work groups, 5 activities with a scheduled start, 17 with due dates, 107 with limitations on the time in which they can be performed, and 39 precedence relationships. Each activity is also characterized by a cost c_i, a duration l_i, and a priority value $\nu_i \in [0, 1]$ defined through MAVT. The interrelationships among activities do not allow reduction into subproblems; in fact, the execution of some activities subject to the precedence constraint is entrusted to different contractors.
The analysis of the model has been developed with the language AMPL and Gurobi 9.1.0 on a computer with an Intel Core i5 dual-core processor, 2.5 GHz, 8 GB of RAM, running macOS Catalina version 10.15.7.

The model's ability to provide useful results for the company's purposes is confirmed by comparison with end users. The final results were presented in the form of easily understandable graphs. Specifically, Figure 2.3a and Figure 2.3b depict two Gantt charts showing the scheduling of activities associated with two contractors over 5 years. The two schedules use one and two operations teams, respectively, as is evident from the overlap of activities in Figure 2.3b. In both graphs, it can be seen that activities with a higher priority (darker colors) are brought forward compared to those with a lower priority (lighter colors) unless there are constraints that dictate that certain activities must be performed at certain times. A comparison of the two graphs shows that the contractor who has two teams available, Figure 2.3b succeeds in completing all assigned activities before the 5-year limit. For this reason, it is also interesting to investigate the results from the perspective of the resources used. Figure 2.4a depicts the utilization of operational teams, highlighting how they are always used by the model at their maximum efficiency, considering that two contractors, with whom two teams are associated, conclude project execution before the end of the time horizon. Figure 2.4b shows the budget utilization, it

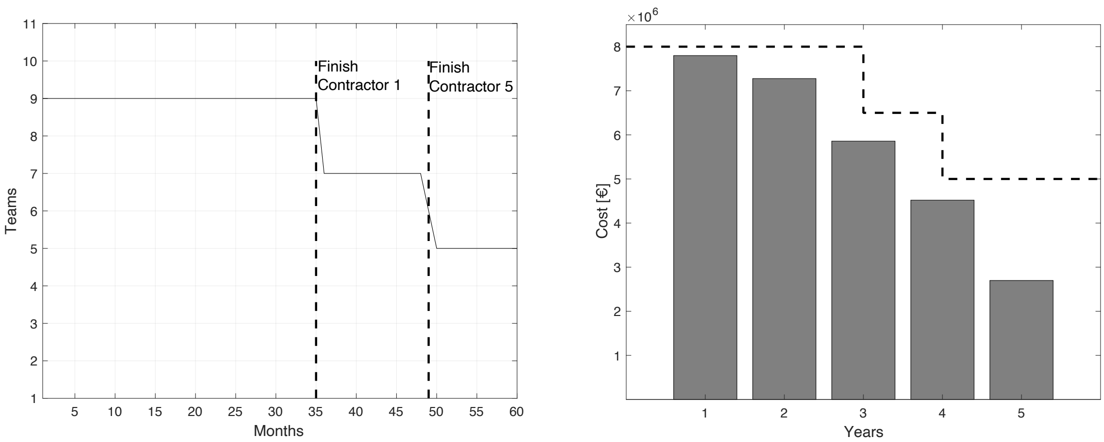


(a) Scheduling of Contractor 4

(b) Scheduling of Contractor 5

Figure 2.3: Scheduling and priorities of Contractors 4 and 5. The colored bar on the right side of the graph represents the values collected by the activities each month.

can be seen that the limit is never reached. Based on these considerations, it is believed that the model can also be used for resource reallocation, e.g., increasing the number of operational teams or assigning more activities to contractors who finish executions early.



(a) Teams occupancy: number of teams employed every month of the time horizon. The two vertical lines mark the end of the activities for contractors 1 and 5, respectively

(b) Budget use: total cost per year of executed activities. The dashed line is the total available budget through years with two steps when contractors 1 and 5 end activities.

Figure 2.4: Resources utilization within the time horizon

From the results obtained, it can be seen that the number of activities and the number of constraints and interrelationships would make it impractical to separate the four core objectives. The approach used, with the contextual aggregation of attributes into a single priority value, makes the model simple and operational and facilitates better understanding by experts, who are successfully involved in the process.

Chapter 3

Entropy measures in Dempster-Shafer Theory

3.1 Introduction

In everyday life, we are used to dealing with uncertainty, making immediate or reasoned decisions, aware that we have information affected by uncertainty. Based on weather forecasts, simple choices such as taking an umbrella or wearing a hat have a low risk of causing harm, but more important decisions require more attention. There are also physical and geological phenomena that cannot be predicted except with enormous uncertainty, such as the prediction of earthquakes or sea quakes, volcanic eruptions, and major weather phenomena. The past and the future thus involve uncertainty, which man has always tried to address by studying the evidence and comparing the conditions under which past events occurred with those that are present or may occur in the future. Sometimes situations are classified to assess risk but aware of uncertainty, as in the case of landslides which are classified according to their state of activity, inactivity, or quiescence, based on a comparison of current conditions with past conditions in which a ground movement phenomenon occurred. These classifications allow action to be taken based on expected risk, but the information on which they are based cannot be considered certain.
Similarly, archaeologists and historians look for historical truth in artifacts and documents, but inferences are made about what has survived to the present day, so there is a lack of complete information about some events

that occasionally leads to correct theories when evidence comes to light. It is clear from these examples that while uncertainty has always been a consciously accepted part of everyday life, in science it is instead a problem that must be addressed, but this was not always the case. Before the 20th century, the general idea was that uncertainty and science did not meet; anything that could not be described accurately in mathematical terms was set aside, leading to significant limitations in discoveries [74].
The first approaches to uncertainty in science occur through probability theory, with the limitations it entails. It will take another 50 years for greater knowledge: thanks to the rapid development of technology during and after World War II scientists have the opportunity to solve increasingly complex problems, requiring more general mathematical theories, and along with them the concept of uncertainty expands.

Before analyzing the theories that arise from this new thinking, it is important to focus on the meaning given to uncertainty. To simplify the concept, one can reason about the simple question of guessing a person's age. One may have several pieces of information that help to narrow the field, such as "he is young," and "he graduated two years ago," but it is still not possible to state an exact age with certainty because there is a lack of information. The link between uncertainty and information is the basis of so-called "uncertainty-based information." Lack of information can present itself in various ways: information may have been provided that is too general or incomplete, thus not achieving the desired goal. A classic example is the diagnosis of a disease: as long as the set of symptoms does not indicate a specific course of action, the physician must evaluate several options that may or may not have a chance of being true. This example is also useful to understand in practice how uncertainty decreases through increased information; the actions that can be performed must be relevant examinations to indicate the right diagnosis. In other fields, one can talk about relevant experiments or the discovery of a historical document containing more accurate information than previously possessed. Uncertainty-based information is calculated as the difference between the uncertainty before the action execution and the uncertainty after it. The measure of information is useful in comparing different systems.

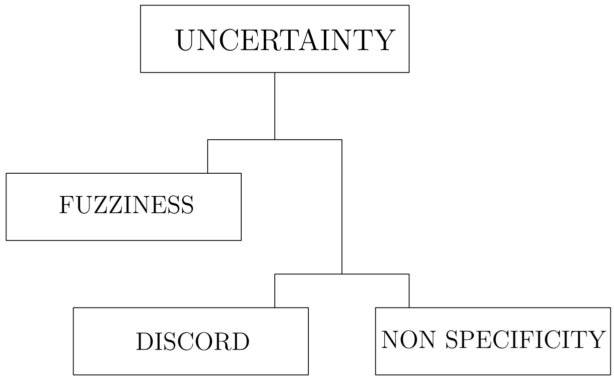


Figure 3.1: Types of uncertainty

3.1.1 Generalized Information Theory

Generalized Information Theory (GIT), a term introduced in 1991 [71], is the product of two generalizations that emerged in the 1980s: the move from classical measure theory to monotone measure theory and the move from classical set theory to fuzzy set theory [73]. GIT collects all uncertainty theories that are generalizations of the two classical theories, namely probability, and possibility. Uncertainty theories have in common the definition of a functional U that measures the amount of uncertainty and satisfies several axioms. The first functional found is within probability theory and is called Shannon's entropy [126].
The different theories are important because they enable us to handle different types of uncertainty and to express "ignorance", in the sense of the absence or incompleteness of information such that the correct answer in a set can be identified without doubt. The different types of uncertainty have been summarized by Klir and Yuan [77] in Figure 3.1 which represents the uncertainty types that can be described by classical set theory, fuzzy set theory, possibility theory, and evidence theory. We can identify the following types of uncertainty:

- **nonspecificity**: related to the cardinality of the set, the larger the set in which the correct answer lies the greater the uncertainty; the more relevant information you have the more you can reduce the set;

- **discord**: expresses a conflict between sets whose information is in conflict with each other;

- **fuzziness**: related to the vagueness of established boundaries and the difficulty of making clear and precise distinctions [70].

3.2 Classical Uncertainty Theories

Before addressing generalized uncertainty theories, in which the study of uncertainty is broader, we recall some of the concepts of classical uncertainty theories, which originated in the first half of the 20th century and are based on probability and possibility theories. In particular, this section will focus on the measures that first addressed the issue of uncertainty measurement.

Classical probability-based uncertainty theory

Generally, in a decision problem, the available alternatives can be included in a finite set X of mutually exclusive alternatives that may represent answers to the problem. This determines the fact that only one of the alternatives will be the correct answer to the question. Within set X we know that there is the correct answer to the question (e.g., a person's age), but with uncertainty, we do not know exactly what it is.
Probability theory allows us to describe random events, and the uncertainty about which alternative is correct to the question is expressed through the probability distribution function:

$$p : X \to [0, 1]$$

where

$$\sum_{x \in X} p(x) = 1$$

Probability distribution function associates with each element x of the set X a value in $[0, 1]$ that represents the probability that it is the correct answer. The so-called probability measure of a subset A of X will be equivalent to the sum of the values associated with each element belonging to A. The probability of the union of two disjoint subsets is equal to the sum of their probabilities; it thus satisfies the additivity property.

Shannon Entropy

The problem of how to measure the amount of uncertainty is addressed by Shannon [126] who defines Shannon Entropy:

$$\mathsf{H}_s(p) = -\sum_{x \in X} p(x) \log_2 p(x) \tag{3.1}$$

where $p(x_i)$ is the probability of $x_i \in X$. It attains its maximum when the probability is uniform and the uncertainty is maximum.

This formula is often referred to as the *conflict* measure because rewriting the function as:

$$\mathsf{H}_s(p) = -\sum_{x \in X} p(x) \log_2 \left[1 - \sum_{y \neq x} p(y)\right] \tag{3.2}$$

and

$$Con(X) = \sum_{y \neq x} p(y)$$

where $Con(X) \in [0,1]$ $\forall x \in X$ is the sum of the probabilities of the elements that are in open contrast to the probability that x occurs or is the correct answer. As this term increases, so does $-\log_2 \left[1 - \sum_{y \neq x} p(y)\right]$ and thus the measured uncertainty.

Classical possibility-based uncertainty theory

The classical-based uncertainty theory based on possibility theory, is older than the one based on probability theory [74]. Possibility theory differs from probability theory in having two set functions (i.e. possibility function and necessity function) and also does not satisfy the additive property. Moreover, while probability describes precise but conflicting phenomena and it is therefore possible to derive a measure of uncertainty that evaluates the conflict, in possibility theory the phenomena are not contradictory but imprecise [36]. In this context, nonspecificity can be measured. *Possibility function*, $Pos(A)$, where $A \in P(X)$, $P(X)$ is the power set of X, takes value

0 when there is no relevant evidence that a possible alternative exists in A, while it takes the value 1 when there is complete certainty.
Thus:

$$Pos : P(X) \to [0, 1]$$

And fulfills the following criteria:

$$Pos(\emptyset) = 0$$

$$Pos(X) = 1$$

Similarly, we define *necessity function* as the function that describes the situation in which an element contained in $A \in P(X)$ must be the correct alternative.
It is deduced that there cannot be a situation such that $Pos(A) = 0$ and $Nec(A) = 1$ and that the logical connection between the two functions is dictated by the fact that if it is not possible for the correct alternative to be contained in the complementary of A, $\overline{A}$, then it must mandatorily, i.e. necessarily, be in A:

$$Nec(A) = 1 - Pos(\overline{A}) \; \forall A \subset X \tag{3.3}$$

From this expression, it is evident that, unlike probability, the functions of possibility and necessity are not self-dual. In addition:

$$Pos(A \cup B) = max(Pos(A), Pos(B)) \; \forall A, B \in P(X), \tag{3.4}$$

means that the possibility function is subadditive. The dual function that allows the complete description of the situation, the necessity function, turns out to be superadditive.

On the basis of the properties described, necessity and possibility measures are respectively particular lower and upper probability measures for which there always exists:

$$Nec(A) \leq Pr(A) \leq Pos(A)$$

To better understand how possibility theory works, a numerical example is proposed. Let it be the case where a physician needs to figure out what disease his patient is suffering from. We have $X = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$. The distribution of possibilities is for example:

$$Pos(\{\mathrm{A}\}) = 0.3 \;\; Pos(\{\mathrm{B}\}) = 1 \;\; Pos(\{\mathrm{C}\}) = 0.2$$

The possibility function is not sufficient to completely describe the uncertainty of a situation, which is why the necessity function is needed. Therefore, we obtain Table 3.1.

set	Pos	Nec
{A}	0.3	0
{B}	1	0.7
{C}	0.2	0
{A,B}	1	0.8
{A,C}	0.3	0
{B,C}	1	0.7
{A,B,C}	1	1

Table 3.1: Numerical example: possibility and necessity function

Hartley measure

The measurement of uncertainty, in particular of nonspecificity is studied by Hartley [51] for finite sets. Hartley's measure is defined as:

$$H(A) = \log_2 |A| \tag{3.5}$$

Hartley's formula depends on the cardinality of the set of possible alternatives and thus indicates nonspecificity, i.e. the larger the set, the more possible alternatives and the less relevant information we have, consequently the uncertainty will be greater. Uncertainty of 0 corresponds to full specificity, that is when the set of alternatives consists of only one element.

3.3 Uncertainty measures

We saw earlier how, with the advent of GIT, broader and more general methods of describing uncertainty were sought. Following the theme of the second publication, we will focus on one of two generalizations: from

classical measure theory to monotone measurement theory, first mentioned by Choquet in [18].
A monotone measure, f, must be a function that, given a nonempty family of subsets E of a universal set X, is $f : E \to [0, \infty]$. It must satisfy certain requirements such as $f(\emptyset) = 0$ and especially monotonicity so that, for any set belonging to E, if $A \subseteq B$, then $f(A) \leq f(B)$. There are two other conditions, called "continuity from above" and "continuity from below," which provide, for decreasing and increasing sequences of nested subsets, that the limit of the monotone measure of the latter is the monotone measure of their intersection and union, respectively. For finite universal sets, the latter two conditions are always satisfied.
From this generalization comes the Dempster-Shafer theory or evidence theory, one of the theories that have attracted more attention due to elements such as *basic probability assignment* that allow for a better representation of uncertainty in information and its adaptivity [29].

3.3.1 Evidence theory

Evidence theory was proposed by Dempster [25] as a formalization of lower and upper probabilities and then developed by Shafer [125]. Like possibility theory, this theory is based on a pair of dual measures that are referred to as Belief and Plausibility measures.
To define these measures, consider a nonempty set $X = \{x_1, x_2, \ldots, x_n\}$ containing a finite number n of exhaustive and mutually exclusive elements, X is generally called, in this context, *frame of discernment* (FOD) and the related power set is $P(X)$. Belief measure Bel is defined as:

$$\mathrm{Bel} : P(X) \to [0, 1]$$

such that:

$\mathrm{Bel}(\emptyset) = 0;$

$\mathrm{Bel}(X) = 1;$

for every family of $A_1, \ldots, A_n$ of subsets of X and for every positive integer

n,

$$\mathrm{Bel}(A_1 \cup \cdots \cup A_n) \geq \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \mathrm{Bel}\left(\bigcap_{i \in I} A_i\right). \tag{3.6}$$

From the last condition, it is denoted that Belief is superadditive.
The dual measure is the Plausibility measure, Pl, defined as:

$$\mathrm{Pl} : P(X) \to [0,1]$$

such that:

$\mathrm{Pl}(\emptyset) = 0$

$\mathrm{Pl}(X) = 1$

for every family of $A_1, \ldots, A_n$ of subsets of X and for every positive integer n,

$$\mathrm{Pl}(A_1 \cap \cdots \cap A_n) \leq \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \mathrm{Pl}\left(\bigcup_{i \in I} A_i\right). \tag{3.7}$$

From the last condition, it is denoted that Plausibility is subadditive.
The two conditions of superadditivity and subadditivity allow this theory to express synergy and incompatibility between subsets [72].
As seen above for possibility theory the duality between the two measures is expressed as:

$$\mathrm{Bel}(A) = 1 - \mathrm{Pl}(\overline{A}) \tag{3.8}$$

Both measures can be expressed in terms of a function, m, called *basic probability assignment* (BPA) or *basic mass assignment*, defined as:

$$m : P(X) \to [0,1]$$

such that:

$m(\emptyset) = 0$

$\sum_{A \in P(X)} m(A) = 1$

Every set for which $m(A) > 0$ is referred to as a *focal element* while the set of all focal elements associated with its BPA is called *body of evidence*. The BPA is a function that assigns a value between 0 and 1 to each element in the power set. The value is based on evidence that supports that the answer to the problem can be found in a particular subset.
An application example of how this function can be used is given by the study of [48], in which the purpose is to identify the points with a higher potential for fire initiation in an area of Chile. The BPA value, for each square into which the area is divided, is derived from the surveys and information collected, thus defining the portion of evidence that supports the assertion that fire may start there. It is important to emphasize that the value of BPA is relative only to the reference subset A and not to its elements or subsets.

The importance of BPA is mainly related to its relationships with Belief and Plausibility measures, which can be expressed as:

$$\mathrm{Bel}(A) = \sum_{B|B\subseteq A} m(B) \quad \forall A \in P(X), \tag{3.9}$$

through which one can better understand how this measure quantifies the level of confidence that the element representing the truth or the answer is within a subset A. This is through the sum of all the evidence and information held that supports the claim, i.e., the BPA value of the set itself and all its subsets.

$$\mathrm{Pl}(A) = \sum_{B|A\cap B\neq\emptyset} m(B) \quad \forall A \in P(X), \tag{3.10}$$

through which it is clarified that the plausibility of an item to be the correct answer derives not only from relevant information that fully supports it but also from evidence that does not contradict it.
Consequently, it will always be the case that $Bel(A) \leq Pl(A) \, \forall A \in P(X)$ so it is more believable that an element is in the subset A the less plausible the opposite.

The evidence theory proves very useful in expressing a concept that was not possible in the other theories: total ignorance. This can be done with the *vacuous mass function* for which $m_?(X) = 1$.
An example can be made by imagining the early scientists of antiquity struggling with the question of whether the Earth was flat or round; until the earliest studies, i.e. the earliest evidence, they could not assign a belief of positive value to either option, even though they knew that one of the two possibilities, i.e., the universal set, could be the correct one.
It is worth noting that evidence theory is a generalization of probability theory and possibility theory. Examining some special cases shows the relationship between these theories. When the focal elements are all the singletons of the set X, i.e. only the singletons $(x_1, x_2, \dots, x_n)$ are assigned $m > 0$, we have the case where Bel $=$ Pl $= Pr$, i.e. the distribution of BPAs corresponds to a probability distribution function because the difference between the two is eliminated, i.e. that the former is defined on the power set while the latter is defined on the set X. When the set of focal elements, $\mathcal{F} = (A_1, A_2, \dots, A_i)$, is composed of subsets that form a nested sequence, i.e. $A_1 \subset A_2 \subset \cdots \subset A_i$, then the rules followed by Belief and Plausibility measures coincide with necessity and possibility measures, respectively.

To better understand these two special cases, Tables 3.2a and 3.2b represent two numerical examples. From these considerations can be deduced

set	m	Bel	Pl
{A}	0.3	0.3	0.3
{B}	0.2	0.2	0.2
{C}	0.5	0.5	0.5
{A,B}	0	0.5	0.5
{A,C}	0	0.8	0.8
{B,C}	0	0.7	0.7
{A,B,C}	0	1	1

(a) Atomic events

set	m	Bel	Pl
{A}	0.3	0.3	1
{B}	0	0	0.7
{C}	0	0	0.5
{A,B}	0.2	0.5	1
{A,C}	0	0.3	1
{B,C}	0	0	0.7
{A,B,C}	0.5	1	1

(b) Nested subsets

Table 3.2: Numerical examples of special cases in evidence theory.

not only the generality of the Theory of Evidence, which allows representing events subject to both random and epistemic uncertainty [33], but also the possibility of representing imprecise and inconsistent elements, i.e., where information may conflict with each other, as is often found when data are collected from experiments and different sources [36, p. 13].

Evidence theory has several practical applications such as the interpretation of seismic parameters [6], decision support on road safety [112], and construction project management [129], quantify uncertainty in multi-criteria decision making [147], in multi-sensor data fusion for fault diagnosis [142], and in pattern classification [148]. Some practical applications also concern civil engineering such as the study of earthquake damage produced on structures [7] and the probability of deterioration of bridges [4].

3.4 Applications in civil engineering

Evidence theory has several applications in the field of civil engineering and particularly in the field of structures and safety. We have already pointed out how uncertainty can affect various aspects of everyday life and experts' judgment [74]. For this reason, applying evidence theory in fields where reliance on expert opinion is imperative can be very useful. In particular, civil engineering, unlike other disciplines, depends heavily on the decisions of the designer and his or her experience [77, p. 419], the presence of uncertainty in this field can be traced back to various processes, from design to evaluation of the level of safety following deterioration of materials and structures [136, p. 404]. Nowadays, engineering problems are increasingly broad and complex, and as a result, it is more and more important to take into account the different types of uncertainty [20]. Probability theory is widely used in this field, but as we have seen, it cannot take into account certain types of uncertainty.
In particular, the ability of evidence theory to consider two types of uncertainty, to represent ignorance and lack of information not on individual items but also on sets can help in the management of uncertainty in the specific field of civil engineering [20]. In addition, another important feature of this theory is that of combining evidence, i.e., the possibility of combining information from different sources (e.g., different experts, mul-

tiple tests) and with different basic probability assignments, m_1 and m_2. The standard way to appropriately aggregate information and obtain a combined BPA is the *Dempster's rule of combination* [74, pp. 170-173]:

$$m_{1,2}(A) = \frac{\sum_{B \cap C = A} m_1(B) \cdot m_2(C)}{1 - K} \tag{3.11}$$

where:

$$K = \sum_{B \cap C = \emptyset} m_1(B) \cdot m_2(C)$$

Thus, evidence theory allows, in addition to the concept of total ignorance, to take into account the concept of *interactivity* [20], i.e. the influence that two disjoint events can have on each other in the presence of both, and the extension to consider situations of nonadditive effects as well. However, there are limitations in using evidence theory, one of them being the complexity of mathematical models [93].

One of the approaches, based on evidence theory, and widely used in civil engineering is *Evidential Reasoning* (ER). This method is developed for multi-criteria decision analysis methods under uncertainty [153]. To support the decision maker in making choices, the "*degree of belief*" is used. This value, in a range $[0, 1]$, is to be assigned by the decision maker, based on his or her knowledge and information. It represents the amount of belief that an alternative, a_i, reaches a certain level for each attribute [128].

Through this concept and the use of an extended decision matrix, it is possible to consider decision-making situations with imprecise, missing, and uncertain information [117] due to the generalization of evidence theory and its ability to represent ignorance [117]. Moreover, ER approach can also assume a hierarchical pattern of attributes [151], imagining that there is a general attribute at the top and several levels of other sets of attributes below [154]. The elements that define an evidential reasoning approach are defined below. Let $A = \{A_1, A_2, \ldots, A_i, \ldots, A_N\}$ be the finite set of alternatives and $C = \{C_1, C_2, \ldots, C_j, \ldots, C_M\}$ be the finite set of criteria or attributes whose relative weight, i.e., relative importance, is denoted by w_j and satisfy the condition:

$$\sum_{j=1}^{M} w_j = 1 \text{ and } 0 \leq w_j \leq 1, \quad j = 1, \ldots, M.$$

Each attribute uses a degree scale $G = \{G_1, G_2, \ldots, G_l, \ldots, G_L\}$ by which it defines the level it could reach for each selected alternative. Each cell of the extended decision matrix will have as its element $S(A_i(C_j)) = \{(\beta_{l,j}, G_l)\}$. Where $\beta_{l,j}$ are the degrees of beliefs, i.e., how much the decision maker believes the alternative A_i can bring to the G_l level of an attribute C_j. For example, the first element of the extended decision matrix is explicitly defined as:

$$S(A_1(C_1)) = \{(\beta_{11}, G_1), (\beta_{21}, G_2), \ldots, (\beta_{l1}, G_l)\}$$

The next step is to transform degrees of belief into BPAs by applying the following equations [112]:

$$m_{l,j} = m_j(G_l) = w_j \times \beta_{l,j} \;\; \forall l = 1, \ldots, L, \; j = 1, \ldots, M \tag{3.12}$$

where $m_{l,j}$ is the basic probability assigned to the attribute j located at level l.

$$m_{G,j} = m_j(G) = 1 - \sum_{l=1}^{L} m_{l,j} = 1 - w_j \sum_{l=1}^{L} \beta_{l,j} \;\; \forall l = 1, \ldots, L, \; j = 1, \ldots, M \tag{3.13}$$

where $m_{G,j}$ is the BPA that aggregates all levels, thus the one assigned to the set of levels G, $m_{G,j}$, can be divided into two parts:

$$\overline{m}_{G,j} = \overline{m}_j(G) = 1 - w_j \;\; j = 1, \ldots, M \tag{3.14}$$

$$\tilde{m}_{G,j} = \tilde{m}_j(G) = w_j \left(1 - \sum_{l=1}^{L} \beta_{l,j}\right) \;\; l = 1, \ldots, L, \; j = 1, \ldots, M \tag{3.15}$$

where $\overline{m}_{G,j}$ expresses the relative importance of attribute j, while $\tilde{m}_{G,j}$ expresses the lack or incompleteness of information held by the expert who defined the degrees of beliefs. ER algorithms can take into account

several axioms that allow for consistent aggregation in terms of accuracy and amount of information among different levels [154].
The analytical algorithm is summarized here, which turns out to be more flexible and simpler than the recursive algorithm [141]. The analytical ER algorithm aggregates attributes that are part of a hierarchical structure [152].

$$m_l = k\left[\prod_{j=1}^{M}(m_{l,j} + \overline{m}_{G,j} + \tilde{m}_{G,j}) - \prod_{j=1}^{M}(\overline{m}_{G,j} + \tilde{m}_{G,j})\right] \tag{3.16}$$

where k is the normalization factor:

$$k = \left[\sum_{l=1}^{L}\prod_{j=1}^{M}(m_{l,j} + \overline{m}_{G,j} + \tilde{m}_{G,j}) - (L-1)\prod_{j=1}^{M}(\overline{m}_{G,j} + \tilde{m}_{G,j})\right]^{-1} \tag{3.17}$$

where m_l is the aggregation of the BPAs of the various elements for each grade of set $\{G\}$.

$$\tilde{m}_G = k\left[\prod_{j=1}^{M}(\overline{m}_{G,j} + \tilde{m}_{G,j}) - \prod_{j=1}^{M}\overline{m}_{G,j}\right] \tag{3.18}$$

$$\overline{m}_G = k\left[\prod_{j=1}^{M}\overline{m}_{G,j}\right] \tag{3.19}$$

where $\tilde{m}_G$ and $\overline{m}_G$ aggregate the factors $\tilde{m}_{G,j}$ and $\overline{m}_{G,j}$ relative to the entire set G. Then the newly aggregated BPAs are transformed back by normalization to obtain the belief structure to be included in the matrix:

$$\beta_l = \frac{m_l}{1 - \overline{m}_G} \tag{3.20}$$

$$\beta_G = \frac{\tilde{m}_G}{1 - \overline{m}_G} \tag{3.21}$$

where β_l is the belief degree for each level l on the attribute at the top of the hierarchy and β_G is the belief degree of set $\{G\}$ and shows the accuracy with which the data is provided, any deficiency or absence, i.e., the level of ignorance. When β_G is equal to 0 it means the information is complete, the greater it is, the less information, the greater the ignorance about the data.

3.4.1 Practical example of ER

Saleh Abu Dabous [121] In the publication “A flexible bridge rating method based on analytical evidential reasoning and Monte Carlo simulation”, Saleh Abu Dabous [121], address the problem of classifying the condition of civil structures, specifically road bridges. They emphasize how very often experts conduct a visual inspection of bridges thus subjectively assigning the state rating of different bridge elements. Using an approach based on evidence theory allows for the uncertainties associated with a subjective process to be taken into account. The authors proceed to an exhaustive literature review and then focus on finding a method that is suitable for the actual situation and flexible to adapt to different realities. They note difficulties in finding a standard method of inspection and classification. The ER approach allows all major bridge elements to be considered hierarchically and uncertainty to be incorporated into the analysis. In addition, the authors incorporate into the process a technique for comparing elements that allows the flexibility of being able to apply the method to different types of existing bridges.

The procedure followed starts by hierarchically dividing the bridge into several corresponding elements as shown in Figure 3.2 In this case, the set of attributes is the set of bridge elements. The condition, i.e., the degree, of the different elements combined through the ER approach algorithm will provide the overall condition of the bridge taking into account the uncertainty due to subjective classification.
The first step is to assign relative weights, w_j, to all attributes. In the case taken as an example, this procedure is done by comparing the different subsets of attributes in pairs and defining matrices in which the results of the comparisons are reported. The weight of each element is equal to the number of times it “wins” the comparison divided by the total number of comparisons.
The second step involves defining the levels that can be attained, it is de-

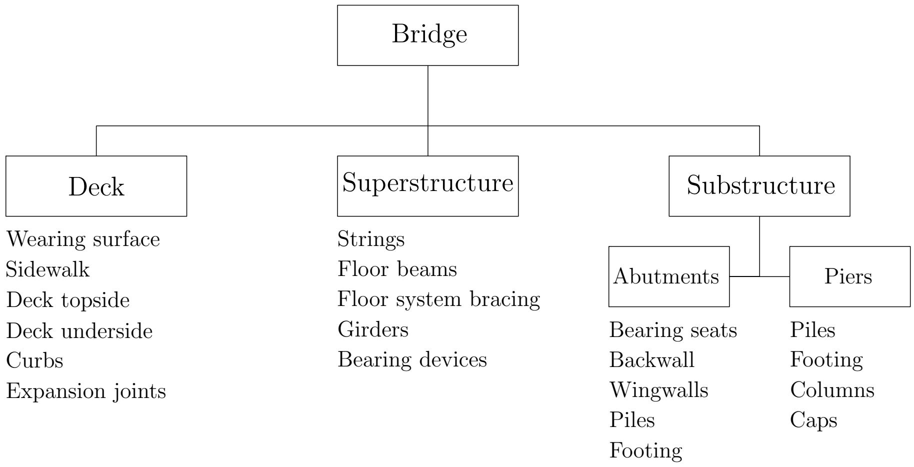


Figure 3.2: Bridge elements.

fined as $G = \{\text{poor}, \text{fair}, \text{good}, \text{excellent}\}$. At this point, degrees of beliefs must be assigned by an expert, i.e., how much the expert, based on the information he or she has from visual inspection and testing of materials, believes that the different items are in the different conditions described by the set G. In the example given in [121], this procedure is carried out with a Monte-Carlo simulation that generates several possible scenarios based on the health index (HI) implemented by the Department of Transportation in California [114], but it is also possible to subject it to the evaluation of different inspectors by combining their opinions through Dempster's rule of combination [20, 8].

Once in possession of all the necessary information, they proceed with the application of the analytic algorithm as described by Equations 3.12-3.21. Please refer to [121] for all computational details, while the results obtained on the overall bridge structure are reported here. After aggregating the ratings of the deck, superstructure, and substructure, the rating of the complete structure turns out to be in poor, fair, and good condition with 26.93%, 42.66% and 30.41%, probability, respectively. These results take into account the fact that the bridge considered in the calculations turns out to be in *good* condition as far as the main structures are concerned, but also that the superstructure is in poor condition, indicating that maintenance is generally needed.

3.5 Generalized uncertainty measures

The two measures of uncertainty seen so far have been defined in probability and possibility theory; in evidence theory, some attempts have been made to generalize them as a function of m.
Hartley's generalized measure was proposed by Dubois and Prade [35]:

$$\mathsf{H}_d(m) = \sum_{A \in \mathcal{F}} m(A) \log_2 (|A|) \,. \tag{3.22}$$

The nonspecificity measure is presented as a weighted average of the Hartley measure for all focal elements, where the weights are the BPA values.

Concerning Shannon's generalization of entropy, several formulations were proposed in the early 1980s, but each of them is deficient in terms of certain properties. The property that is most frequently not satisfied is the subadditivity [74].

Höhle [54] proposed the so called *confusion* measure that represents the conflict between evidences only when $B \cap A = \emptyset$.

$$\mathsf{H}_O(m) = -\sum_{A \in \mathcal{F}} m(A) \log_2 \mathrm{Bel}(A), \tag{3.23}$$

While Yager [149] proposed the *dissonance* measure that includes a broader definition of conflict, but does not correctly scale the level of conflict.

$$\mathsf{H}_y(m) = -\sum_{A \in \mathcal{F}} m(A) \log_2 \mathrm{Pl}(A), \tag{3.24}$$

In the early 1990s the measure of *discord* was proposed by Klir [71], as an improvement of the previous two formulas:

$$\mathsf{H}_k(m) = -\sum_{A \in \mathcal{F}} m(A) \log_2 \left(\sum_{B \in \mathcal{F}} m(B) \frac{|A \cap B|}{|B|} \right) \tag{3.25}$$

In this case, the logarithm argument describes the conflict between relevant evidence related to focal elements. The conflict is appropriately scaled through the denominator cardinality.

Equation 3.25 is further improved by Klir and Parviz [76] to adequately represent the conflict between subset A and the other focal elements with Equation 3.26, named *strife*.

$$\mathsf{H}_p(m) = -\sum_{A\in\mathcal{F}} m(A)\log_2\left(\sum_{B\in\mathcal{F}} m(B)\frac{|A\cap B|}{|A|}\right). \tag{3.26}$$

Before the research moved on to focus on a measure that can describe both uncertainties, two other measures were presented. The first uses the commonality function, defined by Shafer [125] as:

$$Q(A) = \sum_{B|A\subseteq B} m(B) \tag{3.27}$$

The entropy proposed by Smets [127] measures the information contained in the BOE:

$$\mathsf{H}_t(m) = -\sum_{A\in\mathcal{F}} c(A)\log_2 Q(A), \tag{3.28}$$

where we set $c(A) = 1$ which is, according to Smets [127, p. 37], the "most natural choice".

The proposal of Nguyen [104] is probably the most direct extension of Shannon's entropy:

$$\mathsf{H}_n(m) = -\sum_{A\in\mathcal{F}} m(A)\log_2 m(A). \tag{3.29}$$

It is since the early 1990s that research in this field has focused on measures that attempt to describe both nonspecificity and conflict. The second publication focuses on this context.

3.6 Publication II

3.6.1 The Problem

Publication II focuses on a numerical comparison of the many formulations that have crowded the research field of entropy measures in recent years. In particular, in the last 40 years, and more specifically in the last 5 years, a large number of measures have been proposed. Their goal is the correct description and calculation of uncertainty within evidence theory, in an attempt, not entirely successful, to define a formulation that is also able to satisfy some properties. Currently, due to a large number of options, it is very complicated to use one measure over another, and thus the choice in a real application context can be difficult to implement.

At present, the study of numerical comparison is useful to properly analyze what has been done so far. In recent years, some measures have already been compared but only on specific cases or certain properties. In addition, this comparison could be useful in the choice of uncertainty measure that affects concrete and real applications, such as in multicriteria decision making [147], multisensor data fusion for fault diagnosis [142], and model classification [148]. Therefore, the main objective is to present a numerical study comparing uncertainty measures by verifying their differences and similarities, as a support for a more informed choice of uncertainty measures.

3.6.2 Mathematical properties

Mathematical properties of uncertainty measures have been proposed by Klir and Wierman [69]. The numerical analyses conducted in this paper do not focus on these properties, but given their importance in defining the measures, it is useful to mention at least the five most important properties [3].

1. *Probabilistic consistency*: measure must degenerate into Shannon entropy when the distribution of BPAs, $m(A)$, represents a probability distribution, i.e., when all focal elements are singletons.

2. *Set consistency*: When the distribution of BPAs is such that it focuses

on a single subset, i.e., $m(A) = 1$, it means that there are no conflicts between subsets because all the evidence points to the presence of the sought-after element in A, consequently the measure must degenerate into Hartley's measure.

3. *Subadditivity*: the total uncertainty when m is a joint BPA on the Cartesian product $X \times Y$ and m_X and m_Y are the respective associated marginal BPAs, cannot be greater than the sum of the uncertainties of m_X and m_Y, i.e:

$$\mathsf{H}(m) \leq \mathsf{H}(m_X) + \mathsf{H}(m_Y)$$

4. *Additivity*: the total uncertainty when m is a joint BPA on the Cartesian product $X \times Y$ and m_X and m_Y are the respective associated marginal BPAs, is equal to the sum of the uncertainties of m_X and m_Y, if and only if the marginal representations are noninteractive:

$$\mathsf{H}(m) = \mathsf{H}(m_X) + \mathsf{H}(m_Y)$$

5. *Monotonicity*: When it is possible to order the values of evidence, meaning, in evidence theory, m_1 and m_2 defined on the FOD X are such that $\mathrm{Bel}_2(A) \leq \mathrm{Bel}_1(A)\ \forall A \subseteq X$, then H is monotone if and only if

$$\mathsf{H}(m_1) \leq \mathsf{H}(m_2)$$

3.6.3 Recent measures

The search for a unique measure of entropy in the field of evidence theory is still one of the active topics in the literature [74, p.417]. Over the years, as shown in Figure 3.3, many options have been proposed. Some of the best-known ones have been studied in Publication II, including those expressed earlier in Section 3.5. Figure 3.3 shows the years in which the measures were published and indicates the names of the researchers who proposed them. It can be seen that the study of entropy measurement has covered the last 40 years, and in the last 7 years, it has returned as a topic of interest. From 1982 to 1987, there are concentrated attempts

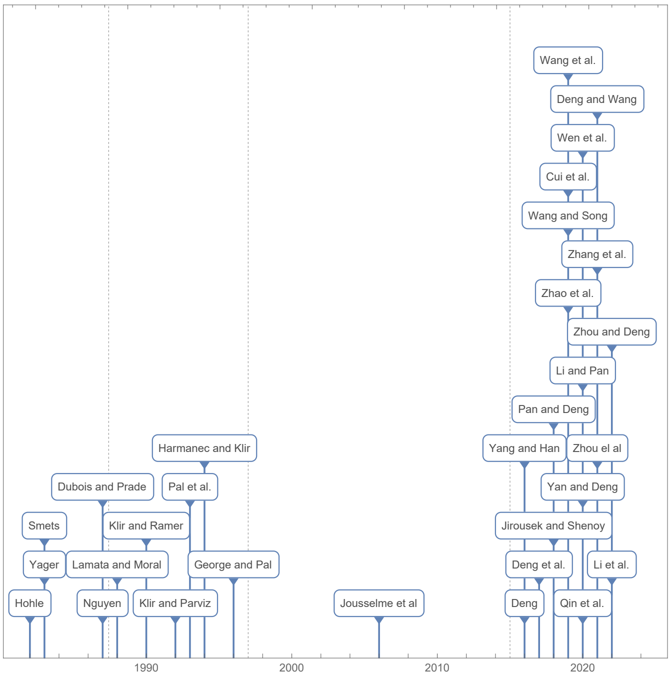


Figure 3.3: Timeline of the seminal papers for various uncertainty measures.

to generalize Hartley's measure and Shannon's entropy. As we shall see in this Section, the measure proposed by Lamata and Moral [83] marks the boundary between measures that address only one type of uncertainty and those that seek to describe both nonspecificity and conflict in a single formula. There is a lack of proposals between 1996 and 2015 in which only the measure of Jousselme et al. [59] is present. In more recent years, we have seen a large number of proposals, some of which find inspiration in existing measures, trying to obtain the satisfaction of the desired properties; however, as pointed out by Dezert and Tchamova [31], the latest proposals have no longer focused efforts on the analysis of formal properties.

After this temporal overview, we move on to analyze the mentioned measures.

Lamata and Moral

Lamata and Moral [83] are the first to introduce a *global measure* that is designed to represent and calculate both types of uncertainty found in evidence theory. The proposed measure turns out to be the sum of the entropies of Yager [149] and Dubois and Prade [35].

$$\mathsf{H}_l(m) = \overbrace{-\sum_{A\in\mathcal{F}} m(A)\log_2\left(\frac{\mathrm{Pl}(A)}{|A|}\right)}^{\mathsf{H}_y(m)+\mathsf{H}_d(m)}. \tag{3.30}$$

Pal et al.

Following the logic of the sum of two measures describing two types of uncertainty, Pal et al. [106, 107] also define total uncertainty as the sum of the measures of Nguyen [104] and Dubois and Prade [35].

$$\mathsf{H}_b(m) = \overbrace{-\sum_{A\in\mathcal{F}} m(A)\log_2\left(\frac{m(A)}{|A|}\right)}^{\mathsf{H}_d(m)+\mathsf{H}_n(m)}. \tag{3.31}$$

Harmanec and Klir

Aggregate Uncertainty is introduced by Harmanec and Klir [50] to measure total uncertainty and can be expressed in terms of belief measures, plausibility measures, or basic probability assignments due to their one-to-one correspondence.

$$\mathsf{AU}(m) = \max_{p\in\mathcal{P}(m)}\left\{-\sum_{i=1}^{n} p(x_i)\log_2 p(x_i)\right\}, \tag{3.32}$$

where $p(x_i)$ are all probability distributions satisfying the following criteria:

$p(x_i) \in [0,1]\ \forall x_i \in X$ and $\sum_{i=1}^{n} p(x_i) = 1$;

$\mathrm{Bel}(A) \leq \sum_{x_i\in A} p(x_i)\ \forall A \subseteq X$.

Thus AU arises as an optimization problem whose aim is to find the maximum Shannon entropy in a set of probability distributions dominating a given belief function [74, p. 231].
Solving the optimization problem would be very complex, but thanks to the study of some relatively simple algorithms it turns out to be a useful

measure. In particular, in the publication we adopted the algorithm proposed by Huynh and Nakamori [56], improvement of the one proposed by Liu et al. [92].

George and Pal

George and Pal [45] depart from the trends of the early 1990s by returning to the search for a measure of uncertainty that describes only conflict. Their reasoning is based on the idea that the difference between two elements in the BOE can be seen as a *distance*, rather than seeking a generalization of Shannon's entropy. The result is a definition of discord as an average conflict of every evidence,

$$\mathsf{TC}(m) = \sum_{A\in\mathcal{F}} m(A) \sum_{B\in\mathcal{F}} m(B) \left(1 - \frac{|A\cap B|}{|A\cup B|}\right). \tag{3.33}$$

Jousselme et al.

The proposed Jousselme et al. [59], called Ambiguity Measure, is a measure of total uncertainty. The name comes from the definitions of Klir and Yuan [77] in Figure 3.1. The proposed measure is similar to the one proposed more than 10 years earlier by Harmanec and Klir [50], it is in fact a measure designed to improve some shortcomings of the former. Moreover, the attempt of the measure is to satisfy the properties proposed by Klir and Wierman [69].
Also in this case, the basis is Shannon entropy and the distribution of belief is transformed into a probability distribution. Specifically, the authors use the pignistic transformation, proposed by Dubois and Prade [34]:

$$\mathrm{Bet}(x_i) = \sum_{A|x_i\in A} \frac{m(A)}{|A|} \quad \forall i = 1, \ldots, n.$$

The result of the transformation is that each basic probability assignment relative to a focal element, $m(A)$ is redistributed uniformly over the singletons, $x_i \in A$.

Accordingly, the proposed measure is:

$$\mathsf{AM}(m) = -\sum_{i=1}^{n} \mathrm{Bet}(x_i) \log_2 \mathrm{Bet}(x_i). \tag{3.34}$$

Yang and Han

Yang and Han [155] propose a measure of total uncertainty based on the distance between belief intervals, meant as $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$, in which they recognize the presence of both uncertainties and which does not require switching from evidence theory measures to probability distribution. Specifically, the distance between the belief interval of each singleton from the interval representing the largest uncertainty, i.e. $[0, 1]$, is used. The greater the distance between the two, the lower the total uncertainty.
The proposed measure is:

$$\mathsf{TU}^I(m) = 1 - \frac{1}{n}\sqrt{3}\sum_{i=1}^{n} d^I([\mathrm{Bel}(x_i), \mathrm{Pl}(x_i)], [0, 1]), \tag{3.35}$$

where $\sqrt{3}$ is the normalization factor and

$$d^I\left([\mathrm{Bel}(x_i), \mathrm{Pl}(x_i)], [0, 1]\right) =$$
$$= \sqrt{\left[\frac{\mathrm{Bel}(x_i) + \mathrm{Pl}(x_i)}{2} - \frac{0+1}{2}\right]^2 + \frac{1}{3}\left[\frac{\mathrm{Pl}(x_i) - \mathrm{Bel}(x_i)}{2} - \frac{1-0}{2}\right]^2}.$$

Deng and Wang

Taking advantage of the idea of using belief distance to measure total uncertainty, Deng and Wang [30] propose an improvement of $\mathsf{TU}^I(m)$, exploiting the Hellinger distance that is applicable in both probability theory and evidence theory.

$$\mathsf{DU}(m) = \sum_{i=1}^{n}\left(1 - \sqrt{\left(\sqrt{\mathrm{Bel}(x_i)} - 0\right)^2 + \left(1 - \sqrt{\mathrm{Pl}(x_i)}\right)^2}\right). \tag{3.36}$$

Li et al.

A further variant on total uncertainty measures based on the distance between belief intervals is proposed by Li et al. [86]. The authors use Euclidean distance by defining:

$$d_E\left([\mathrm{Bel}(x_i), \mathrm{Pl}(x_i)], [0,1]\right) = \sqrt{[\mathrm{Bel}(x_i) - 0]^2 + [\mathrm{Pl}(x_i) - 1]^2}.$$

A linear distance transformation is performed and summed with respect to all elements of the FOD, resulting in the measurement:

$$\mathsf{TU}(m) = \sum_{i=1}^{n} \left(\frac{2}{1 + d_E} - 1 \right). \tag{3.37}$$

It is denoted that Deng et al. [27] also use Euclidean distance to define a measure of total uncertainty.

Wang and Song

Wang and Song [140] point out that it is complex to determine what is the correct distance to use to adequately describe uncertainty. In addition, the authors emphasize the need to develop a measure of uncertainty that can be used in both evidence theory and probability theory. The defined measure is based on Shannon entropy and central values of intervals,

$$\frac{\mathrm{Bel}(x_i) + \mathrm{Pl}(x_i)}{2},$$

to measure discord and on the difference $\mathrm{Bel}(A) - \mathrm{Pl}(A)$ to measure non-specificity:

$$\mathsf{SU}(m) = \sum_{i=1}^{n} \left[-\frac{\mathrm{Bel}(x_i) + \mathrm{Pl}(x_i)}{2} \log_2 \frac{\mathrm{Bel}(x_i) + \mathrm{Pl}(x_i)}{2} + \frac{\mathrm{Pl}(x_i) - \mathrm{Bel}(x_i)}{2} \right]. \tag{3.38}$$

Deng

Deng [28] proposes a measure of total uncertainty that takes inspiration from the measure of Pal et al. [106]

$$\mathsf{H}_g(m) = -\sum_{A\in\mathcal{F}} m(A)\log_2\left(\frac{m(A)}{2^{|A|}-1}\right). \tag{3.39}$$

The formula can be divided into two parts, one for each type of uncertainty. The measure is based on focusing more attention, and thus an increase in uncertainty, as the nonspecificity increases, so when the cardinality of A increases [1]. While the conflict part coincides with the proposal of Nguyen [104].
The properties of measurement have been extensively and critically analyzed by Abellán [1] e [100].

Pan and Deng

The proposal of Pan and Deng [108] uses both the concept of central values from the Wang and Song [140] measure and the Deng [28] measure to define a new entropy measure:

$$\mathsf{H}_{bel}(m) = -\sum_{A\in\mathcal{F}} \frac{\mathrm{Bel}(A)+\mathrm{Pl}(A)}{2}\log_2\frac{\mathrm{Bel}(A)+\mathrm{Pl}(A)}{2(2^{|A|}-1)}. \tag{3.40}$$

Jiroušek and Shenoy

Jiroušek and Shenoy [57] propose a measure of entropy that is different from the others. For this reason, they introduce a list of six properties that the measure should have. The properties are identified partly on the basis of Shannon's entropy and partly on the requirements needed to be aligned with evidence theory. The properties identified by the authors are half the same as those proposed by Klir and Wierman [69].
Like other total uncertainty measures, this one consists of two parts, one being Shannon entropy in which probabilities are obtained by plausibility

transformation,

$$\mathrm{Pl_T}(x_i) = \frac{\mathrm{Pl}(x_i)}{\sum_{j=1}^{n} \mathrm{Pl}(x_j)} \; \forall x_i \in X,$$

the other is the measure defined by Dubois and Prade [35]. The entropy measure obtained turns out to be:

$$\mathsf{H}_j(m) = \overbrace{\sum_{x_i \in X} \mathrm{Pl_T}(x_i) \log_2 \left(\frac{1}{\mathrm{Pl_T}(x_i)} \right)}^{\mathsf{H}_s(\mathrm{Pl_T})} + \overbrace{\sum_{A \in \mathcal{F}} m(A) \log_2(|A|)}^{\mathsf{H}_d(m)} . \tag{3.41}$$

The use of a probabilistic transformation may resemble the formula of Jousselme et al. [59], so much so that the authors compare the two different transformations used showing that the plausibility transformation is better in use within evidence theory.

Zhou, Tang and Jiang

Zhou et al. [163] are the first authors to include the cardinality of the FOD in the entropy measure and to take into account the relationship between the cardinality of the focal elements and the FOD. The inspiration for the formula comes from the entropy of Deng [28].

$$\mathsf{E}_{Md}(m) = -\sum_{A \subseteq X} m(A) \log_2 \left(\frac{m(A)}{2^{|A|} - 1} e^{\frac{|A|-1}{|X|}} \right) \tag{3.42}$$

Compared with Deng's formula, the only addition is the element $\frac{|A|-1}{|X|}$.

Cui et al.

Other new formulations decide to take into account the cardinality of the FOD, in this case also Cui et al. [21] define a measure of entropy based on the measure of Deng [28], in fact, the defined measure degenerates into Deng entropy when the intersection between the focal elements is equal to the empty set.

$$\mathsf{E}(m) = -\sum_{A \subseteq X} m(A) \log_2 \left(\frac{m(A)}{2^{|A|}-1} \exp \left(\sum_{\substack{B \in \mathcal{F} \\ B \neq A}} \frac{|A \cap B|}{2^{|X|}-1} \right) \right). \tag{3.43}$$

Yan and Deng

The proposal of Zhou et al. [163] is modified by Yan and Deng [150] with the inclusion of the belief measure and the replacement of the cardinality of the FOD with the number of singletons with $\mathrm{Pl} > 0$, indicated with $|S|$.

$$\mathsf{H}_{Md}(m) = -\sum_{A \subseteq X} m(A) \log_2 \left(\frac{m(A) + \mathrm{Bel}(A)}{2(2^{|A|}-1)} e^{\frac{|A|-1}{|S|}} \right) \tag{3.44}$$

Qin et al.

Qin et al. [111] propose a new measure of uncertainty in which the cardinality of the FOD, $|X|$, is entered to scale the cardinality of the focal elements with respect to it in the formula of Dubois and Prade [35] while conflict is measured by Nguyen [104].

$$\mathsf{H}_q(m) = \sum_{A \in \mathcal{F}} \frac{|A|}{|X|} m(A) \log_2(|A|) + \sum_{A \in \mathcal{F}} m(A) \log_2 \left(\frac{1}{m(A)} \right), \tag{3.45}$$

Li and Pan

Li and Pan [85] propose a measure that sums the measure of Nguyen [104], and that of Dubois and Prade [35]. The latter is in this case multiplied by the cardinality of the FOD to account for the impact of the universal set, X. The resulting measure is expressed as:

$$\mathsf{H}_{B\&F}(m) = \sum_{A \in \mathcal{F}} m(A) \log_2 \frac{|A|^{|X|}}{m(A)}. \tag{3.46}$$

Zhou and Deng

Zhou and Deng [164] propose a measure of entropy based on a probability transformation which takes inspiration from the theory of fractals:

$$\mathsf{E}_{FB} = -\sum_{A \in \mathcal{F}} m_F(A) \log_2 m_F(A) \tag{3.47}$$

where m_F is defined as follows,

$$m_F(A) = \sum_{B|A \subseteq B} \frac{m(B)}{2^{|B|} - 1} .$$

Other measures

In recent years there has been a lot of research on entropy measurements, the measurements presented are not the only ones in the literature, there are a few others ranging from the more technical, e.g. Wen et al. [146] and Zhao et al. [159], to parametric ones, e.g. Zhang et al. [158] and Wang et al. [139]. In particular, the latter were excluded from the analysis because the need to set an a priori value would have made the comparison complex.

3.6.4 Mathematical properties on analyzed measures

Having described all the measures that will be analyzed below, let us take a quick look at which mathematical properties they satisfy, among those listed in Section 3.6.2. Section 3.3 summarizes the main properties and their satisfaction or non-satisfaction, as well as a classification according to the type of uncertainty measured, i.e., discordance or conflict and non-specificity, and the method of computation used, i.e., generalized entropy measures or belief intervals.

Eq.	Proponent	P. cons.	S. cons.	Add.	Subadd.	Monot.	Source	Type	Content
(3.23)	Höhle	Y	N	Y	N	N	[109, 111]	D	EB
(3.24)	Yager	Y	N	Y	N	N	[109, 111]	D	EB
(3.28)	Smets	N	N	Y	N	N	[109]	D	EB
(3.29)	Nguyen	Y	N	Y	N	N	[109, 111]	D	EB
(3.22)	Dubois and Prade	N	Y	Y	Y	Y	[109, 111]	NS	EB
(3.25)	Klir and Ramer	Y	Y	Y	N	Y	[109, 111]	TU	EB
(3.26)	Klir and Parviz	Y	Y	Y	N	Y	[109, 111]	TU	EB
(3.30)	Lamata and Moral	Y	Y	Y	N	Y	[57]	TU	EB
(3.31)	Pal et al.	Y	Y	Y	N	Y	[109]	TU	EB
(3.32)	Harmanec and Klir	Y	Y	Y	Y	Y	[109, 1]	TU	EB
(3.33)	George and Pal	Y	Y	Y	N	N	[29, 109]	TU	
(3.34)	Jousselme et al.	Y	Y	Y	N	Y	[109, 75]	TU	EB
(3.35)	Yang and Han	N	N	N	N	N	[26, 2]	TU	IB
(3.36)	Deng and Wang	N	N	?	?	Y	[30]	TU	IB
(3.37)	Li et al.	N	N	N	N	?	[86]	TU	IB
(3.38)	Wang and Song	Y	Y	?	?	N	[140, 26]	TU	EB, IB
(3.39)	Deng entropy	Y	N	N	N	N	[1]	TU	EB
(3.40)	Pan and Deng	Y	?	N	N	Y	[108, 109]	TU	EB
(3.41)	Jirousek and Shenoy	Y	N	Y	N	Y	[57, 111]	TU	EB
(3.42)	Zhou et al.	N	N	N	N	N	[100]	TU	EB
(3.43)	Cui et al.	N	N	N	N	N	[100]	TU	EB
(3.44)	Yan and Deng	Y	Y	N	N	?	[150]	TU	EB
(3.45)	Qin at al.	Y	N	N	?	?	[111]	TU	EB
(3.46)	Li and Pan	Y	?	N	N	?	[85]	TU	EB
(3.47)	Zhou and Deng	Y	N	Y	Y	?	[164]	TU	EB

Table 3.3: Measures of uncertainty and their properties: a review of the literature. Y: the property is satisfied, N: the property is not satisfied, ?: the property was not studied or evidence seems inconclusive. The column ‘source’ refers to the publications where the satisfaction of the properties is discussed. In the column ‘Content’, measures are labelled according the category they belong to, i.e., total uncertainty (TU), discord (D), non specificity (NS). The last column indicates whether the measure is based on the entropy formulation (EB) and/or is based on uncertainty intervals (IB).

Looking at Table 3.3 it is clear how the measure of Harmanec and Klir [50] is the only one that satisfies all the properties, also confirmed by Moral-García and Abellán [101]. Despite this, the measure is among the most complex to use and insensitive to the variation of m [140]. It is interesting to mention the proposal of Moral-García and Abellán [101] to pay special attention to certain desiderata that uncertainty measures must satisfy. These recommendations are based on the sensitivity and practical use of uncertainty measures.

Nowadays, the context of the properties of uncertainty measures is still much debated, the most recent proposals have fewer in-depth studies than the previous ones, and some authors, e.g. Yang and Han [155] and Deng and Wang [30] question some properties by emphasizing the importance of requirements specifically suited to evidence theory.

3.6.5 The problem and the methodology

The purpose of the publication is to numerically analyze the differences and similarities among all the measures mentioned. Numerical analysis requires a method for generating the universal set, focal elements, and appropriate BPAs. To this end, the procedure involves defining all necessary elements:

1. a priori definition of the n cardinality of X;
2. the set of focal elements, $\mathcal{F}$, consists of subsets randomly selected by the FOD. The selection stops when the union of the focal elements equals the set X;
3. the definition of BPAs follows the proposal of Burger and Destercke [12] by assigning m to each focal element using the Dirichlet function with parameter vector $\mathbf{1}_{|\mathcal{F}|}$.

Using the Dirichlet function makes it possible to obtain a uniform distribution of m on the focal elements and to satisfy, without further procedure, the rule that the sum of the values of the BPAs is equal to 1.
The simulation starts from the definition of n, then the procedure is repeated $s = 10,000$ times. Each repetition corresponds to measuring the

uncertainty as a function of the BPAs by all the formulas presented. The simulation is summarized in Algorithm 1. The analysis of uncertainty

Algorithm 1 The simulation procedure to compare the selected uncertainty measures.

$X \leftarrow \{x_1, \ldots, x_n\}$ ▷ Define the frame of discernment
$N \leftarrow s$ ▷ Initialize N
$Q \leftarrow \{q_1, \ldots, q_m\}$ ▷ Define a list of entropy measures
$\mathcal{D} \leftarrow \emptyset$ ▷ Create an empty dataset
$i \leftarrow 1$
while $i \leq N$ **do**
  $R \leftarrow \emptyset$
  while $\bigcup_{A \in R} A \subsetneq X$ **do**
    $A \leftarrow sample(2^X \setminus R)$ ▷ Sample a set $A \subseteq 2^X \setminus R$
    $R \leftarrow R \cup \{A\}$
  end while
  $\mathcal{F} \leftarrow \emptyset$
  $B \leftarrow Dirichlet(|R|)$ ▷ Sample $|R|$ values using Dirichlet so that $\sum_{b \in B} b = 1$
  for $b \in B$ **do**
    $\mathcal{F} \leftarrow \mathcal{F} \cup \{\langle A, b \rangle\}$
  end for
  $\mathcal{D} \leftarrow \mathcal{D} \cup \{\langle q_1(\mathcal{F}), \ldots, q_m(\mathcal{F}) \rangle\}$ ▷ Compute a new array of entropy measures
  $i \leftarrow i + 1$
end while

measures is based on three procedures that will be described below:

- Similarity analysis through comonotonicity;
- Hierarchical clustering;
- Centrality analysis.

Some of them are also used by other authors such as Jousselme and Maupin [58].

3.6.6 Results

Similarity analysis

Similarity analysis is conducted by looking for comonotonicity between the measures through the use of Spearman rank correlation coefficient ρ [157], which provides a value in $[-1, 1]$ based on the ordering of the uncertainty

of the pairs of uncertainty measures analyzed. More precisely, when the value is $\rho(\mathbf{a},\mathbf{b}) = 1$ denotes perfect positive comonotonicity, $\rho(\mathbf{a},\mathbf{b}) = -1$ denotes perfect negative comonotonicity, and $\rho(\mathbf{a},\mathbf{b}) = 0$ no comonotonicity. Intermediate values denote intermediate degrees of comonotonicity.
In practice, when comparing two measurements, two lists of uncertainty values $\mathbf{a},\mathbf{b} \in \mathbb{R}^s$ are obtained from the simulation. The values a_i, b_i are initially transformed into ranks $R(a_i), R(b_i)$, i.e., the values are ranked from smallest to largest with numbers from 1 to i as shown in Table 3.4. The

a_i	b_i	$R(a_i)$	$R(b_i)$
2	6	1	3
9	2	3	1
7	5	2	2

Table 3.4: Example of rank transformation of two variables using Spearman rank correlation coefficient ρ

values obtained are entered into the following formula:

$$\rho(\mathbf{a},\mathbf{b}) = \frac{\operatorname{cov}(R(\mathbf{a}), R(\mathbf{b}))}{\sigma_{R(\mathbf{a})}\sigma_{R(\mathbf{b})}}$$

where cov is the covariance of the two ranking vectors and σ is the standard deviation e $R(\mathbf{a}), R(\mathbf{b})$ are the vectors containing the values $R(a_i), R(b_i)$.

The choice of this coefficient, unlike Pearson's linear correlation, is useful for accounting for nonlinear relationships between the values of uncertainty measures. Comonotonicity allows us to assess the similarity between two uncertainty measures compared based on the order given to the BPAs from least to most uncertain. The results of this analysis obtained by fixing $n = 4$ are depicted in Figure 3.4. It is emphasized that the measures defined by Equations 3.22-3.26, Equations 3.28-3.29 and Equation 3.33 because they are all measures of conflict or nonspecificity only, and, in this context, comparison with measures of total uncertainty would not be useful.

The analysis represented in this way is very broad and widespread, so it may be useful to focus on specific confrontations for better evaluation.

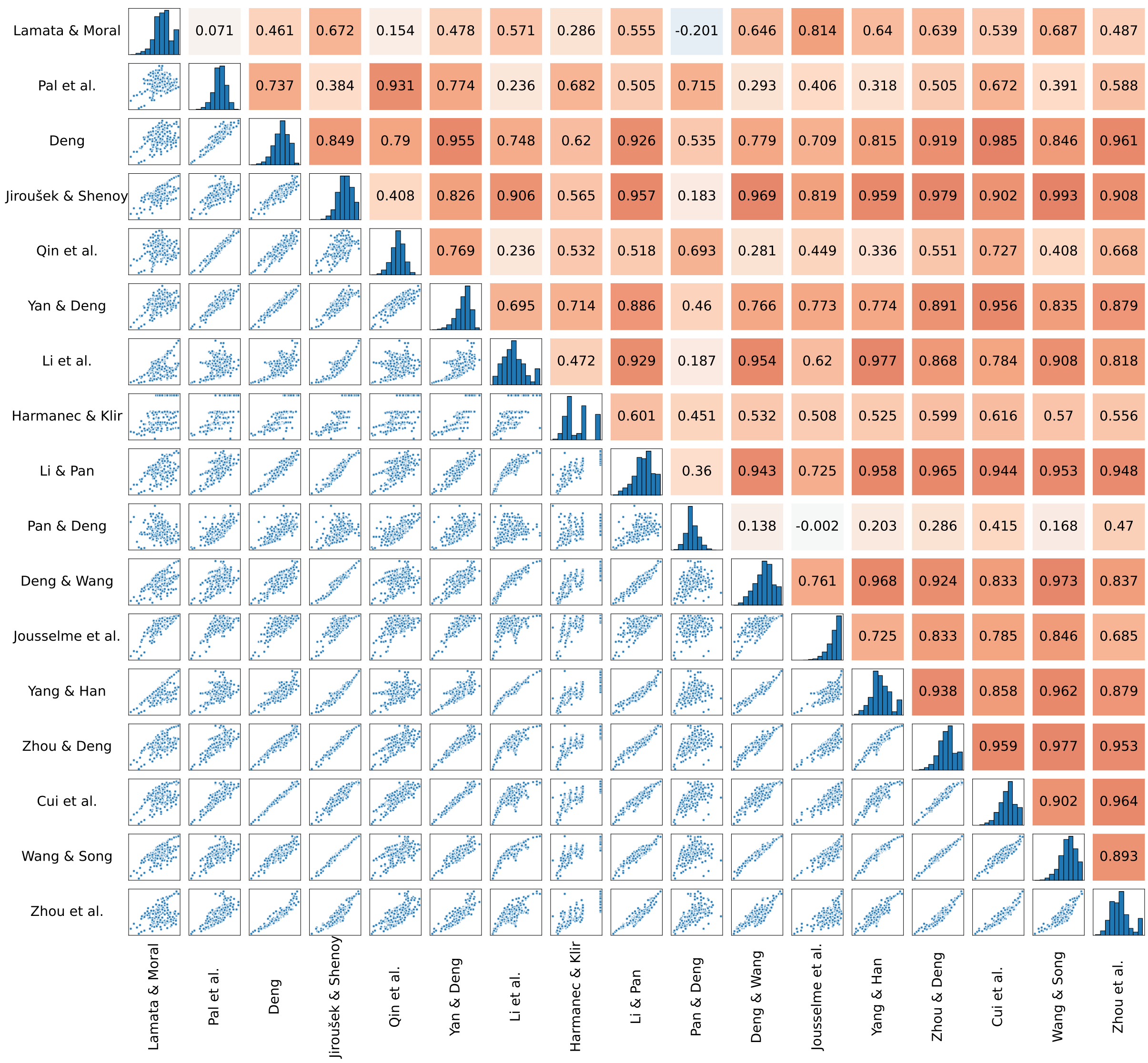
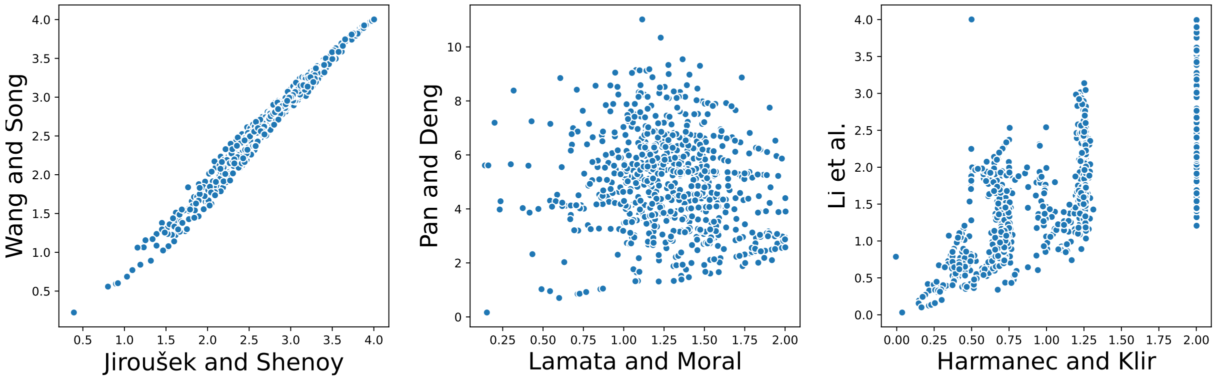


Figure 3.4: Pairwise scatter plots for pairs of uncertainty measures and their values of Spearman rank correlation coefficient for $n = 4$. To enhance readability the scatter plots refer to $s = 300$ whereas, for greater stability, the values in the upper triangular part were obtained with $s = 10,000$.

Figure 3.5a presents a comparison between the measures of Jiroušek and Shenoy [57] and Wang and Song [140]. The two measures have a value of $\rho \approx 0.99$, so according to the chosen coefficient these are two measures with almost identical order of values and high similarity. However, these are two measures based on very different concepts, the first on Shannon's measures, through the plausibility transformation, and Dubois and Prade,



(a) Jiroušek and Shenoy (H_j) vs. Wang and Song (SU). (b) Pan and Deng (H_{bel}) vs. Lamata and Moral (H_l). (c) Harmanec and Klir (AU) vs. Li et al. (TU).

Figure 3.5: Three representative scatter plots with $s = 1000$.

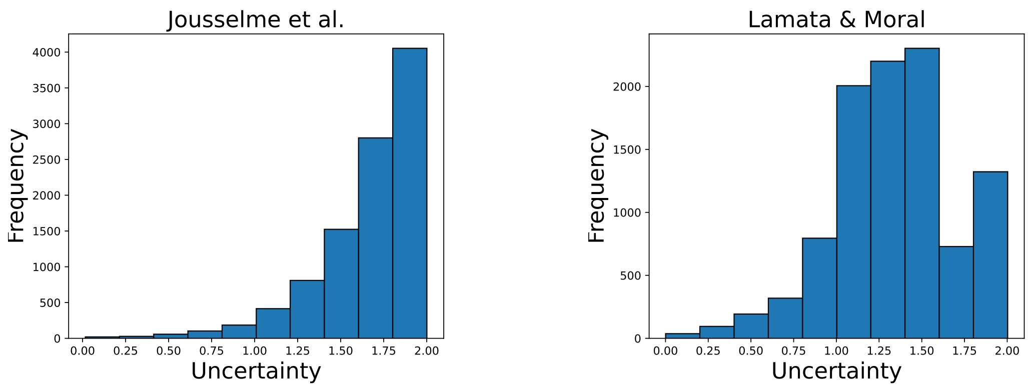
the second on the central values of belief intervals. Moreover, it is clear from Table 3.3 that the two measures are also very different in the properties they satisfy. The numerical result thus emphasizes that in practice these two measures can be used indifferently.

Figure 3.5b represents two measures with a negative ρ value, $\rho \approx -0.204$. The value also turns out to be very close to 0, so it can be asserted that the two measures appear to be unrelated; even visually the graph presents a cloud of points. A practical application of them could therefore lead to conflicting measures that are difficult to compare. The two measures also appear very different from the point of view of the basis on which they are developed, and it is complicated to find an adequate explanation for this behavior.

Figure 3.5c highlights the peculiar behavior of the measure of Harmanec and Klir [50]. As mentioned earlier this measure has been criticized in the literature for its insensitivity to changes in m, the presence in the scatter plot of a maximum value (right vertical alignment) that attracts the closest values seems to confirm this feature once again. Indeed, when comparing with the other measurements, we see that values of uncertainty that turn out to be very different, for the equation of Harmanec and Klir [50] all have the same maximum value. The magnitude of the measurement insensitivity had never been studied, but from the numerical comparison, it appears to be non-negligible.

Interpretation uncertainty values

As pointed out by Klir [74, p. 8], uncertainty measures should provide a useful and intuitive quantitative value for an understanding of the amount of uncertainty present. Thus from a numerical simulation, we expect to obtain a set of values that are comparable to each other.
Take, for example, the values obtained for $n = 4$ from the measurements of Jousselme et al. [59] and Lamata and Moral [83], together with their distributions in the diagonal in Figures 3.4 and analyzable in Figure 3.6. Considering a value of 1.25, out of a range of values contained in $[0, 2]$,



(a) The barchart of the values obtained using the uncertainty measure by Jousselme et al. [59].

(b) The barchart of the values obtained using the uncertainty measure by Lamata and Moral [83].

Figure 3.6: Two representative barcharts with $n = 4$ and $s = 10,000$.

we would be tempted to consider it among the high values, because it lies beyond the middle of the range. But looking at the distribution in Figure 3.6a we see that it is part of the decile containing the least uncertain mass assignments and consequently should be considered a low value. Comparison with the distribution of values depicted in Figures 3.6b, relating to the same range of values, underscores the importance of comparing the values and distributions obtained. Indeed, for the measure of Lamata and Moral [83], as well as for others in Figure 3.4, the value 1.25 represents a high uncertainty. One could solve this problem by returning the distributions to the normal distribution so that a simpler and more meaningful comparison of the measurements could be made.

According to the considerations in this section, when it is necessary to calculate the amount of uncertainty related to a BOE, several measures need to be evaluated. The choice of measures must be careful, since two similar measures may not provide meaningful results, whereas if two different measures both provide a result of higher or lower uncertainty, they may reinforce the concept.

Hierarchical clustering

The second method selected to analyze the measures is hierarchical clustering. It places different items, in our case measures of uncertainty, into different clusters, starting with the individual elements and joining them as we go along according to certain criteria. The result is a grouping in each cluster of items with common characteristics. Visually, a dendrogram is usually used to represent the results.
The data required for this type of analysis may be those contained in a distance matrix, which is square and symmetrical and reports the distances between pairs of elements. In the present case, the Spearman rank correlation coefficient can be used for the measurements. If we denote with $\rho_{\mathsf{H}_i,\mathsf{H}_j}$ the value of the Spearman index calculated for two measures of uncertainty H_i and H_j, then we can consider $d_{ij} = |1 - \rho_{\mathsf{H}_i,\mathsf{H}_j}| < 2$ a measure of their dissimilarity.
Linkages between measures are based on the following criteria:

- *Single linkage*: the value at which the two clusters X and Y are merged corresponds to the minimum distance between an element of X and one of Y, i.e., $\min\{d_{ij}|i \in X, j \in Y\}$
- *Complete linkage*: the value at which the two clusters X and Y are merged corresponds to the maximum distance between an element of X and one of Y, i.e., $\max\{d_{ij}|i \in X, j \in Y\}$
- *Average linkage*: the value at which the two clusters X and Y are

merged corresponds to the average distance between elements belonging to the two clusters, i.e.,

$$\frac{1}{|X||Y|}\sum_{i\in X}\sum_{j\in Y} d_{ij}$$

- *Ward linkage*: the value at which two clusters X and Y are merged corresponds to the cluster distance. The distance between two clusters X and Y is

$$d(X,Y) = \sqrt{\frac{|Y|+|Z|}{T}d(Y,Z)^2 + \frac{|Y|+|W|}{T}d(Y,W)^2 - \frac{|Y|}{T}d(Z,W)^2},$$

  where X is the newly joined cluster consisting of Z and W, Y is the cluster to be merged with, $T = |Y| + |Z| + |W|$. When two clusters contain each a single element, $d(X,Y) = d_{XY}$. The Ward variance minimization algorithm [143] is used to perform clustering[1].

The results of applying the criteria to all the measures considered in Figure 3.4 are shown in Figure 3.7.

The outcomes of the dendrograms turn out to be peculiar and different from what might be expected; in fact, the measures seem to cluster by year of proposal, but not by type, i.e. by base from which they were developed. In particular, it is noticeable that the most recent measures are part of a single cluster this result is probably a consequence of evolving the measurements and gradually basing new measurements on the latest findings.

Centrality analysis

In the context of Social Network Analysis (SNA), *measures of centrality* have been developed with the aim of quantifying connections between actors [84]. There are different approaches to define what is considered centrality [84, 105]. Particularly in the context in which we want to exploit this method, the most suitable approach is *eigenvector centrality*. This ap-

[1] https://docs.scipy.org/doc/scipy/reference/generated/scipy.cluster.hierarchy.linkage.html

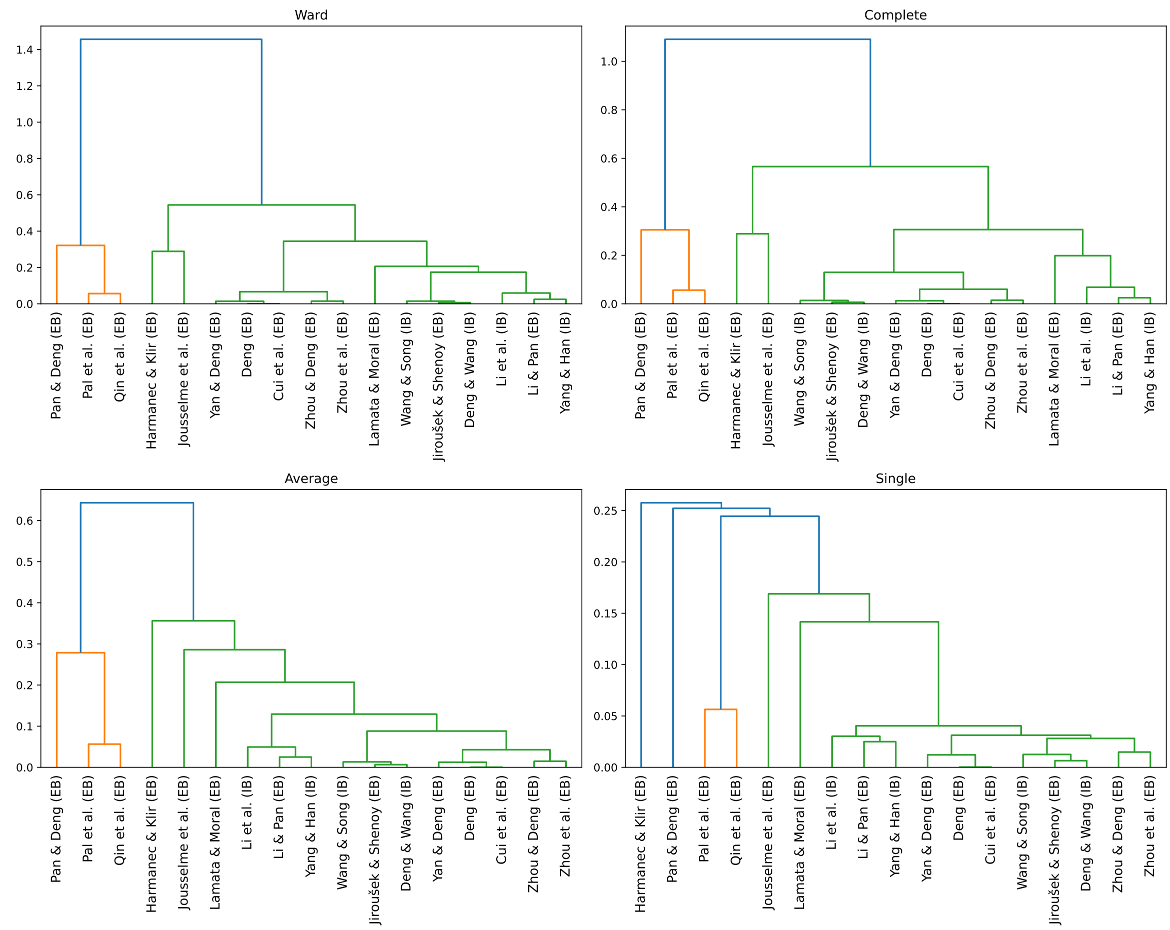


Figure 3.7: A comparison of dendrograms obtained using different clustering heuristics. The values on y-axis correspond to the distance between clusters according to the chosen heuristic. Aside from the name of a measure, there is the EB/IB classification reported from Table 3.3; all measures express total uncertainty.

proach considers a central element when the elements directly connected to it have many connections to other elements. Declined to measures of uncertainty, mean being able to consider all measures that have elements in common with measures that therefore become central. As defined by Bonacich and Lloyd [11], the centrality values of each element taken into account are the items of the vector w solving the eigensystem $\mathbf{Aw} = \lambda_{max}\mathbf{w}$ where λ_{max} is the Perron-Frobenius eigenvalue of A. Note that w is unique up to multiplication by a positive scalar. While A is the adjacency matrix. In the uncertainty measure case, the Spearman rank correlation coefficient, always in the range $[-1, 1]$, can be considered a degree of distance between the different measures, and the matrix in Figure 3.4 as a weighted adjacency matrix, A. One can interpret the values of $a_{i,j}$, elements of A, by considering that the higher they are, the more similar the two measures to which they refer. Figure 3.8 reports the normalized centrality of the

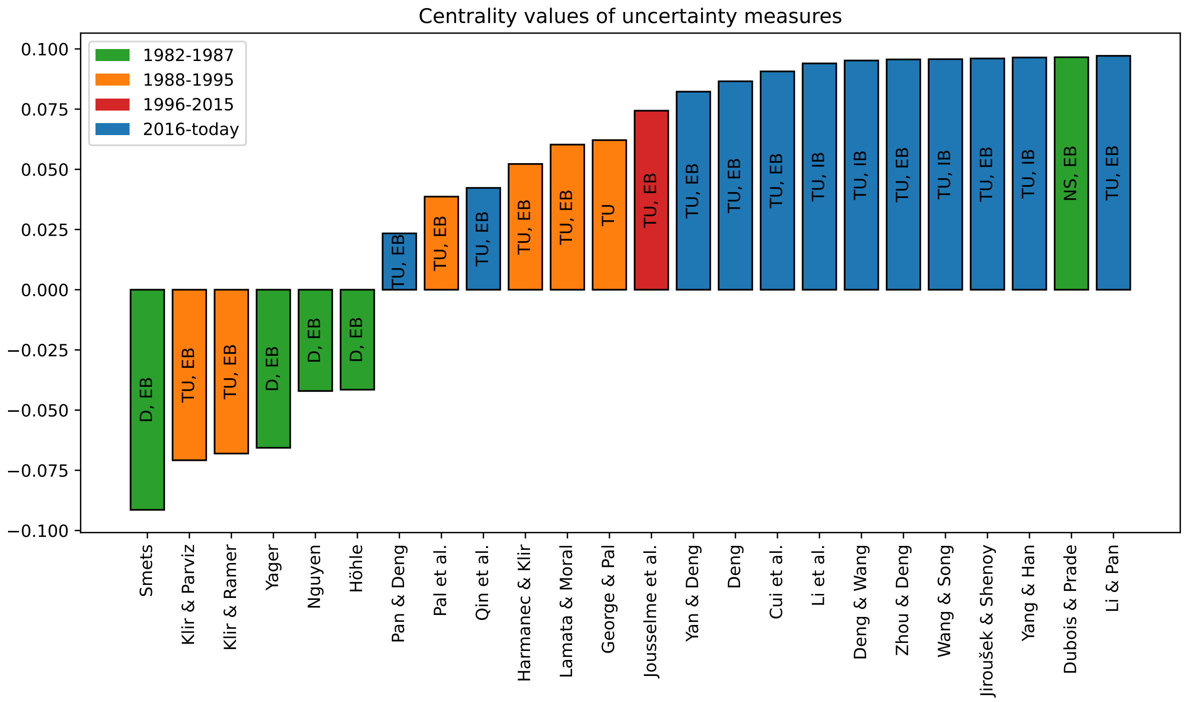


Figure 3.8: A bar chart representing the values of normalized eigenvector centrality of each uncertainty measure.

eigenvectors for all the uncertainty measures considered in this study
The results that can be extrapolated from this analysis are: certainly the most obvious, related to the measures with high values of centrality, symptomatic of being the one most "supported," and thus being able to represent the other measures. But as we were able to reflect earlier, even measures that report a low centrality value and thus are more distant, can provide a measure that allows for greater strength in the results obtained. Another observation of the results relates to measures with a negative centrality value. All the measures related to them are measures of conflict uncertainty alone. This result suggests that when considering a measure of total uncertainty, the addition of the nonspecificity measure has an impact in the opposite direction from the conflict measure: the lower the

nonspecificity the higher the conflict, and vice versa. This behavior is illustrated in Figure 3.9 where the measure of Dubois and Prade [35] is compared with four measures of conflict and the Spearman coefficients are negative. In addition, it can be seen that, with few exceptions, the most

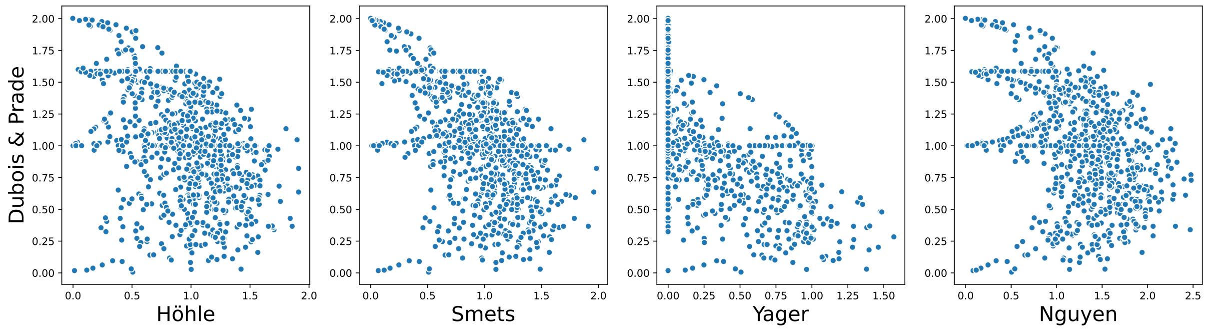


Figure 3.9: The measure of non-specificity by Dubois and Prade [35] compared with four measures of conflict. The respective Spearman coefficients are -0.488, -0.933, -0.705, and -0.525.

recent data have high centrality values. This may be symptomatic of a recent "convergence" of research toward a shared definition of a measure of uncertainty. Among the measures with the highest centrality values, the measure of nonspecificity of Dubois and Prade [35] stands out. One reason may be that the formulation of this uncertainty measure is incorporated in many of the more recent measures.

Sensitivity analysis

All analyses performed were carried out with $n = 4$, as this is the most usual value in applications in the literature. But for a broader view, a sensitivity analysis of 30 values of Spearman's coefficient of randomly chosen pairs of measures was also performed.

The analysis was performed for cases with $n = 3, \ldots, 8$ and the new similarities were eventually collected and compared. The results in Figures 3.10 are qualitative, but allow us to identify trends in the coefficients. There is no particular change in comparisons between measures, with some exceptions such as the lower levels tending to have a decrease.

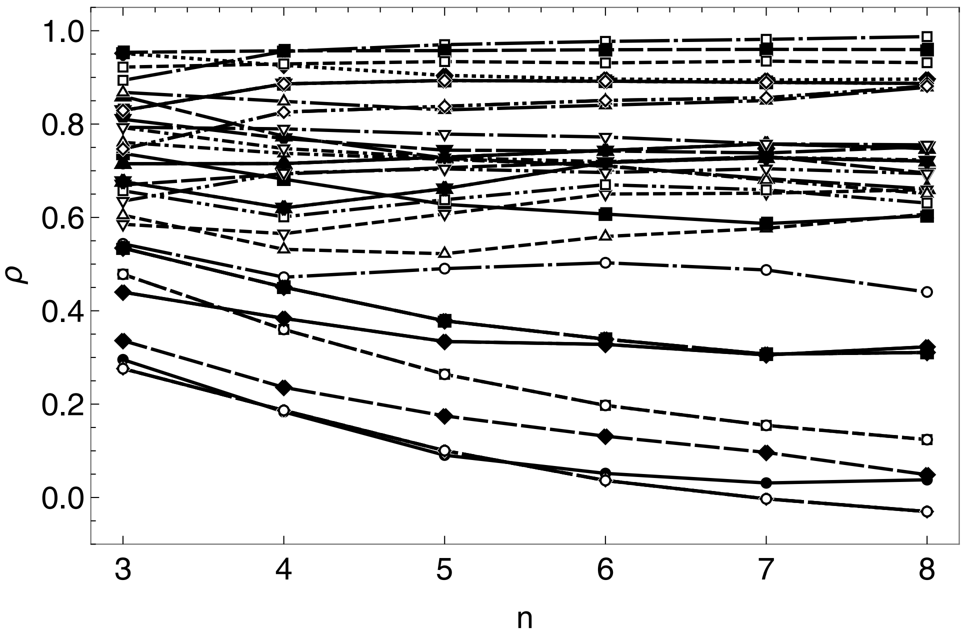


Figure 3.10: Sensitivity analysis for randomly selected similarity values between uncertainty measures. Each line represents the similarity of a different pair of uncertainty measures and how it changes with respect to the cardinality of the FOD.

Chapter 4

Discussion and conclusions

4.1 Research results and prospective future

Chapters 2 and 3 addressed two problems, one more practical, related directly to a company in the energy distribution network sector, and another more theoretical. In the first case, the problem is in the context of resource allocation and scheduling of projects with a known time duration. The request for decision-making support was made directly by the company, which consequently provided us with all the data necessary to apply the solution to a realistic problem. The company had several constraints, including an annual budget and manpower constraints. The projects that had to be scheduled were characterized by time duration, cost, and execution priority. The company's goal was to obtain a decision-support program that would be able to optimize prioritization, that is, to schedule the start of activities by prioritizing those with higher priority. In the course of defining the optimization program, it became clear how the creation of prioritization support could help make the process even more methodical and rapid. The solution to the problem therefore took place in two stages: the first in which thanks to multi-attribute value theory, one of the foremost methodologies of PDA, we defined a value function that thanks to some simple input data can associate a priority value with each project, and the second in which the optimization program was defined and used. The result obtained is easily usable by the company due to the speed of calculation of the program, which allows a reiteration of the same whenever new, or more accurate, information is available. The optimiza-

tion of activity scheduling is satisfactory both with the period required, 5 years, and with the amount of activity provided 368, and with variations in this data.
The model presented seems to be easy to use and intuitive even for those without specific skills. This aspect could prompt further research on this feature to enable greater dialogue between scientific research and the real world of business, with the goal of benefit for both parties. In addition, the use of multi-period PDA has many potentials; it can be used in different areas, including, for example, the so-called Sustainable Development Goals (SDGs) [23]: the relevant SDGs can be used as targets while the indicators can be considered the attributes.

The second problem that was addressed concerned the measures of uncertainty or entropy within evidence theory, which currently appear to be in such large numbers that it is difficult to make an informed choice taking into account all their characteristics and properties. The second publication aims to numerically compare different measures to provide a useful tool for application. To achieve this goal, a descriptive list of selected measures was made. The comparison between measures is made on the results of Monte Carlo simulations using rank correlation, hierarchical clustering, and eigenvector centrality. The results obtained provided insight into when two measures may give conflicting or similar results and the need to identify an appropriate threshold for each measure to understand the degree of uncertainty expressed. Several comparisons can be found in the literature, but no one had yet approached the problem from the numerical point of view, which proved very useful in better understanding entropy measurements.

In the future, it would be interesting to further investigate the comparison of new measures with a numerical approach both to adequately understand similarities and dissimilarities and to understand whether the new measures perform better in calculating entropy. This will also make it easier to choose in the application field. Uncertainty measures are used in various practical fields among which the most frequent are Multi-sensor information fusion and Fault diagnosis, but also in Decision-making [29].

In addition, the application of evidence theory in the field of bridge inspection and maintenance in civil engineering has been presented in depth in the thesis.

A possible inspiration that would be interesting to explore from the very two main themes of this thesis is to address a multiperiod problem, such as the one described in Chapter 2, by considering uncertainty, for example of setup time attribute, and addressing it through the Demspter-Shafer Theory uncertainty measure. An example of research in this direction is the study of Cinfrignini et al. [19].

4.2 Research questions discussion

To summarize the research results more specifically, the research questions in Section 1.2 are recalled in this section.

- **Question 1.** The real-world adaptation of the problem posed in Publication I required that we address issues such as accounting for the duration of different activities in a multi-period problem. It was then demonstrated that the mathematical models could be adapted to real-world needs by creating a model suited to a company's needs until a positive result was achieved in terms of both input processing time and output produced.

- **Question 2.** Also in Publication I, it was shown how collaboration with the company technicians involved was fundamental to the successful development of the model. Using models that were too sophisticated or not suited to the understanding of the company technicians would have limited the results obtained. Even more important was making it possible for technicians to interpret the results obtained. Special attention was paid to the choice of which graphs to create and what type of outputs to allow a generalized, objective view of the results obtained. The benefit was twofold: not only did the technicians find the results valuable and the model created useful because of the ease of finding the inputs and the comprehensibility of the results, but also allowed the author to be assured of having an objective positive view of the results and the model.

- **Question 3.** Publication II addresses with numerical simulations the problem of how to choose among different measures of uncertainty existing in the literature. The result obtained does not make it possible to select one measure as better than the others, but it does make it possible to understand the similarity or dissimilarity of the results of the measures and thus provides the tools to more consciously evaluate the choice of how to measure uncertainty.

4.3 Research process and limitations

The first publication continued work begun in a master's thesis project in the field of industrial engineering, with an expansion of topics that led to the use of portfolio decision analysis. The knowledge required prompted the author to obtain the necessary background to address the topics (since previous studies were in the field of civil engineering). The study started with theory and a review of current literature to the concrete application of which process was reported in the publication. This provided knowledge of different techniques that can be applied to structures and civil engineering. The research described in the publication was applied to a very specific real case with business constraints and requirements and was validated by experts. However, it was not possible to see its practical effects and actual use in subsequent years. One limitation of the research is that it has not been possible to see how it is actively used by the company and whether the results of budget and labor management have been taken into account. One limitation to the possibility of understanding how the model works is the time factor, since it is a model with 5 years, it has not yet been possible to see a general effect. Despite this, an occasional check in the last few years would have allowed the author to understand the short-term effects of the model and the actual possibility of the company using it several times as information increased or characteristics of the activities changed. Also being able to view the outcome in terms of the final budget could be essential to focus more attention on obtaining that output and the program's ability to take economic aspects into account. Another limitation was testing the generality of the model. In theory, the model was developed to be applied to different realities, but, in particular, the MAVT part requires interaction with the company's experts. To understand in the real world whether the application to different companies in the same or a similar application sector is indeed easy and fast, it would have been interesting to have different data and to be able to propose it to different realities.

The second publication ranged in a more theoretical area. However, one of the goals was to offer support for practical applications, as well as to highlight the need for a study comparing measures of uncertainty in evidence theory, the quantity of which is rapidly increasing. In fact, since the publication of the article, in less than two months, four more research papers [22, 38, 17, 60] defining as many new measures have been published. In preparation for Publication II, an extensive literature search was conducted to understand the state of the art in the field of entropy measures in evidence theory and to select appropriate measures for study. The search is intended to offer support in the selection of appropriate measures by providing a more practical and useful tool for understanding the behavior of each measure relative to others. However, the research was limited to the numerical study of measures; it would have been interesting to see its application in a more practical field in which uncertainty measures play an essential role. The study of the theory of evidence has enabled the author to broaden his knowledge of various mathematical theories in addition to the already well-known theory of probability. Again, the deepening of knowledge has enabled the author to understand and study applications to civil engineering that have great potential but are currently underutilized. Without this intensive deepening of the literature, it would have been impossible to understand theories such as probabilistic research that can instead support civil infrastructure condition monitoring.

4.4 Conclusions

The topics covered in the thesis are very different from each other but have allowed the author to broaden her knowledge in different fields. Furthermore, the skills learned through research have made it possible to understand how it can also be applied to a field such as civil engineering. Precisely from this point of view, it could be interesting to investigate some aspects, in particular, in recent years, in many European countries and in particular in Italy there has been a substantial increase in attention to the maintenance of engineering structures that require planning of activities and automated ways to take current conditions into account. Given the large number of roads, bridges, and buildings, it is clearly very difficult for them to be adequately managed by manual methods. Therefore, the acquired knowledge is helpful to the author for future applications in work and practice.

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