EXPLICIT EQUATIONS AND BOUNDS FOR THE NAKAI–NISHIMURA–DUBOIS–EFROYMSON DIMENSION THEOREM

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Abstract

The Nakai–Nishimura–Dubois–Efroymson dimension theorem asserts the following: "Let R be an algebraically closed field or a real closed field, let X be an irreducible algebraic subset of R^n and let Y be an algebraic subset of X of codimension $s \geq 2$ (not necessarily irreducible). Then, there is an irreducible algebraic subset W of X of codimension 1 containing Y". In this paper, making use of an elementary construction, we improve this result giving explicit polynomial equations for W. Moreover, denoting by $\overline{R}$ the algebraic closure of R and embedding canonically W into the projective space $\mathbb{P}^n(\overline{R})$, we obtain explicit upper bounds for the degree and the geometric genus of the Zariski closure of W in $\mathbb{P}^n(\overline{R})$. In future papers, we will use these bounds in the study of morphism space between algebraic varieties over real closed fields.

Key words: Dimension theorems, Irreducible algebraic subvarieties, Upper bounds for the degree of algebraic varieties, Upper bounds for the geometric genus of algebraic varieties.

1 The theorems

Let R be an algebraically closed field or a real closed field. Equip each affine space R^n with the Zariski topology. By algebraic subset of R^n, we mean a closed subspace of R^n. Let X be such a subset of R^n. A point p of X is nonsingular of dimension d if the ring of germs of regular functions on X at p is a regular local ring of dimension d. The dimension $\dim(X)$ of X is the largest dimension of nonsingular points of X and $\text{Nonsing}(X)$ indicates the set of all nonsingular points of X of dimension $\dim(X)$. If $X = \text{Nonsing}(X)$, then X is called nonsingular. We denote by $\mathcal{I}_{R^n}(X)$ the ideal of $R[x_1, \ldots, x_n]$ of polynomials vanishing on X. By an algebraic subset of X, we mean a closed subspace of X. As usual, the codimension of an algebraic subset Y of X is the difference between $\dim(X)$ and $\dim(Y)$. Let

Z be an open subset of X and let S be a non–void subset of Z. We indicate by $\mathcal{R}(Z)$ the ring of regular functions on Z and by $\mathcal{I}_Z^{\mathcal{R}}(S)$ the ideal of $\mathcal{R}(Z)$ of regular functions vanishing on S. The previous notions can be defined similarly in the projective case.

The results presented below improve in several directions the Nakai–Nishimura–Dubois–Efroymson dimension theorem [9], [3] (see also [8]).

Theorem 1.1 *Let X be an irreducible algebraic subset of R^n, let Y be an algebraic subset of X of codimension $s \geq 2$ and let $f_1, \ldots, f_\nu$ be generators of $\mathcal{I}_{R^n}(Y)$ in $R[x_1, \ldots, x_n]$. Then, there exist polynomials $g_1, \ldots, g_{s-1}$ in $R[x_1, \ldots, x_n]$ with the following properties:*

a) *For each $k \in \{1, \ldots, s-1\}$ and for each $j \in \{1, \ldots, \nu\}$, there is a linear polynomial p_{kj} in $R[x_1, \ldots, x_n]$ such that $g_k = \sum_{j=1}^{\nu} p_{kj} f_j$.*

b) *Denote by X^* the set* $\mathrm{Nonsing}(X) \setminus Y$ *and, for each $k \in \{1, \ldots, s-1\}$, define $Y_k := \{x \in X \,|\, g_1(x) = \cdots = g_k(x) = 0\}$ and $Y_k^* := Y_k \cap X^*$. Then, for each $k \in \{1, \ldots, s-1\}$, Y_k is an irreducible algebraic subset of X of codimension k containing Y such that $Y_k^* \neq \emptyset$, $Y_k^* \subset \mathrm{Nonsing}(Y_k)$ and $\mathcal{I}_{X^*}^{\mathcal{R}}(Y_k^*)$ is generated in $\mathcal{R}(X^*)$ by the restrictions of $g_1, \ldots, g_k$ to X^*.*

When R is a real closed field, we can say some more about the polynomials p_{kj}. We recall that the topology of R^n induced by the ordering structure on R is called euclidean topology.

Theorem 1.1′. *Let R be a real closed field and let X, Y, s and $f_1, \ldots, f_\nu$ be as above. Let $(x_{kji})_{k \in \{1,\ldots,s-1\}, j \in \{1,\ldots,\nu\}, i \in \{0,1,\ldots,n\}}$ be the coordinates of $R^{(s-1)\nu(n+1)}$. Then, it is possible to determinate by a constructive argument an element $a = (a_{kji})_{k,j,i} \in R^{(s-1)\nu(n+1)}$ such that, for each euclidean neighborhood $\mathcal{U}$ of a in $R^{(s-1)\nu(n+1)}$, there is $(b_{kji})_{k,j,i} \in \mathcal{U}$ which satisfies the following assertion: For each $k \in \{1, \ldots, s-1\}$, define the polynomial g_k in $R[x_1, \ldots, x_n]$ by*

$$g_k(x) := \sum_{j=1}^{\nu} \left(b_{kj0} + \sum_{i=1}^{n} b_{kji} x_i\right) f_j(x).$$

Then, using such polynomials $g_1, \ldots, g_{s-1}$, point b) *of Theorem 1.1 is verified.*

Let X be an algebraic subset of R^n of dimension r. First, suppose $r < n$. We define the complete intersection degree $\mathrm{cideg}(X, R^n)$ of X in R^n as the minimum integer c such that there are a point $p \in \mathrm{Nonsing}(X)$ and polynomials $P_1, \ldots, P_{n-r}$ in $\mathcal{I}_{R^n}(X)$ with independent gradients at p and $c = \prod_{i=1}^{n-r} \deg(P_i)$. Moreover, we define the upper degree $\mathrm{udeg}(X, R^n)$ of X in R^n as the minimum integer u such that there is a finite set of non–zero generators

$f_1, \dots, f_\nu$ of $\mathcal{I}_{R^n}(X)$ in $R[x_1, \dots, x_n]$ with $u = \max_{j \in \{1,\dots,\nu\}} \deg(f_j)$. If $r = n$, then we consider $\mathrm{cideg}(X, R^n)$ and $\mathrm{udeg}(X, R^n)$ equal to 1. Ler $\overline{R}$ be the algebraic closure of R. We identify canonically R^n with a subset of $\mathbb{P}^n(R)$ and $\mathbb{P}^n(R)$ with a subset of $\mathbb{P}^n(\overline{R})$ so each subset of R^n is a subset of $\mathbb{P}^n(\overline{R})$ also.

Theorem 1.2 *Let X be an irreducible algebraic subset of R^n of dimension r, let $c := \mathrm{cideg}(X, R^n)$, let Y be an algebraic subset of X of codimension $s \geq 2$ and let $u := \mathrm{udeg}(Y, R^n)$. Then, there is a chain of inclusions $Y \subset Y_{s-1} \subset \cdots \subset Y_1 \subset Y_0 = X$ such that, for each $k \in \{0, 1, \dots, s-1\}$, Y_k is an irreducible algebraic subset of X of codimension k, $\emptyset \neq Y_k \cap (\mathrm{Nonsing}(X) \setminus Y) \subset \mathrm{Nonsing}(Y_k)$ and, setting $\overline{Y}_k$ equal to the Zariski closure of Y_k in $\mathbb{P}^n(\overline{R})$, the degree $\deg(\overline{Y}_k)$ of $\overline{Y}_k$ in $\mathbb{P}^n(\overline{R})$ and the geometric genus $p_g(\overline{Y}_k)$ of $\overline{Y}_k$ satisfy the following inequalities:*

$$\deg(\overline{Y}_k) \leq c(u+1)^k$$

and

$$p_g(\overline{Y}_k) \leq \binom{c(u+1)^k - 1}{r-k+1}$$

where the binomial coefficient $\binom{a}{b}$ is considered null if $a < b$.

Let X be an algebraic subset of R^n (resp. $\mathbb{P}^n(R)$). An algebraic subset of X of dimension 1 is called algebraic curve of X.

Corollary 1.3 *Let X be an irreducible algebraic subset of R^n of dimension $r \geq 1$ and let F be a finite subset of X formed by m distinct points. Define $c := \mathrm{cideg}(X, R^n)$. Then, there is an irreducible algebraic curve D of X containing F such that $\emptyset \neq D \cap (\mathrm{Nonsing}(X) \setminus F) \subset \mathrm{Nonsing}(D)$ and, setting $\overline{D}$ equal to the Zariski closure of D in $\mathbb{P}^n(\overline{R})$, it holds:*

$$\deg(\overline{D}) \leq c(m+1)^{r-1}$$

and

$$p_g(\overline{D}) \leq \tfrac{1}{2}\left(c(m+1)^{r-1} - 1\right)\left(c(m+1)^{r-1} - 2\right).$$

In the next theorem, we will improve Corollary 1.3 in the case F is a single point. Before stating this result, we recall some classical notions and give a definition. Let S be a subset of R^n. S is said to be a cone of R^n with vertex $p \in R^n$ if, for each $q \in S \setminus \{p\}$, the affine line through p and q is contained in

S. Moreover, S is said to be nondegenerate in R^n if it is not contained in any affine hyperplane of R^n. Similar definitions can be given in the projective case also. Indicate by $\mathbb{N}$ the set of all non–negative integers. We call Castelnuovo function the function $\mathrm{Castel} : (\mathbb{N} \setminus \{0\}) \times (\mathbb{N} \setminus \{0\}) \longrightarrow \mathbb{N}$ defined as follows: for each (d, n) with d or n equal to 1, $\mathrm{Castel}(d, n) := 0$ and, for each (d, n) with $d \geq 2$ and $n \geq 2$, $\mathrm{Castel}(d, n) := \frac{1}{2}a(a-1)(n-1) + ab$ where a and b are the unique non–negative integers such that $d - 1 = a(n-1) + b$ and $b \in \{0, 1, \ldots, n-2\}$. We use this nomenclature because, when $d = n = 1$ or $d, n \in \mathbb{N} \setminus \{0, 1\}$, $\mathrm{Castel}(d, n)$ is the well–known Castelnuovo bound for the genus of a nondegenerate irreducible complex algebraic curve of $\mathbb{P}^n(\mathbb{C})$ of degree d.

Theorem 1.4 *Let X be a nondegenerate irreducible algebraic subset of R^n of dimension $r \geq 1$ and let p be a point of X such that X is not a cone of R^n with vertex p. Define $c := \mathrm{cideg}(X, R^n)$ and denote by d^* the degree of the Zariski closure of X in $\mathbb{P}^n(\overline{R})$. Then, there exists a non–void Zariski open subset Ω of $\mathrm{Nonsing}(X) \setminus \{p\}$ with the following properties: for each $q \in \Omega$, there is an irreducible algebraic curve D_q of X containing p and q such that $D_q \cap (\mathrm{Nonsing}(X) \setminus \{p\}) \subset \mathrm{Nonsing}(D_q)$ and, setting $\overline{D}_q$ equal to the Zariski closure of D_q in $\mathbb{P}^n(\overline{R})$, it holds:*

$$\deg(\overline{D}_q) = d^*$$

and

$$p_g(\overline{D}_q) \leq \mathrm{Castel}(d^*, n - r + 1) \leq \mathrm{Castel}(c, n - r + 1).$$

Remark 1.5 *Suppose R is a real closed field. An algebraic subset of $\mathbb{P}^n(\overline{R})$ is said to be defined over R if it is the vanishing set of some homogeneous polynomials in $R[x_0, x_1, \ldots, x_n]$. Let D be an irreducible algebraic curve of R^n. It is well–known that there is a nonsingular irreducible algebraic curve $\widetilde{D}$ of some $\mathbb{P}^N(\overline{R})$ defined over R such that the real part $\widetilde{D} \cap \mathbb{P}^n(R)$ of $\widetilde{D}$ is birationally isomorphic to D. Such a curve $\widetilde{D}$ is unique up to biregular isomorphism. In this way, it is possible to define the genus $g(D)$ of D as the genus of $\widetilde{D}$. Remark that, if $\overline{D}$ is the Zariski closure of D in $\mathbb{P}^n(\overline{R})$, then $g(D) = p_g(\overline{D})$ so, in the statements of Corollary 1.3 and Theorem 1.4, $p_g(\overline{D})$ and $p_g(\overline{D}_q)$ can be replaced by $g(D)$ and $g(D_q)$ respectively.*

In future papers, we will use the previous bounds in the study of morphism space between algebraic varieties over real closed fields (see the announcement [4] and [5]). For example, the following result is a consequence of Theorem 1.4.

Theorem 1.6 ([5]) *Let R be a real closed field (resp. algebraically closed field). Let X be a nondegenerate irreducible algebraic subset of R^n of dimension $r \geq 1$, let $c := \mathrm{cideg}(X, R^n)$ and let Y be an algebraic subset of R^m of positive dimension. Indicate by $e(Y)$ the minimum genus (resp. geometric genus) of an irreducible algebraic curve of Y. Then, if $\mathrm{Castel}(c, n-r+1) < e(Y)$, every regular map from X to Y is constant.*

2 The proofs

We will give the proofs only in the case R is a real closed field. When R is an algebraically closed field, the proofs are similar, but very easier. We need some preliminaries. Fix a real closed field R. The ring $R[i] = R[X]/(X^2+1)$ in an algebraically closed field (see section 1.2. of [2]) so $\overline{R} = R[i]$. For convenience, we will use the symbol C in place of $\overline{R}$. Let Z be a Zariski locally closed subset of $\mathbb{P}^n(R)$ (resp. $\mathbb{P}^n(C)$). The notions of $\dim(Z)$, $\mathrm{Nonsing}(Z)$, algebraic subset of Z and codimension of an algebraic subset of Z can be defined as in the case Z is Zariski closed in $\mathbb{P}^n(R)$ (resp. $\mathbb{P}^n(C)$). Denote by $\mathrm{Sing}(Z)$ the set $Z \setminus \mathrm{Nonsing}(Z)$. Indicate by $\sigma_n : \mathbb{P}^n(C) \longrightarrow \mathbb{P}^n(C)$ the conjugation map and identify canonically $\mathbb{P}^n(R)$ with the fixed point set of σ_n. Let S be a subset of $\mathbb{P}^n(C)$. Define the real part $S(R)$ of S by $S(R) := S \cap \mathbb{P}^n(R)$. Recall that S is said to be defined over R if it is σ_n–invariant, i.e, $\sigma_n(S) = S$. Suppose that S has this property and fix a subset T of some $\mathbb{P}^m(C)$. A map $f : S \longrightarrow T$ is said to be defined over R if $\sigma_m \circ f = f \circ \sigma_n|_S$. Remark that, if f is a regular morphism defined over R, then $f(S(R)) \subset T(R)$ and the restriction of f from $S(R)$ to $T(R)$ is a regular morphism.

Let k be the field R or the field C. Let $\mathcal{L} = (v_0, v_1, \ldots, v_r)$ be a $(r+1)$–uple of independent vectors of k^{n+1}. For each $l \in \{0, 1, \ldots, r\}$, write $v_l := (v_{0l}, v_{1l}, \ldots, v_{nl})$ and define the linear polynomial p_l in $k[x_0, x_1, \ldots, x_n]$ by $p_l(x) := \sum_{i=0}^n v_{il} x_i$. Let L be the $(n-r+1)$–dimensional linear subspace of $\mathbb{P}^n(k)$ defined as the vanishing set of $p_0, p_1, \ldots, p_r$. The regular map $\pi_{\mathcal{L}} : \mathbb{P}^n(k) \setminus L \longrightarrow \mathbb{P}^r(k)$ defined by $\pi_{\mathcal{L}}([x]) := [p_0(x), p_1(x), \ldots, p_r(x)]$ is called a projection of $\mathbb{P}^n(k)$ with center L. Remark that $\pi_{\mathcal{L}}$ is uniquely determinated by L up to composition with a projective automorphism of $\mathbb{P}^r(k)$. For simplicity, we indicate $\pi_{\mathcal{L}}$ by π_L and say that π_L is the projection of $\mathbb{P}^n(k)$ with center L. Moreover, if $k = C$ and L is defined over R, then we assume that $\pi_L : \mathbb{P}^n(C) \setminus L \longrightarrow \mathbb{P}^r(C)$ is defined over R also. Let now X be an algebraic subset of $\mathbb{P}^n(k)$ of dimension r and let $p \in \mathrm{Nonsing}(X)$. Let $\sigma : k^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n(k)$ be the natural projection and let $p' \in \sigma^{-1}(p)$. Choose homogeneous polynomials $P_1, \ldots, P_{n-r}$ in $k[x_0, x_1, \ldots, x_n]$ vanishing on X

with independent gradients at p' and, for each $j \in \{1, \dots, n-r\}$, define the linear polynomial $g_j(x)$ in $k[x_0, x_1, \dots, x_n]$ by $g_j(x) := \sum_{i=0}^{n} x_i \cdot (\partial P_j / \partial x_i)(p')$. We define the projective tangent space $\mathbb{P}T_p(X)$ of X at p in $\mathbb{P}^n(k)$ as the vanishing set of $g_1, \dots, g_{n-r}$. It is easy to verify that $\mathbb{P}T_p(X)$ does not depend on the choice of p' and $P_1, \dots, P_{n-r}$ and has dimension r. Moreover, it always contains p.

Lemma 2.1 *Let N and P be linear subspaces of $\mathbb{P}^n(C)$ of dimension d and r respectively and let $\pi_N : \mathbb{P}^n(C) \setminus N \longrightarrow \mathbb{P}^{n-d-1}(C)$ be the projection of $\mathbb{P}^n(C)$ with center N. Indicate by h the dimension of $N \cap P$ where $h = -1$ if $N \cap P = \emptyset$. Then, $\pi_N(P \setminus N)$ is a linear subspace of $\mathbb{P}^{n-d-1}(C)$ of dimension $r - h - 1$.*

Proof. Easy exercise of Linear Algebra. □

Lemma 2.2 *Let X be an algebraic subset of $\mathbb{P}^n(R)$ of dimension $r < n$, let $p \in \mathrm{Nonsing}(X)$ and let X_C be the Zariski closure of X in $\mathbb{P}^n(C)$. Then, it is possible to determinate by a constructive argument a linear subspace L of $\mathbb{P}^n(R)$ of dimension $n - r - 1$ such that the Zariski closure of L in $\mathbb{P}^n(C)$ is disjoint from X_C and, denoting by $\pi_L^* : \mathrm{Nonsing}(X) \longrightarrow \mathbb{P}^r(R)$ the restriction to $\mathrm{Nonsing}(X)$ of the projection of $\mathbb{P}^n(R)$ with center L, p is a regular point of π_L^*.*

Proof. It suffices to find a $(n-r)$–dimensional linear subspace N of $\mathbb{P}^n(C)$ defined over R such that $N \cap \mathbb{P}T_p(X_C) = \{p\}$ and $N \cap X_C$ is finite. We will prove, by induction on $d \in \{0, 1, \dots, n-r\}$, that there is a d–dimensional linear subspace N_d of $\mathbb{P}^n(C)$ defined over R such that $N_d \cap \mathbb{P}T_p(X_C) = \{p\}$ and $N_d \cap X_C$ is finite. The case $d = 0$ is evident. Let $d \in \{1, \dots, n-r\}$. By induction, there is a $(d-1)$–dimensional linear subspace N_{d-1} of $\mathbb{P}^n(C)$ with the prescribed properties. Let $\pi_{d-1} : \mathbb{P}^n(C) \setminus N_{d-1} \longrightarrow \mathbb{P}^{n-d}(C)$ be the projection of $\mathbb{P}^n(C)$ with center N_{d-1} and, for each $z \in \mathbb{P}^{n-d}(C)$, let $N_{d,z}$ be the d–dimensional linear subspace of $\mathbb{P}^n(C)$ defined by $N_{d,z} := N_{d-1} \sqcup \pi_{d-1}^{-1}(z)$. Define $Z := \{z \in \mathbb{P}^{n-d}(C) \,|\, N_{d,z} \cap \mathbb{P}T_p(X_C) \neq N_{d-1} \cap \mathbb{P}T_p(X_C)\} = \pi_{d-1}(\mathbb{P}T_p(X_C) \setminus N_{d-1})$. Since $N_{d-1} \cap \mathbb{P}T_p(X_C) = \{p\}$, Lemma 2.1 ensures that $\dim(Z) = r - 1$. Let X_C^* be the Zariski closure of $\pi_{d-1}(X_C \setminus N_{d-1})$ in $\mathbb{P}^{n-d}(C)$. If $\dim(X_C^*) < n - d$ (for example, when $d < n - r$), then the set $\mathbb{P}^{n-d}(R) \setminus (Z \cup X_C^*)$ is non–void. Fix a point z in such a set. It is easy to see that $N_{d,z}$ has the desired properties. Suppose $d = n - r$ and $\dim(X_C^*) = r$, i.e., $X_C^* = \mathbb{P}^r(C)$. Let W^* be the Zariski closure of $\pi_{d-1}(\mathrm{Sing}(X_C) \setminus N_{d-1})$ in $\mathbb{P}^r(C)$ and let $\Omega := X_C \cap \pi_{d-1}^{-1}(\mathbb{P}^r(C) \setminus W^*)$. Remark that $\dim(W^*) < r$ so Ω is a non–void Zariski open subset of $\mathrm{Nonsing}(X_C) \setminus N_{d-1}$. Applying Sard's theorem to the restriction of π_{d-1} to Ω, we find a point $z \in \mathbb{P}^r(R) \setminus (Z \cup W^*)$ such that $N_{d,z} \cap (X_C \setminus N_{d-1}) \subset \Omega$ and $N_{d,z}$ intersects transversally Ω in $\mathbb{P}^n(C)$. In particular, $N_{d,z}$ is defined over R, $N_{d,z} \cap \mathbb{P}T_p(X_C) = \{p\}$ and $N_{d,z} \cap X_C$ is finite. □

Lemma 2.3 *Let X_C be an irreducible algebraic subset of C^n, let Y_C be an algebraic subset of X_C of codimension $s \geq 1$ and let p be a nonsingular point of Y_C of some dimension. Denote by $\rho_C : \widetilde{X}_C \longrightarrow X_C$ the blowing up of X_C with center Y_C. Then, $\dim(\rho_C^{-1}(p)) \geq s - 1$.*

Proof. Let $r := \dim(X_C)$. Let $\pi : P \longrightarrow C^n$ be the blowing up of C^n with center Y_C. We may suppose that P is an irreducible Zariski locally closed subset of some $\mathbb{P}^N(C)$ and ρ_C is the strict transform of X_C along π so $\widetilde{X}_C \subset P$, $\pi(\widetilde{X}_C) = X_C$ and the restriction of π from $\widetilde{X}_C$ to X_C coincides with ρ_C. Pick a Zariski open neighborhood U of p in C^n such that $U \cap Y_C$ is a nonsingular irreducible algebraic subset of U of some dimension $d \leq r - s$. Remark that: $\dim(P) = n$, $\pi^{-1}(U)$ is a Zariski open subset of $\mathrm{Nonsing}(P)$, $\widetilde{X}_C$ is an irreducible algebraic subset of P of dimension r and $\pi^{-1}(p)$ is a nonsingular irreducible algebraic subset of $\pi^{-1}(U)$ of dimension $n - d - 1$. Since ρ_C is surjective and $\rho_C^{-1}(p) = \widetilde{X}_C \cap \pi^{-1}(p)$, we have that $\dim(\rho_C^{-1}(p)) \geq r + (n - d - 1) - n = r - d - 1 \geq s - 1$. □

The following is a strong version of Theorem 1.1$'$. Recall that the ordering structure on R induces, in a natural way, a topology on R^n and $\mathbb{P}^n(R)$ (and hence on any of their subsets) called euclidean topology.

Theorem 1.1$''$. *Let X be an irreducible algebraic subset of R^n, let Y be an algebraic subset of X of codimension $s \geq 2$, let $f_1, \ldots, f_\nu$ be generators of $\mathcal{I}_{R^n}(Y)$ in $R[x_1, \ldots, x_n]$, let $X^* := \mathrm{Nonsing}(X) \setminus Y$ and let U be a non–void euclidean open subset of X^*. Let $(x_{kji})_{k \in \{1,\ldots,s-1\}, j \in \{1,\ldots,\nu\}, i \in \{0,1,\ldots,n\}}$ be the coordinates of $R^{(s-1)\nu(n+1)}$. Then, it is possible to determinate by a constructive argument an element $a = (a_{kji})_{k,j,i} \in R^{(s-1)\nu(n+1)}$ such that, for each euclidean neighborhood $\mathcal{U}$ of a in $R^{(s-1)\nu(n+1)}$, there is $(b_{kji})_{k,j,i} \in \mathcal{U}$ which satisfies the following assertion: For each $k \in \{1, \ldots, s-1\}$, define the polynomial g_k in $R[x_1, \ldots, x_n]$ by*

$$g_k(x) := \textstyle\sum_{j=1}^{\nu} \left(b_{kj0} + \sum_{i=1}^{n} b_{kji} x_i\right) f_j(x),$$

the polynomial map $G_k : X \longrightarrow R^k$ by $G_k(x) := (g_1(x), \ldots, g_k(x))$ and the subset Y_k of X by $Y_k := G_k^{-1}(0_k)$ where 0_k is the origin of R^k. Then, for each $k \in \{1, \ldots, s-1\}$, Y_k is an irreducible algebraic subset of X of codimension

k containing Y, $Y_k \cap U \neq \emptyset$ and 0_k is a regular value of the restriction of G_k to X^*.

Proof. Let $r := \dim(X)$. Let X_C and Y_C be the Zariski closures of X and Y in C^n respectively and let $f_{1,C}, \dots, f_{\nu,C}$ be the polynomials $f_1, \dots, f_\nu$ viewed as elements of $C[x_1, \dots, x_n]$. Define

$$X'_C := \mathrm{Zcl}_{C^n \times \mathbb{P}^{\nu-1}(C)}\big\{(x, [f_{1,C}(x), \dots, f_{\nu,C}(x)]) \in C^n \times \mathbb{P}^{\nu-1}(C) \,|\, x \in X_C \setminus Y_C\big\}$$

and $\rho'_C : X'_C \longrightarrow X_C$ by $\rho'_C(x,[y]) := x$ where $\mathrm{Zcl}_{C^n \times \mathbb{P}^{\nu-1}(C)}$ indicates the Zariski closure operator of $C^n \times \mathbb{P}^{\nu-1}(C)$. Let $N := \nu(n+1)-1$, let $(y_{ji})_{j \in \{1,\dots,\nu\}, i \in \{0,1,\dots,n\}}$ be the coordinates of $C^{\nu(n+1)}$ and let $[(y_{ji})_{j,i}]$ be the corresponding homogeneous coordinates of $\mathbb{P}^N(C)$. Let $\psi_C : C^n \times \mathbb{P}^{\nu-1}(C) \longrightarrow \mathbb{P}^N(C)$ be the restriction to $C^n \times \mathbb{P}^{\nu-1}(C)$ of the Segre embedding $\mathbb{P}^n(C) \times \mathbb{P}^{\nu-1}(C) \ni ([(x_i)_i],[(y_j)_j]) \longmapsto [(x_i y_j)_{j,i}] \in \mathbb{P}^N(C)$, let $\widetilde{X}_C := \psi_C(X'_C)$ and let $\varphi_C : \widetilde{X}_C \longrightarrow X'_C$ be the inverse of the restriction of ψ_C from X'_C to $\widetilde{X}_C$. Define: the regular map $\rho_C : \widetilde{X}_C \longrightarrow X_C$ defined over R by $\rho_C := \rho'_C \circ \varphi_C$, $\widetilde{X}$ as the real part of $\widetilde{X}_C$ and the regular map $\rho : \widetilde{X} \longrightarrow X$ as the restriction of ρ_C from $\widetilde{X}$ to X. Evidently, ρ_C is the blowing up of X_C with center Y_C and ρ is the blowing up of X with center Y. In particular, $\widetilde{X}_C$ (resp. $\widetilde{X}$) is an irreducible Zariski locally closed subset of $\mathbb{P}^N(C)$ (resp. $\mathbb{P}^N(R)$) of dimension r. Let $\eta : X \setminus Y \longrightarrow \widetilde{X}$ be the regular map such that $\rho(\eta(x)) = x$ for each $x \in X \setminus Y$, i.e., the map which sends $x \in X \setminus Y$ into $[(x_i \cdot f_j(x))_{j,i}] \in \mathbb{P}^N(R)$ where x_0 is considered equal to 1. Fix $p \in U$ and define $p^* := \eta(p)$ and $U^* := \eta(U) \subset \mathrm{Nonsing}(\widetilde{X})$. By Lemma 2.2, it is possible to determinate by a constructive argument a $(N-r-1)$–dimensional linear subspace L of $\mathbb{P}^N(R)$ such that the Zariski closure L_C of L in $\mathbb{P}^N(C)$ is disjoint from $\widetilde{X}_C$ and, denoting by π^*_L the restriction to $\mathrm{Nonsing}(\widetilde{X})$ of the projection $\pi_{L,C} : \mathbb{P}^N(C) \setminus L_C \longrightarrow \mathbb{P}^r(C)$ of $\mathbb{P}^N(C)$ with center L_C, p^* is a regular point of π^*_L. Thanks to this property, we can choose an euclidean neighborhood V^* of p^* in U^* such that $E^* := \pi^*_L(V^*)$ is an euclidean open subset of $\mathbb{P}^r(R)$ and the restriction of π^*_L from V^* to E^* is a diffeomorphism. Define $e^* := \pi^*_L(p^*)$. For each $k \in \{1, \dots, s-1\}$, choose a point $\beta_k = (\beta_{k0}, \beta_{k1}, \dots, \beta_{kr}) \in R^{r+1} \setminus \{0\}$ in such a way that, defining the hyperplane $H_{k,C}$ of $\mathbb{P}^r(C)$ by the linear equation $\sum_{l=0}^r \beta_{kl} z_l = 0$, the following is true: each $H_{k,C}$ contains e^* and, for each $k \in \{2, \dots, s-1\}$, $H_{k,C}$ is transverse to $\bigcap_{h=1}^{k-1} H_{h,C}$ in $\mathbb{P}^r(C)$. Let us write explicitly $\pi_{L,C}$. For each $l \in \{0,1,\dots,r\}$, choose an element $(\alpha_{lji})_{j,i} \in R^{\nu(n+1)}$ such that, defining the linear polynomial $\xi_l((y_{ji})_{j,i}) := \sum_{j,i} \alpha_{lji} y_{ji}$, L_C is the vanishing set of $\xi_0, \xi_1, \ldots, \xi_r$ in $\mathbb{P}^N(C)$. Define the point $a = (a_{kji})_{k,j,i} \in R^{(s-1)\nu(n+1)}$ by setting $a_{kji} := \sum_{l=0}^{r} \beta_{kl}\alpha_{lji}$ for each $k \in \{1, \ldots, s-1\}$, $j \in \{1, \ldots, \nu\}$ and $i \in \{0, 1, \ldots, n\}$. We will prove that such a point of $R^{(s-1)\nu(n+1)}$ has the desired properties. By repeated applications of Bertini's theorem to $\pi_{L,C}|_{\widetilde{X}_C}$ and $\pi_{L,C}|_{\mathrm{Nonsing}(\widetilde{X}_C)\setminus\rho_C^{-1}(Y_C)}$ (see Theorem 6.3, 2) and 4) of [7], page 67), for each $k \in \{1, \ldots, s-1\}$, we find a point $\beta'_k = (\beta'_{k0}, \beta'_{k1}, \ldots, \beta'_{kr})$ of $R^{r+1} \setminus \{0\}$ arbitrarily close to β_k with respect to the euclidean topology such that, defining the hyperplane $H'_{k,C}$ of $\mathbb{P}^r(C)$ by the linear equation $\sum_{l=0}^{r} \beta'_{kl} z_l = 0$, the following four properties are verified:

1) for each $k \in \{2, \ldots, s-1\}$, $H'_{k,C}$ is transverse to $\bigcap_{h=1}^{k-1} H'_{h,C}$ in $\mathbb{P}^r(C)$,

2) $E^* \cap \bigcap_{h=1}^{s-1} H'_{h,C} \neq \emptyset$,

3) for each $k \in \{1, \ldots, s-1\}$, $Y'_{k,C} := \widetilde{X}_C \cap \pi_{L,C}^{-1}(\bigcap_{h=1}^{k} H'_{h,C})$ is an irreducible algebraic subset of $\widetilde{X}_C$ of codimension k,

4) for each $k \in \{1, \ldots, s-1\}$, the restriction of $\pi_{L,C}$ to $\mathrm{Nonsing}(\widetilde{X}_C) \setminus \rho_C^{-1}(Y_C)$ is transverse to $\bigcap_{h=1}^{k} H'_{h,C}$ in $\mathbb{P}^r(C)$.

For each $k \in \{1, \ldots, s-1\}$, denote by Y'_k the real part of $Y'_{k,C}$ and by Y_k the Zariski closure of $\rho(Y'_k)$ in X. Fix $k \in \{1, \ldots, s-1\}$. From 2) and 4), it follows that $V^* \cap \mathrm{Nonsing}(Y'_{k,C}) \neq \emptyset$ so, using 3) also, we have that Y'_k is an irreducible algebraic subset of $\widetilde{X}$ of codimension k and Y_k is an irreducible algebraic subset of X of codimension k. Let us show that $Y \subset Y_k$. Denote by $\mathrm{Nonsing}^{(*)}(Y)$ the set of all nonsingular points of Y of some dimension. By Lemma 2.3, we know that, for each $p \in \mathrm{Nonsing}^{(*)}(Y)$, the dimension of the algebraic subset $\rho_C^{-1}(p)$ of $\mathbb{P}^N(C)$ is at least $s-1$. Let $N_{k,C}$ be the linear subspace of $\mathbb{P}^N(C)$ of codimension k defined by $N_{k,C} := L_C \sqcup \pi_{L,C}^{-1}(\bigcap_{h=1}^{k} H'_{k,C})$. Remark that, for each $p \in \mathrm{Nonsing}^{(*)}(Y)$, $\rho_C^{-1}(p) \cap Y'_{k,C} = \rho_C^{-1}(p) \cap N_{k,C}$ so, being $\dim(N_{k,C}) \geq N - s + 1$, it follows that $\rho_C^{-1}(p) \cap Y'_{k,C} \neq \emptyset$. Since $\rho_C(Y'_{k,C}) \cap X \subset Y_k$, $\mathrm{Nonsing}^{(*)}(Y) \subset \rho_C(Y'_{k,C})$ and $\mathrm{Nonsing}^{(*)}(Y)$ is Zariski dense in Y, we have that $Y \subset Y_k$ as desired. We can now complete the proof. Let $\pi_L : \mathbb{P}^N(R) \setminus L \longrightarrow \mathbb{P}^r(R)$ be the projection of $\mathbb{P}^N(R)$ with center L. Remark that $Y_k \setminus Y = \eta^{-1}(Y'_k) = \bigcap_{h=1}^{k} (\pi_L \circ \eta)^{-1}(H'_{k,C})$ and, for each $h \in \{1, \ldots, k\}$, $(\pi_L \circ \eta)^{-1}(H'_{h,C})$ coincides with the set of points $x = (x_1, \ldots, x_n) \in X \setminus Y$ such that

$$\sum_{j=1}^{\nu} ((\sum_{l=0}^{r} \beta'_{hl}\alpha_{lj0}) + \sum_{i=1}^{n} (\sum_{l=0}^{r} \beta'_{hl}\alpha_{lji}) x_i) \cdot f_j(x) = 0.$$

For each $k \in \{1, \ldots, s-1\}$, $j \in \{1, \ldots, \nu\}$ and $i \in \{0, 1, \ldots, n\}$, define $(b_{kji})_{k,j,i} \in R^{(s-1)\nu(n+1)}$ by $b_{kji} := \sum_{l=0}^{r} \beta'_{kl}\alpha_{lji}$, the linear polynomial g_k in

$R[x_1, \ldots, x_n]$ by

$$g_k(x) := \sum_{j=1}^{\nu} (b_{kj0} + \sum_{i=1}^{n} b_{kji} x_i)\, f_j(x)$$

and the polynomial map $G_k : X \longrightarrow R^k$ by $G_k(x) := (g_1(x), \ldots, g_k(x))$. Remark that $(b_{kji})_{k,j,i}$ is arbitrarily close to a in $R^{(s-1)\nu(n+1)}$. Fix $k \in \{1, \ldots, s-1\}$. We have that $Y_k \setminus Y = G_k^{-1}(0_k)$. Moreover, the explicit form of G_k ensures that $Y \subset G_k^{-1}(0_k)$ so $Y_k = G_k^{-1}(0_k)$. From property 2), it follows that $Y_k \cap U \neq \emptyset$ and, from properties 1) and 4), it follows that 0_k is a regular values of $G_k|_{X^*}$. $\square$

Lemma 2.4 *Let X be an irreducible algebraic subset of R^n of dimension r and let X_C be the Zariski closure of X in $\mathbb{P}^n(C)$. Define $c := \mathrm{cideg}(X, R^n)$. Then,* $\deg(X_C) \leq c$ *and* $p_g(X_C) \leq \binom{c-1}{r+1}$.

Proof. Define $d^* := \deg(X_C)$ and $g^* := p_g(X_C)$. Let $p \in \mathrm{Nonsing}(X)$ and let $P_1, \ldots, P_{n-r} \in \mathcal{I}_{R^n}(X)$ with independent gradient at p such that $c = \prod_{i=1}^{n-r} \deg(P_i)$. Let $P_1^*, \ldots, P_{n-r}^*$ be the homogeneous polynomials of $R[x_0, x_1, \ldots, x_n]$ obtained by homogenization of $P_1, \ldots, P_{n-r}$ respectively and let $P_{1,C}^*, \ldots, P_{n-r,C}^*$ be the polynomials $P_1^*, \ldots, P_{n-r}^*$ viewed as elements of $C[x_0, x_1, \ldots, x_n]$. By the properties of $P_1, \ldots, P_{n-r}$, it follows at once the existence of a Zariski open neighborhood Z of p in $\mathbb{P}^n(C)$ such that $X_C \cap Z$ is the vanishing set of $P_{1,C}^*|_Z, \ldots, P_{n-r,C}^*|_Z$. Remark that $X_C \setminus Z$ is a proper algebraic subset of X_C so $\dim(X_C \setminus Z) < \dim(X_C) = r$. By the Noether Normalization Theorem, there is a $(n-r-1)$–dimensional linear subspace L of $\mathbb{P}^n(C)$ disjoint from X_C such that the restriction $\pi : X \longrightarrow \mathbb{P}^r(C)$ of the projection $\pi_L : \mathbb{P}^n(C) \setminus L \longrightarrow \mathbb{P}^r(C)$ of $\mathbb{P}^n(C)$ with center L is a finite–to–one surjective regular map. Let W be the Zariski closure of $\pi((X_C \setminus Z) \cup \mathrm{Sing}(X_C))$ in $\mathbb{P}^r(C)$. Evidently, we have that $\dim(W) < r$. Applying Sard's theorem to $\pi|_{X_C \setminus \pi^{-1}(W)}$, we find a point $z \in \mathbb{P}^r(C) \setminus W$ such that the $(n-r)$–dimensional linear subspace $N_z := L \sqcup \pi_L^{-1}(z)$ of $\mathbb{P}^n(C)$ intersects transversally $\mathrm{Nonsing}(X_C)$ in $\mathbb{P}^n(C)$ and $N_z \cap X_C \subset X_C \setminus \pi^{-1}(W) \subset \mathrm{Nonsing}(X_C)$. It is well–known that the cardinality of $N_z \cap X_C$ is exactly d^*. Let $\{Q_j = 0\}_{j=1}^{r}$ be linear polynomial equations for N_z in $\mathbb{P}^n(C)$. The points of $N_z \cap X_C$ are the solutions of the following system of equations: $\{P_{i,C}^* = 0\}_{i=1}^{n-r}$ and $\{Q_j = 0\}_{j=1}^{r}$. In this way, Bezout's theorem ensures that $d^* \leq \prod_{i=1}^{n-r} \deg(P_{i,C}^*) = c$. It remains to prove that $g^* \leq \binom{c-1}{r+1}$. If X is an affine subspace of R^n, then $g^* = 0$ so there is nothing to prove. Suppose X is not an affine linear subspace of R^n. Recall that, by the Castelnuovo–Harris Bound Theorem [6], we know that $g^* \leq \binom{a}{r+1}(m-r) + \binom{a}{r} b$ where m is the minimum dimension of a linear subspace of $\mathbb{P}^n(C)$ containing X_C and a and

b are the unique non–negative integers such that $d^* - 1 = a(m-r) + b$ and $b \in \{0, 1, \ldots, m-r-1\}$. By elementary considerations, it is easy to see that $\binom{a}{r+1}(m-r) + \binom{a}{r}b \leq \binom{d^*-1}{r+1} \leq \binom{c-1}{r+1}$. □

Proof of Theorem 1.2. Let $P_1, \ldots, P_{n-r}$ be polynomials of $\mathcal{I}_{R^n}(X)$ such that $c = \prod_{i=1}^{n-r} \deg(P_i)$ and, for some $p \in \mathrm{Nonsing}(X)$, their gradients $\nabla P_1(p), \ldots, \nabla P_{n-r}(p)$ at p are independent. Let U be the Zariski open subset of X formed by points $x \in X^* := \mathrm{Nonsing}(X) \setminus Y$ such that $\nabla P_1(x), \ldots, \nabla P_{n-r}(x)$ are independent. Fix a finite set of non–zero generators $f_1, \ldots, f_\nu$ of $\mathcal{I}_{R^n}(Y)$ in $R[x_1, \ldots, x_n]$ such that $u = \max_{j \in \{1,\ldots,\nu\}} \deg(f_j)$. For each $k \in \{1, \ldots, s-1\}$, let g_k be a polynomial in $R[x_1, \ldots, x_n]$, let $G_k : X \longrightarrow R^k$ be a polynomial map and let Y_k be a subset of X with the properties described in the statement of Theorem 1.1$''$. Fix $k \in \{1, \ldots, s-1\}$. The fact that $Y_k \cap U \neq \emptyset$ and 0_k is a regular value of $G_k|_{X^*}$ implies that $\mathrm{cideg}(Y_k, R^n) \leq (\prod_{i=1}^{n-r} \deg(P_i)) \cdot (u+1)^k = c(u+1)^k$. Theorem 1.2 now follows from Lemma 2.4. □

Lemma 2.5 *Let X be a nondegenerate irreducible algebraic subset of $\mathbb{P}^n(R)$ of dimension $r < n$, let $q \in \mathrm{Nonsing}(X)$ and let N be a linear subspace of $\mathbb{P}^n(R)$ of dimension $d \in \{0, 1, \ldots, n-r-1\}$ such that $N \cap \mathbb{P}T_q(X) = \{q\}$. Let $\pi_N : \mathbb{P}^n(R) \setminus N \longrightarrow \mathbb{P}^{n-d-1}(R)$ be the projection of $\mathbb{P}^n(R)$ with center N. Then, the Zariski closure of $\pi_N(X \setminus N)$ in $\mathbb{P}^n(R)$ has dimension r.*

Proof. It suffices to prove the following version of the lemma: "Let X be as above, let $q \in X$, let N be a linear subspace of $\mathbb{P}^n(R)$ of dimension $d \in \{0, 1, \ldots, n-r-1\}$ and let $\pi_N : \mathbb{P}^n(R) \setminus N \longrightarrow \mathbb{P}^{n-d-1}(R)$ be the corresponding projection. Suppose that there is a Nash submanifold M of $\mathbb{P}^n(R)$ of dimension r containing q such that M is connected with respect to the euclidean topology, $M \subset X$ and $N \cap \mathbb{P}T_q(M) = \{q\}$ where, making use of Nash functions, $\mathbb{P}T_q(M)$ can be defined similarly to the projective tangent space $\mathbb{P}T_p(X)$ presented at page 6 (for the notions of Nash function and Nash submanifold of $\mathbb{P}^n(R)$, see [10] and [2]). Then, the restriction $\pi_N^* : M \setminus N \longrightarrow \mathbb{P}^{n-d-1}(R)$ of π_N to $M \setminus N$ has rank r". First, consider the case $d = 0$. Indicate by $\pi_q : \mathbb{P}^n(R) \setminus \{q\} \longrightarrow \mathbb{P}^{n-1}(R)$ the projection of $\mathbb{P}^n(R)$ with center $\{q\}$ and by $\pi_q^* : M \setminus \{q\} \longrightarrow \mathbb{P}^{n-1}(R)$ its restriction to $M \setminus \{q\}$. We must prove that the rank $\mathrm{rnk}(\pi_q^*)$ of π_q^* is r. Suppose on the contrary that $\mathrm{rnk}(\pi_q^*) < r$. This condition implies that M is contained in $\mathbb{P}T_q(M)$. Since X is irreducible, it follows that $X \subset \mathbb{P}T_q(M)$ which is impossible because X is assumed to be nondegenerate in $\mathbb{P}^n(R)$. Let us complete the proof by induction on $n \geq r+1$. Let $n = r+1$. Since d must be null, we just know that $\mathrm{rnk}(\pi_{\{q\}}^*) = r$. Let $n > r+1$ and $d \in \{1, \ldots, n-r-1\}$. Fix $y \in N \setminus (M \cup \mathbb{P}T_q(M))$

and denote by $\pi_y^* : M \longrightarrow \mathbb{P}^{n-1}(R)$ the restriction to M of the projection $\pi_y : \mathbb{P}^n(R) \setminus \{y\} \longrightarrow \mathbb{P}^{n-1}(R)$ of $\mathbb{P}^n(R)$ with center $\{y\}$. By Lemma 2.1, we know that π_y^* is an immersion at q so, restricting M around q if needed, we may suppose that: $N \cap M = \{q\}$, π_y^* is an immersion and $M^* := \pi_y^*(M)$ is a Nash submanifold of $\mathbb{P}^{n-1}(R)$ of dimension r. Let X^* be the Zariski closure of $\pi_y(X \setminus \{y\})$ in $\mathbb{P}^{n-1}(R)$, let $q^* := \pi_y(q)$ and let $N^* := \pi_y(N \setminus \{y\})$. It is easy to see that X^* is a nondegenerate irreducible algebraic subset of $\mathbb{P}^{n-1}(R)$ of dimension r, $q^* \in M^* \subset X^*$, N^* is a linear subspace of $\mathbb{P}^{n-1}(R)$ of dimension $d-1$ and $\mathbb{P}T_{q^*}(M^*) = \pi_y(\mathbb{P}T_q(M))$. In particular, we have that $N^* \cap \mathbb{P}T_{q^*}(M^*) = \{q^*\}$. Let $\pi_{N^*} : \mathbb{P}^{n-1}(R) \setminus N^* \longrightarrow \mathbb{P}^{n-d-1}(R)$ be the projection of $\mathbb{P}^{n-1}(R)$ with center N^* and let $\pi_{N^*}^* : M^* \setminus N^* \longrightarrow \mathbb{P}^{n-d-1}(R)$ be the restriction of π_{N^*} to $M^* \setminus N^*$. By induction, it follows that $\mathrm{rnk}(\pi_{N^*}^*) = r$. Since $\pi_N^* = \pi_{N^*}^* \circ \pi_y^*|_{M \setminus N}$, we have that $\mathrm{rnk}(\pi_N^*) = \mathrm{rnk}(\pi_{N^*}^*) = r$. □

Proof of Theorem 1.4. We subdivide the proof into three steps.

Step I. We may suppose that X is an algebraic subset of $\mathbb{P}^n(R)$. Let $\pi_p : \mathbb{P}^n(R) \setminus \{p\} \longrightarrow \mathbb{P}^{n-1}(R)$ be the projection of $\mathbb{P}^n(R)$ with center $\{p\}$. Since X is not a cone of $\mathbb{P}^n(R)$ with vertex p, the Zariski closure X^* of $\pi_p(X \setminus \{p\})$ in $\mathbb{P}^{n-1}(R)$ has dimension r. Let W_1^* be the Zariski closure of $\mathrm{Sing}(X^*) \cup \pi_p(\mathrm{Sing}(X) \setminus \{p\})$ in $\mathbb{P}^{n-1}(R)$. Remark that $\dim(W_1^*) < r$ so $A := X \cap \pi_p^{-1}(X^* \setminus W_1^*)$ is a non–void Zariski open subset of $\mathrm{Nonsing}(X) \setminus \{p\}$. Let $\pi_p^* : A \longrightarrow \mathrm{Nonsing}(X^*)$ be the restriction of π_p from A to $\mathrm{Nonsing}(X^*)$ and let W_2^* be the Zariski closure in $\mathbb{P}^{n-1}(R)$ of the set of critical values of π_p^*. By Sard's theorem, we know that $\dim(W_2^*) < r$. Define the non–void Zariski open subset Ω of $\mathrm{Nonsing}(X) \setminus \{p\}$ by $\Omega := (\pi_p^*)^{-1}(\mathrm{Nonsing}(X^*) \setminus W_2^*)$. Remark that, for each $q \in \Omega$, the line L_q of $\mathbb{P}^n(R)$ containing p and q has the following property: $L_q \cap (X \setminus \{p\})$ is a finite subset of $\mathrm{Nonsing}(X) \setminus \{p\}$ and, for each $y \in L_q \cap (X \setminus \{p\})$, $L_q \cap \mathbb{P}T_y(X) = \{y\}$. Fix $q \in \Omega$.

Step II. Let X_C be the Zariski closure of X in $\mathbb{P}^n(C)$. We will prove that, for each $d \in \{1, \ldots, n-r\}$, there is a d–dimensional linear subspace N_d of $\mathbb{P}^n(C)$ defined over R such that, defining $F_d := X_C \cap N_d$ and $F_{d,R}$ as the real part of F_d, the following is true: F_d is finite, contains $\{p, q\}$ and generates N_d in $\mathbb{P}^n(C)$ (i.e., the smallest linear subspace of $\mathbb{P}^n(C)$ containing F is N_d). Moreover, $F_{d,R} \setminus \{p\} \subset \mathrm{Nonsing}(X)$ and, for each $y \in F_{d,R} \setminus \{p\}$, $N_d \cap \mathbb{P}T_y(X_C) = \{y\}$. Let us proceed by induction on d. Let $d = 1$. It suffices to define N_1 equal to the Zariski closure of L_q in $\mathbb{P}^n(C)$. Let $d \in \{2, \ldots, n-r\}$. By induction, there is a $(d-1)$–dimensional linear subspace N_{d-1} of $\mathbb{P}^n(C)$ with the prescribed properties. Let $\pi_{d-1} : \mathbb{P}^n(C) \setminus N_{d-1} \longrightarrow \mathbb{P}^{n-d}(C)$ be the projection of $\mathbb{P}^n(C)$ with center N_{d-1} and, for each $z \in \mathbb{P}^{n-d}(C)$, let $N_{d,z}$ be the d–dimensional linear subspace of $\mathbb{P}^n(C)$ defined by $N_{d,z} := N_{d-1} \sqcup \pi_{d-1}^{-1}(z)$. Define $Z := \bigcup_{y \in F_{d-1,R} \setminus \{p\}} \{z \in \mathbb{P}^{n-d}(C) \,|\, N_{d,z} \cap \mathbb{P}T_y(X_C) \neq N_{d-1} \cap$

$\mathbb{P}T_y(X_C)\} = \bigcup_{y\in F_{d-1,R}\setminus\{p\}} \pi_{d-1}(\mathbb{P}T_y(X_C)\setminus N_{d-1})$. Since $N_{d-1}\cap \mathbb{P}T_y(X_C) = \{y\}$ for each $y \in F_{d-1,R}\setminus\{p\}$, by Lemma 2.1, we know that $\dim(Z) = r-1 < n-d$. Let X_C^* be the Zariski closure of $\pi_{d-1}(X_C \setminus N_{d-1})$ in $\mathbb{P}^{n-d}(C)$ and let $\pi_{d-1}^* : X\setminus N_{d-1} \longrightarrow \mathbb{P}^{n-d}(R)$ be the restriction of π_{d-1} from $X\setminus N_{d-1}$ to $\mathbb{P}^{n-d}(R)$. Lemma 2.5 ensures that the Zariski closure of $\pi_{d-1}^*(X\setminus N_{d-1})$ in $\mathbb{P}^{n-r}(R)$ has dimension r. In particular, it follows that $\dim(X_C^*) = r$. Let W_1^* be the Zariski closure of $\mathrm{Sing}(X_C^*)\cup \pi_{d-1}(\mathrm{Sing}(X_C)\setminus N_{d-1})$ in $\mathbb{P}^{n-d}(C)$, let $A := X_C \cap \pi_{d-1}^{-1}(X_C^*\setminus W_1^*)$ and let W_2^* be the Zariski closure in $\mathbb{P}^{n-d}(C)$ of the set of critical values of the restriction of π_{d-1} from A to $\mathrm{Nonsing}(X_C^*)$. By Sard's theorem, it follows that $\dim(W_2^*) < r$ so $\dim(Z\cup W_1^*\cup W_2^*) < r$ also. In this way, the set $\pi_{d-1}^*(X\setminus N_{d-1})\setminus(Z\cup W_1^*\cup W_2^*)$ is non–void. Fix a point z in such a set. It is easy to see that $N_{d,z}$ has the desired properties. The induction is complete.

Step III. We have just proved the existence of a $(n-r)$–dimensional linear subspace N of $\mathbb{P}^n(C)$ defined over R such that, defining $F := X_C\cap N$ and F_R as the real part of F, the following is true:

a) F is finite, contains $\{p,q\}$ and generates N in $\mathbb{P}^n(C)$,

b) $F_R\setminus\{p\}\subset \mathrm{Nonsing}(X)$ and, for each $y\in F_R\setminus\{p\}$, $N\cap\mathbb{P}T_y(X_C) = \{y\}$.

Let $\pi_N : \mathbb{P}^n(C)\setminus N \longrightarrow \mathbb{P}^{r-1}(C)$ be the projection of $\mathbb{P}^n(C)$ with center N and let $\pi_N' : X_C\setminus N \longrightarrow \mathbb{P}^{r-1}(C)$ be its restriction to $X_C\setminus N$. Following the argument used in the proof of Lemma 2.5, it is easy to see that $\pi_N(X\setminus N)$ contains a non–void euclidean open subset of $\mathbb{P}^{r-1}(R)$. Applying Bertini's theorem to π_N' and to $\pi_N'|_{\mathrm{Nonsing}(X_C)\setminus N}$, we find a point $z\in\pi_N(X\setminus N)$ such that, defining $N_z := N \sqcup \pi_N^{-1}(z)$ and $D_C' := N_z\cap(X_C\setminus N)$, D_C' is an irreducible algebraic curve of $X_C\setminus N$ defined over R, $D_C'\cap X\neq\emptyset$ and $D_C'\cap(\mathrm{Nonsing}(X_C)\setminus N)\subset\mathrm{Nonsing}(D_C')$. Let $D_C := N_z\cap X_C$. Remark that D_C coincides with the Zariski closure of D_C' in X_C because $D_C\setminus D_C'$ is equal to F (which is finite) and each irreducible component of D_C has dimension greater than or equal to $r+(n-r+1)-n = 1$. In this way, D_C is an irreducible algebraic curve of X_C defined over R and containing F. Bearing in mind previous properties a) and b) of F and F_R, we have that D_C generates N_z in $\mathbb{P}^n(C)$ and $D_C\cap(\mathrm{Nonsing}(X)\setminus\{p\})\subset\mathrm{Nonsing}(D_C)$. In particular, denoting by D_q the real part of D_C, it follows that D_q is an irreducible algebraic curve of X containing $\{p,q\}$ such that $D_q\cap(\mathrm{Nonsing}(X)\setminus\{p\})\subset\mathrm{Nonsing}(D_q)$ and the Zariski closure $\overline{D}_q$ of D_q in $\mathbb{P}^n(C)$ is equal to D_C. Since $\overline{D}_q = N_z\cap X_C$, by applying Bezout's theorem, we obtain that $\deg(\overline{D}_q) = \deg(X_C) = d^*$. Moreover, by the Castelnuovo Bound Theorem (see [6] or [1], page 116), we have that $p_g(\overline{D}_q)\leq \mathrm{Castel}(d^*,\dim(N)) = \mathrm{Castel}(d^*, n-r+1)$. It remains to prove that $\mathrm{Castel}(d^*,n-r+1)\leq\mathrm{Castel}(c,n-r+1)$. Lemma 2.4 ensures that $d^* \leq c$ so, by a direct calculation, it is easy to verify the truthfulness of the previous inequality. □

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