About limit cycle's uniqueness for a class of generalized Liénard systems

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Abstract

A uniqueness theorem for limit cycles of a class of generalized Liénard systems is proved. The main result is applicable to generalized Liénard O.D.E.'s with dissipative term depending both on the position and the velocity. [0]

1 Introduction

A classical problem in the study of plane differential systems,

$$x' = P(x, y), \qquad y' = Q(x, y),$$

consist in finding limit cycles, i. e. isolated cycles. A relevant subproblem is that of counting them, or estimating their number, as requested by XVI Hilbert's problem. A very special case is that of systems having a single limit cycle. In this case, if the cycle is stable and the solutions are ultimately bounded, then the cycle dominates the global dynamics of the system. Results about existence and uniqueness of limit cycles have been proved since the very beginning of the study of second order differential equations. The most popular class of systems, those equivalent to Liénard equation,

$$x'' + f(x)x' + g(x) = 0, \tag{1}$$

was considered by several researchers (see [1], [2], [3], [4] and references therein). Quite often the results were concerned with more general classes of systems, containing (1) as a special case, as

$$x' = \beta(x)\big[\varphi(y) - F(x)\big], \qquad y' = -\alpha(y)g(x). \tag{2}$$

Such a class of systems also contain Lotka-Volterra systems and systems equivalent to Rayleigh equation

$$x'' + f(x') + g(x) = 0, \tag{3}$$

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as special cases. Usually, $\alpha(y)$ and $\beta(x)$ are assumed to be positive, without loss of generality, since a limit cycle cannot cross the lines $\alpha(y) = 0$ and $\beta(x) = 0$. More general systems were considered in [3] (lemma 3.2),

$$x' = \beta(x)\left[\varphi(y) - F(x)\nu(y)\right], \qquad y' = -\alpha(y)g(x). \tag{4}$$

where $\nu(y) > 0$. Such systems may be transformed into a system of the form (2) as in [3], provided $\nu(y)$ does not vanish on its domain.

In this paper we prove a uniqueness theorem which applies to systems of the form

$$x' = \beta(x)\left[\varphi(y) - F(x,y)\right], \qquad y' = -\alpha(y)g(x). \tag{5}$$

In particular, we can prove the uniqueness of the limit cycle for a class of systems of the type (4), with $\nu(y)$ vanishing at some point.

Our results are in the line of [1], [3], [4], concerned with Liénard systems,

$$x' = y - F(x), \qquad y' = -g(x).$$

Such papers are based on the observation that the integral $\int_0^T F(x)g(x)dt$ vanishes on every cycle. Comparing the value of such an integral on different cycles allows to prove, under suitable hypotheses, that at most one limit cycle exists. The argument presented in [4] and [2] can be adapted to the case of F depending on both variables, also replacing y with an increasing function $\varphi(y)$.

2 Results

If h is a function defined on a (possibly generalized) interval, we say that $h \in S$ if $th(t) > 0$ for $t \neq 0$ and $h(0) = 0$. If k is a function defined on a domain D, we say that $k \in L(D)$ if k is lipschitzian on D.

Throughout all of this section we assume that there exist $\bar{a} < 0 < \bar{b}$ such that

o) $\alpha, \varphi \in L(\mathbb{R})$, $\beta, g \in L\left((\bar{a}, \bar{b})\right)$, $F \in L\left((\bar{a}, \bar{b}) \times \mathbb{R}\right)$,

oo) $g, \varphi \in S$, $\alpha > 0$, $\beta > 0$ on their domains.

The assumption oo) guarantees that the trajectories wind clockwise around the origin. The assumption on the sign of α and β is not a significant restriction in dealing with limit cycles. In fact, if $\beta(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then the line $x = x_0$ is an invariant line for (5) and no cycle may cross it. Similarly for $\alpha(y)$. Hence we can reparametrize the orbits of (5) multiplying the vector field by $\frac{1}{\alpha(y)\beta(x)}$. The new system is

$$x' = \frac{\varphi(y) - F(x,y)}{\alpha(y)}, \qquad y' = -\frac{g(x)}{\beta(x)},$$

whose orbits coincide with those ones of (5). In particular, uniqueness of limit cycles for such a system is equivalent to uniqueness of limit cycles for (5). The new system can be written as

$$x' = \tilde{\varphi}(y) - \tilde{F}(x,y), \qquad y' = -\tilde{g}(x),$$

with $\tilde{\varphi}(y) = \frac{\varphi(y)}{\alpha(y)}$, $\tilde{F}(x,y) = \frac{F(x,y)}{\alpha(y)}$, $\tilde{g}(x) = \frac{g(x)}{\beta(x)}$. Hence we can restrict, without loss of generality, to the following class of systems

$$x' = \varphi(y) - F(x,y), \qquad y' = -g(x). \tag{6}$$

Let us set

$$G(x) = \int_0^x g(s)\ ds, \qquad \Phi(y) = \int_0^y \varphi(s)\ ds.$$

We denote by y^+ the positive half-line $\{y \in \mathbb{R} : y > 0\}$, by y^- the negative half-line $\{y \in \mathbb{R} : y < 0\}$.

Theorem 1 *Assume there exist $a, b \in (\bar{a}, \bar{b})$, $a < 0 < b$, such that:*

i) for all $x \in (a,0)$ the function $y \mapsto \frac{F(x,y)}{\varphi(y)}$ is strictly decreasing both on y^+ and on y^-;

for all $x \in (0,b)$ the function $y \mapsto \frac{F(x,y)}{\varphi(y)}$ is strictly increasing both on y^+ and on y^-;

ii) for all $x \in (a,b)$, for all $y \in \mathbb{R}$, one has $g(x)F(x,y) \le 0$;

iii) for all $x \notin (a,b)$, for all $y \in \mathbb{R}$, one has $F(x,y) \ge 0$; for all $y \in \mathbb{R}$ the function $x \mapsto F(x,y)$ is increasing out of (a,b).

Then the system (6) has at most one limit cycle meeting both the lines $x = a$, $x = b$.

Proof. The above hypotheses, together with the assumption oo), imply that the origin is the unique critical point of the system (6). In fact, by *ii)* and *iii)*, one has $F(a,y) = F(0,y) = F(b,y) = 0$, for all $y \in \mathbb{R}$. Moreover, $y' = -g(x)$ vanishes only at $x = 0$, and $\varphi(y) - F(0,y) = \varphi(y)$, which vanishes at $y = 0$, is strictly increasing.

As a consequence, cycles wind clockwise around the origin. We claim that the intersection of a cycle γ with the semi-strip $S^* = \{(x,y) : x \in (a,b), y > 0\}$ is contained in the region $x' \ge 0$.

In order to prove this claim, let us first observe that every half-line $x = c, y > 0$, with $a < c < b$, $c \ne 0$, contains at most one point $y = \mu^*(c)$ where $x' = \varphi(y) - F(c,y)$ vanishes. In fact, if $\varphi(y) - F(c,y) = 0$, then $\varphi(y) = F(c,y)$, so that $\frac{F(c,y)}{\varphi(y)} = 1$, which may occur only at most at one point, by the strict monotonicity of $\frac{F(x,y)}{\varphi(y)}$. The strict monotonicity of $\frac{F(x,y)}{\varphi(y)}$ also implies that on the half-line $x = c, y > 0$, for all $y > \mu^*(c)$ one has $x' > 0$, for all $0 < y < \mu^*(c)$ one has $x' < 0$.

Now, let us assume by absurd that there exists a point $(\bar{x}, \bar{y}) \in \gamma \cap S^*$ where $x' < 0$. Since γ rotates clockwise, γ meets the line $x = \bar{x}$ at least at three points, say $(\bar{x}, y_1)$, $(\bar{x}, \bar{y})$, $(\bar{x}, y_2)$, with $y_1 < \bar{y} < y_2$ and $x' \ge 0$ at $(\bar{x}, y_1)$, $(\bar{x}, y_2)$. Then the half-line $x = \bar{x}$ contains at least two points, in the intervals $(y_1, \bar{y})$ and $(\bar{y}, y_2)$, where x' vanishes, contradicting what proved above.

A similar argument shows that the intersection of a cycle γ with the semi-strip $S_* = \{(x,y) : x \in (a,b), y < 0\}$ is contained in the region $x' \le 0$. As a consequence, every cycle γ meets a half-line $x = c, y > 0$, as well as a half-line $x = c, y < 0$, at most at one point.

Let us set $V(x, y) = G(x) + \Phi(y)$. By the sign assumptions on g and φ, the function V is positive definite at the origin.

Let $\gamma(t) = (x_\gamma(t), y_\gamma(t))$ be a T-periodic cycle of (6). Denoting by $\dot{V}$ the derivative of V along the solutions of (6), we have

$$0 = V(\gamma(T)) - V(\gamma(0)) = \int_0^T \dot{V}(\gamma(t))\ dt = -\int_0^T F(x_\gamma(t), y_\gamma(t)) g(x_\gamma(t))\ dt.$$

We denote concisely by $-\int_0^T F(x, y)g(x)dt$ the last integral in the above formula. As in [2], [3], [4], we prove the cycle's uniqueness assuming, by absurd, the existence of a second cycle, and showing that $-\int_0^T F(x, y)g(x)dt$ cannot vanish on both cycles. Let γ_1, γ_2 be distinct cycles of (6) both meeting the lines $x = a$, $x = b$. Let T_j, $j = 1, 2$, be the period of, respectively, γ_j, $j = 1, 2$. As already mentioned, the system (6) has a unique critical point, hence γ_1 and γ_2 are concentric. Let γ_1 be contained in the interior of γ_2. Let I_j, $j = 1, 2$ be the value of $\int_0^{T_j} F(x, y)g(x)dt$ computed along γ_j. We claim that $I_1 < I_2$.

It is sufficient to show that the proof of theorem 1 in [4] can be performed replacing y with $\varphi(y)$ and $F(x)$ with $F(x, y)$.

The decomposition of the integral $\int_0^T F(x, y)g(x)dt$ can be done just as in [4], [2]. First one integrates with respect to x on the interval (a, b), separately along $\gamma_1 \cap S^*$ and along $\gamma_1 \cap S_*$.

The comparison between the integrals I_1 and I_2 along those arcs gives the desired inequality. In fact, $F(x, y)$ does not vanish on the region $a < x < 0$, $y > 0$, since otherwise, by the strict monotonicity of $y \mapsto \frac{F(x,y)}{\varphi(y)}$, $F(x, y)$ should have negative values on $a < x < 0$, $y > 0$, contradicting *ii*). Hence we can write

$$\frac{g(x)F(x, y)}{\varphi(y) - F(x, y)} = \frac{g(x)}{1 - \frac{\varphi(y)}{F(x,y)}},$$

which shows that $\frac{g(x)F(x,y)}{\varphi(y)-F(x,y)}$ is a strictly increasing function of y for all $x \in (a, 0)$. One can work in the same way for $x \in (0, b)$. This step gives the strict inequality between integrals.

The other portions of arc of γ_1 and γ_2 can also be treated as in [4], since, when integrating with respect to y, one uses the sign of $F(x, y)$ and its monotonicity with respect to x outside the strip $a < x < b$. Strict inequalities are not needed at this step, because already obtained in the previous step. ♣

The limit cycles of (5) do not necessarily meet both lines $x = a$, $x = b$. In [1], [2], [3], additional conditions were considered in order to ensure such a property. The same kind of condition, which can actually be found in [2], is used in next corollary.

Corollary 1 *Under the hypotheses of theorem 1, assume additionally $G(a) = G(b)$. Then (5) has at most one limit cycle.*

Proof. The level set $V(x, y) = G(a) = G(b)$ contains the points $(a, 0)$ and $(b, 0)$. The monotonicity of φ and g implies that the sublevel set $V_{G(a)} = \{(x, y) :$

$V(x,y) \leq G(a)\}$ is entirely contained in the strip $a < x < b$. Using $V(x,y)$ as a Liapunov function, one has

$$\dot{V}(x,y) = -F(x,y)g(x) > 0$$

on the strip $a < x < b$, hence also on $V_{G(a)}$. This shows that no cycles can be contained in $V_{G(a)}$, and that every cycle has to meet both the lines $x = a$ and $x = b$. ♣

The condition $G(a) = G(b)$ is in particular satisfied when $g(-x) = -g(x)$, $F(-x,y) = -F(x,y)$, taking $a = -b$.

The theorem (1) only gives an upper bound on the number of cycles of (6), without guaranteeing the existence of such a cycle. In order to get an existence and uniqueness result one can impose a set of hypotheses which ensure the ultimate boundedness of the orbits of (6), together with the negative asymptotic stability of the critical point. Results in this direction may be found in [5].

As an example, let us consider a system of the type (4), with $\nu(y) = y^2$ vanishing at 0, so that the result in [3] cannot be applied,

$$x' = y^3 - (x^3 - x)y^2, \qquad y' = -x. \tag{7}$$

Applying standard Liénard-plane arguments, and in particular the fact that it is possible to find a suitable point $(\bar{x}, \bar{y})$ in the third quadrant, such that the positive semi-trajectory passing at $(\bar{x}, \bar{y})$ meets the curve $y = x^3 - x$ at $x > 0$, while the negative semi-trajectory does not intersect such a curve, as in [6], one can show the existence of a positively compact trajectory. This, together with the repulsivity of the origin, ensures the existence of a limit cycle. Since $a = -1$, $b = 1$, $G(x) = \frac{x^2}{2}$, one has $G(a) = G(b)$, so that the hypotheses of corollary 1 hold, and the system (7) has exactly one limit cycle.

Another class of examples can be obtained as follows,

$$x' = y - (x^3 - x)R(x,y), \qquad y' = -x, \tag{8}$$

where $R(x,y)$ is any function such that $R(x,y) \geq 1$ and, for all x, the function $y \mapsto \frac{y}{R(x,y)}$ is increasing. Since $R(x,y) > 1$, out of the strip $-1 < x < 1$, the orbits of (8) enter the orbits of

$$x' = y - (x^3 - x), \qquad y' = -x,$$

which is the Van der Pol system. Applying phase-plane techniques, comparing the orbits of (8) to those of the Van der Pol oscillator, one shows the existence of a limit cycle of (8). The conditions on $R(x,y)$ guarantee that the hypotheses of theorem 1 hold. Since $G(-1) = G(1)$, also in this case the system (8) has exactly one limit cycle.

We get a simple example by choosing $R(x,y) = 1 + |xy|$,

$$x' = y - (x^3 - x)(1 + |xy|), \qquad y' = -x.$$

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