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Consensus dynamics under asymmetric interactions in low dimensions

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Abstract

We consider a set $N = \{1, \ldots, n\}$, $n \geq 2$, of interacting agents whose individual opinions x_i, with $i \in N$, take values in some domain $\mathbb{D} \subseteq \mathbb{R}$. The interaction among the agents represents the degree of reciprocal influence which the agents exert upon each other and it is expressed by a general asymmetric interaction matrix with null diagonal and off-diagonal coefficients in the open unit interval. The present paper examines the asymmetric generalization of the linear consensus dynamics model discussed in previous publications by the same authors, in which symmetric interaction was assumed. We are mainly interested in determining the form of the asymptotic convergence towards the consensual opinion. In this respect, we present some general results plus the study of three particular versions of the linear consensus dynamics, depending on the relation between the interaction structure and the degrees of proneness to evaluation review of the various individual opinions. In the general asymmetric case, the analytic form of the asymptotic consensual solution $\tilde{x}$ is highly more complex than that under symmetric interaction, and we have obtained it only in two low-dimensional cases. Nonetheless, we are able to write those complex analytic forms arranging the numerous terms in an intelligible way which might provide useful clues to the open quest for the analytic form of the asymptotic consensual solution in higher-dimensional cases.

Keywords Linear dynamical models · Multiagent interaction · Symmetric and asymmetric cases · Consensus reaching

1 Introduction

An extensive literature originating in the 50s has investigated the question of multiagent interaction and opinion aggregation by means of linear dynamical models. A variety of methodological contexts have been considered, many of which refer to the central notion of asymptotic convergence towards a consensual opinion.

Fundamental contributions in this research strand have been made by several authors, among which: Shapley (1953) on cooperative game theory; French (1956) and Harary (1959) on social power theory; DeGroot (1974), Chatterjee (1975), Chatterjee and Chatterjee and Seneta (1977), Berger (1981), Kelly (1981), and French (1981) on DeGroot's consensus formation model; Sen (1982) on models of choice and welfare; Lehrer (1975), Wagner (1978, 1982), Lehrer and Lehrer and Wagner (1981), and Nurmi (1985) on the rational choice model; Anderson and Graesser (1976), Anderson (1981, 1991), and Graesser (1991) on the information integration model; Davis (1973, 1996) on the social decision scheme model; and Abelson (1964), Taylor (1968), Friedkin (1990, 1991, 1993, 1998, 1999, 2001), Friedkin and Johnsen (1990, 1997, 1999, 2011), and Marsden and Friedkin (1993, 1994) on social influence network theory.

In its general formulation, the linear consensus dynamics of our model follows the classical DeGroot's framework, with consensual convergence properties based on Perron's and Frobenius' theorems, plus extensions thereof. Recent reviews on such network models of linear consensus dynamics can be found in Jia et al. (2015), Dong et al. (2018), Dong et al. (2018), and Ureña et al. (2019).

Let $N = \{1, \ldots, n\}$, with $n \geq 2$, be a set of interacting agents whose individual opinions are denoted by $x_i \in \mathbb{D} \subseteq \mathbb{R}$, with $i \in N$. Consider a positive (thus irreducible) row stochastic matrix $\mathbf{C} = [c_{ij},\ i, j \in N]$ and the general convex linear dynamical law $x_i \longmapsto x_i' = \sum_{j \in N} c_{ij} x_j$, where

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$\sum_{j \in N} c_{ij} = 1$, with $i, j \in N$. The linear consensus dynamics in our model is defined in terms of the interaction structure between the agents, expressed by the interaction coefficients $v_{ii} = 0$ and independent $v_{ij}, v_{ji} \in (0, 1)$, with $i \neq j \in N$, plus an individual degree of proneness to evaluation review $u_i \in (0, 1)$, with $i \in N$.

Each iteration obtains the new opinion x'_i of agent $i \in N$ as a convex combination of his/her present opinion x_i and the present opinions x_j, with $j \neq i \in N$, of the remaining agents. Their present opinions $x_{j \neq i}$ are weighted with the coefficients c_{ij}, with $j \neq i \in N$, which constitute the $n - 1$ degrees of freedom of the convex combination associated with agent $i \in N$. As a result, the weight of the agent's own present opinion x_i, with $i \in N$, expressed by the coefficient c_{ii}, is constrained to be one minus the sum of the remaining coefficients c_{ij}, with $j \neq i \in N$.

The distinctive element of our linear consensus model is the way in which the transition coefficients c_{ij} are defined in terms of the interaction coefficients v_{ij} plus the individual degree of proneness to evaluation review $u_i \in (0, 1)$, with $i, j \in N$. The details of this construction have been introduced in Bortot et al. (2020a, b) and the asymptotic convergence to a consensual solution, indicated by $\tilde{x}$, has been discussed on the basis of a symmetry assumption regarding the interaction structure.

In particular, in Bortot et al. (2020a, b), the authors discuss three versions of our linear model of consensus dynamics, depending on the relation between the interaction structure and the degrees of proneness to evaluation review of the various individual opinions. In the first version of the consensus dynamics, we assume a uniform proneness to evaluation review, in the second version, we assume that the proneness to evaluation review is aligned with the interaction structure, and in the third version, we assume that the proneness to evaluation review counter-aligned with the interaction structure.

The present paper investigates the asymmetric generalization of the linear consensus dynamics model discussed in Bortot et al. (2020a, b), where the interaction structure was assumed to be symmetric, that is, identical coefficients $v_{ij} = v_{ji} \in (0, 1)$, with $i \neq j \in N$. In the asymmetric case, on the other hand, with independent coefficients $v_{ij}, v_{ji} \in (0, 1)$ with $i \neq j \in N$, the analytic form of the asymptotic consensual solution $\tilde{x}$ is highly more complex than that under symmetric interaction, and we have obtained it only in the low-dimensional cases $n = 3, 4$. Nonetheless, we are able to write those complex analytic forms arranging the numerous terms in an intelligible way which might provide useful clues to the open quest for the analytic form of the asymptotic consensual solution in higher-dimensional cases.

Over the years, some nonlinear models of consensus dynamics have been also investigated, either with exogenous or endogenous (and thus dynamic) definitions of the interaction structure. This is, for instance, the case of the soft consensus model proposed in Fedrizzi et al. (1999, 2007), and Fedrizzi et al. (2008, 2010), where the symmetric interaction coefficients $v_{ij} = v_{ji} \in (0, 1)$, with $i \neq j \in N$, are defined by filtering the square difference values $(x_i - x_j)^2$ with a decreasing sigmoid function. In the soft consensus model, agents with similar opinions interact strongly, whereas agents with dissimilar opinions interact weakly. A similar idea has inspired the more recent models of bounded confidence, see Deffuant et al. (2000), Dittmer (2001), and Hegselmann and Krause (2002).

The paper is organized as follows: In Sect. 2, we introduce the linear consensus dynamics of our multiagent network model and we discuss its general properties, as well as the asymptotic convergence to a consensual opinion in the general asymmetric case. We also examine in detail the three versions of the consensus dynamics introduced in Bortot et al. (2020a, b).

In Sects. 3 and 4, we focus on the cases $n = 3, 4$ and we examine the asymptotic convergence both in the symmetric and asymmetric cases, for each of the three versions of the consensus dynamics mentioned above. In these sections, we provide the analytic form of the consensual solutions for $n = 3, 4$ and we illustrate each of them with a numerical example.

The final section contains some concluding remarks.

2 The consensus dynamics model

Consider a set $N = \{1, \ldots, n\}$, with $n \geq 2$, of interacting agents indexed by $i \in N$ whose individual opinions are denoted by $x_i \in \mathbb{D} \subseteq \mathbb{R}$. The general asymmetric interaction among the agents is expressed by the interaction matrix $\mathbf{V} = [v_{ij},\ i, j \in N]$, with null diagonal coefficients $v_{ii} = 0$ and independent off-diagonal coefficients $v_{ij}, v_{ji} \in (0, 1)$, with $i \neq j \in N$. Each agent, moreover, has his/her own individual proneness to opinion review, which is expressed by the coefficient $u_i \in (0, 1)$, with $i \in N$.

The interaction matrix $\mathbf{V}$ is the adjacency matrix of a complete directed graph in which each node $i \in N$ represents an individual agent and encodes the corresponding opinion $x_i \in \mathbb{D}$. Moreover, each pair of edges (i, j) and (j, i), with $i \neq j \in N$, represents the interaction between two individuals agents, which is expressed by the independent coefficients v_{ij} and v_{ji} with values in the open unit interval.

Consider now a general positive (thus irreducible) row stochastic matrix $\mathbf{C} = [c_{ij},\ i, j \in N]$, with $c_{ij} \geq 0$ for all $i, j \in N$ and $\sum_{j \in N} c_{ij} = 1$ for $i \in N$. Moreover, consider the general convex linear dynamical law

$$\boldsymbol{x} \longmapsto \boldsymbol{x}' = \mathbf{C}\boldsymbol{x} \qquad x_i \longmapsto x'_i = \sum_{j \in N} c_{ij}\, x_j \qquad i \in N \tag{1}$$

where $\mathbf{C} \geq 0$, $\mathbf{C1} = \mathbf{1}$ and $\mathbf{1} = (1 \ \ldots \ 1)$. In this linear dynamical law, the coefficient c_{ij} represents the influential weight accorded by agent i to agent j, with $i, j \in N$.

In each iteration, the new opinion x_i' of agent $i \in N$ is a convex combination of his/her present opinion x_i and the present opinions x_j, with $i \neq j \in N$, of the remaining agents. Their present opinions $x_{j \neq i}$ are weighted with the coefficients c_{ij}, with $i \neq j \in N$, which constitute the $n-1$ degrees of freedom of the convex combination associated with agent $i \in N$. As a result, the weight of the present opinion x_i, with $i \in N$, expressed by the coefficient c_{ii}, is constrained to be one minus the sum of the remaining coefficients c_{ij}, with $i \neq j \in N$,

$$x_i' = c_{ii}\, x_i + \sum_{j \in N \setminus \{i\}} c_{ij}\, x_j \qquad i \in N \tag{2}$$

$$c_{ii} = 1 - \sum_{j \in N \setminus \{i\}} c_{ij} \qquad i \in N. \tag{3}$$

In our model of consensus dynamics, the coefficients of the transition of matrix $\mathbf{C}$ are expressed as follows:

$$c_{ij} = \varepsilon\, u_i\, \frac{v_{ij}}{v_i} \qquad c_{ii} = 1 - \varepsilon u_i \qquad i \neq j \in N \tag{4}$$

where $\varepsilon \in (0, 1)$, $v_i = \sum_{j \in N} v_{ij}$, and $u_i \in (0, 1)$ denotes the degree of proneness to evaluation review of agent $i \in N$. Note that, in general, $v_j \neq \sum_{i \in N} v_{ij}$.

The matrix $\mathbf{C}$ is in any case not symmetric, even when the interaction matrix $\mathbf{V}$ is symmetric. The influential weight that agent i assigns to agent j is given by $c_{ij} = \varepsilon\, u_i\, v_{ij}/v_i$, whereas, the influential weight that agent j assigns to agent i is given by $c_{ji} = \varepsilon\, u_j\, v_{ji}/v_j \neq c_{ij}$, with $i \neq j \in N$.

The linear dynamical law associated with (2)–(4) can thus be written as

$$\begin{aligned} x_i' &= (1 - \varepsilon\, u_i)\, x_i + \varepsilon\, u_i \sum_{j \in N \setminus \{i\}} \frac{v_{ij}}{v_i}\, x_j \\ &= (1 - \varepsilon\, u_i)\, x_i + \varepsilon\, u_i\, \bar{x}_i \qquad i \in N \end{aligned} \tag{5}$$

where $\bar{x}_i = \sum_{j \in N \setminus \{i\}} v_{ij}\, x_j / v_i$ denotes the context opinion around agent $i \in N$.

The present formulation of the linear dynamical law (5) generalizes the one under symmetric interaction described in Bortot et al. (2020a, b). Analogous formulations of the linear dynamics under symmetric interaction have also been discussed in Dong et al. (2017), and Ding et al. (2019).

Let as now introduce the matrix $\mathbf{B} = [b_{ij},\ i, j \in N]$, with null diagonal elements and off-diagonal elements given by v_{ij}/v_i, with $i \neq j \in N$,

$$\mathbf{B} = \begin{bmatrix} 0 & v_{12}/v_1 & \cdots & v_{1n}/v_1 \\ v_{21}/v_2 & 0 & \cdots & v_{2n}/v_2 \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1}/v_n & v_{n2}/v_n & \cdots & 0 \end{bmatrix}. \tag{6}$$

The matrix $\mathbf{B}$ is nonnegative irreducible and row stochastic. Therefore, by Frobenius' theorem, $\mathbf{B}$ has a simple leading unit eigenvalue and an associated leading left eigenvector $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$, meaning the unique normalized positive left eigenvector $\boldsymbol{\zeta}$ of the matrix $\mathbf{B}$, with $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$ and $\sum_{i \in N} \zeta_i = 1$. Equivalently, we can write $\mathbf{B}^{\mathrm{T}}\boldsymbol{\zeta} = \boldsymbol{\zeta}$, where $\boldsymbol{\zeta}$ is the leading right eigenvector of the matrix $\mathbf{B}^{\mathrm{T}}$ associated with the leading unit eigenvalue.

Proposition 1 *The row stochastic transition matrix $\mathbf{C}$ is positive and thus irreducible. Therefore, by Perron's theorem, $\mathbf{C}$ has a simple leading unit eigenvalue and an associated leading left eigenvector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, with $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$, given by*

$$\xi_i = \frac{\zeta_i / u_i}{\sum_{j \in N} (\zeta_j / u_j)} \qquad i \in N \qquad \sum_{i \in N} \xi_i = 1 \tag{7}$$

where $\boldsymbol{\zeta}$ is the leading left eigenvector of the matrix $\mathbf{B}$, with $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$.

Proof We know that $\boldsymbol{\zeta}$ satisfies $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$, which means

$$\sum_{i \in N} \zeta_i\, v_{ij}/v_i = \zeta_j \qquad j \in N. \tag{8}$$

It follows that

$$\begin{aligned} \sum_{i \in N} \frac{\zeta_i}{u_i}\, c_{ij} &= \frac{\zeta_j}{u_j}\, c_{jj} + \sum_{i \in N \setminus \{j\}} \frac{\zeta_i}{u_i}\, c_{ij} \\ &= \frac{\zeta_j}{u_j}\left(1 - \varepsilon\, u_j\right) + \varepsilon \sum_{i \in N \setminus \{j\}} \frac{\zeta_i}{u_i} \frac{u_i\, v_{ij}}{v_i} \\ &= \frac{\zeta_j}{u_j} - \varepsilon \left[\zeta_j - \sum_{i \in N \setminus \{j\}} \frac{\zeta_i\, v_{ij}}{v_i} \right] \\ &= \frac{\zeta_j}{u_j} \qquad j \in N. \end{aligned} \tag{9}$$

Finally, we obtain $\sum_{i \in N} \xi_i\, c_{ij} = \xi_j$, which means that $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$. □

The leading left eigenvector $\boldsymbol{\xi}$ of the matrix $\mathbf{C}$, which can be obtained from the leading left eigenvector $\boldsymbol{\zeta}$ of the matrix $\mathbf{B}$ as described by the previous Proposition 1, has a central role in the consensus dynamics model, which is the subject of the following Proposition 2.

Proposition 2 *The convex linear dynamical law $\boldsymbol{x} \longmapsto \boldsymbol{x}' = \mathbf{C}\boldsymbol{x}$ leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and converges to the consensual solution $\tilde{x}\,\mathbf{1} = (\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x})\,\mathbf{1}$. In other words, the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion*

$$\tilde{x} = \sum_{i \in N} \xi_i \, x_i = \frac{\sum_{i \in N} \frac{\zeta_i}{u_i} \, x_i}{\sum_{j \in N} \frac{\zeta_j}{u_j}} \tag{10}$$

where $\boldsymbol{x}$ can be either the initial opinion vector or any of the subsequent opinion vectors $\boldsymbol{x}(t)$, due to the invariance property of $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$.

Proof The invariance of $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ is immediate,

$$\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}' = \boldsymbol{\xi}^{\mathrm{T}}(\mathbf{C}\boldsymbol{x}) = (\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C})\,\boldsymbol{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x} \tag{11}$$

and the convergence follows from the positivity of the transition matrix $\mathbf{C}$, see DeGroot (1974) and references therein. In any case, the invariance of $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ implies that $\boldsymbol{x} = (\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x})\,\mathbf{1}$ is the only possible consensual solution. In fact, if $\boldsymbol{x} = c\,\mathbf{1}$ we obtain $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{\xi}^{\mathrm{T}}(c\,\mathbf{1}) = c\,(\boldsymbol{\xi}^{\mathrm{T}}\,\mathbf{1}) = c\,.$ □

We will now discuss the following three versions of the general dynamical law (5). In the first version, we consider a uniform proneness to evaluation review and therefore the free coefficients c_{ij}, with $i \neq j \in N$, are assumed to be proportional to a local normalization of the corresponding interaction coefficients v_{ij}, with $i \neq j \in N$. In the second version, we consider a proneness to evaluation review aligned with the interaction structure and thus the coefficients c_{ij}, with $i \neq j \in N$, are assumed to be simply proportional to the corresponding interaction coefficients v_{ij}, with $i \neq j \in N$. In the third version, on the other hand, we consider a proneness to evaluation review counter-aligned with the interaction structure and thus the coefficients c_{ij}, with $i \neq j \in N$, are assumed to be proportional to the corresponding interaction coefficients v_{ij}, with $i \neq j \in N$, scaled by the local factor $(1 - v_i/v)$ in the unit interval.

2.1 Uniform proneness to evaluation review

In relation with the general form of the linear dynamical law (5), this version of the linear dynamics corresponds to the choice $u_i = u \in (0, 1)$ for all $i \in N$. The coefficients of the transition matrix $\mathbf{C}$ are thus

$$c_{ij} = \varepsilon\, u\, v_{ij}/v_i \qquad c_{ii} = 1 - \varepsilon\, u \qquad i \neq j \in N \tag{12}$$

where $\varepsilon \in (0, 1)$, $\sum_{j \in N\setminus\{i\}} c_{ij} = \varepsilon\, u$ and $\sum_{j \in N} c_{ij} = 1$ for all $i \in N$.

In this case, therefore, the coefficients c_{ij}, with $i \neq j \in N$, are assumed to be proportional to v_{ij}/v_i, with $i \neq j \in N$, which amounts to a local normalization of the corresponding interaction coefficients v_{ij}, with $i \neq j \in N$.

The dynamical law can be written as

$$x_i \;\longmapsto\; x_i' = (1 - \varepsilon\, u)\, x_i + \varepsilon\, u \sum_{j \in N\setminus\{i\}} \frac{v_{ij}}{v_i}\, x_j \qquad i \in N \tag{13}$$

where $\varepsilon \in (0, 1)$. For $\varepsilon \in (0, 1)$ the row stochastic transition matrix $\mathbf{C}$ is positive and thus irreducible. It follows (Perron' theorem) that it has a simple leading unit eigenvalue and an associated leading left eigenvector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, with $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$, given by

$$\xi_i = \frac{\zeta_i}{\sum_{j \in N} \zeta_j} = \zeta_i \qquad i \in N \qquad\qquad \sum_{i \in N} \xi_i = 1 \tag{14}$$

where ζ_i with $i \in N$ are the components of the leading left eigenvector of the matrix $\mathbf{B}$, associated with the leading unit eigenvalue.

This dynamical law leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and, according to Proposition 2, the individual opinions converge asymptotically to the weighted mean of the initial opinions $\tilde{x} = \sum_{i \in N} \xi_i \, x_i$.

2.2 Proneness to evaluation review aligned with the interaction structure

In relation with the general form of the linear dynamical law (5), this version of the linear dynamics corresponds to the choice $u_i = v_i/v \in (0, 1)$ for all $i \in N$, where $v = \sum_{i=1}^{n} v_{ij}$. The coefficients of the transition matrix $\mathbf{C}$ are thus

$$c_{ij} = \varepsilon\, v_{ij}/v \qquad c_{ii} = 1 - \varepsilon\, v_i/v \qquad i \neq j \in N \tag{15}$$

where $\varepsilon \in (0, 1)$, $\sum_{j \in N\setminus\{i\}} c_{ij} = \varepsilon\, v_i/v$ and $\sum_{j \in N} c_{ij} = 1$ for all $i \in N$.

In this case, therefore, the coefficients c_{ij}, with $i \neq j \in N$, are assumed to be proportional to the corresponding interaction coefficients v_{ij}, with $i \neq j \in N$. The dynamical law can be written as

$$x_i \;\longmapsto\; x_i' = (1 - \varepsilon\, \frac{v_i}{v})\, x_i + \varepsilon \sum_{j \in N\setminus\{i\}} \frac{v_{ij}}{v}\, x_j \qquad i \in N \tag{16}$$

where $\varepsilon \in (0, 1)$. For $\varepsilon \in (0, 1)$, the row stochastic transition matrix $\mathbf{C}$ is positive and thus irreducible. It follows (Perron's theorem) that it has a simple leading unit eigenvalue and an associated leading left eigenvector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, with $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$, given by

$$\xi_i = \frac{\zeta_i/v_i}{\sum_{j \in N} \zeta_j/v_j} \qquad i \in N \qquad\qquad \sum_{i \in N} \xi_i = 1 \tag{17}$$

where ζ_i with $i \in N$ are the components of the leading left eigenvector of the matrix $\mathbf{B}$, associated with the leading unit eigenvalue.

This dynamical law leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and, according to Proposition 2, the individual opinions converge asymptotically to the weighted mean of the initial opinions $\tilde{x} = \sum_{i\in N} \xi_i\, x_i$.

2.3 Proneness to evaluation review counter-aligned with the interaction structure

In relation with the general form of the linear dynamical law (5), this version of the linear dynamics corresponds to the choice $u_i = 1 - v_i/v \in (0, 1)$ for all $i \in N$. The coefficients of the transition matrix $\mathbf{C}$ are thus

$$c_{ij} = \varepsilon\, \frac{(v - v_i)\, v_{ij}}{v\, v_i} \qquad c_{ii} = 1 - \varepsilon\, \frac{v - v_i}{v} \qquad i \neq j \in N \tag{18}$$

where $\varepsilon \in (0, 1)$, $\sum_{j\in N\setminus\{i\}} c_{ij} = \varepsilon\,(v - v_i)/v$ and $\sum_{j\in N} c_{ij} = 1$ for all $i \in N$.

In this case, therefore, the coefficients c_{ij}, with $i \neq j \in N$, are assumed to be proportional to v_{ij}/v_i multiplied by the factor $(1 - v_i/v)$, with $i \neq j \in N$, which amounts to a local normalization of the corresponding interaction coefficients v_{ij}, with $i \neq j \in N$, scaled by the local factor $(1 - v_i/v)$ in the unit interval. The dynamical law can be written as

$$\begin{aligned} x_i \;\longmapsto\; x_i' &= (1 - \varepsilon\, \frac{v - v_i}{v})\, x_i \\ &+ \varepsilon \sum_{j\in N\setminus\{i\}} \frac{(v - v_i)\, v_{ij}}{v\, v_i}\, x_j \qquad i \in N. \end{aligned} \tag{19}$$

For $\varepsilon \in (0, 1)$, the row stochastic transition matrix $\mathbf{C}$ is positive and thus irreducible. It follows (Perron's theorem) that it has a simple leading unit eigenvalue and an associated leading left eigenvector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, with $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$, given by

$$\xi_i = \frac{\zeta_i/(v - v_i)}{\sum_{j\in N} \zeta_j/(v - v_j)} \qquad i \in N \qquad \sum_{i\in N} \xi_i = 1 \tag{20}$$

where ζ_i with $i \in N$ are the components of the leading left eigenvector of the matrix $\mathbf{B}$, associated with the leading unit eigenvalue.

This dynamical law leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and, according to Proposition 2, the individual opinions converge asymptotically to the weighted mean of the initial opinions $\tilde{x} = \sum_{i\in N} \xi_i\, x_i$.

Summarizing, we have considered three versions of the consensus dynamics, each of which converges to an asymptotic solution corresponding to a different weighted mean of the initial opinions.

3 Consensus dynamics: the case $n = 3$

In this section, we present the case $n = 3$ of the consensus dynamics. We consider a network of three interacting agents whose initial individual opinions are $\boldsymbol{x}(t = 0) = (x_1^0, x_2^0, x_3^0)$.

The general asymmetric interaction matrix $\mathbf{V} = [v_{ij}, i, j \in N]$, with null diagonal coefficients $v_{ii} = 0$ and independent off-diagonal coefficients $v_{ij}, v_{ji} \in (0, 1)$, with $i \neq j \in N$, is

$$\mathbf{V} = \begin{bmatrix} 0 & v_{12} & v_{13} \\ v_{21} & 0 & v_{23} \\ v_{31} & v_{32} & 0 \end{bmatrix} \tag{21}$$

and the associated matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} 0 & v_{12}/v_1 & v_{13}/v_1 \\ v_{21}/v_2 & 0 & v_{23}/v_2 \\ v_{31}/v_3 & v_{32}/v_3 & 0 \end{bmatrix}. \tag{22}$$

Proposition 3 *The matrix $\mathbf{B}$ is nonnegative irreducible and row stochastic. Therefore, by Frobenius' theorem, $\mathbf{B}$ has a simple leading unit eigenvalue and an associated leading left eigenvector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$, $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$, which is given by*

$$\zeta_i = \frac{p_i}{\sum_{i=1}^{3} p_i} \qquad i \in N \tag{23}$$

$$\begin{aligned} p_1 &= v_1\,(v_{21}\, v_3 + v_{31}\, v_2 - v_{21}\, v_{31}) \\ p_2 &= v_2\,(v_{12}\, v_3 + v_{32}\, v_1 - v_{12}\, v_{32}) \\ p_3 &= v_3\,(v_{13}\, v_2 + v_{23}\, v_1 - v_{13}\, v_{23}) \end{aligned} \tag{24}$$

with $\sum_{i=1}^{3} \zeta_i = 1$.

Proof This explicit form of the leading left eigenvector $\boldsymbol{\zeta}$ of the matrix $\mathbf{B}$, satisfying $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$, can be proved straightforwardly by direct inspection.

The coefficients of the transition matrix $\mathbf{C}$ are

$$c_{ij} = \varepsilon\, u_i\, \frac{v_{ij}}{v_i} \qquad c_{ii} = 1 - \varepsilon\, u_i \qquad i \neq j \in N \tag{25}$$

where the coefficients $u_i \in (0, 1)$ denote the degree of proneness to evaluation review of the various agents and $v_i = \sum_{j\in N} v_{ij}$, with $i \in N$. Note that, in general, $v_j \neq \sum_{i\in N} v_{ij}$, with $j \in N$.

The associated convex linear dynamical law is written as

$$x_i' = c_{ii}\, x_i + \sum_{j\in N\setminus\{i\}} c_{ij}\, x_j \qquad i \in N \tag{26}$$

$$x_i' = (1 - \varepsilon\, u_i)\, x_i + \varepsilon\, u_i \sum_{j\in N\setminus\{i\}} \frac{v_{ij}}{v_i}\, x_j \qquad i \in N \tag{27}$$

with $\varepsilon \in (0,1)$, where each transition coefficient $c_{ij} \in (0,1)$ of the matrix $\mathbf{C}$ represents the influential weight accorded by agent i to agent j, with $i, j \in N$.

For $\varepsilon \in (0,1)$ the row stochastic transition matrix $\mathbf{C}$ is positive, and thus irreducible. It follows (Perron's theorem) that it has a simple leading unit eigenvalue and an associated leading left eigenvector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$ which, by means of Proposition 2, is given by

$$\xi_i = \frac{p_i/u_i}{\sum_{i=1}^{3} p_i/u_i} \qquad i \in N \tag{28}$$

and $\sum_{i=1}^{3}\xi_i = 1$. The dynamical law leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and converges to the consensual opinion $\tilde{x}\,\mathbf{1} = (\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x})\,\mathbf{1}$. In other words, the individual opinions converge asymptotically to weighted mean of the initial opinions $\tilde{x} = \sum_{i\in N}\xi_i x_i$.

In the particular case in which the interaction is symmetric, with identical coefficients $v_{ij} = v_{ji}$ for $i \neq j \in N$, the leading left eigenvector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ of matrix $\mathbf{B}$ reduces to

$$\zeta_i = \frac{v_i}{\sum_{j=1}^{3} v_j} \qquad i \in N \tag{29}$$

with $\sum_{i=1}^{3}\zeta_i = 1$. Therefore, the leading left eigenvector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ of the matrix $\mathbf{C}$ becomes

$$\xi_i = \frac{v_i/u_i}{\sum_{j=1}^{3} v_j/u_j} \qquad i \in N \tag{30}$$

with $\sum_{i=1}^{3}\xi_i = 1$. The dynamical law leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and converges to the consensual opinion $\tilde{x}\,\mathbf{1} = (\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x})\,\mathbf{1}$. In other words, the individual opinions converge asymptotically to weighted mean of the initial opinions $\tilde{x} = \sum_{i\in N}\xi_i x_i$.

We now present an illustrative example of the consensual dynamics in the case $n = 3$. We consider a network of three interacting agents whose initial individual opinions are assumed to be $\boldsymbol{x}(t = 0) = (1, 3, 9)$. We consider two degrees of interaction intensity between the agents: weak or strong, corresponding to interaction coefficients $1/3$ and $2/3$ respectively.

Example. Let us consider the case in which we have the following interaction graph,

In this example, the interaction coefficients v_{ij} with $i, j \in N$ are expressed by the following asymmetric interaction matrix

$$\mathbf{V} = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 0 \end{bmatrix}. \tag{31}$$

As we can see, the first agent assigns a strong influential weight to the opinion of the second agent (strong interaction), and a weak influential weight to the opinion of the third agent (weak interaction). The same interaction pattern follows the second agent, assigning a strong influential weight to the opinion of the third agent (strong interaction), and a weak influential weight to the opinion of the first agent (weak interaction). On the other hand, the third agent assigns a weak influential weight to both the other agents (weak interaction).

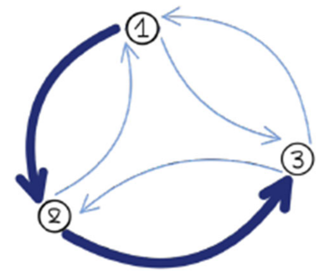

Fig. 1 Interaction graph for the network of 3 agents

The associated matrix $\mathbf{B}$ with null diagonal elements and off-diagonal elements v_{ij}/v_i, where $v_i = \sum_{j\in N} v_{ij}$, is given by

$$\mathbf{B} = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}. \tag{32}$$

The matrix $\mathbf{B}$ is nonnegative irreducible and row stochastic. The leading left eigenvector of the matrix $\mathbf{B}$ is $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3) = (12/41, 15/41, 14/41)$, such that $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$.

The row stochastic positive matrix $\mathbf{C} = [c_{ij},\ i, j \in N]$ and the associated convex linear dynamical law are as (25) and (26)–(27).

We consider the three versions of the consensus dynamics introduced in Sect. 2, in both the asymmetric and the symmetric versions. The latter can be defined in various ways, here we take, for instance $v_{ij}^S = v_{ji}^S = \max\{v_{ij}, v_{ji}\}$. The interaction matrix in the symmetric case is thus given by

$$\mathbf{V}_S = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 2/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{bmatrix} \tag{33}$$

and, consequently, the associated matrix $\mathbf{B}_S$ matrix is expressed as follows,

$$\mathbf{B}_S = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \end{bmatrix}. \tag{34}$$

The matrix $\mathbf{B}_S$ is nonnegative irreducible and row stochastic. The leading left eigenvector of the matrix $\mathbf{B}_S$ is $\boldsymbol{\zeta}_S = (\zeta_1^S, \zeta_2^S, \zeta_3^S) = (3/10, 2/5, 3/10)$.

From Proposition 2, it follows that the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion $\tilde{x}$ given by

$$\tilde{x} = \sum_{i \in N} \xi_i \, x_i = \frac{\sum_{i \in N} \frac{\zeta_i}{u_i} x_i}{\sum_{j \in N} \frac{\zeta_j}{u_j}} \tag{35}$$

where ζ_i, with $i \in N$, are the components of the eigenvector of the matrix $\mathbf{B}$, associated with the unit leading eigenvalue. Note that in the symmetric case, the ζ_i^S, with $i \in N$, are the components of the eigenvector of the matrix $\mathbf{B}_S$.

Uniform proneness to evaluation review. In this version of the consensus dynamics, we assume that $u_i = u = 0.2$ for all $i \in N$ and $\varepsilon = 0.25$. The corresponding row stochastic matrices $\mathbf{C}_S$ and $\mathbf{C}$ are given by

$$\mathbf{C}_S = \begin{bmatrix} 19/20 & 1/30 & 1/60 \\ 1/40 & 19/20 & 1/40 \\ 1/60 & 1/30 & 19/20 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 19/20 & 1/30 & 1/60 \\ 1/60 & 19/20 & 1/30 \\ 1/40 & 1/40 & 19/20 \end{bmatrix} \tag{36}$$

and their leading left eigenvectors are, respectively, $\boldsymbol{\xi}_S = (3/10, 2/5, 3/10)$ and $\boldsymbol{\xi} = (12/41, 15/41, 14/41)$. Thus, from Eq. (35), we obtain that in the symmetric case the individual opinions x_i converge asymptotically to the consensual opinion $\tilde{x}_S = 4.2$, whereas in the general asymmetric case, the opinions x_i converge asymptotically to the consensual solution $\tilde{x} = 4.46$.

In Fig. 2, we present the time evolution of the individual opinions $x_i(t)$, $i = 1, 2, 3$ and the consensual opinion $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{x}(t)$ under uniform proneness to evaluation review, both in the symmetric and asymmetric cases.

As we can see, the simulation results are consistent with Proposition 2:

- in the symmetric case, we observe that the convex dynamical law leaves invariant the weighted mean $\tilde{x}_S = \boldsymbol{\xi}_S^{\mathrm{T}} \boldsymbol{x}_S(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}_S$ and the individual opinions converge to the consensual opinion $\tilde{x}_S = 4.2$;
- in the general asymmetric case, the convex dynamical law leaves invariant the weighted mean $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{x}(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}$ and the individual opinions converge to the consensual solution $\tilde{x} = 4.46$.

To examine the change in the consensual solution from the symmetric to the asymmetric case, we consider the following parametric function regarding the interaction coefficients,

$$v_{ij}(\lambda) = (1 - \lambda)\, v_{ij}^S + \lambda\, v_{ij} \qquad i, j \in N \tag{37}$$

where $\lambda \in [\,0, 1\,]$ and $v_{ij}^S = \max\{v_{ij}, v_{ji}\}$. We can see that when $\lambda = 0$ we obtain the interaction coefficients v_{ij}^S in the symmetric case (33), while when $\lambda = 1$, we obtain the interaction coefficients v_{ij} in the asymmetric case given by the matrix $\mathbf{V}$ in (31).

In Fig. 3, we illustrate the change of the components $\xi_i(\lambda)$ of the leading left eigenvector $\boldsymbol{\xi}(\lambda)$, and the change of the consensual opinion $\tilde{x}(\lambda)$ with respect to the parameter $\lambda \in [\,0, 1\,]$.

When the parameter $\lambda = 0$, we obtain the leading left eigenvector $\boldsymbol{\xi}_S = (3/10, 2/5, 3/10)$ associated with the symmetric case, while when $\lambda = 1$, we obtain the leading left eigenvector $\boldsymbol{\xi} = (12/41, 15/41, 14/41)$ associated with the asymmetric case. Regarding the components of the eigenvector $\boldsymbol{\xi}$, we can observe that as the parameter λ increases and the value of ξ_2 decreases, ξ_1 slightly decreases, whereas the value of ξ_3 increases. Finally, we note that the consensual opinion $\tilde{x}(\lambda)$ increases as the parameter λ increases.

Proneness to evaluation review aligned with the interaction structure. In this version of the consensus dynamics, where $u_i = v_i/v$, it follows that $u_1 = u_2 = 3/8$ and $u_3 = 1/4$ for the asymmetric case, while for the symmetric case $u_1 = u_3 = 3/10$ and $u_2 = 2/5$. Moreover, we assume that $\varepsilon = 0.15$.

The corresponding row stochastic matrices $\mathbf{C}_S$ and $\mathbf{C}$ are given by

$$\mathbf{C}_S = \begin{bmatrix} 191/200 & 3/100 & 3/200 \\ 3/100 & 47/50 & 3/100 \\ 3/200 & 3/100 & 191/200 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 151/160 & 3/80 & 3/160 \\ 3/160 & 151/160 & 3/80 \\ 3/160 & 3/160 & 77/80 \end{bmatrix} \tag{38}$$

and their leading left eigenvectors are, respectively, $\boldsymbol{\xi}_S = (1/3, 1/3, 1/3)$ and $\boldsymbol{\xi} = (1/4, 5/16, 7/16)$. Thus, from Eq. (35), we obtain that in the general asymmetric case the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion $\tilde{x} = 5.13$, whereas in the symmetric case, the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual solution $\tilde{x}_S = 4.33$.

In Fig. 4, we present the time evolution of the individual opinions $x_i(t)$, $i = 1, 2, 3$ and the consensual opinion $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{x}(t)$ under proneness to evaluation review aligned with the interaction structure, both in the symmetric and asymmetric cases.

As we can see, the simulation results are consistent with Proposition 2:

- in the symmetric case, we observe that the convex dynamical law leaves invariant the weighted mean $\tilde{x}_S = \boldsymbol{\xi}_S^{\mathrm{T}} \boldsymbol{x}_S(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}_S$ and the individual opinions converge to the consensual

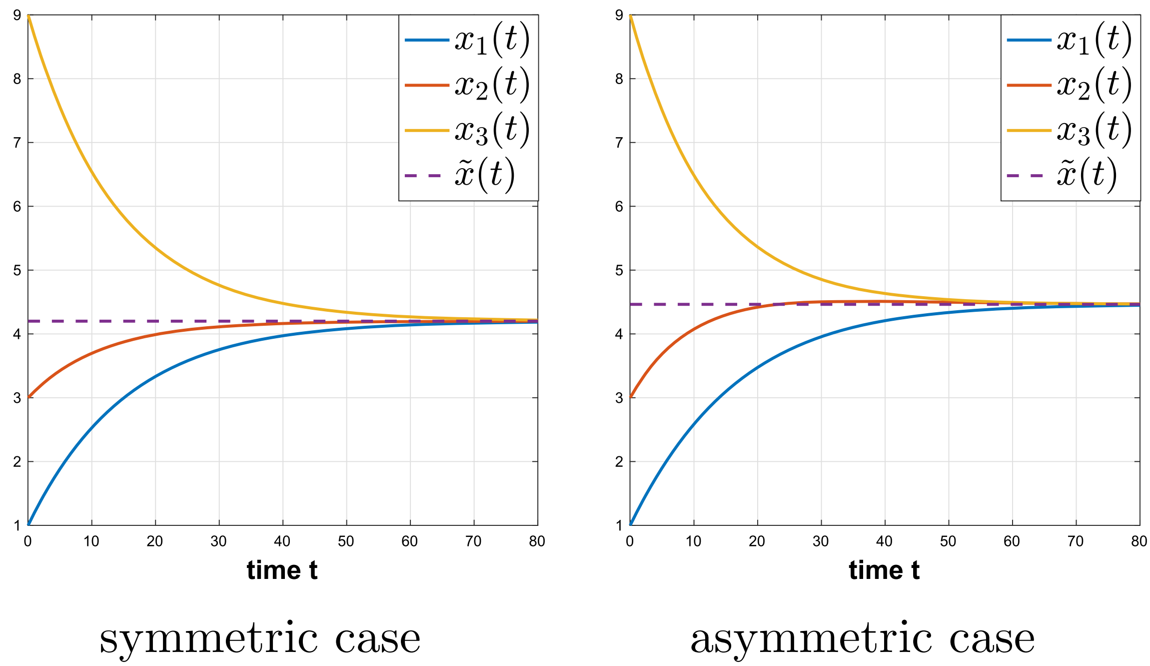

Fig. 2 Time evolution of the opinions $x_1(t)$, $x_2(t)$, $x_3(t)$ and $\tilde{x}$

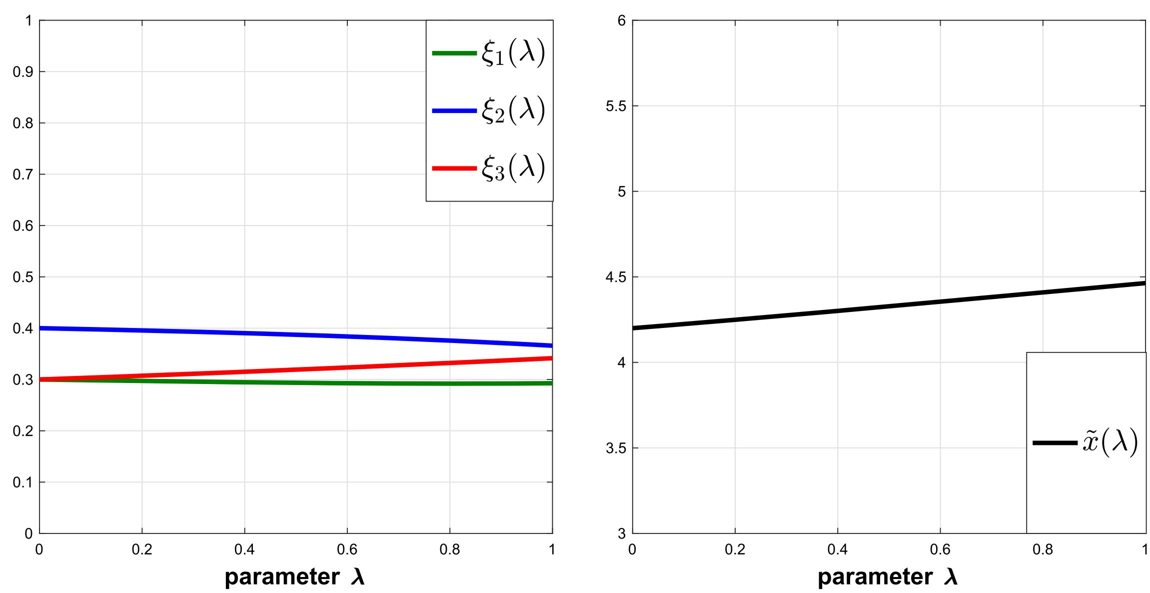

Fig. 3 The change of $\xi_1(\lambda)$, $\xi_2(\lambda)$, $\xi_3(\lambda)$ and the change of the consensual opinion $\tilde{x}(\lambda)$ as the parameter λ increases

opinion $\tilde{x}_S = 4.33$. In the symmetric case, the weighted mean $\tilde{x}_S$ corresponds to the plain mean;

- in the general asymmetric case, the convex dynamical law leaves invariant the weighted mean $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}$ and the individual opinions converge to the consensual solution $\tilde{x} = 5.13$.

To examine the change in the consensual solution from the symmetric to the asymmetric case, we use the parametric function $v_{ij}(\lambda)$ in (37) regarding the interaction coefficients v_{ij}, with $\lambda \in [0, 1]$ and $v_{ij}^S = \max\{v_{ij}, v_{ji}\}$.

In Fig. 5, we illustrate the change of the components $\xi_i(\lambda)$ of the leading left eigenvector $\boldsymbol{\xi}(\lambda)$, and the change of the consensual opinion $\tilde{x}(\lambda)$ with respect to the parameter $\lambda \in [0, 1]$.

When the parameter $\lambda = 0$, we obtain the leading left eigenvector $\boldsymbol{\xi}_S = (1/3, 1/3, 1/3)$ of matrix $\mathbf{C_S}$ associated with the symmetric case, while when $\lambda = 1$, we obtain the leading left eigenvector $\boldsymbol{\xi} = (1/4, 5/16, 7/16)$ associated with the asymmetric case. Regarding the components of the leading left eigenvector $\boldsymbol{\xi}$, we can observe that the values of ξ_1 and ξ_2 decrease, whereas the value of ξ_3 increases.

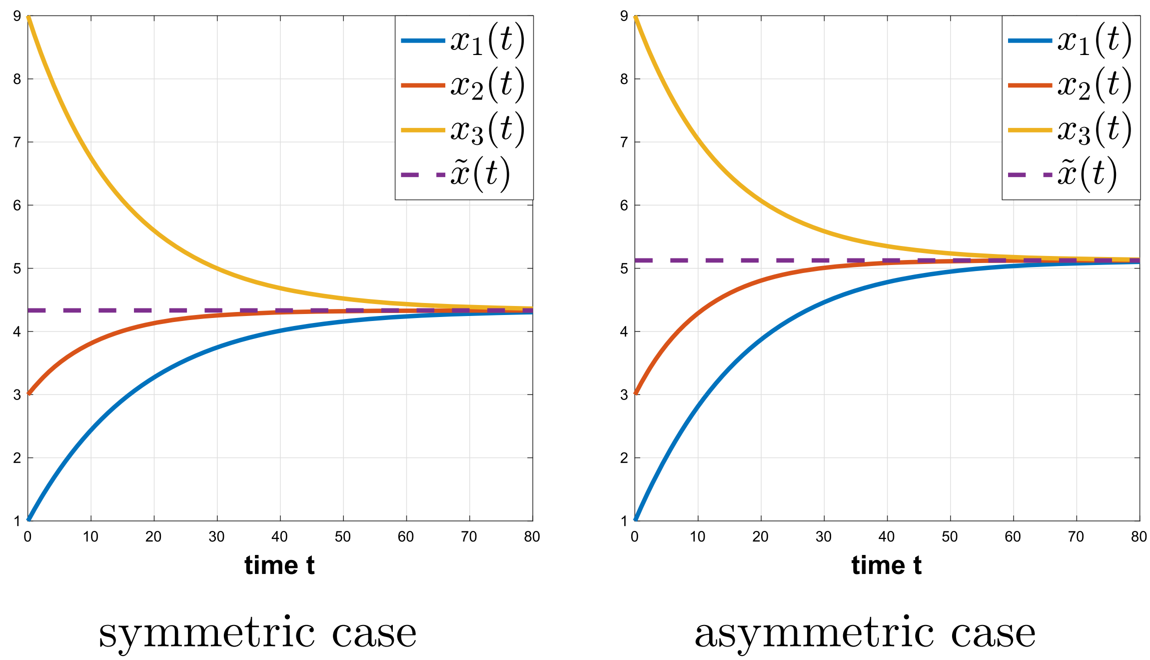

Fig. 4 Time evolution of the opinions $x_1(t)$, $x_2(t)$, $x_3(t)$ and $\tilde{x}$

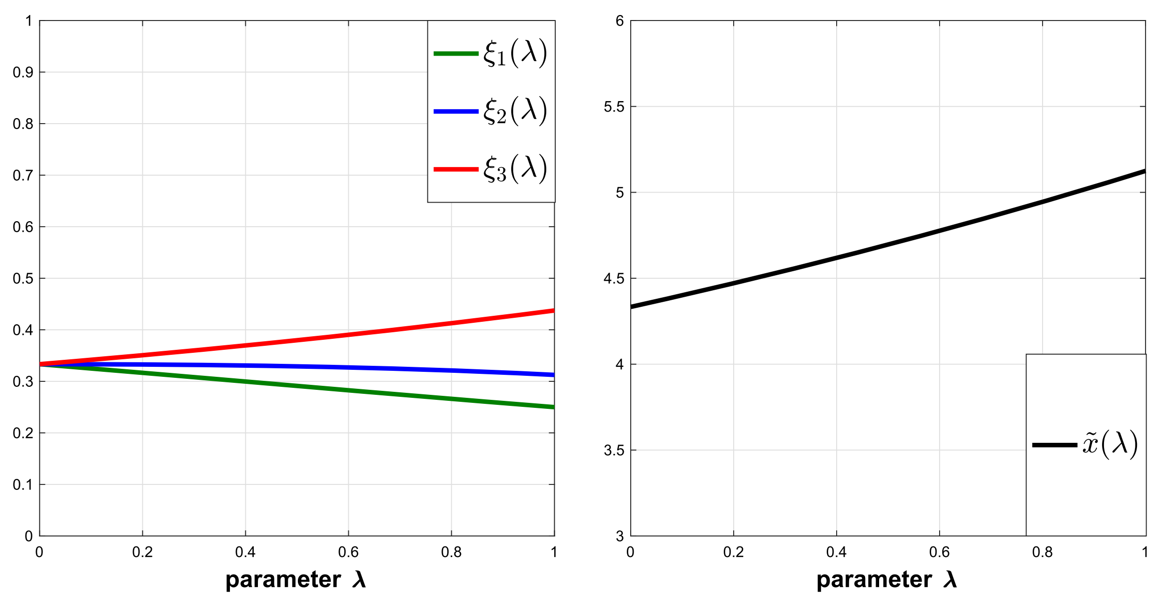

Fig. 5 The change of $\xi_1(\lambda)$, $\xi_2(\lambda)$, $\xi_3(\lambda)$ and the change of the consensual opinion $\tilde{x}(\lambda)$ as the parameter λ increases

Moreover, we notice that the components of the leading left eigenvector $\boldsymbol{\xi}$ follow the same tendency as in the uniform proneness to evaluation review dynamics. Finally, we note that the consensual opinion $\tilde{x}(\lambda)$ increases as the parameter λ increases.

Proneness to evaluation review counter-aligned with the interaction structure. In this version of the consensus dynamics, where $u_i = 1 - v_i/v$, it follows that $u_1 = u_2 = 5/8$ and $u_3 = 3/4$ for the asymmetric case, while for the symmetric case $u_1 = u_3 = 7/10$ and $u_2 = 3/5$. Moreover, we assume that $\varepsilon = 0.07$.

The corresponding row stochastic matrices $\mathbf{C}_S$ and $\mathbf{C}$ are given by

$$\mathbf{C}_S = \begin{bmatrix} 951/1000 & 49/1500 & 49/3000 \\ 21/1000 & 479/500 & 21/1000 \\ 49/3000 & 49/1500 & 951/1000 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 153/160 & 7/240 & 7/480 \\ 7/480 & 153/160 & 7/240 \\ 21/800 & 21/800 & 379/400 \end{bmatrix} \quad (39)$$

and their leading left eigenvectors are, respectively, $\boldsymbol{\xi}_S = (9/32, 7/16, 9/32)$ and $\boldsymbol{\xi} = (9/29, 45/116, 35/116)$. Thus, from Eq. (35), we obtain that in the general asymmetric case the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion $\tilde{x} = 4.19$, whereas in the symmetric case, the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual solution $\tilde{x}_S = 4.13$.

In Fig. 6, we present the time evolution of the individual opinions $x_i(t)$, $i = 1, 2, 3$ and the consensual opinion $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ under proneness to evaluation review counter-aligned with the interaction structure, both in the symmetric and asymmetric cases.

As we can see, the simulation results are consistent with Proposition 2:

- in the symmetric case, we observe that the convex dynamical law leaves invariant the weighted mean $\tilde{x}_S = \boldsymbol{\xi}_S^{\mathrm{T}}\boldsymbol{x}_S(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}_S$ and the individual opinions converge to the consensual opinion $\tilde{x}_S = 4.13$;
- in the general asymmetric case, the convex dynamical law leaves invariant the weighted mean $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}$ and the individual opinions converge to the consensual solution $\tilde{x} = 4.19$.

Finally, to examine the change in the consensual solution from the symmetric to the asymmetric case, we use the parametric function $v_{ij}(\lambda)$ in (37) regarding the interaction coefficients v_{ij}, with $\lambda \in [0, 1]$ and $v_{ij}^S = \max\{v_{ij}, v_{ji}\}$.

In Fig. 7, we illustrate the change of the components $\xi_i(\lambda)$ of the leading left eigenvector $\boldsymbol{\xi}(\lambda)$, and the change of the consensual opinion $\tilde{x}(\lambda)$ with respect to the parameter $\lambda \in [0, 1]$.

When the parameter $\lambda = 0$, we obtain the leading left eigenvector $\boldsymbol{\xi}_S = (9/32, 7/16, 9/32)$ associated with the symmetric case, while when $\lambda = 1$, we obtain the leading left eigenvector $\boldsymbol{\xi} = (9/29, 45/116, 35/116)$ associated with the asymmetric case. Regarding the components of the leading left eigenvector $\boldsymbol{\xi}$, we can observe that the values of ξ_1 and ξ_3 increase, whereas ξ_2 decrease as the parameter λ increases. Finally, we note that the consensual opinion $\tilde{x}(\lambda)$ increases as the parameter λ slightly increases.

4 Consensus dynamics: the case $n = 4$

In this section, we present the case $n = 4$ of the consensus dynamics. We consider a network of four interacting agents whose initial individual opinions are $\boldsymbol{x}(t = 0) = (x_1^0, x_2^0, x_3^0, x_4^0)$.

The general asymmetric interaction matrix $\mathbf{V} = [\, v_{ij}, i, j \in N\,]$, with null diagonal coefficients $v_{ii} = 0$ and independent off-diagonal coefficients $v_{ij}, v_{ji} \in (0, 1)$, with $i \neq j \in N$, is

$$\mathbf{V} = \begin{bmatrix} 0 & v_{12} & v_{13} & v_{14} \\ v_{21} & 0 & v_{23} & v_{24} \\ v_{31} & v_{32} & 0 & v_{34} \\ v_{41} & v_{42} & v_{43} & 0 \end{bmatrix} \quad (40)$$

and the associated matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} 0 & v_{12}/v_1 & v_{13}/v_1 & v_{14}/v_1 \\ v_{21}/v_2 & 0 & v_{23}/v_2 & v_{24}/v_2 \\ v_{31}/v_3 & v_{32}/v_3 & 0 & v_{34}/v_3 \\ v_{41}/v_4 & v_{42}/v_4 & v_{43}/v_4 & 0 \end{bmatrix}. \quad (41)$$

Proposition 4 *The matrix* $\mathbf{B}$ *is nonnegative irreducible and row stochastic. Therefore, by Frobenius' theorem,* $\mathbf{B}$ *has a simple leading unit eigenvalue and an associated leading left eigenvector* $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$, *which is given by*

$$\zeta_i = \frac{p_i}{\sum_{i=1}^4 p_i} \qquad i \in N \quad (42)$$

$$\begin{aligned} p_1 &= v_1 \big[v_{21}\,(v_3\,v_4 - v_{34}\,v_{43}) \\ &\quad + v_{31}\,(v_2\,v_4 - v_{24}\,v_{42}) + v_{41}\,(v_2\,v_3 - v_{23}\,v_{32}) \\ &\quad - v_{21}\,v_{31}\,v_4 - v_{21}\,v_{41}\,v_3 - v_{31}\,v_{41}\,v_2 + v_{21}\,v_{31}\,v_{41}\big] \\ p_2 &= v_2 \big[v_{12}\,(v_3\,v_4 - v_{34}\,v_{43}) \\ &\quad + v_{32}\,(v_1\,v_4 - v_{14}\,v_{41}) + v_{42}\,(v_1\,v_3 - v_{13}\,v_{31}) \\ &\quad - v_{12}\,v_{32}\,v_4 - v_{12}\,v_{22}\,v_3 - v_{32}\,v_{42}\,v_1 \\ &\quad + v_{12}\,v_{32}\,v_{42}\big] \\ p_3 &= v_3 \big[v_{13}\,(v_2\,v_4 - v_{24}\,v_{42}) \\ &\quad + v_{23}\,(v_1\,v_4 - v_{14}\,v_{41}) + v_{43}\,(v_1\,v_2 - v_{12}\,v_{21}) \\ &\quad - v_{13}\,v_{23}\,v_4 - v_{13}\,v_{43}\,v_2 - v_{23}\,v_{43}\,v_1 + v_{13}\,v_{23}\,v_{43}\big] \\ p_4 &= v_4 \big[v_{14}\,(v_2\,v_3 - v_{23}\,v_{32}) \\ &\quad + v_{24}\,(v_1\,v_3 - v_{13}\,v_{31}) + v_{34}\,(v_1\,v_2 - v_{12}\,v_{21}) \\ &\quad - v_{14}\,v_{24}\,v_3 - v_{14}\,v_{34}\,v_2 - v_{24}\,v_{34}\,v_1 + v_{14}\,v_{24}\,v_{34}\big] \end{aligned}$$

with $\sum_{i=1}^4 \zeta_i = 1$.

Proof This explicit form of the leading left eigenvector $\boldsymbol{\zeta}$ of the matrix $\mathbf{B}$, satisfying $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$, can be proved straightforwardly by direct inspection. □

The coefficients of the transition matrix $\mathbf{C}$ are

$$c_{ij} = \varepsilon\, u_i\, \frac{v_{ij}}{v_i} \qquad c_{ii} = 1 - \varepsilon\, u_i \qquad i \neq j \in N \quad (43)$$

where the coefficients $u_i \in (0, 1)$ denote the degree of proneness to evaluation review of the various agents and

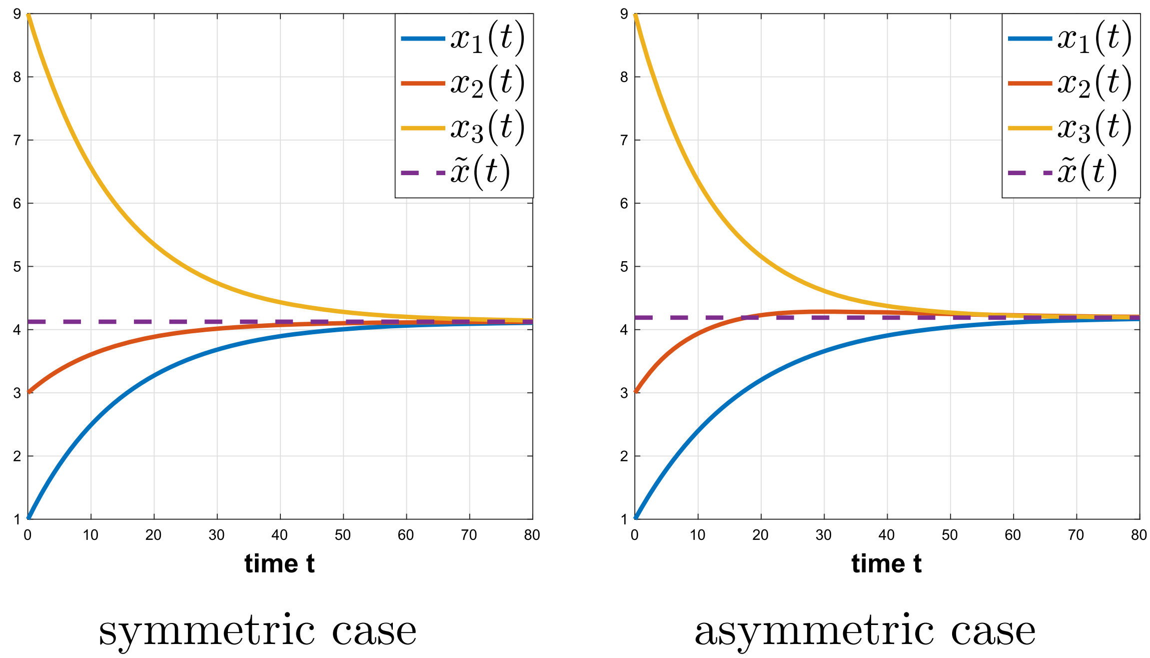

Fig. 6 Time evolution of the opinions $x_1(t)$, $x_2(t)$, $x_3(t)$ and $\tilde{x}$

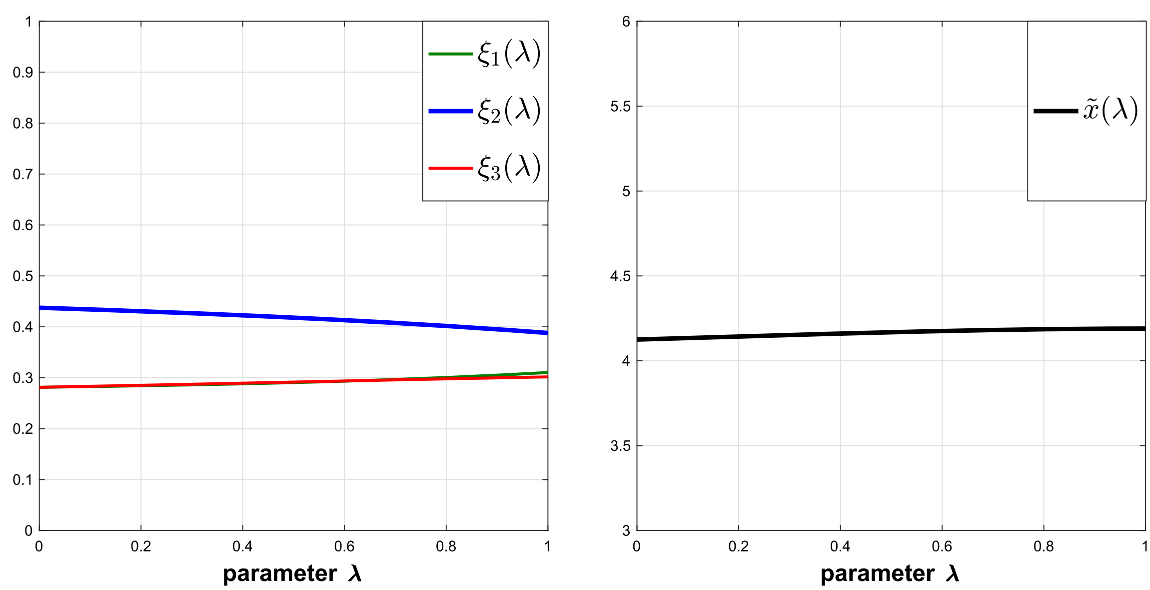

Fig. 7 The change of $\xi_1(\lambda)$, $\xi_2(\lambda)$, $\xi_3(\lambda)$ and the change of the consensual opinion $\tilde{x}(\lambda)$ as the parameter λ increases

$v_i = \sum_{j \in N} v_{ij}$. Note that, in general, $v_j \neq \sum_{i \in N} v_{ij}$, with $j \in N$.

The associated convex linear dynamical law is written as

$$x_i' = c_{ii}\, x_i + \sum_{j \in N \setminus \{i\}} c_{ij}\, x_j \qquad i \in N \tag{44}$$

$$x_i' = (1 - \varepsilon\, u_i)\, x_i + \varepsilon\, u_i \sum_{j \in N \setminus \{i\}} \frac{v_{ij}}{v_i}\, x_j \qquad i \in N \tag{45}$$

with $\varepsilon \in (0, 1)$, where each transition coefficient $c_{ij} \in (0, 1)$ of the matrix $\mathbf{C}$ represents the influential weight accorded by agent i to agent j, with $i, j \in N$.

For $\varepsilon \in (0, 1)$, the row stochastic transition matrix $\mathbf{C}$ is positive, and thus irreducible. It follows (Perron's theorem) that it has a simple leading unit eigenvalue and an associated unique normalized positive left eigenvector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$, $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$ which, by means of Proposition 2, is given by

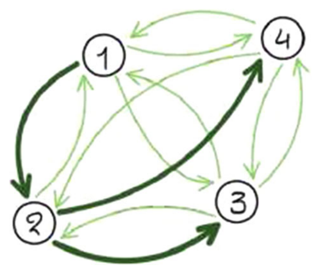

Fig. 8 Interaction graph for the network of 4 agents

$$\xi_i = \frac{p_i/u_i}{\sum_{i=1}^4 p_i/u_i} \qquad i \in N \tag{46}$$

and $\sum_{i=1}^4 \xi_i = 1$. The dynamical law leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and converges to the consensual opinion $\tilde{x}\,\mathbf{1} = (\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x})\,\mathbf{1}$. In other words, the individual opinions converge asymptotically to weighted mean of the initial opinions $\tilde{x} = \sum_{i\in N} \xi_i\, x_i$.

In the particular case in which the interaction is symmetric, with identical coefficients $v_{ij} = v_{ji}$ for $i \neq j \in N$, the leading left eigenvector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ of matrix **B** reduces to

$$\zeta_i = \frac{v_i}{\sum_{j=1}^4 v_j} \qquad i \in N \tag{47}$$

with $\sum_{i=1}^4 \zeta_i = 1$. Therefore, the leading left eigenvector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$ of the matrix **C** becomes

$$\xi_i = \frac{v_i/u_i}{\sum_{j=1}^4 v_j/u_j} \qquad i \in N \tag{48}$$

with $\sum_{i=1}^4 \xi_i = 1$. The dynamical law leaves $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ invariant and converges to the consensual opinion $\tilde{x}\,\mathbf{1} = (\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x})\,\mathbf{1}$. In other words, the individual opinions converge asymptotically to weighted mean of the initial opinions $\tilde{x} = \sum_{i\in N} \xi_i\, x_i$.

We now present an illustrative example of the consensual dynamics in the case $n = 4$. We consider a network of four interacting agents whose initial individual opinions are assumed to be $\boldsymbol{x}(t = 0) = (1, 3, 5, 9)$. We consider again two degrees of interaction intensity between the agents: weak or strong, corresponding to interaction coefficients $1/4$ and $3/4$ respectively.

Example. Let us consider the case in which we have the following interaction graph,

In this example, the interaction coefficients v_{ij} with $i, j \in N$ are expressed by the following asymmetric interaction matrix

$$\mathbf{V} = \begin{bmatrix} 0 & 3/4 & 1/4 & 1/4 \\ 1/4 & 0 & 3/4 & 3/4 \\ 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}. \tag{49}$$

As we can see, the first agent assigns a strong influential weight only to the opinion of the second agent, and a weak influential weight to the opinion of the remaining agents. The second agent assigns a strong influential weight to the opinion of the third and fourth agent, and a weak influential weight to the opinion of the first agent. On the other hand, the agents 3 and 4 assign a weak influential weight to the opinion of the remaining agents.

The associated matrix **B** with null diagonal elements and off-diagonal elements v_{ij}/v_i, where $v_i = \sum_{j\in N} v_{ij}$, is given by

$$\mathbf{B} = \begin{bmatrix} 0 & 3/5 & 1/5 & 1/5 \\ 1/7 & 0 & 3/7 & 3/7 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}. \tag{50}$$

The matrix **B** is nonnegative irreducible and row stochastic. The leading left eigenvector of the matrix **B** is $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (5/24, 7/24, 1/4, 1/4)$, such that $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$.

The row stochastic positive matrix $\mathbf{C} = [c_{ij},\ i, j \in N]$ and the associated convex linear dynamical law are as (43) and (44)–(45).

We consider the three special versions of consensus dynamics introduced in Sect. 2, in both the asymmetric and the symmetric versions. The latter can be defined in various ways, here we take, for instance $v_{ij}^S = v_{ji}^S = \max\{v_{ij}, v_{ji}\}$.

Thus, the interaction matrix in the symmetric case is given by

$$\mathbf{V}_S = \begin{bmatrix} 0 & 3/4 & 1/4 & 1/4 \\ 3/4 & 0 & 3/4 & 3/4 \\ 1/4 & 3/4 & 0 & 1/4 \\ 1/4 & 3/4 & 1/4 & 0 \end{bmatrix} \tag{51}$$

and, consequently, the associated $\mathbf{B}_S$ matrix is expressed as follows,

$$\mathbf{B}_S = \begin{bmatrix} 0 & 3/5 & 1/5 & 1/5 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/5 & 3/5 & 0 & 1/5 \\ 1/5 & 3/5 & 1/5 & 0 \end{bmatrix}. \tag{52}$$

The matrix $\mathbf{B}_S$ is nonnegative irreducible and row stochastic. The leading left eigenvector of the matrix $\mathbf{B}_S$ is $\boldsymbol{\zeta}_S = (\zeta_1^S, \zeta_2^S, \zeta_3^S, \zeta_4^S) = (5/24, 3/8, 5/24, 5/24)$.

From Proposition 2, it follows that the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion $\tilde{x}$ given by

$$\tilde{x} = \sum_{i\in N} \xi_i x_i = \frac{\sum_{i\in N} \frac{\zeta_i}{u_i} x_i}{\sum_{j\in N} \frac{\zeta_j}{u_j}} \tag{53}$$

where ζ_i with $i \in N$ are the components of the eigenvector of the matrix $\mathbf{B}$, associated with the unit leading eigenvalue. Note that in the symmetric case, the ζ_i^S, $i \in N$ are the components of the eigenvector of the matrix $\mathbf{B}_S$.

Uniform proneness to evaluation review. In this version of the consensus dynamics, we assume that $u_i = u = 0.2$ for all $i \in N$ and $\varepsilon = 0.3$.

The corresponding row stochastic matrices $\mathbf{C}_S$ and $\mathbf{C}$ are given by

$$\mathbf{C}_S = \begin{bmatrix} 47/50 & 9/250 & 3/250 & 3/250 \\ 1/50 & 47/50 & 1/50 & 1/50 \\ 3/250 & 9/250 & 47/50 & 3/250 \\ 3/250 & 9/250 & 3/250 & 47/50 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 47/50 & 9/250 & 3/250 & 3/250 \\ 3/350 & 47/50 & 9/350 & 9/350 \\ 1/50 & 1/50 & 47/50 & 1/50 \\ 1/50 & 1/50 & 1/50 & 47/50 \end{bmatrix} \tag{54}$$

and their leading left eigenvectors are, respectively, $\boldsymbol{\xi}_S = (5/24, 3/8, 5/24, 5/24)$ and $\boldsymbol{\xi} = (5/24, 7/24, 1/4, 1/4)$. Thus, from Eq. (53), we obtain that in the general asymmetric case the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion $\tilde{x} = 4.58$, whereas in the symmetric case the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual solution $\tilde{x}_S = 4.25$.

In Fig. 9, we present the time evolution of the individual opinions $x_i(t)$, $i = 1, 2, 3, 4$ and the consensual opinion $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ under uniform proneness to evaluation review, both in the symmetric and asymmetric cases.

As we can see, the simulation results are consistent with Proposition 2:

- in the symmetric case, we observe that the convex dynamical law leaves invariant the weighted mean $\tilde{x}_S = \boldsymbol{\xi}_S^{\mathrm{T}}\boldsymbol{x}_S(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}_S$ and the individual opinions converge to the consensual opinion $\tilde{x}_S = 4.25$;
- in the general asymmetric case, the convex dynamical law leaves invariant the weighted mean $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}$ and the individual opinions converge to the consensual solution $\tilde{x} = 4.58$.

To examine the change in the consensual solution from the symmetric to the asymmetric case, we use the parametric function $v_{ij}(\lambda)$ in (37) regarding the interaction coefficients v_{ij}, with $\lambda \in [0, 1]$ and $v_{ij}^S = \max\{v_{ij}, v_{ji}\}$.

In Fig. 10, we illustrate the change of the components $\xi_i(\lambda)$ of the leading left eigenvector $\boldsymbol{\xi}(\lambda)$, and the change of the consensual opinion $\tilde{x}(\lambda)$ with respect to the parameter $\lambda \in [0, 1]$. Note that the components $\xi_3(\lambda)$ and $\xi_4(\lambda)$, indicated with the same color, are indeed equal for all $\lambda \in [0, 1]$.

When the parameter $\lambda = 0$, we obtain the leading left eigenvector $\boldsymbol{\xi}_S = (5/24, 3/8, 5/24, 5/24)$ associated with the symmetric case, while when $\lambda = 1$, we obtain the leading left eigenvector $\boldsymbol{\xi} = (5/24, 7/24, 1/4, 1/4)$ associated with the asymmetric case. Regarding the components of the leading left eigenvector $\boldsymbol{\xi}$, we can observe that the values of ξ_3 and ξ_4 coincide with respect to the parameter $\lambda \in [0, 1]$. Moreover, we can see that as the parameter λ increases, the values of ξ_3 and ξ_4 increase, whereas ξ_2 decreases and ξ_1 remains mostly stable (initially slightly decreasing and then slightly increasing). Finally, we note that the consensual opinion $\tilde{x}(\lambda)$ increases as the parameter λ increases.

Proneness to evaluation review aligned with the interaction structure. In this version of the consensus dynamics, where $u_i = v_i/v$, it follows that $u_1 = 5/18$, $u_2 = 7/18$ and $u_3 = u_4 = 1/6$ for the asymmetric case, while for the symmetric case $u_1 = u_2 = u_3 = u_4 = 1/4$. Moreover, we assume that $\varepsilon = 0.25$.

The corresponding row stochastic matrices $\mathbf{C}_S$ and $\mathbf{C}$ are given by

$$\mathbf{C}_S = \begin{bmatrix} 91/96 & 1/32 & 1/96 & 1/96 \\ 1/32 & 29/32 & 1/32 & 1/32 \\ 1/96 & 1/32 & 91/96 & 1/96 \\ 1/96 & 1/32 & 1/96 & 91/96 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 67/72 & 1/24 & 1/72 & 1/72 \\ 1/72 & 65/72 & 1/24 & 1/24 \\ 1/72 & 1/72 & 23/24 & 1/72 \\ 1/72 & 1/72 & 1/72 & 23/24 \end{bmatrix} \tag{55}$$

and their leading left eigenvectors are, respectively, $\boldsymbol{\xi}_S = (1/4, 1/4, 1/4, 1/4)$ and $\boldsymbol{\xi} = (1/6, 1/6, 1/3, 1/3)$. Thus, from Eq. (53), we obtain that in the general asymmetric case the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion $\tilde{x} = 5.33$, whereas in the symmetric case the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual solution $\tilde{x}_S = 4.5$.

In Fig. 11, we present the time evolution of the individual opinions $x_i(t)$, $i = 1, \ldots, 4$ and the consensual opinion $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ under proneness to evaluation review aligned with the interaction structure, both in the symmetric and asymmetric cases.

As we can see, the simulation results are consistent with Proposition 2:

- in the symmetric case, we observe that the convex dynamical law leaves invariant the weighted mean $\tilde{x}_S = \boldsymbol{\xi}_S^{\mathrm{T}}\boldsymbol{x}_S(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}_S$ and the individual opinions converge to the consensual opinion $\tilde{x}_S = 4.5$. In the symmetric case, the weighted mean $\tilde{x}^S(t)$ corresponds to the plain mean;

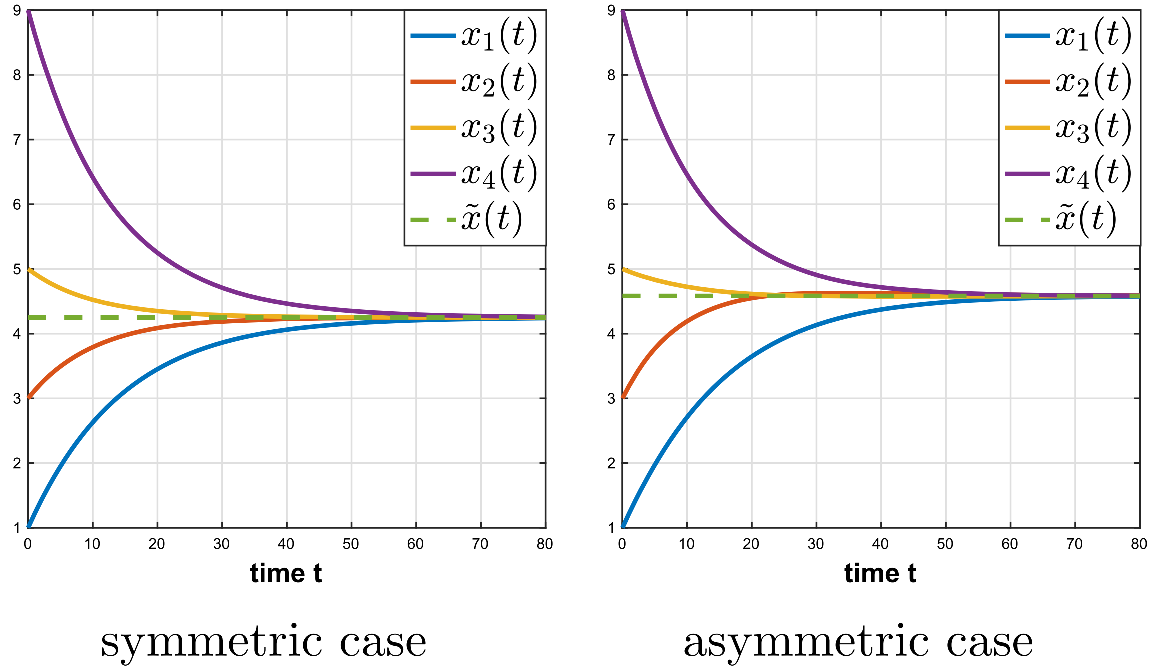

Fig. 9 Time evolution of the opinions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $\tilde{x}$

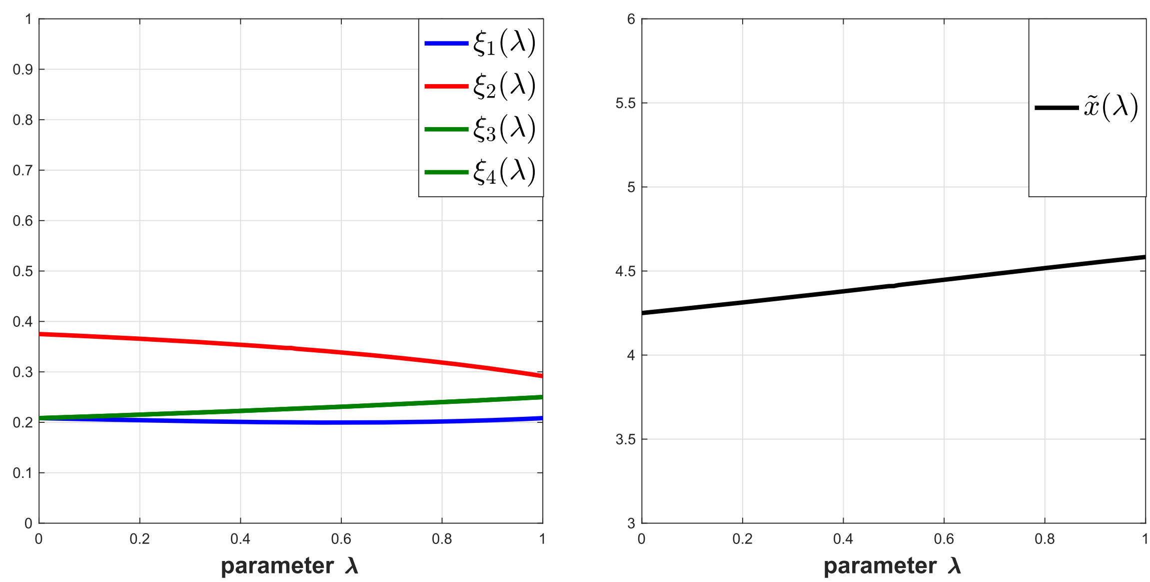

Fig. 10 The change of $\xi_1(\lambda)$, $\xi_2(\lambda)$, $\xi_3(\lambda)$, $\xi_4(\lambda)$ and the change of the consensual opinion $\tilde{x}(\lambda)$ as the parameter λ increases

- in the general asymmetric case, the convex dynamical law leaves invariant the weighted mean $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}$ and the individual opinions converge to the consensual solution $\tilde{x} = 5.33$.

To examine the change in the consensual solution from the symmetric to the asymmetric case, we use the parametric function $v_{ij}(\lambda)$ in (37) regarding the interaction coefficients v_{ij}, with $\lambda \in [\,0, 1\,]$ and $v_{ij}^S = \max\{v_{ij}, v_{ji}\}$.

In Fig. 12, we illustrate the change of the components $\xi_i(\lambda)$ of the leading left eigenvector $\boldsymbol{\xi}(\lambda)$, and the change of the consensual opinion $\tilde{x}(\lambda)$ with respect to the parameter $\lambda \in [\,0, 1\,]$. Note that the components $\xi_3(\lambda)$ and $\xi_4(\lambda)$, indicated with the same color, are indeed equal for all $\lambda \in [0, 1]$.

When the parameter $\lambda = 0$, we obtain the leading left eigenvector $\boldsymbol{\xi}_S = (1/4, 1/4, 1/4, 1/4)$ of matrix $\mathbf{C_S}$ associated with the symmetric case, while when $\lambda = 1$ we obtain the leading left eigenvector $\boldsymbol{\xi} = (1/6, 1/6, 1/3, 1/3)$ associated with the asymmetric case. Regarding the components of

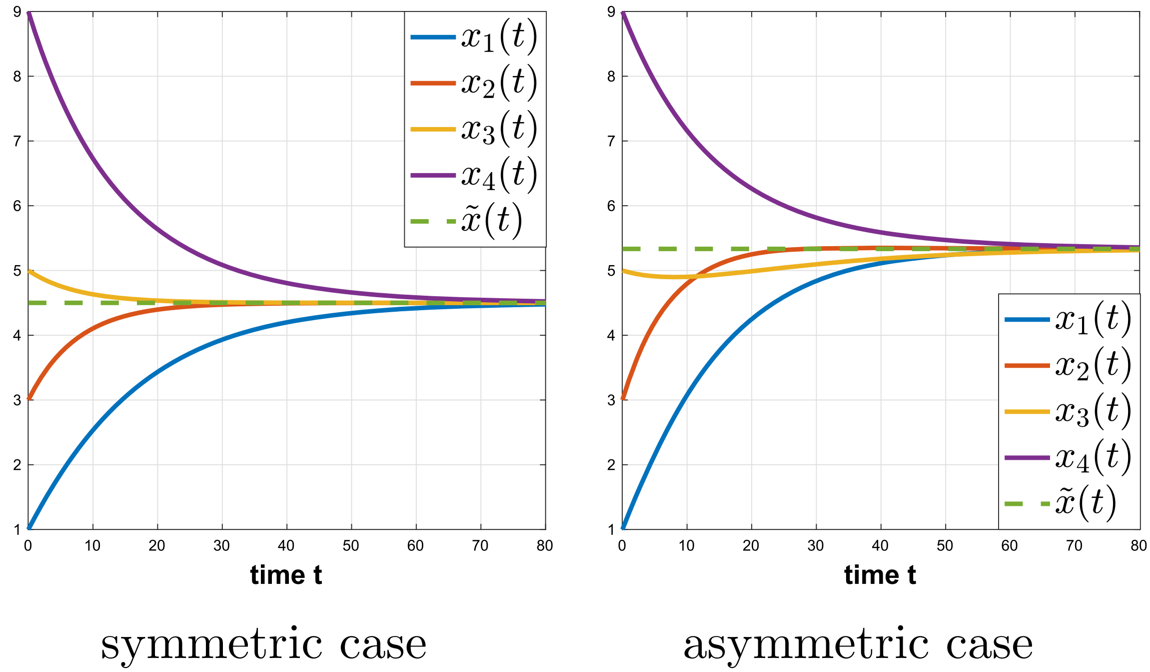

Fig. 11 Time evolution of the opinions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $\tilde{x}$

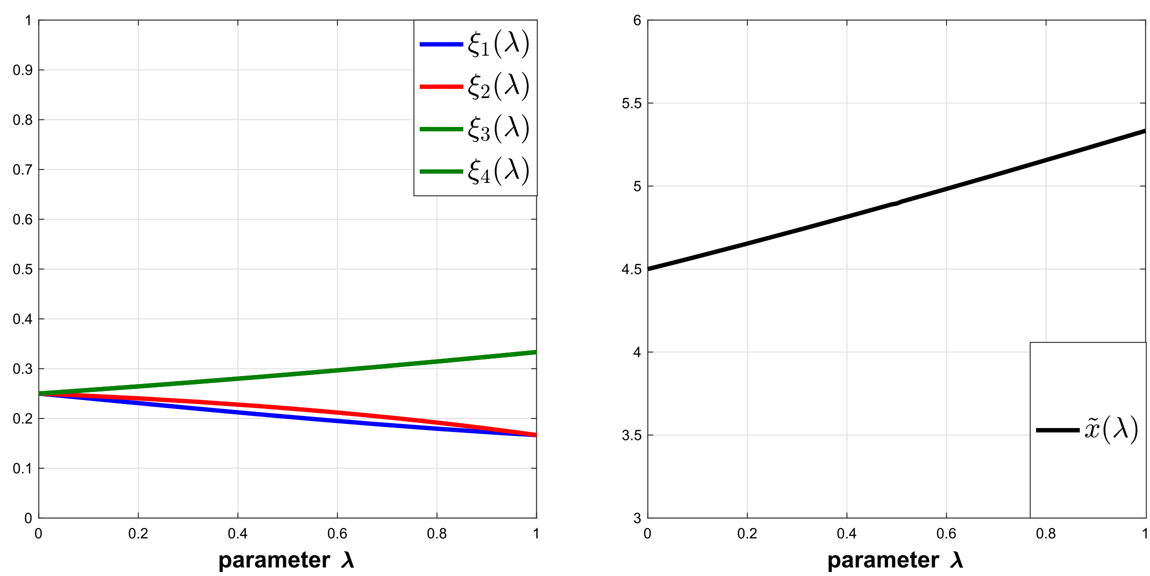

Fig. 12 The change of $\xi_1(\lambda)$, $\xi_2(\lambda)$, $\xi_3(\lambda)$, $\xi_4(\lambda)$ and the change of the consensual opinion $\tilde{x}(\lambda)$ as the parameter λ increases

the leading left eigenvector $\boldsymbol{\xi}$, we can observe that the values of ξ_3 and ξ_4 coincide with respect to the parameter $\lambda \in [0, 1]$. Moreover, we can see that as the parameter λ increases the values of ξ_1 and ξ_2 decrease, whereas the values of ξ_3, ξ_4 increase. Finally, we note that the consensual opinion $\tilde{x}(\lambda)$ increases as the parameter λ increases.

Proneness to evaluation review counter-aligned with the interaction structure. In this version of the consensus dynamics, where $u_i = 1 - v_i/v$, it follows that $u_1 = 13/18$, $u_2 = 11/18$ and $u_3 = u_4 = 5/6$ for the asymmetric case, while for the symmetric case $u_1 = u_3 = u_4 = 19/24$ and $u_2 = 5/8$. Moreover, we assume that $\varepsilon = 0.08$.

The corresponding row stochastic matrices $\mathbf{C}_S$ and $\mathbf{C}$ are given by

$$\mathbf{C}_S = \begin{bmatrix} 281/300 & 19/500 & 19/1500 & 19/1500 \\ 1/60 & 19/20 & 1/60 & 1/60 \\ 19/1500 & 19/500 & 281/300 & 19/1500 \\ 19/1500 & 19/500 & 19/1500 & 281/300 \end{bmatrix} \tag{56}$$

$$\mathbf{C} = \begin{bmatrix} 212/225 & 13/375 & 13/1125 & 13/1125 \\ 11/1575 & 214/225 & 11/525 & 11/525 \\ 1/45 & 1/45 & 14/15 & 1/45 \\ 1/45 & 1/45 & 1/45 & 14/15 \end{bmatrix} \tag{57}$$

and their leading left eigenvectors are, respectively, $\boldsymbol{\xi}_S = (25/132, 19/44, 25/132, 25/132)$ and $\boldsymbol{\xi} = (275/1302, 65/186, 143/651, 143/651)$. Thus, from Eq. (53), we obtain that in the general asymmetric case the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual opinion $\tilde{x} = 4.33$, whereas in the symmetric case, the individual opinions x_i, with $i \in N$, converge asymptotically to the consensual solution $\tilde{x}_S = 4.14$.

In Fig. 13, we present the time evolution of the individual opinions $x_i(t)$, $i = 1, 2, 3, 4$ and the consensual opinion $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ under proneness to evaluation review counter-aligned with the interaction structure, both in the symmetric and asymmetric cases.

As we can see, the simulation results are consistent with Proposition 2:

- in the symmetric case, we observe that the convex dynamical law leaves invariant the weighted mean $\tilde{x}_S = \boldsymbol{\xi}_S^{\mathrm{T}}\boldsymbol{x}_S(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}_S$ and the individual opinions converge to the consensual opinion $\tilde{x}_S = 4.14$;
- in the general asymmetric case, the convex dynamical law leaves invariant the weighted mean $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}(t)$ associated with the leading left eigenvector of matrix $\mathbf{C}$ and the individual opinions converge to the consensual solution $\tilde{x} = 4.33$.

To examine the change in the consensual solution from the symmetric to the asymmetric case, we use the parametric function $v_{ij}(\lambda)$ in (37) regarding the interaction coefficients v_{ij}, with $\lambda \in [\,0, 1\,]$ and $v_{ij}^S = \max\{v_{ij}, v_{ji}\}$.

In Fig. 14, we illustrate the change of the components $\xi_i(\lambda)$ of the leading left eigenvector $\boldsymbol{\xi}(\lambda)$, and the change of the consensual opinion $\tilde{x}(\lambda)$ with respect to the parameter $\lambda \in [\,0, 1\,]$. Note that the components $\xi_3(\lambda)$ and $\xi_4(\lambda)$, indicated with the same color, are indeed equal for all $\lambda \in [0, 1]$.

When the parameter $\lambda = 0$, we obtain the leading left eigenvector $\boldsymbol{\xi}_S = (25/132, 19/44, 25/132, 25/132)$ associated with the symmetric case, while when $\lambda = 1$, we obtain the leading left eigenvector $\boldsymbol{\xi} = (275/1302, 65/186, 143/651, 143/651)$ associated with the asymmetric case. Regarding the components of the leading left eigenvector $\boldsymbol{\xi}$, we can observe that the values of ξ_3 and ξ_4 coincide with respect to the parameter $\lambda \in [0, 1]$. Moreover, we can see that as the parameter λ increases, the values of ξ_1, ξ_3, ξ_4 increase, while ξ_2 decreases. Finally, we note that the consensual opinion $\tilde{x}(\lambda)$ increases as the parameter λ increases.

5 Concluding remarks

We have examined a linear model of consensus dynamics for a network of $n \geq 2$ interactive agents whose individual opinions are denoted by $x_i \in \mathbb{D} \subseteq \mathbb{R}$, with $i \in N = \{1, \ldots, n\}$. The distinctive element of our linear consensus model is the way in which the transition coefficients c_{ij}, with $i, j \in N$, are defined in terms of the null diagonal coefficients $v_{ii} = 0$ and independent off-diagonal coefficients $v_{ij}, v_{ji} \in (0, 1)$, with $i \neq j \in N$, plus the individual degree of proneness to evaluation review $u_i \in (0, 1)$, with $i, j \in N$.

The details of this construction have been introduced in Bortot et al. (2020a, b) and the asymptotic convergence to a consensual solution, indicated by $\tilde{x}$, has been discussed on the basis of a symmetry assumption regarding the interaction structure.

The present paper investigates the asymmetric generalization of the linear consensus dynamics model discussed in Bortot et al. (2020a, b), where the interaction structure was assumed to be symmetric, that is, identical coefficients $v_{ij} = v_{ji} \in (0, 1)$, with $i \neq j \in N$. In the general asymmetric case, on the other hand, with independent coefficients $v_{ij}, v_{ji} \in (0, 1)$, with $i \neq j \in N$, the analytic form of the asymptotic consensual solution $\tilde{x}$ is highly more complex than that under symmetric interaction, and we have obtained it only in the low-dimensional cases $n = 3, 4$. Nonetheless, we are able to write those complex analytic forms arranging the numerous terms in an intelligible way which might provide useful clues to the open quest for the analytic form of the asymptotic consensual solution in higher dimensional cases.

Considering the general asymmetric case of our linear consensus dynamics model, we have discussed its general properties and presented the main results on the asymptotic convergence to a consensual opinion given by $\tilde{x} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{x}$ which is a dynamical invariant constructed from the leading left eigenvector $\boldsymbol{\xi}$ of the transition matrix $\mathbf{C}$, with $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C} = \boldsymbol{\xi}^{\mathrm{T}}$, as in Proposition 2. The leading left eigenvector $\boldsymbol{\xi}$, in turn, is constructed from the leading left eigenvector $\boldsymbol{\zeta}$ of the matrix $\mathbf{B}$, with $\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\zeta}^{\mathrm{T}}$, plus the individual degrees of proneness, as in Proposition 1.

In this respect, in the general asymmetric case, we have obtained the analytic form of the leading left eigenvector $\boldsymbol{\zeta}$ in the low-dimensional cases $n = 3$ and $n = 4$, as in Proposition 3 and Proposition 4. Under symmetric interaction, the leading left eigenvectors $\boldsymbol{\zeta}$ reduce to those obtained in Bortot et al. (2020a, b), as expected.

In the context of the general asymmetric case of our linear consensus dynamics model, we have illustrated our results by examining in detail three versions of our linear model of consensus dynamics, depending on the relation between the interaction structure and the degrees of proneness to evaluation review of the various individual opinions. In the first

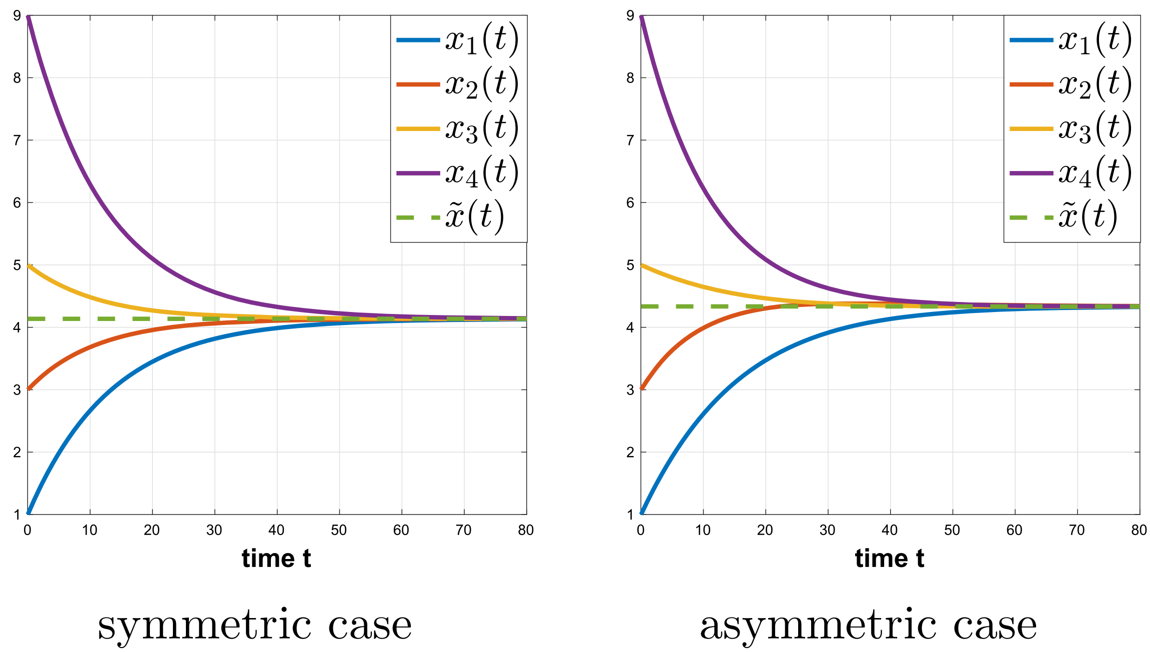

Fig. 13 Time evolution of the opinions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $\tilde{x}$

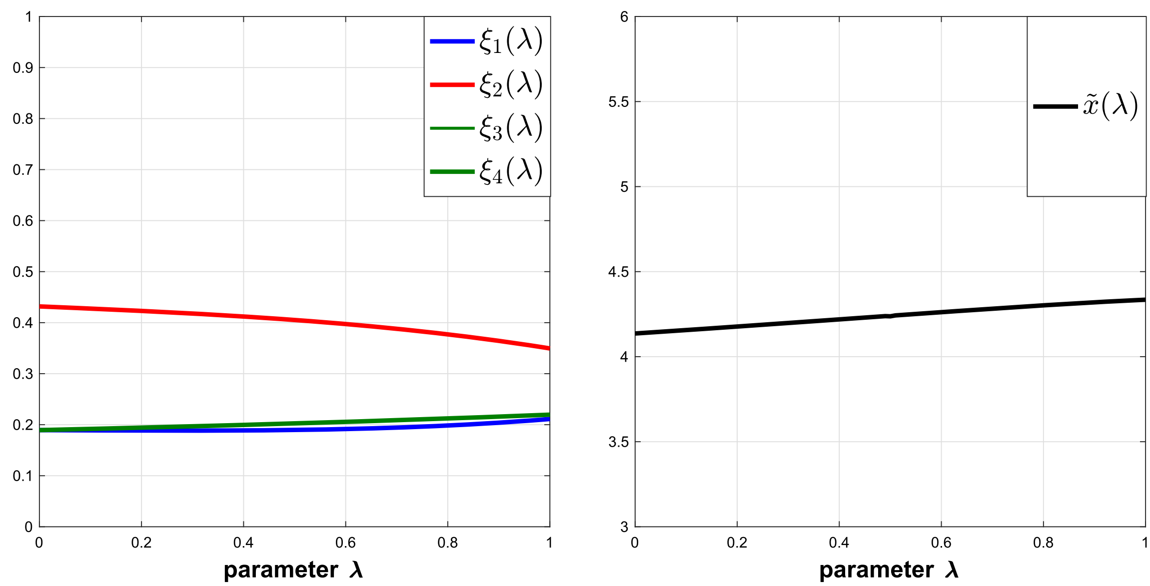

Fig. 14 The change of $\xi_1(\lambda)$, $\xi_2(\lambda)$, $\xi_3(\lambda)$, $\xi_4(\lambda)$ and the change of the consensual opinion $\tilde{x}(\lambda)$ as the parameter λ increases

version of the consensus dynamics, we assume a uniform proneness to evaluation review, in the second version, we assume that the proneness to evaluation review is aligned with the interaction structure, and in the third version, we assume that the proneness to evaluation review counter-aligned with the interaction structure.

Finally, we have presented some numerical simulations which are consistent with the theoretical framework and reveal interesting opinion dynamics under asymmetric interactions, with respect to the simpler symmetric case described in Bortot et al. (2020a, b).

In order to pursue this line of research, we need to have a better understanding of the combinatorial scheme which underlies the analytic form of the components of the leading left eigenvector $\boldsymbol{\zeta}$ of the row stochastic matrix $\mathbf{B}$, constructed from the general asymmetric interaction matrix $\mathbf{V}$. On the basis of the results in Proposition 3 and Proposition 4, it is now clear that the numerators p_i, with $i = 1, \ldots, n$, do not only depend on the corresponding contracted interaction coefficients $v_i = \sum_{j \in N} v_{ij}$ as in the symmetric case. In the general asymmetric case, we see that the numerators p_i, with $i = 1, \ldots, n$, also depend on the interaction coefficients v_{ji}, with $j \neq i \in N$. The precise form of this dependency in higher dimensions is now being investigated.

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Data availability The current study does not use any dataset. The graphs illustrated in the various figures use values which are generated in the simulations according to the formulas and parameters indicated in the text.

Declarations

Conflict of interest The authors declare that they have no relevant financial or non-financial interests to disclose.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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