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New and old sub-Riemannian challenges bridging analysis and geometry

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A chi si è posato sulla mia vita
come uccelli sulle spalle di Francesco d'Assisi

Introduction

The aim of this thesis is to propose a systematic exposition of some analytic and geometric problems arising from the study of sub-Riemannian geometry, Carnot-Carathéodory spaces and, more broadly, anisotropic metric and differential structures.

Carnot-Carathéodory spaces emerge naturally in connection with issues related to different areas of analysis and geometry, as well as in relation with applicative needs. Being a detailed historical introduction of the subject beyond the scope of this thesis, we limit ourselves to emphasise the relevance of the above, for instance, in the study of *degenerate elliptic* and *hypoelliptic PDEs* (cf. [172, 253, 232]), *analysis in metric spaces* (cf. [168, 170]), *geometric measure theory* (cf. [76, 136, 138, 150, 140]), *differential geometry* (cf. [215, 237, 166, 218]), *control theory* (cf. [5, 63, 3]) and *nonholonimic mechanics* (cf. [55, 284]). On the other hand, we refer to [186] for an interesting selection of instances where sub-Riemannian geometry finds application in concrete problems (cf. e.g. [179, 211, 38, 57, 282, 255, 56] for further insights).

Roughly speaking, Carnot-Carathéodory spaces are a particular class of *length spaces* whose distance is induced by suitable families of Lipschitz continuous vector fields. More precisely, consider an open set $\Omega \subseteq \mathbb{R}^n$ and a family of locally Lipschitz continuous vector fields

$$X = (X_1, \dots, X_m).$$

The heuristic nature of Carnot-Carathéodory spaces is quite intuitive: in a Carnot-Carathéodory space, the only movements allowed in Ω are those whose direction is tangent to the distribution generated by X. Accordingly, an absolutely continuous curve $\gamma : [0,1] \longrightarrow \Omega$ is *admissible*, or *horizontal*, if there exists a measurable function $a = (a_1, \dots, a_m) : [0,1] \longrightarrow \mathbb{R}^m$ such that

$$\dot\gamma(t) = \sum_{j=1}^{m} a_j(t) X_j(\gamma(t))$$

for almost every $t \in [0,1]$. The *Carnot-Carathéodory distance* is then defined by

$$d_\Omega(x,y) = \inf\left\{ \int_0^1 |a(t)|\, dt \;:\; \gamma : [0,1] \longrightarrow \Omega \text{ is horizontal, } \gamma(0) = x,\, \gamma(1) = y \right\} \tag{CC}$$

for any $x, y \in \Omega$. The well-posedness of (CC) relies on the existence, for any fixed $x, y \in \Omega$, of a horizontal curve *connecting* them. This property strongly depends both on the domain Ω and on the family X. For instance, it is clear that (CC) is not well-posed whenever Ω is not connected. On the other hand, when $\Omega = \mathbb{R}^2_{x_1,x_2}$ and $X = (X_1)$, where $X_1 = \frac{\partial}{\partial x_1}$, (CC) is not well-posed, since it is impossible to connect two points with different ordinate. When instead (CC) is well-posed, d_Ω is indeed a distance, and the couple (Ω, d_Ω) is called a *Carnot-Carathéodory space*. The well-posedness of (CC) translates intuitively into understanding to which extent X can *control* Ω. This process typically declines into distinguishing certain differential and algebraic properties that guarantee connectivity. For instance, when Ω is connected and X is *bracket-generating*, meaning that each X_j is smooth and

$$\mathrm{Lie}(\mathrm{span}\{X_1(p), \ldots, X_m(p)\}) = \mathbb{R}^n \tag{BG}$$

for any $p \in \Omega$, then (CC) is well-posed in view of the celebrated *Chow-Rashevskii connectivity theorem* (cf. [88, 232, 166]). Couples (Ω, X) for which (BG) holds constitute a special class of Carnot-Carathéodory spaces which becomes crucial to introduce sub-Riemannian geometry. Indeed, a *sub-Riemannian manifold* (cf. [4]) is, broadly speaking, a couple (Ω, X) which satisfies (BG) and for which a suitable *sub-Riemannian metric* is assigned on the distribution generated by X. Looking for models which allow for a richer structure, a particular class of sub-Riemannian manifolds, and hence of Carnot-Carathéodory spaces, consists of *Carnot groups* (cf. [54]), i.e. connected and simply connected Lie groups whose Lie algebra $\mathfrak{g}$ admits a suitable *stratification*. Namely, $\mathfrak{g}$ is *stratified* if there exist linear subspaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_k \neq \{0\}, \quad [\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}$$

for any $i = 2, \ldots, k$. In this case X consists in a basis of the *first layer*, $\mathfrak{g}_1$, of $\mathfrak{g}$. Heuristically, Carnot groups hold the same importance for sub-Riemannian geometry as the Euclidean space does for Riemannian geometry, since for instance they arise as suitable tangent cones to sub-Riemannian manifolds (cf. [215]). Although there are many interesting models for which (CC) is satisfied, and which consequently may fall within the general framework of analysis in metric spaces, differential geometry or Lie groups theory, we already know from our previous considerations that (CC) may fail in general. Consequently, as nicely explained in [219], when a couple (Ω, X) is fixed it is important to understand the relations occurring between those properties and objects that owe to the metric structure (Ω, d_Ω), and those which instead can be deduced and defined relying directly on X. This latter viewpoint can be tackled introducing a suitable notion of *X-gradient*, or *horizontal gradient*, which generalizes the Euclidean gradient reflecting the anisotropic nature of X. More precisely, to each function $u \in L^1_{loc}(\Omega)$ it is possible to associate the distribution Xu defined by

$$\langle Xu, \varphi \rangle := -\int_\Omega u \,\mathrm{div}(\varphi \cdot \mathcal{C})\, dx$$

for any $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$. Here $\mathcal{C}$ is the coefficient matrix associated with X, that is

$$\mathcal{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n} \\ c_{2,1} & c_{2,2} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m,1} & c_{m,2} & \dots & c_{m,n} \end{bmatrix}, \tag{CM}$$

where

$$X_j = c_{j,1}\frac{\partial}{\partial x_1} + \ldots + c_{j,n}\frac{\partial}{\partial x_n}$$

for any $j = 1, \ldots, m$, and where $c_{j,i}$ is a locally Lipschitz continuous function on Ω for any $j = 1, \ldots, m$ and any $i = 1, \ldots, n$. The X-gradient allows to propose a functional framework suitable for the problems of calculus of variations and PDEs. Indeed, by requiring suitable either boundedness or regularity properties of Xu, many classical functional spaces, e.g. $C^k(\Omega), W^{k,p}(\Omega)$ and $BV(\Omega)$, generalize respectively to $C^k_X(\Omega), W^{k,p}_X(\Omega)$ and $BV_X(\Omega)$. Consequently, notions such as perimeter, gradient, total variation, energy and area, as well as many related differential problems, do exist independently of a possible underlying metric structure. However, when X induces a Carnot-Carathéodory structure on Ω, it is reasonable to expect that this would affect the above functional setting, enriching it with some properties that are fundamental in the applications to calculus of variations and PDEs. For instance, the validity of appropriate *isoperimetric inequalities* and *Sobolev embeddings* (cf. [235, 76, 150]), as well matters of *differentiability* and *rectifiability* (cf. [237, 151, 140, 141]), relies heavily on the well-posedness of (CC), as well as on further metric and algebraic properties of the given space.

In what follows, we address some problems related to the general framework we have just introduced. Accordingly, this thesis consists of six parts. While Part I and Part II are in a sense preliminary to the subsequent ones, Part III, Part IV, Part V and Part VI constitute the core of this manuscript. The presentation follows a general-to-particular approach, as well as it is based, as far as possible, on a distinction of the macro-areas of belonging.

The starting point, namely Part I, proposes a short introduction to our anisotropic setting. More precisely, in Chapter 1 we introduce the general functional framework arising from a family of vector fields. We begin introducing the relevant horizontal differential operators in Section 1.1, such as the aforementioned horizontal gradient (cf. Definition 1.1.1) and the *horizontal divergence* (cf. Definition 1.1.2). Owing to these differential notions, in Section 1.2 we introduce the spaces of horizontally differentiable functions (cf. Definition 1.2.1), together with further relevant horizontal operators, such as the *horizontal Hessian* (cf. Definition 1.2.2) and the *horizontal Laplacian* (cf. Definition 1.2.4). In Section 1.3 we introduce the *horizontal Sobolev spaces* (cf. Definition 1.3.1), recalling some of their main properties. Finally, in Section 1.4 we briefly discuss the horizontal versions of notions like *bounded variation* (cf. Definition 1.4.1), *perimeter* (cf. Definition 1.4.4) and *unit normal* (cf. Definition 1.4.5).

In Chapter 2 we specialize our exposition to cover the Carnot-Carathéodory setting. Namely, in Section 2.1 we introduce Carnot-Carathéodory spaces (cf. Definition 2.1.2), together with some of their basic features. In Section 2.2, after a more precise discussion around the bracket-generating condition (BG) (cf. Definition 2.2.4), we briefly introduce the notion of sub-Riemannian manifold (cf. Definition 2.2.5) and the aforementioned connectivity theorem (cf. Theorem 2.2.6). Moreover, we provide an explicit instance which justifies the greater generality of the Carnot-Carathéodory setting with respect to the sub-Riemannian one (cf. Example 2.2.7). To conclude Chapter 2, in Section 2.3 we introduce the relevant Lipschitz (cf. Definition 2.3.1) and Hölder (cf. Definition 2.3.3) spaces, discussing their connection with Sobolev spaces (cf. Proposition 2.3.2) and the validity of appropriate *embedding theorems* (cf. Proposition 2.3.4) and Poincaré inequalities (cf. Theorem 2.3.1).

Finally, in Chapter 3 we lay out the basic background in order to present the Carnot groups environment. After some preliminaries from the Lie groups theory (cf. Section 3.1), we introduce the notions of *stratified algebra* (cf. Definition 3.2.1) and of *Carnot group* (cf. Definition 3.2.3). A relevant feature of Carnot groups is that, differently from an arbitrary Lie group, they are isomorphic in an appropriate sense to their Lie algebra via *exponential map* (cf. Theorem 3.2.5). Accordingly, as explained in Section 3.3, Carnot groups can be naturally identified with non-Abelian structures of $\mathbb{R}^n$ via *exponential coordinates* (cf. Definition 3.2.6). Moreover, they are naturally associated with suitable *intrinsic dilations* (cf. Definition 3.3.1), which are well-behaved with their Lie algebra stratification. Consequently, this rich structure allows to consider some special distances and norms (cf. Section 3.4), namely *invariant distances* (cf. Definition 3.4.1) and *homogeneous norms* (cf. Definition 3.4.2), some of which are particularly relevant in the following developments. We conclude Chapter 3 recalling some basic notions of geometric measure theory in Carnot groups. Precisely, after introducing some relevant measures (cf. Section 3.5), we discuss an intrinsic notion of *rectifiability* (cf. Definition 3.6.3) introduced in the pioneering work [140], together with the related notion of $\mathbb{G}$-*regular hypersurface* (cf. Definition 3.6.1).

In Part II we present some new differentiability results in the Carnot-Carathéodory setting which, despite being propaedeutic to many subsequent discussions, may have an independent interest. First of all, in Chapter 4, we introduce the notion of (X, N)-*subgradient* of functions in $W^{1,\infty}_{X,loc}(\Omega)$ (cf. Definition 4.1.1), a set-valued map which arises as a generalization of the classical Clarke's *subdifferential* (cf. [92]). More precisely, given $u \in W^{1,\infty}_{X,loc}(\Omega)$ and a Lebesgue-negligible set $N \subseteq \Omega$ containing the non-Lebesgue points of Xu, the (X, N)-subgradient of u is defined by

$$\partial_{X,N}u(x) := \overline{co}\left\{\lim_{n\to\infty} Xu(y_n) \,:\, y_n \to x,\, y_n \notin N \text{ and } \lim_{n\to\infty} Xu(y_n) \text{ exists}\right\}, \qquad (X\text{-SUB})$$

where $\overline{co}$ denote the closure of the convex envelope (cf. Chapter 4 for more precise definitions). As a motivating feature of the (X, N)-subgradient, which reduces to the X-gradient for functions in $C^1_X(\Omega)$ (cf. Proposition 4.1.4), we show that it is the right object to deal with derivatives of functions along horizontal curves (cf. Proposition 4.2.1 and Proposition 4.2.2). In addition, in Section 4.3 we discuss some of its properties in connection with optimization problems, among which we mention a suitable anisotropic weak version of Fermat theorem (cf. Proposition 4.3.3).

Subsequently, in Chapter 5, we introduce a notion of differentiability for functions in $C^1_X(\Omega)$ which fits into a series of parallel findings in related settings (cf. [237, 210, 225]). Roughly speaking, given a family X of linearly independent vector fields satisfying (BG), $u \in C^1_X(\Omega)$ and $x \in \Omega$, we prove the existence of a linear mapping $d_X u(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\lim_{d_\Omega(x,y)\to 0} \frac{u(y) - u(x) - d_X u(x)(y-x)}{d_\Omega(x,y)} = 0. \tag{X-DIF}$$

Moreover, we provide an explicit characterization of $d_X u(x)$ (cf. Theorem 5.3.1), namely

$$d_X u(x)(z) = \langle Xu(x) \cdot \tilde{\mathcal{C}}(x), z \rangle,$$

where $\tilde{\mathcal{C}}$ is an appropriate left-inverse of $\mathcal{C}^T$, the transpose of the coefficient matrix (CM), This new notion of X-*differential* (cf. Definition 5.1.4) generalizes the celebrated *Pansu differential* (cf. [237]) beyond the Carnot group setting. Anyway, differently from the latter, it may be non-unique in general (cf. Remark 5.3.2).

Part III is devoted to some issues of *integral representation* and Γ-*convergence* within the general anisotropic setting introduced in Chapter 1. Since its introduction in the seminal papers [114, 115], the variational tool of Γ-convergence has proved to be of fundamental importance in the development of modern analysis ([59, 60, 105]) and in solving problems arising from applications, including phase transitions, elasticity and the theory of fractures ([58, 116, 146]). Roughly speaking, if M is a first-countable metric space and $(F_h)_h$ and F are extended real-valued functions defined on M, then F_h Γ-convergence to F whenever

$$F(x) \leqslant \liminf_{h\to\infty} F_h(x_h)$$

for any $x \in M$ and any sequence $(x_h)_h \subseteq M$ converging to x, and if

$$F(x) \geq \limsup_{h\to\infty} F_h(x_h)$$

for any $x \in M$ and for a suitable sequence $(x_h)_h \subseteq M$ converging to x. A remarkable instance can be found in [104, 66, 68, 67], where the authors studied properties of integral representation and Γ-convergence of *local functionals* defined over classical Euclidean functional spaces. By integral representation one means finding conditions under which an arbitrary functional $F(u, A)$, being u a function and A a set, can be expressed as the integral of a suitable *Lagrangian*

$f(x, u, Du)$ over the set A, meaning that

$$F(u, A) = \int_A f(x, u(x), Du(x))\, dx. \tag{IR}$$

On the other hand, by Γ-compactness one usually means showing that, as soon as a sequence $(F_h)_h$ belongs to a certain class of functionals, then it converges, up to a subsequence, to a functional belonging to the same class. Specializing this notion to our specific interests, a sequence of integral functionals $(F_h)_h$ as in (IR) is Γ-compact as soon as it converges, up to a subsequence, to an integral functional as in (IR). Starting from [138, 150], these and many other typical problems of the calculus of variations have been transposed into the context of variational functional defined starting from suitable families of vector fields (cf. [139, 225, 24, 50, 51]), so that they can be tackled owing to the functional framework introduced in Chapter 1. Recently, the authors of [205, 206] generalized the integral representation and Γ-compactness results proved in [68] to the anisotropic setting, showing properties of integral representation and Γ-compactness of *translation invariant* functionals depending on vector fields X satisfying the so-called *linear independence condition.* More precisely, assuming that

$$F(u + c, A) = F(u, A) \tag{TI}$$

for any $c \in \mathbb{R}$, and that

$$X_1(x), \ldots, X_m(x) \text{ are linearly independent} \tag{LIC}$$

for almost every $x \in \Omega$, they showed the existence of a suitable anisotropic Lagrangian $f(x, \eta)$ for which, under standard conditions,

$$F(u, A) = \int_A f(x, Xu(x))\, dx. \tag{X-IR}$$

Under the same assumptions, the authors of [205, 206] provides Γ-compactness properties for some classes of integral functionals as in (X-IR). Our aim is to extend the results of [205, 206] by avoiding both (TI) and (LIC).

We begin our exposition with Chapter 6, where, after a detailed introduction to our setting, we propose some well-known definitions and preliminaries concerning local functionals (cf. Section 6.2) and Γ-convergence (cf. Section 6.4).

In Chapter 7 we deal with integral representation issues, avoiding the translation invariant property (TI) but still assuming (LIC). Our general strategy, as discussed in detail in Section 7.1, while relying on the corresponding Euclidean results is structured along a different path. The main reason for this difference is that, unlike in the Euclidean setting, it is in general not possible to approximate horizontal Sobolev functions by means of suitable *horizontally affine functions* (cf. Section 7.2). Therefore, according to the original approach of [205], our strategy is divided into three main steps. First, given an abstract local functional, we achieve an Euclidean integral representation as in (IR), whence in terms of an Euclidean Lagrangian f_e. In the second crucial step, exploiting some algebraic properties of the coefficient matrix (CM) provided by (LIC) (cf. Section 7.3), we show the existence of a suitable anisotropic Lagrangian f such that

$$f(x,u,Xu) = f_e(x,u,Du) \tag{E→ X}$$

for any sufficiently smooth function u. Finally, in the third step, we extend the anisotropic integral representation for regular functions provided by (IR) and (E→ X) to the whole anisotropic Sobolev space considered. This procedure is carried out for three classes of functionals. Precisely, in Section 7.4, we begin characterizing convex local functionals in terms of integral functionals associated with a convex Lagrangian (cf. Theorem 7.4.1 and Theorem 7.4.2). Moreover, in Section 7.5 we avoid the convexity assumption, dealing with *weakly-* sequentially lower semicontinuous* functionals (cf. Theorem 7.5.2 and Theorem 7.5.3). Finally, in Section 7.6 we treat general non-convex functionals (cf. Theorem 7.6.1 and Theorem 7.6.2). As explained more in details in Chapter 7, the non-convex environment is by far the most challenging, since many techniques employed in [205] do not apply. To this aim, we introduce a suitable notion of *X-convexity* (cf. Definition 7.6.3) together with a new *zig-zag* argument (cf. Lemma 7.6.5).

The generalization of [205] continues in Chapter 8, where we address issues of Γ-compactness for sequences of integral functionals, with respect to the metric topologies of $L^p(\Omega)$ and $W^{1,p}_X(\Omega)$, again avoiding (TI) and assuming (LIC). More precisely, we achieve two Γ-compactness results for sequences of integral functionals associated with a convex Lagrangian, with respect to both the strong L^p-topology (cf. Theorem 8.2.1) and the strong $W^{1,p}_X$-topology (cf. Theorem 8.3.1). In addition, in the $W^{1,p}_X$-environment, we also cover the non-convex case (cf. Theorem 8.3.2). Our approach require the introduction of some new *ad hoc* notions, among which suitable versions of the well-known *fundamental estimate* (cf. Definition 8.3.4) and *strong conditions* (cf. Definition 8.3.1). Some of our main results and considerations, such as e.g. the Γ-limiting behaviour of the strong condition (cf. Proposition 8.3.9) are, to the best of our knowledge, new even in the Euclidean setting.

Finally, in Chapter 9 we generalize the results proved in Chapter 7 and Chapter 8 avoiding the linear independence condition (LIC). The crucial point in this generalization consists in showing the validity of (E→ X) without exploiting (LIC). Indeed, when (LIC) fails, it is still possible to associate an Euclidean Lagrangian f_e with an anisotropic counterpart f (cf. Section 9.3). This generalization, which is carried out in Section 9.3.2, relies on some new properties of the so-called *Moore-Penrose pseudo-inverse* associated with the coefficient matrix (CM). In Section 9.4 we provide our integral representation and Γ-compactness results in the prototypical (TI) setting, and, most importantly, we discuss some differences between the (LIC) framework and our greater generality. The fairly general perspective that we can propose in light of the efforts of Chapter 9 allows to consider functionals driven by arbitrary anisotropies, thus extending the previous considerations to cover, for instance, the greatest generality of the Carnot-Carathéodory setting. Another main reason to avoid (LIC) becomes evident in the investigation of Γ-compactness properties of sequences of integral functionals $(F_h)_h$ when a fixed family X of vector fields is replaced by a sequence $(X_h)_h$ of families of vector fields, as explained in detail in Example 9.2.2.

Part IV, which enjoys a more PDE-based flavor, proposes some topics related to the theory of viscosity and Monge solutions to first and second-order PDEs in the Carnot-Carathéodory setting. The classical theory of *viscosity solutions*, which we briefly introduce in Chapter 10, is an extremely flexible tool which allows to deal with a broad class of *fully nonlinear* first and second-order partial differential equations (cf. [97, 98, 95, 96]). On the one hand, its strength lies in the possibility of admitting merely continuous solutions. On the other hand, the very mild assumptions on the structure of the equations guarantee flexibility in the existence, comparison and stability results. The typical continuous function $F(x, u, p, A)$ associated with the equation

$$F(x, u, Du.D^2u) = 0$$

is assumed to be decreasing in A, or *degenerate elliptic* (cf. [96]), meaning that

$$F(x, u, p, A) \leqslant F(x, u, p, B)$$

as soon as $A - B$ is positive semi-definite. This weak ellipticity assumption clearly includes the degenerate case of *Hamilton-Jacobi equations*, i.e. first-order equations of the form

$$H(x, u, Du) = 0.$$

Despite the power of the theory of viscosity solutions, in many cases this framework is not the proper one to establish existence results even in the study of Hamilton-Jacobi equations. Indeed, if the *Hamiltonian* H is not continuous, the classical *Hopf-Lax formula* for the Dirichlet problem (cf. [198]) may fail to provide a solution (cf. [233]). To this aim, the authors of [233] introduced the notion of *Monge solution* to discontinuous *Eikonal-type* equations. The latter notion, metric in spirit, is defined starting from the so-called *optical length function*. In [62] the authors generalized the previous results by considering general Hamilton-Jacobi equations

$$H(x, Du) = 0.$$

Here, the Hamiltonian H satisfies typical convexity and coercivity assumptions. More recently, the theory of viscosity solutions has been generalized to the Carnot-Carathéodory framework (cf. [208, 31, 40, 278] and references therein) under suitable *horizontal degenerate ellipticity* assumptions (cf. Definition 10.2.1). Accordingly, if for instance $H : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous function, we say that a lower semicontinuous function $u : \Omega \longrightarrow \mathbb{R}$ is a *viscosity* *subsolution* to

$$H(x, u, Xu) = 0 \tag{HJ}$$

if

$$H(x_0, u(x_0), X\varphi(x_0)) \leqslant 0$$

for any $x_0 \in \Omega$ and for any $\varphi \in C^1_X(\Omega)$ such that

$$u(x_0) - \varphi(x_0) \geq u(x) - \varphi(x)$$

for any x in a neighborhood of x_0. The definitions of supersolution and solution (cf. Definition 10.2.3), as well as the extension of the latter to the second-order case (cf. Definition 10.2.4), are similar and can be found in more detail in Section 10.2. The importance of this generalization, apart from its intrinsic theoretical interest, consists in the fact that a sub-Riemannian viewpoint typically allows to avoid certain coercivity conditions which are usually required in the Euclidean framework (cf. [269]).

In Chapter 11 we extend some well-known Euclidean results relating different notions of solution to Hamilton-Jacobi equations, namely viscosity solutions, *jet solutions* and *almost everywhere solutions*. While the concept of almost everywhere solution is very intuitive and easily extendable to the anisotropic setting, the notion of jet (cf. Definition 11.1.1), as happens in the Euclidean setting (cf. [96]), is based on the differentiability structure related to the chosen environment, whence our new notion of differentiability (X-DIF) introduced in Definition 5.1.4 plays a crucial role. In the great generality of the Carnot-Carathéodory setting, owing to the differentiability result proved in Theorem 5.3.1, we show that jet solutions to (HJ) are viscosity solutions to (HJ) (cf. Proposition 11.1.3). More remarkably, under the additional requirement (BG) and assuming some mild convexity properties on the Hamiltonian, we prove that almost everywhere subsolutions to (HJ) are indeed jet subsolutions to (HJ) (cf. Theorem 11.1.1). Te proof of Theorem 11.1.1, which exploits an appropriate lifting argument à la Rothschild-Stein (cf. [253]), strongly rely on the interplay which occurs between X-differentiability and the (X, N) subgradient (cf. Proposition 11.1.4). Finally, when we restrict to the Carnot group setting, we show that jets solutions to (HJ) and viscosity solutions to (HJ) coincide (cf. Proposition 11.2.1), and that, under the same convexity assumptions as above, viscosity subsolutions to (HJ) are actually almost everywhere subsolutions to (HJ) (cf. Proposition 11.2.2). Finally, again in a Carnot group, we show that the sub-Riemannian definition of viscosity solution is equivalent to its Euclidean counterpart (cf. Proposition 11.2.3).

Chapter 12 and Chapter 13 are devoted to a relevant class of second-order differential equations to which the viscosity theory applies, which arises in the study of the so-called *absolute minimizers* of *supremal functionals* (cf. [37]). Broadly speaking, a supremal functional

$$F(u) = \|f(x, u, Du)\|_\infty$$

with respect to a continuous integrand $f = f(x, u, p)$ (cf. Definition 12.2.1) is the L^∞ version of a variational integral functional, and an absolute minimizer (cf. Definition 12.2.2) is a Lipschitz function which locally minimizes the supremal functional among all Lipschitz functions with the same boundary datum. The most and earliest studied supremal functional is $\||\nabla u|^2\|_\infty$, in connection with the so-called *Lipschitz extension problem* (cf. [20, 21]). Its absolute minimizers, which are known as *absolute minimizing Lipschitz extensions*, are viscosity solutions to the well-known *infinite Laplace equation*

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0 \tag{∞-LAPLACE}$$

(cf. [175, 23], and cf. also [23] for a survey about absolutely minimizing Lipschitz extensions). In [37, 94], the authors generalized the previous result considering general supremal functionals, and showed that absolute minimizers are viscosity solutions to the so-called *Aronsson equation* (cf. Definition 12.2.3)

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f(x, u, Du)) \frac{\partial f}{\partial p_i}(x, u, Du) = 0,$$

which again can be seen as an L^∞ counterpart of the classical *Euler-Lagrange equation* associated to an integral functional. The appropriate preliminaries regarding supremal functionals, absolute minimizers and Aronsson equations are provided in Section 12.2. This result was later generalized in [278, 279] to the setting of Carnot-Carathéodory spaces. More precisely, the authors of [278, 279] considered supremal functionals associated with integrands $f = f(x, p)$, thus avoiding the function dependence, and derived an Aronsson equation of the form

$$-\sum_{i=1}^{m} X_i(f(x, Xu)) \frac{\partial f}{\partial p_i}(x, Xu) = 0$$

under some convexity and homogeneity assumptions on f. In Chapter 12, we generalize these results allowing complete integrands $f = f(x, u, p)$ of class C^2, thus providing a complete extension of [37, 94] to the anisotropic setting (cf. Theorem 12.3.1). More precisely, avoiding the aforementioned homogeneity assumptions, we show that absolute minimizers of the supremal functional

$$F(u) = \|f(x, u, Xu)\|_\infty$$

are viscosity solutions to the anisotropic Aronsson equation

$$-\sum_{i=1}^{m} X_i(f(x, u, Xu)) \frac{\partial f}{\partial p_i}(x, u, Xu) = 0.$$

Allowing the second variable dependence of f prevents the corresponding Aronsson equation from being *proper* in the sense of [96, (0,2)] (cf. Definition 10.2.2). This last property is often crucial in many results of the viscosity theory (cf. [96]). In order to prove Theorem 12.3.1, we strongly rely on our new notion of (X, N)-subgradient (X-SUB) introduced in Definition 4.1.1, an namely on the aforementioned possibility of exploiting the latter to differentiate Lipschitz function along horizontal curves, as stated in Proposition 4.2.1.

In Chapter 13 we specialize our study of anisotropic Aronsson equations considering the anisotropic infinite Laplace equation

$$\Delta_{X,\infty} u = \sum_{i,j=1}^{m} X_i X_j u X_i u X_j u = 0. \tag{X-∞-LAPLACE}$$

In [22], the author derived the Euclidean infinite Laplace equation (∞-LAPLACE) as a formal limit of *p-Laplace equations*

$$-\operatorname{div}(|Du|^{p-2} Du) = 0$$

when $p \to \infty$. This approach was made rigorous in [39], where the authors exploited the viscosity theory to study the limiting behaviour of weak solutions to the Dirichlet problem associated with the *p-Poisson equation*

$$-\operatorname{div}(|Du|^{p-2} Du) = f$$

when $p \to \infty$. The aim of Chapter 13 is to generalize the results in [39] to the setting of Carnot-Carathéodory spaces generated by a family of vector fields satisfying (BG). More precisely, in Section 13.3 we prove that a sequence of weak solutions to the *sub-elliptic* p-Laplace equation

$$-\operatorname{div}_{\mathrm{X}}(|Xu|^{p-2} Xu) = 0,$$

being $\operatorname{div}_{\mathrm{X}}$ the aforementioned horizontal divergence (cf. Definition 1.1.2), converges, up to a subsequence and with respect to suitable topologies, to an absolutely minimizing Lipschitz extension which solves in the viscosity sense the sub-elliptic infinite Laplace equation (X-∞-LAPLACE) (cf. Theorem 13.1.1). Moreover, in Section 13.4 we address the non-homogeneous case, that is understanding the limiting behavior of weak solutions to the anisotropic *p-Poisson equation*

$$-\operatorname{div}_{\mathrm{X}}(|Xu|^{p-2} Xu) = f, \tag{X-p-POISSON}$$

where f is a non-negative source, obtaining in the limit a viscosity solution to a mixed first and second-order differential problem involving the so-called *Eikonal equation* (cf. Theorem 13.1.2), namely

$$\begin{cases} \Delta_{X,\infty} u = 0 & \text{on } \overline{\{f > 0\}}^c, \\ |Xu| = 1 & \text{on } \{f > 0\}. \end{cases}$$

The proof of this last result, which is by far the most demanding, exploits techniques which are different from the Euclidean approach of [39]. To this aim, a crucial role is played by the above-mentioned differentiability result stated in Theorem 5.3.1, as well as the comparison between viscosity and almost everywhere solutions shown in Theorem 11.1.1. Moreover, a relevant part of Chapter 13, namely Section 13.2, is devoted to establish some basic properties of the Dirichlet problem associated to (X-p-POISSON), among which we recall the existence of weak solutions (cf. Proposition 13.2.1), maximum and comparison principles (cf. Lemma 13.2.2) and a comparison between weak and viscosity solutions (cf. Proposition 13.2.3).

To conclude Part IV, in Chapter 14 we generalize the aforementioned notion of Monge solution for possibly discontinuous Hamilton-Jacobi equations to the setting of Carnot groups (cf. Definition 14.1.1), by considering Hamilton-Jacobi equations of the general form

$$H(x, Xu) = 0.$$

After deriving a Hopf-Lax formula for the Dirichlet problem under suitable compatibility conditions (cf. Theorem 14.1.3), we show a comparison principle (cf. Theorem 14.1.4) and a stability result (cf. Theorem 14.1.5). Moreover, we show that the notions of Monge and viscosity solution coincide as soon as the Hamiltonian is continuous (cf. Theorem 14.1.2). This generalization has required a considerable effort, since in many cases most of the Euclidean approach does not work. The main reason is that the sub-Riemannian optical length function (14.1.5) is not globally geodesic in general (cf. Section 14.2). This fact has required the introduction of some delicate localization arguments (cf. e.g. Proposition 14.3.2), and strongly relies on some of the results presented in Chapter 11.

Part V and Part VI constitute a path with a more geometric flavour, and are focused on structural properties of minimizers of suitable geometric functionals in the Carnot groups setting. Despite Part V contains only one chapter, namely Chapter 15, we preferred to keep it separate from Part VI, both because of the difference between the approaches that we employed and because the latter focuses exclusively on the Heisenberg group. In Part V, and hence in Chapter 15, we deal with regularity properties for almost minimizer of the anisotropic perimeter associated to a special class of Carnot groups of step 2. In order to introduce our results, we recall that a key step in the regularity theory for Euclidean perimeter minimizers is the validity of the so-called *Lipschitz approximation theorem* (cf. [202]), which, roughly speaking, states that boundaries of perimeter minimizers are close in measure to graphs of suitable Lipschitz functions. This result has been generalized in [221, 227] to the setting of Heisenberg groups. Our main result, namely Theorem 15.1.1, generalizes the results of [221] to a class of step-2 Carnot groups which we called *plentiful groups* (cf. Definition 15.6.1). Roughly speaking, a step-2 Carnot group is plentiful if, denoting by $\mathfrak{g}$ its Lie algebra, then

$$\mathrm{Lie}(V) = \mathfrak{g}$$

for any 1-codimensional linear subspace of the first layer $\mathfrak{g}_1$. This new class comprehends several relevant classes of Carnot groups, among which we mention the class of *H-type groups* (cf. Theorem 15.6.3) and some more general groups (cf. Example 15.6.4). The main issue in general Carnot groups is that, unlike in $\mathbb{R}^n$, sets which locally constant horizontal normal (Definition 1.4.5) are not locally flat in the intrinsic sense of the geometry of Carnot groups. However, as we show in Theorem 15.6.6, this degeneration cannot occur in plentiful groups. Hence, in light of an appropriate intrinsic area formula (cf. Theorem 15.7.8) the Euclidean approach to regularity can be carried out even in this anisotropic setting.

Finally, in Part VI we specialize our treatment to the Heisenberg group, dealing with matters of existence, uniqueness and rigidity for hypersurfaces of prescribed mean curvature, both from a Riemannian and a sub-Riemannian point of view. The Heisenberg group $\mathbb{H}^n$, for $n \geq 1$, is $\mathbb{R}^{2n+1}$ endowed with the group law

$$p \cdot p' = (\bar{x}, \bar{y}, t) \cdot (\bar{x}', \bar{y}', t') = \left(\bar{x} + \bar{x}', \bar{y} + \bar{y}', t + t' + \sum_{j=1}^{n} \left(x_j' y_j - x_j y_j' \right) \right),$$

where we denoted points $p \in \mathbb{R}^{2n+1}$ by $p = (\bar{x}, \bar{y}, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$, which realizes it as a Carnot group. Its Lie algebra is generated by the left-invariant vector fields

$$Z_j = X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \qquad Z_{n+j} = Y_j = \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial t} \qquad \text{and} \qquad T = \frac{\partial}{\partial t}.$$

for $j = 1, \ldots, n$. Consequently, $\mathbb{H}^n$ is associated with a bracket-generating *horizontal distribution*

$$\mathcal{H} = \operatorname{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}.$$

$\mathbb{H}^n$ can be endowed with a Riemannian structure by choosing, for any $\varepsilon \neq 0$, the unique Riemannian metric g_ε which makes $X_1, \ldots, X_n, Y_1, \ldots, Y_n, \varepsilon T$ orthonormal at every point. The importance of $(\mathbb{H}^n, g_\varepsilon)$ in the Riemannian framework is supported by several reasons. For instance, it appears in the classification of homogeneous 3-spaces with isometry group of dimension 4 (cf. [1]). When ε goes to 0, the space $(\mathbb{H}^n, g_\varepsilon)$ converges in Gromov-Hausdorff sense to the sub-Riemannian Heisenberg group $(\mathbb{H}^n, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the restriction of any of the metric g_ε to the horizontal distribution $\mathcal{H}$. The study of algebraic and geometric properties of the sub-Riemannian Heisenberg group is itself of fundamental importance in various settings, since $\mathbb{H}^n$ constitutes the prototypical model in the context of Carnot groups, sub-Riemannian manifolds and *CR manifolds* (cf. [77]). Likewise, the rich algebraic and geometric structure of $\mathbb{H}^n$ makes it a frenetic crossroads in between geometric measure theory, differential geometry, geometric analysis, calculus of variations and partial differential equations.

In Chapter 16 we begin collecting some basic preliminaries around the Heisenberg setting. After introducing the relevant algebraic and differential structure in Section 16.2, we present the main properties of *Riemannian Heisenberg groups* (cf. Section 16.3), with special regard to certain features of embedded Riemannian hypersurfaces (cf. Section 16.4), for which the classical tools of Riemannian geometry are specialised to our context. Afterwords, we introduce the relevant sub-Riemannian structure (cf. Section 16.5). One of the major tools in this setting is the so-called *pseudohermitian connection* $\nabla^{\mathbb{H}}$ (cf. e.g. [251]), which plays the role of the classical *Levi-Civita* connection and may be viewed as a *flat* connection on $\mathbb{H}^n$ (cf. (16.5.2)). As in the Riemannian case, we then turn our attention to the study of sub-Riemannian hypersurfaces (cf. Section 16.6). A crucial role for a sufficiently smooth sub-Riemannian hypersurface $S \subseteq \mathbb{H}^n$ of class C^1 is played by its *characteristic set* S_0 (cf. Definition 16.6.2). Roughly speaking, S_0 is the sets of points where the tangent space of S coincides with the horizontal distribution, i.e.

$$S_0 := \{p \in S : \mathcal{H}_p = T_p S\} .$$

Accordingly, a hypersurface is *non-characteristic* whenever $S_0 = \emptyset$. Outside the characteristic set, it is possible to give a pointwise meaning to the aforementioned notion of horizontal normal, which we denote by $\nu^{\mathbb{H}}$ in this particular setting. Roughly speaking, $\nu^{\mathbb{H}}$ is a unit horizontal vector field which is orthogonal to the horizontal part of the tangent space. Exploiting these notions, it is customary to mimic the Riemannian approach to introduce and study some relevant sub-Riemannian tensorial objects, among which the *horizontal mean curvature* $H^{\mathbb{H}}$ (cf. Definition 16.6.6) and the *horizontal second fundamental form* $h^{\mathbb{H}}$ (cf. Definition 16.6.5). A first striking difference with the Riemannian environment is the lack of symmetry of the horizontal second fundamental form $h^{\mathbb{H}}$ (cf. e.g. [106]), which led many authors to deal with a related symmetrized form $\tilde{h}^{\mathbb{H}}$. This and other properties of $h^{\mathbb{H}}$ and $\tilde{h}^{\mathbb{H}}$ are discussed in detail in Section 16.6.3. In Section 16.7, we introduce some relevant classes of hypersurfaces, namely *t-graphs* (cf. Section 16.7.1), *intrinsic graphs* (cf. Section 16.7.2) and *intrinsic cones* (cf. Section 16.7.3). The sub-Riemannian structure of $\mathbb{H}^n$, according to the general notion introduced in Definition 1.4.4, allows to define a variational notion of *horizontal perimeter* $P_{\mathbb{H}}$ (cf. [140]), according to the classical De Giorgi's definition in the Euclidean setting (cf. [111]). More precisely, if $\Omega \subseteq \mathbb{H}^n$ is open and $E \subseteq \mathbb{H}^n$ is measurable, we recall (cf. [140, 150]) that the $\mathbb{H}$-*perimeter* of E in Ω is defined by

$$P_{\mathbb{H}}(E, \Omega) := \sup \left\{ \int_E \mathrm{div}_{\mathbb{H}}(\varphi)\, d\mathcal{L}^{2n+1} \; : \; \varphi \in C^1_c(\Omega, \mathcal{H}),\, \langle \varphi, \varphi \rangle \leqslant 1 \right\},$$

where by $C^1_c(\Omega, \mathcal{H})$ we denote the class of compactly supported C^1 sections of the horizontal distribution $\mathcal{H}$, and $\mathrm{div}_{\mathbb{H}}$ is the so called *horizontal divergence* associated with the sub-Riemannian Heisenberg group (cf. Definition 1.1.2). The main properties of $P_{\mathbb{H}}$, also in connection with the perimeter functionals arising from both the Riemannian and the Euclidean structures of $\mathbb{H}^n$, are discussed in Section 16.8.

In Chapter 17, we deal with some Riemannian and sub-Riemannian variational properties of non-characteristic hypersurfaces. Although the results of this section do not constitute a novelty in the existing literature (cf. [261, 152, 216, 217]), we preferred to provide an exposition that, when compared to the original sources, is better tailored to the specific setting of the Heisenberg group. The aim of Section 17.2 and Section 17.3 is to show how the sub-Riemannian first and second variation formulas for the perimeter functional arise as limits of corresponding

Riemannian formulas. As it is well-known (cf. e.g [270]), the two key ingredients of the second variation formula for the area of a smooth hypersurface S in the Riemannian Heisenberg group $(\mathbb{H}^n, g_\varepsilon)$ are its squared mean curvature, say $(H^\varepsilon)^2$, and the term

$$\mathrm{Ric}_\varepsilon(\nu^\varepsilon, \nu^\varepsilon) + |h^\varepsilon|^2, \tag{SV}$$

being Ric_ε the *Ricci tensor* of $(\mathbb{H}^n, g_\varepsilon)$, $|h^\varepsilon|^2$ the squared norm of the *second fundamental form* of S and ν^ε the Riemannian unit normal to S. In Section 17.2, (SV) is explicitly computed for any $\varepsilon \neq 0$. An interesting consequence of these computations is that, although both terms appearing in (SV), when considered separately, diverges when $\varepsilon \to 0$, Theorem 17.1.2 ensures their convergence to the sub-Riemannian term

$$q = \sum_{h,k=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) Z_k(\nu_h^{\mathbb{H}}) + 4\langle J(\nu^{\mathbb{H}}), \nabla^{\mathbb{H}}(Td^{\mathbb{H}})\rangle + 4n(Td^{\mathbb{H}})^2. \tag{q}$$

In (q), J is a suitable sub-Riemannian rotation operator (cf. (16.2.2), while $d^{\mathbb{H}}$ is the horizontal Carnot-Carathéodory distance from S induced by the horizontal distribution $\mathcal{H}$. The crucial role of (q) in the sub-Riemannian second variation formula is discussed in Section 17.3, where an explicit proof of the latter is provided by Theorem 17.1.1. In Section 17.4 and Section 17.5, we show how the quantities (SV) and (q) appears, respectively, in the Riemannian and in the sub-Riemannian *Jacobi equation*, a suitable second-order differential equation satisfied by the vertical component of the unit normal to a sufficiently smooth hypersurface. More precisely, the Riemannian Jacobi equation (cf. Theorem 17.1.3) reads as

$$\Delta^{\varepsilon,S}(\nu_{2n+1}^\varepsilon) = g_\varepsilon\left(\nabla^{\varepsilon,S} H^\varepsilon, \varepsilon T\right) - \nu_{2n+1}^\varepsilon\left(\mathrm{Ric}_\varepsilon(\nu^\varepsilon, \nu^\varepsilon) + |h^\varepsilon|^2\right), \tag{ε-JE}$$

where $\nabla^{\varepsilon,S}$ and $\Delta^{\varepsilon,S}$ are, respectively, the Laplace-Beltrami operator and the Riemannian gradient associated with the Riemannian manifold $(S, g_\varepsilon|_S)$ (cf. Section 16.4.3). On the other hand, the sub-Riemannian Jacobi equation (cf. Theorem 17.1.4) reads as

$$\hat{\Delta}^{\mathbb{H},S}(Td^{\mathbb{H}}) = TH^{\mathbb{H}} - Td^{\mathbb{H}}\langle \nabla^{\mathbb{H}} H^{\mathbb{H}}, \nu^{\mathbb{H}}\rangle - qTd^{\mathbb{H}}, \tag{$\mathcal{H}$-JE}$$

being $\hat{\Delta}^{\mathbb{H},S}$ a suitable sub-Riemannian Laplacian associated with S (cf. (16.6.12)). Further applications of (ε-JE) and ($\mathcal{H}$-JE), as well as their relation when $\varepsilon \to 0$ (cf. Theorem 17.1.5), are discussed in Section 17.1.

Chapter 18 and Chapter 19 are devoted to the study of hypersurfaces of prescribed mean curvature in both the Riemannian and the sub-Riemannian setting. One of the main reasons to deal with this kind of issues can be found in connection with the so-called *Pansu conjecture*, which we shall now briefly describe. As already mentioned, exploiting the notion of horizontal perimeter it is possible to prove the validity of an appropriate isoperimetric inequality in Heisenberg groups (cf. [136, 147, 150, 235, 236] for results concerning the isoperimetric inequality in

$\mathbb{H}^n$ and related settings). More precisely, it is true that

$$P_{\mathbb{H}}(E, \mathbb{H}^n) \geq C|E|^{\frac{Q-1}{Q}}$$

for any measurable set E with finite Lebesgue measure $|E|$, where $Q = 2n + 2$ is the so-called *homogeneous dimension* of $\mathbb{H}^n$ (cf. Definition 3.2.4) and $C > 0$ is the sharp isoperimetric constant, that is

$$C = \{P_{\mathbb{H}}(E, \mathbb{H}^n) \,:\, E \text{ is measurable and } |E| = 1\}.$$

In [191], the authors proved the existence of an open bounded isoperimetric set, that is a set realizing the sharp isoperimetric constant. However, the identification of the isoperimetric set still remains an open problem. In [236], the author conjectured as possible solution those sets whose boundaries are now called *Pansu spheres*. This long-standing conjecture has to date been solved only assuming *a priori* regularity, symmetry or convexity hypotheses on the candidate isoperimetric set (cf. [220, 224, 250, 252]). In analogy with the Riemannian setting, sub-Riemannian isoperimetric sets in $\mathbb{H}^n$ can be described in terms the horizontal mean curvature. Indeed, (cf. e.g. [250]), if E is an isoperimetric set with sufficiently smooth boundary $S = \partial E$, then $H^{\mathbb{H}}$ is constant on $S \setminus S_0$. Moreover, when S is either an intrinsic graph or a t-graph (cf. Section 16.7), the prescribed (not necessarily constant) horizontal mean curvature condition can be translated in terms of PDEs (cf. [264, 85]). In the particular case of t-graphs, which are Euclidean graphs over the hyperplane $\{t = 0\}$ of the form

$$\{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y}))\},$$

the *prescribed horizontal mean curvature equation* reads as

$$\operatorname{div}\left(\frac{Du + (-\bar{y}, \bar{x})}{|Du + (-\bar{y}, \bar{x})|}\right) = H \tag{$\mathcal{H}$-PMC}$$

For a suitable prescribed datum H. ($\mathcal{H}$-PMC) is difficult to handle, since the possible presence of characteristic points may make it both degenerate elliptic and singular.

The aim of Chapter 18 is to solve the Dirichlet problem for ($\mathcal{H}$-PMC) when the prescribed horizontal mean curvature is constant. Anyway, due to the recent growing interest in anisotropic geometric structures which led for instance to the generalization of Pansu spheres to the so-called *Pansu-Wulff spheres* (cf. [244, 135]), our problem is settled in the more general setting of *sub-Finsler Heisenberg groups*. Roughly speaking, a sub-Finsler structure on $\mathbb{H}^n$ (cf. Section 18.4) is provided by means of a (possibly asymmetric) left-invariant norm $\|\cdot\|_{K_0}$ on the horizontal distribution of $\mathbb{H}^n$ associated to a convex body $K_0 \subseteq \mathbb{R}^{2n}$. When such a norm is induced by a left-invariant sub-Riemannian metric, then this general framework reduces to the sub-Riemannian one. After introducing the appropriate preliminaries about asymmetric norms (cf. Section 18.2) and Finsler geometry of hypersurfaces (cf. Section 18.3), we derive the sub-Finsler counterpart of ($\mathcal{H}$-PMC), that is

$$\operatorname{div}(\pi_{K_0}(Du + (-\bar{y}, \bar{x}))) = H, \tag{F-$\mathcal{H}$-PMC}$$

where π_{K_0} is a suitable 0-homogeneous map associated with the sub-Finsler norm $\|\cdot\|_{K_0}$. As its sub-Riemannian counterpart ($\mathcal{H}$-PMC), (F-$\mathcal{H}$-PMC) might present both degenerancies and singularities. Therefore, the Dirichlet problem associated with (F-$\mathcal{H}$-PMC) is formulated by looking for minimizers of the sub-Finsler functional

$$\int_\Omega \|Du + (-\bar{y}, \bar{x})\|_{K_0,*}\, dz + \int_\Omega Hu\, dz \tag{F-$\mathcal{H}$-FUN}$$

in the class $W^{1,1}_\varphi(\Omega)$, where $\|\cdot\|_{K_0,*}$ is the dual norm of $\|\cdot\|_{K_0}$ (cf. (18.2.3)), φ is a sufficiently smooth boundary datum and Ω is a suitable bounded open set satisfying appropriate curvature conditions. Our main result, namely Theorem 18.9.1, provides existence of a Lipschitz continuous minimizer of (F-$\mathcal{H}$-FUN) when H is constant and satisfies standard compatibility assumptions. Our approach to the proof of Theorem 18.9.1 required a Finsler regularization procedure (cf. Section 18.6) which allows to deal with a family of second-order elliptic equations approximating (F-$\mathcal{H}$-PMC) in a suitable sense (cf. Section 18.7). Exploiting the classical Leray-Schauder fixed point theory for quasilinear elliptic equations (cf. [157, 194]), it is possible to provide *a priori* estimates for solutions to the aforementioned regularized equations which are independent of the approximating problem (cf. Section 18.8), whence a compactness procedure allows to conclude the proof (cf. Section 18.9). Finally, in Section 18.10, we provide a sharper existence result in the sub-Riemannian setting. In this specific case, ($\mathcal{H}$-PMC) can be approximated by the family of equations

$$\operatorname{div}\left(\frac{Du + (-\bar{y}, \bar{x})}{\sqrt{\varepsilon^2 + |Du + (-\bar{y}, \bar{x})|^2}}\right) = H, \tag{ε-PMC}$$

which in turn correspond to the prescribed mean curvature equations for t-graphs in the Riemannian Heisenberg groups $(\mathbb{H}^n, g_\varepsilon)$. When H satisfies the standard Euclidean assumption

$$\max_{\partial\Omega} |H| < \min_{\partial\Omega} H_{\partial\Omega}, \tag{SERRIN}$$

being $H_{\partial\Omega}$ the Euclidean mean curvature of $\partial\Omega$, and in addition it is constant, Theorem 18.10.2 provides both a classical solution to the Riemannian Dirichlet problem associated with (ε-PMC) and a Lipschitz minimizer for the sub-Riemannian counterpart of (F-$\mathcal{H}$-FUN), namely

$$\int_\Omega |Du + (-\bar{y}, \bar{x})|\, dz + \int_\Omega Hu\, dz. \tag{$\mathcal{H}$-FUN}$$

Condition (SERRIN) was originally introduced in [265] to study the Euclidean constant mean curvature equation, and, at least in a slightly weaker version, is necessary to provide solutions to the Dirichlet problem for the prescribed mean curvature equation for any given sufficiently smooth boundary datum. Nevertheless, in the Euclidean setting, in view of the celebrated paper [162], it is still possible to provide solutions to the prescribed mean curvature equation avoiding curvature conditions in the spirit of (SERRIN), but provided no boundary conditions are imposed *a priori*. Accordingly, the study of t-graphs of prescribed mean curvature continues in Chapter 18. Here our aim is at least twofold. From one hand, relying again on (SERRIN), we extend the existence result for the Riemannian Dirichlet problem associated to (ε-PMC) allowing non-constant prescribed data (cf. Theorem 19.1.3). On the other hand, we extend the aforementioned existence results of [162] to overcome (SERRIN). More precisely, we show that the existence of classical solutions to (ε-PMC) is characterized, as in the Euclidean setting, by the condition

$$\left|\int_{\tilde{\Omega}} H\, dz\right| < P(\tilde{\Omega}) \tag{GIUSTI}$$

for any set $\tilde{\Omega} \subseteq \Omega$ such that $0 < |\tilde{\Omega}| < |\Omega|$, where $P(\tilde{\Omega})$ is the Euclidean perimeter of $\tilde{\Omega}$. In addition, when equality in (GIUSTI) holds with $\tilde{\Omega} = \Omega$, we provide sufficient conditions to guarantee uniqueness of solutions up to vertical translations (cf. Theorem 19.1.4). A crucial step in our approach consists in providing *interior* and *global gradient estimates* for solutions to (ε-PMC) (cf. Theorem 19.1.2 and Theorem 19.2.3), whose proof strongly relies on the Riemannian Jacobi equation (17.1.3) and which may have an independent interest. Finally, owing again to (GIUSTI), we provide existence in $BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ of local minimizers of the sub-Riemannian functional ($\mathcal{H}$-FUN) (cf. Theorem 19.1.5) via approximation by solutions to (ε-PMC), extending some previous results obtained when $H = 0$ (cf. [264]).

Our last chapter, namely Chapter 20, is devoted to the study of some rigidity properties of sub-Riemannian hypersurfaces in the Heisenberg group, in connection with the so-called *Bernstein problem*. Typically, by Bernstein problem we mean the characterization of global minimizers of a perimeter functional. Accordingly, in the Heisenberg setting we say that a measurable set E is an $\mathbb{H}$*-perimeter minimizer* whenever

$$P_{\mathbb{H}}(E, \Omega) \leqslant P_{\mathbb{H}}(F, \Omega)$$

for any $\Omega \Subset \mathbb{H}^n$ and for any measurable set F such that $E \Delta F \Subset \Omega$. The classical Euclidean Bernstein theorem says that a global minimizer of the Euclidean perimeter in $\mathbb{R}^n$ is a half-space, provided that $n \leqslant 7$ (cf. [163] and references therein). The goal of characterizing global perimeter minimizers in more general settings has been pursued exhaustively in first Heisenberg group $\mathbb{H}^1$ (cf. e.g. [83, 252, 173, 107, 108, 249, 35, 148, 234, 226]). In this framework, a key role is played by non-characteristic surfaces. Indeed, every sufficiently regular non-characteristic boundary of a global minimizer of the horizontal perimeter is a *vertical plane* (cf. [35]), i.e. an Euclidean plane tangent to T at every point. On the other hand, differently from the Euclidean setting, there are minimal smooth surfaces which are not vertical planes. Moreover, when $n \geq 5$, Bernstein theorem in $\mathbb{H}^n$ is false even for smooth non-characteristic minimal hypersurfaces (cf. [35]). To conclude, Bernstein problem is still open in the remaining cases

$\mathbb{H}^2$, $\mathbb{H}^3$ and $\mathbb{H}^4$. A crucial step to approach Bernstein problem in $\mathbb{H}^1$ consists in showing that area-stationary surfaces are *ruled* by horizontal line segments (cf. [148, 283]). The importance of this property is supported by [283], where the author showed a Bernstein theorem in the class of ruled minimal intrinsic graphs. The aim of Chapter 20 is twofold. From one hand, motivated by the previous considerations, in Section 20.2 we propose a generalization of the ruling property of $\mathbb{H}^1$ to higher dimensional Heisenberg groups (cf. Definition 20.2.1). Roughly speaking, a hypersurface $S \subseteq \mathbb{H}^n$ of class C^1 is *ruled* if every non-characteristic point p has a neighborhood U such that

$$p \cdot \mathcal{H}T_pS \cap U \subseteq S.$$

First, we discuss some properties of this new class, among which we mention a global equivalent definition (cf. Definition 20.2.2 and Proposition 20.2.3) and the fact that the ruling property is well-behaved with respect to the intrinsic rigid motions of the Heisenberg group (cf. Theorem 20.2.1), the so-called *pseudohermitian transformations* (cf. [87]). Subsequently, in Section 20.3 we provide a first rigidity property of ruled hypersurfaces under a constraint on the size of their characteristic set. Roughly speaking, when $n \geq 2$, ruled hypersurfaces with countable characteristic set are hyperplanes (cf. Theorem 20.1.3). This result highlights a first relevant difference with $\mathbb{H}^1$, where it is possible to provide instances of ruled, smooth non-characteristic surfaces which are not hyperplanes (cf. Example 20.3.1). Our second aim consists in translating the ruling property, which is differential in spirit, by a sub-Riemannian viewpoint. To this aim, in Section 20.4 we introduce the notion of *horizontally totally geodesic* hypersurface (cf. Definition 20.4.2), i.e. a hypersurface whose symmetric second fundamental form $\tilde{h}^{\mathbb{H}}$ is globally vanishing. This property, which is much weaker than requiring that the non-symmetric form $h^{\mathbb{H}}$ vanishes (cf. Example 20.4.3) is related to the ruling property in Section 20.5, where we show that hypersurfaces of class C^2 are ruled if and only if they are horizontally totally geodesic. In the end, combining the previous effort, we are able to conclude that the unique horizontally totally geodesic hypersurfaces of class C^2 are hyperplanes (cf. Theorem 20.1.1). Finally, although already covered by Theorem 20.1.1, in Section 20.6 we propose some results in the class of ruled intrinsic cones, since in this specific setting it is interesting to observe how many computations can be made more explicitly. It is worth mentioning some by-products of our approach. The first one is an existence result for a particular geodesic-type Cauchy problem on non-characteristic hypersurfaces (cf. Theorem 20.5.4). The second one constitutes another remarkable difference between $\mathbb{H}^1$ and the higher dimensional setting. Indeed, by means of our characterization, it is easy to provide, at least in the characteristic setting, instances of smooth minimal hypersurfaces which are not horizontally totally geodesic (cf. Theorem 20.1.5). An interesting issue arising from our previous considerations consists in understanding whether minimal, non-characteristic hypersurfaces are horizontally totally geodesic when $n = 2, 3$ of 4, since, in view of our results, an affirmative answer would solve positively the Bernstein problem. Being an approach based on estimates for the second fundamental form of minimal hypersurfaces already available in the Riemannian setting, by means of the celebrated paper [260], we hope to continue this path in light of similar considerations.

Credits

The results of Chapter 4 have been obtained, in collaboration with A. Pinamonti and C. Wang, in [243]. The results of Chapter 5 have been obtained, in collaboration with L. Capogna, G. Giovannardi and A. Pinamonti, in [78]. The results of Chapter 7 have been obtained, in collaboration with F. Essebei and A. Pinamonti, in [128]. The results of Chapter 8 have been obtained, in collaboration with F. Essebei, in [129]. The results of Chapter 9 have been obtained in [276]. The results of Section 11.1 have been obtained in [78]. The results of Section 11.2 have been obtained, in collaboration with F. Essebei and G. Giovannardi, in [126]. The results of Chapter 12 have been obtained in [243]. The results of Chapter 13 have been obtained in [78]. The results of Chapter 14 have been obtained in [126]. The results of Chapter 15 have been obtained, in collaboration with A. Pinamonti and G. Stefani, in [241]. Example 16.7.2 and the proof of Proposition 16.7.3 have been suggested us by D. Vittone. The proof of Proposition 16.7.8 has been suggested us by R. Young. Theorem 17.1.3 have been obtained, in collaboration with J. Pozuelo, in [245]. Proposition 17.3.1 and Theorem 17.3.2 are due to G. Giovannardi and F. Serra Cassano. The results of Chapter 18 have been obtained, in collaboration with G. Giovannardi, A. Pinamonti and J. Pozuelo, in [158]. The results of Chapter 19 have been obtained in [245]. The results of Chapter 20 have been obtained, in collaboration with A. Pinamonti, in [242].

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Contents

Notation

For the reader's convenience, we collect some notation that we adopt throughout the thesis. The following shall apply unless otherwise specified.

Framework. We let $m, n \in \mathbb{N} \setminus \{0\}$, with $m \leqslant n$. We adopt the convention $\infty = +\infty$, and we denote by either $\overline{\mathbb{R}}$ or $[-\infty, +\infty]$ the set of extended real numbers. We denote by Ω an open subset of $\mathbb{R}^n$. We denote by $\mathcal{A}$ the family of all open subsets of Ω and by $\mathcal{B}$ the family of all Borel subsets of Ω. If A, B are two open subsets of $\mathbb{R}^n$, we write $A \Subset B$ whenever $\overline{A}$ is compact and $\overline{A} \subseteq B$. If $A, B \subseteq \mathbb{R}^n$ are two arbitrary sets, we let

$$A \pm B = \{a \pm b \, : \, a \in A, \, b \in B\}.$$

Linear algebra. For $\alpha, \beta \in \mathbb{N} \setminus \{0\}$, we denote by $M(\alpha, \beta)$ the set of matrices with α rows and β columns. If α, β are as above and $L : \mathbb{R}^\alpha \longrightarrow \mathbb{R}^\beta$ is a linear map, we denote by $\ker(L) \subseteq \mathbb{R}^\alpha$ and $\mathrm{Im}(L) \subseteq \mathbb{R}^\beta$ respectively its kernel and its range. In the following, we mean vectors in $\mathbb{R}^\alpha$ either as matrices in $M(\alpha, 1)$ or as matrices in $M(1, \alpha)$. We let S^α be the class of all $\alpha \times \alpha$ symmetric matrices with real coefficients. Moreover, we let $A \cdot B$ be the usual matrix product, which may sometimes be denoted simply by AB. If $A, B \in S^m$, we write $A \leqslant B$ whenever

$$p \cdot A \cdot p^T \leqslant p \cdot B \cdot p^T$$

for any $p \in \mathbb{R}^m$. If A is a squared matrix, we denote by $\tau(A)$ its trace. If V, W are two vector spaces, we denote by $\mathcal{L}(V, W)$ the class of linear maps from V to W.

Functions. For $1 \leqslant p \leqslant \infty$, we denote by $L^p(\Omega)$ and $L^p_{loc}(\Omega)$ the Euclidean Lebesgue spaces, and by $W^{1,p}(\Omega)$ and $W^{1,p}_{loc}(\Omega)$ the Euclidean first-order Sobolev spaces (cf. [61]). We let $USC(\Omega)$ and $LSC(\Omega)$ be respectively the sets of upper semicontinuous and lower semicontinuous functions on Ω, and we denote by $C_0(\overline{\Omega})$ the set of continuous functions on $\overline{\Omega}$ which vanish on $\partial\Omega$. If $I \subseteq \mathbb{R}$ is an interval, we denote by $AC(I, \Omega)$ the set of absolutely continuous curves defined over I with values in Ω. If $g \in L^1_{loc}(\Omega)$ and $x \in \Omega$ is a Lebesgue point of g, when we write $g(x)$ we always mean that

$$g(x) = \lim_{r \to 0^+} \fint_{B_r(0)} g(y) dy.$$

If $u \in L^1_{loc}(\Omega)$, we denote by Du its (classical, weak or distributional) Euclidean gradient.

Moreover, denoting points in $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$, the j-th partial derivative of u, for $j = 1, \ldots, n$, is denoted equivalently by

$$\frac{\partial u}{\partial j} = D_j u = \partial_j u = u_{x_j}.$$

The Euclidean Hessian matrix of u is denoted by $D^2 u$. If $f(x, s, p)$ is a regular function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^m$, we denote by

$$D_x f = (D_{x_1} f, \ldots, D_{x_n} f), \qquad D_s f \qquad \text{and} \qquad D_p f = (D_{p_1} f, \ldots, D_{p_m} f)$$

the partial gradients of f with respect to the variables x, s and p respectively.

Normed vector spaces. If $(E, \|\cdot\|)$ is a normed vector space, we denote by $\rightharpoonup$ the weak convergence in E. Moreover, if E^* is the dual space of E, we denote by $\rightharpoonup^\star$ the weak-$\star$ convergence in $E^\star$.

Distances. If (M, d) is a metric space, $x \in M$ and $r > 0$, we let

$$B_d(x, r) = \{y \in M \,:\, d(x, y) < r\}.$$

The only exception to this notation occurs when $M = \mathbb{R}^n$ and d is the Euclidean distance, in which case we let

$$B_r(x) = \{y \in \mathbb{R}^n \,:\, |x - y| < r\}.$$

Part I

A brief introduction to Carnot-Carathéodory structures

Chapter 1

A functional framework arising from vector fields

1.1 Horizontal differential operators

As main references for this section, we refer the reader to [134, 150, 138]. Let Ω be an open subset of $\mathbb{R}^n$. In what follows it is sometimes convenient to identify an arbitrary vector field

$$\sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}$$

with the vector-valued map $(c_1, \dots, c_n)$. Given a family $X := (X_1, \dots, X_m)$ of locally Lipschitz continuous vector fields on Ω, we denote by $\mathcal{C}(x)$ the $m \times n$ matrix defined by

$$\mathcal{C}(x) = \begin{bmatrix} c_{1,1}(x) & c_{1,2}(x) & \dots & c_{1,n}(x) \\ c_{2,1}(x) & c_{2,2}(x) & \dots & c_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ c_{m,1}(x) & c_{m,2}(x) & \dots & c_{m,n}(x) \end{bmatrix}$$

for any $x \in \Omega$, where

$$X_j = c_{j,1} \frac{\partial}{\partial x_1} + \dots + c_{j,n} \frac{\partial}{\partial x_n}$$

for any $j = 1, \dots, m$, and where $c_{j,i}$ is a locally Lipschitz continuous function on Ω for any $j = 1, \dots, m$ and any $i = 1, \dots, n$. It is quite natural to associate to X a suitable notion of gradient. More precisely, if u is a smooth function over Ω, we define its *horizontal gradient* by letting

$$Xu = (X_1 u, \dots, X_m u) = Du \cdot \mathcal{C}^T.$$

This notion generalizes the classical Euclidean gradient in the sense that choosing as X the canonical basis of $\mathbb{R}^n$ reduces the horizontal gradient to the Euclidean gradient. Notice that, given $u \in C^\infty(\Omega)$ and $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, the Euclidean divergence theorem implies that

$$\int_\Omega \langle Xu, \varphi \rangle \, dx = \int_\Omega \langle Du \cdot \mathcal{C}^T, \varphi \rangle \, dx = \int_\Omega \langle Du, \varphi \cdot \mathcal{C} \rangle \, dx = -\int_\Omega u \, \mathrm{div}(\varphi \cdot \mathcal{C}) \, dx.$$

Therefore the previous definition extends to its distributional counterpart as follows.

Definition 1.1.1 (Horizontal gradient)**.** *Let $u \in L^1_{loc}(\Omega)$. We define its distributional* horizontal gradient *by*

$$\langle Xu, \varphi \rangle := -\int_\Omega u \, \mathrm{div}(\varphi \cdot \mathcal{C}) \, dx$$

for any $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$.

Hidden in the previous definition we can find another differential notion associated to X. Indeed, if $u \in C^1(\Omega, \mathbb{R}^m)$, we can define its *horizontal divergence* simply by letting

$$\mathrm{div}_X(u) = \mathrm{div}(u \cdot \mathcal{C}).$$

Arguing as above, this definition extend as follows.

Definition 1.1.2 (Horizontal divergence)**.** *Let $u \in L^1_{loc}(\Omega, \mathbb{R}^m)$. We define its distributional* horizontal divergence *by*

$$\langle \mathrm{div}_X(u), \varphi \rangle = -\int_\Omega u \cdot X\varphi \, dx$$

for any $\varphi \in C_c^\infty(\Omega)$.

In this way, we can rephrase Definition 1.1.1 in the suggestive formula

$$\langle Xu, \varphi \rangle := -\int_\Omega u \, \mathrm{div}_X(\varphi) \, dx. \tag{1.1.1}$$

1.2 Differentiable functions

Once the notion of horizontal gradient is available, it is natural to retrieve in this general setting all the classical functional spaces, such as differentiable, Sobolev and BV spaces.

Definition 1.2.1 (Horizontal differentiable spaces)**.** *For a given $k \geq 1$, we define the space $C_X^k(\Omega)$ recursively by*

$$C_X^k(\Omega) := \{u \in C(\Omega) \, : \, X_i u \in C^{k-1}(\Omega) \text{ for any } i = 1, \ldots, m\}.$$

When $u \in C_X^1(\Omega)$, the horizontal gradient is represented by the continuous vector-valued function

$$(X_1 u, \ldots, X_m u).$$

Moreover, when $u \in C_X^1(\Omega, \mathbb{R}^m)$, its horizontal divergence can be expressed by

$$\mathrm{div}_X(u) := \sum_{j=1}^m X_j u_j + \sum_{j=1}^m \sum_{i=1}^n u_j \frac{\partial c_{j,i}}{\partial x_i}$$

When $u \in C^2_X(\Omega)$, it is possible to define some well-known second-order differential objects.

Definition 1.2.2 (Horizontal Hessian)**.** *If $u \in C^2_X(\Omega)$, its* horizontal Hessian $X^2u \in C(\Omega, S^m)$ *is defined by*

$$(X^2u)_{ij} := \frac{X_iX_ju + X_jX_iu}{2}$$

for any $i, j = 1, \ldots, m$.

The reader may be confused by this definition, since it might seem not the natural extension of the Euclidean one. Our choice, which is common in literature, let the horizontal Hessian be symmetric. Indeed, unlike in the Euclidean setting, a *horizontal* Schwarz lemma is false in this general framework.

Example 1.2.3. Let us consider the vector fields X_1, Y_1 defined over the whole $\mathbb{R}^3$ by

$$X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t} \qquad \text{and} \qquad Y_1 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t},$$

where we denote points of $\mathbb{R}^3$ by (x, y, t). If we define the function $u \in C^\infty(\mathbb{R}^3)$ by

$$u(x, y, t) = t,$$

then $X_1u(x, y, t) = y$ and $Y_1u(x, y, t) = -x$, so that $X_1Y_1u(x, y, t) = -1$ and $Y_1X_1u(x, y, t) = 1$.

The failure of Schwarz lemma can be rephrased in terms of lack of commutation between vector fields. More precisely, if X, Y are vector fields of class C^2, their *Lie bracket* or *commutator* is defined as the continuous vector field $[X, Y]$ acting as

$$[X, Y]\varphi = X(Y\varphi) - Y(X\varphi) \tag{1.2.1}$$

for any sufficiently smooth function φ. Therefore Example 1.2.3 tells us that our chosen vector fields may not commute in general. This fact constitutes one of the most striking differences between the Euclidean and the anisotropic setting that we have just introduced. We conclude this section providing another generalization of a classical second-order differential operator which will be thoroughly discussed in the following chapters.

Definition 1.2.4 (Horizontal Laplacian)**.** *If $u \in C^2_X(\Omega)$ we define the* horizontal Laplacian *of u by*

$$\Delta_X u := \operatorname{div}_X(Xu) = \sum_{j=1}^m X_jX_ju + \sum_{j=1}^m\sum_{i=1}^n X_ju\frac{\partial c_{j,i}}{\partial x_i}. \tag{1.2.2}$$

1.3 Sobolev spaces

According to the previous set of definitions, we introduce the relevant horizontal Sobolev spaces.

Definition 1.3.1 (Horizontal Sobolev spaces). *If* $p \in [1, +\infty]$, *we define the* horizontal Sobolev spaces *by letting*

$$W_X^{1,p}(\Omega) := \{u \in L^p(\Omega) \,:\, Xu \in L^p(\Omega, \mathbb{R}^m)\},$$

$$W_{X,loc}^{1,p}(\Omega) := \left\{u \in L_{loc}^p(\Omega) \,:\, u|_{\tilde{\Omega}} \in W_X^{1,p}(\tilde{\Omega}), \quad \textit{for any } \tilde{\Omega} \Subset \Omega\right\}$$

and

$$W_{X,0}^{1,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W_X^{1,p}(\Omega)}},$$

where

$$\|u\|_{W_X^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)}.$$

Moreover, when $g \in W_X^{1,p}(\Omega)$, *we let*

$$W_{X,g}^{1,p}(\Omega) := \left\{u \in W_X^{1,p}(\Omega) \,:\, u - g \in W_{X,0}^{1,p}(\Omega)\right\}.$$

These first-order horizontal Sobolev spaces are well-studied in literature. For the reader's convenience, we recall some of their basic properties. First, horizontal Sobolev spaces are Banach spaces (cf. [134]).

Proposition 1.3.2. $\left(W_X^{1,p}(\Omega), \|\cdot\|_{W_X^{1,p}(\Omega)}\right)$ *is a Banach space, reflexive if* $1 < p < \infty$.

Moreover, similarly to the Euclidean case, a Meyers-Serrin approximation result holds (cf. [138]).

Theorem 1.3.3 (Meyers-Serrin). *Let* Ω *be an open subset of* $\mathbb{R}^n$, *and let* $1 \leqslant p < +\infty$. *Then*

$$\overline{W_X^{1,p}(\Omega) \cap C^\infty(\Omega)} = W_X^{1,p}(\Omega),$$

where the closure is with respect to the metric topology of $\left(W_X^{1,p}(\Omega), \|\cdot\|_{W_X^{1,p}(\Omega)}\right)$.

The above result has many useful consequences. As an instance, we quote the following Leibniz-type property of the horizontal gradient (cf. [129]).

Proposition 1.3.4 (Leibniz rule). *For any* $u, v \in W_X^{1,p}(\Omega)$, *it holds that*

$$X(uv) = (Xu)v + u(Xv).$$

Proof. Assume first that $u, v \in W_X^{1,p}(\Omega) \cap C^\infty(\Omega)$. Then it follows that

$$\begin{aligned} X(uv) &= D(uv) \cdot \mathcal{C}^T \\ &= [(Du)v + u(Dv)] \cdot \mathcal{C}^T \\ &= Du \cdot \mathcal{C}^T v + uDv \cdot \mathcal{C}^T \\ &= (Xu)v + u(Xv) \end{aligned} \tag{1.3.1}$$

everywhere on Ω. Let now $A' \Subset \Omega$, $u \in W_X^{1,p}(\Omega)$ and $v \in W_X^{1,p}(\Omega) \cap C^\infty(\Omega)$. From Theorem 1.3.3 we know in particular that there exists a sequence $(u_h)_h \subseteq W_X^{1,p}(\Omega) \cap C^\infty(\Omega)$ converging to u

in the strong topology of $W^{1,p}_X(A')$, and clearly $v \in C^\infty(\overline{A'})$. It is easy to see that the sequence $(vu_h)_h$ belongs to $W^{1,p}_X(A') \cap C^\infty(\overline{A'})$ and converges to uv in the strong topology of $W^{1,p}_X(A')$. This fact, together with (1.3.1) and recalling that $\sup_{A'} |Xv| < +\infty$ since $\sup_{A'} |Dv| < +\infty$, yields that

$$\begin{aligned}\|X(uv) - (Xu)v - u(Xv)\|_{L^p(A',\mathbb{R}^m)} &\leqslant \|X(uv) - X(u_h v)\|_{L^p(A',\mathbb{R}^m)} \\ &\quad + \|(Xv)u_h + vX(u_h) - (Xu)v - u(Xv)\|_{L^p(A',\mathbb{R}^m)},\end{aligned}$$

and so, passing to the limit as $h \to \infty$, we conclude that

$$X(uv) = (Xu)v + u(Xv) \qquad \text{a.e. on } A'.$$

Since Ω can be approximated by a countable family of open sets $A' \Subset \Omega$, we conclude that

$$X(uv) = (Xu)v + u(Xv) \qquad \text{a.e. on } \Omega$$

for any $u \in W^{1,p}_X(\Omega)$ and $v \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$. Repeating once more the same procedure, the thesis follows. □

Another similitude with the Euclidean setting is evidenced by the following Riesz-type theorem, which can be proved *verbatim* as in [193, Theorem 10.41].

Proposition 1.3.5. *Let $1 \leqslant p < \infty$, and let $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ and $u \in W^{1,p}_X(\Omega)$. The following conditions are equivalent.*

(i) $u_h \rightharpoonup u$ *in* $W^{1,p}_X(\Omega)$.

(ii) *For $1/p' + 1/p = 1$ and for any $(g_0, \ldots, g_m) \in (L^{p'}(\Omega))^{m+1}$ it holds that*

$$\lim_{h\to\infty} \left(\int_\Omega u_h \cdot g_0 \, dx + \sum_{j=1}^m \int_\Omega X_j u_h \cdot g_j \, dx \right) = \int_\Omega u \cdot g_0 \, dx + \sum_{j=1}^m \int_\Omega X_j u \cdot g_j \, dx.$$

In the statement of Proposition 1.3.5, $\rightharpoonup$ denotes the classical weak convergence in normed vector spaces (cf. [61]). In the following chapters it will be useful to compare X-Sobolev spaces to classical Sobolev spaces (cf. [205, 128]).

Proposition 1.3.6. *Let Ω be an open subset of $\mathbb{R}^n$. Assume that X is made of Lipschitz continuous vector fields. Then $W^{1,p}(\Omega) \subseteq W^{1,p}_X(\Omega)$, the inclusion map*

$$W^{1,p}(\Omega) \hookrightarrow W^{1,p}_X(\Omega)$$

is continuous (and possibly strict) and

$$Xu(x) = Du(x) \cdot C(x)^T \tag{1.3.2}$$

for every $u \in W^{1,p}_{loc}(\Omega)$ and a.e. $x \in \Omega$.

If, in addition to the assumptions of Proposition 1.3.6, Ω is bounded, then clearly

$$W^{1,\infty}(\Omega) \subseteq W^{1,p}(\Omega) \subseteq W_X^{1,p}(\Omega).$$

For further convenience, we need to understand how weak convergence in $W_X^{1,p}$ is related to the weak*- convergence in $W^{1,\infty}$. The following proposition is a direct consequence of [61, Theorem 3.10].

Proposition 1.3.7. *Let Ω be an open and bounded subset of $\mathbb{R}^n$, let $1 \leqslant p < +\infty$ and assume that X is made of Lipschitz continuous vector fields. For any sequence $(u_h)_h \subseteq W^{1,\infty}(\Omega)$ and any $u \in W^{1,\infty}(\Omega)$, it follows that*

$$u_h \rightharpoonup^* u \text{ in } W^{1,\infty}(\Omega) \quad \Longrightarrow \quad u_h \rightharpoonup u \text{ in } W_X^{1,p}(\Omega).$$

Despite many similarities with the Euclidean Sobolev spaces, the anisotropic structure also reveals some notable differences. Let us quote two remarkable instances.

Example 1.3.8 (Approximation by piecewise affine functions)**.** Classical Sobolev functions can be approximated in the Sobolev norm by means of piecewise affine functions (cf. [125]). If we call a smooth function u *X-affine* as soon as Xu is constant, it is natural to wonder whether an analogous property holds for horizontal Sobolev functions. However, as shown in [205], the answer is negative. In this setting, we simply call u piecewise X-affine on Ω if $u \in C(\Omega)$ and if Ω can be divided into a negligible set and a finite collection of open sets on which u is X-affine. Let Ω be a bounded open subset of $\mathbb{R}^3$. Let X_1, Y_1 and u be as in Example 1.2.3. Clearly $u \in C^\infty(\overline{\Omega})$, and so in particular $u \in W_X^{1,p}(\Omega)$ for any $p \geq 1$. An easy computation reveals that a function v is X-affine if and only if

$$v(x, y, t) = ax + by + c$$

for some $a, b, c \in \mathbb{R}$. Therefore, since X-affine functions do not depend on t, u cannot be pointwise approximated by piecewise X-affine functions.

Example 1.3.9 (Lusin-type property)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $1 \leqslant p \leqslant +\infty$. A well-known Lusin-type property of Euclidean Sobolev functions (cf. [69]) states that if $u \in W^{1,p}(\Omega)$, then, for any $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ and $v \in C^1(\overline{\Omega})$ such that $|A_\varepsilon| \leqslant \varepsilon$ and $u|_{\Omega \setminus A_\varepsilon} = v|_{\Omega \setminus A_\varepsilon}$. We show that the same conclusion is false in our general setting. In the following we speak about *approximate differentiability* and *approximate partial derivatives* according e.g. to [131]. Let us choose $n = 2$, $m = 1$, $\Omega = (0,1) \times (0,1)$ and $X = (X_1) = \left(\frac{\partial}{\partial x}\right)$. Let us consider a function $w : (0,1) \longrightarrow \mathbb{R}$ which is bounded, continuous but which is not approximately differentiable for a.e. $x \in (0,1)$ (cf. e.g. [254]), and define the function $u : \Omega \longrightarrow \mathbb{R}$ by

$$u(x, y) := w(y).$$

Then $u \in L^\infty(\Omega)$ and it is constant with respect to x. Thus, for any $\varphi \in C_c^\infty(\Omega)$, we have that

$$-\int_\Omega u \frac{\partial \varphi}{\partial x} dx = -\int_0^1 dy\, w(y) \int_0^1 dx\, \frac{\partial \varphi}{\partial x} = 0,$$

and so $Xu = 0$. Hence $u \in W_X^{1,\infty}(\Omega)$ and in particular we have that $u \in W_X^{1,p}(\Omega)$ for any $p \in [1, +\infty]$. If it was the case that u satisfies the desired property, then we would have that, for a.e. (x, y) in Ω, u is approximately differentiable at (x, y) (cf. [199]). Thus, according to [254, Theorem 12.2] and to the fact that u is constant with respect to x, we would have that for any $x \in (0, 1)$ and for a.e. $y \in (0, 1)$, the function $z \mapsto u(x, z) = w(z)$ is approximately differentiable at y, a contradiction with our choice of w.

1.4 BV-functions and perimeter

To conclude this chapter, we recall the notion of horizontal BV function, with a special regard to its connection with the so-called *horizontal perimeter*. For the main definitions and results of this section, we mainly refer to [138].

Definition 1.4.1 (Bounded X-variation). *If $u \in L^1_{loc}(\Omega)$, we say that u has* locally bounded X-variation*, or $u \in BV_{X,loc}(\Omega)$, whether*

$$\sup\left\{\int_{\tilde\Omega} u \operatorname{div}_X(\varphi)\, dx \,:\, \varphi \in C_c^1(\tilde\Omega, \mathbb{R}^m),\ \|\varphi\|_\infty \leqslant 1\right\} < +\infty$$

*for any open set $\tilde\Omega \Subset \Omega$, and we define its X-*total variation *$|Xu|$ by*

$$|Xu|(\tilde\Omega) = \sup\left\{\int_{\tilde\Omega} u \operatorname{div}_X(\varphi)\, dx \,:\, \varphi \in C_c^1(\tilde\Omega, \mathbb{R}^m),\ \|\varphi\|_\infty \leqslant 1\right\}$$

for any $\tilde\Omega \subseteq \Omega$. If in addition $u \in L^1(\Omega)$ and

$$|Xu|(\Omega) < +\infty,$$

we say that $u \in BV_X(\Omega)$.

As in the Euclidean setting, Riesz theorem implies that $u \in BV_{X,loc}(\Omega)$ (respectively $u \in BV_X(\Omega)$) if and only if the distributional horizontal gradient Xu can be represented by a m-valued Radon (respectively finite) measure over Ω. Moreover, denoting such a measure by Xu, it follows that its total variation measure coincides with the X-total variation $|Xu|$. Therefore, as in the Euclidean setting, Radon-Nikodym theorem implies the existence of a $|Xu|$-a.e unique measurable function $\sigma_u : \Omega \longrightarrow \mathbb{S}^m$, satisfying

$$|\sigma_u(x)| = 1$$

for $|Xu|$-a.e. $x \in \Omega$, such that

$$Xu(\varphi) = \int_\Omega \langle \varphi, dXu \rangle = \int_\Omega \langle \sigma_u, \varphi \rangle \, d|Xu| \tag{1.4.1}$$

for any $\varphi \in C_c(\Omega, \mathbb{R}^m)$. As its Euclidean counterpart, the X-total variation is lower semicontinuous with respect to the convergence of $L^1_{loc}(\Omega)$.

Proposition 1.4.2. *Let $u \in BV_{X,loc}(\Omega)$ and $(u_h)_h \subseteq BV_{X,loc}(\Omega)$ be such that $u_h \to u$ with respect to the L^1_{loc}-convergence. Then*

$$|Xu|(\tilde{\Omega}) \leqslant \liminf_{h \to \infty} |Xu_h|(\tilde{\Omega})$$

for any open set $\tilde{\Omega} \subseteq \Omega$.

Moreover, the anisotropic Meyers-Serrin approximation result Theorem 1.3.3 extends to its Anzellotti-Giaquinta version (cf. [19]) as follows.

Theorem 1.4.3 (Anzellotti-Giaquinta)**.** *Let $u \in BV_X(\Omega)$. Then there exists a sequence $(u_h)_h \subseteq BV_X(\Omega) \cap C^\infty(\Omega)$ such that*

$$\lim_{h \to \infty} \|u - u_h\|_{L^1(\Omega)} = 0 \qquad \text{and} \qquad |Xu|(\Omega) = \lim_{h \to \infty} |Xu_h|(\Omega).$$

The notion of horizontal perimeter is a direct consequence of Definition 1.4.1.

Definition 1.4.4 (Horizontal perimeter)**.** *A measurable set $E \subseteq \mathbb{R}^n$ is* of locally finite X-perimeter *in Ω, or equivalently a X*-Caccioppoli set *in Ω, whether $\chi_E \in BV_{X,loc}(\Omega)$, that is*

$$\sup \left\{ \int_E \operatorname{div}_X(\varphi) \, dx \; : \; \varphi \in C^1_c(\tilde{\Omega}, \mathbb{R}^m), \; \|\varphi\|_\infty \leqslant 1 \right\} < +\infty$$

for any open set $\tilde{\Omega} \Subset \Omega$. In addition, E is of finite X-perimeter *in Ω if $\chi_E \in BV_X(\Omega)$, meaning again that*

$$\sup \left\{ \int_E \operatorname{div}_X(\varphi) \, dx \; : \; \varphi \in C^1_c(\Omega, \mathbb{R}^m), \; \|\varphi\|_\infty \leqslant 1 \right\} < +\infty.$$

In the situations described in Definition 1.4.4, in accordance with the Euclidean framework, we adopt the notation

$$P_X(E, \cdot) = |X\chi_E|(\cdot),$$

and we call $P_X(E, \cdot)$ the *X-perimeter* (or *horizontal perimeter*) of E. Moreover, (1.4.1) allows to introduce another crucial notion, that is the so-called *horizontal normal*.

Definition 1.4.5 (Horizontal normal)**.** *Let E be an X-Caccioppoli set in Ω. Then, following (1.4.1), we denote by $\nu_X : \Omega \longrightarrow \mathbb{S}^m$ the $|X\chi_E|$-a.e unique measurable function satisfying*

$$|\nu_X(x)| = 1$$

for $|Xu|$-a.e. $x \in \Omega$ and

$$\langle X\chi_E, \varphi \rangle = -\int_\Omega \langle \nu_X, \varphi \rangle \, d|X\chi_E|$$

for any $\varphi \in C^1_c(\Omega, \mathbb{R}^m)$. ν_X *is called the* (measure-theorethic outward unit) horizontal normal *to* E *in* Ω.

To justify the above definition, assume that E is a set of locally finite Euclidean perimeter in Ω, and denote by N its measure theoretic outward Euclidean unit normal. If $\varphi \in C^1_c(\Omega, \mathbb{R}^m)$, then $\varphi \cdot \mathcal{C} \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ and has compact support in Ω. Therefore, by Gauss-Green formula for Lipschitz continuous vector fields (cf. [131]) we infer that

$$\int_\Omega \langle \nu_X, \varphi \rangle \, d|X\chi_E| = -\langle X\chi_E, \varphi \rangle = \int_E \mathrm{div}(\varphi \cdot \mathcal{C}) \, dx = \int_{\partial^* E} \langle N, \varphi \cdot \mathcal{C} \rangle \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \langle N \cdot \mathcal{C}^T, \varphi \rangle \, d\mathcal{H}^{n-1},$$

so that

$$P_X(E, \Omega) = \int_{\partial^* E \cap \Omega} |N \cdot \mathcal{C}^T| \, d\mathcal{H}^{n-1} \tag{1.4.2}$$

and

$$\nu_X(x) = \begin{cases} \frac{N(x) \cdot \mathcal{C}(x)^T}{|N(x) \cdot \mathcal{C}(x)^T|} & \text{if } N(x) \cdot \mathcal{C}(x)^T \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the horizontal unit normal of Euclidean Caccioppoli sets is, in a sense, the normalized projection of the Euclidean unit normal, whenever the latter does not vanish, onto the distribution generated by X. When ∂E enjoys better regularity properties, it is possible to provide a pointwise distinction between points where the latter projection vanishes and points where the horizontal unit normal is well-defined. Since a general treatment of these issues goes beyond our scopes, we postpone it to Chapter 16, where we will focus on a very specific, albeit already relevant, setting. To conclude this section, we refer the interested reader to [138, Theorem 2.3.5] for the anisotropic version of the well-known *Coarea formula* and to [138, Corollary 2.3.6] for an approximation result of finite X-perimeter by means of smooth sets. Finally, we point out that, under additional hypotheses on X, it is possible to provide appropriate *blow-up and implicit function theorems* (cf. e.g. [140, 203, 141, 142, 90, 14]) and *isoperimetric inequalities* (cf. e.g. [150, 191, 189]).

Chapter 2

Carnot-Carathéodory spaces and sub-Riemannian manifolds

2.1 Carnot-Carathéodory spaces

Let us fix a family X of locally Lipschitz continuous vector fields defined over an open set $\Omega \subseteq \mathbb{R}^n$. Beside horizontal functions, it is also possible to talk about *horizontal curves.*

Definition 2.1.1 (Horizontal curves)**.** *If $\gamma : [0,T] \longrightarrow \Omega$ is an absolutely continuous curve, we say that it is* horizontal *when there exists $a = (a_1, \dots, a_m) \in L^\infty([0,T], \mathbb{R}^m)$ such that*

$$\dot{\gamma}(t) = \sum_{j=1}^{m} a_j(t) X_j(\gamma(t)) \tag{2.1.1}$$

for a.e. $t \in [0,T]$, and we say that it is sub-unit *whenever it is horizontal with*

$$\|a\|_{L^\infty([0,T],\mathbb{R}^m)} \leqslant 1.$$

Notice that an absolutely continuous curve $\gamma : [0,T] \longrightarrow \mathbb{R}$ is horizontal if and only if

$$\dot{\gamma}(t) = \mathcal{C}(\gamma(t))^T \cdot a(t) \tag{2.1.2}$$

for a.e. $t \in [0,T]$. Roughly speaking, horizontal curves describe the admissible path along which we allow movements. Our choice of admissible directions gives rise to a candidate distance which generalize, for instance, the usual Riemannian distance in a Riemannian manifold.

Definition 2.1.2. *We define the* Carnot-Carathéodory distance *on Ω by*

$$d_\Omega(x,y) := \inf\{T \,:\, \gamma : [0,T] \longrightarrow \Omega \text{ is sub-unit, } \gamma(0) = x \text{ and } \gamma(T) = y\}.$$

If d_Ω is a distance on Ω, then (Ω, d_Ω) is called a Carnot-Carathéodory space*.*

The reader should note the imprecision in the definition above. We called d_Ω Carnot-Carathéodory *distance*, while it is not always the case that d_Ω is a distance. Indeed, while it is easy to check that d_Ω is non-degenerate, symmetric and satisfies the triangle inequality, it may be not well-defined for each couple of points $x, y \in \Omega$. In other words, not every couple of points in Ω can necessarily be joined by a horizontal curve.

Example 2.1.3. Let us consider on $\mathbb{R}^2$ the vector field $X = \frac{\partial}{\partial x}$, where we fixed coordinates (x, y). Then horizontal curves are clearly parallel to the x-axis, so that it is impossible to join two points with, say, same x and different y.

In addition, a hidden necessary condition to ensure that (Ω, d_Ω) is a metric space requires that Ω is connected, so that we shall assume *a priori* this property. The following proposition express the Carnot-Carathéodory distance in an equivalent form, which gives a plainer intuition of the Carnot-Carathéodory distance as a measure of the length of horizontal curves (cf. [232]).

Proposition 2.1.4. *Assume that d_Ω is a distance. Then*

$$d_\Omega(x, y) = \inf \left\{ \int_0^1 |a(t)| dt \,:\, \gamma : [0,1] \longrightarrow \Omega \text{ is horizontal, } \gamma(0) = x \text{ and } \gamma(1) = y \right\},$$

where $a(t) = (a_1(t), \dots, a_m(t))$ is as in (2.1.1).

A natural question at this stage is how the Carnot-Carathéodory structure relates to the Euclidean one, both from the metric and the topological point of view. The following first property follows from [219]

Proposition 2.1.5. *Let (Ω, d_Ω) be a Carnot-Carathéodory space. Let $K \Subset \Omega$. Then there exists $\beta > 0$ such that*

$$d_\Omega(x, y) \geq \beta |x - y|$$

for any $x, y \in K$.

In particular, the Euclidean distance is continuous with respect to the Carnot-Carathéodory distance. Without further assumptions on X, the converse implication is false, as the following example shows (cf. [219]).

Example 2.1.6. In $\mathbb{R}^2$ with coordinates (x, y), consider the vector fields

$$X = \frac{\partial}{\partial x} \qquad \text{and} \qquad Y = a(x)\frac{\partial}{\partial y},$$

where

$$a(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that $(\mathbb{R}^2, d_{\mathbb{R}^2})$ is a Carnot-Carathéodory space, where $d_{\mathbb{R}^2}$ is the Carnot-Carathéodory distance induced by X and Y. Nevertheless, if we consider the sequence $\left(-1, \frac{1}{n}\right)_{n \geq 1}$, then

$$d_{\mathbb{R}^2}\left(\left(-1, \frac{1}{n}\right), (-1, 0)\right) \geq 2.$$

Hence $d_{\mathbb{R}^2}$ is not continuous with respect to the Euclidean topology.

In the following, we will see that under very mild assumptions, the Carnot-Carathéodory topology is continuous with respect to - and hence equivalent with - the Euclidean topology. For the sake of future clarity, although not standard in literature, we give the following definition.

Definition 2.1.7 (Continuous Carnot-Carathéodory space)**.** *We say that* (Ω, d_Ω) *is a* continuous *Carnot-Carathéodory space if it is a Carnot-Carathéodory space whose topology is equivalent to the Euclidean one.*

We point out that, although a Carnot-Carathéodory space is in many situations Euclidean from the topological standpoint, the same cannot be said from the metric point of view, even in the most pleasant situations. We will come back on this shortly. When a family of vector fields X of class C^1 induces a continuous Carnot-Carathéodory space, a stronger L^∞ version of Theorem 1.3.3 is available in the following sense.

Proposition 2.1.8. *Let X be a family of vector fields of class C^1. Assume that (Ω, d_Ω) is a continuous Carnot-Carathéodory space. If $v \in C^1_X(\Omega)$, then for any open set $A \Subset \Omega$ there exists a sequence $(v_h)_h \in C^\infty(\Omega)$ such that $v_h \to u$ and $Xv_h \to Xu$ uniformly on $\overline{A}$.*

Proof. Let v and A as in the statement, and take an open set B such that $A \Subset B \Subset \Omega$, and a smooth cut-off function φ between B and Ω. Define the function $\bar{v} := \varphi v$, and extend it to be zero outside Ω. It is clear that $\bar{v} \in C(\mathbb{R}^n)$ and that $\bar{v}$ coincides with v on $\overline{A}$. Let us define then $v_h := \bar{v} \star \varrho_h$, where ϱ_h is the standard spherically symmetric h-mollifier in $\mathbb{R}^n$ (cf. [61]). It is clear that $v_h \in C^2(\Omega) \subseteq C^2_X(\Omega)$, where the previous inclusion follows as X is C^1. Moreover, as $\bar{v}$ is continuous, from standard properties of mollification we have that $v_h \to \bar{v}$ uniformly on $\overline{A}$, and so $v_h \to v$ uniformly on $\overline{A}$. We are left to show that $Xv_h \to Xv$ uniformly on $\overline{A}$. To this aim, thanks to [278] we know that there exists a modulus of continuity ω such that

$$\|X(\bar{v} \star \varrho_h) - X\bar{v} \star \varrho_h\|_{L^\infty(\overline{A})} \leqslant \omega\left(\frac{1}{h}\right).$$

Moreover, as $X\bar{v}$ is continuous, it holds that $X\bar{v} \star \varrho_h \to X\bar{v}$ uniformly on $\overline{A}$. Therefore we can conclude that

$$\lim_{h\to\infty} \|X(\bar{v} \star \varrho_h) - X\bar{v}\|_{L^\infty(\overline{A})} \leqslant \lim_{h\to\infty} \|X(\bar{v} \star \varrho_h) - X\bar{v} \star \varrho_h\|_{L^\infty(\overline{A})} + \lim_{h\to\infty} \|X\bar{v} \star \varrho_h - X\bar{v}\|_{L^\infty(\overline{A})} = 0.$$

Again, as $X\bar{v} = Xv$ on $\overline{A}$, the proof is complete. □

2.2 Sub-Riemannian manifolds

In this section we introduce a nice and well-known geometric condition on X which ensures, among the other things, that X generates a - even continuous - Carnot-Carathéodory space. For the sake of generality, we deal first with a smooth manifold M. To proceed, we need to recall some basic concepts of the theory of Lie algebras, for which we refer the reader to [54].

Definition 2.2.1 (Lie algebra)**.** *A (real)* Lie algebra $\mathfrak{g}$ *is a (real) vector space endowed with an inner operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *which satisfies the following conditions.*

- $[\cdot,\cdot]$ *is bilinear.*
- $[\cdot,\cdot]$ *is alternating, i.e.*
$$[x, x] = 0$$
for any $x \in \mathfrak{g}$*.*
- $[\cdot,\cdot]$ *satisfies the* Jacobi identity*, i.e.*
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$
for any $x, y, z \in \mathfrak{g}$*.*

Definition 2.2.2 (Lie algebra homomorphism)**.** *Let* $(\mathfrak{g}_1, [\cdot,\cdot]_1)$ *and* $(\mathfrak{g}_2, [\cdot,\cdot]_2)$ *be two real Lie algebras. A linear map* $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ *is a* Lie algebra homomorphism *if*

$$\varphi\left([x, y]_1\right) = [\varphi(x), \varphi(y)]_2$$

for any $x, y \in \mathfrak{g}_1$*. A* Lie algebra isomorphism *is a bijective Lie algebra homomorphism.*

Assume that M is a smooth manifold. A first relevant instance of Lie algebra is that of vector fields over M, say $\Gamma(TM)$, seen as a vector space over $\mathbb{R}$ and endowed with the Lie bracket introduced in (1.2.1).

Definition 2.2.3 (Generated Lie algebra)**.** *If* $G \subseteq \mathfrak{g}$ *is a set. we let*

$$\mathrm{Lie}(G) := \bigcap \left\{\mathfrak{h} \,:\, \mathfrak{h} \subseteq \mathfrak{g}, (\mathfrak{h}, [\cdot,\cdot]) \text{ is a Lie algebra and } G \subseteq \mathfrak{h}\right\}.$$

Equivalently, $\mathrm{Lie}(G)$ *is the smallest Lie algebra containing* G*.*

Assume that an m-dimensional subbundle (or distribution) of TM, say Δ, is fixed.

Definition 2.2.4 (Hörmander condition)**.** *We say that* Δ *satisfies the* Hörmander condition *over* M*, or is* bracket-generating *over* M*, if*

$$\dim\left(\mathrm{Lie}(\Gamma(\Delta))(p)\right) = n$$

for any $p \in M$*.*

Of course, when Δ is generated by a family X of vector fields, meaning that

$$\Delta_p = \mathrm{span}\{X_1|_p, \dots, X_m|_p\} \tag{2.2.1}$$

for any $p \in M$, and moreover

$$\dim\left(\mathrm{Lie}(X_1, \dots, X_m)(p)\right) = n \tag{2.2.2}$$

for any $p \in M$, we simply say that X satisfies the Hörmander condition over M, or that X is bracket-generating over M. We point out that, when Δ is bracket generating and $X_1, \dots, X_m$ is a frame of Δ as in (2.2.1), it is not always the case that (2.2.2) holds for $X_1, \dots, X_m$ (cf. [186]), so that, in this regard, a particular attention should be paid. We can now define the central notion of sub-Riemannian geometry.

Definition 2.2.5 (Sub-Riemannian manifold)**.** *A sub-Riemannian manifold is a triple*

$$(M, \Delta, g),$$

where M is a smooth manifold, Δ is a bracket-generating distribution and g is a metric over Δ.

Roughly speaking, Hörmander condition tells us that, although X may not exhaust the entire tangent space by itself, we can bridge this degeneration by using commutations. This heuristic interpretation suggests that, under the Hörmander condition, we may have enough horizontal curves to join points in M. This statement is the content of the celebrated *Chow–Rashevskii Theorem*. We refer the reader to [88] for the original reference, while [232, 166] offer a more modern treatment. Since in the following we will be mainly interested in the case $M = \Omega$, we continue the exposition in this particular case.

Theorem 2.2.6 (Chow-Rashevskii)**.** *Let Ω be a domain, and assume that X satisfies the Hörmander condition on Ω. The following properties hold.*

(i) (Ω, d_Ω) *is a Carnot-Carathéodory space.*

(ii) *For any domain $\tilde{\Omega} \subseteq \Omega$ there exists $C_{\tilde{\Omega}} > 0$ and $r \in \mathbb{N}$, $r \geq 1$, such that*

$$C_{\tilde{\Omega}}^{-1}|x-y| \leqslant d_\Omega(x,y) \leqslant C_{\tilde{\Omega}}|x-y|^{\frac{1}{r}} \qquad \text{for any } x, y \in \tilde{\Omega}.$$

In addition, if $m < n$, then $r > 1$.

Combining properties (i) and (ii), we see that a family of Hörmander vector fields generates a continuous Carnot-Carathéodory space. Remarkably, when $m < n$, property (ii) states that d_Ω is not Euclidean at any scale, so that again, from the metric viewpoint, our frameworks cannot be reduced to the Euclidean one. At this stage, the reader may wonder whether any continuous Carnot-Carathéodory space generated by even smooth vector fields falls within Definition 2.2.4. As the next example shows, the answer is negative. The reader is referred to [137] for other examples in the non-smooth setting.

Example 2.2.7. Let us consider the two linearly independent vector fields X, Y defined on $\mathbb{R}^3$ by

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial y} + \varphi(x)\frac{\partial}{\partial z},$$

where $\varphi(x) := \psi(x) + \psi(-x)$ and $\psi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\psi(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varphi^{(k)}(0) = 0$ for any $k \in \mathbb{N}$, it is easy to see that

$$[X, [\ldots, [X, Y] \ldots](0, y, z) = [Y, [\ldots, [X, Y] \ldots](0, y, z) = 0$$

for any $y, z \in \mathbb{R}$, so that X, Y do not satisfy the Hörmander condition. It is not difficult to show that they induces a continuous Carnot-Carathéodory distance d on $\mathbb{R}^3$. Indeed, let $A = (x, y, z)$ and $B = (x_1, y_1, z_1)$ in $\mathbb{R}^3$. We construct an horizontal curve joining them whose horizontal length tends to zero as A tends to B in the Euclidean topology. First, notice that moving along the X direction the induced Carnot-Carathéodory distance is comparable with the Euclidean one. Hence, without loss of generality, we can assume that $x = x_1 = 0$. Moreover, since $Y = \frac{\partial}{\partial y}$ on $\{x = 0\}$, then moving along the Y direction inside $\{x = 0\}$ the induced Carnot-Carathéodory distance is comparable with the Euclidean one. Hence we assume that $y_1 = y$. The last step is to join $(0, y, z)$ and $(0, y, z_1)$. We assume, without loss of generality, that $z_1 > z$. Let us set

$$\delta := -\frac{1}{\log(\sqrt{z_1 - z})}$$

then $\delta \to 0^+$ as $z_1 \to z$. Let us define the curves $\gamma_1, \ldots, \gamma_4 : [0, 1] \to \mathbb{R}^3$ by

$$\gamma_1(t) = (0, y, z) + t(\delta, 0, 0),$$

$$\gamma_2(t) = (\delta, y, z) + t\left(0, \frac{z_1 - z}{\varphi(\delta)}, z_1 - z\right),$$

$$\gamma_3(t) = \left(\delta, y + \frac{z - z_1}{\varphi(\delta)}, z_1\right) + t(-\delta, 0, 0)$$

and

$$\gamma_4(t) = \left(0, y + \frac{z_1 - z}{\varphi(\delta)}, z_1\right) + t\left(0, \frac{z - z_1}{\varphi(\delta)}, 0\right)$$

it is easy to see that they are horizontal and that they connect $(0, y, z)$ and $(0, y, z_1)$. Moreover, a quick computation shows that

$$d((0, y, z), (0, y, z_1)) \leqslant 2\delta + \frac{z_1 - z}{\varphi(\delta)} = -\frac{2}{\log(\sqrt{z_1 - z})} + \sqrt{z_1 - z}.$$

As the right hand side tends to zero as $z_1 \to z$, the conclusion follows.

2.3 Lipschitz spaces, Hölder spaces and embeddings

When (Ω, d_Ω) is a Carnot-Carathéodory space, we have at our disposal the tools and methods of analysis in metric spaces. For instance we can talk about *Lipschitz* and *Hölder* functions.

Definition 2.3.1 (Lipschitz spaces)**.** *The* horizontal Lipschitz space *is defined by*

$$\mathrm{Lip}(\Omega, d_\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \, : \, \sup_{x \neq y,\, x,y \in \Omega} \frac{|u(x) - u(y)|}{d_\Omega(x,y)} < +\infty \right\}.$$

Moreover, we say that $u \in \mathrm{Lip}_{loc}(\Omega, d_\Omega)$ if every point $x \in \Omega$ has a neighbourhood U such that $u \in \mathrm{Lip}(U, d_\Omega)$.

As in the Euclidean setting, Lipschitz function are strictly related to $W^{1,\infty}_X$-functions, as the next characterization shows (cf. [151]).

Proposition 2.3.2. *If (Ω, d_Ω) is a Carnot-Carathéodory space, then*

$$W^{1,\infty}_{X,loc}(\Omega) = \mathrm{Lip}_{loc}(\Omega, d_\Omega).$$

In particular, when (Ω, d_Ω) is a continuous Carnot-Carathéodory space, each function $u \in W^{1,\infty}_{X,loc}(\Omega)$ admits a continuous representative, that is

$$W^{1,\infty}_{X,loc}(\Omega) \subseteq C(\Omega). \tag{2.3.1}$$

Indeed, if $u \in W^{1,\infty}_{X,loc}(\Omega)$ and $x, y \in \Omega$, then, if $x, y \in K \Subset \Omega$, it holds that

$$|u(x) - u(y)| = d_\Omega(x,y) \frac{|u(x) - u(y)|}{d_\Omega(x,y)} \leqslant d_\Omega(x,y) \sup_{z \neq w \in K} \frac{|u(z) - u(w)|}{d_\Omega(z,w)},$$

and the right side goes to zero as $x \to y$. Therefore, in the following we will identify functions $u \in W^{1,\infty}_{X,loc}(\Omega)$ with their continuous representatives, of course provided that the underlying Carnot-Carathéodory space is continuous.

Definition 2.3.3 (Hölder spaces)**.** *If $\alpha \in (0,1)$, we define the* Folland-Stein Hölder spaces *as*

$$C^{0,\alpha}_X(\Omega) := \left\{ u : \Omega \longrightarrow \overline{\mathbb{R}} \, : \, \sup_{x \neq y,\, x,y \in \Omega} \frac{|u(x) - u(u)|}{d_\Omega(x,y)^\alpha} < +\infty \right\}$$

and

$$C^{0,\alpha}_{X,loc}(\Omega) := \left\{ u : \Omega \longrightarrow \overline{\mathbb{R}} \, : \, \sup_{x \neq y,\, x,y \in K} \frac{|u(x) - u(u)|}{d_\Omega(x,y)^\alpha} < +\infty \text{ for any compact set } K \Subset \Omega \right\}.$$

Moreover, when $E \subseteq \Omega$ and $u : \Omega \longrightarrow \overline{\mathbb{R}}$ we set

$$\|u\|_{0,\alpha,E} := \sup_{x \in E} |u(x)| + \sup_{x \neq y,\, x,y \in E} \frac{|u(x) - u(u)|}{d_\Omega(x,y)^\alpha}.$$

From these definitions, arguing as in the Lipschitz case, it is clear that

$$C^{0,\alpha}_X(\Omega) \subseteq C^{0,\alpha}_{X,loc}(\Omega) \subseteq C(\Omega).$$

As usual, in order to define a notion of convergence on $C^{0,\alpha}_{X,loc}(\Omega)$, we say that a sequence $(u_h)_h \subseteq C^{0,\alpha}_{X,loc}(\Omega)$ converges to $u \in C^{0,\alpha}_{X,loc}(\Omega)$ if it holds that

$$\lim_{h\to\infty} \|u_h - u\|_{0,\alpha,K} = 0$$

for any compact set $K \Subset \Omega$. If we fix an increasing sequence $(\Omega_k)_k$ of open subsets of Ω such that $\Omega_k \Subset \Omega_{k+1} \Subset \Omega$ and $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$, and for any $u, v \in C^{0,\alpha}_{X,loc}(\Omega)$ we define

$$\varrho(u,v) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|u-v\|_{0,\alpha,\Omega_k}\},$$

it is easy to see that ϱ is a translation-invariant distance on $C^{0,\alpha}_{X,loc}(\Omega)$ which induces the above-defined convergence. When the Carnot-Carathéodory structure is induced by a system of Hörmander vector fields, beside Proposition 2.3.2 we have the following Morrey-Campanato-type embedding results.

Proposition 2.3.4 (Morrey-Campanato embeddings). *Assume that X satisfies the Hörmander condition on Ω. There exists $Q \in (1,\infty)$, which depends only on n, Ω and X, such that the following facts hold.*

(i) $W^{1,p}_X(\Omega) \subseteq C^{0,1-\frac{Q}{p}}_{X,loc}(\Omega)$ *for any $p > Q$, and the inclusion is continuous.*

(ii) *The inclusion $W^{1,p}_X(\Omega) \subseteq C^{0,\beta}_{X,loc}(\Omega)$ is compact for any $p > Q$ and for any $\beta \in [0, 1-\frac{Q}{p})$.*

(iii) $W^{1,p}_{X,0}(\Omega) \subseteq C^{0,1-\frac{Q}{p}}_X(\Omega) \cap C(\overline{\Omega})$ *for any $p > Q$.*

In the setting of Hörmander vector fields, these results were first proved in [201], and it was later realized that they continue to hold in the general setting of metric measure spaces satisfying a doubling property and a Poincaré inequality (cf. [170, Lemma 9.2.12]). Beside these embeddings, we also recall a Poincaré-type inequality for trace zero functions. We refer the reader to [75, 206] for the Carnot-Carathéodory setting and to [52, Theorem 6.21] for a version in PI spaces.

Theorem 2.3.1. *Let $X = (X_1, ..., X_m)$ be a smooth family of Hörmander vector fields in $\Omega_0 \subseteq \mathbb{R}^n$. Let $\Omega \Subset \Omega_0$ be a bounded domain and let $1 \leqslant p < \infty$. Then there exists a constant $c = c(\Omega, p) > 0$ such that*

$$\int_\Omega |u|^p\,dx \leqslant c \int_\Omega |Xu|^p\,dx$$

for any $u \in W^{1,p}_{X,0}(\Omega)$.

We give a simple corollary which will be very useful in the sequel.

Corollary 2.3.5. *Under the same hypotheses as above, for every* $g \in W^{1,p}_X(\Omega)$ *there exists a constant* $K = K(\Omega, p, g) > 0$ *such that*

$$\int_\Omega |u|^p \, dx \leqslant K \left(1 + \int_\Omega |Xu|^p \, dx\right)$$

for any $u \in W^{1,p}_{X,g}(\Omega)$.

Proof. Let $u \in W^{1,p}_{X,g}(\Omega)$. Then by definition $u - g \in W^{1,p}_{X,0}(\Omega)$. Therefore, thanks to the previous result, we can estimate as follows.

$$\begin{aligned}\int_\Omega |u|^p dx &\leqslant 2^{p-1} \int_\Omega |u-g|^p dx + 2^{p-1} \int_\Omega |g|^p dx \\ &\leqslant c2^{p-1} \int_\Omega |Xu - Xg|^p dx + 2^{p-1} \int_\Omega |g|^p dx \\ &\leqslant c2^{2p-2} \int_\Omega |Xu|^p dx + c2^{2p-2} \int_\Omega |Xg|^p dx + 2^{p-1} \int_\Omega |g|^p dx.\end{aligned}$$

The thesis follows setting $K = c2^{2p-2} + +c2^{2p-2} \int_\Omega |Xg|^p dx + 2^{p-1} \int_\Omega |g|^p dx$. □

Chapter 3

Carnot groups

As main reference for this chapter, we refer the reader to [54, Chapter 2].

3.1 Lie groups

Let us briefly recall some basic preliminaries form the theory of Lie groups.

Definition 3.1.1 (Lie groups)**.** *A Lie group* $(\mathbb{G}, \cdot)$ *is a smooth manifold* $\mathbb{G}$ *which is endowed with a group law* $\cdot$ *and which satisfies the following properties.*

- *The map* $p \mapsto p^{-1}$ *is smooth.*
- *The map* $(p, q) \mapsto p \cdot q$ *is smooth.*

The unit of $\mathbb{G}$ *is denoted by* e*.*

Definition 3.1.2 (Lie group homomorphisms)**.** *A* Lie group homomorphism *between two Lie groups* $(\mathbb{G}_1, \cdot)$ *and* $(\mathbb{G}_2, \star)$ *is a smooth group homomorphism. A* Lie group isomorphism *is a bijective Lie group homomorphism.*

Definition 3.1.3 (Left translations)**.** *Let* $(\mathbb{G}, \cdot)$ *be a Lie group. For any* $p \in \mathbb{G}$*, we define the* left-translation by p *as the diffeomorphism* $\tau_p : \mathbb{G} \longrightarrow \mathbb{G}$ *given by*

$$\tau_p(q) := p \cdot q$$

for any $q \in \mathbb{G}$*.*

Definition 3.1.4 (Left-invariant vector field)**.** *Let* $(\mathbb{G}, \cdot)$ *be a Lie group. A vector field* X *is called* left-invariant *if*

$$d\tau_p|_e (X|_e) = X|_p$$

for any $p \in \mathbb{G}$*.*

Given a Lie group $(\mathbb{G}, \cdot)$, we denote by $\mathfrak{g}$ the set of left-invariant vector fields. We already know that the set of all vector fields, endowed with the Lie bracket (1.2.1), is a real Lie algebra. The next proposition shows that the same holds for $\mathfrak{g}$.

Proposition 3.1.5 (Lie algebra of a Lie group)**.** *Let* $(\mathbb{G}, \cdot)$ *be a Lie group. Let us define*

$$\mathfrak{g} := \{X \in \Gamma(T\mathbb{G}) \,:\, X \text{ is left-invariant}\}.$$

Then $(\mathfrak{g}, [\cdot,\cdot])$ *is a real Lie algebra, which is called the* Lie algebra of $(\mathbb{G}, \cdot)$*. Moreover, the map*

$$X \mapsto X|_e$$

is a vector space isomorphism from $\mathfrak{g}$ *to* $T_e\mathbb{G}$*.*

Despite the definition of Lie algebra of a Lie group may seem ambiguous, it is the case that every finite dimensional Lie algebra arises as Lie algebra of a (connected and simply connected) Lie group (cf. [54, Theorem 2.2.14]). Lie groups and their Lie algebras are related in the following sense (cf. [54, Theorem 2.1.50]).

Proposition 3.1.6. *Let* $(\mathbb{G}_1, \cdot)$ *and* $(\mathbb{G}_2, \star)$ *be two Lie groups. Let*

$$\varphi : \mathbb{G}_1 \longrightarrow \mathbb{G}_2$$

be a Lie group homomorphism. Then

$$d\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$$

is a Lie algebra homomorphism.

In particular, isomorphic Lie groups have isomorphic Lie algebras. While the converse is false in general, we will be shortly interested in a particular situation in which Lie groups are completely characterized by their Lie algebras. Given $X \in \mathfrak{g}$, we let γ_X be the *integral curve* of X starting from e, i.e. the unique smooth curve (cf. [54, Theorem 2.1.56]) which solves the initial value problem

$$\begin{cases} \dot{\gamma}_X(t) = X|_{\gamma_X(t)} \\ \gamma_X(0) = e \end{cases}$$

By the left-invariance of X and standard ODE results, we know that γ_X is defined on the whole $\mathbb{R}$ (cf. [54, Proposition 2.1.53]). We are hence allowed to give the following fundamental definition.

Definition 3.1.7 (Exponential map)**.** *Let* $(\mathbb{G}, \cdot)$ *be a Lie group. We define the* exponential map $\exp : \mathfrak{g} \longrightarrow \mathbb{G}$ *by*

$$\exp(X) := \gamma_X(1).$$

3.2 Lie algebra stratifications and abstract Carnot groups

Definition 3.2.1 (Stratified Lie algebra). *We say that a Lie algebra* $\mathfrak{g}$ *admits a* stratification of step k *if there exist linear subspaces* $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ *of* $\mathfrak{g}$ *such that*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_k \neq \{0\}, \quad [\mathfrak{g}_1, \mathfrak{g}_k] = \{0\} \tag{3.2.1}$$

for any $i = 2, \ldots, k$*, where* $[\mathfrak{g}_1, \mathfrak{g}_i]$ *is the subspace of* $\mathfrak{g}$ *generated by the commutators* $[X, Y]$ *with* $X \in \mathfrak{g}_1$ *and* $Y \in \mathfrak{g}_i$*. We denote by* k *the* step *of* $\mathfrak{g}$*, by* $m := \dim(\mathfrak{g}_1)$ *its* rank *and by* $n := \dim(\mathfrak{g})$ *its* dimension*.*

Definition 3.2.2 (Adapted basis). *If* $\mathfrak{g}$ *is a stratified Lie algebra. we say that a basis* $X = (X_1, \ldots, X_n)$ *of* $\mathfrak{g}$ *is* adapted to the stratification *whether*

$$(X_{h_{j-1}+1}, \ldots, X_{h_j}) \text{ is a basis of } \mathfrak{g}_j \text{ for any } j = 1, \ldots, k,$$

where $h_0 := 0$ *and* $h_j := \sum_{i=1}^{j} \dim(\mathfrak{g}_i)$*.*

We are ready to provide the main definition of this chapter.

Definition 3.2.3 (Carnot group). *A* Carnot Group *is a connected and simply connected Lie group* $(\mathbb{G}, \cdot)$ *whose Lie algebra admits a stratification. If the stratification of* $\mathfrak{g}$ *is as in Definition 3.2.1, we say that* $\mathbb{G}$ *has* step k, rank m *and by* dimension n*.*

The Carnot group definition is well-posed, since two stratification of a Lie algebra share the same step and rank (cf. [54, Proposition 2.2.8]). In particular the following *homogeneous dimension* is well defined.

Definition 3.2.4 (Homogeneous dimension). *Let* $\mathbb{G}$ *be a Carnot group. We call*

$$Q := \sum_{i=1}^{k} i \dim(\mathfrak{g}_i)$$

its homogeneous dimension*.*

We just point out that Q may in general differ from the topological dimension n, and that should be seen as the metric dimension of a Carnot group, as explained more in detail in the forthcoming Section 3.5. The property of being a Carnot group, as well as the step, the dimension and the rank, are preserved by Lie group isomorphisms (cf. [54, Proposition 2.2.10]). Even better, since the differential of a Lie group isomorphism is a Lie algebra isomorphism, then Lie group isomorphisms preserve stratifications. With the following crucial result, we will be allowed to identify Carnot groups which special polynomial groups over the Euclidean space. Loosely speaking, this identification is allowed since the stratification of the Lie algebra let the exponential map be a Lie group isomorphism between a Carnot group and its Lie algebra, as soon as on the latter is defined a suitable group law given in terms of the so-called *Campbell–Hausdorff formula* (cf. [54, Definition 2.2.11]). The following statement is a summary of [54, Corollary 2.2.15], [54, Proposition 2.2.17] and [54, Proposition 2.2.18].

Theorem 3.2.5. *If $(\mathbb{G}, \cdot)$ is a Carnot group, then $\mathfrak{g}$ can be equipped with a suitable group law $\star$ which realizes $(\mathfrak{g}, \star)$ as a Carnot group, and for which*

$$\exp : (\mathfrak{g}, \star) \longrightarrow (\mathbb{G}, \cdot)$$

is a Lie group isomorphism.

As an interesting consequence of Theorem 3.2.5, it can be proved that every Lie algebra isomorphism between two stratified Lie algebras arises as differential of a Lie group isomorphism between the corresponding stratified Lie groups. Finally, to identify a Carnot group with the Euclidean space, we introduce the so-called *exponential coordinates of the first kind.*

Definition 3.2.6 (Exponential coordinates of the first kind)**.** *Let $(\mathbb{G}, \cdot)$ be a Carnot group. Let $X_1, \ldots, X_n$ be an adapted basis of $\mathfrak{g}$. Then we identify $\mathbb{R}^n$ with $\mathfrak{g}$ via the diffeomorphism*

$$(x_1, \ldots, x_n) \mapsto x_1 X_1 + \ldots + x_n X_n.$$

These coordinates are called exponential coordinates of the first kind*.*

At this stage we just need to pull-back the group law of $\mathfrak{g}$ to turn our coordinate map into a Lie group isomorphism. Hence, in the following, we will always see Carnot groups as $\mathbb{R}^n$ endowed with pull-backs of group laws given by the *Campbell–Hausdorff formula.* According with this identification, we denote by π both the smooth section defined by

$$\pi(y) = \sum_{j=1}^{m} y_j X_j(y)$$

for any $y \in \mathbb{G}$ and the vector valued map

$$\pi(y) = (y_1, \ldots, y_m) \tag{3.2.2}$$

for any $y \in \mathbb{G}$. Let us list some relevant instances of Carnot groups.

Example 3.2.7 (The Euclidean space)**.** The most trivial Carnot group is the additive Euclidean group $(\mathbb{R}^n, +)$, for any $n \geq 1$. These are the unique examples of Abelian Carnot groups, whose Lie algebra admits the trivial stratification

$$\mathfrak{g} = \mathfrak{g}_1 = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}.$$

Therefore, $(\mathbb{R}^n, +)$ is a Carnot group of dimension n, rank n and step 1.

Example 3.2.8 (The Heisenberg group)**.** The first non-trivial family of examples is constituted by the class of *Heisenberg groups* $(\mathbb{H}^n, \cdot)$, for any $n \geq 1$, where we adopted the standard convention $\mathbb{H}^n = \mathbb{R}^{2n+1}$ and where $\cdot$ is the non-Abelian the group law defined by

$$p \cdot p' = \left(\bar{x} + \bar{x}', \bar{y} + \bar{y}', t + t' + \sum_{j=1}^{n}(x'_j y_j - x_j y'_j)\right).$$

Here we denoted points $p \in \mathbb{R}^{2n+1}$ by $p = (\bar{x}, \bar{y}, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t)$. A stratification of $\mathfrak{g}$ is given by

$$\mathfrak{g}_1 = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \qquad \text{and} \qquad \mathfrak{g}_2 = \operatorname{span}\{T\},$$

where

$$X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial t} \qquad \text{and} \qquad T = \frac{\partial}{\partial t}$$

for $j = 1, \dots, n$. Therefore, $(\mathbb{H}^n, \cdot)$ is a Carnot group of dimension $2n+1$, rank $2n$ and step 2.

We point out that what we called a *Carnot group* in Definition 3.2.3 is what is often referred to in the literature as *stratified Lie group*. Following this convention, a Carnot group is a stratified Lie group which is endowed with a particular sub-Riemannian metric, which is well-behaved with respect to the Lie group structure. More precisely, let us give the following definition.

Definition 3.2.9 (Left-invariant Riemannian metric)**.** *Let $\mathbb{G}$ be a Lie group. A Riemannian metric $\langle \cdot, \cdot \rangle$ is called* left-invariant *if*

$$\langle d\tau_q|_p(u), d\tau_q|_p(v) \rangle_{p \cdot q} = \langle u, v \rangle_p$$

for any $p, q \in \mathbb{G}$ and any $u, v \in T_p\mathbb{G}$.

Nevertheless, since every stratified Lie group can be easily equipped with a left-invariant Riemannian metric, the two notions are actually equivalent. To avoid confusion, in the following we will talk about *Riemannian Carnot groups* to specify that a suitable left-invariant Riemannian metric is fixed. Since the Lie algebra of a Carnot group $\mathbb{G}$ is by definition stratified, then $\mathbb{G}$ turns out to be a sub-Riemannian manifold. Indeed, if an adapted basis $X_1, \dots, X_n$ is fixed, the m-dimensional distribution $\mathcal{G}$ defined by

$$\mathcal{G}|_p = \operatorname{span}\{X_1|_p, \dots, X_m|_p\} \tag{3.2.3}$$

for any $p \in \mathbb{G}$ is by definition bracket-generating. To conclude, it suffices to endow $T\mathbb{G}$ with the unique left-invariant Riemannian metric which makes $X_1, \dots, X_n$ orthonormal, and to restrict the latter to the horizontal distribution, obtaining in turn a *left-invariant sub-Riemannian metric*. Again, we will talk about *sub-Riemannian Carnot groups* when a left-invariant sub-Riemannian metric is assigned on $\mathcal{G}$.

3.3 Homogeneous Carnot groups

It is common in literature to call a Carnot group *homogeneous* when it is endowed with a family of suitable *dilations*. Let $\mathbb{G}$ be a Carnot group, and let us fix an adapted basis $X_1, \dots, X_n$.

Moreover, we choose a coordinate system $e_1, \dots, e_n$ on $T_e\mathbb{G}$ in such a way that

$$X_i|_e = e_i$$

for any $i = 1, \dots, n$. In the following, we refer to $X_1, \dots, X_m$ as *generating vector fields.* As already pointed out, from now we identify $\mathbb{G}$ with $\mathbb{R}^n$ by means of exponential coordinates of the first kind $(x_1 \dots, x_n)$ associated with $X_1, \dots, X_n$. Moreover, for the sake of notational simplicity we will write

$$x^{(j)} = (x_{h_{j-1}+1}, \dots, x_{h_j})$$

for any $j = 1, \dots, k$ in such a way that

$$p = (x_1, \dots, x_n) = (x^{(1)}, \dots, x^{(k)})$$

for any $p \in \mathbb{G}$. In these coordinates, the aforementioned dilations can be defined as follows.

Definition 3.3.1 (Dilations)**.** *For any $\lambda > 0$, we define the map $\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}$ by*

$$\delta_\lambda(x^{(1)}, x^{(2)}, \dots, x^{(k)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^k x^{(k)})$$

for any $p = (x^{(1)}, \dots, x^{(k)}) \in \mathbb{G}$. Moreover, if we let

$$\alpha_i = j$$

whenever $h_{j-1} + 1 \leqslant i \leqslant h_j$ and $j = 1, \dots, k$, then α_j is called the homogeneity degree *of the variable x_i.*

As the differential $d\delta_\lambda : \mathfrak{g} \longrightarrow \mathfrak{g}$ is easily seen to be a Lie algebra isomorphism, then δ_λ is a Lie group isomorphism. Given $N \in \mathbb{N}$, we call a function $f : \mathbb{G}^N \longrightarrow \mathbb{R}$ *homogeneous of degree* α, for a given $\alpha \in \mathbb{R}$, if

$$f(\delta_\lambda(p_1), \dots, \delta_\lambda(p_N)) = \lambda^\alpha f(p_1, \dots, p_N)$$

for any $p_1, \dots, p_N \in \mathbb{G}$. Let us describe some properties of $(\mathbb{G}, \cdot)$ (cf. [54, Proposition 2.2.22]), [262, Proposition 2.3] and [262, Proposition 2.4]).

Proposition 3.3.2. *Let $(\mathbb{G}, \cdot)$ be a Carnot group of step k, rank m and dimension n. Let us fix coordinates $(x_1, \dots, x_n)$. The following facts hold.*

- *If $p = (x^{(1)}, x^{(2)}, \dots, x^{(k)})$ and $q = (y^{(1)}, x^{(2)}, \dots, y^{(k)})$, then*

  $$p \cdot q = \left(x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)} + \mathcal{Q}^{(2)}(p, q), \dots, x^{(k)} + y^{(k)} + \mathcal{Q}^{(k)}(p, q)\right),$$

  where

  $$\mathcal{Q}^j(p, q) = \left(\mathcal{Q}_{h_{j-1}+1}(p, q), \dots, \mathcal{Q}_{h_j}(p, q)\right)$$

  and each Q_i is a homogeneous polynomial of degree α_i.

- *If* $m < i \leqslant n$, *then*
$$\mathcal{Q}_i(p,q) = -\mathcal{Q}_i(-q,-p)$$
for any $p, q \in \mathbb{G}$.

- *If* $m < i \leqslant n$, *then*
$$\mathcal{Q}_i(p,0) = \mathcal{Q}_i(0,q) = \mathcal{Q}_i(p,p) = \mathcal{Q}_i(p,-p) = 0$$
for any $p, q \in \mathbb{G}$. *In particular,*
$$p^{-1} = (-x_1, \ldots, -x_n) \qquad \text{and} \qquad e = 0$$
for any $p = (x_1, \ldots, x_n) \in \mathbb{G}$.

- *If* $1 < i \leqslant k$ *and* $h_{i-1} < j \leqslant h_i$, *then*
$$\mathcal{Q}_j(p,q) = \mathcal{Q}_i(x_1, \ldots, x_{h_{i-1}}, y_1, \ldots, y_{h_{i-1}})$$
for any $p = (x_1, \ldots, x_n) \in \mathbb{G}$ *and any* $q = (y_1, \ldots, y_n) \in \mathbb{G}$.

- *If* $j = 1, \ldots, n$, *then* X_j *has polynomial coefficients. Moreover, when* $1 \leqslant l \leqslant k$ *and* $h_{l-1} < j < h_l$, *then*
$$X_j = \frac{\partial}{\partial x_j} + \sum_{i > h_l} q_{i,j} \frac{\partial}{\partial x_i},$$
where
$$q_{i,j}(p) = \frac{\partial Q_i}{\partial y_j}(p,q)\Big|_{q=0}.$$
In particular,
$$q_{i,j}(x_1, \ldots, x_n) = q_{i,j}(x_1, \ldots, x_{h_{l-1}})$$
and
$$q_{i,j}(0) = 0.$$

3.4 Invariant distances and homogeneous norms

As main reference for this section, we refer the reader to [262]. As we already know, a Carnot group $\mathbb{G}$ equipped with the Carnot-Carathéodory distance induced by the family $X = (X_1, \ldots, X_m)$ of generating vector fields becomes a Carnot-Carathéodory space. Nevertheless, the rich algebraic structure of a Carnot group allows to consider different distances, which typically have the advantage to be more explicitly calculable.

Definition 3.4.1 (Invariant distance)**.** *A distance* $d : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty)$ *is called* invariant *if the following properties hold.*

- *d is continuous with respect to the Euclidean topology.*

- *d is invariant under left translations, i.e.*

$$d(\tau_p(q_1), \tau_p(q_2)) = d(q_1, q_2)$$

for any $p, q_1, q_2 \in \mathbb{G}$.

- *d is 1-homogeneous with respect to dilations, i.e. if*

$$d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$$

for any $\lambda > 0$ and any $p, q \in \mathbb{G}$.

Invariant distances in a Carnot group are in one-to-one correspondence with *homogeneous norms*.

Definition 3.4.2. *A function $\| \cdot \| : \mathbb{G} \longrightarrow [0, +\infty)$ is called a* homogeneous norm *if the following properties hold.*

- *$\| \cdot \|$ is continuous with respect to the Euclidean topology.*
- *It holds that*

$$\|p\| = \|p^{-1}\|$$

for any $p \in \mathbb{G}$.

- *$\| \cdot \|$ is 1-homogeneous with respect to dilations, i.e. if*

$$\|\delta_\lambda(p)\| = \lambda \|p\|$$

for any $\lambda > 0$ and any $p \in \mathbb{G}$.

- *$\| \cdot \|$ satisfies*

$$\|p \cdot q\| \leqslant \|p\| + \|q\|$$

for any $p, q \in \mathbb{G}$.

It is a simple exercise to check that, given a homogeneous norm $\| \cdot \|$, then

$$d(p, q) = \|p^{-1} q\| \tag{3.4.1}$$

is an invariant distance. In the same way, if d is an invariant distance, then

$$\|p\| = d(p, 0)$$

is a homogeneous norm. The following result motivates our initial statement. We refer the readers to [54, Proposition 5.1.4], [54, Proposition 5.2.4] and [54, Theorem 5.2.8], although they should be careful, since the authors of [54] gave there a slightly different definition of homogeneous norm.

Proposition 3.4.3. *Let* $(\mathbb{G}, \cdot)$ *be a Carnot group, and let* $d_{\mathbb{G}}$ *be its Carnot-Carathéodory distance. Then* $d_{\mathbb{G}}$ *is an invariant distance. Moreover, every other invariant distance is equivalent to* $d_{\mathbb{G}}$*.*

Another interesting feature of the Carnot-Carathéodory distance $d_{\mathbb{G}}$ is that it is a *complete*, or *geodesic*, distance in the sense of [64]. More precisely, for any $x, y \in \mathbb{G}$, there always exists a horizontal curve $\gamma : [0,1] \longrightarrow \mathbb{G}$ such that

$$\dot{\gamma}(t) = \sum_{j=1}^{m} a_j(t) X_j(\gamma(t))$$

and

$$d_{\mathbb{G}}(x,y) = \int_0^1 |a(t)|\, dt.$$

Such curves are usually known as *optimal curves*, *geodesics* or *shortest paths*. We conclude this section introducing two interesting homogeneous norms.

Example 3.4.4 (Gauge-Koranyi norm)**.** The well-known *Gauge-Koranyi* norm is defined by

$$\|(x^{(1)}, \dots, x^{(k)})\| = \left(\sum_{j=1}^{k} |x^{(j)}|^{\frac{2k!}{j}} \right)^{\frac{1}{2k!}}$$

for any $p \in \mathbb{G}$. This homogeneous norm has the remarkable advantage of being smooth outside the origin (cf. [54, Example 5.1.2]). In the following, we denote by $d_{\mathfrak{g}}$ the invariant distance induced by the Gauge-Koranyi norm.

Example 3.4.5 (∞-norm)**.** Following [141, Theorem 5.1], we let

$$\|p\|_\infty = \max\left\{ \varepsilon_j |x^{(j)}|^{1/j} : j = 1, \dots, k \right\}$$

for any $p \in \mathbb{G}$, with constants $\varepsilon_1 = 1$ and $\varepsilon_j \in (0,1)$ for any $j = 2, \dots, s$ depending on the structure of $\mathbb{G}$.

3.5 Measures

In this section, following [262], we briefly recall some relevant measures that can be defined in a Carnot group. According to Section 3.3, we identify a Carnot group $(\mathbb{G}, \cdot)$ with $\mathbb{R}^n$ endowed with a polynomial group law as in Proposition 3.3.2. A first special role is played by the standard Lebesgue measure $\mathcal{L}^n$. The latter turns out to be a *Haar measure* of $(\mathbb{G}, \cdot)$, i.e. a Radon measure which is invariant under left-translations (cf. [133] for a thorough account on Haar measures). More precisely, the following statement holds (cf. [262, Proposition 2.19]).

Proposition 3.5.1. *Let* $(\mathbb{G}, \cdot)$ *be a Carnot group. Then* $\mathcal{L}^n$ *is a Haar measure of* $\mathbb{G}$*, meaning that*

$$|\tau_p(E)| = |E|$$

for any Borel set $E \subseteq \mathbb{G}$ and any $p \in G$. Moreover, if Q is the homogeneous dimension of $\mathbb{G}$ as in Definition 3.2.4, then

$$|\delta_\lambda(E)| = \lambda^Q |E|$$

for any Borel set $E \subseteq \mathbb{G}$ and any $\lambda > 0$. Finally, if d is any invariant distance, then

$$|B_d(p,r)| = |\overline{B_d(p,r)}| = r^Q |B_d(p,1)| = r^Q |B_d(0,1)| \tag{3.5.1}$$

for any $p \in G$ and any $r > 0$.

A first consequence of Proposition 3.5.1 is that, in view of (3.5.1), the metric measure space $(\mathbb{G}, d, \mathcal{L}^n)$ is *Ahlfors-regular* in the sense of [262, Definition 2.25] (cf. [17] for further insights about Ahlfors-regularity). Since Part V and Part VI are devoted to the study of hypersurfaces in Carnot groups, beside $\mathcal{L}^n$ and the anisotropic perimeter measure in Definition 1.4.4, we introduce suitable surface measures according to the general theory of Hausdorff measures in metric measure spaces. More precisely, if an invariant distance d is fixed and $s \in [0, n]$, the standard Carathéodory's construction (cf. [131]) allows to introduce, respectively, the *s-dimensional Hausdorff measure $\mathcal{H}^s_d$* and the *s-dimensional spherical Hausdorff measure $\mathcal{S}^s_d$*. For a precise definition, we refer to [262, Section 2.3]. As a general fact, $\mathcal{H}^s_d$ and $\mathcal{S}^s_d$ are Carathéodory outer measures. In addition, as for $\mathcal{L}^n$, they are-left invariant and s-homogeneous with respect to intrinsic dilations. Moreover, they are equivalent one to the other, and this property is not affected by choosing in the definition any other invariant distance. Therefore, the general notion of *Hausdorff dimension* of the metric space $(\mathbb{G}, d)$ is independent on the chosen invariant distance. As already mentioned, the latter differs from the topological dimension of $(\mathbb{G}, d)$. Indeed, from [262, Theorem 2.30], we infer that the Hausdorff dimension of $(\mathbb{G}, d)$ coincides with its homogeneous dimension Q. It is clear from Definition 3.2.4 that, in general $n \leqslant Q$, and that equality holds if and only if $\mathbb{G}, \cdot)$ is the Abelian Euclidean group $(\mathbb{R}^n, +)$.

3.6 Rectifiability

In a general metric setting, Hausdorff measure are strictly related to the notion of rectifiability. This notion, which is now part of the current vocabulary of a large community, has been thoroughly discussed in the Euclidean setting in [131], and has been systematically studied in a general metric space in [13], to which we refer to for the relevant definitions. However, as already pointed out in [13], the standard metric rectifiablility is not well-suited for Carnot groups, since, for instance, the first Heisenberg group $\mathbb{H}^1$ (cf. Example 3.2.8) does not contain non-trivial s-rectifiable sets when $s = 2, 3$ or 4 (cf. [13] for more precise statements and definitions). Accordingly, motivated by the study of sets of finite horizontal perimeter in Heisenberg groups (cf. [140]) and in general step-2 Carnot groups (cf. [141]), the authors of these papers introduced a suitable notion of *intrinsic (Q-1)-rectifiability*, which, morally speaking, is tailored to deal with *good* hypersurfaces in Carnot groups. To state precisely what we mean by good hypersurface, we provide the following definition (cf. [262, Definition 4.20]).

Definition 3.6.1 ($\mathbb{G}$-regular hypersurfaces)**.** *We say that $S \subseteq \mathbb{G}$ is a G-*regular hypersurface *if, for any $p \in S$, there exists an open neighborhood U of p and a function $f \in C^1_{\mathbb{G}}(U)$ such that*

$$S \cap U = \{q \in \mathbb{G} \,:\, f(q) = 0\} \qquad \text{and} \qquad \nabla^{\mathbb{G}} f \neq 0 \text{ on } U.$$

Here, by $C^1_{\mathbb{G}}$ and $\nabla^{\mathbb{G}}$, we mean respectively the space C^1_X and the X-gradient X, where $X = (X_1, \ldots, X_m)$ is a chosen basis of the first layer $\mathfrak{g}_1$. A good motivating feature of $\mathbb{G}$-regular hypersurfaces is that they are the right hypersurfaces to ensure the validity of suitable *intrinsic implicit function theorems* (cf. [142] and [262, Theorem 4.24]). Moreover, as stated in [262, Proposition 4.21], $\mathbb{G}$-regular hypersurfaces admits, at every point, a unique *intrinsic tangent hyperplane.* Intrinsic tangent hyperplanes are a special class of Euclidean hyperplanes which arise as blow up of $\mathbb{G}$-regular hypersurfaces S via intrinsic dilations, and can be described with the following crucial definition.

Definition 3.6.2 (Vertical hyperplanes)**.** *Let $(\mathbb{G}, \cdot)$ be a Carnot group, and let $X_1, \ldots, X_n$ be an adapted basis of $\mathfrak{g}$, with associated exponential coordinates $x_1, \ldots, x_n$. A hyperplane $S \subseteq \mathbb{G}$ is* vertical *whether there exists $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$, with $a \neq 0$, and $c \in \mathbb{R}$ such that*

$$S = \left\{ (x_1, \ldots, x_n) \in \mathbb{G} \,:\, \sum_{j=1}^{m} a_j x_j = c \right\}.$$

Let S be a vertical hyperplane, and let a be as in Definition 3.6.2. Letting $f(x) = \sum_{j=1}^{m_1} a_j x_j$, it is clear that S is itself a $\mathbb{G}$-regular hypersurface, since $S = \{f = 0\}$ and $\nabla^{\mathbb{G}} f = a$. Moreover, a simple computation shows that the horizontal unit normal $\nu^{\mathbb{G}}$ associated with the half-space delimited by S satisfies

$$\nu^{\mathbb{G}} \equiv \frac{a}{|a|},$$

whence vertical hyperplanes are hypersurfaces with constant horizontal normal. We point out that, as discussed thoroughly in the forthcoming Part V, the converse implication may fail in general. Once Definition 3.6.1 is available, we basically expect that a big portion of an intrinsic $(Q-1)$-rectifiable set coincides with $\mathbb{G}$-regular hypersurfaces. The accurate definition reads as follows (cf. [262, Definition 4.101]).

Definition 3.6.3 (Intrinsic rectifiability)**.** *A set $E \subseteq \mathbb{G}$ is* intrinsically $(Q-1)$-rectifiable *if there exists a sequence $(S_j)_j$ of $\mathbb{G}$-regular hypersurfaces such that*

$$\mathcal{H}_d^{Q-1}\left(E \setminus \bigcup_{j=1}^{\infty} S_j \right) = 0,$$

being d an arbitrary invariant distance on $\mathbb{G}$.

As already pointed out, Definition 3.6.3 plays a crucial role in the study of structural properties of sets of finite horizontal perimeter in Carnot groups. To motivate this assertion, let us give some further definitions. In the following, we denote by $P_{\mathbb{G}}$ the horizontal perimeter as in Definition 1.4.4 induced by the family $X = (X_1, \dots, X_m)$, and by $\nu^{\mathbb{G}}$ the associated horizontal normal as in Definition 1.4.5.

Definition 3.6.4. *Let d be an invariant distance. Let E be a set of locally finite horizontal perimeter. A point $p \in \mathbb{G}$ belongs to the* $\mathbb{G}$-reduced boundary *$\partial^*_{\mathbb{G}} E$ of E if*

$$P_{\mathbb{G}}(E, B_d(p, r)) > 0$$

for any $r > 0$, the limit

$$\lim_{r \to 0^+} \fint_{B_d(p,r)} \nu^{\mathbb{G}} \, dP_{\mathbb{G}}(E, \cdot)$$

exists and its Euclidean norm is equal to 1.

As happens for its Euclidean counterpart in view of the celebrated De Giorgi's structure and blow up theorems (cf. [111, 112]), the $\mathbb{G}$-reduced boundary of a $\mathbb{G}$-Caccioppoli set in a Carnot group of step 2 is intrinsically $(Q-1)$-rectifiable and supports the $\mathbb{G}$-perimeter measure. We refer to [140, 141, 262] and references therein for accurate statements. Moreover, we refer to [9, 204, 262] for further connections between the $\mathbb{G}$-perimeter measure and the $(Q-1)$-dimensional Hausdorff measures, some of which are discussed in Part V. Finally, we refer to [209] for an extension of the previous results beyond the step 2 setting.

3.7 Carnot-Carathéodory distances on domains

When $X = (X_1, \dots X_m)$ is a basis of the first layer $\mathfrak{g}_1$, it is natural to consider the Carnot-Carathéodory distance $d_{\mathbb{G}}$. On the other hand, if $\Omega \subseteq \mathbb{G}$ is a given domain, in order to endow Ω with a metric structure we may either restrict $d_{\mathbb{G}}$ to Ω or consider directly d_Ω. It is clear that

$$d_{\mathbb{G}} \leqslant d_\Omega,$$

since basically $\mathbb{G}$ contains more horizontal curves than Ω. However, these two distances may not be equivalent in general. To see this, it is sufficient to consider a horseshoe-shaped domain whose ends touch tangentially. Nevertheless, the following local result still holds (cf. [126]).

Proposition 3.7.1. *Let $\mathbb{G}$ be a Carnot group, and let $\Omega \subseteq \mathbb{G}$ be open and connected. Then d_Ω and $d_{\mathbb{G}}$ are locally equivalent on Ω.*

To prove Proposition 3.7.1, we need the following result.

Lemma 3.7.2. *Let $\mathbb{G}$ be a Carnot group, and let $\Omega \subseteq \mathbb{G}$ be open and connected. Then, for any $x_0 \in \Omega$, there exists $r > 0$ such that, for any $x, y \in B_{d_{\mathbb{G}}}(x_0, r)$, any optimal curve for $d_{\mathbb{G}}(x, y)$ lies in Ω.*

Proof. Assume by contradiction that there exists $x_0 \in \Omega$ and sequences $(x_h)_h$, $(y_h)_h$, $(\gamma_h)_h$ such that $d_{\mathbb{G}}(x_0, x_h), d_{\mathbb{G}}(x_0, y_h) < \frac{1}{h}$, $\gamma_h : [0, T_h] \longrightarrow \mathbb{G}$ is sub-unit and is optimal for $d_{\mathbb{G}}(x_h, y_h)$, and there exists $0 < t_h < T_h$ such that $z_h := \gamma_h(t_h) \in \partial\Omega$. Up to a subsequence, there exists $R > 0$ such that $(x_h)_h, (y_h)_h \subseteq B_{d_{\mathbb{G}}}(x_0, R) \Subset \Omega$. Set

$$D = \inf\{d_{\mathbb{G}}(z, w) \, : \, z \in \partial\Omega, \; w \in \partial B_{d_{\mathbb{G}}}(x_0, R)\}.$$

Since $B_{d_{\mathbb{G}}}(x_0, R) \Subset \Omega$, then $D > 0$. On one hand

$$d_{\mathbb{G}}(x_h, y_h) \leqslant d_{\mathbb{G}}(x_0, x_h) + d_{\mathbb{G}}(x_0, y_h) \to 0$$

as $h \to \infty$. On the other hand, in view of the choice of γ_h,

$$d_{\mathbb{G}}(x_h, y_h) = d_{\mathbb{G}}(x_h, z_h) + d_{\mathbb{G}}(y_h, z_h) \geq 2D > 0.$$

A contradiction then follows. □

Proof of Proposition 3.7.1. Assume that $\mathbb{G}$ nhas step k. In view of the previous considerations, we are left to show that for any domain $\tilde{\Omega} \Subset \Omega$ there exists $K_{\tilde{\Omega}} > 0$ such that $d_{\mathbb{G}} \geqslant K_{\tilde{\Omega}} d_\Omega$. Assume by contradiction that there exists a domain $\tilde{\Omega} \Subset \Omega$ and two sequences $(x_h)_h, (y_h)_h \subseteq \tilde{\Omega}$ such that

$$d_{\mathbb{G}}(x_h, y_h) < \frac{1}{h} d_\Omega(x_h, y_h)$$

for any $h \in \mathbb{N}$. Let D be the Euclidean diameter of $\tilde{\Omega}$. Since $\tilde{\Omega}$ is bounded, then $D < \infty$. Thanks to Theorem 2.2.6, we have that

$$d_{\mathbb{G}}(x_h, y_h) < \frac{1}{h} d_\Omega(x_h, y_h) \leqslant \frac{1}{h} \sup_{x,y \in \tilde{\Omega}} d_\Omega(x, y) \leqslant \frac{C_{\tilde{\Omega}}}{h} \sup_{x,y \in \tilde{\Omega}} |x - y|^{\frac{1}{k}} \leqslant \frac{C_{\tilde{\Omega}} D^{\frac{1}{k}}}{h}.$$

This implies that $d_{\mathbb{G}}(x_h, y_h) \to 0$. Therefore, up to a subsequence, we can assume that $x_h, y_h \to x_0$ for some $x_0 \in \Omega$. Choose r as in Lemma 3.7.2, and assume up to a subsequence that $(x_h)_h, (y_h)_h \subseteq B_{d_{\mathbb{G}}}(x_0, r)$. Then Lemma 3.7.2 implies that

$$d_{\mathbb{G}}(x_h, y_h) = d_\Omega(x_h, y_h),$$

a contradiction. □

Part II

Some differentiability results in Carnot-Carathéodory spaces

Chapter 4

Subgradients

4.1 Definition and first properties

We refer the reader to [243, 78] as main references for the definitions and the results of this chapter. In this chapter we study the so-called (X, N)-*subgradient* of a function $u \in W^{1,\infty}_{X,loc}(\Omega)$, introduced in [243] as a generalization of the classical Clarke's subdifferential (cf. [92]). Let us fix a little notation. If $E \subseteq \mathbb{R}^n$, we set $\overline{co}E$ to be the closure of

$$coE := \bigcap \{C \,:\, C \text{ is convex and } E \subseteq C\}.$$

It is easy to see that coE is convex and that $\overline{co}E$ is closed and convex. Moreover we set

$$\Lambda_n := \left\{ (\lambda_1, \ldots, \lambda_n) \,:\, 0 \leqslant \lambda_j \leqslant 1, \sum_{j=1}^{n} \lambda_j = 1 \right\}. \tag{4.1.1}$$

Definition 4.1.1. *Let X be a family of locally Lipschtiz continuous vector fields. Let $u \in W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$. We define the* (X, N)-subgradient *of u by*

$$\partial_{X,N} u(x) := \overline{co} \left\{ \lim_{n\to\infty} Xu(y_n) \,:\, y_n \to x,\, y_n \notin N \text{ and } \lim_{n\to\infty} Xu(y_n) \text{ exists} \right\}$$

for any $x \in \Omega$, where $N \subseteq \Omega$ is any Lebesgue-negligible set containing the non-Lebesgue points of Xu.

As we said, this notion generalizes Clarke's subdifferential. The latter, for a fixed $u \in W^{1,\infty}_{loc}(\Omega)$ and $x \in \Omega$, is defined by

$$\overline{co} \left\{ \lim_{n\to\infty} Du(y_n) \,:\, y_n \to x,\, y_n \notin N \text{ and } \lim_{n\to\infty} Du(y_n) \text{ exists} \right\}$$

where $N \subseteq \Omega$ is any Lebesgue-negligible set containing the non-differentiability points of Du. Notice that, in view of Rademacher's theorem, the above definition is well posed. Let us fix $u \in W^{1,\infty}_{loc}(\Omega)$. Then Morrey's inequality implies that any Lebesgue point of Du is a point of differentiability of u (cf. [193, Corollary 11.36]). Therefore, when $X = (\partial_1, \ldots, \partial_n)$, $\partial_{X,N} u$

coincides with Clarke's subdifferential. We begin by proving some properties of the (X, N)-subgradient with the help of the two following lemmas, for whose proof we refer to Section 4.5.

Lemma 4.1.2. *Let*

$$S := \left\{ \lim_{n\to\infty} Xu(y_n) \,:\, y_n \to x,\, y_n \notin N \text{ and } \exists \lim_{n\to\infty} Xu(y_n) \right\}$$

and, for any $k \geq 1$, let

$$A_k = \{Xu(y) \,:\, y \in B_{1/k}(x) \setminus N\}.$$

Then it follows that

$$\bigcap_{k=1}^{\infty} \overline{A}_k \subseteq S.$$

Lemma 4.1.3. *Let $(A_k)_k$ be a decreasing sequence of non-empty bounded subsets of $\mathbb{R}^m$, and let S be a non-empty, bounded subset of $\mathbb{R}^m$. Assume that*

$$\bigcap_{k=1}^{\infty} \overline{A}_k \subseteq S.$$

Then it follows that

$$\bigcap_{k=1}^{\infty} \overline{co}(\overline{A}_k) \subseteq \overline{co}(S).$$

Proposition 4.1.4. *Let u and N be as above. Then the following facts hold.*

(i) *$\partial_{X,N}u(x)$ is a non-empty, convex, closed and bounded subset of $\mathbb{R}^m$ for any $x \in \Omega$;*

(ii) *for any $x \in \Omega$*

$$\partial_{X,N}u(x) = \bigcap_{k=1}^{\infty} \overline{co}\{Xu(y) \,:\, y \in B_{1/k}(x) \setminus N\};$$

(iii) *if $u \in C^1_X(\Omega)$, then*

$$\partial_{X,N}u(x) = \{Xu(x)\}$$

for any $x \in \Omega$.

Proof. We start by proving (i). We fix $x \in U$ and show that $\partial_{X,N}u(x) \neq \emptyset$. Let $r > 0$ be small enough to have $B_r(x) \Subset U$. Then $u \in W^{1,\infty}_X(B_r(x))$. So we set $L := \|Xu\|_{L^\infty(B_r(x))}$. Let $(r_n)_n \subseteq (0, r)$ with $r_n \searrow 0$. Then, for any $n \in \mathbb{N}$, take $y_n \in B_{r_n}(x) \setminus N$. Then clearly y_n tends to x. Moreover, being y_n a Lebesgue point of Xu, it follows that

$$|Xu(y_n)| = \left| \lim_{s\to 0^+} \fint_{B_s(y_n)} Xu(z)\,dz \right| \leqslant \lim_{s\to 0^+} \fint_{B_s(y_n)} |Xu(z)|\,dz \leqslant L,$$

and so $(Xu(y_n))_n$ is bounded in $\mathbb{R}^m$. Therefore, up to a subsequence, we can assume that its limit exists, that is $\partial_{X,N}u(x)$ is non-empty. From the above proof it is easy to see that $\partial_{X,N}u(x)$ is bounded, while convexity and closure follows directly from its definition. Let us prove (ii).

We fix $x \in \Omega$ and start by proving the left-to-right inclusion. As the right set is convex and closed, it is sufficient to show that any z of the form

$$z = \lim_{n\to\infty} Xu(y_n),$$

with $y_n \to x$ and $y_n \notin N$, belongs to

$$\overline{co}\{Xu(y) \,:\, y \in B_{1/k}(x) \setminus N\}$$

for any $k \in \mathbb{N}\setminus\{0\}$. As y_n tends to x we get that $y_n \in B_{1/k}(x)\setminus N$ for n sufficiently large. Therefore, as the conclusion follows for each $X(y_n)$ and the right set is closed, we have proved the desired inclusion. The proof of the converse inclusion follows from Lemma 4.1.2 and Lemma 4.1.3. Now we prove (iii). Let $x \in \Omega$ and let $(y_n)_n \subseteq \Omega\setminus N$ converges to x. Then from the continuity of Xu it follows that $\lim_{n\to\infty} Xu(y_n) = Xu(x)$. Since $\{Xu(x)\}$ is convex and closed, this implies that $\partial_{X,N}u(x) \subseteq \{Xu(x)\}$. Conversely, being N null, there exists a sequence $(y_n)_n \subseteq \Omega \setminus N$ which converges to x. Again thanks to the continuity of Xu, the converse inclusion follows. □

Proposition 4.1.5. *Let $u, v \in W^{1,\infty}_{X,loc}(\Omega)$ and let N be a negligible set which contains the non-Lebesgue points of Xu and Xv. Then*

$$\partial_{X,N}(u - v)(x) \subseteq \partial_{X,N}u(x) - \partial_{X,N}v(x)$$

for any $x \in \Omega$.

Proof. Fix $x \in \Omega$. Since $\partial_{X,N}u(x) - \partial_{X,N}v(x)$ is convex and closed, it suffices to show that the set

$$\left\{\lim_{n\to\infty} X(u - v)(y_n) \,:\, y_n \notin N,\, y_n \to x\right\}$$

is contained in $\partial_{X,N}u(x) - \partial_{X,N}v(x)$. Therefore let $(y_n)_n \subseteq \mathbb{R}^n \setminus N$ be such that $y_n \to x$. Since $u, v \in W^{1,\infty}_{X,loc}(\Omega)$ we can assume that, up to a subsequence, both the limits of $(Xu(y_n))_n$ and $(Xv(y_n))_n$ exist. Therefore it follows that

$$\lim_{n\to\infty} X(u - v)(y_n) = \lim_{n\to\infty}(Xu(y_n) - Xv(y_n)) = \lim_{n\to\infty} Xu(y_n) - \lim_{n\to\infty} Xv(y_n).$$

Since the right hand side belongs to $\partial_{X,N}u(x) - \partial_{X,N}v(x)$, the thesis follows. □

4.2 Derivation along horizontal curves

When the distribution X generates a continuous Carnot-Carathéodory space, the (X, N)-subgradient proves to be the right tool, in analogy with the Euclidean setting, to deal with differentiability of horizontal Lipschitz functions along horizontal curves.

Proposition 4.2.1. *Assume that Ω is a continuous Carnot-Carathéodory space. Assume that*

$1 \leqslant p \leqslant +\infty$, *let* $u \in W^{1,\infty}_{X,loc}(\Omega)$ *and let* $\gamma \in \mathrm{AC}([-\beta,\beta],\Omega)$ *be a horizontal curve with*

$$\dot{\gamma}(t) = C(\gamma(t))^T \cdot A(t).$$

Then the curve $t \mapsto u(\gamma(t))$ *belongs to* $W^{1,\infty}(-\beta,\beta)$, *and there exists a function* $g \in L^\infty((-\beta,\beta),\mathbb{R}^m)$ *such that*

$$\frac{du(\gamma(t))}{dt} = g(t) \cdot A(t)$$

for a.e. $t \in (-\beta,\beta)$. *Moreover*

$$g(t) \in \partial_{X,N} u(\gamma(t))$$

for a.e. $t \in (-\beta,\beta)$.

Proof. Let $(\varrho_\delta)_\delta$ be a sequence of spherically symmetric mollifiers, and let N be any null set which contains all the non-Lebesgue points of Xu. If δ is sufficiently small and we define u_δ and $(Xu)_\delta$ to be the standard convolutions, we have that these functions are smooth on a bounded open set, say V, such that $V \Subset \Omega$ and V contains the support of γ. Moreover, since X induces a continuous Carnot-Carathéodory space, from [278] we know that there exists a non-negative and non-decreasing function $w(\delta)$ (depending on the chosen function u) defined in a right neighborhood of 0, such that

$$\lim_{\delta \to 0^+} w(\delta) = 0$$

and moreover

$$|X(u_\delta)(x) - (Xu)_\delta(x)| \leqslant w(\delta) \tag{4.2.1}$$

for any $x \in V$. As u_δ is C^1 and γ is absolutely continuous, from standard calculus we have that

$$\begin{aligned} u_\delta(\gamma(t)) - u_\delta(\gamma(0)) &= \int_0^t D(u_\delta)(\gamma(s)) \cdot \dot{\gamma}(s)\, ds \\ &= \int_0^t D(u_\delta)(\gamma(s)) \cdot C(\gamma(s))^T \cdot A(s)\, ds \\ &= \int_0^t X(u_\delta)(\gamma(s)) \cdot A(s)\, ds. \end{aligned} \tag{4.2.2}$$

Let us consider now the sequence of functions $X(u_{1/n})(\gamma(\cdot))$. It is easy to see that it is bounded in $L^\infty((-\beta,\beta),\mathbb{R}^m)$. Therefore (up to a subsequence) there exists a function $g \in L^\infty((-\beta,\beta),\mathbb{R}^m)$ such that

$$X(u_{1/n})(\gamma(\cdot)) \rightharpoonup^* g(\cdot) \qquad \text{in } L^\infty((-\beta,\beta),\mathbb{R}^m) \tag{4.2.3}$$

as n goes to infinity, and so in particular

$$X(u_{1/n})(\gamma(\cdot)) \rightharpoonup g(\cdot) \qquad \text{in } L^2((-\beta,\beta),\mathbb{R}^m) \tag{4.2.4}$$

as n goes to infinity. Since u is continuous, then u_δ converges uniformly to u on V (cf. [61]).

Therefore, passing to the limit in (4.2.2), noticing in particular that $A \in L^1((-\beta,\beta),\mathbb{R}^m)$ and exploiting (4.2.3), we obtain that

$$u(\gamma(t)) - u(\gamma(0)) = \int_0^t g(s)\cdot A(s)ds.$$

We are left to show that $g(t) \in \partial_{X,N} u(\gamma(t))$ for a.e. $t \in (-\beta,\beta)$. Let us notice that, since for any $x \in V$ we have that

$$(Xu)_\delta(x) = \int_{B_\delta(x)\setminus N} \varrho_\delta(y-x)Xu(y)dy,$$

it follows that

$$(Xu)_\delta(x) \in \overline{co}\{Xu(y)\,:\, y \in B_\delta(x)\setminus N\} \tag{4.2.5}$$

for any $x \in V$. Thanks to (4.2.4) and Mazur's Lemma (cf. e.g. [61, Corollary 3.9]), for each $m \in \mathbb{N}$ there are convex combinations of $X(u_{1/n})(\gamma(\cdot))$ converging strongly to g in $L^2((-\beta,\beta),\mathbb{R}^m)$, that is

$$v_m(\cdot) := \sum_{n=M_m}^{N_m} a_{m,n} X(u_{1/n})(\gamma(\cdot)) \longrightarrow g(\cdot) \qquad \text{in } L^2((-\beta,\beta),\mathbb{R}^m),$$

with $M_m < N_m$ and $\lim_{m\to\infty} M_m = +\infty$. Moreover (again up to a subsequence) we can assume that the above convergence holds pointwise for a.e. $t \in (-\beta,\beta)$. Let us define now

$$z_m(\cdot) := \sum_{n=M_m}^{N_m} a_{m,n}(Xu)_{1/n}(\gamma(\cdot)).$$

Then, (4.2.1) implies that

$$\begin{aligned}
|z_m(t) - g(t)| &\leqslant \sum_{n=M_m}^{N_m} a_{m,n}|X(u_{1/n})(\gamma(t)) - (Xu)_{1/n}(\gamma(t))| + |v_m(t) - g(t)| \\
&\leqslant \sum_{n=M_m}^{N_m} a_{m,n} w(1/n) + |v_m(t) - g(t)| \\
&\leqslant \sum_{n=M_m}^{N_m} a_{m,n} w(1/M_m) + |v_m(t) - g(t)| \\
&= w(1/M_m) + |v_m(t) - g(t)|,
\end{aligned}$$

which implies that z_m converges to g pointwise for a.e. $t \in (-\beta,\beta)$ as $m \to \infty$. Moreover, from (4.2.5) and the definition of z_m it follows easily that

$$z_m(t) \in \overline{co}\{Xu(y)\,:\, y \in B_{1/M_m}(\gamma(t))\setminus N\} \subseteq \overline{co}\{Xu(y)\,:\, y \in B_{1/k}(\gamma(t))\setminus N\}$$

for any $t \in (-\beta,\beta)$ and for any $k \leqslant M_m$. Therefore, thanks to the pointwise convergence as $m \to \infty$, we get that

$$g(t) \in \bigcap_{k=1}^{\infty} \overline{co}\{Xu(y)\,:\, y \in B_{1/k}(\gamma(t))\setminus N\}.$$

for a.e. $t \in (-\beta, \beta)$. Finally, from Proposition 4.1.4, the thesis follows. □

As a corollary of the previous proposition we have the following result.

Proposition 4.2.2. *Assume that Ω is a continuous Carnot-Carathéodory space. Let $u \in C^1_X(\Omega)$ and let $\gamma \in C^1([-\beta, \beta], \Omega)$ be a horizontal curve with*

$$\dot{\gamma}(t) = C(\gamma(t))^T \cdot A(t)$$

and $A \in C([-\beta, \beta], \mathbb{R}^m)$. Then the curve $t \mapsto u(\gamma(t))$ belongs to $C^1(-\beta, \beta)$ and

$$\frac{du(\gamma(t))}{dt} = Xu(\gamma(t)) \cdot A(t)$$

for any $t \in (-\beta, \beta)$.

4.3 Some consequences of Proposition 4.2.1

Let us list a bunch of simple consequences of Proposition 4.2.1 and Proposition 4.2.2. We begin with the following expected property, whose proof in the smooth case goes back to [177].

Corollary 4.3.1. *Assume that (Ω, d_Ω) is a continuous Carnot-Carathéodory space, and let $u \in W^{1,\infty}_{X,loc}(\Omega)$. If $Xu = 0$ on Ω, then u is constant on Ω.*

Proof. Let $x, y \in \Omega$ and let $\gamma : [0, T] \longrightarrow \Omega$ be a sub-unit curve connecting x and y. This in particular implies that

$$\dot{\gamma}(t) = C(\gamma(t))^T \cdot A(t)$$

with $A \in L^\infty((0, T), \mathbb{R}^m)$. Therefore, thanks to Proposition 4.2.1 we know that the curve $t \mapsto u(\gamma(t))$ belongs to $W^{1,\infty}(0, T)$, and there exists a function $g \in L^\infty((0, T), \mathbb{R}^m)$ such that

$$\frac{du(\gamma(t))}{dt} = g(t) \cdot A(t)$$

for a.e. $t \in (0, T)$, and moreover

$$g(t) \in \partial_{X,N} u(\gamma(t))$$

for a.e. $t \in (0, T)$, where N is the set of non-Lebesgue points of Xu. It is easy to notice that, since $Xu = 0$, then $\partial_{X,N} u(x) = \{0\}$ for any $x \in \Omega$. Therefore we conclude that

$$u(y) - u(x) = u(\gamma(T)) - u(\gamma(0)) = \int_0^T g(t) \cdot A(t) dt = 0.$$

□

Another interesting consequence of Proposition 4.2.2 consists in the first and second-order behaviour of functions of class C^2_X at local extremal points.

Proposition 4.3.2. *Let (Ω, d_Ω) be a continuous Carnot-Carathéodory space. Let $u \in C^2_X(\Omega)$. Let x_0 be a local maximum (minimum) point of u. Then $Xu(x_0) = 0$ and $X^2u(x_0) \leqslant (\geq)\ 0$.*

Proof. We assume that x_0 is a local maximum, being the other case analogous. Let γ be a horizontal curve defined in a neighborhood of 0, such that $\gamma(0) = x_0$ and $\dot\gamma(t) = \mathcal{C}(\gamma(t))^T \cdot A(t)$. Fix $i = 1, \ldots, m$ and choose $A(t) = e_i$ where e_i denotes the $i-$th element in the canonical basis of $\mathbb{R}^m$. Let $g(t) := u(\gamma(t))$. Then g is of class C^2 in a neighborhood of 0, whence $g'(0) = 0$ and $g''(0) \leqslant 0$. Thanks to Proposition 4.2.2, we know that

$$g'(t) = Xu(\gamma(t)) \cdot A(t).$$

Hence, thanks to the choice of A, we conclude that $X_iu(x_0) = 0$, and so $Xu(x_0) = 0$. To conclude, let us fix $\xi \in \mathbb{R}^m$ and let $A(t) = \xi$. Then, arguing as above,

$$g'(t) = Xu(\gamma(t)) \cdot \xi,$$

which implies that

$$g''(t) = \sum_{i,j=1}^{m} X_iX_ju(\gamma(t))\xi_i\xi_j.$$

Evaluating the previous identity in $t = 0$ allows to conclude that $X^2u(x_0) \leqslant 0$. □

If we restrict our attention to the first-order behavior near extremal points, the (X, N)-subdifferential provides a weak form of Fermat theorem even for functions in $W^{1,\infty}_{X,loc}(\Omega)$.

Proposition 4.3.3. *Assume that (Ω, d_Ω) is a continuous Carnot-Carathéodory space. Let $u \in W^{1,\infty}_{X,loc}(\Omega)$ and assume that $x_0 \in \Omega$ is either a point of local minimum or a point of local maximum for u. Then $0 \in \partial_{X,N}u(x_0)$.*

Proof. We prove the statement assuming that x_0 is a minimum point, since the argument for the other case is analogous. Assume by contradiction that $0 \notin \partial_{X,N}u(x_0)$. Since $\partial_{X,N}u(x_0)$ is convex and compact, then by Hahn-Banach theorem (cf. [61]) there exists $a \in \mathbb{R}^m$ and $\alpha > 0$ such that

$$\max_{p \in \partial_{X,N}u(x_0)} \langle p, a\rangle < -\alpha. \tag{4.3.1}$$

Now we claim that there exists $r > 0$ such that

$$\langle p, a\rangle \leqslant -\alpha \tag{4.3.2}$$

for any $p \in \partial_{X,N}u(y)$ and for any $y \in B_r(x_0)$. To prove this fact we first show that there exists $r > 0$ such that

$$\langle Xu(y), a\rangle < -\alpha$$

for any $y \in B_r(x_0) \setminus N$. If it is not the case, then there is a sequence $(y_n)_n \subseteq \mathbb{R}^n \setminus N$ such that $y_n \to x_0$ and

$$\langle Xu(y_n), a\rangle \geq -\alpha. \tag{4.3.3}$$

Moreover, since $u \in W^{1,\infty}_{X,loc}(\Omega)$ we can assume that up to a subsequence

$$\exists \lim_{n\to\infty} Xu(y_n) =: p,$$

and by construction we have that $p \in \partial_{X,N} u(x_0)$. Therefore, recalling (4.3.1) and (4.3.3), we conclude that

$$-\alpha \leqslant \lim_{n\to\infty} \langle Xu(y_n), a\rangle = \langle p, a\rangle < -\alpha,$$

which is a contradiction. Let us now define

$$A := \{p \in \mathbb{R}^m \,:\, \langle p, a\rangle \leqslant -\alpha\},$$

and, for any $y \in B_r(x_0)$, the set

$$S_y := \left\{\lim_{n\to\infty} Xu(y_n) \,:\, y_n \to y,\, y_n \notin N\right\}$$

so that $\partial_{X,N} u(y) = \overline{co}(S_y)$. Since A is convex and closed, our claim is proved if we show that $S_y \subseteq A$. Let us take a sequence $(y_n)_n$ converging to y and such that $y_n \notin N$ and the sequence $Xu(y_n)$ has a limit. Then up to a subsequence we have that $(y_n)_n \subseteq B_r(x_0) \setminus N$, and so thanks to the previous claim we conclude that

$$\lim_{n\to\infty} \langle Xu(y_n), a\rangle \leqslant -\alpha.$$

Hence $S_y \subseteq A$, and so (4.3.2) is proved. Let now $\gamma : [0,1] \longrightarrow \Omega$ be a solution to

$$\begin{cases} \dot\gamma(t) = \mathcal{C}(\gamma(t))^T \cdot a \\ \gamma(0) = x_0. \end{cases} \tag{4.3.4}$$

Then by construction γ is a horizontal curve. Moreover, if we define $x_n := \gamma(\frac{1}{n})$, it follows that $x_n \to x_0$, and so up to a subsequence we can assume that $(x_n)_n \subseteq \gamma([0,\delta]) \subseteq B_r(x_0)$ for some $\delta > 0$ small enough. Therefore, thanks to these facts, Proposition 4.2.1 and (4.3.2), there exists $g \in L^\infty(0,1)$ such that $g(t) \in \partial_{X,N} u(\gamma(t))$ for a.e. $t \in (0,1)$ and

$$u(x_n) - u(x_0) = u\left(\gamma\left(\frac{1}{n}\right)\right) - u(\gamma(0)) = \int_0^{\frac{1}{n}} \langle g(t), a\rangle\, dt \leqslant -\frac{\alpha}{n} < 0.$$

Therefore we conclude that $u(x_0) > u(x_n)$ for any $n \in \mathbb{N}$, which is a contradiction with the fact that x_0 is a point of local minimum. □

4.4 Subgradient and sublevel sets

We conclude this chapter with another feature of the (X,N)-subgradient which will be useful in the sequel.

Proposition 4.4.1. *Let $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^m)$. Let $K \in \mathbb{R}$ be such that*

$$\{\xi \in \mathbb{R}^m \,:\, f(x,u,\xi) \leqslant K\} \text{ is convex} \tag{4.4.1}$$

for any $x \in \Omega$ and any $u \in \mathbb{R}$. Let $u \in W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$ and let $\tilde{\Omega} \subseteq \Omega$ be an open set such that

$$f(x,u(x),Xu(x)) \leqslant K \tag{4.4.2}$$

for a.e. $x \in \tilde{\Omega}$. Let N be a Lebesgue-null subset of $\tilde{\Omega}$ containing all the points where (4.4.2) fails and all the non-Lebesgue points of Xu. Then it follows that

$$f(x,u(x),w) \leqslant K$$

for any $x \in \tilde{\Omega}$ and for any $w \in \partial_{X,N}u(x)$.

Proof. Let $x \in \tilde{\Omega}$ be fixed and let $w \in \partial_{X,N}u(x)$. Then there exists a sequence

$$(w_h)_h \subseteq co\left\{\lim_{n\to\infty} Xu(y_n) \,:\, y_n \to x,\, y_n \notin N \text{ and } \exists \lim_{n\to\infty} Xu(y_n)\right\}$$

converging to w in $\mathbb{R}^m$. If we are able to prove the claim for each w_h, the thesis follows from the continuity of f in the third argument. Fix then h. Thanks to Carathéodory Theorem (cf. [100, Theorem 1.2]) there are $(\lambda_1^h,\dots,\lambda_{n+1}^h) \in \Lambda_{n+1}$ and $w_1^h,\dots,w_{n+1}^h$ such that

$$w_j^h \subseteq \left\{\lim_{n\to\infty} Xu(y_n) \,:\, y_n \to x,\, y_n \notin N \text{ and } \exists \lim_{n\to\infty} Xu(y_n)\right\}$$

for any $j = 1,\dots,n+1$ and

$$w^h = \sum_{j=1}^{n+1} \lambda_j^h w_j^h.$$

Again, if we are able to show the claim for each w_j^h, we are done because of (4.4.1). Let us fix j and take a sequence $(y_s)_s \subseteq V \setminus N$ converging to x and such that $w_j^h = \lim_{s\to\infty} X(y_s)$. As the the map $(x,\eta) \mapsto f(x,u(x),\eta)$ is continuous, and thanks again to the global continuity of f, we conclude that

$$f(x,u(x),w_j^h) = \lim_{s\to\infty} f(x,u(x),Xu(y_s)) = \lim_{s\to\infty} f(y_s,u(y_s),Xu(y_s)) \leqslant K.$$

□

4.5 Proof of Lemma 4.1.2 and Lemma 4.1.3

Proof of Lemma 4.1.2. Let $z \in \overline{A}_k$ for any $k \geq 1$. Then for any $k \geq 1$ there exists a sequence $(z_h^k)_h \subseteq A_k$ converging to z as h goes to infinity. Therefore we can select a subsequence $(z^k)^k \subseteq (z_h^k)_h^k$ which converges to z as k goes to infinity and such that $z^k \in A_k$ for any $k \geq 1$. Since $z^k \in A_k$, then there exists $y^k \in B_{1/k}(x) \setminus N$ such that $Xu(y^k) = z^k$. It follows that y^k

converges to x as k goes to infinity, $y^k \notin N$ and

$$z = \lim_{k\to\infty} z^k = \lim_{k\to\infty} Xu(y^k).$$

We conclude that $z \in S$. □

Proof of Lemma 4.1.3. Let $z \in \overline{co}(\overline{A}_k)$ for any $k \geq 1$. Then for any $k \geq 1$ there exists a sequence $(z_h^k)_h \subseteq co(\overline{A}_k)$ converging to z as h goes to infinity. As in the previous proof, let $(z^k)^k \subseteq (z_h^k)_h^k$ be a sequence which converges to z as k goes to infinity and such that $z^k \in co(\overline{A}_k)$ for any $k \geq 1$. Therefore, for each $k \geq 1$, there exist $(\lambda_1^k \ldots, \lambda_{m+1}^k) \in \Lambda_{m+1}$, where Λ_{m+1} is as in (4.1.1), and $y_1^k, \ldots, y_{m+1}^k$ belonging to $\overline{A}_k$ such that

$$z^k = \sum_{j=1}^{m+1} \lambda_j^k y_j^k.$$

Up to subsequences, we assume that

$$\lambda_j^k \to \lambda_j \qquad \text{as } k \to \infty$$

and

$$y_j^k \to y_j \qquad \text{as } k \to \infty$$

for any $j = 1, \ldots, m+1$. It is easy to see that $(\lambda_1, \ldots, \lambda_{m+1}) \in \Lambda_{m+1}$ and that y_j belongs to $\overline{A}_k$ for any $k \geq 1$. Therefore, thanks to our hypotheses, we have that $y_j^h \in S$. If we set

$$x := \sum_{j=1}^{m+1} \lambda_j y_j,$$

then $x \in co(S)$. Moreover, it holds that

$$x = \sum_{j=1}^{m+1} \lambda_j y_j = \sum_{j=1}^{m+1} \lim_{k\to\infty} \lambda_j^k y_j^k = \lim_{k\to\infty} \sum_{j=1}^{m+1} \lambda_j^k y_j^k = \lim_{k\to\infty} z^k = z,$$

which implies that $z \in co(S)$. □

Chapter 5

Differentiability in Carnot-Carathéodory spaces

5.1 Definition and motivations

We refer to [78] as main reference for this chapter. In this chapter we introduce a notion of differentiability for C^1_X functions which is a generalization of the ones introduced in [237, 225] to prove a Rademacher-type theorem for Lipschitz functions on suitable families of Carnot-Carathéodory spaces. We refer to [210] for similar results in related settings. The main result of the section is Theorem 5.3.1, which yields the differentiability and an explicit form for the differential of C^1_X functions. The first celebrated attempt to extend the classical notion of differentiability beyond the Euclidean setting is due to Pansu (cf. [237]), with the introduction of the so-called *Pansu differential* for functions acting between Carnot groups. For the sake of completeness, let us recall some basic definitions regarding differentiability in Carnot groups.

Definition 5.1.1 (Homogeneous homomorphisms). *A map $L : \mathbb{G} \longrightarrow \mathbb{R}$ is called a* homogeneous homomorphism *if*

$$L(x \cdot y) = L(x) + L(y) \quad \textit{and} \quad L(\delta_\lambda(x)) = \lambda L(x) \quad \textit{for every } x, y \in \mathbb{G} \quad \textit{and} \quad \lambda > 0.$$

According to the above definition, Pansu's differentiability reads as follows.

Definition 5.1.2 (Pansu differential). *Let $\mathbb{G}$ be a Carnot group. Let $\Omega \subset \mathbb{G}$ be an open subset. A map $f : \Omega \longrightarrow \mathbb{R}$ is* Pansu differentiable *at $x \in \Omega$ if there exists a homogeneous homomorphism $L : \mathbb{G} \longrightarrow \mathbb{R}$ such that*

$$\lim_{y \longrightarrow x} \frac{f(y) - f(x) - L(x^{-1} \cdot y)}{d_{\mathbb{G}}(y, x)} = 0.$$

The map $L := D_{\mathbb{G}} f(x) : \mathbb{G} \to \mathbb{R}$ is called Pansu differential *of f at x.*

Accordingly, the following anisotropic version of Rademacher's theorem holds.

Theorem 5.1.3 (Pansu-Rademacher). *Let $\Omega \subset \mathbb{G}$ be an open set. Let $u \in W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$. Then u is* Pansu-differentiable *at almost every $x_0 \in \Omega$, that is*

$$\lim_{x \to x_0} \frac{u(x) - u(x_0) - \langle Xu(x_0), \pi(x_0^{-1} \cdot x)\rangle}{d_\Omega(x_0, x)} = 0$$

for almost every $x_0 \in \Omega$, where

$$\pi(y) = (y_1, \dots, y_m)$$

for any $y \in \mathbb{G}$.

In our general setting, we extend Definition 5.1.2 in the following way.

Definition 5.1.4 (X-differentiability). *Let (Ω, d_Ω) be a Carnot-Carathéodory space. We say that a function $u \in C(\Omega)$ is* X-differentiable *at $x \in \Omega$ if there exists a linear mapping $L_x : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that*

$$\lim_{d_\Omega(x,y) \to 0} \frac{u(y) - u(x) - L_x(y - x)}{d_\Omega(x, y)} = 0.$$

In such a case we say that $d_X u(x) := L_x$ is a X-differential of u at x.

In order to guarantee the existence of a X-differential for a C^1_X function, we assume that (Ω, d_Ω) is a continuous Carnot-Carathéodory space. In addition, we require the following additional condition on X.

$$X_1(x), \dots, X_m(x) \text{ are linearly independent for any } x \in \Omega. \tag{LIC}$$

The previous conditions embraces many relevant families of vector fields, such as for instance Carnot Groups and the class of Carnot-type groups considered in [219] for the same purposes.

5.2 Some technical results

The additional hypothesis (LIC) implies that the matrix $\mathcal{C}(x)^T$ admits a left-inverse matrix for any $x \in \Omega$.

Proposition 5.2.1. *Assume that X satisfies (LIC). Then, if we define $\tilde{\mathcal{C}}$ by*

$$\tilde{\mathcal{C}}(x) := (\mathcal{C}(x) \cdot \mathcal{C}(x)^T)^{-1} \cdot \mathcal{C}(x)$$

for any $x \in \Omega$, then $\tilde{\mathcal{C}}$ is well defined and continuous on Ω. Moreover it holds that

$$\tilde{\mathcal{C}}(x) \cdot \mathcal{C}(x)^T = I_m$$

for any $x \in \Omega$. Here I_m denotes the $m \times m$ identity matrix.

Proof. Let us define $B(x) := \mathcal{C}(x) \cdot \mathcal{C}(x)^T$ for any $x \in \Omega$. Thanks to (LIC) we know that $\mathcal{C}(x)$ and $\mathcal{C}(x)^T$ have maximum rank, and so by standard linear algebra we know that $B(x)$ is a square matrix with maximum rank. Thus $B(x)$ is invertible and $\tilde{\mathcal{C}}(x)$ is well defined. Moreover it holds that

$$\tilde{\mathcal{C}}(x) := \frac{Adj(B)(x) \cdot \mathcal{C}(x)}{\det(B(x))},$$

being $Adj(B)(x)$ the *adjugate* matrix of $(B)(x)$ (cf. [272]), and so it is continuous on Ω. Finally, a trivial calculation shows that $\tilde{\mathcal{C}}$ is a left-inverse of $\mathcal{C}^T$. □

Lemma 5.2.2. *Assume that* (Ω, d_Ω) *is a continuous Carnot-Carathéodory space. Let* $x, y \in \Omega$ *and* $\varepsilon > 0$. *Assume that* $\gamma \in AC([0,T], \Omega)$ *is a sub-unit curve such that* $\gamma(0) = x$, $\gamma(T) = y$ *and* $T < d_\Omega(x,y) + \varepsilon$. *Then it holds that*

$$\gamma([0,T]) \subseteq B_{d_\Omega}(x, d_\Omega(x,y) + \varepsilon). \tag{5.2.1}$$

Proof. Let x, y, γ and ε as above. Assume by contradiction that there exists $\bar{t} \in (0,T)$ such that $d_\Omega(x, \gamma(\bar{t})) \geq d_\Omega(x,y) + \varepsilon$. Then it follows that

$$d_\Omega(x,y) + \varepsilon \leqslant d_\Omega(x, \gamma(\bar{t})) \leqslant \bar{t} < T < d_\Omega(x,y) + \varepsilon,$$

which is a contradiction. □

Proposition 5.2.3. *Assume that* (Ω, d_Ω) *is a continuous Carnot-Carathéodory space. Let* $g \in C^1_X(\Omega)$ *and let* $x \in \Omega$. *Then*

$$\limsup_{y \to x} \frac{|g(y) - g(x)|}{d_\Omega(x,y)} \leqslant |Xg(x)|.$$

Proof. Let x and g be as in the statement. Let $\tilde{\Omega} \Subset \Omega$ be an open and connected neighborhood of x, and let $\beta = C^{-1}_{\tilde{\Omega}}$ be as in Proposition 2.1.5. Let $R > 0$ be such that $\overline{B_{d_\Omega}(x, 2R)} \subseteq \tilde{\Omega}$. Choose now $y \in B_{d_\Omega}(x, R)$ and $0 < \varepsilon \leqslant R$. Then, thanks to Proposition 2.1.5, it follows that

$$\overline{B_{d_\Omega}(x, d_\Omega(x,y) + \varepsilon)} \subseteq \overline{B_{\beta d_\Omega(x,y) + \beta\varepsilon}(x)}. \tag{5.2.2}$$

Moreover, if we let M be the family of all sub-unit curves $\gamma : [0,T] \longrightarrow \Omega$ connecting x and y and such that $T < d_\Omega(x,y) + \varepsilon$, then it is clear that

$$d_\Omega(x,y) = \inf\{T \,:\, \gamma : [0,T] \longrightarrow \Omega,\, \gamma \in M\}.$$

Fix now a curve $\gamma : [0,T] \longrightarrow \Omega$, $\gamma \in M$ with horizontal derivative A. Then, thanks to (5.2.2), [243, Proposition 2.6] and Lemma 5.2.2, it follows that

$$|g(y) - g(x)| = \left| \int_0^T \langle Xg(\gamma(t)), A(t) \rangle dt \right| \leqslant T \|Xg\|_{L^\infty\left(\overline{B_{\beta d_\Omega(x,y) + \beta\varepsilon}(x)}\right)} \tag{5.2.3}$$

Therefore, passing to the infimum over M, it follows that

$$\frac{|g(y)-g(x)|}{d_\Omega(x,y)} \leqslant \|Xg\|_{L^\infty\left(\overline{B_{\beta d_\Omega(x,y)+\beta\varepsilon}(x)}\right)}.$$

The conclusion follows letting $\varepsilon \to 0^+$ and $y \to x$, together with the continuity of Xg. □

5.3 The differentiability theorem

Now we state our main differentiability result.

Theorem 5.3.1. *Assume that (Ω, d_Ω) is a continuous Carnot-Carathéodory space and that X satisfies (LIC). Let $u \in C^1_X(\Omega)$ and $x \in \Omega$. Then u is X-differentiable at x and*

$$d_X u(x)(z) = \langle Xu(x) \cdot \tilde{\mathcal{C}}(x), z\rangle,$$

where $\tilde{\mathcal{C}}$ is as in Proposition 5.2.1 and $z \in \mathbb{R}^n$.

Proof. Let $x \in \Omega$ be fixed. Define $g : \Omega \longrightarrow \mathbb{R}$ as $g(y) := u(y) - h(y)$, where

$$h(y) = \langle Xu(x) \cdot \tilde{\mathcal{C}}(x), y - x\rangle.$$

Then clearly $g \in C^1_X(\Omega)$. Moreover, by explicit computations, we get that

$$\begin{aligned} Xg(y) &= Xu(y) - X(\langle Xu(x) \cdot \tilde{\mathcal{C}}(x), y - x\rangle) \\ &= Xu(y) - D(\langle Xu(x) \cdot \tilde{\mathcal{C}}(x), y - x\rangle) \cdot \mathcal{C}(y)^T \\ &= Xu(y) - Xu(x) \cdot \tilde{\mathcal{C}}(x) \cdot \mathcal{C}(y)^T, \end{aligned}$$

which in particular implies that

$$Xg(x) = 0.$$

The conclusion then follows by invoking Proposition 5.2.3. □

Remark 5.3.2. It is clear from the proof of Theorem 5.3.1 that the X-differential is non-unique in general. Indeed, Theorem 5.3.1 remains true if we let

$$d_X u(x)(z) = \langle Xu(x) \cdot D(x), z\rangle,$$

where $D(x)$ is any left-inverse matrix of $\mathcal{C}^T(x)$. Since for a non-squared matrix the left-inverse matrix is non-unique in general, the non-uniqueness of the X-differential follows. As an instance, consider the first Heisenberg group $\mathbb{H}^1$ (cf. Example 3.2.8). It is easy to see that the matrices

$$\tilde{\mathcal{C}}(x,y) = \frac{1}{1+x^2+y^2}\begin{bmatrix} 1+x^2 & xy & y \\ xy & 1+y^2 & -x \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

are both left-inverse matrices of

$$\mathcal{C}(x,y)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & -x \end{bmatrix}$$

Nevertheless, if in a Carnot group we require in addition that the X-differential is a homogeneous homomorphism as in Definition 5.1.1, then it is unique and it coincides with the classical Pansu differential (cf. [237, 262]). Finally, we point out that when $n = m$ and $X_1(x), \ldots, X_n(x)$ are linearly independent for any x, i.e. the Riemannian case, then the X-differential is unique since $\tilde{\mathcal{C}}(x) = (\mathcal{C}(x)^T)^{-1}$.

Part III

Local functionals depending on vector fields

Chapter 6

Introduction and preliminaries

6.1 Introduction and motivations

Starting from the seminal works by E. De Giorgi and T. Franzoni ([114, 115]), the study of Γ-convergence has pervaded the evolution of modern mathematical analysis, and has developed in several different directions, exhibiting important applications to many branches of calculus of variations, such as homogenization, minimal surfaces and partial differential equations. For an exhaustive introduction to this topic, we refer to the monographs [105, 59, 60].
Since the late 1970s, G. Buttazzo and G. Dal Maso has investigated Γ-convergence in the framework of Lebesgue spaces, Sobolev spaces and BV spaces (cf. e.g. [66, 68, 104]). A key tool in the study of Γ-convergence properties in these framework consists in the so-called *integral representation of local functionals.* By integral representation we mean finding conditions under which an arbitrary functional $F(u, A)$, being u a function and A a set, can be expressed as

$$F(u, A) = \int_A f(x, u(x), Du(x))\, dx$$

for a suitable *Lagrangian* f. Again, in the Euclidean setting this problem is very well understood, and we refer the interested reader to the papers [6, 65, 66, 68, 67] for a complete overview of the subject. More recently, in the seminal papers [138, 150], the authors started the study of variational functionals driven by a family of Lipschitz vector fields X, while developing the functional framework introduced in Chapter 1. Since [138, 150], the possibility to extend the classical results of the calculus of variations to the setting of variational functionals driven by vector fields has been the object of study of many papers. For instance, many homogenization problems have been considered in the Heisenberg group (cf. [118, 50, 145]). The authors of [205, 206] investigated integral representation and Γ−convergence properties of functionals $F(u, A)$ modelled by suitable families of Lipschitz continuous vector fields $X = (X_1, \ldots, X_n)$, basically assuming that

$$X_1(x), \ldots, X_m(x) \text{ are linearly independent for a.e. } x \in \Omega \tag{6.1.1}$$

and

$$F(u+c,A) = F(u,A) \text{ for any function } u, \text{ any set } A \text{ and any constant } c. \tag{6.1.2}$$

According to [205], we call a functional *translation-invariant* whether it satisfies (6.1.2), while we say that X satisfies the *linear independence condition* if it satisfies (6.1.1). Let us be more precise about this last feature.

Definition 6.1.1 (Linear independence condition). *We say that X satisfies the* linear independence condition *on Ω if the set*

$$N_X := \{x \in \Omega \,:\, X_1(x), \dots, X_m(x) \text{ are linearly dependent}\} \tag{a.e. LIC}$$

is such that $|N_X| = 0$. In this case we set $\Omega_X := \Omega \setminus N_X$.

Let us point out that (a.e. LIC), which is the weaker almost everywhere counterpart of the aforementioned (LIC), embraces many relevant families of vector fields studied in literature (cf. [206] for some instances). In particular neither the Hörmander condition nor the weaker assumption that X induces a Carnot-Carathéodory metric in Ω are requested. Although (LIC) and (a.e. LIC) are clearly very similar and may appear as the same notion at a first glance, there are many relevant settings in which (a.e. LIC) is satisfied and (LIC) fails.

Example 6.1.2 (The Grushin plane). The *Grushin plane* (cf. [167, 137]) is $\mathbb{R}^2_{x_1,x_2}$ together with the family $X = (X_1, X_2)$ of smooth Lipschitz continuous vector fields

$$X_1(x_1,x_2) = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad X_2(x_1,x_2) = x_1 \frac{\partial}{\partial x_2}$$

for any $(x_1,x_2) \in \mathbb{R}^2$. Notice that

$$[X_1, X_2] = \frac{\partial}{\partial x_2}$$

everywhere in $\mathbb{R}^2$, whence X is bracket-generating in $\mathbb{R}^2$. Moreover, it is clear that X satisfies (a.e. LIC), but clearly it does not satisfy (LIC).

In [205, Theorem 3.12], the authors found conditions under which F can be represented as

$$F(u,A) = \int_A f(x, Xu(x))\, dx \tag{6.1.3}$$

for any $A \subseteq \Omega$ open and $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}_{X,loc}(A)$, and for a suitable $f : \Omega \times \mathbb{R}^m \to [0,\infty)$. Moreover, they applied this characterization to prove a Γ-compactness theorem for integral functionals of the form (6.1.3), when $1 < p < \infty$. We refer to [207] for similar results under stronger conditions on the family X. The aim of Part III is to generalize the above-mentioned results in order to avoid both (6.1.2) and (a.e. LIC). More precisely, in Chapter 7 we extend the integral representation result of [205] avoiding (6.1.2) but still assuming (a.e. LIC). Again assuming (a.e. LIC), in Chapter 8 we deal with Γ compactness properties of integral functionals for which (6.1.2) fails. Finally, in Chapter 9, we show how to avoid (a.e. LIC). Before enter into further details, it is convenient to recall some relevant preliminaries, for which we mainly refer to [105].

6.2 Local functionals

Throughout Part III, Unless otherwise specified, we let $1 \leqslant p < \infty$, and we denote by Ω an open and bounded subset of $\mathbb{R}^n$. We set $\mathcal{A}_0$ to be the subfamily of $\mathcal{A}$ of all the open subsets A of Ω such that $A \Subset \Omega$ and by $\mathcal{B}_0$ the subfamily of $\mathcal{B}$ of all the Borel subsets B of Ω such that $B \Subset \Omega$. In this section we collect some definitions about increasing set functions and local functionals (cf. [105]). The latter, heuristically, behave like variational functionals in the first entry and like measures in the second one. Accordingly, it is worth introducing some general definitions regarding set functions.

Definition 6.2.1. *Let $\alpha : \mathcal{A} \longrightarrow [0, +\infty]$ be a function. We say that α is*

(i) increasing *if it holds that*

$$\alpha(A) \leqslant \alpha(B)$$

for any $A, B \in \mathcal{A}$ such that $A \subseteq B$;

(ii) inner regular *if it is increasing and*

$$\alpha(A) = \sup\{\alpha(A') \,:\, A' \Subset A\}$$

for any $A \in \mathcal{A}$;

(iii) subadditive *if it is increasing and*

$$\alpha(A) \leqslant \alpha(B) + \alpha(C)$$

for any $A, B, C \in \mathcal{A}$ with $A \subseteq B \cup C$;

(iv) superadditive *if it is increasing and*

$$\alpha(C) \geq \alpha(A) + \alpha(B)$$

for any $A, B, C \in \mathcal{A}$ with $A \cap B = \emptyset$ and $A \cup B \subseteq C$;

(v) a measure *if it is increasing and the restriction to $\mathcal{A}$ of a non-negative Borel measure.*

On the other hand, the following definition adapts some standard notation to our anisotropic setting.

Definition 6.2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set, let $1 \leqslant p < +\infty$ and Let X be a family of Lipschitz vector fields. Consider a functional*

$$F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty].$$

We say that F is:

(*i*) a measure *if, for any $u \in W^{1,p}_{X,loc}(\Omega)$, $F(u, \cdot) : \mathcal{A} \longrightarrow [0, +\infty]$ is a measure;*

(*ii*) local *if*

$$u|_{A'} = v|_{A'} \implies F(u, A') = F(v, A');$$

for any $A' \in \mathcal{A}_0$ and $u, v \in W^{1,p}_{X,loc}(\Omega)$;

(*iii*) convex *if, for any $A' \in \mathcal{A}_0$, the function $F(\cdot, A') : W^{1,p}_X(\Omega) \longrightarrow [0, +\infty]$ is convex;*

(*iv*) p-bounded *if there exist $a \in L^1_{loc}(\Omega)$ and $b, c > 0$ such that*

$$F(u, A') \leqslant \int_{A'} a(x) + b|Xu|^p + c|u|^p dx$$

for any $A' \in \mathcal{A}_0$ and for any $u \in W^{1,p}_X(\Omega)$

(*v*) lower semicontinuous *if*

$$u_h \to u \text{ in } W^{1,p}_X(\Omega) \qquad \Longrightarrow \qquad F(u, A') \leqslant \liminf_{h\to+\infty} F(u_h, A')$$

for any $A' \in \mathcal{A}_0$, $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ and $u \in W^{1,p}_X(\Omega)$;

(*vi*) weakly (sequentially) lower semicontinuous *if*

$$u_h \rightharpoonup u \text{ in } W^{1,p}_X(\Omega) \qquad \Longrightarrow \qquad F(u, A') \leqslant \liminf_{h\to+\infty} F(u_h, A')$$

for any $A' \in \mathcal{A}_0$, $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ and $u \in W^{1,p}_X(\Omega)$;

(*vii*) weakly-* sequentially lower semicontinuous *if*

$$u_h \rightharpoonup^* u \text{ in } W^{1,\infty}(\Omega) \qquad \Longrightarrow \qquad F(u, A') \leqslant \liminf_{h\to+\infty} F(u_h, A')$$

for any $A' \in \mathcal{A}_0$, $(u_h)_h \subseteq W^{1,\infty}(\Omega)$ and $u \in W^{1,\infty}(\Omega)$.

The very same definitions apply in the case in which a local functional is defined on $W^{1,p}_X(\Omega) \times \mathcal{A}$ or on $L^p(\Omega) \times \mathcal{A}$.

Definition 6.2.3. *If we have a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ (respectively $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$), we say that F is:*

(*i*) a measure *if $F(u, \cdot)$ is a measure for any $u \in L^p(\Omega)$ (respectively $u \in W^{1,p}_X(\Omega)$);*

(*ii*) local *if, for any* $A \in \mathcal{A}$ *and* $u, v \in L^p(\Omega)$ *(respectively* $u, v \in W^{1,p}_X(\Omega)$*), then*

$$u|_A = v|_A \implies F(u, A) = F(v, A);$$

(*iii*) convex *on* $W^{1,p}_X(\Omega)$ *if* $F(\cdot, A)$ *restricted to* $W^{1,p}_X(\Omega)$ *is convex for any* $A \in \mathcal{A}$*;*

(*iv*) L^p-lower semicontinuous *(respectively* $W^{1,p}_X$-lower semicontinuous*) if* $F(\cdot, A)$ *is* L^p*-lower semicontinuous (respectively* $W^{1,p}_X$*-lower semicontinuous) for any* $A \in \mathcal{A}$*;*

(*v*) weakly-* sequentially lower semicontinuous *if if* $F(\cdot, A)$ *restricted to* $W^{1,\infty}(\Omega)$ *is seq. l.s.c. with respect to the weak-** *topology of* $W^{1,\infty}(\Omega)$ *for any* $A \in \mathcal{A}$*.*

To conclude, the following definitions introduce some standard conditions (cf. [67]) in order to avoid convexity assumptions on the functionals.

Definition 6.2.4. *We say that* $\omega : \Omega \times [0, +\infty) \longrightarrow [0, +\infty)$ *is a* locally integrable modulus of continuity *if and only if*

$$r \mapsto \omega(x, r) \text{ is increasing, continuous and } \omega(x, 0) = 0 \text{ for a.e. } x \in \Omega$$

and

$$x \mapsto \omega(x, r) \in L^1_{loc}(\Omega) \quad \forall r \geq 0.$$

Definition 6.2.5. *Let us consider a functional* $F : \mathcal{F} \times \mathcal{A} \longrightarrow [0, +\infty]$*, where* $\mathcal{F}$ *is a functional space such that* $C^1(\overline{\Omega}) \subseteq \mathcal{F}$*. We say that:*

(*i*) *F satisfies the* strong condition (ω) *if there exists a sequence* $(\omega_k)_k$ *of locally integrable moduli of continuity such that*

$$|F(v, A') - F(u, A')| \leqslant \int_{A'} \omega_k(x, r)\, dx \tag{6.2.1}$$

for any $k \in \mathbb{N}$*,* $A' \in \mathcal{A}_0$*,* $r \in [0, \infty)$*,* $u, v \in C^1(\overline{\Omega})$ *such that*

$$\begin{aligned} &|u(x)|, |v(x)|, |Du(x)|, |Dv(x)| \leqslant k \\ &|u(x) - v(x)|, |Du(x) - Dv(x)| \leqslant r; \end{aligned}$$

for all $x \in A'$*.*

(*ii*) *F satisfies the* weak condition (ω) *if there exists a sequence* $(\omega_k)_k$ *of locally integrable moduli of continuity such that*

$$|F(u + s, A') - F(u, A')| \leqslant \int_{A'} \omega_k(x, |s|) dx$$

for any $k \in \mathbb{N}$*,* $A' \in \mathcal{A}_0$*,* $s \in \mathbb{R}$*,* $u \in C^1(\overline{\Omega})$ *such that*

$$|u(x)|, |u(x) + s|, |s| \leqslant k \quad \forall x \in A'.$$

6.3 Carathéodory functions

In order to ensure that an integral functional of the form

$$F(u, A) = \int_A f(x, u(x), Xu(x))\, dx$$

is well-defined, we need to impose some conditions on the Lagrangian f in such a way that the function

$$x \mapsto f(x, u(x), Xu(x))$$

is integrable on Ω for any $u \in W^{1,1}_{X,loc}(\Omega)$. To this aim, it is customary to work in the class of *Carathéodory functions* (cf. [100]).

Definition 6.3.1 (Carathéodory functions). *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, \infty]$ be a function. We say that f is a* Carathéodory function *if:*

(i) $f(\cdot, u, \eta)$ is measurable for any $u \in \mathbb{R}$ and any $\eta \in \mathbb{R}^m$;

(ii) $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$.

Carathéodory functions constitute the right class of Lagrangians, as the next proposition shows (cf. e.g. [125])

Proposition 6.3.2. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, \infty]$ be a Carathéodory function. Let $u \in W^{1,1}_{X,loc}(\Omega)$. Then the function*

$$x \mapsto f(x, u(x), Xu(x)) \tag{6.3.1}$$

is measurable for any $u \in W^{1,1}_{X,loc}(\Omega)$. In particular, being f non-negative, it is integrable on Ω.

6.4 Basic notions of Γ-convergence

In this section we collect some basic definition and results about Γ-convergence, for which we refer to [105]. We recall that, if (X, τ) is a first-countable topological space and $(F_h)_h$ is a sequence of functions defined on (X, τ) with values in $\overline{\mathbb{R}}$, we define the Γ-*lower limit* and Γ-*upper limit* respectively as

$$\Gamma - \liminf_{h\to\infty} F_h(u) := \inf \left\{ \liminf_{h\to\infty} F_h(u_h) \; : \; u_h \to u \right\}$$

and

$$\Gamma - \limsup_{h\to\infty} F_h(u) := \inf \left\{ \limsup_{h\to\infty} F_h(u_h) \; : \; u_h \to u \right\},$$

and we say that $(F_h)_h$ Γ-*converges* to $F : (X, \tau) \longrightarrow \overline{\mathbb{R}}$ if it holds that

$$\Gamma - \liminf_{h\to\infty} F_h(u) = \Gamma - \limsup_{h\to\infty} F_h(u) = F(u)$$

for any $u \in X$. In this case we say that F is the $\Gamma-$limit of $(F_h)_h$ and we write $F = \Gamma - \lim_{h\to\infty} F_h$. The next proposition presents some basic properties of Γ-limits which will be useful later on.

Proposition 6.4.1. *The following facts hold.*

(i) For any $u \in X$ and for any sequence $(u_h)_h$ converging to u in X, it holds that

$$\Gamma - \liminf_{h\to\infty} F_h(u) \leqslant \liminf_{h\to\infty} F_h(u_h) \quad \text{and} \quad \Gamma - \limsup_{h\to\infty} F_h(u) \leqslant \limsup_{h\to\infty} F_h(u_h).$$

(ii) For any $u \in X$ there exist two sequences $(u_h)_h$ and $(v_h)_h$, converging to u in X, which we call recovery sequences*, such that*

$$\Gamma - \liminf_{h\to\infty} F_h(u) = \liminf_{h\to\infty} F_h(u_h) \quad \text{and} \quad \Gamma - \limsup_{h\to\infty} F_h(u) = \limsup_{h\to\infty} F_h(v_h).$$

(iii) For any $u \in X$ and for any sequence $(u_h)_h$ converging to u in X, it holds that

$$\Gamma - \lim_{h\to\infty} F_h(u) \leqslant \liminf_{h\to\infty} F_h(u_h); \tag{6.4.1}$$

(iv) For any $u \in X$ there exists a sequence $(u_h)_h$ converging to u in X, which we call recovery sequence*, such that*

$$\Gamma - \lim_{h\to\infty} F_h(u) = \lim_{h\to\infty} F_h(u_h). \tag{6.4.2}$$

Beside the notion of Γ-convergence there is a related one, which is more suitable to deal with sequences of local functionals, usually known as $\bar{\Gamma}$-*convergence*. If we have a sequence of increasing functionals (cf. [105]) such that $F_h : X \times \mathcal{A} \longrightarrow \overline{\mathbb{R}}$ for any $h \in \mathbb{N}$, and we define

$$F'(\cdot, A) := \Gamma - \liminf_{h\to\infty} F_h(\cdot, A) \quad \text{and} \quad F''(\cdot, A) := \Gamma - \limsup_{h\to\infty} F_h(\cdot, A)$$

for any $A \in \mathcal{A}$, we say that F_h $\bar{\Gamma}$-*converges* to a functional $F : X \times \mathcal{A} \longrightarrow \bar{\mathbb{R}}$ if it holds that

$$F(\cdot, A) = \sup\{F'(\cdot, A') \,:\, A' \in \mathcal{A},\, A' \Subset A\} = \sup\{F''(\cdot, A') \,:\, A' \in \mathcal{A},\, A' \Subset A\}.$$

In other words we say that $(F_h)_h$ $\bar{\Gamma}-$converges to F whenever the *inner regular envelopes* of F' and F'' coincide and are equal to F. It is easy to check (cf. [105, Remark 16.3]) that any $\bar{\Gamma}-$limit is increasing, inner regular and lower semicontinuous. In the sequel, when we will deal with Γ-convergence with respect to the strong topology of L^p or with respect to the strong topology of $W^{1,p}_X$, we will refer respectively to $\Gamma(L^p)-$convergence or $\Gamma(W^{1,p}_X)-$convergence. The notions of $\Gamma-$convergence and $\bar{\Gamma}$-convergence, as one could expect, are strongly related. Indeed, assume for instance that a sequence of increasing functionals $F_h : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ is such that

$$F(\cdot, A) = \Gamma(L^p) - \lim_{k\to\infty} F_h(\cdot, A) \tag{6.4.3}$$

for any $A \in \mathcal{A}$ and for a suitable measure functional $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$. Then F is L^p-lower semicontinuous, since it is a Γ-limit (cf. [105, Proposition 6.8], and also increasing and inner regular, since it is a non-negative measure (cf. [105, Theorem 14.23]). Therefore, [105, Proposition 16.4] allows to conclude that

$$F = \bar{\Gamma}(L^p) - \lim_{h\to\infty} F_h. \tag{6.4.4}$$

The converse implication is usually more delicate because, in general, the $\bar{\Gamma}(L^p)-$limit is not a measure. Indeed, even if the $\bar{\Gamma}-$limit is always increasing and inner regular, and even if superadditivity behaves usually well, there are examples (cf. [105, Example 16.13]) in which F fails to be subadditive. Therefore, when dealing with these issues, it is practise to work within milder classes of local functionals. To this aim, the so-called *uniform fundamental estimates* are introduced. These estimates, although depending in their definition on the chosen topological space, are usually sufficient conditions for the subadditivity of the $\bar{\Gamma}-$limit. To give an instance, we introduce here the standard notion of uniform fundamental estimate (cf. [105, Definition 18.2]) for functional defined on $L^p(\Omega) \times \mathcal{A}$. We recall here that, as we will deal also with functionals defined on $W^{1,p}_X(\Omega) \times \mathcal{A}$, in the forthcoming Chapter 8 we will need to slightly modify the current notion to guarantee a better compatibility with the new functional setting.

Definition 6.4.2. *If we have a class $\mathcal{F}$ of non-negative local functionals defined on $L^p(\Omega)\times\mathcal{A}$, we say that $\mathcal{F}$ satisfies the* uniform fundamental estimate on $L^p(\Omega)$ *if, for any $\varepsilon > 0$ and for any $A', A'', B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ such that for any $u, v \in L^p(\Omega)$ and for any $F \in \mathcal{F}$, there exists a smooth cut-off function φ between A'' and A', such that*

$$\begin{aligned} F\Big(\varphi u + (1-\varphi)v, A' \cup B\Big) &\leqslant (1+\varepsilon)\Big(F(u, A'') + F(v, B)\Big) + \\ &+ \varepsilon\Big(\|u\|^p_{L^p(S)} + \|v\|^p_{L^p(S)} + 1\Big) + M\|u - v\|_{L^p(S)}, \end{aligned} \tag{6.4.5}$$

where $S = (A'' \setminus A') \cap B$.

The following result, which can be found in [105, Theorem 18.7], tells us that (6.4.4) is sufficient to guarantee (6.4.3), provided that our sequence satisfies the uniform fundamental estimate and that some reasonable boundedness properties hold.

Theorem 6.4.1. *Let $F_h : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ be a sequence of functionals for which there exists a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ such that (6.4.4) holds. Assume in addition that $(F_h)_h$ satisfies the uniform fundamental estimate and that there exist constants $e_1 \geq 1$ and $e_2 \geq 0$, a non-negative increasing functional $G : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ and a finite measure μ on Ω such that*

$$G(u, A) \leqslant F_h(u, A) \leqslant e_1 G(u, A) + e_2\|u\|^p_{L^p(A)} + \mu(A)$$

for any $u \in L^p(\Omega)$, $A \in \mathcal{A}$ and $h \in \mathbb{N}$. Then (6.4.3) holds.

Chapter 7

Integral representation

7.1 Introduction

We refer to [128] as main reference for this chapter. Inspired by the results proved in [66, 67], the aim of the present chapter is to extend the results achieved in [205] when the assumption of translation-invariance (6.1.2) is dropped. More precisely, we find some sufficient and necessary conditions under which a local functional

$$F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$$

admits an integral representation of the form

$$F(u, A) = \int_A f(x, u(x), Xu(x))\, dx \quad \forall\, u \in W^{1,p}_{X,loc}(\Omega),\, \forall\, A \in \mathcal{A}, \tag{7.1.1}$$

for a suitable Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty)$. We point out that in this new framework, due to the lack of translation-invariance, a dependence of the integrand with respect to the function is expected. Let us observe that if F is defined on $L^p_{loc}(\Omega) \times \mathcal{A}$ instead of $W^{1,p}_{X,loc}(\Omega) \times \mathcal{A}$, under reasonable improvements of some assumptions it is easy to extend the integral representation to get

$$F(u, A) = \int_A f(x, u(x), Xu(x))\, dx \quad \forall\, A \in \mathcal{A},\, \forall u \in L^p_{loc}(\Omega) \text{ such that } u|_A \in W^{1,p}_{X,loc}(A).$$

Our main goal is to obtain a representation formula as in (7.1.1) for the following three different classes of functionals:

(i) convex functionals (cf. Theorem 7.4.1);

(ii) $W^{1,\infty}$ weakly-* sequentially lower semicontinuous functionals (cf. Theorem 7.5.2);

(iii) none of the above (cf. Theorem 7.6.1).

As already pointed out in Chapter 1, unlike in Euclidean Sobolev spaces, in this context no analogue of approximation results by a reasonable notion of piecewise X-affine function holds in general (cf. [205, Section 2.3]). To overcome this difficulty we rely on the method employed in [205], consisting of three steps.

1. Apply one of the classical results for Sobolev spaces ([66, 67]) to the functional, obtaining an integral representation with respect to a Euclidean Lagrangian f_e of the form

$$F(u, A) = \int_A f_e(x, u(x), Du(x))dx \quad \forall u \in W^{1,p}_{loc}(\Omega),\ \forall A \in \mathcal{A}.$$

2. Find sufficient conditions on f_e that guarantee the existence of am anisotropic Lagrangian f such that

$$\int_A f_e(x, u(x), Du(x))dx = \int_A f(x, u(x), Xu(x))dx \quad \forall A \in \mathcal{A},\ \forall u \in C^\infty(A). \tag{7.1.2}$$

3. Extend the previous equality to the whole space $W^{1,p}_{X,loc}(\Omega)$.

The second step crucially exploits the third-argument convexity of the Euclidean Lagrangian f_e. Indeed, convexity of $f_e(x, u, \cdot)$ is sufficient to guarantee (7.1.2) (cf. Proposition 7.4.2). This is shown in [205], and the same ideas can be adapted to the cases (i) and (ii) of convex and weakly-* sequentially lower semicontinuous funtionals, for which the convexity of $f_e(x, u, \cdot)$ is granted. On the contrary, due to the weaker assumptions on the functional, case (iii) is more demanding and requires a further step. In Section 7.6 we show that the convexity of $f_e(x, u, \cdot)$ is not necessary for (7.1.2). Thus, in order to find a more suitable notion of convexity, we define the weaker concept of *X-convexity* (cf. Definition 7.6.3), which strongly depends on the chosen family of vector fields. We show that, under a classical growth assumption on the functional, this new condition is equivalent to (7.1.2) (cf. Proposition 7.6.4). Finally, by sligthly modifying a well known zig-zag argument (cf. [67, Lemma 2.11]), we show that X-convexity is a consequence of a reasonable lower semicontinuity assumption (cf. Lemma 7.6.5). This procedure allows to generalize the final case as well. Finally, for each of the previous results we show that our hypotheses are also necessary, in order to give a complete characterization of the classes of functionals studied.

7.2 When the Euclidean approach fails

7.2.1 Approximation by regular and affine functions

When dealing with representation theorems for local functionals defined on classical Sobolev spaces, a typical strategy is to exploit classical differentiation theorems for measures to get an integral representation of the form

$$F(u, A) = \int_A f_e(x, u, Du)dx$$

for classes of simple functions, that is for instance linear or affine functions of the form

$$\varphi_{x,u,\xi}(y) := u + \langle \xi, y - x \rangle \tag{7.2.1}$$

for given $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Then one can combine some semicontinuity properties of the functional together with approximation results by means of piecewise affine functions (see for instance [125, Chapter X, Proposition 2.9]), in order to extend the integral representation to all Sobolev functions. We already know from Example 1.3.8 that, in our general setting, this last approximation property fails. On the contrary, we saw from Theorem 1.3.3 that a Meyers-Serrin type property is still available. We want now to reformulate Theorem 1.3.3 in a way that is more suitable for our purposes. Let us begin with the following simple proposition.

Proposition 7.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set, take $1 \leqslant p < +\infty$ and consider a function $u \in W^{1,p}_{X,loc}(\Omega)$. Then, Given $u \in W^{1,p}_{X,loc}(\Omega)$ and $A' \Subset \Omega$, then there exists a function $v \in W^{1,p}_X(\Omega)$ which coincides with u on A'.*

Proof. Let φ be a smooth cut-off function between A' and Ω. It is straightforward to verify that the function $v(x) := \varphi(x)u(x)$ satisfies the desired requirements. □

The previous proposition, together with Theorem 1.3.3, allows to prove the following result.

Proposition 7.2.2. *Take a function $u \in W^{1,p}_{X,loc}(\Omega)$ and an open set $A' \Subset \Omega$. Then there exists a sequence $(u_\varepsilon)_\varepsilon \subseteq W^{1,p}_X(\Omega)$ such that*

$$u_\varepsilon|_{A'} \in W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_\varepsilon|_{A'} \longrightarrow u|_{A'} \text{ in } W^{1,p}_X(A').$$

Proof. Let us fix $u \in W^{1,p}_{X,loc}(\Omega)$ and $A' \in \mathcal{A}_0$. By Proposition 7.2.1 we can find a function $\tilde{u} \in W^{1,p}_X(\Omega)$ such that $u|_{A'} = \tilde{u}|_{A'}$, and by Theorem 1.3.3 there exists a sequence $(u_\varepsilon)_\varepsilon \subseteq W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ converging to $\tilde{u}$ in $W^{1,p}_X(\Omega)$. It is easy to see that $(u_{\varepsilon|A'})_\varepsilon \subseteq W^{1,p}_X(A') \cap C^\infty(A')$; moreover, since $u|_{A'} = \tilde{u}|_{A'}$, we conclude that $u_\varepsilon|_{A'} \longrightarrow u|_{A'}$ in $W^{1,p}_X(A')$. □

7.2.2 Failure of a Lusin-type theorem

When dealing with integral representation in classical Sobolev spaces one might exploit the following Lusin-type result (cf. [69, Theorem 13]):

Proposition 7.2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $1 \leqslant p \leqslant +\infty$ and $u \in W^{1,p}(\Omega)$. Then, for any $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ and $v \in C^1(\overline{\Omega})$ such that $|A_\varepsilon| \leqslant \varepsilon$ and $u|_{\Omega \setminus A_\varepsilon} = v|_{\Omega \setminus A_\varepsilon}$.*

Under reasonable assumptions (cf. [67, Lemma 2.7]) this result allows to extend an integral representation result from $C^1(\overline{\Omega}) \times \mathcal{A}$ to $W^{1,p}(\Omega) \times \mathcal{A}$. Again, Example 1.3.9 gives a counterexample to the validity an analogous of Proposition 7.2.3 in our general setting.

7.3 The coefficient matrix

Here we present some algebraic properties of the coefficient matrix $\mathcal{C} : \Omega \longrightarrow \mathbb{R}^{m\times n}$, for which we refer to [205, Section 3.2]. The careful reader will certainly notice several similarities with Chapter 5. Although some of the following results are actually analogous to some of the former, we present them in their entirety for the sake of clarity. For any $x \in \Omega$, we define the linear map $L_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by

$$L_x(\xi) = \mathcal{C}(x) \cdot \xi$$

for any $\xi \in \mathbb{R}^n$. Moreover, we let

$$N_x = \ker(L_x) \qquad \text{and} \qquad V_x = \left\{\mathcal{C}(x)^T \cdot \eta \ : \ \eta \in \mathbb{R}^m\right\}.$$

From standard linear algebra (cf. e.g. [272]), we know that $\mathbb{R}^n = N_x \oplus V_x$. Hence, for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$, there are unique $\xi_{N_x} \in N_x$ and $\xi_{V_x} \in V_x$ such that

$$\xi = \xi_{N_x} + \xi_{V_x}.$$

Therefore, the map $\Pi_x : \mathbb{R}^n \to V_x$ defined by $\Pi_x(\xi) = \xi_{V_x}$ is well-posed. These definitions make sense for a generic family of Lipschitz vector fields. Nevertheless, under the additional assumption (a.e. LIC), the following invertibility and continuity properties hold.

Proposition 7.3.1. *Let X be a family of Lipschitz vector fields satisfying (a.e. LIC) on Ω. Then the following facts hold.*

(i) $\dim V_x = m$ *for each* $x \in \Omega_X$ *and* $L_x(V_x) = \mathbb{R}^m$.
In particular $L_x : V_x \to \mathbb{R}^m$ *is an isomorphism.*

(ii) Let

$$B(x) := \mathcal{C}(x)\, C^T(x) \quad x \in \Omega\,.$$

Then, for each $x \in \Omega_X$, $B(x)$ is a symmetric invertible matrix of order m. Moreover the map $B^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, defined by

$$B^{-1}(x)(z) := B(x)^{-1}z \quad \textit{if } z \in \mathbb{R}^m\,,$$

is continuous.

(iii) For each $x \in \Omega_X$, the projection Π_x can be represented as

$$\Pi_x(\xi) = \xi_{V_x} = \mathcal{C}(x)^T B(x)^{-1}\mathcal{C}(x)\,\xi, \quad \forall\, \xi \in \mathbb{R}^n\,.$$

Remark 7.3.2. It is easy to see that $N_X = \{x \in \Omega \ : \ \det B(x) = 0\}$. Hence N_X is closed in Ω.

Proposition 7.3.3. *Let X be a family of Lipschitz vector fields satisfying* (a.e. LIC) *on* Ω. *Then the map* $L_x : V_x \to \mathbb{R}^m$ *is invertible and the map* $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ *defined by*

$$L^{-1}(x) := L_x^{-1} \text{ if } x \in \Omega_X$$

belongs to $\mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$.

We refer the reader to Chapter 9 for weaker statements outside the (a.e. LIC) setting.

7.4 Convex functionals

In this section we completely characterize a class of convex local functionals defined on $W^{1,p}_X$. As announced, we exploit [66, Lemma 4.1] to get an integral representation of the form

$$F(u, A) = \int_A f_e(x, u, Du)dx \quad \forall A \in \mathcal{A}, \forall u \in W^{1,p}(\Omega).$$

Then the forthcoming Proposition 7.4.1 and Proposition 7.4.2 guarantee the existence of a non Euclidean Lagrangian f such that

$$\int_A f(x, u, Xu)dx = \int_A f_e(x, u, Du)dx \quad \forall A \in \mathcal{A}, \forall u \in C^\infty(A).$$

Finally, we extend the integral representation to the whole $W^{1,p}_{X,loc}(\Omega)$. The following propositions, which are almost totally inspired by [205, Theorem 3.5] and [205, Lemma 3.13], allow us to pass from an Euclidean to a non Euclidean integral representation.

Proposition 7.4.1. *Let* $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ *be a Carathéodory function. Let us define* $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ *by*

$$f(x, u, \eta) := \begin{cases} f_e(x, u, L^{-1}(x)(\eta)) & \text{if } (x, u, \eta) \in \Omega_X \times \mathbb{R} \times \mathbb{R}^m \\ 0 & \text{otherwise.} \end{cases} \tag{7.4.1}$$

Then the following facts hold:

(i) *f is a Carathéodory function;*

(ii) *if $f_e(x, \cdot, \cdot)$ is convex for a.e. $x \in \Omega$, then $f(x, \cdot, \cdot)$ is convex for a.e. $x \in \Omega$;*

(iii) *if $f_e(x, u, \cdot)$ is convex for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$, then $f(x, u, \cdot)$ is convex for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$;*

(iv) *If we assume that*

$$f_e(x, u, \xi) = f_e(x, u, \Pi_x(\xi)) \quad \text{for a.e. } x \in \Omega, \ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{7.4.2}$$

then it follows that

$$\int_A f_e(x,u,Du)\,dx = \int_A f(x,u,Xu)\,dx \quad \forall A \in \mathcal{A},\, \forall u \in C^\infty(A). \tag{7.4.3}$$

Proof. (i) First we want to show that, for any $(u,\eta) \in \mathbb{R}\times\mathbb{R}^m$, the function $x \mapsto f(x,u,\eta)$ is measurable. Let us fix then $(u,\eta) \in \mathbb{R}\times\mathbb{R}^m$, define the function $\Phi : \Omega_X \longrightarrow \mathbb{R}\times\mathbb{R}^n$ as $\Phi(x) := (u, L^{-1}(x)(\eta))$ and extend it to be zero on $\Omega \setminus \Omega_X$. By Proposition 7.3.3, $\Phi|_{\Omega_X}$ is continuous, and so in particular Φ is measurable. Noticing that

$$f(x,u,\eta) = f_e(x,\Phi(x)) \quad \forall x \in \Omega_X,$$

being f_e a Carathéodory function and recalling [100, Proposition 3.7] we conclude that $x \mapsto f(x,u,\eta)$ is measurable. Let us define now the function $\Psi : \Omega_X \times \mathbb{R}\times\mathbb{R}^m \longrightarrow \Omega_X\times\mathbb{R}\times\mathbb{R}^n$ as $\Psi(x,u,\eta) := (x,u,L^{-1}(x)(\eta))$. Since on Ω_X we have that $f = f_e\circ\Psi$, then, for any fixed $x \in \Omega_X$ such that $f_e(x,\cdot,\cdot)$ is continuous, $f(x,\cdot,\cdot)$ is the composition of a continuous function and a linear function, and so it is continuous.
(ii) If $x \in \Omega_X$ is such that $f_e(x,\cdot,\cdot)$ is convex, then $f = f_e\circ\Psi$ is the composition of a convex function and a linear function, and so it is convex.
(iii) Follows as (ii).
(iv) Assume that (7.4.2) holds. Let us fix $A \in \mathcal{A}$ and $u \in C^\infty(A)$. From the regularity of u we have that $Xu(x) = \mathcal{C}(x)Du(x)$. By Proposition 7.3.1 we get

$$\begin{aligned} L_x(\Pi_x(Du)) &= L_x(\mathcal{C}(x)^T B(x)^{-1}\mathcal{C}(x)Du) \\ &= \mathcal{C}(x)\mathcal{C}(x)^T B(x)^{-1}\mathcal{C}(x)Du \\ &= B(x)B(x)^{-1}\mathcal{C}(x)Du \\ &= \mathcal{C}(x)Du \\ &= L_x(Du), \end{aligned}$$

and

$$\begin{aligned} f(x,u,Xu) &= f(x,u,\mathcal{C}(x)Du) \\ &= f(x,u,L_x(Du)) \\ &= f(x,u,L_x(\Pi_x(Du))) \\ &= f_e(x,u,L_x^{-1}(L_x(\Pi_x(Du)))) \\ &= f_e(x,u,\Pi_x(Du)) \\ &= f_e(x,u,Du). \end{aligned}$$

Now (7.4.3) follows by integrating over A.

□

In the following result we provide some sufficient conditions to guarantee (7.4.2).

Proposition 7.4.2. *Let $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0, +\infty]$ be a Carathéodory function such that*

(i) *$f_e(x, u, \cdot)$ is convex for a.e $x \in \Omega$, for any $u \in \mathbb{R}$;*

(ii) *there exist $a \in L^1_{loc}(\Omega)$ and $b, c > 0$ such that*

$$f_e(x, u, \xi) \leqslant a(x) + b|\mathcal{C}(x)\xi|^p + c|u|^p \tag{7.4.4}$$

for a.e. $x \in \Omega$, for any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Then f_e satisfies (7.4.2).

Proof. Follows with some trivial modifications as in [205, Lemma 3.13]. □

Let us now state and prove the main result of this section.

Theorem 7.4.1. *Let $F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ be such that:*

(i) *F is a measure;*

(ii) *F is local;*

(iii) *F is convex;*

(iv) *F is p-bounded.*

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ such that

$$(u, \xi) \mapsto f(x, u, \xi) \text{ is convex for a.e. } x \in \Omega, \tag{7.4.5}$$

$$f(x, u, \xi) \leqslant a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m \tag{7.4.6}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu)dx \quad \forall u \in W^{1,p}_{X,loc}(\Omega),\ \forall A \in \mathcal{A}. \tag{7.4.7}$$

Moreover, if $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ are two Carathéodory functions satisfying (7.4.5), (7.4.6) and (7.4.7), then there exists $\tilde{\Omega} \subseteq \Omega$ such that $|\tilde{\Omega}| = |\Omega|$ and

$$f_1(x, u, \xi) = f_2(x, u, \xi) \quad \forall x \in \tilde{\Omega},\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m. \tag{7.4.8}$$

Proof. We divide the proof into several steps.
Step 1. Let

$$C := \max\{\sup\{|c_{j,i}(x)| \,:\, x \in \Omega\} \,:\, i = 1, \dots, n,\ j = 1, \dots, m\}.$$

Then from our assumptions on X it follows that $0 < C < +\infty$. Let $\tilde{b} := C^p b$. Using (iv) and recalling that for all $u \in W^{1,p}(\Omega)$ we have that $Xu(x) = \mathcal{C}(x)Du(x)$ it follows that

$$F(u,A') \leqslant \int_{A'} a(x) + c|u|^p + \tilde{b}|Du|^p dx \quad \forall A' \in \mathcal{A}_0,\ \forall u \in W^{1,p}(\Omega). \tag{7.4.9}$$

Thus we can apply [66, Lemma 4.1] to get a Carathéodory function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0,+\infty]$ such that

$$F(u,A) = \int_A f_e(x,u,Du)dx \quad \forall A \in \mathcal{A},\ \forall u \in W^{1,p}_{loc}(\Omega), \tag{7.4.10}$$

$$f_e(x,u,\xi) \leqslant a(x) + \tilde{b}|\xi|^p + c|u|^p \qquad \text{for a.e. } x \in \Omega,\ \forall (u,\xi) \in \mathbb{R} \times \mathbb{R}^n \tag{7.4.11}$$

and

$$f_e(x,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \to [0,\infty] \text{ is convex for a.e. } x \in \Omega. \tag{7.4.12}$$

Step 2. We want to prove that f_e satisfies (7.4.2). By Proposition 7.4.2 and (7.4.12) we only need to prove (7.4.4). Let us take then $\Omega' \subseteq \Omega$ such that $|\Omega'| = |\Omega|$ and

$$(u,\xi) \mapsto f_e(x,u,\xi) \text{ is convex and finite } \forall x \in \Omega', \tag{7.4.13}$$

and fix $x \in \Omega'$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$. By (7.4.10), for any $R > 0$ small enough to ensure that $B_R(x) \Subset \Omega$, we have that

$$F(\varphi_{x,u,\xi}, B_R(x)) = \int_{B_R(x)} f_e(y, u + \langle \xi, y-x\rangle, \xi)\, dy$$

and from (iv) we have that

$$F(\varphi_{x,u,\xi}, B_R(x)) \leqslant \int_{B_R(x)} a(y) + c|u + \langle \xi, y-x\rangle|^p + b|C(y)\xi|^p\, dy,$$

where $\varphi_{x,u,\xi}$ is as in (7.2.1). Combining these two facts and dividing by $|B_R(x)|$ we obtain that

$$\fint_{B_R(x)} f_e(y, u + \langle \xi, x-y\rangle, \xi) dy \leqslant \fint_{B_R(x)} a(y) + c|u + \langle \xi, y-x\rangle|^p + b|C(y)\xi|^p dy. \tag{7.4.14}$$

Since the right integrand is in $L^1_{loc}(\Omega)$, and (7.4.14) holds indeed for all $A' \in \mathcal{A}_0$, the left one is in $L^1_{loc}(\Omega)$ as well. Therefore, thanks to Lebesgue theorem (cf. [285]) we can find $\Omega_{u,\xi} \subseteq \Omega'$ such that $|\Omega_{u,\xi}| = |\Omega|$ and

$$f_e(x,u,\xi) \leqslant a(x) + c|u|^p + b|\mathcal{C}(x)\xi|^p \quad \forall x \in \Omega_{u,\xi}.$$

Setting $\tilde{\Omega} := \bigcap_{(u,\xi)\in\mathbb{Q}\times\mathbb{Q}^n} \Omega_{u,\xi}$, it holds that $|\tilde{\Omega}| = |\Omega|$ and

$$f_e(x,u,\xi) \leqslant a(x) + c|u|^p + b|\mathcal{C}(x)\xi|^p \quad \forall x \in \tilde{\Omega},\ \forall (u,\xi) \in \mathbb{Q} \times \mathbb{Q}^n.$$

Since the map $(u,\xi) \mapsto f_e(x,u,\xi)$ is continuous for any $x \in \tilde{\Omega}$ and $\mathbb{Q} \times \mathbb{Q}^n$ is dense in $\mathbb{R} \times \mathbb{R}^n$ then (7.4.4) holds and the conclusion follows.

Step 3. Thanks to the previous step we can apply (iv) of Proposition 7.4.1. Hence we get

$$\int_A f_e(x,u,Du)dx = \int_A f(x,u,Xu)dx \quad \forall A \in \mathcal{A},\ u \in C^\infty(A), \tag{7.4.15}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty]$ is the function defined in (7.4.1). First of all we can assume that f is finite up to modifying it on a set of measure zero. Moreover, thanks to (7.4.12) and (ii) of Proposition 7.4.1 we have that f satisfies (7.4.5). Now we want to prove that f satisfies (7.4.6). Let us fix $x \in \Omega$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$: by (iv), (7.4.10) and (7.4.15) we have that

$$\begin{aligned}\int_{B_R(x)} f(y,\varphi_{x,u,\xi},X\varphi_{x,u,\xi})\,dy &\leqslant \int_{B_R(x)} a(y) + c|\varphi_{x,u,\xi}|^p + b|X\varphi_{x,u,\xi}|^p dy \\ &= \int_{B_R(x)} a(y) + c|u + \langle \xi, y-x\rangle|^p + b|C(y)\xi|^p\,dy,\end{aligned}$$

and so, dividing by $|B_R(x)|$, we get that

$$⨍_{B_R(x)} f(y,u+\langle\xi,y-x\rangle,C(y)\xi)dy \leqslant ⨍_{B_R(x)} a(y) + c|u+\langle\xi,y-x\rangle|^p + b|C(y)\xi|^p dy.$$

Arguing as in the second step we can conclude that

$$f(x,u,\mathcal{C}(x)\xi) \leqslant a(x) + b|\mathcal{C}(x)\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u,\xi) \in \mathbb{R}\times\mathbb{R}^n.$$

Finally, recalling that for $x \in \Omega_X$ the map $L_x : V_x \to \mathbb{R}^m$ is surjective, (7.4.6) follows.

Step 4. Here we want to prove that (7.4.7) holds. Let us fix $u \in W^{1,p}_X(\Omega)$ and $A' \in \mathcal{A}_0$, and consider the two functionals

$$F_{A'}, G_{A'} : (\{v|_{A'} \,:\, v \in W^{1,p}_X(\Omega)\}, \|\cdot\|_{W^{1,p}_X(A')}) \longrightarrow [0,+\infty]$$

defined by $F_{A'}(v|_{A'}) := F(v,A')$ and $G_{A'}(v|_{A'}) := \int_{A'} f(x,v,Xv)dx$ respectively. Thanks to (iii), (iv), (7.4.5) and (7.4.6), they are convex and bounded on bounded sets on $\{v|_{A'} \,:\, v \in W^{1,p}_X(\Omega)\}$. Hence, they are continuous (cf. [125, Lemma 2.1]). Moreover, from Proposition 7.2.2 we can find a sequence $(u_\varepsilon)_\varepsilon \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_\varepsilon|_{A'})_\varepsilon \subseteq W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_\varepsilon|_{A'} \longrightarrow u|_{A'} \text{ in } W^{1,p}_X(A').$$

From (7.4.10) and (7.4.15) we get that

$$F(u,A') = \lim_{\varepsilon\to 0} F(u_\varepsilon,A') = \lim_{\varepsilon\to 0}\int_{A'} f_e(x,u_\varepsilon,Du_\varepsilon = \lim_{\varepsilon\to 0}\int_{A'} f(x,u_\varepsilon,Xu_\varepsilon) = \int_{A'} f(x,u,Xu)dx,$$

and so we assert that

$$F(u,A') = \int_A f(x,u,Xu)dx \quad \forall u \in W^{1,p}_X(\Omega),\ \forall A' \in \mathcal{A}_0. \tag{7.4.16}$$

Let us take now $u \in W^{1,p}_{X,loc}(\Omega)$, $A \in \mathcal{A}$ and $A' \Subset A$, and, thanks to Proposition 7.2.1, take a function $v \in W^{1,p}_X(\Omega)$ such that $u|_{A'} = v|_{A'}$. Thus, from hypothesis (*ii*) and from (7.4.16), we have that

$$F(u, A') = F(v, A') = \int_{A'} f(x, v, Xv)dx = \int_{A'} f(x, u, Xu)dx. \tag{7.4.17}$$

Since by hypothesis the function $B \mapsto F(u, B)$ is inner regular (cf. [105, Theorem 14.23]), and noticing that the function $B \mapsto \int_B f(x, u, Xu)dx$ is inner regular, thanks to (7.4.17) we have that

$$F(u, A) = \sup\{F(u, A') \,:\, A' \Subset A\} = \sup\left\{\int_{A'} f(x, u, Xu)dx \,:\, A' \Subset A\right\} = \int_A f(x, u, Xu)dx,$$

and so we can conclude that (7.4.7) holds.

Step 5. Let us show the uniqueness of the Lagrangian. Fix then $x \in \Omega$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$: since (7.4.7) holds both for f_1 and f_2, for any $R > 0$ small enough we have that

$$\fint_{B_R(x)} f_1(y, u + \langle \xi, y - x\rangle, C(y)\xi)dy = \fint_{B_R(x)} f_2(y, u + \langle \xi, y - x\rangle, C(y)\xi)dy$$

Since both integrand functions satisfy (7.4.6), then they are both in $L^1_{loc}(\Omega)$. Again, thanks to Lebesgue theorem, there exists $\Omega_{u,\xi} \subseteq \Omega$ such that $|\Omega_{u,\xi}| = |\Omega|$ and

$$f_1(x, u, \mathcal{C}(x)\xi) = f_2(x, u, \mathcal{C}(x)\xi) \quad \forall x \in \Omega_{u,\xi}.$$

If we set

$$\tilde{\Omega} := \bigcap_{(u,\xi)\in\mathbb{Q}\times\mathbb{Q}^n} \Omega_{u,\xi} \cap \{x \in \Omega \,:\, (7.4.5) \text{ and } (7.4.6) \text{ hold for } f_1 \text{ and } f_2\} \cap \Omega_X,$$

clearly we have $|\tilde{\Omega}| = |\Omega|$ and it holds that

$$f_1(x, u, \mathcal{C}(x)\xi) = f_2(x, u, \mathcal{C}(x)\xi) \quad \forall x \in \tilde{\Omega},\ \forall (u, \xi) \in \mathbb{Q} \times \mathbb{Q}^n. \tag{7.4.18}$$

Since $(u, \xi) \mapsto f_1(x, u, \xi)$ and $(u, \xi) \mapsto f_2(x, u, \xi)$ are continuous for any $x \in \tilde{\Omega}$, and recalling again that for any $x \in \Omega_X$ L_x is surjective, then (7.4.8) follows. □

The following theorem tells us that all the hypotheses of Theorem 7.4.1 are also necessary.

Theorem 7.4.2. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ be a Carathéodory function such that*

$$(u, \xi) \mapsto f(x, u, \xi) \text{ is convex for a.e. } x \in \Omega, \tag{7.4.19}$$

$$f(x, u, \xi) \leqslant a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m \tag{7.4.20}$$

for some $b, c > 0$ and $a \in L^1_{loc}(\Omega)$. If we set the functional $F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ as

$$F(u, A) := \int_A f(x, u, Xu)dx \quad \forall u \in W^{1,p}_{X,loc}(\Omega),\ \forall A \in \mathcal{A},$$

then F satisfies hypotheses $(i)-(iv)$ of Theorem 7.4.1.

Proof. Let us fix $u \in W^{1,p}_{X,loc}(\Omega)$: our aim is to prove that $\alpha(A) := F(u,A)$ is a measure. Notice that, being $f \geq 0$, α is increasing, and of course $\alpha(\emptyset) = 0$. Then, according to [105, Theorem 14.23], it suffices to show that α is subadditive, superadditive and inner regular. The first two properties are trivial, so let us focus on the third one. Let us fix $A \in \mathcal{A}$ and define the sequence of sets $(A_h)_h$ as $A_h := \{x \in A : \text{dist}(x, \partial A) > \frac{1}{h}\}$. We have that $(A_h)_h \subseteq \mathcal{A}_0$, $A_h \Subset A_{h+1} \Subset A$ and $\bigcup_{h\in\mathbb{N}_+} A_h = A$. Thus by the Monotone Convergence Theorem we conclude that

$$\int_A f(x,u,Xu)dx = \int_A \lim_{h\to+\infty} \chi_{A_h} f(x,u,Xu)dx = \lim_{h\to+\infty} \int_{A_h} f(x,u,Xu)dx,$$

and so α is a measure. Property (ii) is straightforward, noticing that the X-gradients of two a.e. equal functions coincide a.e. Finally, (iii) and (iv) follow from (7.4.19) and (7.4.20). □

7.5 Weakly-* sequentially semicontinuous functionals

In this section we characterize a class of local functionals defined on $W^{1,p}_X$ for which we do not require neither translation-invariance nor convexity, but which are weakly-* sequentially lower semicontinuous in $W^{1,\infty}$. It is well known (cf. [2]) that, for an integral functional of the form

$$F(u,A) := \int_A f_e(x,u,Du)dx,$$

the weak-* lower semicontinuity is equivalent to the convexity in the third entry of f_e. Therefore we can adopt the same strategy employed in the previous section, exploiting [67, Theorem 1.10] to get an Euclidean integral representation of the form

$$F(u,A) = \int_A f_e(x,u,Du)dx \quad \forall A \in \mathcal{A}, \, , \forall u \in W^{1,p}(\Omega).$$

Again, Proposition 7.4.1 and Proposition 7.4.2 guarantee the existence of a non Euclidean Lagrangian f such that

$$\int_A f(x,u,Xu)dx = \int_A f_e(x,u,Du)dx \quad \forall A \in \mathcal{A}, \, , \forall u \in C^\infty(A).$$

We start by proving an useful continuity result in $W^{1,p}_X$, whose classical version is usually known as *Carathéodory continuity theorem* (cf. [100]).

Theorem 7.5.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty]$ be a Carathéodory function such that there exist $a \in L^1_{loc}(\Omega)$ and $b, c > 0$ such that*

$$f(x,u,\xi) \leqslant a(x) + b|\xi|^p + c|u|^p \quad \textit{for a.e. } x \in \Omega, \, \forall (u,\xi) \in \mathbb{R} \times \mathbb{R}^m. \tag{7.5.1}$$

Then it holds that, for any $A' \in \mathcal{A}_0$, the functional

$$F : W^{1,p}_X(A') \longrightarrow [0, +\infty)$$

defined by

$$F(u) := \int_{A'} f(x, u, Xu)dx$$

is continuous with respect to the strong topology of $W^{1,p}_X(A')$.

Proof. Let us prove that F is lower semicontinuous. Fix $u \in W^{1,p}_X(A')$ and take a sequence $(u_h)_h \subseteq W^{1,p}_X(A')$ converging to u and such that

$$\exists \lim_{h\to+\infty} F(u_h) < +\infty.$$

Up to a subsequence we assume that $(u_h(x))_h$ converges to $u(x)$ and $(Xu_h(x))_h$ converges to $Xu(x)$ for a.e. $x \in A'$. Being f Carathéodory, $\lim_{h\to\infty} f(x, u_h(x), Xu_h(x)) = f(x, u(x), Xu(x))$ for a.e. $x \in \Omega$. Thanks to Fatou's lemma (cf. [285]) we conclude that

$$F(u) = \int_{A'} f(x, u, Xu)dx = \int_{A'} \liminf_{h\to+\infty} f(x, u_h, Xu_h) \leqslant \liminf_{h\to+\infty} \int_{A'} f(x, u_h, Xu_h) = \lim_{h\to+\infty} F(u_h).$$

We are left to prove that F is upper semicontinuous. Again, fix $u \in W^{1,p}_X(A')$ and take a sequence $(u_h)_h \subseteq W^{1,p}_X(A')$ converging to u and such that

$$\exists \lim_{h\to+\infty} F(u_h) > -\infty.$$

Up to a subsequence, we can assume that $(u_h(x))_h$ converges to $u(x)$ and $(Xu_h(x))_h$ converges to $Xu(x)$ for almost every $x \in A'$. Let us define the sequence of functions

$$g_h(x) := -f(x, u_h, Xu_h) + C(|Xu_h|^p + |u_h|^p)$$

where $C := \max\{b, c\} > 0$. Using (7.5.1) we get

$$g_h(x) \geq -a(x) \text{ for a.e. } x \in A',$$

and so, since the right side belongs to $L^1(A')$, we can apply Fatou's Lemma and get that

$$\begin{aligned}
\int_{A'} -f(x, u, Xu)dx + \|u\|_{W^{1,p}_X(A')} &= \int_{A'} \liminf_{h\to+\infty} g_h(x, u, Xu)dx \\
&= \int_{A'} \liminf_{h\to+\infty} (-f(x, u_h, Xu_h) + C(|Xu_h|^p + |u_h|^p))dx \\
&\leqslant \liminf_{h\to+\infty} \int_{A'} -f(x, u_h, Xu_h) + C(|Xu_h|^p + |u_h|^p))dx \\
&= \lim_{h\to+\infty} \int_{A'} -f(x, u_h, Xu_h) + C \lim_{h\to+\infty} \|u_h\|_{W^{1,p}_X(A')} \\
&= \lim_{h\to+\infty} \int_{A'} -f(x, u_h, Xu_h) + \|u\|_{W^{1,p}_X(A')}.
\end{aligned}$$

□

In the following proposition we prove that the notion of lower semicontinuity introduced in Definition 6.2.2 is actually equivalent to a more useful condition.

Proposition 7.5.1. *Let* $F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ *be such that:*

(i) F *is a measure;*

(ii) F *is local.*

Then the following conditions are equivalent:

(a) F *is lower semicontinuous;*

(b) $\forall A' \in \mathcal{A}_0$, $F_{A'} : (\{u|_{A'} : u \in W^{1,p}_X(\Omega)\}, \|\cdot\|_{W^{1,p}_X(A')}) \to [0, +\infty]$ *defined by* $F_{A'}(u|_{A'}) := F(u, A')$ *is lower semicontinuous.*

Proof. $(b) \implies (a)$**.** It is straightforward.
$(a) \implies (b)$**.** Fix an open set $A' \in \mathcal{A}_0$ and take $(u_h)_h, u$ in $W^{1,p}_X(\Omega)$ such that $\|u_h|_{A'} - u|_{A'}\|_{W^{1,p}(A')} \to 0$. Now, for any $k \in \mathbb{N}$, take an open set A_k such that $A_k \Subset A_{k+1} \Subset A'$ and $\bigcup_{k=0}^{+\infty} A_k = A'$, and a smooth cut-off function φ_k between A_k and A'. For any $h, k \in \mathbb{N}$, define the functions $v^k := \varphi_k u$ and $v^k_h := \varphi_k u_h$. We have that, for any $h, k \in \mathbb{N}$, v^k_h, v^k belong to $W^{1,p}_X(\Omega)$, $v^k_h|_{A_k} = u_h|_{A_k}$, $v^k|_{A_k} = u|_{A_k}$ and moreover $\lim_{h\to\infty} \|v^k_h - v^k\|_{W^{1,p}_X(\Omega)} = 0$ for any $k \in \mathbb{N}$. Using (i) and (ii) we get

$$\begin{aligned}
F(u, A') &= \lim_{k\to\infty} F(u, A_k) \\
&= \lim_{k\to\infty} F(v^k, A_k) \\
&\leqslant \lim_{k\to\infty} \liminf_{h\to\infty} F(v^k_h, A_k) \\
&= \lim_{k\to\infty} \liminf_{h\to\infty} F(u_h, A_k) \\
&\leqslant \lim_{k\to\infty} \liminf_{h\to\infty} F(u_h, A') \\
&= \liminf_{h\to\infty} F(u_h, A').
\end{aligned}$$

□

We are ready to state the main result of this section.

Theorem 7.5.2. *Let* $F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ *be such that:*

(i) F *is a measure;*

(ii) F *is local;*

(iii) F *satisfies the weak condition* (ω)*;*

(iv) F *is* p*-bounded;*

(v) *F is weakly-** *sequentially lower semicontinuous;*

(vi) *F is lower semicontinuous.*

Then there exists a unique Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ *such that*

$$\xi \mapsto f(x, u, \xi) \text{ is convex for a.e. } x \in \Omega,\ \forall u \in \mathbb{R}, \tag{7.5.2}$$

$$f(x, u, \xi) \leqslant a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m \tag{7.5.3}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu)dx \quad \forall u \in W^{1,p}_{X,loc}(\Omega),\ \forall A \in \mathcal{A}. \tag{7.5.4}$$

Remark 7.5.2. If we substitute hypotheses (v) and (vi) with

(v′) F is weakly sequentially lower semicontinuous,

then the conclusions of Theorem 7.5.2 still hold. Indeed, thanks to Proposition 1.3.7 the latter is stronger than both (v) and (vi), even if not equivalent in general.

Proof. Arguing as in the first step of the proof of Theorem 7.4.1, the restriction of F to $W^{1,p}_{loc}(\Omega) \times \mathcal{A}$ satisfies all the hypotheses of [67, Theorem 1.10]. Thus there exist $\tilde{b} > 0$ and a Carathéodory function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0, +\infty]$ such that

$$F(u, A) = \int_A f_e(x, u, Du)dx \quad \forall A \in \mathcal{A},\ \forall u \in W^{1,p}_{loc}(\Omega), \tag{7.5.5}$$

$$f_e(x, u, \xi) \leqslant a(x) + \tilde{b}|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n \tag{7.5.6}$$

and

$$f_e(x, u, \cdot) : \mathbb{R}^n \to [0, \infty] \text{ is convex} \quad \text{for a.e. } x \in \Omega,\ \forall u \in \mathbb{R}. \tag{7.5.7}$$

Now, arguing as in the second step of the proof of Theorem 7.4.1, from (7.5.6) and (7.5.7) and recalling Proposition 7.4.1 and Proposition 7.4.2, we obtain that

$$\int_A f_e(x, u, Du)dx = \int_A f(x, u, Xu)dx \quad \forall A \in \mathcal{A},\ u \in C^\infty(A), \tag{7.5.8}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty]$ is the Carathéodory function defined in (7.4.1). Up to modifying f on a set of measure zero, we can assume that it is finite. Moreover, arguing as in the third step of the proof of Theorem 7.4.1, f satisfies (7.5.2) and (7.5.3).
Let us prove that (7.5.4) holds. Let us start by fixing $u \in W^{1,p}_X(\Omega)$ and $A' \in \mathcal{A}_0$. Thanks to Proposition 7.2.2 we can find a sequence $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_h|_{A'})_h \subseteq W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_h|_{A'} \longrightarrow u|_{A'} \text{ in } W^{1,p}_X(A').$$

From this, (vi), (7.5.5), (7.5.8), Theorem 7.5.1 and Proposition 7.5.1 it follows that

$$\begin{aligned} F(u,A') &\leqslant \liminf_{h\to+\infty} F(u_h,A') \\ &= \liminf_{h\to+\infty} \int_{A'} f_e(x,u_h,Du_h)dx \\ &= \lim_{h\to+\infty} \int_{A'} f(x,u_h,Xu_h)dx \\ &= \int_{A'} f(x,u,Xu)dx, \end{aligned}$$

and hence we obtain that

$$F(u,A') \leqslant \int_{A'} f(x,u,Xu)dx \quad \forall A' \in \mathcal{A}_0,\, \forall u \in W^{1,p}_X(\Omega). \tag{7.5.9}$$

To prove the converse inequality, fix $u_0 \in W^{1,p}_X(\Omega)$ and set $H : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ as $H(u,A) := F(u+u_0,A)$. It is straightforward to check that H satisfies all the hypotheses of the theorem. Hence there exist a Carathéodory function $h : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty)$, $a_H \in L^1_{loc}(\Omega)$ and $b_H, c_H > 0$ such that

$$h(x,u,\xi) \leqslant a_H(x) + b_H|\xi|^p + c_H|u|^p \quad \text{for a.e. } x \in \Omega,\, \forall (u,\xi) \in \mathbb{R} \times \mathbb{R}^m.$$

Moreover, it holds that

$$H(u,A) = \int_A h(x,u,Xu)dx \quad \forall A \in \mathcal{A},\, \forall u \in C^\infty(A) \tag{7.5.10}$$

and

$$H(u,A') \leqslant \int_{A'} h(x,u,Xu)dx \quad \forall A' \in \mathcal{A}_0,\, \forall u \in W^{1,p}_X(\Omega). \tag{7.5.11}$$

Fix then $A' \in \mathcal{A}_0$. Arguing as before we can find a sequence $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_h|_{A'})_h \subseteq W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_h|_{A'} \longrightarrow u_0|_{A'} \text{ in } W^{1,p}_X(A').$$

Thus, thanks to Theorem 7.5.1, and the following chain of inequalities we get that

$$
\begin{aligned}
\int_{A'} h(x,0,0) &\overset{(7.5.10)}{=} H(0,A') \\
&= F(u_0,A') \\
&\overset{(7.5.9)}{\leqslant} \int_{A'} f(x,u_0,Xu_0)dx \\
&= \lim_{h\to+\infty} \int_{A'} f(x,u_h,Xu_h)dx \\
&= \lim_{h\to+\infty} F(u_h,A') \\
&= \lim_{h\to+\infty} H(u_h-u_0,A') \\
&\overset{(7.5.11)}{\leqslant} \lim_{h\to+\infty} \int_{A'} h(x,u_h-u_0,Xu_h-Xu_0)dx \\
&= \int_{A'} h(x,0,0)dx,
\end{aligned}
$$

and all inequalities are indeed equalities. Being u_0 arbitrarily chosen, we conclude that

$$
F(u,A') = \int_{A'} f(x,u,Xu)dx \quad \forall u \in W^{1,p}_X(\Omega),\ \forall A' \in \mathcal{A}_0. \tag{7.5.12}
$$

The rest of the proof follows as in the proof of Theorem 7.4.1. □

The following theorem shows that the hypotheses of Theorem 7.5.2 are also necessary.

Theorem 7.5.3. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty)$ be a Carathéodory function such that*

$$
\xi \mapsto f(x,u,\xi) \text{ is convex for a.e. } x \in \Omega,\ \forall u \in \mathbb{R}, \tag{7.5.13}
$$

$$
f(x,u,\xi) \leqslant a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u,\xi) \in \mathbb{R} \times \mathbb{R}^m \tag{7.5.14}
$$

for $b, c > 0$ and $a \in L^1_{loc}(\Omega)$, and define the functional $F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ as

$$
F(u,A) := \int_A f(x,u,Xu)dx \quad \forall u \in W^{1,p}_{X,loc}(\Omega),\ \forall A \in \mathcal{A}.
$$

Then F satisfies hypotheses $(i)-(vi)$ of Theorem 7.5.2.

Proof. (i) follows as in the proof of Theorem 7.4.2, while (ii) is trivial. In order to prove (iii) let us show that F satisfies the strong property (ω). This suffices, according to [67]. Since f is Carathéodory, then the set $\Omega' := \{x \in \Omega : (u,\xi) \mapsto f(x,u,\xi) \text{ is continuous}\}$ satisfies $|\Omega'| = |\Omega|$. For any $k \in \mathbb{N}$ and $\varepsilon > 0$ set $E^k_\varepsilon \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ as

$$
E^k_\varepsilon := \{(u,v,\xi,\eta) : |u|,|v|,|\xi|,|\eta| \leqslant k,\ |u-v|,|\xi-\eta| \leqslant \varepsilon\}
$$

and the function

$$\omega_k(x,\varepsilon) := \begin{cases} \sup\{|f(x,u,\xi) - f(x,v,\eta)| \,:\, (u,v,\xi,\eta) \in E^k_\varepsilon\} & \text{if } x \in \Omega', \\ 0 & \text{otherwise.} \end{cases}$$

We show that, for any k, ω_k is a locally integrable modulus of continuity. Let us fix then $\varepsilon \geq 0$: since $(u,\xi) \mapsto f(x,u,\xi)$ is continuous for almost every $x \in \Omega$, then the supremum in the definition of ω_k can be taken over a countable subset of E^k_ε. Since for any (u,v,ξ,η) the function $x \mapsto |f(x,u,\xi) - f(x,v,\eta)|$ is measurable, then $\omega_k(\cdot,\varepsilon)$ is measurable. We are left to show that it belongs to $L^1_{loc}(\Omega)$. Observe that by (7.5.14) it follows that, for any $(u,v,\xi,\eta) \in E^k_\varepsilon$,

$$\begin{aligned} |f(x,u,\xi) - f(x,v,\eta)| &\leqslant 2|a(x)| + b|\xi|^p + b|\eta|^p + c|u|^p + c|v|^p \\ &\leqslant 2|a(x)| + 4k(b+c). \end{aligned}$$

Since the right side does not depend on $(u,v,\xi,\eta) \in E^k_\varepsilon$, we conclude that

$$\omega_k(x,\varepsilon) \leqslant 2|a(x)| + 4k(b+c).$$

Hence $\omega_k(\cdot,\varepsilon) \in L^1_{loc}(\Omega)$. Fix now $x \in \Omega'$. Since $E^k_\varepsilon \subseteq E^k_\delta$ for any $\varepsilon \leqslant \delta$, then $\omega_k(x,\cdot)$ is increasing, and $\omega_k(x,0) = 0$. Finally its continuity follows from the continuity of $f(\cdot,u,\xi)$. Then $(\omega_k)_k$ is a sequence of locally integrable moduli of continuity. Let us recall that, if we define $C := \max\{\sup\{|c_{j,i}(x)| \,:\, x \in \Omega\} \,:\, i = 1,\dots,n,\ j = 1,\dots,m\}$, it holds that $0 < C < +\infty$. Let us define now, for any $k \in \mathbb{N}$, the function

$$\tilde{\omega}_k(x,\varepsilon) := \omega_{(\lfloor C \rfloor + 1)k}(x, C\varepsilon) \quad \forall x \in \Omega,\ \forall \varepsilon \geq 0.$$

Of course we have that $(\tilde{\omega}_k)_k$ is still a sequence of locally integrable moduli of continuity: we show that such a sequence satisfies (6.2.1). Take $A' \in \mathcal{A}_0$, $k \in \mathbb{N}$, $\varepsilon \geq 0$, $u, v \in C^1(\overline{\Omega})$ such that

$$|u(x)|, |v(x)|, |Du(x)|, |Dv(x)| \leqslant k,\ |u(x) - v(x)|, |Du(x) - Dv(x)| \leqslant \varepsilon \quad \forall x \in A'.$$

Then it follows that

$$|Xu(x)| = |\mathcal{C}(x)Du(x)| \leqslant C|Du(x)| \leqslant Ck \leqslant (\lfloor C \rfloor + 1)k,$$

$$|Xv(x)| = |\mathcal{C}(x)Dv(x)| \leqslant C|Dv(x)| \leqslant Ck \leqslant (\lfloor C \rfloor + 1)k$$

and

$$|Xu(x) - Xv(x)| = |\mathcal{C}(x)(Du(x) - Dv(x))| \leqslant C|Du(x) - Dv(x)| \leqslant C\varepsilon.$$

Thus we conclude that

$$|F(u,A') - F(v,A')| \leqslant \int_{A'} |f(x,u,Xu) - f(x,v,Xv)| dx \leqslant \int_{A'} \tilde{\omega}_k(x,\varepsilon) dx,$$

and so also (iii) is proved. (iv) follows easily from (7.5.14), while (vi) is a direct consequence of Theorem 7.5.1. Let us now define $H : W^{1,\infty}(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ as the restriction to $W^{1,\infty}(\Omega) \times \mathcal{A}$ of F. Then, since for every $u \in W^{1,\infty}(\Omega)$ it holds that $Xu(x) = \mathcal{C}(x)Du(x)$, if we define $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0,+\infty)$ as

$$f_e(x,u,\xi) := f(x,u,\mathcal{C}(x)\xi)$$

we can easily notice that f_e is a Caratheodory function, convex in the third argument and such that

$$H(u,A) = \int_A f_e(x,u,Du)dx.$$

Applying [2, Theorem 2.1], condition (v) holds for H and hence for F. □

7.6 Non-convex functionals

In this section we want to exploit [67, Theorem 1.8] to characterize a class of local functionals for which again we do not require neither translation-invariance nor convexity, and for which we want to weaken the assumption of weak-* sequential lower semicontinuity in Theorem 7.5.2. Convexity was a crucial assumption in Proposition 7.4.2 to guarantee the validity of (7.4.2), which can be easily seen to fail if we drop it. To justify this assertion let us give an instance of this problem.

Example 7.6.1. Let us take $\Omega = B_1(0) \subseteq \mathbb{R}^2$, $m = 1$ and

$$X_1 := x\frac{\partial}{\partial y}.$$

Then X_1 is a Lipschitz continuous vector field satisfying (a.e. LIC) on Ω, with $N_X := \{(x,y) \in \Omega : x = 0\}$. Clearly, for all $(x,y) \in \Omega_X$ we have

$$C((x,y))^T \cdot B^{-1}((x,y)) \cdot C((x,y)) = \begin{bmatrix} 0 \\ x \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{x^2} \end{bmatrix} \cdot \begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

thus by Proposition 7.3.1 it follows that

$$\Pi_{(x,y)}(\xi_1,\xi_2) = (0,\xi_2) \quad \forall (\xi_1,\xi_2) \in \mathbb{R}^2, \forall (x,y) \in \Omega_X. \tag{7.6.1}$$

Let us define the map $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow [0,+\infty)$ by

$$f_e((x,y),u,(\xi_1,\xi_2)) := \begin{cases} 1 - \xi_1^2 - \xi_2^2 & \text{if } \xi_1^2 + \xi_2^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, f_e is a bounded Carathéodory function, not convex in the third entry. Moreover, for any $(x, y) \in \Omega_X$ and $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $\xi_1^2 + \xi_2^2 \leqslant 1$, thanks to (7.6.1) it holds that

$$f_e((x, y), u, \Pi_{(x,y)}(\xi_1, \xi_2)) = 1 - \xi_2^2.$$

We conclude that (7.4.2) does not hold.

On the other hand it is easy to see that there are cases when Proposition 7.4.2 still holds even if the Lagrangian is not convex in the third argument, as the following example shows.

Example 7.6.2. Let us take n, m, X and Ω as in the previous example, and define the function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow [0, +\infty)$ by

$$f_e((x, y), u, (\xi_1, \xi_2)) := \begin{cases} 1 - \xi_2^2 & \text{if } |\xi_2| \leqslant 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then f_e is again a bounded Carathéodory function which is not convex in the third entry. Anyway we can easily see that f_e satisfies (7.4.2).

At this point we may ask ourselves if there is a way to weaken the convexity of f_e in the third entry which is still able to guarantee the validity of (7.4.2). In the previous example we see that, even if f_e is not globally convex in the third entry, it is anyway convex along the direction indicated by N_x. This leads us to the following

Definition 7.6.3 (X-convexity)**.** *We say that a Carathéodory function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0, +\infty]$ is X-*convex *if, for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$, $t \in (0, 1)$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ such that $\xi_2 - \xi_1 \in N_x$, it holds that*

$$f_e(x, u, t\xi_1 + (1 - t)\xi_2) \leqslant t f_e(x, u, \xi_1) + (1 - t) f_e(x, u, \xi_2).$$

The following proposition tells us that X-convexity is the proper requirement that we have to assume on the Euclidean Lagrangian.

Proposition 7.6.4. *Let $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0, +\infty]$ be a Carathéodory function such that there exist $a \in L^1_{loc}(\Omega)$ and $b, c > 0$ such that*

$$f_e(x, u, \xi) \leqslant a(x) + b|\mathcal{C}(x)\xi|^p + c|u|^p \text{ for a.e. } x \in \Omega, \ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{7.6.2}$$

Then the following facts are equivalent:

(i) f_e *is X-convex;*

(ii) for a.e. $x \in \Omega$ and for any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$, the function $g : N_x \longrightarrow [0, +\infty]$ defined by $g(\eta) := f_e(x, u, \xi + \eta)$ is constant;

(iii) $f_e(x, u, \xi) = f_e(x, u, \Pi_x(\xi))$ *for a.e. $x \in \Omega$, $\forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n$;*

Proof. $(ii) \Leftrightarrow (iii)$ Fix $x \in \Omega$ such that (ii) holds. For any $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$, we have that

$$f_e(x,u,\xi) = f_e(x,u,\xi_{N_x} + \Pi_x(\xi)) = f_e(x,u,\Pi_x(\xi)).$$

Conversely, take $x \in \Omega$ such that (iii) holds. For any $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and $\eta \in N_x$, it holds that

$$f_e(x,u,\xi+\eta) = f_e(x,u,\Pi_x(\xi+\eta)) = f_e(x,u,\Pi_x(\xi)) = f_e(x,u,\xi).$$

$(i) \Leftrightarrow (ii)$ The right implication is trivial. Conversely, assume (i) and fix $x \in \Omega$ such that (i) holds and $a(x) < +\infty$. Thanks to (7.6.2) we have that, for any fixed $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $\eta \in N_x$,

$$\begin{aligned} g(\eta) = f_e(x,u,\xi+\eta) &\leqslant a(x) + b|\mathcal{C}(x)\xi + \mathcal{C}(x)\eta|^p + c|u|^p \\ &= a(x) + b|\mathcal{C}(x)\xi|^p + c|u|^p < +\infty. \end{aligned}$$

Since the right side does not depend on η, then g is bounded on N_x. Since by assumption it is also convex on N_x, then g is constant. $\square$

In order to guarantee the X-convexity of the Euclidean Lagrangian we establish a zig-zag argument according to it celebrated Euclidean counterpart available in [67, Lemma 2.11].

Lemma 7.6.5. *Let $F : W^{1,p}_{loc}(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ be such that*

(i) $\forall u \in W^{1,p}_{loc}(\Omega)$*, the map* $A \mapsto F(u,A)$ *is a measure;*

(ii) $\forall u,v \in W^{1,p}_{loc}(\Omega), \forall A' \in \mathcal{A}_0,\ u|_{A'} = v|_{A'} \implies F(u,A') = F(v,A')$;

(iii) F satisfies the weak condition (ω);

(iv) For any $A' \in \mathcal{A}_0$ and $(u_h)_h \subseteq W^{1,p}(\Omega), u \in W^{1,p}(\Omega)$ such that $\lim_{h\to\infty} \|u_h - u\|_{W^{1,p}_X(\Omega)} = 0$, then $F(u,A') \leqslant \liminf_{h\to\infty} F(u_h,A')$;

Then, if for any $x \in \Omega$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we define

$$f_e(x,u,\xi) := \limsup_{R\to 0} \frac{F(\varphi_{x,u,\xi}, B_R(x))}{|B_R(x)|} \tag{7.6.3}$$

it holds that f_e is X-convex.

Proof. A slight modification of [67, Lemma 2.10] ensures the existence of a sequence $(\omega_k)_k$ of locally integrable moduli of continuity and a set $\Omega' \subseteq \Omega$ such that $|\Omega'| = |\Omega|$ and all the points in Ω' are Lebesgue points of $x \mapsto \omega_k(x,r)$ for any $k \in \mathbb{N}$ and for any $r \geq 0$. Moreover

$$|f_e(x,u,\xi) - f_e(x,v,\xi)| \leqslant \omega_k(x,|u-v|) \tag{7.6.4}$$

for any $x \in \Omega'$, $k \in \mathbb{N}$, $u,v \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ such that

$$|\xi|, |u|, |v| \leqslant k.$$

Take $x \in \Omega'$, $z \in \mathbb{R}$, $t \in (0,1)$, $\xi_1 \neq \xi_2$ in $\mathbb{R}^n$ such that $\xi_2 - \xi_1 \in N_x$, and set $\xi := t\xi_1 + (1-t)\xi_2$. We want to prove that

$$f_e(x,z,\xi) \leqslant t f_e(x,z,\xi_1) + (1-t) f_e(x,z,\xi_2). \tag{7.6.5}$$

Let us define

$$\xi_0 := \frac{\xi_2 - \xi_1}{|\xi_2 - \xi_1|},$$

and, for any $h \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i = 1,2$, set

$$\Omega^1_{h,k} := \left\{ y \in \Omega \,:\, \frac{k-1}{h} \leqslant \langle \xi_0, y \rangle < \frac{k-1+t}{h} \right\};$$

$$\Omega^2_{h,k} := \left\{ y \in \Omega \,:\, \frac{k-1+t}{h} \leqslant \langle \xi_0, y \rangle < \frac{k}{h} \right\};$$

$$\Omega^i_h := \bigcup_{k \in \mathbb{Z}} \Omega^i_{h,k};$$

$$u(y) := z + \langle \xi, y - x \rangle \quad \forall y \in \Omega;$$

$$v_h(y) := \begin{cases} (1-t)\frac{k-1}{h}|\xi_2 - \xi_1| + z + \langle \xi_1, y - x\rangle & \text{if } y \in \Omega^1_{h,k} \\ -t\frac{k}{h}|\xi_2 - \xi_1| + z + \langle \xi_2, y - x \rangle & \text{if } y \in \Omega^2_{h,k} \end{cases}.$$

Arguing as in the proof of [66, Lemma 2.11] we have that $v_h \to u$ uniformly on Ω. Hence, in particular, $v_h \to u$ strongly in $L^p(\Omega)$. Moreover, since $\xi_2 - \xi_1$ belongs to N_x and ξ is a convex combination of ξ_1 and ξ_2, then both $\xi - \xi_1$ and $\xi - \xi_2$ belong to N_x. Thus for $i = 1,2$ and for any $y \in \Omega^i_{h,k}$ we have that

$$|Xu(y) - Xv_h(y)| = |\mathcal{C}(x)\xi - \mathcal{C}(x)\xi_i| = |\mathcal{C}(x)(\xi - \xi_i)| = 0.$$

Therefore v_h converges to u strongly in $W^{1,p}_X(\Omega)$. Take now $k \in \mathbb{N}_+$ such that, for any $y \in \Omega$ and for any $h \in \mathbb{N}_+$,

$$|\xi_1|, |\xi_2|, |u_1(y)|, |u_2(y)|, |v_h(y)| \leqslant k.$$

Then, thanks to (7.6.4) and noticing that (see [67, Lemma 2.4])

$$F(u,A) = \int_A f_e(x,u,Du)dx \quad \forall u \text{ affine on } \Omega,\ \forall A \in \mathcal{A},$$

arguing as in [66, Lemma 2.11] and setting $B^i_{h,R}(x) := B_R(x) \cap \Omega^i_h$ for $i = 1,2$ and for any $R > 0$ such that $B_R(x) \Subset \Omega$, it holds that

$$F(v_h, B_R(x)) \leqslant \int_{B^1_{h,R}(x)} f_e(y,u_1,Du_1)dy + \int_{B^2_{h,R}(x)} f_e(y,u_2,Du_2)dy + \int_\Omega w_k\left(y, aR + \frac{b}{h}\right),$$

with $a := |\xi_2 - \xi_1|$ and $b := at(1-t)$. Since v_h converges to u strongly in $W^{1,p}_X(\Omega)$ and thanks to hypothesis (iv) it is easy to see that

$$F(u, B_R(x)) \leqslant tF(u_1, B_R(x)) + (1-t)F(u_2, B_R(x)) + \int_\Omega w_k(y,\varepsilon),$$

where this inequality holds for any $\varepsilon > 0$ and for any $R \in (0, \frac{\varepsilon}{a}]$. Dividing both sides by $|B_R(x)|$, passing to the limsup and recalling that x is a Lebesgue point of $y \mapsto w_k(y,\varepsilon)$, we have that

$$f_e(x,z,\xi) \leqslant tf_e(x,z,\xi_1) + (1-t)f_e(x,z,\xi_2) + w_k(x,\varepsilon).$$

Letting ε go to zero, the thesis is proved. □

We are now ready to state and prove the main result of this section.

Theorem 7.6.1. *Let $F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ be such that:*

(i) F is a measure;

(ii) F is local;

(iii) F satisfies the strong condition (ω);

(iv) F is p-bounded;

(v) F is lower semicontinuous.

Then there exists a unique Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty)$ such that

$$f(x,u,\xi) \leqslant a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u,\xi) \in \mathbb{R} \times \mathbb{R}^m \tag{7.6.6}$$

and the following representation formula holds:

$$F(u,A) = \int_A f(x,u,Xu)dx \quad \forall u \in W^{1,p}_{X,loc}(\Omega),\ \forall A \in \mathcal{A}. \tag{7.6.7}$$

Proof. Let us consider the restriction of F to $W^{1,p}_{loc}(\Omega) \times \mathcal{A}$. Arguing as in the first step of the proof of Theorem 7.4.1 it is easy to see that it satisfies all the hypotheses of [67, Theorem 1.8]. Thus, if f_e is defined as in (7.6.3), it is a Carathéodory function and moreover there exists $\tilde{b} > 0$ such that

$$F(u,A) = \int_A f_e(x,u,Du)dx \quad \forall A \in \mathcal{A},\ \forall u \in W^{1,p}_{loc}(\Omega)$$

and

$$f_e(x,u,\xi) \leqslant a(x) + \tilde{b}|\xi|^p + c|u|^p \qquad \text{for a.e. } x \in \Omega,\ \forall (u,\xi) \in \mathbb{R} \times \mathbb{R}^m.$$

Moreover, thanks to Lemma 7.6.5, f_e is X-convex. So, recalling Proposition 7.6.4 and (iv) of Proposition 7.4.1, we get that

$$\int_A f_e(x,u,Du)dx = \int_A f(x,u,Xu)dx \quad \forall A \in \mathcal{A},\ u \in C^\infty(A),$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty]$ is the function defined in (7.4.1). Such an f can be supposed to be finite up to modifying it on a set of measure zero. Arguing as in the third step of the proof of Theorem 7.4.1, (7.6.6) holds, while (7.6.7) follows exactly as in the last step of the proof or Theorem 7.5.2. Finally, uniqueness follows as usual. □

Proceeding exactly as in Theorem 7.5.3 we have the following

Theorem 7.6.2. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ be a Carathéodory function such that*

$$f(x, u, \xi) \leqslant a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m,$$

for $b, c > 0$ and $a \in L^1_{loc}(\Omega)$. Setting the functional $F : W^{1,p}_{X,loc}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ as

$$F(u, A) := \int_A f(x, u, Xu)dx \quad \forall u \in W^{1,p}_{X,loc}(\Omega),\ \forall A \in \mathcal{A},$$

then F satisfies hypotheses $(i) - (v)$ of Theorem 7.6.1.

Chapter 8

Γ-compactness

8.1 Introduction

We refer to [129] as main reference for this chapter. In this chapter we continue the generalization of [205], providing Γ-compactness results for different classes of integral functionals which are not assumed to be translation-invariant in the sense of (6.1.2), and which can be defined both on $L^p(\Omega)$ and on $W^{1,p}_X(\Omega)$. To be more precise, we show the three following results.

- $\Gamma(L^p)$-compactness, under standard boundedness and coercivity requirements, for a class of non-negative convex integral functionals defined on $L^p(\Omega) \times \mathcal{A}$.
- $\Gamma(W^{1,p}_X)$-compactness, under standard boundedness requirements, for a class of non-negative convex integral functionals defined on $W^{1,p}_X(\Omega) \times \mathcal{A}$.
- $\Gamma(W^{1,p}_X)$-compactness, under standard boundedness requirements, for a class of non-negative and possibly non-convex integral functionals defined on $W^{1,p}_X(\Omega) \times \mathcal{A}$ which satisfies the *strong condition* (ωX) uniformly on the class.

Here the strong condition (ωX) is a suitable continuity condition which is strongly inspired by its Euclidean counterpart introduced in Definition 6.2.5, and is properly defined in Section 8.3. Again, the lack of translation-invariance implies in general a dependence of the Lagrangian on the function variable. Moreover we point out that in the last two cases no coercivity assumption is requested, and differently from the L^p situation, we can allow also the case in which $p = 1$.

Our general strategy is classical and consists of two main steps:

Step 1 given a sequence $(F_h)_h$ in an appropriate class of integral functional $\mathcal{I}$, find a subsequence $(F_{h_k})_k$ and a local functional F such that

$$F(\cdot, A) = \Gamma - \lim_{k\to\infty} F_{h_k}(\cdot, A)$$

for any $A \in \mathcal{A}$, and moreover show that such an F satisfies some structural properties;

Step 2 choose a suitable subclass $\mathcal{J} \subseteq \mathcal{I}$ and show that, whenever $(F_h)_h$ belongs to $\mathcal{J}$, then F belongs to $\mathcal{J}$.

Working in the L^p framework the approach is quite standard. Indeed, for achieving Step 1 we exploits classical results of Γ-convergence in L^p, for which we refer to [105], and some properties of the X-gradient, for which we refer to Chapter 1. On the other hand, Step 2 is based on the new integral representation result for convex local functionals introduced in Chapter 7, and consists in verifying that the abstract Γ-limit F satisfies the hypotheses relative to $\mathcal{J}$.

When instead we consider functionals defined on $W^{1,p}_X(\Omega)$ and we perform the Γ-limit with respect to the strong topology of $W^{1,p}_X(\Omega)$, the situation is more delicate. In particular, in order to achieve Step 1, we need to understand how to modify some arguments of [105]. More precisely, we introduce a suitable notion of *uniform fundamental estimate* which is inspired by the classical notion of fundamental estimate but which turns out to be more useful for our purposes. Indeed it allows us to drop the coercivity assumptions, and to mimic the results that allowed the conclusion of Step 1 in the L^p case, adapting them to this new framework. Again, Step 2 relies on the possibility to exploit Theorem 7.4.1 and the integral representation result for non-convex integral functionals proved in Theorem 7.6.1 to represent the $\Gamma(W^{1,p}_X)-$limit in integral form. To this aim, we show that the strong condition (ωX) well behaves with respect to the passage to the Γ-limit, provided we perform this operation with respect to the strong topology of $W^{1,p}_X(\Omega)$. For the sake of completeness, we want to point out that some of the results achieved in the non-Euclidean framework were, to our knowledge, unsolved even in the classical Sobolev setting $W^{1,p}(\Omega)$. To conclude, in the final section we list some remarks and problems that are still open. In particular, we show some critical aspects and we prove some results with the hope that they may be useful to anyone who will try to handle this analysis.

8.2 Γ-compactness in L^p

In this section we prove a Γ-compactness result for a class of convex integral functionals defined on $L^p(\Omega)$ with respect to the strong topology of $L^p(\Omega)$. Our strategy is based on classical results of Γ-convergence in L^p spaces and on the possibility to exploit Theorem 7.4.1. First of all we introduce a large class of integral functionals for which some important properties, such as the uniform fundamental estimate introduced in Definition 6.4.2, are satisfied. Therefore we let $1 < p < \infty$, and we fix $a \in L^1(\Omega)$ and constants $0 < c_0 \leqslant c_1$ and $c_2 \geq 0$. We say that a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ belongs to $\mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ if there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, \infty]$ such that

$$c_0|\eta|^p \leqslant f(x, u, \eta) \leqslant a(x) + c_1|\eta|^p + c_2|u|^p \tag{8.2.1}$$

for any $(u, \eta) \in \mathbb{R} \times \mathbb{R}^m$, for a.e. $x \in \Omega$, and it holds that

$$F(u, A) = \begin{cases} \int_A f(x, u(x), Xu(x))\, \mathrm{d}x & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ +\infty & \text{otherwise} \end{cases}.$$

In particular, we say that $F \in \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$ whenever $F \in \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ and it holds that

$$f(x, \cdot, \cdot) \text{ is convex for a.e. } x \in \Omega. \tag{8.2.2}$$

As announced, the main result of this section is the Γ-compactness for the class of convex integral functionals.

Theorem 8.2.1. *For any sequence $(F_h)_h \subseteq \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$ there exists a subsequence $(F_{h_k})_k$ and a local functional $F \in \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$ such that*

$$F(\cdot, A) = \Gamma(L^p) - \lim_{k \to +\infty} F_{h_k}(\cdot, A) \qquad \text{for any } A \in \mathcal{A}.$$

In order to prove the latter we first describe some properties of $\Gamma(L^p)$-limits within the class $\mathcal{I}_{m,p}(a, c_0, c_1, c_2)$. To this aim, we recall the following result, which can be found in [205, Lemma 4.15].

Proposition 8.2.1. *Let us define the functional $\Psi_p : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ by*

$$\Psi_p(u, A) := \begin{cases} \|Xu\|^p_{L^p(A)} & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ +\infty & \text{otherwise} \end{cases}.$$

Then Ψ_p is a L^p-lower semicontinuous measure.

Proposition 8.2.2. *For any sequence $(F_h)_h \subseteq \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ such that*

- *F is a measure*
- *F is local*
- *F is L^p-lower semicontinuous*
- *For any $u \in W^{1,p}_X(\Omega)$ and $A \in \mathcal{A}$ it holds that*

$$\int_A c_0 |Xu(x)|^p \mathrm{d}x \leqslant F(u, A) \leqslant \int_A a(x) + c_1 |Xu(x)|^p + c_2 |u(x)|^p \mathrm{d}x \tag{8.2.3}$$

and moreover it holds that

$$F(\cdot, A) = \Gamma(L^p) - \lim_{k \to +\infty} F_{h_k}(\cdot, A) \tag{8.2.4}$$

for any $A \in \mathcal{A}$.

Proof. The proof is based on general results of [105]. Indeed, according to [105, Theorem 19.4], we introduce a suitable superclass of $\mathcal{I}_{m,p}(a, c_0, c_1, c_2)$. To this aim, we say that a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ belongs to $\mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$ if F is a measure and if there exist $d_1 \geq 1$, $d_2, d_3, d_4 \geq 0$, a finite measure μ, independent of F, and a measure $G : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$, which may depend on F, such that

$$G(u, A) \leqslant F(u, A) \leqslant d_1 G(u, A) + d_2 \|u\|_{L^p(A)} + \mu(A) \tag{8.2.5}$$

and

$$G(\varphi u + (1-\varphi)v, A) \leqslant d_4(G(u, A) + G(v, A)) + d_3 d_4 (\max |D\varphi|^p) \|u - v\|_{L^p(A)} + \mu(A), \tag{8.2.6}$$

for any $u, v \in L^p(\Omega)$, $A \in \mathcal{A}$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$. We are going to show that $\mathcal{I}_{m,p}(a, c_0, c_1, c_2) \subseteq \mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$. For this purpose, let us define $\mu : \mathcal{B} \longrightarrow [0, +\infty]$ as

$$\mu(B) := \int_B |a(x)| \, \mathrm{d}x.$$

Then μ is a finite measure on Ω. Moreover, thanks to Proposition 8.2.1, the non-negative local functional $G : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ defined by

$$G(u, A) := c_0 \Psi_p(u, A) \qquad \text{for any } u \in L^p(\Omega),\ A \in \mathcal{A}$$

is a measure. Let us show (8.2.5). Let us set $d_1 := \frac{c_1}{c_0}$ and $d_2 := c_2$. If $A \in \mathcal{A}$ and $u \notin W_X^{1,p}(A)$, the estimate is trivial, while if $u \in W_X^{1,p}(A)$, it follows from the definition of $I_{m,p}(a, c_0, c_1, c_2)$. So we are left to show (8.2.6). Fix then $A \in \mathcal{A}$. If either $u \notin W_X^{1,p}(A)$ or $v \notin W_X^{1,p}(A)$ the estimate is trivial. Hence assume that $u, v \in W_X^{1,p}(A)$ and take $\varphi \in C_c^\infty(\Omega)$ such that $0 \leqslant \varphi \leqslant 1$. Then, recalling Proposition 1.3.6, Proposition 1.3.4, the fact that $\eta \mapsto |\eta|^p$ is convex on $\mathbb{R}^m$, and setting

$$C := \max\{\|c_{j,i}\|_\infty \, : \, j = 1, \ldots, m,\ i = 1, \ldots, n\}$$

it follows that $0 < C < \infty$ and

$$\begin{aligned}
G(\varphi u + (1-\varphi)v, A) &= c_0 \int_A |X\varphi(u-v) + \varphi Xu + (1-\varphi)Xv|^p \, \mathrm{d}x \\
&= c_0 2^p \int_A \left| \frac{X\varphi(u-v)}{2} + \frac{\varphi Xu + (1-\varphi)Xv}{2} \right|^p \mathrm{d}x \\
&\leqslant c_0 2^{p-1} \int_A |X\varphi(u-v)|^p \, \mathrm{d}x + c_0 2^{p-1} \int_A |\varphi Xu + (1-\varphi)Xv|^p \, \mathrm{d}x \\
&\leqslant c_0 2^{p-1} \int_A |X\varphi(u-v)|^p \, \mathrm{d}x + 2^{p-1}(G(u, A) + G(v, A)) \\
&\leqslant c_0 2^{p-1} (C\sqrt{m})^p (\max |Du|^p) \|u - v\|_{L^p(A)} + 2^{p-1}(G(u, A) + G(v, A)).
\end{aligned}$$

Thus (8.2.6) follows. Hence $\mathcal{I}_{m,p}(a, c_0, c_1, c_2) \subseteq \mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$. Therefore, thanks to [105, Theorem 19.5], there exist a subsequence of $(F_h)_h$, still denoted by $(F_h)_h$, and a L^p-lower semicontinuous functional $F \in \mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$ such that $(F_h)_h$ $\overline{\Gamma}(L^p)$-converges to F. In particular F is a measure. By [105, Proposition 16.4] and [105, Proposition 16.15], F is also local. Furthermore, by Proposition 8.2.1 G is a L^p-lower semicontinuous measure and, since $(F_h)_h$ satisfies the uniform fundamental estimate on $L^p(\Omega)$ according to [105, Theorem 19.4], we can apply Theorem 6.4.1 to conclude that (8.2.4) holds. Finally, we show that F satisfies (8.2.3). Let us fix $A \in \mathcal{A}$ and $u \in W^{1,p}_X(\Omega)$, and a sequence $(u_h)_h$ such that

$$F(u, A) = \lim_{h\to+\infty} F_h(u_h, A). \tag{8.2.7}$$

Arguing as above we can assume that $(u_h)_h \subset W^{1,p}_X(A)$. Therefore, thanks to (8.2.7) and Proposition 8.2.1, it follows that

$$c_0 \int_A |Xu|^p \,\mathrm{d}x \leqslant \liminf_{h\to+\infty} \int_A |Xu_h|^p \,\mathrm{d}x \leqslant \liminf_{h\to+\infty} F_h(u_h, A) = F(u, A), \tag{8.2.8}$$

and so the first inequality follows. Finally we have that

$$\begin{aligned} F(u, A) &\leqslant \liminf_{h\to+\infty} F_h(u, A) \\ &\leqslant \liminf_{h\to+\infty} \int_A a(x) + c_1|Xu|^p + c_2|u|^p \,\mathrm{d}x \\ &= \int_A a(x) + c_1|Xu|^p + c_2|u|^p \,\mathrm{d}x. \end{aligned}$$

This proves the thesis. □

In order to represent the Γ-limit achieved in the previous proposition in an integral form, we wish to exploit Theorem 7.4.1. Following a previous remark, we present here a slight variant which is more suitable to our purposes.

Theorem 8.2.2. *Let $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ be such that:*

(i) F is a measure;

(ii) F is local;

(iii) F is convex on $W^{1,p}_X(\Omega)$;

(iv) F satisfies (8.2.3).

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ which satisfies (8.2.1) and (8.2.2), and such that

$$F(u, A) = \int_A f(x, u(x), Xu(x))\mathrm{d}x \tag{8.2.9}$$

for any $A \in \mathcal{A}$ and for any $u \in W^{1,p}_X(A)$.

Proof. We point out that in Chapter 7 we did not take into account the possible equivalence between the bound from below of the Lagrangian and the bound from below of the functional, as the latter is actually not necessary to represent an abstract convex local functional in integral form. On the other hand, it is clear from the previous proofs that such an equivalence is trivial, so that we take it for granted. From [128] we know that there exists a a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ which satisfies (8.2.1) and (8.2.2), and such that

$$F(u, A) = \int_A f(x, u(x), Xu(x))\mathrm{d}x$$

for any $A \in \mathcal{A}$ and for any $u \in W^{1,p}_X(\Omega)$. Fix now $A \in \mathcal{A}$, $A' \in \mathcal{A}_0$ with $A' \Subset A$ and $u \in L^p(\Omega) \cap W^{1,p}_X(A)$, and let $v := \varphi u$, where φ is a smooth cut-off function between A' and A. Then clearly $v \in W^{1,p}_X(\Omega)$ and $v|_{A'} = u$. As F is local, it follows that

$$F(u, A') = F(v, A') = \int_{A'} f(x, v(x), Xv(x))\mathrm{d}x = \int_{A'} f(x, u(x), Xu(x))\mathrm{d}x.$$

Since F is a measure, it is in particular inner regular, and so we conclude that (8.2.9) holds. □

We are now ready to prove Theorem 8.2.1.

Proof of Theorem 8.2.1. As $F \in \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$, thanks to Proposition 8.2.2 there exists a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ which is a measure, local, satisfies (8.2.3) and such that (8.2.4) holds. Let us show that F is convex on $W^{1,p}_X(\Omega)$. Fix then $A \in \mathcal{A}$ and take $t \in (0, 1)$ and $u, v \in W^{1,p}_X(\Omega)$. Let $(u_h)_h$ and $(v_h)_h$ be two sequences converging respectively to u and v in $L^p(\Omega)$ and such that

$$F(u, A) = \lim_{h\to+\infty} F_h(u_h, A), \qquad F(v, A) = \lim_{h\to+\infty} F_h(v_h, A). \tag{8.2.10}$$

Since $F(u, A)$ and $F(v, A)$ are finite we can assume that the sequences $(u_h)_h, (v_h)_h$ belong to $W^{1,p}_X(A)$. Therefore, since each $F_h(\cdot, A)$ is convex on $W^{1,p}_X(A)$, recalling (8.2.10) and the fact that $(tu_h + (1 - t)v_h)_h$ converges to $tu + (1 - t)v$ in $L^p(\Omega)$, it follows that

$$\begin{aligned} F(tu + (1 - t)v, A) &\leqslant \liminf_{h\to+\infty} F_h(tu_h + (1 - t)v_h, A) \\ &\leqslant \liminf_{h\to+\infty} (tF_h(u_h, A) + (1 - t)F_h(v_h, A)) \\ &= t \lim_{h\to+\infty} F_h(u_h, A) + (1 - t) \lim_{h\to+\infty} F_h(v_h, A) \\ &= tF(u, A) + (1 - t)F(v, A). \end{aligned}$$

Therefore we are in position to apply Theorem 8.2.2. Finally, we notice that if $A \in \mathcal{A}$ and $u \in L^p(\Omega) \setminus W^{1,p}_X(A)$, arguing as in (8.2.8) we conclude that $+\infty = c_0\Psi_p(u, A) \leqslant F(u, A)$, which implies that

$$\{u \in L^p(\Omega) : F(u, A) < +\infty\} = W^{1,p}_X(A),$$

and so the thesis follows. □

8.3 Γ-compactness in $W_X^{1,p}$

In this section we show two Γ-compactness results for suitable classes of integral functionals defined on $W_X^{1,p}(\Omega)$ and with respect to the strong topology of $W_X^{1,p}(\Omega)$. Working in this new framework has surely some advantages. For instance we do not have to assume any coercivity assumptions on the sequence of Lagrangians, and we can allow the case $p = 1$, since, among the other things, Proposition 8.2.1 is not needed anymore. Therefore, throughout this section, we let $1 \leqslant p < +\infty$ and, as in the previous one, we fix $a \in L^1(\Omega)$, $c_1, c_2 \geq 0$. We say that a functional $F : W_X^{1,p}(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ belongs to $\mathcal{U}_{m,p}(a, c_1, c_2)$ if there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, \infty]$ such that

$$f(x, u, \eta) \leqslant a(x) + c_1|\eta|^p + c_2|u|^p \tag{8.3.1}$$

for any $(u, \eta) \in \mathbb{R} \times \mathbb{R}^m$ and for a.e. $x \in \Omega$, and it holds that

$$F(u, A) = \int_A f(x, u(x), Xu(x))dx$$

for any $A \in \mathcal{A}$ and any $u \in W_X^{1,p}(\Omega)$. Similarly to the previous section, we will show that this large class of functionals satisfies many nice properties, among which a suitable notion of uniform fundamental estimate that will be introduced below. However, this class is too general to hope to achieve Γ-compactness. Therefore we define two sub-classes which will be shown to be Γ-compact. For the first case we consider the sub-class of the convex functionals belonging to $\mathcal{U}_{m,p}(a, c_1, c_2)$, i.e. we say that $F \in \mathcal{V}_{m,p}(a, c_1, c_2)$ whenever $F \in \mathcal{U}_{m,p}(a, c_1, c_2)$ and

$$f(x, \cdot, \cdot) \text{ is convex for a.e. } x \in \Omega.$$

In the second case we want to drop the convexity assumption. To this aim, we introduce a notion of strong condition which is strongly inspired by the Euclidean condition introduced in Definition 6.2.5.

Definition 8.3.1. *We say that* $\omega = (\omega_s)_{s \geq 0}$ *is a (generalized)* family of locally integrable moduli of continuity *if* $\omega_s : \Omega \times [0, +\infty) \longrightarrow [0, +\infty)$ *and*

$$r \mapsto \omega_s(x, r) \text{ is increasing, continuous and } \omega(x, 0) = 0 \tag{8.3.2}$$

for a.e. $x \in \Omega$ *and for any* $s \geq 0$,

$$s \mapsto \omega_s(x, r) \text{ is increasing and continuous} \tag{8.3.3}$$

for a.e. $x \in \Omega$ *and for any* $r \geq 0$, *and*

$$x \mapsto \omega_s(x, r) \in L^1_{loc}(\Omega) \quad \text{for any } r, s \geq 0.$$

Moreover we say that a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ satisfies the strong condition (ωX) with respect to ω *if there exists a family $\omega = (\omega_s)_{s\geq 0}$ of locally integrable moduli of continuity such that*

$$|F(v, A') - F(u, A')| \leqslant \int_{A'} \omega_s(x, r)\, \mathrm{d}x$$

for any $s \geq 0$, $A' \in \mathcal{A}_0$, $r \geq 0$, $u, v \in W^{1,p}_X(\Omega)$ such that

$$\begin{aligned} &|u(x)|, |v(x)|, |Xu(x)|, |Xv(x)| \leqslant s \\ &|u(x) - v(x)|, |Xu(x) - Xv(x)| \leqslant r \end{aligned}$$

for a.e. $x \in A'$.

This new notion seems to be more flexible and to fit better with our non-Euclidean setting, and allows to deal with more general classes of functions. On the other hand, it is quite easy to see that our condition is stronger than the one introduced in Definition 6.2.5, and so all the integral representation results proved in Chapter 7 remain valid. Moreover, we point out that our family of moduli of continuity, unlike in [67], is indexed over a continuous set, and the assumption on the behaviour of $s \mapsto \omega_s(x, r)$ is completely new. Nevertheless we will see in a while that, at least when dealing with integral functionals, this new requirement is quite natural. Indeed the following fact holds.

Proposition 8.3.2. *Let $F \in \mathcal{U}_{m,p}(a, c_1, c_2)$. Then F satisfies the strong condition (ωX).*

Proof. Since f is Carathéodory, then the set $\Omega' := \{x \in \Omega : (u, \xi) \mapsto f(x, u, \xi) \text{ is continuous}\}$ satisfies $|\Omega'| = |\Omega|$. For any $s, r \geq 0$, set $E^s_r \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ as

$$E^s_r := \{(u, v, \xi, \eta) : |u|, |v|, |\xi|, |\eta| \leqslant s, |u - v|, |\xi - \eta| \leqslant r\}$$

and the function

$$\omega_s(x, r) := \begin{cases} \sup\{|f(x, u, \xi) - f(x, v, \eta)| : (u, v, \xi, \eta) \in E^s_r\} & \text{if } x \in \Omega', \\ 0 & \text{otherwise.} \end{cases}$$

We show that $(\omega_s)_{s\geq 0}$ is a family of locally integrable moduli of continuity. Let us fix then $s, r \geq 0$: since $(u, \xi) \mapsto f(x, u, \xi)$ is continuous for almost every $x \in \Omega$, then the supremum in the definition of ω_s can be taken over a countable subset of E^k_ε. Since for any (u, v, ξ, η) the function $x \mapsto |f(x, u, \xi) - f(x, v, \eta)|$ is measurable, then $\omega_s(\cdot, r)$ is measurable. Moreover, thanks to (8.3.1), it follows that, for any $(u, v, \xi, \eta) \in E^k_\varepsilon$,

$$\begin{aligned} |f(x, u, \xi) - f(x, v, \eta)| &\leqslant 2|a(x)| + c_1|\xi|^p + c_1|\eta|^p + c_2|u|^p + c_2|v|^p \\ &\leqslant 2|a(x)| + 4s(c_1 + c_2). \end{aligned}$$

Since the right side does not depend on $(u, v, \xi, \eta) \in E^s_r$, we conclude that

$$\omega_s(x, r) \leqslant 2|a(x)| + 4s(c_1 + c_2).$$

Hence $\omega_k(\cdot,\varepsilon) \in L^1_{loc}(\Omega)$. Fix now $x \in \Omega'$ and $s \geq 0$. Since $E^s_r \subseteq E^s_t$ for any $r \leqslant t$, then $\omega_s(x,\cdot)$ is increasing, $\omega_k(x,0) = 0$ and the continuity follows from the continuity of $f(\cdot,u,\xi)$. Finally, taking $x \in \Omega'$ and $r \geq 0$ we have again that $E^s_r \subseteq E^t_r$ for any $r \leqslant t$, hence $s \mapsto \omega_s(x,r)$ is increasing. Once more, from the continuity of $f(\cdot,u,\xi)$ we conclude that $s \mapsto \omega_s(x,r)$ is continuous. Then $(\omega_s)_s$ is a family of locally integrable moduli of continuity. It is straightforward to check that F satisfies the strong condition (ωX) with respect to $(\omega_s)_{s\geq 0}$. □

On the other hand, if $(F_h)_h \subseteq \mathcal{U}_{m,p}(a,c_1,c_2)$, even if each F_h satisfies the strong condition (ωX), in general the family of moduli of continuity strongly depends on h. Therefore we introduce suitable subclasses of $\mathcal{U}_{m,p}(a,c_1,c_2)$ which present uniformity in the choice of the family of moduli of continuity. Hence, if a family $\omega = (\omega_s)_{s\geq 0}$ is fixed, we say that a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0,\infty]$ belongs to $\mathcal{W}_{m,p}(a,c_1,c_2,\omega)$ if $F \in \mathcal{U}_{m,p}(a,c_1,c_2)$ and it satisfies the strong condition (ωX) with respect to ω.

Remark 8.3.3. Let $(F_h)_h \subseteq \mathcal{U}_{m,p}(a,c_1,c_2)$ be such that there exists $K \in L^1_{loc}(\Omega)$ such that

$$|f_h(x,u,\xi) - f_h(x,v,\eta)| \leqslant |K(x)|(|u-v| + |\xi-\eta|) \tag{8.3.4}$$

for any $u,v \in \mathbb{R}$, $\xi,\eta \in \mathbb{R}^m$ and $h \in \mathbb{N}$. If for any $s,r \geq 0$ we define E^s_r as in Proposition 8.3.2 and

$$\tilde{\omega}_s(x,r) := |K(x)| \sup\{(|u-v| + |\xi-\eta|) \,:\, (u,v,\xi,\eta) \in E^s_r\},$$

then it is easy to see that $(F_h)_h$ belongs to $\mathcal{W}_{m,p}(a,c_0,c_1,c_2,\tilde{\omega})$.

We are ready now to state the two main results of this section.

Theorem 8.3.1. *For any sequence $(F_h)_h \subseteq \mathcal{V}_{m,p}(a,c_1,c_2)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F \in \mathcal{V}_{m,p}(a,c_1,c_2)$ such that*

$$F(\cdot,A) = \Gamma(W^{1,p}_X) - \lim_{k\to+\infty} F_{h_k}(\cdot,A) \qquad \textit{for any } A \in \mathcal{A}.$$

Theorem 8.3.2. *For any sequence $(F_h)_h \subseteq \mathcal{W}_{m,p}(a,c_1,c_2,\omega)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F \in \mathcal{W}_{m,p}(a,c_1,c_2,\omega)$ such that*

$$F(\cdot,A) = \Gamma(W^{1,p}_X) - \lim_{k\to+\infty} F_{h_k}(\cdot,A) \qquad \textit{for any } A \in \mathcal{A}.$$

As already said, one of the key step for the proof of these results is introducing a suitable notion of uniform fundamental estimate. Therefore, inspired by the classical notion stated in [105], we give the following definition.

Definition 8.3.4. *Let $\mathcal{F}$ be a class of non-negative local functionals defined on $W^{1,p}_X(\Omega) \times \mathcal{A}$. We say that $\mathcal{F}$ satisfies the* uniform fundamental estimate *on $W^{1,p}_X(\Omega)$ if, for any $\varepsilon > 0$ and for any $A', A'', B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ and a finite family $\{\varphi_1,\ldots,\varphi_k\}$ of smooth cut-off functions between A' and A'' such that for any u, $v \in W^{1,p}_X(\Omega)$*

and for any $F \in \mathcal{F}$, we can choose $\varphi \in \{\varphi_1, \dots, \varphi_k\}$ such that

$$F\Big(\varphi u + (1-\varphi)v, A' \cup B\Big) \leqslant \Big(F(u, A'') + F(v, B)\Big) + \\ + \varepsilon\Big(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\Big) + M\|u-v\|_{L^p(S)},$$

where $S = (A'' \setminus A') \cap B$.

Let us point out the differences between the two definitions. From one hand, this estimate is stronger, since it requires that the choice of the cut-off function must be done among a finite family of candidates which depends only on ε, A', A'' and B. This requirement, as we will see, is crucial to guarantee a uniform estimate for the X-gradients of the test functions. However, we replace some of the L^p norms on the right hand side of (6.4.5) with $W^{1,p}_X$-norms, thus weakening some of the requirements. This choice, as we will see, is crucial to avoid the coercivity assumptions on the Lagrangians. The following results and their proofs are respectively the counterparts of [105, Proposition 19.1] and [105, Proposition 18.3].

Proposition 8.3.5. *$\mathcal{U}_{m,p}(a, c_1, c_2)$ satisfies the uniform fundamental estimate on $W^{1,p}_X(\Omega)$.*

Proof. Let us set $d_1 := c_1$, $d_2 := c_2$ and $d_4 := 2^{p-1}$ and $\sigma(C) := \int_C |a(x)|dx$ for any $C \in \mathcal{B}$. Fix $\varepsilon > 0$, $B \in \mathcal{A}$ and $A', A'' \in \mathcal{A}$ with $A' \Subset A''$. Choose $A \in \mathcal{A}$ with $A' \Subset A \Subset A''$ and $k \in \mathbb{N}$ with

$$\max\left\{\frac{d_1 + d_2 d_4}{k}, \frac{\sigma(A \setminus \overline{A'})}{k}\right\} < \varepsilon.$$

Moreover, choose open sets $A_1, \dots, A_{k+1}$ such that $A' \Subset A_1 \Subset \dots \Subset A_{k+1} \Subset A$, and, for any $i = 1, \dots, k$ take a smooth cut-off function φ_i between A_i and A_{i+1}. Finally, set

$$M := \frac{d_1 d_4}{k} \max_{1 \leqslant i \leqslant k} \max_{x \in \Omega} |X\varphi_i(x)|^p.$$

Let $F \in \mathcal{U}_{m,p}(a, c_1, c_2)$ and $u, v \in W^{1,p}_X(\Omega)$. Then, for any $i = 1, \dots, k$, from the choice of φ_i it follows that

$$F(\varphi_i u + (1-\varphi_i)v, A' \cup B) \leqslant F(u, (A' \cup B) \cap \overline{A_i}) + F(v, B \setminus A_{i+1}) + F(\varphi_i u + (1-\varphi_i)v, S_i), \quad (8.3.5)$$

where $S_i := B \cap (A_{i+1} \setminus \overline{A_i})$. Setting $I_i := F(\varphi_i u + (1-\varphi_i)v, S_i)$, from the bound on the

Lagrangian and arguing as in the proof of Proposition 8.2.2, we get that

$$\begin{aligned}
I_i &\leqslant d_1 \int_{S_i} |X(\varphi_i u + (1-\varphi_i)v)|^p \mathrm{d}x + d_2 \int_{S_i} |\varphi_i u + (1-\varphi_i)v|^p \mathrm{d}x + \sigma(S_i) \\
&= d_1 \int_{S_i} |uX\varphi_i + \varphi_i Xu - vX\varphi_i + (1-\varphi_i)Xv)|^p \mathrm{d}x + d_2 \int_{S_i} |u|^p \mathrm{d}x + d_2 \int_{S_i} |v|^p \mathrm{d}x + \sigma(S_i) \\
&= d_1 \int_{S_i} |(\varphi_i Xu + (1-\varphi_i)Xv) + X\varphi_i(u-v)|^p \mathrm{d}x + d_2 \int_{S_i} (|u|^p + |v|^p)\mathrm{d}x + \sigma(S_i) \\
&\leqslant d_1 d_4 \left[\int_{S_i} |\varphi_i Xu + (1-\varphi_i)Xv|^p + \int_{S_i} |X\varphi_i|^p |u-v|^p \mathrm{d}x\right] + d_2 \int_{S_i} (|u|^p + |v|^p)\mathrm{d}x + \sigma(S_i) \\
&\leqslant d_1 d_4 \left[\int_{S_i} |Xu|^p \mathrm{d}x + \int_{S_i} |Xv|^p \mathrm{d}x\right] + kM \int_{S_i} |u-v|^p \mathrm{d}x + d_2 \int_{S_i} (|u|^p + |v|^p)\mathrm{d}x + \sigma(S_i) \\
&\leqslant (d_2 + d_1 d_4) \left(\|u\|^p_{W^{1,p}_X(S_i)} + \|v\|^p_{W^{1,p}_X(S_i)}\right) + kM\|u-v\|^p_{L^p(S_i)} + \sigma(S_i).
\end{aligned}$$

Noticing that σ is a measure and that

$$S_1 \cup \ldots \cup S_k \subseteq (A \setminus \overline{A'}) \cap B \subseteq S,$$

and recalling the choice of k, it follows that

$$\begin{aligned}
\min_{1\leqslant i \leqslant k} I_i &\leqslant \frac{1}{k}\sum_{i=1}^k I_k \leqslant \frac{d_2 + d_1 d_4}{k}\left(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)}\right) + M\|u-v\|^p_{L^p(S)} + \frac{\sigma(A\setminus \overline{A'})}{k} \\
&\leqslant \varepsilon\left(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\right) + M\|u-v\|^p_{L^p(S)}.
\end{aligned} \tag{8.3.6}$$

Therefore, if $\varphi_i \in \{\varphi_1, \ldots, \varphi_k\}$ is chosen to realize the minimum, observing that F is a measure, $(A' \cup B) \cap \overline{A_i} \subseteq A''$ and $B \setminus A_{i+1} \subseteq B$, thanks to (8.3.5) and (8.3.6) the thesis follows. □

Proposition 8.3.6. *Let $(F_h)_h \in \mathcal{U}_{m,p}(a, c_1, c_2)$. Then it holds that*

$$F''(u, A' \cup B) \leqslant F''(u, A'') + F''(u, B) \tag{8.3.7}$$

for any $u \in W^{1,p}_X(\Omega)$, $B \in \mathcal{A}$ and $A', A'' \in \mathcal{A}$ with $A' \Subset A''$.

Proof. Let u, A', A'', B as above fix $\varepsilon > 0$, and let $(u_h)_h, (v_h)_h \subseteq W^{1,p}_X(\Omega)$ be two recovery sequences for u with respect to $F''(\cdot, A'')$ and $F''(\cdot, B)$ respectively. From Proposition 8.3.5 we know that $(F_h)_h$ satisfies the uniform fundamental estimate on $W^{1,p}_X(\Omega)$. Therefore there exists $M > 0$ and a finite family $\{\varphi^1, \ldots, \varphi^k\}$ of smooth cut-off functions between A' and A'', depending only on ε, A', A'' and B, and a sequence $(\varphi_h)_h \subseteq \{\varphi^1, \ldots, \varphi^k\}$, such that

$$\begin{aligned}
F_h\Big(\varphi_h u_h + (1-\varphi_h)v_h, A' \cup B\Big) &\leqslant \Big(F_h(u_h, A'') + F_h(v_h, B)\Big) + \\
&\quad + \varepsilon\Big(\|u_h\|^p_{W^{1,p}_X(S)} + \|v_h\|^p_{W^{1,p}_X(S)} + 1\Big) + M\|u_h - v_h\|_{L^p(S)},
\end{aligned} \tag{8.3.8}$$

where $S = (A'' \setminus A') \cap B$. Let us define $w_h := \varphi_h u_h + (1-\varphi_h)v_h$. Then it follows that

$$\|w_h - u\|_{L^p(\Omega)} = \|\varphi_h(u_h - v_h)\|_{L^p(\Omega)} + \|v_h - u\|_{L^p(\Omega)} \leqslant \|u_h - v_h\|_{L^p(\Omega)} + \|v_h - u\|_{L^p(\Omega)},$$

and moreover

$$\begin{aligned}\|Xw_h - Xu\|_{L^p(\Omega)} &= \|X\varphi_h \cdot u_h + \varphi_h X u_h - X\varphi_h \cdot v_h + (1-\varphi_h)Xv_h - Xu\|_{L^p(\Omega)} \\ &\leqslant \|X\varphi_h(u_h - v_h)\|_{L^p(\Omega)} + \|\varphi_h(Xu_h - Xv_h)\|_{L^p(\Omega)} + \|Xv_h - Xu\|_{L^p(\Omega)} \\ &\leqslant \max_{1\leqslant i\leqslant k} \||X\varphi^k|^p\|_{L^\infty(\Omega)} \cdot \|u_h - v_h\|_{L^p(\Omega)} + \|Xu_h - Xv_h\|_{L^p(\Omega)} + \|Xv_h - Xu\|_{L^p(\Omega)}.\end{aligned}$$

Therefore we conclude that w_h converges to $u \in W^{1,p}_X(\Omega)$. This fact, the choices of u_h and v_h and (8.3.8) allow to conclude that

$$\begin{aligned}F''(u, A' \cup B) &\leqslant \limsup_{h\to\infty} F''(w_h, A' \cup B) \\ &\leqslant \limsup_{h\to\infty} F''(u_h, A'') + \limsup_{h\to\infty} F''(v_h, B) \\ &\quad + \varepsilon\Big(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\Big) \\ &= F''(u, A'') + F''(u, B) + \varepsilon\Big(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\Big).\end{aligned}$$

Being ε arbitrary, the thesis follows. □

We are ready to complete **Step 1** of our general scheme.

Proposition 8.3.7. *For any sequence $(F_h)_h \subseteq \mathcal{U}_{m,p}(a, c_1, c_2)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ such that*

- *F is a measure;*
- *F is local;*
- *F is $W^{1,p}_X$–lower semicontinuous;*
- *for any $u \in W^{1,p}_X(\Omega)$ and $A \in \mathcal{A}$ it holds that*

$$F(u, A) \leqslant \int_A a(x) + c_1|Xu(x)|^p + c_2|u(x)|^p \mathrm{d}x \tag{8.3.9}$$

and moreover we have that

$$F(\cdot, A) = \Gamma(W^{1,p}_X) - \lim_{k\to+\infty} F_{h_k}(\cdot, A) \tag{8.3.10}$$

for any $A \in \mathcal{A}$.

Proof. Since $(W^{1,p}_X(\Omega), \|\cdot\|_{W^{1,p}_X(\Omega)})$ is a metric space, by [105, Theorem 16.9] we know that, up to a subsequence, $(F_h)_h$ $\bar{\Gamma}(W^{1,p}_X)$-converges to a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow \overline{\mathbb{R}}$. Being F a $\bar{\Gamma}$-limit, we know from [105, Remark 16.3] that F is increasing, inner regular and $W^{1,p}_X$– lower semicontinuous. Moreover, thanks to [105, Proposition 16.12], we know that F is superadditive. Let us show that F is non-negative. Indeed, fix $A \in \mathcal{A}$ and $u \in W^{1,p}_X(\Omega)$, then we know that

$$F(u,A) = \sup\Big\{\inf\{\liminf_{h\to\infty} F_h(u_h,A') \,:\, u_h \to u \text{ in } W^{1,p}_X(\Omega)\} \,:\, A' \in \mathcal{A},\, A' \Subset A\Big\}.$$

As each $F_h(u_h,A')$ is non-negative, then $F(u,A) \geq 0$. Moreover, in the same way we can see that $F(u,\emptyset) = 0$ for any $u \in W^{1,p}_X(\Omega)$. Now, adapting the proof of [105, Proposition 16.15], we show that F is local. Let us fix $A \in \mathcal{A}$ and $u, v \in W^{1,p}_X(\Omega)$ coinciding a.e. on A. Fix $A' \Subset A$, take a smooth cut-off function φ between A' and A and let $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ be a recovery sequence for u with respect to $F'(\cdot, A')$. We define a new sequence $(v_h)_h$ requiring that

$$v_h := \varphi u_h + (1-\varphi)v.$$

It is clear that

$$\|v_h - v\|_{L^p(\Omega)} = \|\varphi(u_h - v)\|_{L^p(\Omega)} = \|\varphi(u_h - v)\|_{L^p(A)} \leqslant \|u_h - u\|_{L^p(A)},$$

and moreover

$$\begin{aligned}\|Xv_h - Xv\|_{L^p(\Omega)} &= \|X\varphi(u_h - v) + \varphi(Xu_h - Xv)\|_{L^p(\Omega)}\\ &\leqslant \|X\varphi(u_h - v)\|_{L^p(A)} + \|\varphi(Xu_h - Xv)\|_{L^p(A)}\\ &\leqslant \||X\varphi|^p\|_{L^\infty(\Omega)}\|u_h - u\|_{L^p(A)} + \|Xu_h - Xu\|_{L^p(A)}.\end{aligned}$$

Therefore we have that v_h converges to v in $W^{1,p}_X(\Omega)$. As each F_h is local and $u_h = v_h$ on A', we conclude that

$$F'(v,A') \leqslant \liminf_{h\to\infty} F_h(v_h,A') = \liminf_{h\to\infty} F_h(u_h,A') = F'(u,A').$$

As the converse inequality can be proved exchanging the roles of u and v, we conclude that $F'(u,A') = F'(v,A')$. Finally, being $A' \Subset A$ arbitrary and recalling the definition of a $\bar{\Gamma}-$limit, we conclude that F is local. Moreover, thanks to Proposition 8.3.6, we can repeat essentially the same steps of the proof of [105, Proposition 18.4] and achieve that F is subadditive. Notice that, thanks to [105, Theorem 14.23] and the previous steps, this suffices to conclude that F is a measure. If we define now $G : W^{1,p}_X(\Omega) \times \mathcal{A} \to [0,+\infty]$ by

$$G(u,A) := \int_A a(x) + c_2|u|^p + c_1|Xu|^p$$

for any $u \in W^{1,p}_X(\Omega)$ and for any $A \in \mathcal{A}$, it is clear that G is a measure and that, thanks to our hypotheses, $F_h \leqslant G$ for any $h \in \mathbb{N}$. Therefore, if $u \in W^{1,p}_X(\Omega)$ and $A \in \mathcal{A}$, it follows that

$$F(u,A) \leqslant \liminf_h F_h(u,A) \leqslant G(u,A).$$

Finally, thanks again to Proposition 8.3.6 and repeating the proof of [105, Theorem 18.7], we conclude that

$$F(\cdot, A) = \Gamma(W^{1,p}_X) - \lim_{h\to+\infty} F_h(\cdot, A), \tag{8.3.11}$$

for any $A \in \mathcal{A}$. □

We have developed all the tools that we need to prove Theorem 8.3.1.

Proof of Theorem 8.3.1. Since $(F_h)_h \subseteq \mathcal{V}_{m,p}(a, c_1, c_2)$, from Proposition 8.3.7 we know that there exists a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ which is a measure, local, satisfies (8.3.9) and such that (8.3.10) holds. Moreover, arguing as in the proof of Theorem 8.2.1, F is convex. Therefore F satisfies all the hypotheses of Theorem 7.4.1, and so we conclude that $F \in \mathcal{V}_{m,p}(a, c_1, c_2)$. □

In order to prove Theorem 8.3.2, we wish to apply Theorem 7.6.1 to a suitable functional F. Anyway, among the other things, we need to guarantee that F satisfies the strong condition (ωX). With the two following propositions we are going to show that, whenever we work in $\mathcal{W}_{m,p}(a, c_1, c_2, \omega)$, the strong condition (ωX) with respect to ω is preserved by the operation of $\Gamma(W^{1,p}_X)$-limit.

Proposition 8.3.8. *If a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ is a measure, it is $W^{1,p}_X$-continuous, it satisfies (8.3.9) for any $u \in W^{1,p}_X(\Omega)$ and for any $B \in \mathcal{B}$ and it satisfies the strong condition (ωX) with respect to ω, then it holds that*

$$|F(v, B') - F(u, B')| \leqslant \int_{B'} \omega_s(x, r) \mathrm{d}x \tag{8.3.12}$$

for any $s \geq 0$, $B' \in \mathcal{B}_0$, $r \geq 0$, $u, v \in W^{1,p}_X(\Omega)$ such that

$$\begin{aligned} &|u(x)|, |v(x)|, |Xu(x)|, |Xv(x)| \leqslant s \\ &|u(x) - v(x)|, |Xu(x) - Xv(x)| \leqslant r \end{aligned} \tag{8.3.13}$$

for a.e. $x \in B'$.

Proof. It is not restrictive to assume that $c_1 = c_2 = 1$. First we show the thesis for regular functions $u, v \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$. Let us fix $B' \in \mathcal{B}_0$, and s, r, such that (8.3.13) holds, and let us take $m, M > 0$. Since $F(u, \cdot)$ and $F(v, \cdot)$ are Borel measures, there exists a decreasing sequence of open sets $(A_n)_n \subseteq \mathcal{A}$ such that $B' = \bigcap_{n=1}^\infty A_n$ and moreover

$$F(u, B') = \lim_{n\to\infty} F(u, A_n) \quad \text{and} \quad F(v, B') = \lim_{n\to\infty} F(v, A_n).$$

Furthermore, as $B' \Subset \Omega$, we can assume that $A_n \Subset \Omega$ for each $n \in \mathbb{N}$. Finally, as $u, v \in C^1(\overline{A_0})$ we can assume that

$$\begin{aligned} &|u(x)|, |v(x)|, |Xu(x)|, |Xv(x)| \leqslant s + \frac{1}{M} \\ &|u(x) - v(x)|, |Xu(x) - Xv(x)| \leqslant r + \frac{1}{m} \end{aligned}$$

for any $x \in A_n$ and for any $n \geq 0$. We obtain that

$$\begin{aligned}|F(u,B') - F(v,B')| &= \lim_{n\to\infty} |F(u,A_n) - F(v,A_n)| \\ &\leqslant \lim_{n\to\infty} \int_{A_n} w_{s+\frac{1}{M}}\left(x, r+\frac{1}{m}\right) \mathrm{d}x \\ &= \int_{B'} w_{s+\frac{1}{M}}\left(x, r+\frac{1}{m}\right) \mathrm{d}x.\end{aligned}$$

Therefore, thanks to (8.3.2), (8.3.3) and the Monotone Convergence Theorem we conclude that

$$\begin{aligned}|F(u,B') - F(v,B')| &\leqslant \lim_{m\to\infty} \lim_{M\to\infty} \int_{B'} w_{s+\frac{1}{M}}\left(x, r+\frac{1}{m}\right) \mathrm{d}x \\ &= \lim_{m\to\infty} \int_{B'} w_s\left(x, r+\frac{1}{m}\right) \mathrm{d}x \\ &= \int_{B'} w_s(x,r) \mathrm{d}x.\end{aligned}$$

Let now $B' \in \mathcal{B}_0$, $u, v \in W^{1,p}_X(\Omega)$ and s, r, such that (8.3.13) holds, and fix again $m, M > 0$. By Theorem 1.3.3 there are two sequences $(u_h)_h, (v_h)_h \subseteq W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ converging respectively to u and v in the strong topology of $W^{1,p}_X(\Omega)$. Therefore, thanks to the previous step and the continuity of the functional, we get that

$$|F(u,B') - F(v,B')| = \lim_{h\to\infty} |F(u_h,B') - F(v_h,B')|.$$

Now we want to estimate the right term. For doing this let us define, for any $h \geq 0$,

$$\begin{aligned} A_h &:= \left\{x \in B' \,:\, |u_h(x)| > s + \frac{1}{M}\right\} \quad B_h := \left\{x \in B' \,:\, |v_h(x)| > s + \frac{1}{M}\right\} \\ C_h &:= \left\{x \in B' \,:\, |Xu_h(x)| > s + \frac{1}{M}\right\} \quad D_h := \left\{x \in B' \,:\, |Xv_h(x)| > s + \frac{1}{M}\right\} \\ E_h &:= \left\{x \in B' \,:\, |u_h(x) - v_h(x)| > r + \frac{1}{m}\right\} \\ F_h &:= \left\{x \in B' \,:\, |Xu_h(x) - Xv_h(x)| > r + \frac{1}{m}\right\}, \end{aligned}$$

and let

$$Z_h := A_h \cup B_h \cup C_h \cup D_h \cup E_h \cup F_h. \tag{8.3.14}$$

We claim that

$$\lim_{h\to\infty} |Z_h| = 0.$$

Here we only show that $\lim_{h\to\infty} |A_h| = 0$, being the other parts of the proof similar. Assume that $x \in A_h$ and assume that (8.3.13) holds in x. Then it follows that

$$|u_h(x) - u(x)| \geq |u_h(x)| - |u(x)| > \frac{1}{M}.$$

and hence

$$x \in \left\{z \in \Omega \,:\, |u(z) - u_h(z)| > \frac{1}{M}\right\}.$$

Since u_h converges to u in $W^{1,p}_X(\Omega)$, then in particular u_h converges to u in measure, and so the measure of the right set goes to zero as h goes to infinity. We can now estimate in this way.

$$
\begin{aligned}
\lim_{h\to\infty}|F(u_h,B')-F(v_h,B')| &\leqslant \limsup_{h\to\infty}|F(u_h,B'\setminus Z_h)-F(v_h,B'\setminus Z_h)|+|F(u_h,Z_h)-F(v_h,Z_h)|\\
&\leqslant \int_{B'} w_{s+\frac{1}{M}}\left(x,r+\frac{1}{m}\right)+\limsup_{h\to\infty}|F(u_h,Z_h)|+|F(v_h,Z_h)|\\
&\leqslant \int_{B'} w_{s+\frac{1}{M}}\left(x,r+\frac{1}{m}\right)\mathrm{d}x+\limsup_{h\to\infty}2\int_{Z_h}|a(x)|\mathrm{d}x\\
&+\limsup_{h\to\infty}\int_{Z_h}|u_h|^p\mathrm{d}x+\int_{Z_h}|v_h|^p\mathrm{d}x+\int_{Z_h}|Xu_h|^p\mathrm{d}x+\int_{Z_h}|Xv_h|^p\mathrm{d}x\\
&\leqslant \int_{B'} w_{s+\frac{1}{M}}\left(x,r+\frac{1}{m}\right)\mathrm{d}x+\limsup_{h\to\infty}2\int_{Z_h}|a(x)|\mathrm{d}x\\
&+\limsup_{h\to\infty}2^{p-1}\left(\int_{Z_h}|u_h-u|^p\mathrm{d}x+\int_{Z_h}|u|^p\mathrm{d}x+\int_{Z_h}|Xu_h-Xu|^p\mathrm{d}x+\int_{Z_h}|Xu|^p\mathrm{d}x\right)\\
&+\limsup_{h\to\infty}2^{p-1}\left(\int_{Z_h}|v_h-v|^p\mathrm{d}x+\int_{Z_h}|v|^p\mathrm{d}x+\int_{Z_h}|Xv_h-Xv|^p\mathrm{d}x+\int_{Z_h}|Xv|^p\mathrm{d}x\right)\\
&\leqslant \int_{B'} w_{s+\frac{1}{M}}\left(x,r+\frac{1}{m}\right)\mathrm{d}x+K\lim_{h\to\infty}\left(\|u-u_h\|_{W^{1,p}_X(\Omega)}+\|v-v_h\|_{W^{1,p}_X(\Omega)}\right)\\
&+\limsup_{h\to\infty}\int_{B'}\chi_{Z_h}b(x)\mathrm{d}x,
\end{aligned}
$$

for a constant $K>0$ and a suitable function $b\in L^1(B')$. Therefore, thanks to the dominated convergence theorem, we conclude that

$$
|F(u,B')-F(v,B')|\leqslant \int_{B'} w_{s+\frac{1}{M}}\left(x,r+\frac{1}{m}\right)\mathrm{d}x.
$$

Arguing as in the first step and letting M,m go to infinity, the thesis follows. □

Proposition 8.3.9. *Let $(F_h)_h$ be a sequence in $\mathcal{W}_{m,p}(a,c_1,c_2,\omega)$ and assume that there exists a functional $F: W^{1,p}_X(\Omega)\times\mathcal{A}\longrightarrow[0,\infty]$ such that*

$$
F(\cdot,A')=\Gamma(W^{1,p}_X)-\lim_{h\to\infty}F_h(\cdot,A')\qquad \textit{for any } A'\in\mathcal{A}_0.
$$

Then F satisfies the strong condition (ωX) with respect to ω.

Proof. Let $A'\in\mathcal{A}_0$, $u,v\in W^{1,p}_X(\Omega)$ and $s,r\geq 0$ such that (8.3.13) holds, and fix $m,M>0$. Let $(u_h)_h$ and $(v_h)_h$ be recovery sequences respectively for u and v. Then it follows that

$$
|F(u,A')-F(v,A')|=\lim_{h\to\infty}|F_h(u_h,A')-F_h(v_h,A')|.
$$

Notice that, since $F_h\in\mathcal{W}_{m,p}(a,c_1,c_2,\omega)$ then it is a measure, it satisfies the strong condition (ωX) with respect to $(\omega_s)_{s\geq 0}$, and thanks to a slight variant of Theorem 7.5.1, it is $W^{1,p}_X$-continuous. Moreover, thanks to (8.3.1), it satisfies (8.3.9) for any $u\in W^{1,p}_X(\Omega)$ and for any $B\in\mathcal{B}$. Therefore it satisfies the hypotheses of Proposition 8.3.8. Hence, repeating exactly the same estimates performed in the proof of Proposition 8.3.8, we conclude that

$$\lim_{h\to\infty} |F_h(u_h, A') - F_h(v_h, A')| \leqslant \int_{A'} \omega_s(x, r)\mathrm{d}x,$$

and so the thesis follows. □

We are now in position to give the proof of Theorem 8.3.2.

Proof of Theorem 8.3.2. Since $(F_h)_h \subseteq \mathcal{W}_{m,p}(a, c_1, c_2, \omega)$, from Proposition 8.3.7 we know that there exists a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ which is a measure, local, $W^{1,p}_X$-lower semicontinuous and satisfies (8.3.9), and such that (8.3.10) holds. Moreover, thanks to Proposition 8.3.9, F satisfies the strong condition (ωX) with respect to ω. Therefore F satisfies all the hypotheses of Theorem 7.6.1, and so we conclude that $F \in \mathcal{W}_{m,p}(a, c_1, c_2, \omega)$. □

8.4 Further remarks and open problems

The classical strong and weak condition (ω) were introduced in [67] in order to guarantee the continuity of the candidate Lagrangian when proving an integral representation result. In particular, the strong condition (ω) guarantees that $f(x, \cdot, \cdot)$ is continuous, while the weak condition (ω) implies the continuity of $f(x, \cdot, \xi)$. Moreover, it is easy to see that the strong condition (ω) implies the weak condition (ω). Anyway, in many situations it is difficult to verify the strong condition (ω), whereas the weak condition (ω) is easier. On the other hand, if we require only the weak condition (ω), we have to add an extra hypothesis in order to get the equivalence, i.e. the weak-*sequential lower semicontinuity of the functional, which is well known (cf. [2]) to be equivalent to the convexity of $f(x, u, \cdot)$. In Chapter 7, inspired by [67], we exploited these ideas in order to achieve two integral representation results when the local functional is not assumed to be convex. In Section 8.2 we obtained a $\Gamma(L^p)-$compactness result for a class of convex integral functionals defined on $L^p(\Omega)$, but we did not generalized it when the convexity assumption is dropped. On the other hand, in Section 8.3 we considered also the non-convex case, working in a suitable class of integral functionals where the strong condition (ωX) is required uniformly on the class. Therefore there are some questions still unsolved. Let us begin by properly extending Definition 8.3.1.

Definition 8.4.1. *If $\omega = (\omega_s)_{s\geq 0}$ is a family of locally integrable moduli of continuity (cf. Definition 8.3.1), we say that a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ satisfies the* weak condition (ωX) with respect to ω *if*

$$|F(u + r, A') - F(u, A')| \leqslant \int_{A'} \omega_s(x, |r|)\,\mathrm{d}x$$

for any $s \geq 0$, $A' \in \mathcal{A}_0$, $r \in \mathbb{R}$, $u \in W^{1,p}_X(\Omega)$ such that

$$|u(x)|, |v(x) + r|, |r| \leqslant s$$

for a.e. $x \in A'$.

Indeed, if ω is a fixed family of moduli of continuity it is reasonable to ask:

- if the subclass of $\mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ of those integral functionals satisfying the strong condition (ωX) with respect to ω is $\Gamma(L^p)-$compact;
- if the subclass of $\mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ of those integral functionals satisfying the weak condition (ωX) with respect to ω and which are weakly-*seq. l.s.c. is $\Gamma(L^p)-$compact;
- if the subclass of $\mathcal{U}_{m,p}(a, c_1, c_2)$ of those integral functionals satisfying the weak condition (ωX) with respect to ω and which are weakly-*seq. l.s.c. is $\Gamma(W^{1,p}_X)-$compact.

In view of Proposition 8.2.2, Proposition 8.3.7 and the integral representation results in [128], the only questions left open are the following.

(a) Is the $\Gamma(L^p)$-limit of a sequence of (possibly not weakly-*seq. l.s.c.) functionals a weakly-*seq. l.s.c functional?

(b) Is the $\Gamma(W^{1,p}_X)-$limit of a sequence of (possibly not weakly-*seq. l.s.c.) functional a weakly-*seq. l.s.c functional?

(c) Does the $\Gamma(L^p)$-limit of a sequence of functionals satisfy the weak condition (ωX) provided that the sequence does satisfy it?

(d) Does the $\Gamma(W^{1,p}_X)-$limit of a sequence of functionals satisfy the weak condition (ωX) provided that the sequence does satisfy it?

(e) Does the $\Gamma(L^p)$-limit of a sequence of functionals satisfy the strong condition (ωX) provided that the sequence does satisfy it?

Unfortunately we have not been able to answer to questions $(b), (c)$ and (e). Anyway we are going to show that the questions (a) and (d) have a positive answer.

Proposition 8.4.2 (Answer to question (d))**.** *Let ω be a family of locally integrable moduli of continuity. Let $(F_h)_h$ be a sequence in $\mathcal{U}_{m,p}(a, c_1, c_2)$ and assume that each F_h satisfies the weak condition (ωX) with respect to ω. Assume in addition that there exists a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ such that*

$$F(\cdot, A') = \Gamma(W^{1,p}_X) - \lim_{h \to +\infty} F_h(\cdot, A') \qquad \text{for any } A' \in \mathcal{A}_0.$$

Then F satisfies the weak condition (ωX) with respect to ω.

Proof. The proof of this result is totally similar to the proofs of Proposition 8.3.8 and Proposition 8.3.9, and so we take it for granted. □

Proposition 8.4.3 (Answer to question (a)). *Let $F_h : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ be a sequence of (not necessary integral) functionals, and assume that there exists a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ which is a measure and such that*

$$F(\cdot, A') = \Gamma(L^p) - \lim_{h \to +\infty} F_h(\cdot, A') \qquad \textit{for any } A' \in \mathcal{A}_0.$$

Then F is weakly-*seq. l.s.c

Proof. Let $A \in \mathcal{A}$, $A' \in \mathcal{A}$ with $A' \Subset A$, $u \in W^{1,\infty}(\Omega)$ and take a sequence $(u_h)_h \subseteq W^{1,\infty}(\Omega)$ which is weakly-*convergent to u. Then, since $A' \Subset A$, it is well known that u_h converges to u strongly in $L^\infty(A')$, and so in particular strongly in $L^p(A')$. Being $F(\cdot, A')$ a $\Gamma(L^p)-$limit, it is L^p-lower semicontinuous. Moreover, being F a measure, it is also increasing. These facts imply that

$$F(u, A') \leqslant \liminf_{h \to \infty} F(u_h, A') \leqslant \liminf_{h \to \infty} F(u_h, A).$$

Since F is inner regular and since $A' \Subset A$ is arbitrary, the conclusion follows. □

Chapter 9

How to avoid the linear independence condition

9.1 Introduction

We refer to [276] as main reference for this chapter. In the previous chapters, as we already pointed out, (a.e. LIC) plays a crucial role to upgrade Euclidean representations as in

$$F(u, A) = \int_A f_e(x, u, Du)\, dx \tag{9.1.1}$$

to suitable anisotropic representations as in

$$F(u, A) = \int_A f(x, u, Xu)\, dx. \tag{9.1.2}$$

More precisely, since (9.1.1) gives rise to a representation depending on an Euclidean Lagrangian $f_e(x, u, \xi)$, in Chapter 7 we exploited (a.e. LIC) to define a new anisotropic Lagrangian $f(x, u, \eta)$ in such a way that

$$f_e(x, u, Du) = f(x, u, Xu) \tag{9.1.3}$$

for any sufficiently regular function u. Further to [128, 129, 205], an interesting open question was whether these results could be generalised beyond the (a.e. LIC) setting.

In this chapter, we provide an affirmative answer to the above issue, showing that all the results in [128, 129, 205] still hold even without requiring (a.e. LIC). The value of this result is at least twofold. On the one hand, avoiding (a.e. LIC) allows to consider anisotropies in the greatest generality. In particular, our results apply to the whole sub-Riemannian framework of Carnot-Carathéodory spaces. Indeed, while (a.e. LIC) is general enough to cover many relevant settings, among which Carnot groups and Grushin spaces (cf. [205] and Example 6.1.2), it is easy to provide instances of Carnot-Carathéodory spaces whose associated generating vector fields do not satisfy (a.e. LIC) (cf. Example 9.2.1). On the other hand, our generality allows to consider the case in which a fixed family X is replaced by a sequence of families $(X^h)_h$

converging to a limiting family X in any reasonable sense. Even assuming that each family X^h satisfies (a.e. LIC), not even the strongest convergence (say, for instance, C^∞) can guarantee in general that X will do the same (cf. Example 9.2.2).

Our approach starts from noticing that (9.1.3) is essentially the only point where (a.e. LIC) proves fundamental in the approach of [128, 129, 205]. Therefore, the crucial part of this chapter is to achieve (9.1.3) without requiring (a.e. LIC). To explain our approach, we briefly recall the strategy that we followed in the previous chapters. To this aim, fix a point $x \in \Omega$ and assume that $X_1(x), \ldots, X_m(x)$ are linearly independent in $\mathbb{R}^m$. This implies that the projection map $\mathcal{C}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ induced by $X_1(x), \ldots, X_m(x)$ is surjective. In this case, it is possible to set

$$f(x,u,\eta) = f_e\left(x,u,\mathcal{C}(x)^{-1}(\eta)\right), \tag{9.1.4}$$

being $\mathcal{C}(x)^{-1}$ a suitable right-inverse map of $\mathcal{C}(x)$, and to show, under additional assumptions on f_e, that (9.1.4) suffices to infer (9.1.3). Since our vector fields may be in general linearly dependent, $\mathcal{C}(x)$ may not be right-invertible. To this aim, we replace $\mathcal{C}(x)^{-1}$ with the so-called *Moore-Penrose pseudo-inverse* of $\mathcal{C}(x)$ (cf. [165]), say $\mathcal{C}_P(x)$, and we set

$$f(x,u,\eta) = f_e\left(x,u,\mathcal{C}_P(x)\cdot\eta\right). \tag{9.1.5}$$

A careful analysis of the properties of $\mathcal{C}_P(x)$ (cf. Proposition 9.3.1) will allow us to exploit (9.1.5) to provide anisotropic representations as in (9.1.3) (cf. Proposition 9.3.2). Once (9.1.3) is achieved, we devote the rest of the chapter to the generalization of the results in [128, 129, 205] (cf. Section 9.4). We decided to make this last part of the exposition as concise as possible, both to emphasise the crucial importance of Proposition 9.3.2, and because, once Proposition 9.3.2 is obtained, the proofs work exactly like their counterparts in [128, 129, 205]. We stress that our results are substantially analogous to those proved in [128, 129, 205]. Nevertheless, the possible non-surjectivity of $\mathcal{C}(x)$ brings out some interesting new phenomena. First of all, a deep look at the shape of f in (9.1.5) reveals that it is constant outside the range of $\mathcal{C}(x)$ (cf. Proposition 9.3.2). More precisely, if we orthogonally decompose any $\eta \in \mathbb{R}^m$ as

$$\eta = \mathcal{C}(x)\cdot\xi_\eta + \eta^\perp$$

for some $\xi_\eta \in \mathbb{R}^n$, then f satisfies

$$f(x,u,\eta) = f(x,u,\mathcal{C}(x)\cdot\xi_\eta). \tag{9.1.6}$$

Anyway, (9.1.6) is verified only if f is defined as in (9.1.5). Indeed, it is possible to provide integral representations as in (9.1.2) by arbitrarily choosing the value of the corresponding anisotropic Lagrangian outside the range of $\mathcal{C}(x)$ (cf. Example 9.4.1 and Theorem 9.4.2). Notwithstanding, we prove that (9.1.6) is a sufficient property to guarantee uniqueness in the integral representation (cf. Theorem 9.4.1). Another consequence of (9.1.6) is that f as in

(9.1.5) cannot inherit from f_e full coercivity in the gradient argument. Nevertheless, one can easily observe how the structural properties of an integral functional depend, in our case, solely on the behaviour of the Lagrangian on the range of $\mathcal{C}(x)$ (cf. Theorem 9.4.2).

9.2 Relevant vector fields

As already known, many relevant families of vector fields can already be found when (a.e. LIC) holds, such as the Euclidean space, Carnot groups and Grushin spaces (cf. [205]). Nevertheless, avoiding (a.e. LIC) is crucial to ensure that the results of [128, 129, 205], gain sufficient generality to be applied, for example, to the Carnot-Carathéodory setting.

Example 9.2.1. As an instance, consider the the family $X = (X_1, X_2)$ of vector fields defined on $\Omega = (-1, 1)^2 \subseteq \mathbb{R}^2$ by

$$X_1(x) = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad X_2(x) = \begin{cases} 0 & \text{if } x_1 \in (-1, 0) \\ x_1 \frac{\partial}{\partial x_2} & \text{if } x_1 \in [0, 1) \end{cases} .$$

for any $x = (x_1, x_2) \in \Omega$. Is is easy to check that Ω, endowed with the Carnot-Carathéodory induced by X, is a Carnot-Carathéodory space. Moreover, X_1, X_2 are Lipschitz continuous on Ω. Nevertheless, they do not satisfy (a.e. LIC).

Moreover, as pointed out in the introduction, (a.e. LIC) may not in general be preserved under even strong notions of convergence.

Example 9.2.2. For any $h \in \mathbb{N} \setminus \{0\}$, consider the the family $X^h = (X_1, X_2^h)$ of vector fields defined on $\mathbb{R}^2$ by

$$X_1(x) = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad X_2^h(x) = \frac{1}{h}\frac{\partial}{\partial x_2}$$

for any $x = (x_1, x_2) \in \mathbb{R}^2$. For any h as above, X^h is made of smooth and globally Lipschitz continuous vector fields which satisfy (a.e. LIC) on $\mathbb{R}^2$. Nevertheless, $(X^h)_h$ convergence uniformly, with all its derivatives, to $X = (X_1, 0)$, which clearly does not satisfy (a.e. LIC).

9.3 Anisotropic representation of Euclidean Lagrangians

This section constitutes the core of this chapter. More precisely, we show how to express a Euclidean Lagrangian in terms of an anisotropic Lagrangian, proving (9.1.3).

9.3.1 Algebraic properties of the Moore-Penrose pseudo-inverse

Let us recall, for the reader's convenience, some notation from Section 7.3. For any $x \in \Omega$, we define the linear map $L_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by

$$L_x(\xi) = \mathcal{C}(x) \cdot \xi$$

for any $\xi \in \mathbb{R}^n$. Moreover, we let

$$N_x = \ker(L_x) \qquad \text{and} \qquad V_x = \left\{ \mathcal{C}(x)^T \cdot \eta \ : \ \eta \in \mathbb{R}^m \right\}.$$

We recall that $\mathbb{R}^n = N_x \oplus V_x$, whence, for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$, there are unique $\xi_{N_x} \in N_x$ and $\xi_{V_x} \in V_x$ such that

$$\xi = \xi_{N_x} + \xi_{V_x}, \tag{9.3.1}$$

and the map $\Pi_x : \mathbb{R}^n \to V_x$ defined by $\Pi_x(\xi) = \xi_{V_x}$ is well-defined. In Section 7.3 we exploited in a crucial way (a.e. LIC) to ensure the existence of a right-inverse map associated to $\mathcal{C}(x)$. Precisely, if $X_1(x), \dots, X_m(x)$ are linearly independent at some $x \in \Omega$, then any $\eta \in \mathbb{R}^m$ can be expressed in the form $\eta = \mathcal{C}(x) \cdot \xi_\eta$ for some $\xi_\eta \in \mathbb{R}^n$. In the general case, we decompose $\eta \in \mathbb{R}^m$ as

$$\eta = \mathcal{C}(x) \cdot \xi_\eta + \eta^\perp,$$

where $\eta^\perp \in \mathrm{Im}(\mathcal{C}(x))^\perp$. We stress that ξ_η is uniquely defined only modulo $\ker(\mathcal{C}(x))$. Since $\mathcal{C}(x)$ may not have full rank, our approach must therefore differ from the one of Section 7.3. Let $\mathcal{C}_P : \Omega \longrightarrow M(n,m)$ be defined so that $\mathcal{C}_P(x)$ is the Moore-Penrose pseudo-inverse of $\mathcal{C}(x)$ (cf. [165]) for any $x \in \Omega$. Precisely, for a fixed $x \in \Omega$, $\mathcal{C}_P(x)$ is the unique matrix in $M(n,m)$ such that (cf. [165])

$$\begin{aligned} &\mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \mathcal{C}_P(x) = \mathcal{C}_P(x), \qquad && \mathcal{C}(x) \cdot \mathcal{C}_P(x) \cdot \mathcal{C}(x) = \mathcal{C}(x), \\ &\mathcal{C}_P(x) \cdot \mathcal{C}(x) = \mathcal{C}(x)^T \cdot \mathcal{C}_P(x)^T, \qquad && \mathcal{C}(x) \cdot \mathcal{C}_P(x) = \mathcal{C}_P(x)^T \cdot \mathcal{C}(x)^T. \end{aligned} \tag{9.3.2}$$

Our anisotropic representation is based on the following properties of $\mathcal{C}_P$.

Proposition 9.3.1. *Let $\mathcal{C}_P$ be the above-defined map. Moreover, for any $x \in \Omega$, let $\mathcal{C}_P(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be the linear map defined by*

$$\mathcal{C}_P(x)(\eta) = \mathcal{C}_P(x) \cdot \eta$$

for any η. Then the map

$$x \mapsto \mathcal{C}_P(x)(\eta)$$

is measurable for any $\eta \in \mathbb{R}^m$. Moreover, for any $x \in \Omega$, the following facts hold.

(i) $\mathrm{Im}(\mathcal{C}_P(x)) = V_x$.

(ii) $\Pi_x(\xi) = \mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \xi$ *for any* $\xi \in \mathbb{R}^n$.

(iii) $\ker(\mathcal{C}_P(x)) = \mathrm{Im}(L_x)^\perp$.

Proof. For a given $\eta \in \mathbb{R}^n$, it is well-known (cf. [165]) that

$$\mathcal{C}_P(x) \cdot \eta = \lim_{h \to +\infty} \left(\mathcal{C}(x)^T \cdot \mathcal{C}(x) + \frac{1}{h} I_n \right)^{-1} \cdot \mathcal{C}(x)^T \cdot \eta.$$

for any $x \in \Omega$. In particular, being $\mathcal{C}$ continuous over Ω, $x \mapsto \mathcal{C}_P(x) \cdot \eta$ is the pointwise limit of continuous functions, and hence it is measurable. Now we fix $x \in \Omega$. Notice that, by (9.3.2),

$$\mathcal{C}_P(x) \cdot \eta = \mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \mathcal{C}_P(x) \cdot \eta = \mathcal{C}(x)^T \cdot \left(\mathcal{C}_P(x)^T \cdot \mathcal{C}_P(x) \cdot \eta\right)$$

for any $\eta \in \mathbb{R}^m$, so that $\mathrm{Im}(\mathcal{C}_P(x)) \subseteq V_x$. To prove the other inclusion, it suffices to show (ii). To this aim, fix $\xi \in \mathbb{R}^n$. by (9.3.1) and (9.3.2),

$$\mathcal{C}(x) \cdot \mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \xi = \mathcal{C}(x) \cdot \xi = \mathcal{C}(x) \cdot (\Pi_x(\xi) + \xi_{N_x}) = \mathcal{C}(x) \cdot \Pi_x(\xi).$$

Since we already know that $\mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \xi \in V_x$, and being $\mathcal{C}(x)$ injective on V_x, (ii) follows. To prove (iii), fix $\eta \in \ker(\mathcal{C}_P(x))$ and $\xi \in \mathbb{R}^n$. Then, by (9.3.2),

$$\eta^T \cdot \mathcal{C}(x) \cdot \xi = \eta^T \cdot \mathcal{C}(x) \cdot \mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \xi = (\mathcal{C}_P(x) \cdot \eta)^T \cdot \mathcal{C}(x)^T \cdot \mathcal{C}(x) \cdot \xi = 0,$$

so that $\eta \in \mathrm{Im}(\mathcal{C}(x))^\perp$. Hence $\ker(\mathcal{C}_P(x)) \subseteq \mathrm{Im}(L_x)^\perp$. Assume by contradiction that there exists $\eta \neq 0$ such that $\eta \in \mathrm{Im}(L_x)^\perp \cap \ker(\mathcal{C}_P(x))^\perp$. In view of (9.3.2),

$$\mathcal{C}_P(x) \cdot \eta = \mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \mathcal{C}_P(x) \cdot \eta. \tag{9.3.3}$$

Since we know that $\ker(\mathcal{C}_P(x)) \subseteq \mathrm{Im}(L_x)^\perp$, then $\mathrm{Im}(L_x) \subseteq \ker(\mathcal{C}_P(x))^\perp$, so that both η and $\mathcal{C}(x) \cdot \mathcal{C}_P(x) \cdot \eta$ belongs to $\ker(\mathcal{C}_P(x))^\perp$. Being $\mathcal{C}_P(x)$ injective on $\ker(\mathcal{C}_P(x))^\perp$, we conclude from (9.3.3) that $\eta = \mathcal{C}(x) \cdot \mathcal{C}_P(x) \cdot \eta$, a contradiction with $\eta \in \mathrm{Im}(L_x)^\perp$. □

9.3.2 The anisotropic representation result

We exploit Proposition 9.3.1 to show that the anisotropic Lagrangian in (9.1.5) satisfies (9.1.3).

Proposition 9.3.2. *Let $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0, +\infty]$ be a Carathéodory function. Assume that*

$$f_e(x, u, \xi) = f_e(x, u, \Pi_x(\xi)) \tag{9.3.4}$$

for a.e. $x \in \Omega$, any $u \in \mathbb{R}$ and any $\xi \in \mathbb{R}^n$. Define the map $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty]$ by

$$f(x, u, \eta) = f_e(x, u, \mathcal{C}_P(x) \cdot \eta) \tag{9.3.5}$$

for any $x \in \Omega$, any $u \in \mathbb{R}$ and any $\eta \in \mathbb{R}^m$. Then f is a Carathéodory function such that

$$f(x, u, \eta) = f(x, u, \mathcal{C}(x) \cdot \xi_\eta) \tag{9.3.6}$$

and

$$f_e(x, u, \xi) = f(x, u, \mathcal{C}(x) \cdot \xi) \tag{9.3.7}$$

for a.e. $x \in \Omega$, any $u \in \mathbb{R}$, any $\eta \in \mathbb{R}^m$ and any $\xi \in \mathbb{R}^n$. Moreover, f enjoys the following *properties.*

(i) *If there exist* $a \in L^1_{loc}(\Omega)$ *and some* $b, c \geq 0$ *such that*

$$f_e(x,u,\xi) \leqslant a(x) + b|u|^p + c|\mathcal{C}(x)\cdot\xi|^p \tag{9.3.8}$$

for a.e. $x \in \Omega$, *any* $u \in \mathbb{R}$ *and any* $\xi \in \mathbb{R}^n$, *then*

$$f(x,u,\mathcal{C}(x)\cdot\xi) \leqslant a(x) + b|u|^p + c|\mathcal{C}(x)\cdot\xi|^p \tag{9.3.9}$$

for a.e. $x \in \Omega$, *any* $u \in \mathbb{R}$ *and any* $\xi \in \mathbb{R}^n$.

(ii) *If there exist* $d > 0$ *such that*

$$d|\mathcal{C}(x)\cdot\xi|^p \leqslant f_e(x,u,\xi) \tag{9.3.10}$$

for a.e. $x \in \Omega$, *any* $u \in \mathbb{R}$ *and any* $\xi \in \mathbb{R}^n$, *then*

$$d|\mathcal{C}(x)\cdot\xi|^p \leqslant f(x,u,\mathcal{C}(x)\cdot\xi) \tag{9.3.11}$$

for a.e. $x \in \Omega$, *any* $u \in \mathbb{R}$ *and any* $\eta \in \mathbb{R}^m$.

(iii) *If* $f_e(x,u,\xi) = f_e(x,\xi)$ *for a.e.* $x \in \Omega$, *any* $u \in \mathbb{R}$ *and any* $\xi \in \mathbb{R}^n$, *then* $f(x,u,\eta) = f(x,\eta)$ *for a.e.* $x \in \Omega$, *any* $u \in \mathbb{R}$ *and any* $\eta \in \mathbb{R}^m$.

(iv) *If*

$$f_e(x,u,\cdot) \text{ is convex} \tag{9.3.12}$$

for a.e. $x \in \Omega$ *and any* $u \in \mathbb{R}$, *then*

$$f(x,u,\cdot) \text{ is convex} \tag{9.3.13}$$

for a.e. $x \in \Omega$ *and any* $u \in \mathbb{R}$.

(v) *If*

$$f_e(x,\cdot,\cdot) \text{ is convex}$$

for a.e. $x \in \Omega$, *then*

$$f(x,\cdot,\cdot) \text{ is convex} \tag{9.3.14}$$

for a.e. $x \in \Omega$.

Proof. Let f be the function in (9.3.5). First we show that f is a Carathéodory function. To this aim, fix $u \in \mathbb{R}$ and $\eta \in \mathbb{R}^m$, and define the function $\Phi_{u,\eta} : \Omega \longrightarrow \mathbb{R}\times\mathbb{R}^n$ by

$$\Phi_{u,\eta}(x) = (u, \mathcal{C}_P(x)\cdot\eta)$$

for any $x \in \Omega$. Being $x \mapsto \mathcal{C}_P(x) \cdot \eta$ measurable by Proposition 9.3.1, then $\Phi_{u,\eta}$ is measurable. Since

$$f(x, u, \eta) = f_e(x, \Phi_{u,\eta}(x)) \tag{9.3.15}$$

for a.e. $x \in \Omega$, and being f_e a Carathéodory function, we deduce from [100, Proposition 3.7] that $x \mapsto f(x, u, \eta)$ is measurable for any $u \in \mathbb{R}$ and any $\eta \in \mathbb{R}^m$. Fix now $x \in \Omega$ and define $\Psi_x : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R} \times \mathbb{R}^n$ by

$$\Psi_x(u, \eta) = \Phi_{u,\eta}(x)$$

for any $u \in \mathbb{R}$ and any $\eta \in \mathbb{R}^m$. Clearly, Ψ_x is a linear function. In particular, by (9.3.15) and being f_e a Carathéodory function, then $(u, \eta) \mapsto f(x, u, \eta)$ is continuous for a.e. $x \in \Omega$, so that f is a Carathéodory function. Moreover, in view of (9.3.15), the linearity of Ψ_x and the definition of f, (iii), (iv) and (v) easily follows. Moreover, (9.3.6) follows directly from (iii) of Proposition 9.3.1. Let us prove (9.3.7). In view of (ii) of Proposition 9.3.1, (9.3.4) and the definition of f, we infer that

$$f(x, u, \mathcal{C}(x) \cdot \xi) = f_e(x, u, \mathcal{C}_P(x) \cdot \mathcal{C}(x) \cdot \xi) = f_e(x, u, \Pi_x(\xi)) = f_e(x, u, \xi)$$

for a.e. $x \in \Omega$, any $u \in \mathbb{R}$ and any $\xi \in \mathbb{R}^n$, so that (9.3.7) follows. Finally, (i) and (ii) are direct consequences of (9.3.7). $\square$

9.4 Avoiding (a.e. LIC) in a prototypical example

In this section we prove integral representation and Γ-compactness results in the setting of translation-invariant local functionals as in (6.1.2) proposed in [68, 205]. As already pointed out, our proofs will be concise and focused on the application of Proposition 9.3.2.

9.4.1 Integral representation

Let us begin with the generalization of [205, Theorem 3.12] to our general setting.

Theorem 9.4.1. *Let $p \in [1, +\infty)$. Let $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ satisfy the following properties.*

(i) F is a measure.

(ii) F is local.

(iii) F is L^p-lower semicontinuous.

(iv) $F(u + k, A) = F(u, A)$ for any $A \in \mathcal{A}$, any $u \in C^\infty(A)$ and any $k \in \mathbb{R}$.

(v) There exist $a \in L^1_{loc}(\Omega)$ and $c \geq 0$ such that

$$F(u, A) \leqslant \int_A a(x) + c|Xu|^p \, dx$$

for any $A \in \mathcal{A}$ and any $u \in C^\infty(A) \cap L^p(\Omega)$.

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R}^m \longrightarrow [0, +\infty)$ such that

$$F(u, A) = \int_A f(x, Xu(x))\, dx \tag{9.4.1}$$

for any $A \in \mathcal{A}$ and any $u \in W^{1,p}_{X,loc}(A) \cap L^p(\Omega)$. Moreover, f satisfies (9.3.6), (9.3.9) and (9.3.13). In addition, if $\tilde{f} : \Omega \times \mathbb{R}^m \longrightarrow [0, +\infty)$ is a Carathéodory function which verifies (9.3.6), (9.3.9) and for which (9.4.1) holds with $\tilde{f}$ in place of f, then

$$\tilde{f}(x, \eta) = f(x, \eta) \tag{9.4.2}$$

for a.e. $x \in \Omega$ and any $\eta \in \mathbb{R}^m$. Finally, if there exists $d > 0$ such that

$$d \int_A |Xu|^p\, dx \leqslant F(u, A) \tag{9.4.3}$$

for any $A \in \mathcal{A}$ and any $u \in C^\infty(A) \cap L^p(\Omega)$, then f satisfies (9.3.10).

Proof. Arguing *verbatim* as in the first step of the proof of [205, Theorem 3.12], our assumptions allow an Euclidean integral representation for F, meaning that

$$F(u, A) = \int_A f_e(x, Du(x))\, dx \tag{9.4.4}$$

for any $A \in \mathcal{A}$ and any $u \in W^{1,p}_{loc}(A)$, where $f_e : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty)$ is a suitable Carathéodory function satisfying (9.3.4), (9.3.8) and (9.3.12). Therefore, by Proposition 9.3.2, $f : \Omega \times \mathbb{R}^m \longrightarrow [0, +\infty)$ defined as in (9.3.5) is a Carathéodory function which satisfies (9.3.6), (9.3.7), (9.3.9) and (9.3.13). Therefore, combining (1.3.2), (9.3.7) and (9.4.4),

$$F(u, A) = \int_A f(x, Xu(x))\, dx \tag{9.4.5}$$

for any $A \in \mathcal{A}$ and any $u \in C^\infty(A)$. In order to achieve (9.4.1), one exploits (9.4.5) to argue *verbatim* as in the third step of the proof of [205, Theorem 3.12]. If (9.4.3) holds, arguing *verbatim* as in the first step of the proof of [205, Theorem 3.12] we infer that f_e satisfies (9.3.10), so that, by Proposition 9.3.2, f verifies (9.3.11). Finally, assume that there exists a Carathéodory function $\tilde{f} : \Omega \times \mathbb{R}^m \longrightarrow [0, +\infty)$ which verifies (9.3.6), (9.3.9) and (9.4.1). By (9.3.9), (9.4.1) and proceeding as in the fifth step of the proof of [128, Theorem 3.3], one infer that

$$\tilde{f}(x, \mathcal{C}(x) \cdot \xi) = f(x, \mathcal{C}(x) \cdot \xi) \tag{9.4.6}$$

for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^n$. Since both f and $\tilde{f}$ satisfy (9.3.6), we conclude by (9.4.6) that

$$\tilde{f}(x, \eta) = \tilde{f}(x, \mathcal{C}(x) \cdot \xi_\eta) = f(x, \mathcal{C}(x) \cdot \xi_\eta) = f(x, \eta)$$

for a.e. $x \in \Omega$ and any $\eta \in \mathbb{R}^m$, so that (9.4.2) follows. □

We point out that the statement of Theorem 9.4.1 is sharp. On the one hand, neither in (9.3.9) nor in (9.3.13) it is reasonable to expect global bounds rather than partial bounds on $\mathrm{Im}(\mathcal{C}(x))$. On the other hand, a uniqueness property as in (9.4.2) may fails dropping (9.3.6). This is to say, roughly speaking, that the structural properties of F translates into structural properties of f only as regards the part of f acting on the image of $\mathcal{C}$. This fact is not surprising. Indeed, for a fixed $x \in \Omega$, we already know that the action of $\mathcal{C}(x)$ is surjective only when $X_1(x), \dots, X_m(x)$ are linearly independent. Since we are not assuming (a.e. LIC), this property may trivially fail in general.

Example 9.4.1. As an instance, consider the the family $X = (X_1, X_2)$ of vector fields defined on $\Omega = (0,1)^2 \subseteq \mathbb{R}^2$ by

$$X_1(x) = X_2(x) = \frac{\partial}{\partial x_1}$$

for any $x = (x_1, x_2) \in \Omega$. Clearly X_1, X_2 are Lipschitz continuous on Ω and linearly dependent for any $x \in \Omega$. The associated matrices $\mathcal{C}$ and $\mathcal{C}_P$ are respectively

$$\mathcal{C}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{C}_P(x) = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$$

for any $x \in \Omega$. In particular,

$$N_x = \left\{(0, \lambda) \in \mathbb{R}^2 \,:\, \lambda \in \mathbb{R}\right\} \qquad \text{and} \qquad \mathrm{Im}(L_x) = \left\{(\lambda, \lambda) \in \mathbb{R}^2 \,:\, \lambda \in \mathbb{R}\right\} \tag{9.4.7}$$

for any $x \in \Omega$. Consider the functions $f_1, f_2 : \Omega \times \mathbb{R}^2 \longrightarrow [0, +\infty)$ defined by

$$f_1(x, \eta) = 2\left(\frac{\eta_1 + \eta_2}{2}\right)^2 \qquad \text{and} \qquad f_2(x, \eta) = 2\left(\frac{\eta_1 + \eta_2}{2}\right)^2 + e^{(\eta_1 - \eta_2)^2} - 1$$

for any $x \in \Omega$ and any $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. They are clearly Carathéodory functions. In view of (9.4.7), they both verify (9.3.9), (9.3.11) and (9.3.13) with $a, b = 0$ and $c, d = 1$. Moreover,

$$f_1(x, \mathcal{C}(x) \cdot \xi) = f_2(x, \mathcal{C}(x) \cdot \xi) \tag{9.4.8}$$

for any $x \in \Omega$ and any $\xi \in \mathbb{R}^2$, but they differ otherwise. In particular f_1 satisfies (9.3.6), while f_2 does not. Consider the local functionals $F_1, F_2 : L^2(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ defined by

$$F_j(u, A) = \begin{cases} \int_A f_j(x, Xu(x))\, dx & \text{if } A \in \mathcal{A},\ u \in W^{1,2}_{X,loc}(A) \\ +\infty & \text{otherwise} \end{cases}.$$

for $j = 1, 2$. Clearly F_1 and F_2 verify (i), (ii) and (iv) in Theorem 9.4.1. By means of the forthcoming Theorem 9.4.2, it holds that

$$F_1(u, A) = F_2(u, A) =: F(u, A) \tag{9.4.9}$$

for any $A \in \mathcal{A}$ and any $u \in L^2(\Omega)$. In particular coupling (9.4.9) with [205, Lemma 4.14], we conclude that F verifies also (iii) and (v) in Theorem 9.4.1, with $a = 0$ and $c = 1$, so that F verifies all the hypotheses of Theorem 9.4.1. In addition, F satisfies (9.4.3) with $d = 1$. Nevertheless, on the one hand we know by (9.4.8) that the integral representation of F drastically lack uniqueness. On the other hand, neither $f_1(x,\eta) \geq |\eta|^2$ for a.e. $x \in \Omega$ and any $\eta \in \mathbb{R}^2$, nor $f_2(x,\eta) \leqslant |\eta|^2$ for a.e. $x \in \Omega$ and any $\eta \in \mathbb{R}^2$.

Despite these differences with respect to the (a.e. LIC) framework, we show that the structural properties of f that one can derive from an integral representation as in Theorem 9.4.1 are essentially the only ones relevant for deducing structural properties of the associated functional. More precisely, the following holds.

Theorem 9.4.2. *Let $p \in [1,+\infty)$. Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty]$ be a Carathéodory function. Let $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ be defined by*

$$F(u,A) = \begin{cases} \int_A f(x,u(x),Xu(x))\,dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_{X,loc}(A) \\ +\infty & \text{otherwise} \end{cases}.$$

The following facts hold.

(i) If f satisfies (9.3.9), then

$$F(u,A) \leqslant \int_A a(x) + b|u(x)|^p + c|Xu|^p\,dx$$

for any $A \in \mathcal{A}$ and any $u \in W^{1,p}_{X,loc}(A) \cap L^p(\Omega)$.

(ii) If f satisfies (9.3.11), then

$$d\int_A |Xu|^p\,dx \leqslant F(u,A)$$

for any $A \in \mathcal{A}$ and any $u \in W^{1,p}_{X,loc}(A) \cap L^p(\Omega)$.

(iii) If $\tilde{f} : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty]$ is another Carathéodory function such that

$$f(x,u,\mathcal{C}(x)\cdot\xi) = \tilde{f}(x,u,\mathcal{C}(x)\cdot\xi) \tag{9.4.10}$$

for a.e. $x \in \Omega$, any $u \in \mathbb{R}$ and any $\xi \in \mathbb{R}^n$, then

$$F(u,A) = \int_A \tilde{f}(x,u(x),Xu(x))\,dx$$

for any $A \in \mathcal{A}$ and any $u \in W^{1,p}_{X,loc}(A) \cap L^p(\Omega)$.

Proof. In view of (1.3.2), the three statements are clearly true for any $A \in \mathcal{A}$ and any $u \in C^\infty(A) \cap L^p(\Omega)$. Noticing that all the involved functionals are continuous with respect to the metric topology of $W^{1,p}_X$, the general statement follows by means of standard localization and continuity arguments (cf. [205, 128]) coupled with Theorem 1.3.3. ☐

9.4.2 Γ-compactness

We conclude this section with the generalization of [205, Theorem 4.10].

Theorem 9.4.3. *Let $p \in (1,+\infty)$. For any $h \in \mathbb{N}$, let $f_h : \Omega \times \mathbb{R}^m \longrightarrow [0,+\infty]$ be a Carathéodory function satisfying (9.3.9), (9.3.11) and (9.3.13) with $a \in L^1(\Omega)$, $b = 0$, $c \geq 0$ and $d > 0$ independent of $h \in \mathbb{N}$. For any $h \in \mathbb{N}$, define the integral functional $F_h : L^p(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ by*

$$F_h(u,A) = \begin{cases} \int_A f_h(x, Xu(x))\,dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ +\infty & \text{otherwise} \end{cases}.$$

Then, up to a subsequence, there exists an integral functional of the form

$$F(u,A) = \begin{cases} \int_A f(x, Xu(x))\,dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ +\infty & \text{otherwise} \end{cases},$$

where $f : \Omega \times \mathbb{R}^m \longrightarrow [0,+\infty)$ is a Carathéodory function which satisfies (9.3.6), (9.3.9), (9.3.11) and (9.3.13) with a, b, c, d as above, for which

$$F(\cdot, A) = \Gamma(L^p) - \lim_{h\to+\infty} F_h(\cdot, A) \tag{9.4.11}$$

for any $A \in \mathcal{A}$.

Proof. By means of (i) and (ii) in Theorem 9.4.2, it is simply a matter of retracing the steps of the proof of [129, Proposition 3.3] to ensure the existence of a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ which verifies (i), (ii), (iii), (v) and (9.4.3) in Theorem 9.4.1 and such that (9.4.11) holds. We claim that F verifies (iv) in Theorem 9.4.1. To this aim, fix $A \in \mathcal{A}$, $u \in C^\infty A$ and $k \in \mathbb{R}$. We only show that $F(u,A) \geq F(u+k,A)$, being the other inequality analogous. If $F(u,A) = +\infty$, the claim is trivial. Assume otherwise that $F(u,A)$ is finite. Let $(u_h)_h \subseteq L^p(\Omega)$ be a recovery sequence for u as in (6.4.2). Since $F(u,A)$ is finite, up to a subsequence $(u_h)_h \subseteq W^{1,p}_X(A) \cap L^p(\Omega)$. Therefore, by our choice of $(u_h)_h$, (6.4.1), (6.4.2) and the definition of $(F_h)_h$,

$$F(u,A) = \liminf_{h\to+\infty} F_h(u_h, A) = \liminf_{h\to+\infty} F_h(u_h + k, A) \geq F(u+k, A).$$

To conclude, F satisfies the hypotheses of Theorem 9.4.1, so that there exists a Carathéodory function $f : \Omega \times \mathbb{R}^m \longrightarrow [0,+\infty)$ which satisfies (9.3.6), (9.3.9), (9.3.11) and (9.3.13) with a, b, c, d as in the statement, such that (9.4.1) holds for any $A \in \mathcal{A}$ and any $u \in W^{1,p}_X(A) \cap L^p(\Omega)$. Finally, fix $A \in \mathcal{A}$ and let $u \in L^p(\Omega) \setminus W^{1,p}_X(A)$. If it was the case that $F(u,A) < +\infty$, then u would admit a recovery sequence $(u_h)_h \subseteq W^{1,p}_X(A) \cap L^p(\Omega)$. But then, in view of (6.4.2), (ii) of Theorem 9.4.2 and [205, Lemma 4.14],

$$F(u,A) = \liminf_{h\to+\infty} F_h(u_h, A) \geq \liminf_{h\to+\infty} d \int_A |Xu_h|^p\,dx \geq d \int_A |Xu|^p\,dx = +\infty,$$

from which a contradiction would follow. □

Part IV

Weak solutions in Carnot-Carathéodory spaces

Chapter 10

Viscosity solutions: introduction and preliminaries

10.1 Introduction and motivations

The theory of *viscosity solutions* to first and second-order partial differential equation provides a powerful tool which allows to overcome some criticalities arising both in the classical and in the distributional theory. The interested reader is refereed to [96] for a wonderful introduction to the second-order viscosity theory in the Euclidean setting. First of all, this framework allows to admit solutions which are, *a priori*, only continuous functions. Moreover, the equations to which the viscosity theory applies need very weak structural assumptions. Indeed, it allows to deal with *fully non-linear* equations of the form

$$F(x, u, Du, D^2u) = 0, \tag{10.1.1}$$

where F is usually only a continuous function. In particular, the viscosity theory circumvents the restrictive *quasi-linearity* conditions imposed for instance by the classical Leray-Schauder theory for quasi-linear elliptic equations (cf. [194, 157]), as well as the the need to work with equations *in divergence form* typical of the weak theory (cf. [61]). Also, the ellipticity assumptions which characterize most of the classical and the weak theory are replaced by a more flexible *degenerate ellipticity* condition. Roughly speaking, we say that F is *degenerate elliptic* if

$$F(x, u, p, A) \leqslant F(x, u, p, B) \tag{10.1.2}$$

as soon as $A - B$ is positive semi-definite. Notice that (10.1.2) is automatically satisfied by first-order equations

$$H(x, u, Du) = 0$$

which are known in literature as *Hamilton-Jacobi equations*. Another remarkable advantage of the viscosity theory consists in the fact that the very mild assumptions on the structure of the equations guarantee a great flexibility in the existence, comparison and stability results. Before entering in the core of the definitions, let us give them an heuristic motivation. Assume that $u \in C^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is open, is a classical *subsolution* to (10.1.1), i.e.

$$F(x, u(x), Du(x), D^2u(x))$$

for any $x \in \Omega$. First, notice that the twice-differentiability properties of u let the property of being a classical subsolution be a pointwise property. Let us fix $x_0 \in \Omega$. Now, instead of considering the behavior of u at x_0, we wish to approximate u via suitable test functions, and to test the value of F over the latter. To this aim, consider a function $\varphi \in C^2(\Omega)$ whit the property that

$$u(x_0) - \varphi(x_0) \geq u(x) - \varphi(x) \tag{10.1.3}$$

for any x in a neighborhood of x_0. The reader can interpret the above property noticing that, in the particular case in which

$$u(x_0) = \varphi(x_0),$$

φ touches u from above at x_0. Since x_0 is a maximum point of $u - \varphi$, and this function is of class C^2, then

$$Du(x_0) = D\varphi(x_0) \qquad \text{and} \qquad D^2u(x_0) - D^2\varphi(x_0) \leqslant 0.$$

Therefore, (10.1.2) implies that

$$\begin{aligned} F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) &= F(x_0, u(x_0), Du(x_0), D^2\varphi(x_0)) \\ &\leqslant F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \\ &\leqslant 0, \end{aligned}$$

so that

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0.$$

Notice that the above inequality does not involve any first or second-order differential property of u, and thus it can be adapted itself as a definition of (sub)solution as soon as we wish to consider merely continuous solutions. So far we have considered equations depending on the Euclidean gradient Du and the Euclidean Hessian D^2u. Nevertheless, the above situation can be clearly generalized by considering more general equations of the form

$$F(x, u(x), Xu(x), X^2u(x)) = 0,$$

being X a suitable family of vector fields. Indeed, assume for instance that X induces a continuous Carnot-Carathéodory space over Ω. Assume in addition that $u \in C^2_X(\Omega)$ satisfies

$$F(x, u(x), Xu(x), X^2u(x)) \leqslant 0$$

for any $x \in \Omega$. Arguing as above, we can exploit Proposition 4.3.2 to infer that

$$F(x_0, u(x_0), X\varphi(x_0), X^2\varphi(x_0)) \leqslant 0$$

for any $\varphi \in \mathcal{C}^2_X(\Omega)$ which satisfies (10.1.3). Therefore the notion of viscosity solution emerges in the Carnot-Carathéodory context as naturally as it does in the classical setting. Apart from its intrinsic interest, there is also a deep Euclidean motivation which justifies the rising interest around the anisotropic viscosity theory. A relevant instance can be already found in the case of Hamilton-Jacobi equations. Indeed, in the study of Hamilton-Jacobi equatios of the form

$$H(x, Du(x)) = 0,$$

a typical structural assumption on H consists in requiring that there exists a positive constant $\beta > 0$ such that

$$H(x, p) \leqslant 0 \quad \Longrightarrow \quad |p| \leqslant \beta \tag{10.1.4}$$

for any $x \in \Omega$ and any $p \in \mathbb{R}^n$. This condition, which can be seen as a weak coercivity requirement, turns out to be fundamental in many situations (cf. [198]). However, there are many interesting situations in which (10.1.4) fails (cf. e.g. [269]). As a significant instance, one can consider the eikonal-type equation

$$|Du \cdot \mathcal{C}(x)^T| = 1 \tag{10.1.5}$$

on $\Omega \subseteq \mathbb{R}^3$, where $\mathcal{C}(x)$ is the coefficient matrix associated to the generating horizontal vector fields of the first Heisenberg group (cf. Example 3.2.8), that is

$$X_1|_q = \frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial t} \qquad \text{and} \qquad Y_1|_q = \frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial t},$$

where we denoted points $q \in \mathbb{R}^3$ by $q = (x_1, y_1, t)$. Being the kernel of $\mathcal{C}(x)^T$ non-trivial, it is easy to notice that the Hamiltonian associated to (10.1.5) does not satisfy (10.1.4). A standard approach to overcome this difficulty consists in changing the underlying geometry of the ambient space, rephrasing (10.1.5) by considering the corresponding anisotropic equation

$$|Xu| = 1, \tag{10.1.6}$$

whence introducing a sub-Riemannian viewpoint.

10.2 Main definitions

In this section we introduce the relevant structural assumptions on the equations that we will consider, together with the definition an the main properties of viscosity solutions.

Definition 10.2.1 (Horizontally elliptic equation)**.** *Given a function*

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^m \times S^m \longrightarrow \mathbb{R},$$

we say that F is horizontally elliptic *if*

$$F(x, s, p, A) \leqslant F(x, s, p, B)$$

whenever $x \in \Omega$, $s \in \mathbb{R}$, $p \in \mathbb{R}^m$ and $A, B \in S^m$ with $B \leqslant A$ (i.e. $A-B$ is positive semidefinite).

As already pointed out, it is clear that when F is independent of A, i.e. it describes a first-order differential operator, then it is automatically horizontally elliptic. Another common assumption in literature requires F to be increasing in the function variable.

Definition 10.2.2 (Proper equation)**.** *Given a function*

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^m \times S^m \longrightarrow \mathbb{R},$$

we say that F is proper *if*

$$F(x, s, p, A) \leqslant F(x, t, p, A) \tag{10.2.1}$$

whenever $x \in \Omega$, $s, t \in \mathbb{R}$, $p \in \mathbb{R}^m$ and $A \in S^m$ with $s \leqslant t$.

Definition 10.2.2 provides a fundamental tool in the establishment of comparison properties (cf. [96]). Nevertheless, although Definition 10.2.2 applies to a broad class of equations, among which we recall all the equations which are independent of the function variable, there are many interesting cases in which (10.2.1) fails. Some relevant instances of this phenomenon will be the object of Chapter 12 and Chapter 13. Let us begin with the definition of viscosity solution to Hamilton-Jacobi equations.

Definition 10.2.3 (Viscosity solutions to first-order PDEs)**.** *Let*

$$H : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$$

be continuous. We say that $u \in C(\Omega)$ is a viscosity subsolution *to*

$$H(x, u, Xu) = 0 \quad \text{in } \Omega \tag{10.2.2}$$

if

$$H(x, u(x_0), X\varphi(x_0)) \leqslant 0$$

for any $x_0 \in \Omega$ and for any $\varphi \in C^1_X(\Omega)$ such that

$$u(x_0) - \varphi(x_0) \geq u(x) - \varphi(x)$$

for any x in a neighborhood of x_0. We say that $u \in C(\Omega)$ is a viscosity supersolution *to* (10.2.2) *if*

$$H(x_0, u(x_0), X\varphi(x_0)) \geq 0$$

for any $x_0 \in \Omega$ and for any $\varphi \in C^1_X(\Omega)$ such that

$$u(x_0) - \varphi(x_0) \leqslant u(x) - \varphi(x)$$

for any x in a neighborhood of x_0. Finally we say that u is a viscosity solution *to* (10.2.2) *if it is both a viscosity subsolution and a viscosity supersolution.*

Similarly, we recall the definition of viscosity solutions to second-order horizontally elliptic PDEs.

Definition 10.2.4. *[Viscosity solutions to second-order PDEs] Let*

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^m \longrightarrow \mathbb{R}$$

be continuous and horizontally elliptic. We say that $u \in C(U)$ is a viscosity subsolution *to the equation*

$$F(x, u, Xu, X^2 u) = 0 \quad in \quad \Omega \tag{10.2.3}$$

if

$$F(x_0, u(x_0), X\varphi(x_0), X^2\varphi(x_0)) \leqslant 0 \tag{10.2.4}$$

for any $x_0 \in \Omega$ and for any $\varphi \in C^2_X(\Omega)$ such that

$$u(x_0) - \varphi(x_0) \geq u(x) - \varphi(x) \tag{10.2.5}$$

for any x in a neighborhood of x_0. We say that $u \in C(\Omega)$ is a viscosity supersolution *to* (10.2.3) *if*

$$F(x_0, u(x_0), X\varphi(x_0), X^2\varphi(x_0)) \geq 0$$

for any $x_0 \in \Omega$ and for any $\varphi \in C^2_X(\Omega)$ such that

$$u(x_0) - \varphi(x_0) \leqslant u(x) - \varphi(x) \tag{10.2.6}$$

for any x in a neighborhood of x_0. Finally we say that u is a viscosity solution *to* (10.2.3) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 10.2.5. As usual, when dealing with viscosity solutions to partial differential equations, there are many equivalent ways to define this notion. For instance, one can check the inequality (10.2.4) only in the more restrictive case when in (10.2.5) x_0 is a strict minimum point. Indeed, assume that (10.2.5) holds, and define $\tilde{\varphi}(x) := \varphi(x) + |x - x_0|^4$. Then it is clear that

$$F(x_0, u(x_0), X\tilde{\varphi}(x_0), X^2\tilde{\varphi}(x_0)) = F(x_0, u(x_0), X\varphi(x_0), X^2\varphi(x_0))$$

and that

$$u(x_0) - \tilde{\varphi}(x_0) > u(x) - \tilde{\varphi}(x)$$

for any x in a neighborhood of x_0. Moreover, one can equivalently require that

$$F(x_0, \varphi(x_0), X\varphi(x_0), X^2\varphi(x_0)) \leqslant 0$$

for any $x_0 \in \Omega$ and for any $\varphi \in C^2_X(\Omega)$ such that

$$0 = u(x_0) - \varphi(x_0) > u(x) - \varphi(x)$$

for any x in a neighborhood of x_0. Similar equivalences hold for the other cases.

Notice that the only difference between Definition 10.2.3 and Definition 10.2.4 is the required regularity of the test function φ. With the following proposition, we formalize the fact that the viscosity theory embeds the classical one.

Proposition 10.2.6. *Assume that X induces a continuous Carnot-Carathéodory space. Let*

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^m \longrightarrow \mathbb{R}$$

be continuous and horizontally elliptic. Then $u \in C^2_X(\Omega)$ is a viscosity solution to Equation (10.2.3) if and only if it is a classical solution to Equation (10.2.3).

Proof. If u is a classical solution, the thesis follows as described above with the help of Proposition 4.3.2. Conversely, if u is a viscosity solution, then it suffices to choose $u = \varphi$ in (10.2.5) and (10.2.6). □

Arguing *verbatim* as in the previous proof, the following first-order statement holds.

Proposition 10.2.7. *Assume that X induces a continuous Carnot-Carathéodory space. Let*

$$H : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

be continuous. Then $u \in C^1_X(\Omega)$ is a viscosity solution to (10.2.2) if and only if it is a classical solution to (10.2.2).

Remark 10.2.8. Beside Definition 10.2.3 and Definition 10.2.4, which provide the natural extension of the Euclidean notion of viscosity solution to the anisotropic setting, it is also common in the literature (cf. e.g. [278]) to consider weaker notions of viscosity solution. Accordingly, we call a function $u \in C(\Omega)$ a *weak viscosity solution* to the Hamilton-Jacobi equation (10.2.2) if Definition 10.2.3 the space $C^1_X(\Omega)$ is replaced by $C^1(\Omega)$. In the second-order case, Definition 10.2.4 has to be modified replacing $C^2_X(\Omega)$ with $C^2(\Omega)$. Since X is made of locally Lipschitz continuous vector fields, it is always the case that

$$C^1_X(\Omega) \subseteq C^1(\Omega),$$

whence viscosity solutions to (10.2.2) are always weak viscosity solutions to (10.2.2). As explained in the forthcoming Chapter 11, there are particular cases in which the two definitions are actually equivalent.

Chapter 11

Further properties of viscosity solutions to Hamilton-Jacobi equations

11.1 Viscosity and almost everywhere solutions

We refer to [78] as main reference for this section. In this section we relate the notion of viscosity solutions to Hamilton-Jacobi equations to solutions defined through horizontal jets, extending the results of [42] to the Carnot-Carathéodory setting. Exploiting this relation we prove that almost everywhere subsolutions to quasiconvex Hamilton-Jacobi equations associated to a family of Hörmander vector fields turn out to be viscosity subsolutions. Before proceeding, we recall that a Hamiltonian $H(x, u, p)$ is *quasiconvex* whenever

$$\{p \in \mathbb{R}^m \,:\, f(x, u, p) \leqslant K\} \text{ is convex} \tag{11.1.1}$$

for any $x \in \Omega$, any $u \in \mathbb{R}$ and any $K \in \mathbb{R}$. (cf. (4.4.1)). In particular, Proposition 4.4.1 applies. In the following, we will actually need (11.1.1) only for $K = 0$, so that our general statement is slightly more general with respect to the quasiconvex one. Our proof is divided in two steps. First we deal with a family $X = (X_1, \ldots, X_m)$ of vector fields, defined over a domain Ω, which satisfies both the bracket-generating condition

$$\mathrm{Lie}(\mathrm{span}\{X_1(p), \ldots, X_m(p)\}) = \mathbb{R}^n \tag{11.1.2}$$

for any $p \in \Omega$, and the additional linear independence condition (LIC) introduced in Chapter 5, in order to exploit Theorem 5.3.1. We recall that X satisfies (LIC) in Ω if

$$X_1(x), \ldots, X_m(x) \text{ are linearly independent for any } x \in \Omega.$$

Then, thanks to a lifting argument à la Rothschild-Stein (cf. [253]) we extend the result to an arbitrary family of Hörmander vector fields, thus avoiding (LIC). We begin by introducing the first-order horizontal subjet and superjet. The main reason for requiring (LIC), and hence the validity of Theorem 5.3.1, lies in the fact that the the definition of first-order jets typically relies on the differentiability structure of the ambient space. Therefore, (LIC) allows to rely on the notion of X-differentiability introduced in Chapter 5.

Definition 11.1.1 (Horizontal jets)**.** *Assume that X satisfies (11.1.2) and (LIC). Let $\tilde{\mathcal{C}}$ be as in Proposition 5.2.1. If $u \in C(\Omega)$ and $x_0 \in \Omega$, we define the* first-order horizontal superjet *of u at x_0 by*

$$Xu^+(x_0) := \left\{ p \in \mathbb{R}^m \,:\, u(x) \leqslant u(x_0) + \langle p \cdot \tilde{\mathcal{C}}(x_0), x - x_0 \rangle + o(d_\Omega(x, x_0)) \text{ as } d_\Omega(x, x_0) \to 0 \right\}.$$

If $u \in C(\Omega)$ and $x_0 \in \Omega$, we define the first-order horizontal subjet *of u at x_0 by*

$$Xu^-(x_0) := \left\{ p \in \mathbb{R}^m \,:\, u(x) \geq u(x_0) + \langle p \cdot \tilde{\mathcal{C}}(x_0), x - x_0 \rangle + o(d_\Omega(x, x_0)) \text{ as } d_\Omega(x, x_0) \to 0 \right\}.$$

In the Euclidean setting, it is well known that the notion of viscosity solution given in terms of comparison with sufficiently smooth tests functions is equivalent to the notion involving jets. Accordingly, we introduce the following definition.

Definition 11.1.2 (Jet solutions)**.** *we say that a function $u \in C(\Omega)$ is a* jet subsolution *to (10.2.2) in Ω if*

$$H(x_0, u(x_0), p) \leqslant 0$$

for every $x_0 \in \Omega$ and every $p \in Xu^+(x_0)$. Similarly, u is a jet supersolution *to (10.2.2) in Ω if*

$$H(x_0, u(x_0), p) \geq 0$$

for every $x_0 \in \Omega$ and every $p \in Xu^-(x_0)$. Finally, u is a jet solution *to (10.2.2) if it is both a jet subsolution and a jet supersolution.*

Even in our general framework, the following result holds.

Proposition 11.1.3. *Assume that X satisfies (11.1.2) and (LIC). Let $u \in C(\Omega)$. If u is a jet subsolution (respectively supersolution) to (10.2.2), then u is a viscosity subsolution (respectively supersolution) to (10.2.2).*

Proof. Since the two statements follow from similar arguments, we prove only the first one. Let $x_0 \in \Omega$ and let $\varphi \in C^1_X(\Omega)$ be an admissible function in the definition of viscosity subsolution. Then, thanks to Theorem 5.3.1, we obtain

$$\begin{aligned} u(x) &= u(x_0) + u(x) - u(x_0) \leqslant u(x_0) + \varphi(x) - \varphi(x_0) \\ &= u(x_0) + \langle X\varphi(x_0) \cdot \tilde{\mathcal{C}}(x_0), x - x_0 \rangle + o(d_X(x, x_0)). \end{aligned}$$

Therefore one has $X\varphi(x_0) \in Xu^+(x_0)$. In view of the hypothesis then one has

$$H(x_0, u(x_0), X\varphi(x_0)) \leqslant 0,$$

concluding the proof. □

To establish our desired implication we need some technical, but still intuitive, preliminary results, which are based on the notion of (X, N)-subgradient introduced in Definition 4.1.1.

Proposition 11.1.4. *Assume that X satisfies (11.1.2) and (LIC). Let $x_0 \in \Omega$, $u \in W^{1,\infty}_{X,loc}(\Omega)$ and let N be a negligible set which contains the non-Lebesgue points of Xu and $d_\Omega(\cdot, x_0)$. Then*

$$Xu^+(x_0) \cup Xu^-(x_0) \subseteq \partial_{X,N} u(x_0).$$

Proof. Fix $x_0 \in \Omega$ and N as in the statement. We only show that $Xu^+(x_0) \subseteq \partial_{X,N} u(x_0)$, being the proof of the other inclusion completely analogous. Let $p \in Xu^+(x_0)$. For any $n \in \mathbb{N} \setminus \{0\}$, we define

$$v_n(x) := u(x) - \langle p \cdot \tilde{\mathcal{C}}(x_0), x - x_0 \rangle - \frac{1}{n} d_\Omega(x, x_0).$$

Owing to [151] it is easy to see that $v_n \in W^{1,\infty}_{X,loc}(\Omega)$ and that $v_n(x_0) = u(x_0)$. Moreover, since $p \in Xu^+(x_0)$, it follows that

$$\begin{aligned} v_n(x) &= v_n(x_0) + u(x) - u(x_0) - \langle p \cdot \tilde{\mathcal{C}}(x_0), x - x_0 \rangle - \frac{1}{n} d_\Omega(x, x_0) \\ &\leqslant v_n(x_0) - \frac{1}{n} d_\Omega(x, x_0) + o(d_\Omega(x, x_0)) \end{aligned}$$

as $d_\Omega(x, x_0) \to 0$, thus

$$\begin{aligned} v_n(x_0) &\geq v_n(x) + \frac{1}{n} d_\Omega(x, x_0) + o(d_\Omega(x, x_0)) \\ &= v_n(x) + \frac{1}{n} d_\Omega(x, x_0) \left[1 + \frac{o(d_\Omega(x, x_0))}{d_\Omega(x, x_0)} \right] \end{aligned}$$

as $d_\Omega(x, x_0) \to 0$. Therefore x_0 is a point of local maximum of v_n which together with Proposition 4.3.3 and Proposition 4.1.5 gives

$$0 \in \partial_{X,N} u(x_0) - \partial_{X,N}(\langle p \cdot \tilde{\mathcal{C}}(x_0), \cdot - x_0 \rangle))(x_0) - \partial_{X,N} \left(\frac{1}{n} d_\Omega(\cdot, x_0) \right) (x_0).$$

We start by noticing that $x \mapsto \langle p \cdot \tilde{\mathcal{C}}(x_0), x - x_0 \rangle$ is in $C^1(\Omega) \subseteq C^1_X(\Omega)$, and so, thanks to Proposition 4.1.4, it follows that

$$\partial_{X,N}(\langle p \cdot \tilde{\mathcal{C}}(x_0), \cdot - x_0 \rangle)(x_0) = \{X(\langle p \cdot \tilde{\mathcal{C}}(x_0), \cdot - x_0 \rangle)(x_0)\} = \{p \cdot \tilde{\mathcal{C}}(x_0) \cdot \mathcal{C}(x_0)^T\} = \{p\}.$$

Moreover, thanks for instance to [151], we know that $|X(\frac{1}{n} d_\Omega(\cdot, x_0))(x)| \leqslant \frac{1}{n}$ for a.e. $x \in \Omega$, and, by the very definition of (X, N)-subgradient, we infer

$$\partial_{X,N} \left(\frac{1}{n} d_\Omega(\cdot, x_0) \right) (x_0) \subseteq B_{\frac{1}{n}}(0).$$

Putting all together we get that

$$0 \in \partial_{X,N} u(x_0) - \{p\} - B_{\frac{1}{n}}(0)$$

for any $n \in \mathbb{N} \setminus \{0\}$. Since $\bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(0) = \{0\}$, we conclude that

$$0 \in \partial_{X,N} u(x_0) - \{p\} - \{0\} = \partial_{X,N} u(x_0) - \{p\},$$

which is the thesis. □

We have developed all the tools that we need to prove the main result assuming (LIC).

Proposition 11.1.5. *Assume that X satisfies (11.1.2) and (LIC). Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function such that (11.1.1) holds for $K = 0$. Let $u \in W^{1,\infty}_{X,loc}(\Omega)$ be such that*

$$H(x, u(x), Xu(x)) \leqslant 0 \tag{11.1.3}$$

for a.e. $x \in \Omega$. Then u is both a jet subsolution and a viscosity subsolution to (10.2.2).

Proof. We already know from (2.3.1) that $u \in C(\Omega)$. In view of Proposition 11.1.3 it suffices to show that

$$H(x_0, u(x_0), p) \leqslant 0$$

for any $x_0 \in \Omega$ and for any $p \in Xu^+(x_0)$. Fix then $x_0 \in \Omega$, and let N be a negligible set which contains the non-Lebesgue points of Xu and of $Xd_\Omega(\cdot, x_0)$ and the points where (11.1.3) is not satisfied. Then thanks to Proposition 4.4.1 and (11.1.1) we know that

$$H(x, u(x), p) \leqslant 0 \tag{11.1.4}$$

for any $x \in \Omega$ and for any $p \in \partial_{X,N} u(x)$. Therefore, thanks to the choice of N, we can apply Proposition 11.1.4, which combined with (11.1.4) allows to conclude that

$$H(x_0, u(x_0), p) \leqslant 0$$

for any $p \in Xu^+(x_0)$. Being x_0 arbitrary, the thesis follows. □

Exploiting the previous result and a lifting scheme proposed in [253], we can finally drop (LIC) and prove the following theorem.

Theorem 11.1.1. *Let X satisfy (11.1.2). Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function such that (11.1.1) holds for $K = 0$. Let $u \in W^{1,\infty}_{X,loc}(\Omega)$ be such that (11.1.3) holds for a.e. $x \in \Omega$. Then u is a viscosity subsolution to (10.2.2).*

Proof. As above, $u \in C(\Omega)$. Let $x_0 \in \Omega$ and let $\varphi \in C^1_X(\Omega)$ be such that there exists an open neighborhood U of x_0 in Ω such that

$$u(x) - u(x_0) \leqslant \varphi(x) - \varphi(x_0) \tag{11.1.5}$$

for any $x \in U$. Invoking an argument as in [253, Part II], there exists an open and connected neighborhood $V \subseteq U$ of x_0, $r \in \mathbb{N}$ with $0 \leqslant r < m$, and $\delta > 0$ such that, setting $V_\delta :=$

$V \times (-\delta, \delta)^r$, $t = (t_1, \ldots, t_r)$,

$$\bar{X}_i(x,t) := X_i(x)$$

for $i = 1, \ldots, m-r$ and

$$\bar{X}_i(x,t) := X_i(x) + \frac{\partial}{\partial t_i}$$

for $i = m-r+1, \ldots, m$, (where we have assumed that, up to reordering, the vector fields $X_1, \ldots, X_{m-r}$ are linearly independent at x_0), then $\bar{X} := (\bar{X}_1, \ldots, \bar{X}_m)$ are linearly independent and satisfy the Hörmander condition at every point $(x,t) \in V_\delta$. Denote by $d_{\bar{X}}$ the Carnot-Carathéodory distance induced by $\bar{X}$ on V_δ. It is clear that given $v \in W^{1,1}_{X,loc}(\Omega)$ and setting $\bar{v}(x,t) := v(x)$ for any $(x,t) \in V_\delta$, then

$$\bar{X}\bar{v}(x,t) = Xv(x). \tag{11.1.6}$$

Therefore it is easy to see that $\bar{u} \in W^{1,\infty}_{\bar{X},loc}(V_\delta)$ and $\bar{\varphi} \in C^1_{\bar{X}}(V_\delta)$. Moreover, (11.1.5) implies that

$$\bar{u}(x,t) - \bar{u}(x_0,0) \leqslant \bar{\varphi}(x,t) - \bar{\varphi}(x_0,0)$$

for any $(x,t) \in V_\delta$, which is an open neighborhood of $(x_0,0)$. Therefore, proceeding as in the proof of Proposition 11.1.3 and using (11.1.5) and (11.1.6) we get that

$$X\varphi(x_0) \in \bar{X}\bar{u}^+(x_0,0), \tag{11.1.7}$$

where the horizontal superjet is considered with respect to the Carnot-Carathéodory distance induced by the family $\bar{X}$, $d_{\bar{X}}$ on V_δ. To conclude the proof, set

$$\bar{H}(x,t,s,p) := H(x,s,p)$$

for any $(x,t) \in V_\delta$, $s \in \mathbb{R}$ and $p \in \mathbb{R}^m$. It is clear that $\bar{H}$ is continuous and that $\{p \in \mathbb{R}^m : \bar{H}(x,t,u,p) \leqslant 0\}$ is convex for any $(x,t) \in V_\delta$ and $s \in \mathbb{R}$. We show that (11.1.3) implies that

$$\bar{H}(x_0,t_0,\bar{u}(x_0,t_0),p) \leqslant 0 \tag{11.1.8}$$

for any $(x_0,t_0) \in V_\delta$ and for any $p \in \bar{X}\bar{u}^+(x_0,t_0)$. This and (11.1.7) allow to conclude. To prove (11.1.8) it suffices to notice that by (11.1.3) it holds that

$$\bar{H}(x,t,\bar{u}(x,t),\bar{X}\bar{u}(x,t)) = H(x,u(x),Xu(x)) \leqslant 0$$

for a.e. $(x,t) \in V_\delta$. Then (11.1.8) follows as in the proof of Proposition 11.1.5. □

11.2 Viscosity solutions in Carnot groups

We refer to [126] as main reference for this section. In this section we derive further properties of viscosity solutions to Hamilton-Jacobi equations in the Carnot groups setting, where a richer algebraic structure allows to strengthening some of the achievements of Section 11.1. Let us fix a Carnot group $\mathbb{G}$ of dimension n and rank m. According to Proposition 3.3.2, we can choose a frame of generating vector fields $X = (X_1, \dots, X_m)$ in such a way that

$$\mathcal{C}(x) = \left[\begin{array}{cc} I_m & D(x) \end{array} \right]$$

for any $x \in \mathbb{G}$ and for a suitable polynomial matrix $D(x)$. In particular, X satisfies (11.1.2) and (LIC), and an easy computation shows that

$$\hat{\mathcal{C}}(x) = \left[\begin{array}{cc} I_m & 0_{n-m,m} \end{array} \right]$$

is a left-inverse of $\tilde{C}(x)^T$ for any $x \in \mathbb{G}$, being $0_{n-m,m}$ the null matrix of $n-m$ rows and m columns. Therefore, according to Remark 5.3.2 our notion of X-differential introduced in Definition 5.1.4 with this particular choice reduces to Pansu differential (cf. Definition 5.1.2), whence both Theorem 5.1.3 and the results of Section 11.1 apply. The jets Xu^+ and Xu^- associated with the chosen particular X-differential will be denoted by $\partial_X^+ u$ and $\partial_X^+ u$ respectively. More precisely,

$$\partial_X^+ u(x_0) = \{v \in \mathbb{R}^m \,:\, u(x) \leqslant u(x_0) + \langle v, \pi(x_0^{-1} \cdot x)\rangle + o(d_\Omega(x_0, x))\}$$

and

$$\partial_X^- u(x_0) = \{v \in \mathbb{R}^m \,:\, u(x) \geq u(x_0) + \langle v, \pi(x_0^{-1} \cdot x)\rangle + o(d_\Omega(x_0, x))\}.$$

In view of Proposition 3.7.1, in the previous definition d_Ω can be equivalently replaced by $d_\mathfrak{g}$ or $d_\mathbb{G}$. We already know from Proposition 11.1.3 that jet solutions to Hamilton-Jacobi equations are viscosity solutions. In Carnot groups, these two notions are actually equivalent, as the next proposition shows (cf. [208, 273] for further insights).

Proposition 11.2.1. *Let $\Omega \subseteq \mathbb{G}$ be open. Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. Then $u \in C(\Omega)$ is a jet subsolution (resp. supersolution) to (10.2.2) if and only if it is a viscosity subsolution (resp. supersolution) to (10.2.2).*

Proof. In view of Proposition 11.1.3, we just need to prove that viscosity solutions are jet solutions. We prove only the half of the claim concerning subsolutions, being the other half analogous. Assume that u is a viscosity subsolution to (10.2.2), let $x_0 \in \Omega$ and $p \in \partial_X^+ u(x_0)$. Let $d_\mathfrak{g}$ be the invariant distance induced (cf. (3.4.1)) by the Gauge-Koranyi norm defined in Example 3.4.4. It is well known that $y \mapsto d_\mathfrak{g}(x_0, y)$ is smooth outside x_0 and its horizontal gradient is bounded near x_0. Since $p \in \partial_X^+ u(x_0)$, then

$$u(x) \leqslant u(x_0) + \langle p, \pi(x_0^{-1} \cdot x)\rangle + o(d_\mathfrak{g}(x_0, x)). \tag{11.2.1}$$

Let $R > 0$ be such that $B_{d_\mathfrak{g}}(x_0, R) \Subset \Omega$, and define $g : (0, R] \longrightarrow \mathbb{R}$ by

$$g(r) := \sup_{x \in B_{d_\mathfrak{g}}(x_0, r)} \frac{\max\{0, u(x) - u(x_0) - \langle p, \pi(x_0^{-1} \cdot x)\rangle\}}{d_\mathfrak{g}(x_0, x)}.$$

Then g is nondecreasing and, by the choice of p, $\lim_{r \to 0} g(r) = 0$, Hence there exists $\tilde{g} \in C([0, R])$ such that $\tilde{g}$ is nondecreasing, $\tilde{g}(0) = 0$ and $\tilde{g} \geq g$. Let $G(r) := \int_0^r \tilde{g}(\tau) d\tau$. Then $G \in C^1([0, R[)$ and $G(0) = G'(0) = 0$. Moreover, for any $0 < r < \frac{R}{2}$, it holds that

$$G(2r) \geq \int_r^{2r} \tilde{g}(\tau) d\tau \geq r\tilde{g}(r) \geq rg(r). \tag{11.2.2}$$

Let us define $\varphi(x) = u(x_0) + \langle p, \pi(x_0^{-1} \cdot x)\rangle + G(2d_\mathfrak{g}(x, x_0))$. Then $\varphi \in C^1_X(B_{d_\mathfrak{g}}\left(x_0, \frac{R}{2}\right))$, $u(x_0) = \varphi(x_0)$ and $X\varphi(x_0) = p$. Finally, notice that (11.2.2) and the definition of g imply that $u(x) \leqslant \varphi(x)$ on $B_{d_\mathfrak{g}}\left(x_0, \frac{R}{2}\right)$. Therefore, being u a viscosity subsolution, we conclude that

$$H(x_0, u(x_0), p) = H(x_0, u(x_0), X\varphi(x_0)) \leqslant 0.$$

□

In addition, the following finer version of Proposition 11.1.5 holds.

Proposition 11.2.2. *Let Ω be an open subset of $\mathbb{G}$. Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function such that (11.1.1) holds for $K = 0$. Let $u \in W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$. Then the following conditions are equivalent.*

(*i*) *u is a viscosity subsolution to (14.2.1).*

(*ii*) *u is a jet subsolution to (14.2.1).*

(*iii*) *$H(x, Xu(x)) \leqslant 0$ for almost every $x \in \Omega$.*

Proof. The implication $(i) \iff (ii)$ follows from Proposition 11.2.1. Moreover, $(iii) \implies (i)$ follows from Proposition 11.1.5. Finally, we prove $(ii) \implies (iii)$. Let $x \in \Omega$ be such that u is Pansu-differentiable at x. Then clearly $Xu(x) \in \partial^+_X u(x)$, and so $H(x, Xu(x)) \leqslant 0$. □

To conclude this section, we point out that the sub-Riemannian Hamilton–Jacobi equation (10.2.2) can be viewed as an Euclidean equation in the following sense. We define the auxiliary Hamiltonian $\tilde{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\tilde{H}(x, u, p) = H(x, u, p \cdot \mathcal{C}(x)^T) \tag{11.2.3}$$

for any $(x, u, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. It is easy to see that $\tilde{H} \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ when H is continuous. With the next result, we show that sub-Riemannian viscosity solutions to (10.2.2) coincides with Euclidean viscosity solutions to the Hamilton–Jacobi equation associated to (11.2.3).

Proposition 11.2.3. *Let Ω be an open subset of $\mathbb{G}$. Let $\tilde{H}$ be as in (11.2.3). Then $u \in C(\Omega)$ is a viscosity solution to*

$$H(x, , u, Xu) = 0 \tag{11.2.4}$$

if and only if u is a viscosity solution to

$$\tilde{H}(x, u, Du) = 0. \tag{11.2.5}$$

Proof. Since $C^1(\Omega) \subseteq C^1_X(\Omega)$, then a viscosity solution to (11.2.4) is a viscosity solution to (11.2.5). To prove the converse implication we only show that viscosity subsolutions to (11.2.5) are viscosity subsolutions to (11.2.4), being the other part of the proof analogous. Therefore, assume that u is a viscosity subsolution to (11.2.5), let $x_0 \in \Omega$ and let $\varphi \in C^1_X(\Omega)$ be such that $u(x_0) = \varphi(x_0)$ and $\varphi(x) > u(x)$ for any $x \in B_{d_\mathfrak{g}}(x_0, 2r)$, for some $r > 0$ small enough to ensure that $B_{d_\mathfrak{g}}(x_0, 2r) \Subset \Omega$. Thanks to Proposition 2.1.8, there exists a sequence $(\varphi_h)_h \subseteq C^\infty(\Omega)$ converging to φ in $C^1_X(B_{d_\mathfrak{g}}(x_0, 2r))$. For any $h \in \mathbb{N}$, let x_h be a maximum point for $u - \varphi_h$ in $\overline{B_{d_\mathfrak{g}}(x_0, r)}$. We claim that $x_h \to x_0$ as $h \to +\infty$. Otherwise, we can assume that, up to a subsequence, $x_h \to x_1$ for some $x_1 \neq x_0$ such that $x_1 \in \overline{B_{d_\mathfrak{g}}(x_0, r)}$. Recalling that $u(x_h) - \varphi_h(x_h) \geq u(x_0) - \varphi_h(x_0)$ for any $h \in \mathbb{N}$, and since $x_h \to x_1$ and $\varphi_h \to \varphi$ uniformly on $B_{d_\mathfrak{g}}(x_0, 2r)$, we pass to the limit and we infer that $u(x_1) - \varphi(x_1) \geq u(x_0) - \varphi(x_0) = 0$. Therefore $\varphi(x_1) \leqslant u(x_1)$, a contradiction. By our choice of x_h, and thanks to (11.2.5), we get that

$$H(x_h, u(x_h), X\varphi_h(x_h)) = \tilde{H}(x_h, u(x_h), D\varphi_h(x_h)) \leqslant 0.$$

Therefore, since H is continuous, $x_h \to x$ and $X\varphi_h \to X\varphi$ uniformly on $B_{d_\mathfrak{g}}(x_0, 2r)$, passing to the limit in the previous inequality we conclude that

$$H(x_0, u(x_0), X\varphi(x_0)) \leqslant 0.$$

Hence u is a viscosity subsolution to (11.2.4). □

Chapter 12

The Aronsson equation for absolute minimizers of supremal functionals

12.1 Introduction

We refer to [243] as main reference for this chapter. The study of variational problems in L^∞ is very often a good starting point to set up problems coming both from theoretical issues and from real applications. The earliest works in this direction are due to Aronsson (cf. [20, 21]). In these seminal papers, the author studied the connection between Lipschitz extension problems and PDEs, introducing the notion of *absolute minimizing Lipschitz extension (AMLE)* and showing that a C^2 function is an AMLE if and only if it satisfies the *infinity Laplace equation*

$$-\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{12.1.1}$$

Aronsson observed (cf. [22]) that there are examples of AMLE which are not of class C^2, and thus solving equation (12.1.1) only in a formal sense. The problem was solved by Jensen in [175], exploiting the machinery of viscosity solutions. Indeed, Jensen showed that being an AMLE is equivalent to being a viscosity solution to (12.1.1). Moreover, he showed that viscosity solutions to (12.1.1) are unique, provided a Dirichlet boundary datum is assigned. We advise the reader that the above-mentioned notions will be extensively discussed in Chapter 13. One step further was made by Barron, Jensen and Wang (cf. [37]), who started the study of L^∞ variational functionals F which are usually known as *supremal functionals*, that is

$$F(u,V) := \|f(x,u(x),Du(x))\|_{L^\infty(V)} \qquad u \in W^{1,\infty}(\Omega), V \in \mathcal{A}.$$

where throughout this chapter Ω is a domain of $\mathbb{R}^n$ and f is a suitable continuous non-negative function. In particular, they generalized the notion of AMLE to the one of *absolute minimizer* of the functional F, that is a function $u \in W^{1,\infty}(U)$ such that

$$F(u,V) \leqslant F(v,V)$$

for any $V \Subset \Omega$ and for any $v \in W^{1,\infty}(V)$ with $v|_{\partial V} = u|_{\partial V}$. The authors of [37] showed that any absolute minimizer of F is a solution, in the viscosity sense, to the so-called *Aronsson equation*

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f(x,u(x),Du(x)))\frac{\partial f}{\partial p_i}(x,u(x),Du(x)) = 0,$$

provided that, among the other things, f is C^2 and $p \mapsto f(x,s,p)$ is *strictly quasiconvex*, where we call a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ strictly quasiconvex whenever

$$g(tp_1 + (1-t)p_2) < \max\{g(p_1), g(p_2)\}$$

for any $p_1, p_2 \in \mathbb{R}^m$ with $p_1 \neq p_2$ and $t \in (0,1)$. This result generalizes the previous ones, in the sense that, in the particular case in which $f(p) = |p|^2$, the notion of absolute minimizer reduces to the one of AMLE and the Aronsson equation becomes the infinity Laplace equation. Many improvements of the results in [37] have been achieved by Crandall (cf. [94]), both weakening some assumptions and exploiting a concise and elegant proof, and by Crandall, Wang and Yu ([99]), dealing with the more natural assumption of C^1 Hamiltonians. More recently, Bieske and Capogna (cf. [40, 44]) studied the derivation of the Aronsson equation, and the question of uniqueness of absolute minimizers, in the setting of Carnot groups and for the case $f(p) = |p|^2$. Later, Wang ([278]) moved the focus on the possibility to extend the previous results to more general frameworks, and started the study of supremal functionals defined in the setting of Carnot-Carathéodory spaces. In [278] the author adapted in the obvious way the notion of absolute minimizer to this framework. He showed, under mild assumptions on the vector fields, that any absolute minimizer of the supremal functional defined by

$$F(u,V) := \|f(x,Xu(x))\|_{L^\infty(V)}$$

is a weak viscosity solution, in the sense of Remark 10.2.8, to

$$-\sum_{i=1}^{m} X_i(f(x,Xu(x)))\frac{\partial f}{\partial p_i}(x,Xu(x)) = 0,$$

provided that $p \mapsto f(x,p)$ is quasiconvex in the sense of (4.4.1), f is homogeneous of degree $\alpha \geq 1$ and $D_p f(0,0) = 0$. Finally, Wang and Yu ([279]) improved the previous result by requiring only C^1 regularity for f and dropping the assumption that $D_p f(0,0) = 0$ (cf. also [123] for some more specific results for the case $f(p) = |p|^2$). However, neither [278] nor [279] studied the problem for Hamiltonian functions f that allow s-variable dependence. Accordingly, the aim of the present chapter is to generalize the results in [94] and [278], showing that any absolute minimizer of the functional

$$F(u,V) := \|f(x,u(x),Xu(x))\|_{L^\infty(V)}$$

is a weak viscosity solution, again in the sense of Remark 10.2.8, to the Aronsson equation

$$-\sum_{i=1}^{m} X_i(f(x,u(x),Xu(x)))\frac{\partial f}{\partial p_i}(x,u(x),Xu(x)) = 0,$$

provided that the following conditions hold.

$(X1)$ (Ω, d_Ω) is a continuous Carnot-Carathéodory space.

$(X2)$ $X_i \in C^2(\Omega, \mathbb{R}^n)$ for any $i = 1, \dots, m$.

$(f1)$ $f \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^m, [0, \infty))$.

$(f2)$ $p \mapsto f(x,s,p)$ is quasiconvex for any $x \in \Omega$ and for any $s \in \mathbb{R}$.

The strategy of our proof, strongly inspired by [94], is divided into five steps.

Step 1. Arguing by contradiction, we assume that there is an absolute minimizer which fails to be a viscosity subsolution to the Aronsson equation. Therefore, without loss of generality, we assume that there exists a function $\phi \in C^2(\Omega)$, which touches u from above in 0, such that

$$-\sum_{i=1}^{m} X_i(f(0,\phi(0),X\phi(0)))\frac{\partial f}{\partial p_i}(0,\phi(0),X\phi(0)) > 0.$$

Step 2. Exploiting ideas from [94, 278], we build a family $(\Psi_\varepsilon)_\varepsilon$ of classical solutions to the Hamilton-Jacobi equation

$$f(x,\Psi_\varepsilon(x),X\Psi_\varepsilon(x)) = f(0,\phi(0)-\varepsilon,X\phi(0)),$$

in order to approximate in a suitable way the behavior of ϕ in 0. We stress that, since this passage strongly relies on the arguments in [94, pages 275-276], the C^2 regularity of f is crucial to guarantee that Ψ_ε is a classical C^2 solution.

Step 3. We find and open set $\mathcal{N}_\varepsilon$ which allows to consider Ψ_ε as a competitor in the definition of absolute minimizer.

Step 4. By an appropriate change of variables we reduce to the case in which $s \mapsto f(x,s,p)$ is non-decreasing in a neighborhood of $(0,\phi(0),X\phi(0))$.

Step 5. We show the solvability of a suitable system of ODEs to get a family of C^1 curves $(\gamma_\varepsilon)_\varepsilon$, and we show that there is a choice among such curves which allows to reach a contradiction.

The previous scheme is formally analogous to the one employed in [94]. Nevertheless, our non-Euclidean framework presents some technical difficulties that require some new tools. In particular, the last step is strongly supported by the new differentiability results proved in Chapter 4. Moreover, differently from [94], the aforementioned system of ODEs cannot be solved by means of the classical Cauchy-Lipschitz existence theorem. From one hand, our result generalizes [94] to the more general setting of Carnot-Carathéodory spaces. Moreover, differently from [278], we allow also the function dependence of the Hamiltonian and we drop the requirement $D_p f(0,0) = 0$. Finally, the results in [279], apart from not allowing the function dependence of the Hamiltonian, are achieved under the Hörmander condition as in Definition 2.2.4, which, as we already know from Theorem 2.2.6 and Example 2.2.7, is a stronger requirement with respect to $(X1)$. On the other hand the construction of $(\Psi_\varepsilon)_\varepsilon$, according to [94, 278], strongly relies on the C^2 regularity of the Hamiltonian, which, on the contrary, is weakened in [279]. We point out that our assumptions are too general to ensure uniqueness for the associated Dirichlet problem, as shown in [176] in the Euclidean setting. Nevertheless, many uniqueness results are available in particular settings and under suitable hypotheses on the Hamiltonian (cf. for instance [175, 278, 176]).

12.2 Supremal functionals and absolute minimizers

In this section we recall the notion of supremal functional associated to suitable Hamiltonian functions, together with the related notions of absolute minimizers and absolute minimizing Lipschitz extensions. We refer to [23, 37, 94, 278] for an extensive account of the topic.

Definition 12.2.1 (Supremal functionals)**.** *Given a non-negative function $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^m)$, we define its associated* supremal functional

$$F : W_X^{1,\infty}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$$

by

$$F(u, V) := \|f(x, u, Xu)\|_{L^\infty(V)}$$

for any $V \in \mathcal{A}, u \in W_X^{1,\infty}(V)$, where we recall that $\mathcal{A}$ is the class of all open subsets of Ω.

Definition 12.2.2 (Absolute minimizers)**.** *We say that $u \in W_X^{1,\infty}(\Omega)$ is an* absolute minimizer *of F if*

$$F(u, V) \leqslant F(v, V)$$

for any $V \Subset \Omega$ and for any $v \in W_X^{1,\infty}(V)$ with $v|_{\partial V} = u|_{\partial V}$.

Definition 12.2.3 (Aronsson equation)**.** *If f belongs to $C^1(\Omega \times \mathbb{R} \times \mathbb{R}^m)$, we can define*

$$A_f : \Omega \times \mathbb{R} \times \mathbb{R}^m \times S^m \longrightarrow \mathbb{R}$$

by

$$A_f(x, s, p, Y) := -(Xf(x, s, p) + D_s f(x, s, p)p + D_p f(x, s, p) \cdot Y) \cdot D_p f(x, s, p),$$

and we say that

$$A_f[\phi](x) := A_f(x, \phi, X\phi, X^2\phi) = 0 \tag{12.2.1}$$

is the Aronsson equation *associated to F.*

It is easy to check that A_f is continuous and horizontally elliptic. Moreover, for any $\phi \in C^2(\Omega)$ and $x \in \Omega$ it holds that

$$A_f[\phi](x) = -X(f(x, \phi, X\phi)) \cdot D_p f(x, \phi, X\phi)^T.$$

Throughout this chapter, unless otherwise specified, we talk about viscosity solutions meaning weak viscoaity solutions in the sense of Remark 10.2.8.

12.3 The main theorem

Before stating and proving the main theorem, we just recall the following straightforward property of quasiconvex functions (cf. [94]).

Lemma 12.3.1. *Let $g \in C^1(\mathbb{R}^m)$ be a quasiconvex function. Then*

$$g(p) \geq g(q) \implies D_p g(p) \cdot (q - p) \leqslant 0$$

for any $p, q \in \mathbb{R}^m$.

Theorem 12.3.1. *Assume that (Ω, d_Ω) is a continuous Carnot-Carathéodory space induced by a family $X = (X_1, \ldots, X_m)$ of class C^2. Assume in addition that $(f1)$ and $(f2)$ hold. Then any absolute minimizer of F is a weak viscosity solution, in the sense of Remark 10.2.8, to the Aronsson equation.*

Proof. We divide the proof into several steps:

Step 1. Let u be an absolute minimizer for F. It suffices to show that u is a viscosity subsolution to (12.2.1), the other half of the proof being completely analogous. Arguing by contradiction, we assume that u fails to be a subsolution, that is there exists $x_0 \in \Omega$, $R_1 > 0$ and $\phi \in C^2(\Omega)$ such that (10.2.5) holds for any $x \in \overline{B_{R_1}(x_0)}$ and

$$A_f[\phi](x_0) > 0. \tag{12.3.1}$$

Without loss of generality we assume that $x_0 = 0 \in \Omega$.

Step 2. We combine ideas form [94] and [278] to achieve the following

Lemma 12.3.2. *There exist $0 < R_2 < R_1$, $\epsilon_1 > 0$, $\mu > 0$ and a continuous function $\Psi : [0, \epsilon_1] \times B_{R_2}(0) \longrightarrow \mathbb{R}$ such that, if we denote $\Psi(\epsilon, x)$ by $\Psi_\epsilon(x)$, it holds that $x \to \Psi_\epsilon(x) \in C^2(B_{R_2}(0))$ for any $\epsilon \in [0, \epsilon_1]$ and*

$$D\Psi_\epsilon \text{ is continuous in } (x, \epsilon) = (0, 0). \tag{12.3.2}$$

Moreover, it holds that

$$\begin{gathered}\Psi_\epsilon(0) = \phi(0) - \epsilon, \qquad D\Psi_\epsilon(0) = D\phi(0), \qquad D^2\Psi_\epsilon(0) - D^2\phi(0) > 2\mu I_n, \\ f(x, \Psi_\epsilon(x), X\Psi_\epsilon(x)) = f(0, \phi(0) - \epsilon, X\phi(0)),\end{gathered} \tag{12.3.3}$$

for any $x \in B_{R_2}(0)$.

Proof of Lemma 12.3.2. Let us define a new function $\overline{f}$ on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ by

$$\overline{f}(x, s, \xi) := f(x, s, \mathcal{C}(x) \cdot \xi) \tag{12.3.4}$$

for any $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Then, since f and X are C^2, it follows that

$$\overline{f} \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n).$$

Moreover, trivial computations show that

$$D_\xi \overline{f}(x, u, \xi) = D_p f(x, u, \mathcal{C}(x) \cdot \xi) \cdot \mathcal{C}(x), \tag{12.3.5}$$

and that

$$f(x, \varphi(x), X\varphi(x)) = \overline{f}(x, \varphi(x), D\varphi(x)) \tag{12.3.6}$$

for any $x \in U$ and any $\varphi \in C^2(\Omega)$. Finally, if we let $\overline{A}_{\overline{f}} \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n)$ be the Euclidean Aronsson operator associated to $\overline{f}$, i.e.

$$\overline{A}_{\overline{f}}(x, s, \xi, Z) := -(D_x\overline{f}(x, s, \xi) + D_s\overline{f}(x, s, \xi)\xi + D_\xi\overline{f}(x, s, \xi) \cdot Z) \cdot D_\xi\overline{f}(x, s, \xi)^T,$$

it follows from (12.3.5) and (12.3.6) that

$$\begin{aligned}\overline{A}_{\overline{f}}[\varphi](x) &= D_x(\overline{f}(x, \varphi(x), D\varphi(x))) \cdot D_\xi\overline{f}(x, s, D\varphi)^T \\ &= D_x(f(x, \varphi(x), X\varphi(x))) \cdot (D_p f(x, \varphi(x), X\varphi(x)) \cdot \mathcal{C}(x))^T \\ &= D_x(f(x, \varphi(x), X\varphi(x))) \cdot \mathcal{C}(x)^T \cdot D_p f(x, \varphi(x), X\varphi(x))^T \\ &= X(f(x, \varphi(x), X\varphi(x))) \cdot D_p f(x, \varphi(x), X\varphi(x))^T = A_f[\varphi](x),\end{aligned}$$

whence $\overline{A}_{\overline{f}}[\varphi](0) > 0$. The claim then follows as in [94, Theorem 1] and thanks to (12.3.6). □

Step 3. Now we want to exploit Ψ_ϵ as a test function in the definition of absolute minimizer on a suitable neighbourhood of 0. For doing this let us notice that (12.3.3) implies that

$$\begin{aligned}\Psi_\epsilon(x) &= \Psi_\epsilon(0) + D\Psi_\epsilon(0) \cdot x + x^T \cdot D^2\Psi_\epsilon(0) \cdot x + o(|x|^2) \\ &= \phi(0) - \epsilon + D\phi(0) \cdot x + x^T \cdot D^2\Psi_\epsilon(0) \cdot x + o(|x|^2) \\ &> \phi(0) - \epsilon + D\phi(0) \cdot x + x^T \cdot D^2\phi(0) \cdot x + 2\mu|x|^2 + o(|x|^2) \\ &= \phi(x) - \epsilon + 2\mu|x|^2 + o(|x|^2)\end{aligned}$$

as x goes to zero. Therefore we have that

$$\Psi_\epsilon(x) > \phi(x) - \epsilon + \mu|x|^2 \tag{12.3.7}$$

for any $x \in \overline{B_{R_3}(0)} \setminus \{0\}$, for any $\epsilon \in [0, \epsilon_1]$ and for some $R_3 < R_2$ sufficiently small. Let now $0 < \epsilon_2 < \epsilon_1$ small enough such that $\sqrt{\frac{\epsilon}{\mu}} < R_3$ for any $\epsilon \in [0, \epsilon_2]$ and define $\mathcal{N}_\epsilon$ as the connected component of

$$\{x \in B_{R_3}(0) \,:\, \Psi_\epsilon(x) < u(x)\}$$

containing zero (note that $\Psi_\epsilon(0) = u(0) - \epsilon < u(0)$ if $\epsilon > 0$). Therefore $\mathcal{N}_\epsilon$ is an open and connected neighborhood of 0 for any $\epsilon \in (0, \epsilon_2]$. Moreover, since (12.3.7) implies that

$$\Psi_\epsilon(x) > \phi(x) \geq u(x) \qquad \text{on } \partial B_{\sqrt{\frac{\epsilon}{\mu}}}(0),$$

it follows that

$$\mathcal{N}_\epsilon \subseteq B_{\sqrt{\frac{\epsilon}{\mu}}}(0) \subsetneqq B_{R_3}(0), \tag{12.3.8}$$

which implies that

$$u|_{\partial \mathcal{N}_\epsilon} = \Psi_\epsilon|_{\partial \mathcal{N}_\epsilon}.$$

Being u an absolute minimizer, and recalling (12.3.3), we conclude that

$$f(x, u(x), Xu(x)) \leqslant F(u, \mathcal{N}_\epsilon) \leqslant F(\Psi_\epsilon, \mathcal{N}_\epsilon) = f(0, \phi(0) - \epsilon, X\phi(0)) = f(x, \Psi_\epsilon(x), X\Psi_\epsilon(x)) \tag{12.3.9}$$

for a.e. $x \in \mathcal{N}_\epsilon$ and for any $\epsilon \in [0, \epsilon_2]$.

Step 4. At this point we wish to achieve the situation in which $s \mapsto f(x, s, p)$ is non-decreasing locally in a neighborhood of $(0, \phi(0), X\phi(0))$. Therefore we follow the strategy of [94] and we show that, via a suitable change of variables, this assumption is possible. Let us define then a new function g by

$$g(x, s, p) := f(x, u(0) + q \cdot x + G(s), q \cdot \mathcal{C}(0)^T + G'(s)p)$$

for any (x, s, p) in a suitable neighborhood of $(0, \phi(0), X\phi(0))$, where $q \in \mathbb{R}^n$ has to be determined and $G \in C^\infty(-\delta, \delta)$ is a local increasing diffeomorphism such that $G(0) = 0$ and $G'(0) > 0$. Let us notice that g is C^2 and quasiconvex in the third argument. Moreover, if we define $\overline{u}$ and $\overline{\phi}$ in a neighborhood of 0 by requiring that

$$u(x) = u(0) + q \cdot x + G(\overline{u}(x)),$$

$$\phi(x) = \phi(0) + q \cdot x + G(\overline{\phi}(x)), \tag{12.3.10}$$

it is easy to see that (10.2.5) holds for $\overline{u}$ and $\overline{\phi}$ and that $\overline{\phi}(0) = \overline{u}(0) = 0$. If H is the supremal functional associated to g it is easy to see that $\overline{u}$ is an absolute minimizer for H (we stress that we are working in a suitable neighborhood of 0). Easy computations show that

$$D_x g = D_x f + D_s f q, \qquad D_s g = G'(s) D_s f + G''(s) D_p f \cdot p^T, \qquad D_p g = G'(s) D_p f.$$

Therefore, noticing that

$$g(x, \overline{\phi}(x), X\overline{\phi}(x)) = f(x, \phi(x), X\phi(x))$$

for any x in the usual neighborhood of 0, we have that

$$\begin{aligned} A_g[\overline{\phi}](x) &= -X(g(x, \overline{\phi}(x), X\overline{\phi}(x))) \cdot D_p g(x, \overline{\phi}(x), X\overline{\phi}(x))^T \\ &= -X(f(x, \phi(x), X\phi(x))) \cdot D_p g(x, \overline{\phi}(x), X\overline{\phi}(x))^T \\ &= -X(f(x, \phi(x), X\phi(x))) \cdot (G'(\overline{\phi}(x)) D_p f(x, \phi(x), X\phi(x))^T) = G'(\overline{\phi}(x)) A_f[\phi](x), \end{aligned}$$

and so $A_g[\overline{\phi}](0) = G'(0) A_f[\phi](0) > 0$. Moreover, (12.3.10) implies that

$$X\overline{\phi}(0) = \frac{X\phi(0) - q \cdot \mathcal{C}(0)^T}{G'(0)}.$$

Therefore we have that

$$D_s g(0, \overline{\phi}(0), X\overline{\phi}(0)) = G'(0) D_s f(0, \phi(0), X\phi(0)) + \frac{G''(0)}{G'(0)} (X\phi(0) - q \cdot \mathcal{C}(0)^T) \cdot D_p f(0, \phi(0), X\phi(0))^T.$$

Hence, if we choose G as $G(s) = s + \frac{\beta}{2} s^2$, where $\beta > 0$, and we choose q as

$$q := D\phi(0) + D_x f(0, \phi(0), X\phi(0)) + D_s f(0, \phi(0), X\phi(0)) D\phi(0) + D_p f(0, \phi(0), X\phi(0)) \cdot B,$$

where B is the $m \times n$ matrix defined by

$$B_{ij} := \frac{\partial}{\partial x_j} X_i \phi(x) \Big|_{x=0}$$

for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and noticing that

$$p \cdot B \cdot \mathcal{C}(0)^T \cdot p^T = p \cdot X^2 \phi(0) \cdot p^T$$

for any $p \in \mathbb{R}^m$, thanks to (12.3.1) we conclude that

$$D_s g(0, \overline{\phi}(0), X\overline{\phi}(0)) = D_s f(0, \phi(0), X\phi(0)) + \beta A_f[\phi](0) > 0,$$

provided we choose β sufficiently big. Therefore in this new setting we can assume that $s \mapsto f(x, s, p)$ is increasing in a neighborhood of $(0, \phi(0), X\phi(0))$. This fact and (12.3.9) allow to find $0 < \epsilon_3 < \epsilon_2$ such that

$$f(x, u(x), Xu(x)) \leqslant f(x, u(x), X\Psi_\epsilon(x)) \tag{12.3.11}$$

for any $\epsilon \in (0, \epsilon_3]$ and for a.e. $x \in \mathcal{N}_\epsilon$.

Step 5. We are going to exploit (12.3.11). For doing this let us consider the first-order system of ODEs

$$\begin{cases} \dot\gamma(t) = -\mathcal{C}(\gamma(t))^T \cdot D_p f(\gamma(t), u(\gamma(t), X\Psi_\epsilon(\gamma(t)))^T \\ \gamma(0) = 0 \end{cases} \tag{12.3.12}$$

and, for any $\epsilon \in [0, \epsilon_3]$ and a suitable $R_4 < R_3$, we define $g_\epsilon : B_{R_4}(0) \longrightarrow \mathbb{R}^n$ as

$$g_\epsilon(x) := -\mathcal{C}(x)^T \cdot D_p f(x, u(x), X\Psi_\epsilon(x))^T.$$

It is easy to see (recall (2.3.1)) that $g_\epsilon \in C(B_{R_4}(0), \mathbb{R}^n)$. If we define

$$\mathcal{C} := \max_{i,j}\{\sup_{B_{R_4}(0)} |c_{ij}|\},$$

it follows from our assumptions that $0 < \mathcal{C} < +\infty$. Moreover, thanks to (2.3.1) and (12.3.2), there exist $0 < \epsilon_4 < \epsilon_3$ and $0 < R_5 < R_4$ such that

$$|D\Psi_\epsilon(x) - D\phi(0)| \leqslant 1$$
$$|u(x) - u(0)| \leqslant 1$$

for any $x \in \overline{B_{R_5}(0)}$ and $\epsilon \in [0, \epsilon_4]$. Therefore, if we let $M_\epsilon := \max\{g_\epsilon(x) : x \in \overline{B_{R_5}(0)}\}$, it follows that

$$\begin{aligned} \|g_\epsilon(x)\|_{L^\infty(B_{R_5}(0))} &\leqslant \mathcal{C}\|D_p f(x, u(x), X\Psi_\epsilon(x))\|_{L^\infty(B_{R_5}(0))} \\ &\leqslant \mathcal{C}\|D_p f(x, s, p)\|_{L^\infty(B_{R_5}(0)\times B_1(u(0))\times B_{\mathcal{C}}(D\phi(0)))} := M \end{aligned}$$

for any $\epsilon \in [0, \epsilon_4]$. Since (12.3.1) implies that $M_\epsilon > 0$, we conclude that $0 < M_\epsilon < M$ for any $\epsilon \in [0, \epsilon_4]$. Therefore, if we let

$$\epsilon_5 := \min\left\{\epsilon_4, \frac{R_5}{M}\right\},$$

Peano's Theorem (cf. e.g. [275, Theorem 2.19]) guarantees the existence, for any $\epsilon \in [0, \epsilon_5]$, of a curve $\gamma_\epsilon \in C^1((-\epsilon_5, \epsilon_5), \mathbb{R}^n)$ which solves (12.3.12). Moreover, from (2.1.2) and the first line of (12.3.12) it follows that γ_ϵ is a horizontal curve. Then, Proposition 4.2.1, Proposition 4.4.1, Lemma 12.3.1 and (12.3.11) imply that

$$\left.\frac{d}{dt}\left(\Psi_\epsilon(\gamma_\epsilon(t)) - u(\gamma_\epsilon(t))\right)\right|_{t=t_0} = D_p f(\gamma_\epsilon(t_0), u(\gamma_\epsilon(t_0)), X\Psi_\epsilon(\gamma(t_0))) \cdot (g(t_0) - X\Psi_\epsilon(\gamma(t_0))) \leqslant 0$$

for a.e. $t_0 \in (-\epsilon_5, \epsilon_5)$ and for any $\epsilon \in [0, \epsilon_5)$, and where $g(t_0)$ is as in Proposition 4.2.1.

Therefore, if we fix $t_0 \in (0, \epsilon_5)$, the previous inequality implies that

$$\begin{aligned}
\Psi_\epsilon(\gamma_\epsilon(t_0)) &= \Psi_\epsilon(0) + \int_0^{t_0} \frac{d\Psi_\epsilon(\gamma_\epsilon(t))}{dt} dt \\
&\leqslant u(0) - \epsilon + \int_0^{t_0} \frac{du(\gamma_\epsilon(t))}{dt} dt \\
&= u(\gamma_\epsilon(t_0)) - \epsilon < u(\gamma_\epsilon(t_0)),
\end{aligned}$$

hence we conclude that $\gamma_\epsilon(t_0) \in \mathcal{N}_\epsilon$, which implies, together with (12.3.8), that

$$\gamma_\epsilon(t_0) \in B_{\sqrt{\frac{\epsilon}{\mu}}}(0) \tag{12.3.13}$$

for any $t_0 \in [0, \epsilon_5)$ and any $\epsilon \in (0, \epsilon_5)$. On the other hand, the classical Taylor's formula applied to γ_ϵ implies that

$$\gamma_\epsilon(t) = -\mathcal{C}(0)^T \cdot (D_p f(0, \phi(0), X\phi(0))^T t + o(t) \tag{12.3.14}$$

as t tends to zero and for any $\epsilon \in (0, \epsilon_5)$. If we let $2K := |\mathcal{C}(0)^T \cdot (D_p f(0, \phi(0), X\phi(0))^T|$, (12.3.1) says that $2K > 0$. Therefore, thanks to (12.3.14), we know that there exists $0 < \epsilon_6 < \epsilon_5$ such that

$$|\gamma_\epsilon(t)| \geq Kt \tag{12.3.15}$$

for any for any $t, \epsilon \in (0, \epsilon_6)$. Let us choose $\bar{\epsilon} \in (0, \epsilon_6)$ such that

$$t_0 := \frac{2}{K}\sqrt{\frac{\bar{\epsilon}}{\mu}} < \epsilon_6.$$

Then (12.3.15) yelds that $|\gamma_{\bar{\epsilon}}(t_0)| \geq 2\sqrt{\frac{\bar{\epsilon}}{\mu}}$, which is a clear contradiction with (12.3.13).

□

Chapter 13

The p-Poisson equation as $p \to \infty$

13.1 Introduction

We refer to [78] as main reference for this chapter. In Chapter 12 we studied the relations occurring between general supremal functionals

$$F(u, V) := \|f(x, u, Xu)\|_{L^\infty(V)}$$

and their associated Aronsson equation

$$-X(f(x, u, Xu)) \cdot D_p f(x, u, Xu)^T = 0.$$

As we already pointed out, in the particular case in which $f(x, u, p) = |p|^2$, then absolute minimizers are known as *absolute minimizing Lipschitz extensions* (AMLE for short). Moreover, their associated Aronsson equation becomes the *infinite Laplace equation*

$$-\Delta_{X,\infty}\phi = 0.$$

As already evidenced, the interest around these topics began to emerge in [20, 21] in connection with the study of the so-called *Lipschitz extension problem* Indeed, the problem of finding the best possible Lipschitz extension of a given sample of a scalar function presents connections with many fields of mathematics and has several real-world applications. Although issues of existence of minimizers date back to the early 30's in the work of McShane and Whitney (cf. [23] and references therein for a detailed history), the work of Aronsson [20, 21] in the mid 60's represented truly a turning point, bringing a PDE point of view in the picture. A key novelty in Aronsson's approach was the notion of *absolutely minimizing Lipschitz extensition* (AMLE): a Lipschitz function u is an AMLE of its boundary datum on the boundary of an open set $\Omega \subset \mathbb{R}^n$ if for every subdomain $V \subset \Omega$ one has $\mathrm{Lip}(u, V) = \mathrm{Lip}(u, \partial V)$, where we have set

$$\mathrm{Lip}(u, V) = \sup_{x \neq y,\ x,y \in V} \frac{u(x) - u(y)}{d(x, y)}.$$

This definition in a sense characterizes a canonical optimal Lipschitz extension for Lipschitz boundary data, since we already know that it provides uniqueness. This notion is meaningful in every metric space, with no additional structure needed. As already mentioned, following in the footprints of Aronsson, who had studied the C^2 case, Jensen proved in [175] that AMLE are viscosity solutions to the infinity Laplacian equation

$$\Delta_\infty u := \sum_{i,j=1}^{n} u_{ij} u_i u_j = 0, \tag{13.1.1}$$

along with a uniqueness theorem for such solutions. The infinity Laplacian operator arose from the work of Aronsson though a formal argument, based on L^p approximation. Namely, for every $p > 1$ Aronsson considered C^2 minimizers u_p of the energy

$$\int_\Omega |Du|^p dx.$$

These minimizers are $p-$harmonic, i.e.

$$\mathrm{div}(|Du_p|^{p-2} Du_p) = 0.$$

Taking the formal limit of this PDE as $p \to \infty$ one obtains (13.1.1). Since $p-$harmonic functions are not in general C^2, it took several years to build a rigorous framework for Aronsson's asymptotic approach. This was eventually accomplished thanks to the work of Bhattacharya, DiBenedetto and Manfredi [39, Propositions 2.1 and 2.2]. In this chapter we prove an extension of [39, Propositions 2.1 and 2.2] to the non-Euclidean setting of Carnot-Carathéodory spaces, and we also extend the non-homogeneous case studied in [39]. Specifically, we are concerned with the asymptotic behavior, as $p \to \infty$, of vanishing trace critical points for the functionals

$$E_p(w, \Omega) = \int_\Omega \frac{1}{p} |Xw|^p dx - \int_\Omega f w dx,$$

where X is a family of smooth vector fields satisfying the Hörmander condition in a neighborhood of an bounded open set $\Omega \subset \mathbb{R}^n$, and $f \in L^{p'}(\Omega)$ is a given datum. More specifically we consider weak solutions $u_p \in W^{1,p}_X(\Omega)$ to the non-homogeneous boundary value problem

$$\begin{cases} \mathrm{div}_{\mathrm{X}}(|Xu_p|^{p-2} Xu_p) = -f & \text{in } \Omega, \\ u_p = 0 & \text{in } \partial\Omega. \end{cases} \tag{13.1.2}$$

In the homogenous case $f = 0$ we will also consider non-zero Lipschtiz boundary values. We will denote by $\{u_p\}_{p>1}$ the net of weak solutions to (13.1.2). As in the Euclidean case, it is plausible to expect that its cluster point(s) u_∞ solve an equation analogue to (13.1.1) which is derived by (13.1.2) in the limit $p \to \infty$. A formal computation, in the special homogeneous case $f = 0$, indicates that a likely candidate for such a limit is the $X - \infty-$Laplace equation

$$\Delta_{X,\infty} u_\infty = 0, \tag{13.1.3}$$

where

$$\Delta_{X,\infty}u = \sum_{i,j=1}^{m} X_iX_juX_iuX_ju = \sum_{i,j=1}^{m} \frac{X_iX_ju + X_jX_iu}{2} X_iuX_ju$$

denotes the subelliptic $\infty-$Laplacian. Our main result in the homogeneous case $f = 0$ is the following

Theorem 13.1.1. *Let $g \in W_X^{1,\infty}(\Omega)$, and for each $p > 1$ consider the weak solution u_p of the boundary value problem*

$$\begin{cases} \mathrm{div_X}(|Xu_p|^{p-2}Xu_p) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{13.1.4}$$

Every sequence $\{u_{p_k}\}$ of weak solutions to (13.1.4) admits a subsequence converging locally uniformly on Ω and weakly in $W_X^{1,m}(\Omega)$, for any $m > 1$, to a function $u_\infty \in W_X^{1,\infty}(\Omega) \cap C(\Omega)$ satisfying:

1. $\|Xu_\infty\|_\infty \leqslant \|Xg\|_\infty$.
2. $u_\infty - g \in W_{X,0}^{1,p}(\Omega)$ *for any* $p \in [1, \infty)$.
3. $u_\infty - g \in C_X^{0,\alpha}(\Omega) \cap C_0(\overline{\Omega})$ *for any* $\alpha \in [0, 1)$.
4. *If* $g \in W_X^{1,\infty}(\Omega) \cap C(\overline{\Omega})$, *then* $u_\infty \in W_X^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ *and* $u_\infty(x) = g(x)$ *for any* $x \in \partial\Omega$.
5. u_∞ *is a viscosity solution to* (13.1.3).
6. u_∞ *is an AMLE.*

In the case of the Heisenberg group, this theorem is due to Bieske [42]. Infinite and p-harmonic functions have been studied by the same author in the setting of Carnot groups [275], Riemannian vector fields [275] and Grushin spaces [41, 43]. Theorem 13.1.1 can also be proved, more indirectly, by invoking results from three earlier papers [278, 180, 123], all of which draw from the geometric significance of equation (13.1.3) in the study of minimal Lipschtz extensions: in 2006, Juutinen and Shanmugalingam [180], studied the asymptotic limits as $p \to \infty$ of $p-$energy minimizers in the setting of metric measure spaces satisfying a doubling condition, a $p-$Poincarè inequalities and a *weak Fubini property*, proving that such limits are AMLE. In that paper, the notion of viscosity solution for the infinity Laplacian was substituted with the notions of comparison with cones and strongly Absolutely Minimizing Lipschitz Extensions (sAMLE), which they prove to be equivalent to AMLE. In the Carnot-Carathéodory setting the notion of sAMLE is equivalent to the notion of Absolutely Minimizing Gradient Extension (AMGS) (cf. [123], i.e. a Lipschitz function u is an AMGS of its boundary data in Ω, if for every subdomain $U \subset \Omega$ and $v \in W_X^{1,\infty}(U)$ with $u - v \in W_{X,0}^{1,\infty}(U)$, one has $\|Xu\|_{L^\infty(U)} \leqslant \|Xv\|_{L^\infty(U)}$. In [123], Dragoni, Manfredi and Vittone prove that Carnot-Carathéodory metrics satisfy the weak Fubini property and that AMGS is equivalent to sAMLE. Since the latter is equivalent to AMLE, it follows that the limits of $p-$energy minimizers u_p as $p \to \infty$ converge to a function u_∞ which is an AMGS. At this point one can invoke Wang's result [278] (cf. also [44] in the case of Carnot groups), where it is proved that AMGS are viscosity solutions to (13.1.3). By contrast, our proof is quite direct and it mirrors the strategy in [39]. It also has the advantage of containing several technical steps upon which the non-homogeneous case rests. Before proceeding to the non-homogenous case, we want to note that the properties of AMLE and comparison by cones are equivalent in every length space [79]. In the presence of a weak Fubini property, they imply sAMLE. In the setting of Riemannian and subriemannian manifolds the latter agrees with AMGS and so it implies the property of being a viscosity solution to the $\infty-$Laplacian. The reverse implication follows from the uniqueness of solutions, and is known only for Carnot groups and Riemannian manifolds. Further connections have been studied in the setting of doubling metric measure space that satisfy a weaker condition, the $\infty-$weak Fubini property (cf. [124]).

In the general non-homogenous case $f \neq 0$, analogously to [39], one can prove that u_∞ solves a hybrid first and second order PDE in the viscosity sense. Our main result is the following

Theorem 13.1.2. *If $f \in L^\infty(\Omega) \cap C(\Omega)$, and $f \geqslant 0$, then every sequence $\{u_{p_k}\}$ of weak solutions to (13.1.2) admits a subsequence converging uniformly on $\bar{\Omega}$ and weakly in $W^{1,m}_X(\Omega)$, for any $m > 1$, to a function $u_\infty \in Lip(\Omega) \cap C(\bar{\Omega})$ vanishing on the boundary. Moreover, u_∞ is a solution of*

$$\begin{cases} \Delta_{X,\infty} u_\infty = 0 & \text{on } \overline{\{f > 0\}}^c, \\ |X u_\infty| = 1 & \text{on } \{f > 0\}, \end{cases} \tag{13.1.5}$$

in the viscosity sense.

To our knowledge, the present paper is the first extension of the results for the non-homogeneous problem in [39] beyond the Euclidean setting. One of the main challenges in this extension comes from the lack of linear structure and its role in the definition of viscosity solutions. Accordingly, a key contribution is provided by the new differentiability results presented in Chapter 5, an namely by Theorem 5.3.1, in view of their applications to the results presented in Chapter 11.

Remark 13.1.1. We note that the property of being a (viscosity) solution of either PDE in the mixed problem (13.1.5) could be separately be expressed in the setting of metric measure spaces: for the first order PDE cf. [200], while for the infinity Laplacian one could use comparison by cones or AMLE, or (with a Fubini property hypothesis) sAMLE. One could then pose the question whether the conclusions of Equation (13.1.2) could continue to hold in the setting of PI spaces satisfying a weak Fubini property. Unfortunately, in our proof of the convergence for the non-homogeneous case $f \neq 0$ we use in a crucial way the differential structure associated to the Hörmander vector fields. More specifically, we rely on the non-divergence form formulation of (13.1.2), which is not allowed in a general metric measure space, even with the additional hypotheses of doubling and Poincaré inequality.

Remark 13.1.2. It is interesting to note that in Theorem 13.1.1 we do not require any regularity of the boundary of the domain. While this is sufficient to guarantee global Lipschitz continuity of u_∞, there is no parallel regularity theory for $p-$harmonic functions. Indeed, even the case $p=2$ is quite involved and boundary regularity may fail even for smooth domains, in connection with their characteristic points (cf. [178]).

13.2 Some properties of the p-Poisson equation

In this section we study some properties of the p-Poisson equation associated to a family X of vector fields. From now on, unless otherwise specified, we assume that X satisfies the Hörmander condition on a domain Ω_0, with $\Omega \Subset \Omega_0$. The reason for which we require the Hörmader condition to be satisfied on Ω_0 is twofold. On the one hand, we will need to exploit Theorem 2.3.1. On the other hand, at some stage we will need to give a meaning to the Carnot-Carathéodory distance from $\partial\Omega$. Let $p \in (1, +\infty)$ and $p' = \frac{p}{p-1}$. We say that a function $u \in W^{1,p}_X(\Omega)$ is a *weak subsolution* (*weak supersolution*) to the p-Poisson equation

$$-\operatorname{div}_X(|Xw|^{p-2}Xw) = f \qquad \text{in } \Omega, \tag{13.2.1}$$

for a given datum $f \in L^{p'}(\Omega)$, if

$$\int_\Omega |Xu|^{p-2}\langle Xu, X\varphi\rangle\, dx \leqslant (\geq) \int_\Omega f\varphi dx$$

for any non-negative $\varphi \in W^{1,p}_{X,0}(\Omega)$. Finally, u is a *weak solution* to the p-Poisson equation if it is both a weak subsolution and a weak supersolution, i.e. if

$$\int_\Omega |Xu|^{p-2}\langle Xu, X\varphi\rangle\, dx = \int_\Omega f\varphi dx \tag{13.2.2}$$

for any $\varphi \in W^{1,p}_{X,0}(\Omega)$. We begin our investigation with an existence result to the minimization problem associated to (13.2.1).

Proposition 13.2.1. *Let $p \in (1,\infty)$, $f \in L^{p'}(\Omega)$, $g \in W^{1,p}_X(\Omega)$ and let us define the functional $I_p : W^{1,p}_{X,g}(\Omega) \longrightarrow \mathbb{R}$ by*

$$I_p(u) := \frac{1}{p}\int_\Omega |Xu|^p dx - \int_\Omega fu\, dx. \tag{13.2.3}$$

Then there exists a unique $u_p \in W^{1,p}_{X,g}(\Omega)$ such that

$$I_p(u_p) = \min_{u \in W^{1,p}_{X,g}(\Omega)} I_p(u). \tag{13.2.4}$$

Moreover, if $p \geq 2$, u_p is the unique weak solution to (13.2.1).

Proof. We wish to apply the direct method of the calculus of variations. To this aim, we notice that $W^{1,p}_{X,g}(\Omega)$ is a closed and convex subset of $W^{1,p}_X(\Omega)$, and so it is weakly closed. Moreover, I_p is strictly convex and strongly lower semicontinuous, and so it is weakly sequentially lower semicontinuous. Finally, thanks to Corollary 2.3.5 and Hölder's inequality it follows that

$$\begin{aligned}\int_\Omega |Xu|^p dx - \int_\Omega fu &\geqslant \min\left\{\frac{1}{2},\frac{1}{2K}\right\}\|u\|^p_{W^{1,p}_X} - \|f\|_{L^{p'}}\|u\|_{L^p} - \frac{1}{2}\\ &\geqslant \min\left\{\frac{1}{2},\frac{1}{2K}\right\}\|u\|^p_{W^{1,p}_X} - \|f\|_{L^{p'}}\|u\|_{W^{1,p}_X} - \frac{1}{2} \to +\infty\end{aligned}$$

as $\|u\|_{W^{1,p}_X} \to +\infty$. Therefore I_p is sequentially weakly coercive. Hence there exists $u_p \in W^{1,p}_{X,g}(\Omega)$ which minimizes I_p. The strict convexity of I_p yields the uniqueness of such a minimizer. It is now standard calculus to observe that a function u minimizes I_p if and only if it is a weak solution to (13.2.1). □

As in the Euclidean setting (cf. [197] for an elementary proof) the following comparison principle holds.

Lemma 13.2.2. *Let $u, v \in C^0(\overline{\Omega})$ be a weak subsolution and a weak supersolution to* (13.2.1) *respectively. Then the following facts hold:*

(i) If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ on Ω.

(ii) It holds that

$$\sup_{x\in\Omega}(u-v) \leqslant \sup_{x\in\partial\Omega}(u-v).$$

Moreover, if u, v are both weak solutions, it holds that

$$\|u-v\|_{\infty,\Omega} \leqslant \|u-v\|_{\infty,\partial\Omega}.$$

Proof. Let us prove (i). Fix $\varepsilon > 0$ and set $D_\varepsilon := \{x \in \Omega : u(x) > v(x) + \varepsilon\}$. Since u and v are continuous, then D_ε is open. Assume by contradiction that $D_\varepsilon \neq \emptyset$. Then let us define $\eta_\varepsilon := \max\{u - v - \varepsilon, 0\}$. Since by assumption $u \leqslant v$ on $\partial\Omega$ it holds that $\eta_\varepsilon \in W^{1,p}_{X,0}(\Omega)$. Therefore, from our assumtpions on u and v, and thanks to Simon's inequality (cf. [267]), it follows that

$$\int_{D_\varepsilon} |Xu - Xv|^p dx \leqslant \int_{D_\varepsilon} \left(|Xu^{p-2}Xu - |Xv|^{p-2}Xv|\right)(Xu - Xv)\, dx \leqslant 0,$$

which implies that $u - v - \epsilon$ is constant and positive on every connected component of D_ε. A contradiction then follows. For proving (ii) it suffices to notice that $v + \alpha$ is still a weak supersolution for any $\alpha \in \mathbb{R}$. Then, noticing that $u \leqslant v + \sup_{\partial\Omega}(u - v)$ on $\partial\Omega$, and thanks to (i), (ii) follows. Finally, the last statement follows exchanging the roles of u and v in (ii). □

In the next result we study the relationships between weak and viscosity solutions to (13.2.1). It is easy to see that when evaluated on $C^2_X(\Omega)$ functions, equation (13.2.1) becomes

$$-|Xw|^{p-2}\Delta_X w - (p-2)|Xw|^{p-4}\Delta_{X,\infty} w = f.$$

The associated differential operator, that is

$$F(x,\xi,X) = -|\xi|^{p-2}\left(\operatorname{trace}(X) + \sum_{j=1}^{m}\sum_{i=1}^{n}\xi_j\frac{\partial c_{j,i}}{\partial x_i}\right) - (p-2)|\xi|^{p-4}\xi\cdot X\cdot\xi^T - f(x),$$

is horizontally elliptic and continuous, provided that $p \geq 4$ and f is continuous. Therefore we require in addition that $p \geq 4$ and that $f \in L^{p'}(\Omega)\cap C(\Omega)$. The proof of the following result is inspired by [212].

Proposition 13.2.3. *Let $p \geq 4$, $f \in L^{p'}(\Omega)\cap C(\Omega)$ and let $u \in W_X^{1,p}(\Omega)\cap C(\Omega)$ be a weak solution to (13.2.1). Then u is a viscosity solution to (13.2.1).*

Proof. We only prove that u is a viscosity subsolution, being the other half of the proof completely analogous. We already know that $u \in C(\Omega)$. Therefore, arguing by contradiction, we assume that there exists $x_0 \in \Omega$, $v \in C_X^2(\Omega)$ and $R > 0$ such that $B_R(x_0) \Subset \Omega$,

$$0 = v(x_0) - u(x_0) < v(x) - u(x) \qquad \text{on } \overline{B_R(x_0)} \tag{13.2.5}$$

and

$$-|Xv(x_0)|^{p-2}\Delta_X v(x_0) - (p-2)|Xv(x_0)|^{p-4}\Delta_{X,\infty}v(x_0) > f(x_0).$$

Hence, thanks to the continuity of the p-Poisson operator, the continuity of f and the fact that $v \in C_X^2(\Omega)$, up to choosing R small enough we can assume that

$$-|Xv(x)|^{p-2}\Delta_X v(x) - (p-2)|Xv(x)|^{p-4}\Delta_{X,\infty}v(x) \geq f(x)$$

for any $x \in B_R(x_0)$. Therefore v is a classical supersolution to the p-Poisson equation on $B_R(x_0)$, and so it is in particular a weak supersolution. Since $u \in C(\overline{B_R(x_0)})$ it is well defined the number $m := \min_{\partial B_R(x_0)}(v-u)$ and by (13.2.5) we get $m > 0$. Now we notice that $v - m$ is still a weak supersolution to the p-Poisson equation and $u \leqslant v - m$ on $\partial B_R(x_0)$. Therefore, thanks to Lemma 13.2.2, we conclude that $u \leqslant v - m$ on $B_R(x_0)$. Recalling that $v(x_0) = u(x_0)$ we get $m \leqslant 0$ which is a contradiction. Hence u is a viscosity subsolution, and the proof is complete. □

13.3 Variational solutions: the homogeneous case

In this section we study the limiting behavior of solutions to (13.1.4) and we prove Theorem 13.1.1.

13.3.1 Existence and properties

Our approach follows the scheme employed in [39]. We fix a function $g \in W_X^{1,\infty}(\Omega)$ and $p \in (4,\infty)$. Let us denote by u_p the unique weak solution to (13.2.1), coming from Proposition 13.2.1, with boundary datum g and $f = 0$. Since $u_p - g$ is an admissible test function in

(13.2.2), it follows from Hölder's inequality that

$$\int_\Omega |Xu_p|^p dx \leqslant \int_\Omega |Xu_p|^{p-1}|Xg|dx \leqslant \left(\int_\Omega |Xu_p|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega |Xg|^p\right)^{\frac{1}{p}},$$

which implies that

$$\int_\Omega |Xu_p|^p dx \leqslant \int_\Omega |Xg|^p dx. \tag{13.3.1}$$

Let us fix a non-decreasing sequence $(m_k)_k \subseteq (4, \infty)$ with $\lim_{k\to\infty} m_k = \infty$. We are going to show that the family $(Xu_p)_{p>m_0}$ is bounded in $L^{m_0}(\Omega)$. Indeed, if $p > m_0$ then using (13.3.1), Hölder's inequality and the fact that $g \in W^{1,\infty}_X(\Omega)$, we get

$$\int_\Omega |Xu_p|^{m_0} dx \leqslant \|Xu_p\|^{m_0}_{L^p(\Omega)} |\Omega|^{\frac{p-m_0}{p}} \leqslant \left(\|Xg\|^p_{L^\infty(\Omega)}|\Omega|\right)^{\frac{m_0}{p}} |\Omega|^{\frac{p-m_0}{p}} = |\Omega| \|Xg\|^{m_0}_{L^\infty(\Omega)}. \tag{13.3.2}$$

Thanks to Corollary 2.3.5 and (13.3.2), we can conclude that the family $(u_p)_{p>m_0}$ is bounded in $W^{1,m_0}_X(\Omega)$. Therefore, by reflexivity (cf. Proposition 1.3.2), we know that there exist a subsequence $(u_{p_h})_h$ and a function $u_\infty \in W^{1,m_0}_X(\Omega)$ such that

$$u_{p_h} \rightharpoonup u_\infty \quad \text{in} \quad W^{1,m_0}_X(\Omega) \quad \text{as } h \to \infty.$$

We call u_∞ a *variational solution* to the ∞-Laplace equation. Next, we prove points (1)-(4) in Theorem 13.1.1.

Proof of (1)-(4) in Theorem 13.1.1. The proof of the weak convergence in $W^{1,m}_X(\Omega)$ for any $m \in (1, \infty)$ follows repeating the same steps employed for finding u_∞ for each $k \in \mathbb{N}$ and by a standard diagonal argument. The uniform convergence follows by the previous fact and thanks to Proposition 2.3.4. Let us prove (1). From the lower semicontinuity of the L^{m_k}-norm with respect to the weak convergence, and the analogous of (13.3.2) with m_k in place of m_0 we get

$$\|Xu_\infty\|_{L^{m_k}(\Omega)} \leqslant |\Omega|^{\frac{1}{m_k}} \|Xg\|_{L^\infty(\Omega)}$$

for any $k \in \mathbb{N}$. Therefore, passing to the limit as k goes to infinity, we conclude that

$$\|Xu_\infty\|_{L^\infty(\Omega)} \leqslant \|Xg\|_{L^\infty(\Omega)}.$$

This, together with Corollary 2.3.5 and Proposition 2.3.4, allows to conclude that $u_\infty \in W^{1,\infty}_X(\Omega) \cap C(\Omega)$. To prove (2) we show that $u_\infty \in W^{1,m_k}_{X,g}(\Omega)$ for any $k \in \mathbb{N}$. Indeed, fix $k \in \mathbb{N}$. For any h with $p_h > m_k$, there exists a sequence $(\varphi^h_j)_j \subseteq C^\infty_c(\Omega)$ converging to $u_{p_h} - g$ strongly in $W^{1,p_h}_X(\Omega)$, and so, since $p_h > m_k$, strongly in $W^{1,m_k}_X(\Omega)$. Therefore we can find a sequence $(\varphi_h) \subseteq (\varphi^h_j)^h_j$ such that

$$\|\varphi_h - (u_{p_h} - g)\|_{W^{1,m_k}_X(\Omega)} < \frac{1}{h} \tag{13.3.3}$$

for any $h > 0$. We claim that $(\varphi_h)_h$ converges weakly to $u_\infty - g$ in $W^{1,m_k}_X(\Omega)$. Indeed, for any

$\psi \in L^{m_k^*}(\Omega)$, thanks to (13.3.3) and Hölder's inequality it follows that

$$\begin{aligned}\left|\int_\Omega \varphi_h \psi dx - \int_\Omega (u_\infty - g)\psi dx\right| &\leqslant \int_\Omega |\varphi_h - (u_{p_h} - g)||\psi| dx + \left|\int_\Omega (u_{p_h} - u_\infty)\psi dx\right| \\ &\leqslant \|\varphi_h - (u_{p_h} - g)\|_{L^{m_k}(\Omega)} \|\psi\|_{m_k^*(\Omega)} + \left|\int_\Omega (u_{p_h} - u_\infty)\psi dx\right| \\ &\leqslant \frac{1}{h}\|\psi\|_{L^{m_k^*}(\Omega)} + \left|\int_\Omega (u_{p_h} - u_\infty)\psi dx\right|.\end{aligned}$$

The conclusion follows letting $h \to \infty$. Reasoning in a similar way for the X-gradients, thanks to Proposition 1.3.5, the claim is proved. Therefore, thanks to Mazur's lemma (cf. e.g. [61, Corollary 3.9]), for each $j \in \mathbb{N}$ there are convex combinations of φ_h converging strongly to $u_\infty - g$ in $W_X^{1,m_k}(\Omega)$, that is, for any $j \in \mathbb{N}$ there exist natural numbers $M_j < N_j$ and real numbers $a_{j,M_j}, \dots, a_{j,N_j}$, with $\lim_{j\to\infty} M_j = +\infty$, $0 \leqslant a_{j,h} \leqslant 1$ and $\sum_{h=M_j}^{N_j} a_{j,h} = 1$, such that

$$\phi_j := \sum_{h=M_j}^{N_j} a_{j,h}\varphi_h \longrightarrow u_\infty - g \qquad \text{in } W_X^{1,m_k}(\Omega).$$

Since each ϕ_j belongs to $C_c^\infty(\Omega)$, it follows that $u_\infty - g \in W_{X,0}^{1,m_k}(\Omega)$. The proof of (3) follows from (2) and thanks to Proposition 2.3.4. Finally, (4) follows trivially from (3). □

The remaining part of this section is dedicated to the proof of the last two statements in Theorem 13.1.1.

13.3.2 Variational solutions are AMLEs

In this section we show that variational solutions, as one might expect, are absolutely minimizing Lipschitz extensions. We point out that this result has already been proved, in greater generality, in [180]. Nevertheless we prefer to give here a shorter and more direct proof.

Proposition 13.3.1. *u_∞ is an AMLE.*

Proof. Let $v \in W_X^{1,\infty}(\Omega)$ and $V \Subset \Omega$ with $v|_{\partial V} = u_\infty|_{\partial V}$. Let $(m_k)_k$ and $(p_h)_h$ as above. For any $h \in \mathbb{N}$, consider the unique weak solution v_p to the problem

$$\begin{cases} -\operatorname{div}_X(|Xu|^{p_h-2}Xu) = 0 & \text{in } V \\ u = v & \text{on } \partial V \end{cases} \tag{13.3.4}$$

Up to a subsequence, we can assume that $(v_{p_h})_h$ converges to a variational solution v_∞ in the sense of Theorem 13.1.1. We claim that $v_\infty = u_\infty$ on V. First of all notice that, for h big enough and thanks to Proposition 2.3.4, being $v \in C(\overline{V})$, it holds that $u_{p_h}, v_{p_h} \in C(\overline{V})$. Moreover, observe that both u_{p_h} and v_{p_h} satisfies the equation

$$\int_V |Xu|^{p-2}Xu \cdot X\varphi dx = 0$$

for any $\varphi \in W^{1,p}_{X,0}(V)$. Therefore, thanks to Lemma 13.2.2 and Theorem 13.1.1, it follows that

$$\|u_{p_h} - v_{p_h}\|_{L^\infty(V)} \leqslant \|u_{p_h} - v_{p_h}\|_{L^\infty(\partial V)} \leqslant \|u_{p_h} - u_\infty\|_{L^\infty(\partial V)} \to 0$$

as h goes to infinity. Therefore, again thanks to Theorem 13.1.1, we conclude that $u_\infty = v_\infty$. On the other hand, arguing as in the proof of Theorem 13.1.1 and thanks to the previous claim, we conclude that

$$\|Xu_\infty\|_{L^\infty(V)} = \|Xv_\infty\|_{L^\infty(V)} \leqslant \|Xv\|_{L^\infty(V)}.$$

The previous equation yields at once that

$$\||Xu_\infty|^2\|_{L^\infty(V)} \leqslant \||Xv|^2\|_{L^\infty(V)},$$

and the thesis follows. □

13.3.3 Variational solutions are ∞-harmonic

To complete the study of variational solutions, we conclude by showing that they are viscosity solutions to the ∞-Laplace equations. We point out that we cannot combine Proposition 13.3.1 with Theorem 12.3.1 to conclude that u_∞, being an AMLE, is ∞-harmonic. Indeed, as mentioned before, since the notion of viscosity solution adopted in Theorem 12.3.1 is the weaker version of Remark 10.2.8. Therefore we need to give a direct proof which exploits again the approximation scheme employed for obtaining u_∞.

Proposition 13.3.2. *u_∞ is a viscosity solution to the ∞-Laplace equation*

$$-\Delta_{X,\infty} u_\infty = 0 \qquad \text{on } \Omega. \tag{13.3.5}$$

Proof. We only show that u_∞ is a viscosity subsolution to (13.3.5), being the other half of the proof analogous. To this aim, let $x_0 \in \Omega$, $v \in C^2_X(\Omega)$ and $R > 0$ be such that $u_\infty - v$ has a strict maximum at x_0 in $B_R(x_0) \Subset \Omega$. If $Xv(x_0) = 0$, the thesis is trivial by the definition of $\Delta_{X,\infty}$. So we can assume that $|Xv(x_0)| > 0$. Let $u_h := u_{p_h}$ be a sequence which allows to define u_∞. We can assume without loss of generality that $p_h > Q$ for any $h \in \mathbb{N}$, where Q is as in Proposition 2.3.4. Then it follows that $u_h \in C^0(\Omega)$. Moreover, thanks to Theorem 13.1.1 we can assume that u_h converges to u_∞ uniformly on $B_R(x_0)$. Let now x_h be a maximum point of $u_h - v$ on $\overline{B_{\frac{R}{2}}(x_0)}$. We claim that x_h has a subsequence, still denoted by x_h, which converges to x_0. If it is not the case, assume without loss of generality that $x_h \to x_1 \neq x_0$, for some $x_1 \in B_R(x_0)$. Then it follows that

$$u_h(x_h) - v(x_h) \geq u_h(x_0) - v(x_0),$$

and so, passing to the limit and thanks to uniform convergence, we get that

$$u_\infty(x_1) - v(x_1) \geq u_\infty(x_0) - v(x_0),$$

which contradicts the strict maximality of x_0. Hence, up to a subsequence, we assume that $x_h \to x_0$. By Proposition 13.2.3 we know that u_h is a viscosity solution to (13.2.1), therefore

$$-|Xv(x_h)|^{p_h-2}\Delta_X v(x_h) - (p_h-2)|Xv(x_h)|^{p_h-4}\Delta_{X,\infty}v(x_h) \leqslant 0.$$

Since $|Xv(x_0)| > 0$, then for h big enough we have that $|Xv(x_h)| > 0$. Therefore we can divide both sides by $(p_h-2)|Xv(x_h)|^{p_h-4}$, and get that

$$-\frac{|Xv(x_h)|^2\Delta_X v(x_h)}{p_h-2} - \Delta_{X,\infty}v(x_h) \leqslant 0.$$

Passing to the limit as $h \to \infty$, the proof is complete. □

13.4 Variational solutions: the non-homogeneous case

In this section we prove Theorem 13.1.2 and study the limiting behavior of weak solutions to the p-Poisson equation as $p \to \infty$ with a non-negative datum $f \in L^\infty(\Omega) \cap C^0(\Omega)$. In analogy with the previous section we introduce the notion of *variational solutions* u_∞ as suitable limits of the sequence $(u_p)_p$. Moreover, we show that u_∞ is the solution of a constrained extremal problem which can be understood as the limiting problem arising from (13.2.4). Finally, we study the limiting partial differential equation satisfied by u_∞. In particular we show that u_∞ is a viscosity supersolution to the ∞-Laplace equation and a viscosity subsolution to the eikonal equation. Unlike the homogeneous case, u_∞ is not in general ∞-harmonic. Nevertheless, it satisfies in the viscosity sense the system (13.1.5).

13.4.1 Existence and properties

We follow again the approach of [39]. From now on we fix $f \in L^\infty(\Omega)$ and we denote by $u_p \in W^{1,p}_{X,0}(\Omega)$ the unique solution to (13.2.1) with $f \geqslant 0$ and $p > 4$. Let us denote by I_∞ the variational functional that we get taking the (formal) limit as $p \to +\infty$ in (13.2.3), namely

$$I_\infty(\varphi) := -\int_\Omega f\varphi dx$$

with $\varphi \in W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$. Clearly, I_∞ does not admit a minimum in $W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$. Nevertheless, in analogy with the Euclidean setting, we are going to show that imposing the extra condition $\|X\varphi\|_{L^\infty(\Omega)} = 1$ is enough to find a solution.

Theorem 13.4.1. *There exists $u_\infty \in W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$ such that*

$$I_\infty(u_\infty) \leqslant I_\infty(\varphi) \tag{13.4.1}$$

for any $\varphi \in W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$ such that $\|X\varphi\|_{L^\infty(\Omega)} = 1$. Moreover, it holds that

$$0 \leqslant u_\infty(x) \leqslant d_{\Omega_0}(x, \partial\Omega) \quad \forall x \in \overline{\Omega}, \tag{13.4.2}$$

where $d_{\Omega_0}(x, \partial\Omega) = \inf_{y\in\partial\Omega} d_{\Omega_0}(x, y)$.

Before proving the theorem we construct the candidate solutions u_∞, in analogy with the previous section, as suitable limits of subsequences of $(u_p)_p$. To this aim, let us define the real number E_p by

$$E_p = E_p(\Omega, f) := \int_\Omega |Xu_p|^p dx.$$

By (13.2.2) and Hölder's inequality we have

$$\left|\int_\Omega f\,\varphi\,dx\right| \leqslant E_p^{\frac{p-1}{p}} \left(\int_\Omega |X\varphi|^p\right)^{\frac{1}{p}}$$

for each $\varphi \in W^{1,p}_{X,0}(\Omega)$. Therefore it holds that

$$\max_{\varphi\in W^{1,p}_{X,0}(\Omega),\varphi\neq 0} \left(\frac{\int_\Omega f\,\varphi\,dx}{\left(\int_\Omega |X\varphi|^p\right)^{1/p}}\right)^{\frac{p}{p-1}} \leqslant E_p, \tag{13.4.3}$$

where by possibly changing φ into $-\varphi$ we have assumed that

$$\int_\Omega f\,\varphi\,dx \geqslant 0.$$

Testing (13.2.2) with $\varphi = u_p$ we get

$$E_p = \int_\Omega |Xu_p|^p dx = \int_\Omega f\,u_p\,dx. \tag{13.4.4}$$

From this we have

$$E_p = \frac{\left(\int_\Omega |Xu_p|^p\right)^{\frac{p}{p-1}}}{\left(\int_\Omega |Xu_p|^p\right)^{\frac{1}{p-1}}} = \left(\frac{\int_\Omega f\,u_p}{\left(\int_\Omega |Xu_p|^p\right)^{1/p}}\right)^{\frac{p}{p-1}} \leqslant \max_{\varphi\in W^{1,p}_{X,0}(\Omega),\varphi\neq 0} \left(\frac{\int_\Omega f\,\varphi dx}{\left(\int_\Omega |X\varphi|^p\right)^{1/p}}\right)^{\frac{p}{p-1}} \tag{13.4.5}$$

which together with (13.4.3) gives

$$E_p = \max_{\varphi\in W^{1,p}_{X,0}(\Omega),\varphi\neq 0} \left(\frac{\int_\Omega f\,\varphi dx}{\left(\int_\Omega |X\varphi|^p\right)^{1/p}}\right)^{\frac{p}{p-1}},$$

that is the anisotropic analogous of the so-called *Thompson principle* (cf. [39]). Using equation (13.4.4) we have

$$E_p = \int_\Omega \langle V, Xu_p\rangle\,dx,$$

where $V \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^m)$ is any vector-valued function satisfying $-\operatorname{div}_X(V) = f$. By Hölder's inequality

$$E_p \leqslant \int_\Omega |V|^{\frac{p}{p-1}}$$

with equality if $V = |Xu_p|^{p-2}Xu_p$. Therefore the Thompson principle is equivalent to the

Dirichlet principle given by

$$E_p = \min\left\{\int_\Omega |V|^{\frac{p}{p-1}} dx \;:\; V \in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^m), \quad -\operatorname{div}_X(V) = f\right\}. \tag{13.4.6}$$

Lemma 13.4.1. *The function $p \to (|\Omega|^{-1}E_p)^{\frac{p-1}{p}}$ is monotonically decreasing as $p \to +\infty$.*

Proof. Let $1 < q < p$. For all V in $L^{\frac{q}{q-1}}(\Omega,\mathbb{R}^m)$ such that $-\operatorname{div}_X(V) = f$, we have

$$(|\Omega|^{-1}E_p)^{\frac{p-1}{p}} \leqslant \left(|\Omega|^{-1}\int_\Omega |V|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \leqslant \left(|\Omega|^{-1}\int_\Omega |V|^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}}.$$

Then we have

$$(|\Omega|^{-1}E_p)^{\frac{p-1}{p}} \leqslant \inf_{V\in L^{q/(q-1)}(\Omega,\mathbb{R}^m),\operatorname{div}_X(V)=-f} \left(|\Omega|^{-1}\int_\Omega |V|^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \leqslant (|\Omega|^{-1}E_q)^{\frac{q-1}{q}},$$

where the last inequality follows by (13.4.6). □

By Lemma 13.4.1 we get that $\{E_p\}_p$ converges and we set $E_\infty = \lim_{p\to+\infty} E_p$. Fix $m > 1$, by Hölder's inequality we have

$$\int_\Omega |Xu_p|^m \, dx \leqslant \left(\int_\Omega |Xu_p|^p \, dx\right)^{\frac{m}{p}} |\Omega|^{1-\frac{m}{p}} = E_p^{\frac{m}{p}} |\Omega|^{1-\frac{m}{p}} \quad \text{for all } p > m. \tag{13.4.7}$$

Let us fix a non-decreasing sequence $(m_k)_k \subseteq (4,+\infty)$ with $\lim_{k\to\infty} m_k = +\infty$. By (13.4.7) and $E_\infty = \lim_{p\to+\infty} E_p$, the family $(u_p)_{p>m_k}$ is bounded in $W^{1,m_k}_{X,0}(\Omega)$ for each $k \in \mathbb{N}$. Therefore, by reflexivity, there exists a subsequence $(u_{p_h})_h$ and a function $u_\infty \in W^{1,m_k}_{X,0}(\Omega)$ such that

$$u_{p_h} \rightharpoonup u_\infty \qquad \text{in } W^{1,m_k}_{X,0}(\Omega)$$

as h goes to infinity for each $k \in \mathbb{N}$. In analogy with the homogeneous case, we call u_∞ a *variational solution*. It is now possible to repeat the same arguments of the previous section to see that $u_{p_h} \rightharpoonup u_\infty$ in $W^{1,p}_X(\Omega)$ for any $p > 4$. Moreover by (13.4.7) we conclude that

$$\|Xu_\infty\|_\infty \leqslant \lim_{p\to+\infty}\left(\frac{E_p}{|\Omega|}\right)^{\frac{1}{p}} = 1. \tag{13.4.8}$$

Therefore $u_\infty \in W^{1,\infty}_X(\Omega)$. Moreover, by Proposition 2.3.4 we know that $u_\infty \in W^{1,\infty}_X(\Omega)\cap C_0(\overline{\Omega})$. Finally, again by Proposition 2.3.4 we conclude that $u_{p_h} \to u_\infty$ uniformly on $\overline{\Omega}$.

Proof of Theorem 13.4.1. Let us consider a variational solution u_∞, relative to sequences $(m_k)_k$ and $(p_h)_h$. For the sake of simplicity, we denote p_h by p and we write $p \to \infty$ meaning that $h \to \infty$. We already know that $u_\infty \in W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$. Therefore, if we extend u_∞ to be zero outside Ω, then clearly $u_\infty \in W^{1,\infty}_X(\Omega_0)$. Hence (cf. [151]) it follows that $u_\infty \in \operatorname{Lip}_{loc}(\Omega_0, d_{\Omega_0})$. Since $\Omega \Subset \Omega_0$, we conclude that $u_\infty \in \operatorname{Lip}(\overline{\Omega}, d_{\Omega_0})$. By (13.4.8) we get

$$|u_\infty(x) - u_\infty(y)| \leqslant d_{\Omega_0}(x,y)$$

for each $x, y \in \overline{\Omega}$. Taking the infimum for $y \in \partial\Omega$ and recalling that $u_\infty(y) = 0$, we obtain

$$|u_\infty(x)| \leqslant d_{\Omega_0}(x, \partial\Omega).$$

On one hand, by (13.4.3) it follows that for $\varphi \in W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$, $\varphi \neq 0$ fixed we have

$$\frac{\int_\Omega f\,\varphi\,dx}{\left(\int_\Omega |X\varphi|^p\,dx\right)^{1/p}} \leqslant E_p^{\frac{p-1}{p}}$$

and letting $p \to +\infty$

$$\frac{\int_\Omega f\,\varphi\,dx}{\|X\varphi\|_\infty} \leqslant E_{L^\infty(\Omega)}. \tag{13.4.9}$$

On the other hand, recalling (13.4.4) and by the weak convergence, we have

$$E_\infty = \int_\Omega f u_\infty\,dx. \tag{13.4.10}$$

Combining (13.4.8), (13.4.9) and (13.4.10) we get that $\|Xu_\infty\|_{L^\infty(\Omega)} = 1$ and that

$$\int_\Omega f u_\infty\,dx \geq \int_\Omega f\varphi\,dx$$

for any $\varphi \in W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$ such that $\|X\varphi\|_{L^\infty(\Omega)} = 1$. This concludes the proof.

□

To conclude this section, in analogy with [39], we show that when $f > 0$ variationals solutions are unique and coincide with the Carnot-Carathéodory distance from the boundary of Ω. Before we need a technical lemma.

Lemma 13.4.2. *The distance function $x \mapsto d_{\Omega_0}(x, \partial\Omega)$ belongs to $W^{1,\infty}_X(\Omega) \cap C_0(\overline{\Omega})$. In particular, $d_{\Omega_0}(\cdot, \partial\Omega)$ belongs to $W^{1,p}_{X,0}(\Omega)$ for all $p \geqslant 1$. Moreover, $\|Xd_{\Omega_0}(\cdot, \partial\Omega)\|_{L^\infty(\Omega)} = 1$.*

Proof. It is well known that $d_{\Omega_0}(\cdot, \partial\Omega) \in \mathrm{Lip}(\Omega, d_{\Omega_0})$ and that $\|Xd_{\Omega_0}(\cdot, \partial\Omega)\|_\infty = 1$ (cf. [151]). Since $\mathrm{Lip}(\Omega, d_{\Omega_0}) \subseteq \mathrm{Lip}(\Omega, d_\Omega)$ and $\mathrm{Lip}(\Omega, d_\Omega) \subseteq W^{1,\infty}_X(\Omega)$ (cf. [151]), we conclude that $d_{\Omega_0}(\cdot, \partial\Omega) \in W^{1,\infty}_X(\Omega)$. Moreover, $d_{\Omega_0}(\cdot, \partial\Omega)$ is continuous and $d_{\Omega_0}(x, \partial\Omega) = 0$ for $x \in \partial\Omega$, thus $d_{\Omega_0}(x, \partial\Omega) \in C_0(\overline{\Omega})$. Finally, in order to prove that $d(x, \partial\Omega)$ belongs to $W^{1,p}_{X,0}(\Omega)$ we argue as in [61, Theorem 9.17]. □

Proposition 13.4.3. *Assume that $f > 0$ in Ω. Then there exists a unique variational solution u_∞. Moreover, every sequence $(u_{p_i})_i \subseteq (u_p)_p$ converges to u_∞ strongly in $W^{1,m}_X(\Omega)$ for any $m \geq 1$. Finally, it holds that*

$$u_\infty(x) = d_{\Omega_0}(x, \partial\Omega), \qquad \forall x \in \overline{\Omega}.$$

Proof. Let u_∞ be as in Theorem 13.4.1, relative to sequences $(m_k)_k$ and $(p_h)_h$. By Lemma 13.4.2, $d_{\Omega_0}(\cdot, \partial\Omega)$ is a suitable test function in (13.4.1), and so

$$\int_\Omega f(x)u_\infty(x)\,dx \geqslant \int_\Omega f(x)d_{\Omega_0}(x, \partial\Omega)\,dx,$$

which together with $f > 0$ in Ω gives $u_\infty(x) \geqslant d_{\Omega_0}(x, \partial\Omega)$ for all x in Ω. This inequality and (13.4.2) imply that $u_\infty = d_{\Omega_0}(\cdot, \partial\Omega)$. Fix now a sequence $(u_{p_i})_i \subseteq (u_p)_p$ and $m \geq 1$. Since every subsequence of $(u_{p_i})_i$ has a subsequence that weakly converges to $d_{\Omega_0}(\cdot, \Omega_0)$ in $W^{1,m}_X(\Omega)$, then the $(u_{p_i})_i$ weakly converges to $u_\infty = d(x, \partial\Omega)$ in $W^{1,m_0}_{X,0}(\Omega)$. In particular we gain that $(u_{p_i})_i$ converges to $d_{\Omega_0}(\cdot, \partial\Omega)$ in $C^{0,\alpha}_X(\overline{\Omega})$ for $\alpha = 1 - Q/m_0$ and $(Xu_{p_i})_i$ converges weakly in L^m to $Xd_{\Omega_0}(\cdot, \partial\Omega)$. The rest of the proof follows exactly as in the proof of [39, Part II, Proposition 2.1]. □

Corollary 13.4.4. *Let Ω_1 be a domain such that $\Omega \Subset \Omega_1 \subseteq \Omega_0$. Then*

$$d_{\Omega_1}(\cdot, \partial\Omega) = d_{\Omega_0}(\cdot, \partial\Omega) \quad \text{on } \overline{\Omega}.$$

13.4.2 The limiting PDE

In this final section, in analogy with [39], we want to understand which is the limiting partial differential equation that variational solutions have to satisfy. As in the Euclidean setting we show that the limiting equations depend on the fact that we are in the support of f or not. Indeed we show that a variational solution is ∞-harmonic outside the support of f and that it satisfies the eikonal equation inside the support of f. We begin our proof with the following result.

Proposition 13.4.5. *u_∞ is a viscosity supersolution to the eikonal equation*

$$|Xu_\infty| = 1 \qquad \text{in} \quad \{f > 0\}.$$

Proof. We begin by showing that it suffices to consider tests functions in $C^2_X(\Omega)$. Indeed, let $x_0 \in \{f > 0\}$ and $v \in C^1_X(\Omega)$ such that $u_\infty - v$ has a strict minimum at x_0 in a ball $B_R(x_0) \Subset \{f > 0\}$. Thanks to Proposition 2.1.8, there exists a sequence $(v_h)_h \in C^2_X(\Omega)$ such that $v_h \to v$ and $Xv_h \to Xv$ uniformly on $\overline{B_R(x_0)}$. Let now x_h be a minimum point of $u_\infty - v_h$ on $\overline{B_{\frac{R}{2}}(x_0)}$. Arguing as in the proof of Proposition 13.3.2, up to a subsequence we can assume that $x_h \to x_0$. Therefore, passing to the limit in

$$|Xv_h(x_h)| \geq 1,$$

thanks to uniform convergence we get that

$$|Xv(x_0)| \geq 1.$$

Hence we can work with tests functions in $C^2_X(\Omega)$. Let $x_0 \in \{f > 0\}$, $v \in C^2_X(\Omega)$ and $R > 0$ be such that $u_\infty - v$ has a strict minimum at x_0 in $B_R(x_0) \Subset \{f > 0\}$. If $u_h := u_{p_h}$ is a sequence which allows to define u_∞, then we can assume that u_h converges to u_∞ uniformly on $B_R(x_0)$. Let now x_h be a minimum point of $u_h - v$ on $\overline{B_{\frac{R}{2}}(x_0)}$. Arguing as above we can assume that, up to a subsequence, $x_h \to x_0$. Let us assume without loss of generality that $p_h > Q$ for any

$h \in \mathbb{N}$, where Q is as in Proposition 2.3.4. Then it follows that $u_h \in C^0(\Omega)$. Therefore we can apply Proposition 13.2.3 and obtain that u_h is a viscosity solution to (13.2.1), i.e.

$$|Xv(x_h)|^{p_h-2}\Delta_X v(x_h) + (p_h-2)|Xv(x_h)|^{p_h-4}Xv(x_h)\cdot X^2v(x_h)\cdot Xv(x_h)^T \leqslant -f(x_h), \quad (13.4.11)$$

and recalling that $x_h \in \{f > 0\}$, we also get $|Xv(x_h)| > 0$ for any $h \in \mathbb{N}$. Assume by contradiction that $|Xv(x_0)| < 1$, then there exists $\delta > 0$ such that $|Xv(x_0)| \leqslant 1-2\delta$ and without loss of generality we can also assume that $|Xv(x_h)| \leqslant 1-\delta$ for any $h \in \mathbb{N}$. Consequently,

$$0 \leqslant \lim_{h\to\infty}(p_h-2)|Xv(x_h)|^{p_h-4} \leqslant \lim_{h\to\infty}(p_h-2)(1-\delta)^{p_h-4} = 0. \quad (13.4.12)$$

Dividing (13.4.11) by $(p_h-2)|Xv(x_h)|^{p_h-4}$ and using (13.4.12) we conclude

$$Xv(x_0)\cdot X^2v(x_0)\cdot Xv(x_0)^T = -\infty$$

which contradicts $v \in C^2_X(\Omega)$. □

Exploiting the previous result we can prove that variational solutions are ∞-superharmonic on the entire domain.

Proposition 13.4.6. *u_∞ is a viscosity supersolution to the ∞-Laplace equation*

$$-\Delta_{X,\infty}u_\infty = 0 \qquad \text{on } \Omega.$$

Proof. Let $x_0 \in \Omega$, $v \in C^2_X(\Omega)$ and $R > 0$ be such that $u_\infty - v$ has a strict minimum at x_0 in $B_R(x_0)$. Assume without loss of generality that $|Xv(x_0)| \neq 0$. We argue exactly as in the previous proof to get that

$$-Xv(x_0)\cdot X^2v(x_0)\cdot Xv(x_0)^T \geq \frac{f(x_0)}{\lim_{h\to\infty}(p_h-2)|Xv(x_h)|^{p_h-4}}.$$

If $f(x_0) = 0$ the thesis is trivial. If instead $x_0 \in \{f > 0\}$, we know by the previous proposition that $\lim_{h\to\infty}(p_h-2)|Xv(x_h)|^{p_h-4} = +\infty$, and so the thesis follows. □

Since the notion of viscosity solution is of local nature then proceeding exactly as in the proof of Proposition 13.3.2 the following result holds.

Proposition 13.4.7. *u_∞ is a viscosity subsolution to the ∞-Laplace equation*

$$-\Delta_{X,\infty}u_\infty = 0 \qquad \text{on} \qquad \overline{\{f > 0\}}^c.$$

To conclude our investigation we show that u_∞ is a viscosity subsolution to the eikonal equation on Ω. For doing this we invoke Theorem 11.1.1, together with the fact that, thanks to (13.4.8), $\|Xu_\infty\|_\infty \leqslant 1$.

Proposition 13.4.8. *u_∞ is a viscosity subsolution to the eikonal equation*

$$|Xu_\infty| = 1 \qquad \textit{on } \Omega.$$

We summarize our results as follows.

Theorem 13.4.2. *Let u_∞ be a variational solution. Then the following facts hold.*

(i) *u_∞ is a viscosity supersolution to the ∞-Laplace equation on Ω.*

(ii) *u_∞ is a viscosity solution to the ∞-Laplace equation on $\overline{\{f > 0\}}^c$.*

(iii) *u_∞ is a viscosity subsolution to the eikonal equation on Ω.*

(iv) *u_∞ is a viscosity solution to the eikonal equation on $\{f > 0\}$.*

Chapter 14

Monge solutions to Hamilton-Jacobi equations

14.1 Introduction

We refer to [126] as main reference for this chapter. So far we have dealt with the study of viscosity solutions to at least continuous equations. The purpose of this chapter is to lay out the framework for the study of discontinuous Hamilton-Jacobi equations in Carnot groups. In this regard, we generalise the Euclidean theory by addressing the major challenges implied by the degenerate structure that characterises sub-Riemannian geometry. The study of Hamilton-Jacobi equations plays an important role in modern analysis, and its applications are related to many research areas, e.g. control theory and mathematical physics. The interested reader can find complete surveys of this topic in the monographs [198, 28, 71]. We already met the anisotropic counterpart of the prototypical Hamilton–Jacobi equation, that is the eikonal equation

$$|Du| = f(x), \tag{14.1.1}$$

where f is a continuous function. The study of this kind of equations is typically carried out in the setting of viscosity solutions (cf. [98, 96]). Thanks to the effort of many authors (cf. [98, 198, 95, 30, 34, 36, 29, 110] and references therein), problem (14.1.1) has been generalized by considering first-order differential equations of the general form

$$H(x, u, Du) = 0$$

on Ω, together with their evolutionary counterparts. Here H is a continuous function which usually satisfies suitable convexity and coercivity properties. A further step has been made by taking into account the case in which the Hamiltonian is not assumed to be continuous (cf. [174, 233, 268, 70, 62]). In all these papers the authors had to adapt the definition of viscosity solutions taking into account the new measurable setting. In particular, in [233] the authors introduced the notion of *Monge solution* to the eikonal-type equation

$$H(Du) = n(x)$$

on Ω, where H is convex and continuous and n is lower semicontinuous. The importance of this notion, which is shown by the authors to be equivalent to the viscosity one when n is continuous, is motivated by the fact that the classical *Hopf–Lax formula* (cf. [198]) does not provide in general a viscosity solution if n is only lower semicontinuous. On the other hand, the setting of Monge solutions is shown to be the right one to establish existence, uniqueness, comparison and stability results. The results in [233] have been later generalized in [62], where the authors extended the notion of Monge solution to discontinuous Hamilton–Jacobi equations of the form

$$H(x, Du) = 0 \tag{14.1.2}$$

on Ω. Here H is only assumed to be Borel measurable, together with some mild assumptions in the gradient variable. As already pointed out, a sub-Riemannian approach to the study of Hamilton-Jacobi equations is very useful in order to avoid typical coercivity assumptions. The sub-Riemannian eikonal equation has been studied in the viscosity setting in [122] in general Carnot–Carathéodory spaces, whereas more general equations has been considered for instance in [208, 273, 49, 269, 78, 45, 31]. In the broad generality of metric spaces the notion of viscosity solution to Hamilton–Jacobi equations has been studied by [156, 10]. In [149] the authors introduced a different notion of metric viscosity solution for continuous Hamiltonians $H(x, u, |Du|)$ based on the local metric slope $|Du|$, that is a generalized notion of the gradient norm of u in metric spaces, and they showed several comparison and existence results. Moreover, in [200] the authors studied the eikonal equation (14.1.1) in complete and rectifiably connected metric spaces, providing the equivalence between their notion of viscosity solutions and Monge solutions when the right hand side f is continuous with respect to the metric distance. To the best of our knowledge, in a general metric space a notion of metric gradient is not available, and only the local metric slope $|Du|$ can be considered (cf. [12]). Accordingly, the last entry of the metric Hamiltonian is a scalar and not a vector. However, in Carnot–Carathéodory spaces, that are examples of length metric spaces, in light of the functional framework introduced in Chapter 1, general stationary discontinuous Hamiltonians $H(x, Xu)$ can be considered. In this chapter, inspired by [62], we study sub-Riemannian Hamilton–Jacobi equations of the form

$$H(x, Xu) = 0 \tag{14.1.3}$$

on Ω, where here and in the following Ω denotes a subdomain of a Carnot group $\mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ of step k, rank m and dimension $n \geq m$. Moreover, we fix an adapted basis $X_1, \ldots, X_n$ that coincides with the canonical basis of $\mathbb{R}^n$ at the origin. We endow $\mathbb{G}$ the standard sub-Riemannian structure associated to $X_1, \ldots, X_n$ (cf. Section 3.2). The Hamiltonian

$$H : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}$$

satisfies the following structural assumptions (H):

(H_1) $H : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is Borel measurable;

(H_2) The set

$$Z(x) := \{p \in \mathbb{R}^m \ : \ H(x,p) \leqslant 0\}$$

is closed, convex and $\partial Z(x) = \{p \in \mathbb{R}^m \ : \ H(x,p) = 0\}$ for any $x \in \Omega$;

(H_3) There exist $\alpha > 1$ such that

$$\hat{B}_{\frac{1}{\alpha}}(0) \subset Z(x) \subset \hat{B}_{\alpha}(0)$$

for any $x \in \Omega$, where $\hat{B}_{\alpha}(0)$ is the Euclidean open ball of radius α centered at the origin in $\mathbb{R}^m$.

The structural assumptions (H) allows us to associate a suitable norm to the Hamiltonian H. More precisely, inspired by [62], we define $\sigma^\star : \Omega \times \mathbb{R}^m \longrightarrow [0, \infty)$ by

$$\sigma^\star(x,p) = \sup\{\langle -\xi, p\rangle \ : \ \xi \in Z(x)\} \tag{14.1.4}$$

for any $x \in \Omega$ and any $p \in \mathbb{R}^m$. It is easy to observe that $\sigma^\star$ is a sub-Finsler norm defined on the *horizontal bundle* $\mathcal{G}\Omega$, that is the subbundle of $T\Omega$ of the horizontal vector fields. Accordingly, we exploit $\sigma^\star$ to induce a distance $d_{\sigma^\star}$ on Ω, whose Euclidean counterpart is known in literature as *optical length function*, by

$$d_{\sigma^\star}(x,y) = \inf\left\{\int_0^1 \sigma^\star(\gamma(t), \dot\gamma(t))\, dt \ : \ \gamma : [0,1] \longrightarrow \Omega,\ \gamma \text{ is horizontal},\ \gamma(0) = x,\ \gamma(1) = y\right\} \tag{14.1.5}$$

for each $x, y \in \Omega$. Being Ω open and connected, Theorem 2.2.6 implies that $d_{\sigma^\star}$ is finite for any $x, y \in \Omega$, since every two points in an open and connected set can be joined by a horizontal curve. Again inspired by [233, 62], we are ready to state our main definition.

Definition 14.1.1 (Monge solution). *Let $\Omega \subset \mathbb{G}$ be an open and connected subset of $\mathbb{G}$. If $u \in C(\Omega)$, we say that u is a* Monge solution *(resp.* subsolution,supersolution*) to* (14.2.1) *in Ω if*

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) + d_{\sigma^\star}(x_0, x)}{d_\Omega(x_0, x)} = 0 \quad (\textit{resp.} \ \ \geq, \leqslant) \tag{14.1.6}$$

for any $x_0 \in \Omega$, where d_Ω is the Carnot-Carathéodory distance *on Ω.*

The aim of this chapter is to investigate the main aspects of this definition in the sub-Riemannian setting, recovering the Euclidean results achieved in [62]. A first step consists in relating this notion to notion of viscosity solution introduced in Chapter 10. To this aim, after describing some properties of the optical length function (14.1.5) (cf. Section 14.2), and exploiting some results from Chapter 11, we show that the theory of Monge solutions embeds the theory of viscosity solutions, proving the equivalence of these two notion as soon as the Hamiltonian is continuous.

Theorem 14.1.2. *Let $\Omega \subseteq \mathbb{G}$ be a domain. Let H be a continuous Hamiltonian satisfying* (H)*. Then $u \in C(\Omega)$ is a Monge subsolution (resp. supersolution) to* (14.1.3) *if and only if it is a viscosity subsolution (resp. supersolution) to* (14.1.3)*.*

Despite some similarities with the Euclidean method, the sub-Riemannian structure requires some adjustments. Indeed, in order to prove Theorem 14.1.2, we first need to recover a suitable Hopf–Lax formula for the Dirichlet problem associated to (14.1.3). In this respect, the first striking difference with the Euclidean environment emerges. Indeed, in the classical theory of Monge solutions (cf. [233, 62]) the optical length function is defined on the whole $\overline{\Omega}$. This possibility relies on the fact that every two points in $\overline{\Omega}$ can be joined by an Euclidean Lipschitz curve as soon as the boundary of Ω is locally Lipschitz. Unfortunately this property is no longer true in our setting, since it is not always the case that two points on $\partial\Omega$ can be connected by a horizontal curve lying in $\overline{\Omega}$. A useful consequence of the Euclidean approach is that the optical length function is a geodesic distance (cf. Section 14.1.1), which is no longer true in our case. The solution to this first major problem relies on some delicate localisation arguments. The key point in which one would like to exploit the fact that the optical length function is defined up to the boundary is the validity of the classical Hopf–Lax formula. To be more precise, in the Euclidean setting it is the case (cf. [62, Theorem 5.3]) that if Ω is a bounded domain with Lipschitz boundary and $g \in C(\partial\Omega)$ satisfies the compatibility condition

$$g(x) - g(y) \leqslant d_{\sigma^\star}(x, y)$$

for any $x, y \in \partial\Omega$. Then (cf. [62, Theorem 5.3]) the function w defined by

$$w(x) = \inf_{y \in \partial\Omega} \{d_{\sigma^\star}(x, y) + g(y)\}$$

is a Monge solution to (14.1.3) and coincides with g on $\partial\Omega$. Since our optical length function is defined only on Ω, this formula would become meaningless. We overcome this difficulty by suitably extending our original Hamiltonian. To this aim, we let

$$\mathcal{K}(H, \Omega) := \{K : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R} \,:\, K \text{ satisfies (H) and } K \equiv H \text{ on } \Omega \times \mathbb{R}^m\}. \qquad (\mathcal{K})$$

Notice that $\mathcal{K}(H, \Omega)$ is always non-empty, as every Hamiltonian can be extended to the whole $\mathbb{R}^n \times \mathbb{R}^m$ by letting $H(x, p) = |\xi| - \alpha$ outside $\Omega \times \mathbb{R}^m$. For any fixed $K \in \mathcal{K}(H, \Omega)$, we consider the associated metric $\sigma^\star_K$ and optical length function $d_{\sigma^\star_K}$. The advantage of this approach consists in the fact that, in view of the aforementioned Chow–Rashevskii connectivity theorem, every two points in $\mathbb{R}^n$ can be connected by a horizontal curve. Therefore, $d_{\sigma^\star_K}$ is actually a finite distance on the whole $\mathbb{R}^n$, whence in particular on $\overline{\Omega}$. Surprisingly (cf. Proposition 14.3.2), the definition of Monge solution on Ω is invariant by replacing H with any $K \in \mathcal{K}(H, \Omega)$. In this way, what *a priori* constitutes a considerable problem ensures *a posteriori* a more accurate understanding of Monge's notion of solution. These facts motivate the following result.

Theorem 14.1.3 (Hopf–Lax formula)**.** *Let $\Omega \subseteq \mathbb{G}$ be a domain and let H satisfy* (H)*. Let*

$g \in C(\partial\Omega)$ be bounded and such that there exists $K \in \mathcal{K}_0(\Omega)$ for which

$$g(x) - g(y) \leqslant d_{\sigma_K^\star}(x, y) \tag{14.1.7}$$

for any $x, y \in \partial\Omega$. Let us define

$$w(x) := \inf_{y \in \partial\Omega} \{d_{\sigma_K^\star}(x, y) + g(y)\}. \tag{14.1.8}$$

Then $w \in \mathrm{Lip}(\Omega, d_\Omega) \cap C(\overline{\Omega})$ and w is a Monge solution to the Dirichlet problem

$$\begin{aligned} H(x, Xw) = 0 \quad & \text{in } \Omega \\ w = g \quad & \text{on } \partial\Omega. \end{aligned}$$

Notice that the compatibility condition (14.1.7) is trivially necessary for the function w given by (14.1.8) to attain the boundary datum g on $\partial\Omega$. After proving Theorem 14.1.2 and Theorem 14.1.3, we continue the study of Hamilton–Jacobi equations in the discontinuous setting. First, we show the validity of the following comparison principle for Monge solutions.

Theorem 14.1.4 (Comparison Principle)**.** *Let $\Omega \subseteq \mathbb{G}$ be a bounded domain. Let H be an Hamiltonian satisfying* (H)*, let $u \in C(\bar{\Omega})$ be a Monge subsolution of* (14.2.1) *and $v \in C(\bar{\Omega})$ be a Monge supersolution of* (14.2.1)*. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.*

Notice that, combining Theorem 14.1.3 and Theorem 14.1.4, we guarantee existence and uniqueness for the Dirichlet problem associated to (14.1.3) under the compatibility condition (14.1.7). Finally, inspired by [62], we show that the notion of Monge solution is stable under suitable notions of convergence for sequences of Hamiltonians and Monge solutions.

Theorem 14.1.5 (Stability)**.** *Let $\Omega \subseteq G$ be a domain. Let $(H_n)_{n\in\mathbb{N}}$ and H_∞ satisfy* (H) *with a uniform choice of α. For any $n \in \mathbb{N}$, let $u_n \in C(\Omega)$ be a Monge solution to*

$$H_n(x, Xu_n(x)) = 0$$

on Ω. Assume that $d_{\sigma_n^\star} \to d_{\sigma_\infty^\star}$ locally uniformly on $\Omega \times \Omega$, where, for any $n \in \mathbb{N}$, $d_{\sigma_n^\star}$ is the optical length function associated to H_n and $d_{\sigma_\infty^\star}$ is the optical length function associated to H_∞. Assume that there exists $u_\infty \in C(\Omega)$ such that $u_n \to u_\infty$ locally uniformly on Ω. Then u_∞ is a Monge solution to

$$H_\infty(x, Xu_\infty(x)) = 0$$

on Ω.

14.1.1 Length and geodesic distances

Let us briefly recall some general facts about metric spaces for the sake of completeness. We refer to [64] as main reference. Let (M, d) be a possibly non-symmetric metric space. We stress that, in light of [64, Remark 2.2.6], the statements of [64, Chapter 2] which we are going to recall hold as well in the non-symmetric setting. If $\gamma : [0,T] \longrightarrow M$ is a continuous curve, we define its *length* by

$$\mathrm{L}_d(\gamma) = \sup\left\{\sum_{j=1}^{s} d\left(\gamma(t_{j-1}), \gamma(t_j)\right) \,:\, 0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_{s-1} \leqslant t_s = T\right\}. \tag{14.1.9}$$

The length functional L_d is lower semicontinuous with respect to the uniform convergence of continuous curves (cf. [64, Proposition 2.3.4], and allows to define a second distance, say d_{L_d}, by letting

$$d_{\mathrm{L}_d}(x,y) = \inf\left\{\mathrm{L}_d(\gamma) \,:\, \gamma : [0,1] \longrightarrow M \text{ is } d\text{-Lipschitz}, \ \gamma(0) = x \text{ and } \gamma(1) = y\right\}. \tag{14.1.10}$$

Accordingly, (M,d) is a *length space* (cf. [64, Definition 2.1.6]) whenever

$$d = d_{\mathrm{L}_d}, \tag{14.1.11}$$

and it is a *geodesic*, or *complete*, space (cf. [64, Definition 2.1.10]) whenever it is a length space such that the infimum in (14.1.10) is attained by a suitable d-Lipschitz curve. We shall refer to such curves as *optimal curves*. We point out (cf. [64, Section 2.5.2]) that, if $x, y \in M$ and $\gamma : [0,1] \longrightarrow M$ is an optimal curve for d connecting x to y, then

$$d(x,y) = d(x,\gamma(t)) + d(\gamma(t),y) \tag{14.1.12}$$

for any $t \in [0,1]$.

14.2 Some properties of $\sigma^\star$ and $d_{\sigma^\star}$

Let us consider the Hamilton–Jacobi equation

$$H(x, Xu) = 0 \tag{14.2.1}$$

on Ω, where Ω is a subdomain of $\mathbb{G}$ and H satisfies the structural assumptions (H).

Since the notion of Monge solution heavily depends on the properties of the associated optical length function, and hence on the properties of $\sigma^\star$, let us make some preliminary considerations on these objects. First, notice that condition (H_3) is equivalent to the estimate

$$\frac{1}{\alpha}|v|_x \leqslant \sigma^\star(x,v) \leqslant \alpha|v|_x \quad \text{for every } (x,v) \in H\mathbb{G}. \tag{14.2.2}$$

Moreover the following simple result, which is the sub-Riemannian analogous of [62, Lemma 4.2], will be useful to state the equivalence between Monge and viscosity solutions in the continuous setting. We refer to [127] for an account of sub-Finsler metrics.

Lemma 14.2.1. *$\sigma^\star : \mathcal{G}\Omega \longrightarrow \mathbb{R}$ is a sub-Finsler convex metric. Moreover, for any $v \in \mathbb{R}^m$, the following hold.*

(i) If H is upper semicontinuous on $\mathcal{G}\Omega$, then $\sigma^\star(\cdot, v)$ is lower semicontinuous on Ω.

(ii) If H is lower semicontinuous on $\mathcal{G}\Omega$, then $\sigma^\star(\cdot, v)$ is upper semicontinuous on Ω.

Regarding the optical length function, an easy computation shows that

$$d_{\sigma^\star}(x, y) = \inf \left\{ \int_0^T \sigma^\star(\gamma(t), \dot\gamma(t))\, dt \ : \ \gamma : [0, T] \longrightarrow \Omega \text{ is sub-unit, } \gamma(0) = x\,, \gamma(T) = y \right\} \tag{14.2.3}$$

for any $x, y \in \Omega$. The quantity (14.2.3) is well-defined, both because the map $t \mapsto \sigma^\star(\gamma(t), \dot\gamma(t))$ is Borel measurable on the horizontal bundle, and because, as already mentioned, every two points in Ω can be connected by a horizontal curve. However, $d_{\sigma^\star}$ can presents some pathological behaviour without some semicontinuity assumptions (cf. [127, Example 5.5]). Let us discuss some properties of $d_{\sigma^\star}$ which will be useful in the sequel.

Lemma 14.2.2. *The following properties hold.*

(i) *$d_{\sigma^\star}$ is a non-symmetric distance on Ω.*

(ii) *$d_{\sigma^\star}$ is equivalent to d_Ω on Ω, i.e.*

$$\frac{1}{\alpha}\, d_\Omega(x, y) \leqslant d_{\sigma^\star}(x, y) \leqslant \alpha\, d_\Omega(x, y)$$

for any $x, y \in \Omega$.

(iii) *$d_{\sigma^\star}$ is d_Ω-Lipschitz on $\Omega \times \Omega$, that is*

$$|d_{\sigma^\star}(x, y) - d_{\sigma^\star}(z, w)| \leqslant \alpha(d_\Omega(x, z) + d_\Omega(y, w))$$

for any $x, y, z, w \in \Omega$.

Proof. The proof of (i) follows as in [127, Lemma 5.7]. (ii) is an easy consequence of estimate (14.2.2). Let us show (iii). To this aim, fix $x, y, z, w \in \Omega$. Being $d_{\sigma^\star}$ a distance and thanks to point (ii), we have that

$$\begin{aligned} d_{\sigma^\star}(x, y) - d_{\sigma^\star}(z, w) &= d_{\sigma^\star}(x, y) - d_{\sigma^\star}(z, y) + d_{\sigma^\star}(z, y) - d_{\sigma^\star}(z, w) \\ &\leqslant d_{\sigma^\star}(x, z) + d_{\sigma^\star}(w, y) \\ &\leqslant \alpha(d_\Omega(x, z) + d_\Omega(y, w)) \end{aligned}$$

and

$$\begin{aligned} d_{\sigma^\star}(z, w) - d_{\sigma^\star}(x, y) &= d_{\sigma^\star}(z, w) - d_{\sigma^\star}(x, w) + d_{\sigma^\star}(x, w) - d_{\sigma^\star}(x, y) \\ &\leqslant d_{\sigma^\star}(z, x) + d_{\sigma^\star}(y, w) \\ &\leqslant \alpha(d_\Omega(x, z) + d_\Omega(y, w)). \end{aligned}$$

□

Therefore $(\Omega, d_{\sigma^\star})$ is a non-symmetric metric space. In view of general results in metric spaces (cf. [64, Proposition 2.4.1], $d_{\sigma^\star}$ is a length distance in the sense of Section 14.1.1. However, we already know that it is not geodesic in general, since, for instance, (Ω, d_Ω) may not be geodesic. Nevertheless, exploiting standard arguments of analysis in metric spaces (cf. [17]) it can be shown that $(\Omega, d_{\sigma^\star})$ is locally geodesic in the following sense.

Proposition 14.2.3. *For any $x_0 \in \Omega$ there exists $r > 0$ such that for any $x, y \in B_{d_\Omega}(x_0, r)$ there exists a horizontal curve $\gamma : [0,1] \longrightarrow \Omega$, $\gamma(0) = x$, $\gamma(1) = y$ and*

$$d_{\sigma^\star}(x, y) = \mathrm{L}_{d_{\sigma^\star}}(\gamma),$$

where $\mathrm{L}_{d_{\sigma^\star}}$ is as in (14.1.9).

We first need the following technical lemma, whose proof is omitted being analogous to the proof of the forthcoming Lemma 14.3.1.

Lemma 14.2.4. *For any $x_0 \in \Omega$, and for any $R > 0$ such that $B_{d_\Omega}(x_0, R) \Subset \Omega$, there exists $0 < r < R$ and $\bar{\varepsilon} > 0$ such that, for any $x, y \in B_{d_\Omega}(x_0, r)$ and for any $0 < \varepsilon < \bar{\varepsilon}$, every horizontal curve $\gamma : [0,1] \longrightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$ and*

$$d_{\sigma^\star}(x, y) \geq \mathrm{L}_{d_{\sigma^*}}(\gamma) - \varepsilon$$

lies in $B_{d_\Omega}(x_0, R)$.

Proof of Proposition 14.2.3. Let $x_0 \in \Omega$ and $R > 0$ be such that $B_{d_\Omega}(x_0, R) \Subset \Omega$. Then let $r > 0$ be as in Lemma 14.2.4. Let $x, y \in B_{d_\Omega}(x_0, r)$ and let $(\gamma_h)_h$ be a sequence of horizontal curves such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$\mathrm{L}_{d_{\sigma^*}}(\gamma_h) \leqslant d_{\sigma^\star}(x, y) + \frac{1}{h}. \tag{14.2.4}$$

In view of Lemma 14.2.4, we can assume that $\gamma_h([0, T_h]) \subseteq \overline{B_{d_\Omega}(x_0, R)} \subseteq \Omega$ for any $h \in \mathbb{N}$. Clearly $(\overline{B_R(x_0, d_\Omega)}, d_{\sigma^\star})$ is a compact metric space and the sequence $(\gamma_h)_{h\in\mathbb{N}}$ is uniformly bounded. Arguing *verbatim* as in the proof of [17, Theorem 4.3.2], $(\gamma_h)_{h\in\mathbb{N}}$ is also equicontinuous with respect to $d_{\sigma^\star}$. Therefore Ascoli–Arzelà's theorem implies the existence of a horizontal curve $\gamma : [0,1] \longrightarrow \Omega$ such that $(\gamma_h)_h$ converges uniformly to γ. In particular, $\gamma(0) = x$ and $\gamma(1) = y$. Hence, combining (14.1.11) and (14.2.4) with the lower semicontinuity of $\mathrm{L}_{d_{\sigma^\star}}$ (cf. Section 14.1.1) we infer that

$$d_{\sigma^\star}(x, y) \leqslant \mathrm{L}_{d_{\sigma^\star}}(\gamma) \leqslant \liminf_{h\to\infty} \mathrm{L}_{d_{\sigma^\star}}(\gamma_h) \leqslant \liminf_{h\to\infty} \left(d_{\sigma^\star}(x, y) + \frac{1}{h} \right) = d_{\sigma^\star}(x, y).$$

Therefore we conclude that γ is optimal for $d_{\sigma^\star}(x, y)$, and the thesis follows. □

Proposition 14.2.5. *Assume that H is upper semicontinuous on $\mathcal{G}\Omega$. Then it holds that*

$$\liminf_{t\to 0^+} \frac{d_{\sigma^\star}(x, x\cdot\delta_t(\xi,\eta))}{t} \geq \sigma^\star(x,\xi)$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^{n-m}$.

Proof. Let us fix $x \in \Omega$, $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^{n-m}$. Since H is upper semicontinuous on $\mathcal{G}\Omega$, then $\sigma^\star(\cdot,\xi)$ is lower semicontinuous on Ω by Lemma 14.2.1. This is equivalent to say that, for any $\varepsilon > 0$ and for any $\tilde{\xi} \in S^{m-1}$, there exists $r = r(x,\epsilon,\tilde{\xi})$ such that $\sigma^\star(y,\tilde{\xi}) \geq \sigma^\star(x,\tilde{\xi}) - \varepsilon$ for any $y \in B_{d_\Omega}(x,r)$. Recalling that $\sigma^\star$ is Lipschitz in the second entry and exploiting a standard compactness argument, we infer that for any $\varepsilon > 0$ there exists $r = r(x,\varepsilon) > 0$ such that

$$\sigma^\star(y,\tilde{\xi}) \geq \sigma^\star(x,\tilde{\xi}) - \varepsilon \tag{14.2.5}$$

for any $y \in B_{d_\Omega}(x,r)$ and any $\tilde{\xi} \in S^{m-1}$. Let us choose a sequence of sub-unit curves $\gamma_h : [0,t_h] \longrightarrow \Omega$ in such a way that $\gamma_h(0) = x$, $\gamma_h(t_h) = x\cdot\delta_{t_h}(\xi,\eta)$ and

$$\liminf_{t\to 0^+} \frac{d_{\sigma^\star}(x, x\cdot\delta_t(\xi,\eta))}{t} = \liminf_{h\to\infty} \fint_0^{t_h} \sigma^\star(\gamma_h(t),\dot{\gamma}_h(t))\,dt.$$

Since $\lim_{t\to 0^+} x\cdot\delta_t(\xi,\eta) = x$, and in view of Lemma 14.2.4, the sequence of curves can be choosen in such a way that $\gamma_h([0,t_h]) \subseteq B_{d_\Omega}(x,r)$ for any $h \in \mathbb{N}$. Therefore, exploiting (14.2.5), we infer that

$$\liminf_{h\to\infty} \fint_0^{t_h} \sigma^\star(\gamma_h(t),\dot{\gamma}_h(t))\,dt \geq \liminf_{h\to\infty} \fint_0^{t_h} \sigma^\star(x,\dot{\gamma}_h(t))\,dt - \varepsilon.$$

For any $h \in \mathbb{N}$, set $\gamma_h = (\gamma_h^1,\ldots,\gamma_h^m,\gamma_h^{m-1},\ldots,\gamma_h^n)$. We recall that in the previous equations $\dot{\gamma}_h$ is the m-tuple of the components of $\dot{\gamma}_h$ along the generating vector fields. In other words, we mean $\dot{\gamma}_h(t) = (a_h^1(t),\ldots,a_h^m(t))$, where $\dot{\gamma}_h(t) = \sum_{j=1}^m a_h^j(t)X_j(\gamma_h(t))$. It is then easy to see that $\dot{\gamma}_h^j = a_h^j$ for any $j = 1,\ldots,m$. Therefore, by Proposition 3.3.2 and the fundamental theorem of calculus for absolutely continuous functions, we infer that

$$\fint_0^{t_h} \dot{\gamma}_h(t)\,dt = \frac{\pi(\gamma_h(t_h)) - \pi(\gamma(0))}{t_h} = \xi \tag{14.2.6}$$

for any $h \in \mathbb{N}$, where π is the projection map defined in (3.2.2). Combining (14.2.6) with the convexity properties of $\sigma^\star$ and Jensen's inequality, we get that

$$\liminf_{h\to\infty} \fint_0^{t_h} \sigma^\star(x,\dot{\gamma}_h(t))\,dt - \varepsilon \geq \liminf_{h\to\infty} \sigma^\star\left(x,\ \fint_0^{t_h} \dot{\gamma}_h(t)\,dt\right) - \varepsilon = \sigma^\star(x,\xi) - \varepsilon.$$

The thesis follows letting ε go to 0. □

14.3 A Hopf-Lax formula for the Dirichlet problem

As already mentioned, the properties of Monge subsolutions and supersolutions strictly depend on those enjoyed by the optical length function $d_{\sigma^\star}$. Moreover, as it happens in the viscosity setting, d_Ω can be equivalently replaced by $d_{\mathfrak{g}}$ or $d_{\mathbb{G}}$. Now we explain how to replace $d_{\sigma^\star}$ with suitable extensions as already explained in the introduction. According to the latter, we recall the family defined in ($\mathcal{K}$), that is

$$\mathcal{K}(H,\Omega) := \{K : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R} \,:\, K \text{ satisfies (H) and } K \equiv H \text{ on } \Omega \times \mathbb{R}^m\}.$$

As already mentioned, $\mathcal{K}(H,\Omega)$ is always non-empty. For instance, it is always the case that

$$K(x,\xi) := \begin{cases} H(x,\xi) & \text{if } (x,\xi) \in \Omega \times \mathbb{R}^m \\ |\xi| - \alpha & \text{otherwise} \end{cases} \tag{14.3.1}$$

belongs to $\mathcal{K}(H,\Omega)$. For a fixed $K \in \mathcal{K}(H,\Omega)$, we can consider the associated $\sigma_K^\star$ and $d_{\sigma_K^\star}$. We want to show that the notion of Monge solution is independent of the choice of $K \in \mathcal{K}$. To this aim, we prove the following preliminary result.

Lemma 14.3.1. *For any $K \in \mathcal{K}$, for any $x_0 \in \Omega$ and for any $R > 0$ such that $B_{d_\Omega}(x_0,R) \subseteq \Omega$ there exists $r > 0$ and $\bar\varepsilon > 0$ such that, for any $x \in B_{d_\Omega}(x_0,r)$ and for any $0 < \varepsilon < \bar\varepsilon$, any curve $\gamma : [0,T] \longrightarrow \mathbb{R}^n$ such that γ is sub-unit, $\gamma(0) = x_0$, $\gamma(T) = x$ and*

$$d_{\sigma_K^\star}(x_0,x) \geq \int_0^T \sigma_K^\star(\gamma(t),\dot\gamma(t))\, dt - \varepsilon$$

lies in $B_{d_\Omega}(x_0,R)$.

Proof. Assume by contradiction that there exists $K : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $K \in \mathcal{K}$, $x_0 \in \Omega$, $R > 0$ with $B_{d_\Omega}(x_0,R) \subseteq \Omega$ and sequences $(x_h)_h$ and $\gamma_h : [0,T_h] \longrightarrow \mathbb{R}^n$ sub-unit such that $\gamma_h(0) = x_0$, $\gamma_h(T_h) = x_h$ and

$$d_{\sigma_K^\star}(x_0,x_h) \geq L_{\sigma_K^\star}(x_0,x_h) - \frac{1}{h}$$

such that $x_h \to x_0$ and for any h there exists $0 < t_h < T_h$ such that $z_h := \gamma(t_h) \in \partial B_{d_\Omega}(x_0,R)$. Since K satisfies (H) on $\mathbb{R}^n$, it follows that

$$d_{\sigma_K^\star}(x_0,x_h) \leqslant \alpha d_{\mathbb{G}}(x_0,x_h) \leqslant \alpha d_\Omega(x_0,x_h) \to 0$$

as $h \to \infty$. On the other hand, in view of the choice of γ_h and Proposition 3.7.1, there exists $C > 0$ such that

$$d_{\sigma^\star}(x_0,x_h) \geq d_{\sigma^\star}(x_0,z_h) + d_{\sigma^\star}(z_h,x_h) - \frac{1}{h} \geq \frac{1}{\alpha} d_{\mathbb{G}}(x_0,z_h) - \frac{1}{h} \geq \frac{C}{\alpha} d_\Omega(x_0,z_h) - \frac{1}{h} \geq \frac{CR}{2\alpha}$$

for any h big enough. A contradiction then follows. $\square$

Proposition 14.3.2. *Let $K : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be such that $K \in \mathcal{K}$. A function $u \in C(\Omega)$ is a* Monge solution *(resp.* subsolution, supersolution*) to* (14.2.1) *in Ω if and only if*

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) + d_{\sigma_K^\star}(x_0, x)}{d_{\mathbb{G}}(x_0, x)} = 0 \quad (\textit{resp.} \;\; \geq, \leqslant) \tag{14.3.2}$$

for any $x_0 \in \Omega$.

Proof. It suffices to observe that, thanks to Lemma 14.3.1 and the definition of $\mathcal{K}(H, \Omega)$, for any $x_0 \in \Omega$ there exists $r > 0$ such that

$$d_{\sigma^\star}(x_0, x) = d_{\sigma_K^\star}(x_0, x)$$

for any $x \in B_{d_\Omega}(x_0, r)$ and for any $K \in \mathcal{K}(H, \Omega)$. □

Thanks to the results of Section 14.2, we are in position to prove Theorem 14.1.3. The proof of this result is inspired by [62].

Proof of Theorem 14.1.3. Let $K : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be as in the statement. First, notice that, since g is bounded, then w is well-defined. Fix $x, z \in \Omega$ and, for any $h \in \mathbb{N}_+$, let $y_h \in \partial\Omega$ be such that $w(z) \geq d_{\sigma_K^\star}(z, y_h) + g(y_h) - \frac{1}{h}$. Then

$$w(x) - w(z) \leqslant d_{\sigma_K^\star}(x, y_h) - d_{\sigma_K^\star}(z, y_h) + \frac{1}{h} \leqslant d_{\sigma_K^\star}(x, z) + \frac{1}{h} \leqslant d_{\sigma^\star}(x, z) + \frac{1}{h} \leqslant \alpha d_\Omega(x, z) + \frac{1}{h}.$$

Letting $h \to \infty$, and since $w(z) - w(x)$ can be estimated similarly, we conclude that $w \in \mathrm{Lip}(\Omega, d_\Omega)$. Fix $x \in \partial\Omega$. Then, by definition of w, it follows that $w(x) \leqslant g(x)$. On the other hand, if $y \in \partial\Omega$, (14.1.7) implies that

$$d_{\sigma_K^\star}(x, y) + g(y) \geq g(x),$$

and so, taking the infimum over $\partial\Omega$, we conclude that $w(x) \geq g(x)$. Therefore $w = g$ on $\partial\Omega$. Let now $x \in \partial\Omega$ and let $(x_h)_h \subseteq \Omega$ be such that $x_h \to x$ as $h \to \infty$. Then

$$w(x_h) - w(x) \leqslant d_{\sigma_K^\star}(x_h, x) + g(x) - g(x) \leqslant \alpha d_{\mathbb{G}}(x_h, x)$$

and there exists $(y_h)_h \subseteq \partial\Omega$ such that

$$w(x) - w(x_h) \leqslant g(x) - g(y_h) - d_{\sigma_K^\star}(x_h, y_h) + \frac{1}{h} \leqslant d_{\sigma_K^\star}(x, y_h) - d_{\sigma_K^\star}(x_h, y_h) + \frac{1}{h} \leqslant d_{\sigma_K^\star}(x, x_h) + \frac{1}{h}.$$

Hence we conclude that $w \in C(\overline{\Omega})$. Let us show that w is a Monge subsolution. To this aim, let $x_0 \in \Omega$ and let $(x_h)_h \subseteq \Omega$ be such that $x_h \to x_0$ as $h \to \infty$. For any $h \in \mathbb{N}$, by definition of w, there exists $y_h \in \partial\Omega$ such that

$$w(x_h) \geq d_{\sigma_K^\star}(x_h, y_h) + g(y_h) - \frac{\|x_0^{-1} \cdot x_h\|}{h},$$

where $\|\cdot\|$ is the homogeneous norm defined in Example 3.4.4. Therefore we infer that

$$\frac{w(x_h) - w(x_0) + d_{\sigma_K^\star}(x_0, x_h)}{\|x_0^{-1} \cdot x_h\|} \geq \frac{d_{\sigma_K^\star}(x_0, y_h) + g(y_h) - w(x_0)}{\|x_0^{-1} \cdot x_h\|} - \frac{1}{h} \geq -\frac{1}{h}.$$

Letting $h \to \infty$, being the sequence $(x_h)_h$ arbitrary and recalling Proposition 14.3.2, we infer that w is a Monge subsolution. Conversely, let $x_0 \in \Omega$ and assume without loss of generality that $B_{d_\mathfrak{g}}(x_0, \frac{1}{h}) \subseteq \Omega$ for any $h \in \mathbb{N}_+$. Fix such an h and choose $y_h \in \partial\Omega$ such that

$$w(x_0) \geq d_{\sigma_K^\star}(x_0, y_h) + g(y_h) - \frac{1}{h^2}.$$

Moreover, for any h, let $\gamma_h : [0, T_h] \longrightarrow \mathbb{R}^n$ be a sub-unit curve with the property that $\gamma_h(0) = x_0$, $\gamma_h(T_h) = y_h$ and

$$d_{\sigma_K^\star}(x_0, y_h) \geq \int_0^{T_h} \sigma_K^\star(\gamma(t), \dot\gamma(t))\, dt - \frac{1}{h^2}.$$

Pick $t_h \in (0, T_h)$ such that $\gamma(t_h) \in \partial B_{d_\mathfrak{g}}(x_0, \frac{1}{h})$ and set $x_h := \gamma(t_h)$. Then clearly $x_h \to x_0$ as $h \to \infty$ and therefore, by definition of w and the choice of $(\gamma_h)_h$, we infer that

$$w(x_h) - w(x_0) + d_{\sigma_K^\star}(x_0, x_h) \leqslant d_{\sigma_K^\star}(x_h, y_h) - d_{\sigma_K^\star}(x_0, y_h) + d_{\sigma_K^\star}(x_0, x_h) + \frac{1}{h^2} \leqslant \frac{2}{h^2}.$$

Noticing that $\|x_0^{-1} \cdot x_h\| = \frac{1}{h}$ for some $C > 0$, we conclude that

$$\liminf_{h\to\infty} \frac{w(x_h) - w(x_0) + d_{\sigma_K^\star}(x_0, x_h)}{\|x_0^{-1} \cdot x_h\|} \leqslant \liminf_{h\to\infty} \frac{2}{h} = 0,$$

and so w is a Monge supersolution. □

14.4 Monge and viscosity solutions

In this section we show that, as in the Euclidean setting (cf. [233, 62]), when H is continuous the notions of Monge and viscosity solution coincide. We begin to prove that Monge solutions are viscosity solutions.

Proposition 14.4.1. *Let H be continuous. If $u \in C(\Omega)$ is a Monge subsolution (resp. supersolution) to (14.2.1), then u is a viscosity subsolution (resp. supersolution) to (14.2.1).*

Proof. Let u be a Monge supersolution to (14.2.1), fix $x_0 \in \Omega$ and $p \in \partial_X^- u(x_0)$. Then it follows that

$$0 \geq \liminf_{x\to x_0} \frac{u(x) - u(x_0) + d_{\sigma^\star}(x_0, x)}{\|x_0^{-1} \cdot x\|} \geq \liminf_{x\to x_0} \frac{\langle p, \pi(x_0^{-1} \cdot x)\rangle + d_{\sigma^\star}(x_0, x)}{\|x_0^{-1} \cdot x\|}$$

Let $(x_h)_h$ be a minimizing sequence for the right hand side. Let us set $t_h := \|x_0^{-1} \cdot x_h\|$ and $\xi_h := \frac{1}{t_h}\pi(x_0^{-1} \cdot x_h)$. In this way, $t_h \to 0^+$ when $h \to \infty$. For any $h \in \mathbb{N}$, let $\eta_h \in \mathbb{R}^{n-m}$ be such that

$$\delta_{\frac{1}{t_h}}(x_0^{-1} \cdot x_h) = (\xi_h, \eta_h).$$

By construction, $(\delta_{\frac{1}{t_h}}(x_0^{-1}\cdot x_h))_h$ is bounded. Then there exists $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^{n-m}$ such that, up to a subsequence, $(\xi_h, \eta_h) \to (\xi, \eta)$ as $h \to \infty$. Then, by Proposition 14.2.5 and the choice of $(x_h)_h$, we infer that

$$\begin{aligned}\liminf_{x\to x_0} \frac{\langle p, \pi(x_0^{-1}\cdot x)\rangle + d_{\sigma^\star}(x_0, x)}{\|x_0^{-1}\cdot x\|} &= \liminf_{h\to\infty}\left(\langle p, \xi_h\rangle + \frac{d_{\sigma^\star}(x_0, x_h)}{t_h}\right)\\ &= \langle p, \xi\rangle + \liminf_{h\to\infty}\frac{d_{\sigma^\star}(x_0, x_0\cdot\delta_{t_h}(\xi_h,\eta_h))}{t_h}\\ &= \langle p, \xi\rangle + \liminf_{h\to\infty}\frac{d_{\sigma^\star}(x_0, x_0\cdot\delta_{t_h}(\xi,\eta))}{t_h}\\ &\geq \langle p, \xi\rangle + \sigma^\star(x_0,\xi).\end{aligned}$$

Therefore we conclude that $\langle -\xi, p\rangle \geq \sigma^\star(x_0,\xi)$. If it was the case that $H(x_0,p) < 0$, then p is an interior point of $Z(x_0)$. But then $\langle -\xi, p\rangle < \sigma^\star(x_0,\xi)$, since $q \mapsto \langle -\xi, q\rangle$ is a linear and non-constant, and so it achieves its maximum on $\partial Z(x_0)$. A contradiction then follows. Assume now that u is a Monge subsolution to (14.2.1), let $x_0 \in \Omega$ and $p \in \partial_X^+ u(x_0)$. Assume by contradiction that $H(x_0,p) > 0$. Hence, by Hahn-Banach theorem (cf. [61]), there exists $\xi \in S^{m-1}$ such that $\langle -\xi, p\rangle > \sigma^\star(x_0,\xi)$. For any $h \in \mathbb{N}\setminus\{0\}$, let $x_h := x_0\cdot\delta_{t_h}(\xi,0)$, where $(t_h)_h \subseteq (0,1)$ goes to 0 as $h \to \infty$. Then $x_h \to x_0$ as $h\to\infty$, and moreover $x_0^{-1}\cdot x_h = (t_h\xi, 0)$. Therefore, being u a Monge subsolution, it follows that

$$\begin{aligned} 0 &\leqslant \liminf_{x\to x_0}\frac{u(x)-u(x_0)+d_{\sigma^\star}(x_0,x)}{\|x_0^{-1}\cdot x\|}\\ &\leqslant \liminf_{x\to x_0}\frac{\langle p, \pi(x_0^{-1}\cdot x)\rangle + d_{\sigma^\star}(x_0,x)}{\|x_0^{-1}\cdot x\|}\\ &\leqslant \liminf_{h\to\infty}\frac{\langle p, \pi(x_0^{-1}\cdot x_h)\rangle + d_{\sigma^\star}(x_0,x_h)}{\|x_0^{-1}\cdot x_h\|}\\ &= \langle p,\xi\rangle + \liminf_{h\to\infty}\frac{d_{\sigma^\star}(x_0, x_0\cdot\delta_{t_h}(\xi,0))}{t_h}.\end{aligned}$$

Let us set $\gamma : [0,1] \longrightarrow \Omega$ by $\gamma(t) := x_0\cdot\delta_t(\xi,0)$. Notice that $\dot\gamma(t) \equiv \xi$, and so γ is sub-unit. Moreover $\gamma(0) = x_0$ and $\gamma(t_h) = x_h$. Hence, since the continuity of H implies the continuity of $\sigma^\star(\cdot,\xi)$, we infer that

$$\liminf_{h\to\infty}\frac{d_{\sigma^\star}(x_0, x_0\cdot\delta_{t_h}(\xi,0))}{t_h} \leqslant \liminf_{h\to\infty} \fint_0^{t_h}\sigma^\star(\gamma(t),\xi)\,dt = \sigma^\star(x_0,\xi).$$

Therefore we conclude that $\langle -\xi, p\rangle \leqslant \sigma^\star(x_0,\xi)$, a contradiction.

□

In order to prove the converse implication, we need some preliminary results.

Proposition 14.4.2. *Let H be continuous. Let $u \in C(\Omega)$ and assume that u is a viscosity subsolution to (14.2.1). Then $u \in W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$.*

Proof. Let $x_0 \in \Omega$ and $p \in \partial_X^+ u(x_0)$ with $p \neq 0$. Then $H(x_0,p) \leqslant 0$, which implies that

$p \in Z(x_0)$. Therefore it holds that $|p| \leqslant \alpha$ by (H_3). Hence u is a viscosity subsolution to

$$|Xu| \leqslant \alpha \tag{14.4.1}$$

on Ω. Thanks to Proposition 11.2.3 and [269, Proposition 2.1], we conclude that $u \in W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$. □

Proposition 14.4.3. *Assume that H is continuous. If u is a viscosity subsolution to* (14.2.1) *in Ω, then*

$$u(x) - u(y) \leqslant d_{\sigma^\star}(x, y) \tag{14.4.2}$$

for any $x, y \in \Omega$.

Proof. Let $x, y \in \Omega$. If $x = y$ the thesis is trivial. If instead $x \neq y$, let $\gamma : [0, T] \longrightarrow \Omega$ be a sub-unit curve such that $\gamma(0) = x$ and $\gamma(T) = y$ for some $T > 0$. Thanks to Proposition 14.4.2 and Proposition 11.2.2 we know that $u \in W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$ and that

$$H(z, Xu(z)) \leqslant 0 \tag{14.4.3}$$

for almost every $z \in \Omega$. Let N be a Lebesgue negligible subset of Ω containing all the non-Lebesgue points of Xu and all the points where (14.4.3) does not hold. Then, in view of Proposition 4.4.1, we infer that $H(z, p) \leqslant 0$ for any $z \in \Omega$ and for any $p \in \partial_{X,N} u(z)$. Therefore, in particular,

$$p \in Z(z) \tag{14.4.4}$$

for any $z \in \Omega$ and for any $p \in \partial_{X,N} u(z)$. Hence, thanks to (14.4.4) and Proposition 4.2.1, we conclude that

$$u(x) - u(y) = \int_0^T \langle \dot\gamma(t), -g(t) \rangle \, dt \leqslant \int_0^T \sigma^\star(\gamma(t), \dot\gamma(t)) \, dt,$$

where g is as in Proposition 4.2.1. Since γ is arbitrary, the thesis follows. □

Proposition 14.4.4. *Let H be a continuous Hamiltonian satisfying* (H). Let $u \in C(\Omega)$ be a viscosity subsolution to (14.2.1). Then u is a Monge subsolution to (14.2.1).

Proof. Let $x_0 \in \Omega$. Hence, in view of (14.4.2), we infer that

$$u(x) - u(x_0) + d_{\sigma^\star}(x_0, x) \geq 0$$

for any $x \in \Omega$, from which the thesis easily follows. □

In order to prove that viscosity supersolutions are Monge supersolutions, we argue as in [233]. To this aim, we combine Theorem 14.1.3 and Proposition 14.4.1 to show the solvability of the Dirichlet problem associated to a continuous Hamiltonian in the setting of viscosity solutions.

Theorem 14.4.5. *Let H be a continuous Hamiltonian satisfying* (H), and let $g \in C(\partial\Omega)$ be bounded and such that (14.1.7) holds. Then the function w defined by (14.1.8) is a viscosity solution to the Dirichlet problem

$$\begin{aligned} H(x, Xw) &= 0 \quad \text{in } \Omega \\ w &= g \quad \text{on } \partial\Omega. \end{aligned}$$

We need also the following sub-Riemannian comparison principle, whose proof is inspired by [33].

Proposition 14.4.6. *Let Ω be a bounded domain. Assume that H is continuous and satisfies* (H). *Assume that $u \in C(\overline{\Omega}) \cap W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$ is a viscosity subsolution to* (14.2.1) *on Ω and that $v \in C(\overline{\Omega})$ is a viscosity supersolution to* (14.2.1) *on Ω. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ on $\overline{\Omega}$.*

Proof. We can assume without loss of generality that $u, v > 0$. Let us fix $\delta \in (0,1)$ and set $w := \delta u$. Clearly $w \in C(\overline{\Omega}) \cap W^{1,\infty}_{X,\mathrm{loc}}(\Omega)$ and $w \leqslant v$ on $\partial\Omega$. If we prove that $w \leqslant v$ on Ω, then the thesis follows letting $\delta \to 1$.

Step 1. We first claim that for any $\tilde{\Omega} \Subset \Omega$ there exists $\eta > 0$ such that w is a viscosity subsolution to

$$H(x, Xw) + \eta = 0 \quad \text{on} \quad \tilde{\Omega}.$$

If it was not the case, then there exists $\tilde{\Omega} \Subset \Omega$ and sequences $(x_h)_h \subseteq \tilde{\Omega}$, $(p_h)_h \subseteq \mathbb{R}^m$ such that $p_h \in \partial^+_X w(x_h)$ and

$$H(x_h, p_h) + \frac{1}{h} > 0$$

for any $h \in \mathbb{N}_+$. Since by assumption $Z(x_h) \subseteq \hat{B}_\alpha(0)$ for any $h \in \mathbb{N}_+$, then we can assume up to a subsequence that $x_h \to \tilde{x} \in \Omega$ and $p_h \to \tilde{p} \in \mathbb{R}^m$. Being H continuous, we infer that $H(\tilde{x}, \tilde{p}) \geq 0$. On the other hand, notice that $\frac{p_h}{\delta} \in \partial^+_X u(x_h)$ for any $h \in \mathbb{N}_+$, and so, being u a subsolution, we infer that $H\left(x_h, \frac{p_h}{\delta}\right) \leqslant 0$. Since H is continuous, we conclude that

$$H\left(\tilde{x}, \frac{\tilde{p}}{\delta}\right) \leqslant 0.$$

The last equation implies that $\frac{\tilde{p}}{\delta} \in Z(\tilde{x})$. But then, being $Z(\tilde{x})$ convex and since $|\tilde{p}| < \frac{|\tilde{p}|}{\delta}$, we conclude that $\tilde{p}$ is an interior point of $Z(\bar{x})$, and so $H(\tilde{x}, \tilde{p}) < 0$, a contradiction.

Step 2. Let us define $M := \max_{\overline{\Omega}}(w - v)$, and assume by contradiction that $M > 0$. Let us define, for any $\varepsilon \in (0,1)$,

$$\varphi_\varepsilon(x, y) := w(x) - v(y) - \frac{d_{\mathfrak{g}}(x, y)^{2k!}}{\varepsilon^2}.$$

Being φ_ε continuous on $\overline{\Omega} \times \overline{\Omega}$, there exists $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ such that

$$M_\varepsilon := \max_{\overline{\Omega} \times \overline{\Omega}} \varphi_\varepsilon = \varphi_\varepsilon(x_\varepsilon, y_\varepsilon).$$

Step 3. We claim the following facts.

(i) $M_\varepsilon \to M$ as $\varepsilon \to 0$.

(ii) $w(x_\varepsilon) - v(y_\varepsilon) \to M$ as $\varepsilon \to 0$.

(iii) $\frac{d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)^{2k!}}{\varepsilon^2} \to 0$ as $\varepsilon \to 0$.

(iv) Let us set

$$p_\varepsilon := \frac{(2k!) d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)^{2k!-1} X d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)}{\varepsilon^2}.$$

Then $(p_\varepsilon)_\varepsilon$ is bounded.

(v) There exists $\tilde{\Omega} \Subset \Omega$ such that $x_\varepsilon, y_\varepsilon \in \tilde{\Omega}$ for any ε small enough.

Indeed, since from the choice of $(x_\varepsilon, y_\varepsilon)$ it is easy to see that $M \leqslant M_\varepsilon$ for any $\varepsilon \in (0,1)$. Let us set $R := \max\{\|w\|_\infty, \|v\|_\infty\}$. Then we have that

$$M \leqslant 2R - \frac{d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)^{2k!}}{\varepsilon^2}.$$

Since we assumed that $M > 0$, we infer that

$$\frac{d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)^{2k!}}{\varepsilon^2} \leqslant 2R.$$

This implies in particular that $d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon) \to 0$ as $\varepsilon \to 0$. This fact, together with the compactness of $\overline{\Omega}$, allows to assume up to a subsequence that there exists $\bar{x} \in \overline{\Omega}$ such that

$$\lim_{\varepsilon \to 0} d_{\mathfrak{g}}(x_\varepsilon, \bar{x}) = \lim_{\varepsilon \to 0} d_{\mathfrak{g}}(y_\varepsilon, \bar{x}) = 0. \tag{14.4.5}$$

Moreover, notice that $M \leqslant M_\varepsilon$ implies that $M \leqslant w(x_\varepsilon) - v(y_\varepsilon)$ for any $\varepsilon > 0$. This last inequality, together with (14.4.5), implies that

$$M \leqslant \liminf_{\varepsilon \to 0} w(x_\varepsilon) - v(y_\varepsilon) \leqslant \limsup_{\varepsilon \to 0} w(x_\varepsilon) - v(y_\varepsilon) \leqslant M. \tag{14.4.6}$$

This proves (ii). The fact that $M \leqslant M_\varepsilon$, combined with (14.4.6), allows to conclude that

$$M \leqslant \liminf_{\varepsilon \to 0} M_\varepsilon \leqslant \lim_{\varepsilon \to 0} w(x_\varepsilon) - v(y_\varepsilon) = M.$$

This proves (i) and (iii). To prove (v), it suffices to observe that

$$M = \lim_{\varepsilon \to 0} w(x_\varepsilon) - v(y_\varepsilon) = w(\bar{x}) - v(\bar{x}),$$

and thus, recalling that $M > 0$ and that $w \leqslant v$ on $\partial\Omega$, (v) follows. Finally, we prove (iv). Indeed, notice that, in view of the choice of $x_\varepsilon, y_\varepsilon$, then

$$w(y_\varepsilon) - v(y_\varepsilon) = \varphi_\varepsilon(y_\varepsilon, y_\varepsilon) \leqslant \varphi(x_\varepsilon, y_\varepsilon) = w(x_\varepsilon) - v(y_\varepsilon) - \frac{d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)^{2k!}}{\varepsilon^2},$$

which implies that

$$\frac{d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)^{2k!}}{\varepsilon^2} \leqslant w(x_\varepsilon) - w(y_\varepsilon) \leqslant C d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon),$$

where $C > 0$ is the $d_{\mathfrak{g}}-$Lipschitz constant of w on $\tilde{\Omega}$. Therefore

$$\frac{d_{\mathfrak{g}}(x_\varepsilon, y_\varepsilon)^{2k!-1}}{\varepsilon^2} \leqslant C.$$

The proof is concluded noticing that $z \mapsto X d_{\mathfrak{g}}(z_0, z)$ is bounded on $\Omega \setminus \{z_0\}$ uniformly with respect to $z_0 \in \Omega$.

Step 4. Let us define

$$\varphi^1_\varepsilon(y) := w(x_\varepsilon) - \frac{d_{\mathfrak{g}}(x_\varepsilon, y)^{2k!}}{\varepsilon^2} \qquad \text{and} \qquad \varphi^2_\varepsilon(x) := v(y_\varepsilon) + \frac{d_{\mathfrak{g}}(x, y_\varepsilon)^{2k!}}{\varepsilon^2}$$

for any $x, y \in \Omega$. These are smooth function on Ω. Moreover, x_ε is a maximum point for $x \mapsto w(x) - \varphi^2_\varepsilon(x)$ and y_ε is a maximum point for $y \mapsto -v(y) + \varphi^1_\varepsilon(y)$. Therefore, if $\eta > 0$ is the constant coming from Step 1 and relative to $\tilde{\Omega}$ as in (v), then

$$H(x_\varepsilon, p_\varepsilon) + \eta \leqslant H(y_\varepsilon, p_\varepsilon).$$

Being $(p_\varepsilon)_\varepsilon$ bounded, we can assume that $p_\varepsilon \to \bar{p}$ as $\varepsilon \to 0$. Therefore we conclude from the previous inequality that

$$H(\bar{x}, \bar{p}) + \eta \leqslant H(\bar{x}, \bar{p}),$$

a contradiction. □

Proposition 14.4.7. *Let H be continuous. Let $u \in C(\Omega)$ be a viscosity supersolution to (14.2.1). Then u is a Monge supersolution to (14.2.1).*

Proof. Let u be as in the statement. If by contradiction u is not a Monge supersolution to (14.2.1), there exists $x_0 \in \Omega$, $r > 0$ and $\delta > 0$ such that

$$u(x) - u(x_0) + d_{\sigma^\star}(x_0, x) \geq \delta \|x_0^{-1} \cdot x\| \tag{14.4.7}$$

for any $x \in B_{d_{\mathfrak{g}}}(x_0, r)$. Notice that, without loss of generality, we can assume that $u(x_0) = 0$. Set $\psi(x) = -d_{\sigma^\star}(x_0, x) + \delta r$. Notice that, as $B_{d_{\mathfrak{g}}}(x_0, R) \Subset \Omega$, then $H \in \mathcal{K}_0(H, B_{d_{\mathfrak{g}}}(x_0, r))$. Moreover, notice that

$$\psi(x) - \psi(y) = d_{\sigma^\star}(x_0, y) - d_{\sigma^\star}(x_0, x) \leqslant d_{\sigma^\star}(x, y)$$

for any $x, y \in \partial B_{d_{\mathfrak{g}}}(x_0, r)$, and so (14.1.7) is satisfied by ψ. Therefore we know from Theorem 14.4.5 that, if we define $w : \overline{B_{d_{\mathfrak{g}}}(x_0, r)} \longrightarrow \mathbb{R}$ as in (14.1.8) with $\Omega = B_{d_{\mathfrak{g}}}(x_0, r)$ and $g = \psi$, then $w \in C(\overline{B_{d_{\mathfrak{g}}}(x_0, r)})$ and w solves in the viscosity sense the Dirichlet problem

$$\begin{aligned} H(x, Xw) = 0 \quad &\text{in } B_{d_{\mathfrak{g}}}(x_0, r) \\ w = \psi \quad &\text{on } \partial B_{d_{\mathfrak{g}}}(x_0, r). \end{aligned}$$

Moreover, in view of (14.4.7), $u \geq \psi$ on $\partial B_{d_{\mathfrak{g}}}(x_0, r)$. Therefore, recalling that $w \in C(\overline{B_{d_{\mathfrak{g}}}(x_0, r)}) \cap W^{1,\infty}_{X,\text{loc}}(B_{d_{\mathfrak{g}}}(x_0, r))$, we conclude from Proposition 14.4.6 that $w(x_0) \leqslant u(x_0) = 0$, but this is impossible, since $w(x_0) = \delta r > 0$. □

Proof of Theorem 14.1.2. It follows from Proposition 14.4.1, Proposition 14.4.4 and Proposition 14.4.7. □

14.5 Comparison Principle

In this section we prove Theorem 14.1.4. This result, as customary, yields uniqueness for the Dirichlet problem associated to (14.2.1). The proof of Theorem 14.1.4, strongly inspired by [109], is based on the validity of the following two properties of Monge subsolutions.

Proposition 14.5.1. *Let $u \in C(\Omega)$. Assume that u is a Monge subsolution to* (14.2.1)*. Then $u \in W^{1,\infty}_{X,\text{loc}}(\Omega)$.*

Proof. Assume that $u \in C(\Omega)$ is a Monge subsolution to (14.2.1). Then

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) + \alpha d_\Omega(x_0, x)}{\|x_0^{-1} \cdot x\|} \geq \liminf_{x \to x_0} \frac{u(x) - u(x_0) + d_{\sigma^\star}(x_0, x)}{\|x_0^{-1} \cdot x\|} \geq 0$$

for any $x_0 \in \Omega$. Let $K(x, \xi) := |\xi| - \alpha$. Then $\sigma_K^\star(x, \xi) = \alpha|\xi|$ and $d_{\sigma_K^\star}(x, y) = \alpha d_\Omega(x, y)$. This implies that u is a Monge subsolution to (14.4.1) on Ω. Since K is continuous, then u is also a viscosity subsolution to (14.4.1), in view of Proposition 14.4.1. The conclusion follows as in the proof of Proposition 14.4.2. □

Proposition 14.5.2. *If u is a Monge subsolution to* (14.2.1) *in Ω, then for any $x_0 \in \Omega$ there exists $r > 0$ such that*

$$u(x) - u(y) \leqslant d_{\sigma^\star}(x, y)$$

for any $x, y \in \overline{B_{d_\Omega}(x_0, r)}$.

Proof. Let $r > 0$ be as in Proposition 14.2.3. Then in particular $B_{d_\Omega}(x_0, r) \Subset \Omega$. Moreover, since u and $d_{\sigma^\star}$ are continuous on $\overline{B_{d_\Omega}(x_0, r)}$, it suffices to consider points in $B_{d_\Omega}(x_0, r)$. Let $x, y \in B_{d_\Omega}(x_0, r)$. If $x = y$ the thesis is trivial. If instead $x \neq y$, in view of Proposition 14.2.3 there exists a sub-unit curve $\gamma : [0, T] \longrightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(T) = y$ for some $T > 0$ and γ is optimal for $d_{\sigma^\star}(x, y)$ in the sense of Section 14.1.1. In particular, (14.1.12) holds for γ. Set $f(t) := d_{\sigma^\star}(x_0, \gamma(t))$ and $g(t) := u(\gamma(t))$. Therefore Proposition 14.5.1 implies that both $f, g \in W^{1,\infty}_{\text{loc}}(0, T)$. We infer that the derivative of $f + g$ exists almost everywhere on $(0, T)$. To conclude, it suffices to show that it is non-negative. To this aim, recalling that u is a Monge subsolution to (14.2.1) and by the choice of γ, we observe that

$$\begin{aligned}\frac{d}{dt}(f+g)\Big|_{t=t_0} &= \lim_{h\to 0^+}\frac{g(t_0+h)-g(t_0)+f(t_0+h)-f(t_0)}{h}\\ &= \lim_{h\to 0^+}\frac{u(\gamma(t_0+h))-u(\gamma(t_0))+d_{\sigma^\star}(x_0,\gamma(t_0+h))-d_{\sigma^\star}(x_0,\gamma(t_0))}{\|\gamma(t_0)^{-1}\cdot\gamma(t_0+h)\|}\cdot\frac{\|\gamma(t_0)^{-1}\cdot\gamma(t_0+h)\|}{h}\\ &\geq \liminf_{h\to 0^+}\frac{u(\gamma(t_0+h))-u(\gamma(t_0))+d_{\sigma^\star}(\gamma(t_0),\gamma(t_0+h))}{\|\gamma(t_0)^{-1}\cdot\gamma(t_0+h)\|}\cdot\frac{\|\gamma(t_0)^{-1}\cdot\gamma(t_0+h)\|}{h}\\ &\geq 0\end{aligned}$$

for almost every $t_0\in(0,T)$. Finally, integrating $\frac{d}{dt}(f+g)$ in $[0,T]$ we get the result. □

Lemma 14.5.3. *Let $\Omega\subseteq\mathbb{G}$ be a bounded domain. Let $H,K:\Omega\times\mathbb{R}^m\longrightarrow\mathbb{R}$ satisfy* (H)*, and assume that there exists $\delta\in(0,1)$ such*

$$Z_K(x)\subseteq\delta Z_H(x) \tag{14.5.1}$$

for any $x\in\Omega$. Assume that $u\in C(\overline{\Omega})$ is a Monge subsolution to $K(x,Xu)=0$ and that $v\in C(\overline{\Omega})$ is a Monge supersolution to $H(x,Xv)=0$. If $u\leqslant v$ on $\partial\Omega$, then $u\leqslant v$ on $\overline{\Omega}$.

Proof. Assume by contradiction that there exists $x_0\in\Omega$ such that $u(x_0)>v(x_0)$. Let us define $\tilde H,\tilde K$ by

$$\tilde H(x,\xi):=\begin{cases}H(x,\xi) & \text{if } (x,\xi)\in\Omega\times\mathbb{R}^m\\ |\xi|-\alpha & \text{otherwise}\end{cases}$$

and

$$\tilde K(x,\xi):=\begin{cases}K(x,\xi) & \text{if } (x,\xi)\in\Omega\times\mathbb{R}^m\\ |\xi|-\frac{1}{\alpha} & \text{otherwise.}\end{cases}$$

Then $\tilde H\in\mathcal{K}(H,\Omega)$ and $\tilde K\in\mathcal{K}(K,\Omega)$. Notice that, since $\tilde H,\tilde K$ are defined on the whole $\mathbb{G}\times\mathbb{R}^m$, then $d_{\sigma^\star_{\tilde H}},d_{\sigma^\star_{\tilde K}}$ are geodesic distances in the sense of Section 14.1.1 (cf. [127]). Moreover, (14.5.1) and the definition of $\tilde H,\tilde K$ imply that

$$Z_{\tilde K}(x)\subseteq\delta Z_{\tilde H}(x) \tag{14.5.2}$$

holds for any $x\in\Omega$. We claim that there exists $\varepsilon>0$ such that

$$f_\varepsilon(x,y):=u(x)-v(y)-\frac{d_{\sigma^\star_{\tilde H}}(x,y)^2}{\varepsilon}$$

achieves its maximum over $\overline{\Omega}\times\overline{\Omega}$ on $\Omega\times\Omega$. If not, then for any $h\in\mathbb{N}_+$ there exists $(x_h,y_h)\in(\overline{\Omega}\times\overline{\Omega})\setminus\Omega\times\Omega$ which realizes the maximum for $f_{\frac{1}{h}}$. Up to a subsequence, we can assume that $x_h\to\bar x$ and that $y_h\to\bar y$. Moreover, we can assume without loss of generality that $\bar x\in\partial\Omega$. Notice that

$$0<f_{\frac{1}{h}}(x_0,x_0)\leqslant u(x_h)-v(y_h)-hd_{\sigma^\star_{\tilde H}}(x_h,y_h)^2. \tag{14.5.3}$$

Therefore $hd_{\sigma^\star_{\tilde H}}(x_h, y_h)$ is bounded, an hence $d_{\mathbb{G}}(x_h, y_h) \to 0$. This implies that $\bar x = \bar y$. Hence, noticing that $f_{\frac{1}{h}}(x_0, x_0)$ does not depend on h, (14.5.3) implies that $u(\bar x) > v(\bar x)$, which is impossible since $\bar x \in \partial\Omega$. Let then $(\tilde x, \tilde y) \in \Omega \times \Omega$ be a maximum point for f_ε, and let $\gamma : [0, T] \longrightarrow \mathbb{G}$ be a sub-unit curve such that $\gamma(0) = \tilde x$, $\gamma(T) = \tilde y$ and, recalling (14.1.12),

$$d_{\sigma^\star_{\tilde H}}(\tilde x, \tilde y) = d_{\sigma^\star_{\tilde H}}(\tilde x, \gamma(t)) + d_{\sigma^\star_{\tilde H}}(\gamma(t), \tilde y) \tag{14.5.4}$$

for any $t \in [0, T]$. Set

$$h(t) := \frac{1}{\varepsilon}(d_{\sigma^\star_{\tilde H}}(\tilde x, \tilde y) + d_{\sigma^\star_{\tilde H}}(\gamma(t), \tilde y)).$$

We claim that $h(0) \leqslant \delta$. If $\tilde x = \tilde y$, the thesis is trivial. So assume $\tilde x \neq \tilde y$. Notice that $f_\varepsilon(\tilde x, \tilde y) \geq f_\varepsilon(\gamma(t), \tilde y)$ for any t small enough, and so

$$u(\tilde x) - u(\gamma(t)) \geq h(t)(d_{\sigma^\star_{\tilde H}}(\tilde x, \tilde y) - d_{\sigma^\star_{\tilde H}}(\gamma(t), \tilde y)) \geq h(t) d_{\sigma^\star_{\tilde H}}(\tilde x, \gamma(t))$$

for any t small enough. Since u is a subsolution to $K(x, Xu) = 0$, we can apply Proposition 14.5.2 to infer that

$$d_{\sigma^\star_{\tilde K}}(\tilde x, \gamma(t)) \geq h(t) d_{\sigma^\star_{\tilde H}}(\tilde x, \gamma(t))$$

for any $t > 0$ small enough. Moreover (14.5.2) implies that $d_{\sigma^\star_{\tilde K}}(\tilde x, \gamma(t)) \leqslant \delta d_{\sigma^\star_{\tilde H}}(\tilde x, \gamma(t))$ for any $t \in [0, T]$. We conclude that

$$\delta d_{\sigma^\star_{\tilde H}}(\tilde x, \gamma(t)) \geq h(t) d_{\sigma^\star_{\tilde H}}(\tilde x, \gamma(t))$$

for any $t > 0$ small enough, which yields the claim. Noticing that $f_\varepsilon(\tilde x, \tilde y) \geq f_\varepsilon(\tilde x, y)$ for any y close enough to $\tilde y$, we see that

$$\begin{aligned}
v(\tilde y) - v(y) &= f_\varepsilon(\tilde x, y) - f_\varepsilon(\tilde x, \tilde y) + \frac{1}{\varepsilon}(d_{\sigma^\star_{\tilde H}}(\tilde x, y)^2 - d_{\sigma^\star_{\tilde H}}(\tilde x, \tilde y)^2) \\
&\leqslant \frac{1}{\varepsilon}(d_{\sigma^\star_{\tilde H}}(\tilde x, y)^2 - d_{\sigma^\star_{\tilde H}}(\tilde x, \tilde y)^2) \\
&\leqslant \frac{1}{\varepsilon}\left(d_{\sigma^\star_{\tilde H}}(\tilde x, y) + d_{\sigma^\star_{\tilde H}}(\tilde x, \tilde y)\right) d_{\sigma^\star_{\tilde H}}(\tilde y, y) \\
&= \left(h(0) + \frac{1}{\varepsilon}\left(d_{\sigma^\star_{\tilde H}}(\tilde x, y) - d_{\sigma^\star_{\tilde H}}(\tilde x, \tilde y)\right)\right) d_{\sigma^\star_{\tilde H}}(\tilde y, y) \\
&\leqslant \left(\delta + \frac{\alpha}{\varepsilon} d_{\mathbb{G}}(y, \tilde y)\right) d_{\sigma^\star_{\tilde H}}(\tilde y, y) \\
&< \frac{1+\delta}{2} d_{\sigma^\star_{\tilde H}}(\tilde y, y)
\end{aligned}$$

for any y in a neighborhood of $\tilde y$, where the last inequality follows provided that y is sufficiently close to $\tilde y$ to ensure that

$$d_{\mathbb{G}}(y, \tilde y) < \frac{\varepsilon(1-\delta)}{2\alpha}.$$

Therefore we can conclude that

$$v(y) - v(\tilde{y}) + d_{\sigma^\star_{\tilde{H}}}(\tilde{y}, y) \geq \frac{1-\delta}{2} d_{\sigma^\star_{\tilde{H}}}(\tilde{y}, y) \geq \frac{1-\delta}{2\alpha} d_{\mathbb{G}}(\tilde{y}, y),$$

which is a contradiction since v is a Monge supersolution to $H(x, Xv) = 0$. □

Proof of Theorem 14.1.4. The proof, in view of Proposition 14.5.1, Proposition 14.5.2 and Lemma 14.5.3, follows with the obvious modifications as in [109, Theorem 5.8]. □

14.6 Stability

Finally, following [62], we prove Theorem 14.1.5, which is the analogue of [62, Theorem 6.4].

Proof of Theorem 14.1.5. Fix $x_0 \in \Omega$ and let $r > 0$ be such that $B_{d_\Omega}(x_0, r) \Subset \Omega$ and Proposition 14.2.3 holds. Then $H_n \in \mathcal{K}_0(H_n, B_{d_\Omega}(x_0, r))$ for any $n \in \mathbb{N}$ and $H_\infty \in \mathcal{K}_0(H_\infty, B_{d_\Omega}(x_0, r))$. Moreover, in view of Proposition 14.5.2, $u_n(x) - u_n(y) \leqslant d_{\sigma^\star_{H_n}}(x, y)$ for any $n \in \mathbb{N}$ and for any $x, y \in \partial B_{d_\Omega}(x_0, r)$. Hence, in view of Theorem 14.1.3,

$$u_n(x) = \inf_{y \in \partial B_r(x_0)} \{d_{\sigma^\star_{H_n}}(x, y) + u_n(y)\}$$

for any $x \in \overline{B_{d_\Omega}(x_0, r)}$ and any $n \in \mathbb{N}$. By the local uniform convergence assumptions we infer that

$$u_\infty(x) = \inf_{y \in \partial B_r(x_0)} \{d_{\sigma^\star_{H_\infty}}(x, y) + u_\infty(y)\}$$

for any $x \in \overline{B_{d_\Omega}(x_0, r)}$, and so we conclude thanks to Theorem 14.1.3. □

Remark 14.6.1. The convergence condition in the hypotheses of Theorem 14.1.5 is based on the optical length functions rather than on the Hamiltonians. Arguing as in [62], one can easily find sufficient conditions on the Hamiltonians in order to guarantee the local uniform convergence of the optical length functions.

Part V

Regularity of almost minimizers in Carnot groups

Chapter 15

Lipschitz approximation of almost perimeter minimizing boundaries

15.1 Introduction

We refer to [241] as main reference for this chapter. The study of geometric measure theory in Carnot groups started from the pioneering work [140], and the regularity of sets that are local minimizers for the horizontal perimeter as defined in Definition 1.4.4 is one of the most important open problems in the field. All regularity results known so far are limited to the *Heisenberg groups* $\mathbb{H}^n$, $n \geqslant 1$, and assume either some additional strong *a priori* regularity or some restrictive geometric structure of the minimizer (cf. [73, 74, 86, 264, 223]). On the other hand, as discussed more in detail in Part VI, there are examples of minimal surfaces in the first Heisenberg group $\mathbb{H}^1$ that are only Euclidean Lipschitz continuous (cf. [239, 249]) or even discontinuous (cf [264]). The first step in the celebrated De Giorgi's regularity theory for sets of finite perimeter in $\mathbb{R}^n$ is based on a good approximation of the boundary of minimizing sets (cf. [163, 202]), namely, the so-called *Lipschitz approximation*. In the original strategy, the approximation is made by convolution and the estimates strongly rely on a *monotonicity formula*. However, the validity of such a formula is an open problem in the sub-Riemannian setting (cf. [106]). A more flexible approach has been proposed in [258] by means of Lipschitz graphs. Although the boundary of sets with finite horizontal perimeter may be quite irregular from an Euclidean point of view (cf. [182]), the natural notion of *intrinsic Lipschitz graph* (cf. [144, 143]) turns out to be effective in the approximation within this framework (cf. [221, 227, 223]). In the present chapter, we provide an extension of the approximation by means of intrinsic Lipschitz graphs in the Heisenberg groups $\mathbb{H}^n$ for $n \geq 2$, achieved in [221, 227], in a more general class of Carnot groups of step 2, that we call *plentiful groups*. In a nutshell, plentiful groups are characterized by the property that any 1-codimensional linear subspace of the first layer of their Lie algebra still generates the second layer (cf. Section 15.6). The class of plentiful groups not only includes (cf. Theorem 15.6.3) the important family of *H-type groups* (cf. [181]), but also other interesting examples (cf. Example 15.6.4). Although our results hold for the general case of *almost minimizers* (cf. Theorem 15.8.2), for the sake of notational simplicity we propose here a statement for minimizers, for whose notation we refer to the following sections.

Theorem 15.1.1 (Intrinsic Lipschitz approximation). *Let $\mathbb{G}$ be a plentiful group. For any $L \in (0,1)$, there exist $\varepsilon, C > 0$, depending on L only, with the following property. If ν is a horizontal direction and $E \subseteq \mathbb{G}$ is a minimizer of the $\mathbb{G}$-perimeter in the cylinder C_{324} with intrinsic cylindrical excess $\mathbf{e}(E,0,324,\nu) \leqslant \varepsilon$ and $0 \in \partial E$, then, letting*

$$M = C_1 \cap \partial E, \quad M_0 = \left\{ q \in M : \sup_{0<r<16} \mathbf{e}(E,q,r,\nu) \leqslant \varepsilon \right\},$$

there exists an intrinsic Lipschitz function $\varphi \colon \mathbb{W} \to \mathbb{R}$ such that

$$\sup_{\mathbb{W}} |\varphi| \leqslant L, \quad \mathrm{Lip}_{\mathbb{W}}(\varphi) \leqslant c_{\mathbb{G}}\, L,$$

$$M_0 \subseteq M \cap \Gamma, \quad \Gamma = \mathrm{graph}_\nu(\varphi; D_1),$$

$$\mathcal{S}^{Q-1}_\infty(M \Delta \Gamma) \leqslant C\, \mathbf{e}(E,0,324,\nu),$$

$$\int_{D_1} |W^\varphi \varphi|^2 \, \mathrm{d}\mathcal{L}^{n-1} \leqslant C\, \mathbf{e}(E,0,324,\nu),$$

where $c_{\mathbb{G}} > 0$ is a structural constant independent of L.

Theorem 15.1.1 perfectly generalizes [221, Theorem 5.1] to the class of plentiful groups, in fact providing a sub-optimal version of the Lipschitz approximation proved in [227, Theorem 3.1] in $\mathbb{H}^n$, for $n \geqslant 2$ (cf. [202, Theorem 23.7] for the analogous result in the Euclidean setting). In Theorem 15.1.1, differently from the corresponding result in [227], the constants ε and C may depend on the chosen Lipschitz constant L. This is due to the current lack of an analog of the deep *height estimate* proved in [228] for $\mathbb{H}^n$, with $n \geqslant 2$, in plentiful groups. However, we believe that the algebraic framework provided by plentiful groups is the correct setting where to possibly extend Theorem 15.1.1 to its optimal version.

15.2 Carnot groups of step 2

We begin specializing some terminology as in Chapter 3 to our specific framework. Let $(\mathbb{G}, \cdot)$ be a Carnot group of step 2. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2, \qquad [\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}, \qquad n = m_1 + m_2 \qquad \text{and} \qquad Q = m_1 + 2m_2.$$

We let $m = m_1$ be the rank of $\mathbb{G}$. We fix an adapted orthonormal basis $X_1, \dots, X_{m_1}, T_1, \dots, T_{m_2}$ of $\mathfrak{g}$, so that $X_1, \dots, X_{m_1}$ and $T_1, \dots, T_{m_2}$ are orthonormal bases of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. As we know, exploiting exponential coordinates of the first kind associated to $X_1, \dots, X_{m_1}, T_1, \dots, T_{m_2}$, we can write

$$p \cdot q = (x,t) \cdot (\xi,\tau) = \Big(x+\xi, t+\tau+\tfrac{1}{2}\langle \mathsf{B}x,\xi\rangle\Big) \tag{15.2.1}$$

for $p,q \in \mathbb{G}$, with $p=(x,t)$, $q=(\xi,\tau)$, $x,\xi \in \mathbb{R}^{m_1}$, $t,\tau \in \mathbb{R}^{m_2}$, where $\mathsf{B} = (\mathsf{B}^1,\dots,\mathsf{B}^{m_2})$ is an m_2-tuple of linearly independent skew-symmetric $m_1 \times m_1$ matrices and

$$\langle \mathsf{B}x,\xi\rangle = \Big(\big\langle \mathsf{B}^1 x,\xi\big\rangle,\dots,\langle \mathsf{B}^{m_2}x,\xi\rangle\Big) \in \mathbb{R}^{m_2}.$$

With this notation, we recognize that

$$\|p\|_\infty = \max\Big\{|x|, \varepsilon_2\sqrt{|t|}\Big\}$$

and

$$\delta_\lambda(p) = \delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$$

for $\lambda \geqslant 0$ and $p=(x,t) \in \mathbb{G}$, where $\|\cdot\|_\infty$ is as in Example 3.4.5. Moreover, we denote by d_∞ the invariant distance induced by $\|\cdot\|_\infty$. Finally, we let $\mathcal{M} \in (0,+\infty)$ be such that

$$|\langle \mathsf{B}x,\xi\rangle| \leqslant \mathcal{M}\,|x|\,|\xi| \quad \text{for all } x,\xi \in \mathbb{R}^{m_1}. \tag{15.2.2}$$

Since our representation of $\mathbb{G}$ in Euclidean coordinates depends on the chosen orthonormal basis of $\mathfrak{g}_1$, it is worthy to have the possibility to change coordinates in a *stratified* way. Let $X_1',\dots,X_{m_1}'$ be another orthonormal basis of $\mathfrak{g}_1$. Given $p \in \mathbb{G}$, let $p=(x',t)$ be the exponential coordinates associated with the adapted basis $X_1',\dots,X_{m_1}',T_1,\dots,T_{m_2}$. Then $x' = Mx$, for a suitable orthogonal $m_1 \times m_1$ matrix M. Being M orthogonal, $\|\cdot\|_\infty$ is not affected by this change of coordinates. Moreover, in these new coordinates,

$$p \cdot q = (x',t)\cdot(\xi',\tau) = \Big(x'+\xi', +\tfrac{1}{2}\big\langle \tilde{\mathsf{B}}x',\xi'\big\rangle\Big),$$

where $\tilde{\mathsf{B}} = (\tilde{\mathsf{B}}^1,\dots,\tilde{\mathsf{B}}^{m_2})$ and $\tilde{\mathsf{B}}^j = M\mathsf{B}^j M^T$ for any $j=1,\dots,m_2$. Notice that

$$\sup_{x'\neq 0}\frac{|\tilde{\mathsf{B}}^j x'|}{|x'|} = \sup_{x'\neq 0}\frac{|M\mathsf{B}^j M^T x'|}{|x'|} = \sup_{x'\neq 0}\frac{|\mathsf{B}^j M^T x'|}{|x'|} = \sup_{x'\neq 0}\frac{|\mathsf{B}^j M^T x'|}{|M^T x'|} = \sup_{x\neq 0}\frac{|\mathsf{B}^j x|}{|x|}$$

for any $j=1,\dots,m_2$, which in turn implies that

$$\Big|\big\langle \tilde{\mathsf{B}}x',\xi'\big\rangle\Big| \leqslant \mathcal{M}\,|x'|\,|\xi'|, \tag{15.2.3}$$

with the same constant $\mathcal{M}$ as in (15.2.2). We stress that, although the above change of coordinates induces an isometry of $\mathfrak{g}$, it may not be a group morphism (cf. [203, Example 2.15]). In fact, a simple computation shows that M induces a group morphism if and only if

$$\mathsf{B}^j M = M\mathsf{B}^j$$

for any $j=1,\dots,m_2$. Following [141, Theorem 5.1], we fix the distance d_∞ as in Example 3.4.5.

15.3 Complementary subgroups

According to [144, Section 4], we give the following fundamental definition.

Definition 15.3.1 (Complementary subgroups)**.** *Let V and W be two subgroups of $\mathbb{H}^n$. They are* complementary *when*

$$V \cap W = \{0\} \qquad \textit{and} \qquad V \cdot W = \mathbb{G}.$$

If V is one-dimensional, then it can be proved that

$$V = \mathrm{span}\{V_1\}$$

for some $V_1 \in \mathfrak{g}_1$ with $|V_1| = 1$. In this case V is called *horizontal*. In the following, in the spirit of Section 15.2, we will often choose an orthonormal basis $X_1, X_2 \dots, X_{m_1}$ of $\mathfrak{g}_1$ adapted to the decomposition $\mathbb{W} \cdot \mathbb{V}$, that is $X_1 = V_1$ and

$$\mathbb{V} = \exp(\mathrm{span}\{X_1\}), \quad \mathbb{W} = \exp(\mathrm{span}\{X_2, \dots, X_{m_1}, T_1, \dots, T_{m_2}\}).$$

We naturally identify

$$\mathbb{V} \equiv \mathbb{R}, \quad \mathbb{W} \equiv \{p = (x,t) \in \mathbb{G} : x_1 = 0\} \equiv \mathbb{R}^{n-1}.$$

In particular, $w \in \mathbb{R}^{n-1}$ is identified with $(0, w) \in \mathbb{G}$. Consequently, given $A \subseteq \mathbb{W} \equiv \mathbb{R}^{n-1}$, any function $\varphi\colon A \subseteq \mathbb{W} \to \mathbb{V}$ can be identified with a function $\varphi\colon A \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$. For a given $\nu \in \mathbb{R}^{m_1}$ with $|\nu| = 1$, we let the group homomorphism

$$\mathfrak{h}\colon \mathbb{G} \to \mathbb{R}, \quad \mathfrak{h}(p) = \langle \nu, x \rangle \quad \text{for } p = (x,t) \in \mathbb{G},$$

be the *height function*. We let $\pi_{\mathbb{V}}\colon \mathbb{G} \to \mathbb{V}$, $\pi_{\mathbb{V}}(p) = \mathfrak{h}(p)\nu$ for $p \in \mathbb{G}$, be the *projection on* $\mathbb{V}$, where, with a slight abuse of notation, we identify ν with $(\nu, 0) \in \mathbb{G}$. Moreover, we let $\pi_{\mathbb{W}}\colon \mathbb{G} \to \mathbb{W}$, uniquely given by the relation

$$p = \pi_{\mathbb{W}}(p) \cdot \pi_{\mathbb{V}}(p) \quad \text{for } p \in \mathbb{G}, \tag{15.3.1}$$

be the *projection on* $\mathbb{W}$. Notice that $\mathbb{V}$ and $\mathbb{W}$ as above are *homogeneous subgroups*, meaning that

$$\delta_\lambda(\mathbb{W}) \subseteq \mathbb{W} \qquad \text{and} \qquad \delta_\lambda(\mathbb{V}) \subseteq \mathbb{V}$$

for any $\lambda > 0$. Therefore, it is easy to verify that

$$\delta_\lambda\left(\pi_{\mathbb{W}}(p)\right) = \pi_{\mathbb{W}}(\delta_\lambda(p)) \qquad \text{and} \qquad \delta_\lambda\left(\pi_{\mathbb{V}}(p)\right) = \pi_{\mathbb{V}}(\delta_\lambda(p)) \tag{15.3.2}$$

for any $p \in \mathbb{G}$ and any $\lambda > 0$. Using the shorthands $x^{\parallel} = \mathfrak{h}(p)\nu$ and $x^{\perp} = x - x^{\parallel}$ for $x \in \mathbb{R}^{m_1}$, and exploiting (15.2.1) we easily get that, for $p = (x, t) \in \mathbb{G}$,

$$\pi_{\mathbb{V}}(p) = (x^{\|}, 0), \quad \pi_{\mathbb{W}}(p) = \left(x^{\perp}, t - \tfrac{1}{2}\left\langle \mathsf{B}x^{\perp}, x^{\|}\right\rangle\right),$$

owing to the fact that $\langle \mathsf{B}y, y\rangle = 0$ for any $y \in \mathbb{R}^{m_1}$ by skew-symmetry. Let us also observe that, for $w \in \mathbb{R}^{n-1}$ and $s \in \mathbb{R}$,

$$w \cdot (s\nu) = \exp(s\nu)(w) \quad \text{in } \mathbb{G},$$

where, again with an abuse of notation, we identify ν with its associated left-invariant vector field. Finally, by definition, we can estimate

$$\|\pi_{\mathbb{V}}(p)\|_\infty = |\langle x, \nu\rangle| \leqslant |x| \leqslant \|p\|_\infty \tag{15.3.3}$$

and, consequently,

$$\|\pi_{\mathbb{W}}(p)\|_\infty = \|p \cdot \pi_{\mathbb{V}}(p)^{-1}\|_\infty \leqslant \|p\|_\infty + \|\pi_{\mathbb{V}}(p)^{-1}\|_\infty = \|p\|_\infty + \|\pi_{\mathbb{V}}(p)\|_\infty \leqslant 2\|p\|_\infty. \tag{15.3.4}$$

We let

$$D_r = \{w \in \mathbb{W} : \|w\|_\infty < r\}$$

be the *open disk centered at* $0 \in \mathbb{W}$ *of radius* $r > 0$, and we set $D_r(w) = w \cdot D_r$ for any $w \in \mathbb{W}$. Note that $\mathcal{L}^{n-1}(D_r(w)) = \mathcal{L}^{n-1}(D_1)\, r^{Q-1}$ for all $r > 0$ and $w \in \mathbb{W}$. We also let

$$C_r = D_r \cdot (-r, r) = \{w \cdot (s\nu) : w \in D_r,\ s \in (-r, r)\}$$

be the *open cylinder with central section* D_r *and height* $2r$, and we set $C_r(p) = p \cdot C_r$ for any $p \in \mathbb{G}$. We also let

$$A \cdot \mathbb{R} = \{w \cdot (s\nu) : w \in A,\ s \in \mathbb{R}\}$$

be the *open infinite cylinder with central section* $A \subseteq \mathbb{W}$. In virtue of (15.3.1), we have that

$$p \in C_r \iff \pi_{\mathbb{W}}(p) \in D_r,\ \mathscr{h}(p) \in (-r, r) \iff \|\pi_{\mathbb{W}}(p)\|_\infty < r,\ |\mathscr{h}(p)| < r.$$

Thanks to the inequalities (15.3.3) and (15.3.4), the left-invariant map $\|\cdot\|_C \colon \mathbb{G} \to [0, +\infty)$,

$$\|p\|_C = \max\left\{\|\pi_{\mathbb{W}}(p)\|_\infty, |\mathscr{h}(p)|\right\} \quad \text{for } p \in \mathbb{G}, \tag{15.3.5}$$

is a quasi-norm such that $C_r = \{p \in \mathbb{G} : \|p\|_C < r\}$ and

$$\|p\|_C \leqslant 2\|p\|_\infty, \quad \|p\|_\infty \leqslant 2\|p\|_C, \quad \text{for } p \in \mathbb{G}. \tag{15.3.6}$$

Consequently, $d_C \colon \mathbb{G} \times \mathbb{G} \to [0, +\infty)$, $d_C(p, q) = \|q^{-1} \cdot p\|_C$ for $p, q \in \mathbb{G}$, is a left-invariant quasi-distance on $\mathbb{G}$ and

$$B_{d_\infty}\left(0, \frac{r}{2}\right) \subseteq C_r(p) \subseteq B_{d_\infty}(0, 2r) \quad \text{for all } p \in \mathbb{G},\ r > 0. \tag{15.3.7}$$

15.4 Perimeter minimizers

In the following, we denote by $\operatorname{div}_{\mathbb{G}}$ the X-divergence as in Definition 1.1.2 associated with the generators $X_1, \ldots, X_m$. Moreover, we denote by $\nu_{\mathbb{G}}\colon \Omega \to \mathbb{R}^{m_1}$ the measure-theoretic *inner* horizontal normal of a $\mathbb{G}$-Caccioppoli set E in Ω, whence, if compared to Definition 1.4.5, we choose the opposite sign. The (*measure-theoretic*) *boundary* of a measurable set $E \subseteq \mathbb{G}$ is

$$\partial E = \{p \in \mathbb{G} : |E \cap B_{d_\infty}(p, r)| > 0 \text{ and } |E^c \cap B_{d_\infty}(p, r)| > 0 \text{ for all } r > 0\}. \tag{15.4.1}$$

Up to modifying a set $E \subseteq \mathbb{G}$ of locally finite $\mathbb{G}$-perimeter in an $\mathcal{L}^n$-negligible way, arguing *verbatim* as in [202, Proposition 12.19], we can always assume that ∂E coincides with the topological boundary of E. Let $\Omega \subseteq \mathbb{G}$ be a (non-empty) open set and let $E \subseteq \mathbb{G}$ be a set with locally finite $\mathbb{G}$-perimeter in $\mathbb{G}$. We say that the set E is a (Λ, r_0)-*minimizer of the* $\mathbb{G}$-*perimeter in* Ω if there exist $\Lambda \in [0, +\infty)$ and $r_0 \in (0, +\infty]$ such that

$$P_{\mathbb{G}}(E; B_{d_\infty}(p, r)) \leqslant P_{\mathbb{G}}(F; B_{d_\infty}(p, r)) + \Lambda\, |E \Delta F|$$

for any measurable set $F \subseteq \mathbb{G}$, $p \in \Omega$ and $r < r_0$ such that $E\Delta F \Subset B_{d_\infty}(p, r) \Subset \Omega$. If $\Lambda = 0$ and $r_0 = \infty$, then E is a *local* $\mathbb{G}$-*perimeter minimizer in* Ω, that is,

$$P_{\mathbb{G}}(E; B_{d_\infty}(p, r)) \leqslant P_{\mathbb{G}}(F; B_{d_\infty}(p, r))$$

for any measurable set $F \subseteq \mathbb{G}$, $p \in \Omega$ and $r > 0$ such that $E\Delta F \Subset B_{d_\infty}(p, r) \Subset \Omega$.

Remark 15.4.1 (Scaling of (Λ, r_0)-minimizers)**.** If the set E is a (Λ, r_0)-minimizer of the $\mathbb{G}$-perimeter in $\Omega \subseteq \mathbb{G}$, then the set $E_{p,r} = \delta_{\frac{1}{r}}(\tau_{p^{-1}}(E))$ is a (Λ', r_0')-minimizer of the $\mathbb{G}$-perimeter in $\Omega_{p,r} = \delta_{\frac{1}{r}}(\tau_{p^{-1}}(\Omega))$ for every $p \in \mathbb{G}$ and $r > 0$, where $\Lambda' = \Lambda r$ and $r_0' = r_0/r$. In particular, the product Λr_0 is invariant by dilation, and thus it is convenient to assume that $\Lambda r_0 \leqslant 1$, as we shall always do in the following.

In a Carnot group $\mathbb{G}$ of step 2, locally finite $\mathbb{G}$-perimeter sets enjoy further regularity properties (cf. [141, 204, 262]). In particular, for any set $E \subseteq \mathbb{G}$ with locally finite $\mathbb{G}$-perimeter,

$$P_{\mathbb{G}}(E; A) = \mathcal{S}_\infty^{Q-1}(\partial^\star_{\mathbb{G}} E \cap A) \quad \text{for each Borel } A \subseteq \mathbb{G}.$$

We can state the following results concerning the properties of (Λ, r_0)-minimizers of the $\mathbb{G}$-perimeter in Carnot groups of step 2. The proofs are straightforward adaptations of those for (Λ, r_0)-minimizers of the Euclidean perimeter in $\mathbb{R}^n$ as in [202, Chapter 12].

Theorem 15.4.2 (Density estimates)**.** *There exist* $c_1, c_2, c_3, c_4 > 0$ *such that, if* $E \subseteq \mathbb{G}$ *is a* (Λ, r_0)-*minimizer of the* $\mathbb{G}$-*perimeter in the open set* $\Omega \subseteq \mathbb{G}$, $\Lambda r_0 \leqslant 1$, $p \in \partial E \cap \Omega$, $B_{d_\infty}(p, r_0) \subseteq \Omega$, *then*

$$c_1 \leqslant \frac{|E \cap B_{d_\infty}(p, r)|}{r^Q} \leqslant c_2 \quad \textit{and} \quad c_3 \leqslant \frac{P_{\mathbb{G}}(E, B_{d_\infty}(p, r))}{r^{Q-1}} \leqslant c_4 \quad \textit{for } r \in (0, r_0).$$

In particular, $\mathcal{S}_\infty^{Q-1}\Big((\partial E \setminus \partial_{\mathbb{G}}^\star E)\cap \Omega\Big) = 0$.

Proof. The result follows by adapting the proof of [202, Theorem 21.11] and invoking[141, Lemma 2.21] and [141, Proposition 2.23]. □

Theorem 15.4.3 (Compactness)**.** *If* $(E_j)_{j\in\mathbb{N}}$ *is a sequence of* (Λ, r_0)*-minimizers of the* $\mathbb{G}$*-perimeter in the open set* $\Omega \subseteq \mathbb{G}$*,* $\Lambda r_0 \leqslant 1$*, then there exist a subsequence* $(E_{j_k})_{k\in\mathbb{N}}$ *and a* (Λ, r_0)*-minimizer of the* $\mathbb{G}$*-perimeter* $E \subseteq \mathbb{G}$ *in* Ω *such that*

$$E_{j_k} \to E \quad in L^1_{\mathrm{loc}}(\Omega) \quad and \quad |P_{\mathbb{G}}(E_{j_k},\cdot)| \overset{*}{\rightharpoonup} |P_{\mathbb{G}}(E,\cdot)| \quad as\ k \to +\infty.$$

Moreover, $(\partial E_{j_k})_{k\in\mathbb{N}}$ *converges to* ∂E *in the sense of Kuratowski, i.e.:*

(i) if $p_{j_k} \in \partial E_{j_k} \cap \Omega$ *and* $p_{j_k} \to p \in \Omega$ *as* $k \to +\infty$*, then* $p \in \partial E$*;*

(ii) if $p \in \partial E \cap \Omega$*, then there exist* $p_{j_k} \in \partial E_{j_k} \cap \Omega$ *such that* $p_{j_k} \to p$ *as* $k \to +\infty$*.*

Proof. The result follows by adapting the proof of [202, Proposition 21.13] and [202, Proposition 21.14], and exploiting the density estimates provided by Theorem 15.4.2. □

15.5 Cylindrical excess

A concept which plays a key role in the regularity theory of (Λ, r_0)-minimizers of the $\mathbb{G}$-perimeter is that of the *cylindrical excess* (cf. [202, Chapter 22] for the Euclidean setting and [221, 228, 223, 227] for the Heisenberg group).

Definition 15.5.1 (Cylindrical excess)**.** *The* cylindrical excess *of a locally finite* $\mathbb{G}$*-perimeter set* $E \subseteq \mathbb{G}$ *at* $p \in \partial E$*, at scale* $r > 0$*, and with respect to the horizontal direction* ν*, is*

$$\begin{aligned}\mathbf{e}(E,p,r,\nu) &= \frac{1}{2\,r^{Q-1}} \int_{C_r(p)} |\nu_{\mathbb{G}}(p) - \nu|^2 \, dP_{\mathbb{G}}(E,\cdot)(p) \\ &= \frac{1}{r^{Q-1}} \int_{C_r(p)\cap\partial_{\mathbb{G}}^\star E} \Big(1 - \langle \nu_{\mathbb{G}}(p), \nu\rangle^2\Big)\, \mathrm{d}\mathcal{S}_\infty^{Q-1}(p).\end{aligned}$$

If no confusion arises, we set $\mathbf{e}(p,r) = \mathbf{e}(E,p,r) = \mathbf{e}(E,p,r,\nu)$ *and* $\mathbf{e}(r) = \mathbf{e}(0,r)$*.*

The basic properties of the cylindrical excess introduced in Definition 15.5.1 can be plainly recovered from the corresponding ones known in the Euclidean setting (cf. [202, Chapter 22]) and in the Heisenberg groups (cf. [221, Section 3] and [228, Section 3B]). The following result corresponds to [228, Lemma 3.4] and [228, Corollary 3.5], which were stated in the setting of the Heisenberg groups $\mathbb{H}^n$, $n \geqslant 2$ (cf. [202, Lemma 22.11] for the Euclidean case). The very same results hold for any Carnot group of step 2, with identical proof.

Lemma 15.5.2 (Excess measure)**.** *Let* $E \subseteq \mathbb{G}$ *be a set with locally finite* $\mathbb{G}$*-perimeter with* $0 \in \partial E$*. If there exists* $s_0 \in (0,1)$ *such that*

$$\sup\Big\{|\not{z}(p)| : p \in C_1 \cap \partial E\Big\} \leqslant s_0,$$

$$\mathcal{L}^{n-1}\left(\left\{p \in E \cap C_1 : \mathfrak{f}(p) > s_0\right\}\right) = 0,$$

$$\mathcal{L}^{n-1}\left(\left\{p \in C_1 \setminus E : \mathfrak{f}(p) < -s_0\right\}\right) = 0,$$

then, for a.e. $s \in (-1, 1)$ *and any* $\phi \in C_c(D_1)$*, letting*

$$M = C_1 \cap \partial^\star_{\mathbb{G}} E, \quad M_s = M \cap \left\{\mathfrak{f} > s\right\}, \quad E_s = \{w \in \mathbb{W} : w \cdot (s\nu) \in E\},$$

we have

$$\int_{E_s \cap D_1} \phi \, \mathrm{d}\mathcal{L}^{n-1} = \int_{M_s} \phi \circ \pi_{\mathbb{W}} \langle \nu_{\mathbb{G}}, \nu \rangle \, \mathrm{d}\mathcal{S}^{Q-1}_\infty.$$

Consequently, for any Borel set $G \subseteq D_1$*,*

$$\mathcal{L}^{n-1}(G) = \int_{M \cap \pi_{\mathbb{W}}^{-1}(G)} \langle \nu_{\mathbb{G}}, \nu \rangle \, \mathrm{d}\mathcal{S}^{Q-1}_\infty,$$

$$\mathcal{L}^{n-1}(G) \leqslant \mathcal{S}^{Q-1}_\infty \left(M \cap \pi_{\mathbb{W}}^{-1}(G)\right). \tag{15.5.1}$$

Moreover, we have

$$0 \leqslant \mathcal{S}^{Q-1}_\infty(M_s) - \mathcal{L}^{n-1}(E_s \cap D_1) \leqslant \mathbf{e}(E, 0, 1) \quad \text{for a.e. } s \in (-1, 1),$$

$$\mathcal{S}^{Q-1}_\infty(M) - \mathcal{L}^{n-1}(D_1) = \mathbf{e}(E, 0, 1).$$

15.6 Plentiful groups

Contrarily to what happens in $\mathbb{R}^n$, the fact that $\mathbf{e}(E, p, r) = 0$ for some $p \in \partial E$ and $r > 0$ does not necessarily imply that ∂E is flat in a neighborhood of p. This indeed happens in the first Heisenberg group $\mathbb{H}^1$ (cf. [226, 221]). Nevertheless, this is not the case for any Heisenberg group $\mathbb{H}^n$ with $n \geqslant 2$, as proved in [221]. Consequently, in order to avoid minimal surfaces with zero excess that are not flat, we need to restrict our attention to a special class of Carnot groups, defined as follows.

Definition 15.6.1 (Plentiful group)**.** *We say that a Carnot group* $\mathbb{G}$ *of step* 2 *is* plentiful *if any* $V \subseteq \mathfrak{g}_1$ *with* $\dim V = m - 1$ *satisfies* $[V, V] = \mathfrak{g}_2$*.*

The property of being plentiful is well behaved with respect to Lie group isomorphisms.

Proposition 15.6.2. *Let* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *be two Carnot groups of step* 2*. If* $\mathbb{G}_1$ *is plentiful and* $\phi\colon \mathbb{G}_1 \to \mathbb{G}_2$ *is a Lie group isomorphism, then also* $\mathbb{G}_2$ *is plentiful.*

Proof. Set $\mathfrak{g}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, with $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1]$ and $\mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1]$. Note that $\mathrm{d}\phi\colon \mathfrak{g} \to \mathfrak{h}$ is an isomorphism preserving the stratification of the corresponding algebras. Hence, letting $W \subseteq \mathfrak{h}_1$ be as in Definition 15.6.1 for $\mathbb{G}_2$, $V = (d\phi)^{-1}(W)$ is an $(m-1)$-dimensional vector subspace of $\mathfrak{g}_1$. Thus, since $\mathbb{G}_1$ is plentiful, we get that

$$[W, W] = [d\phi(V), d\phi(V)] = d\phi([V, V]) = d\phi(\mathfrak{g}_2) = \mathfrak{h}_2,$$

proving that also $\mathbb{G}_2$ is plentiful. $\square$

We observe that the first Heisenberg group $\mathbb{H}^1$ is not plentiful. More generally, every *free* Carnot group of step 2 (cf. [54, Section 3.3] for the precise definition) is not plentiful. On the other hand, the Heisenberg group $\mathbb{H}^n$ is plentiful for any $n \geq 2$. More in general, we have the following result.

Theorem 15.6.3. *An H-type group is plentiful if and only if it is not isomorphic to $\mathbb{H}^1$.*

We recall that a Carnot group $\mathbb{G}$ of step 2 is of *H-type* if, for any $Z \in \mathfrak{g}_2$, the map $J_Z\colon \mathfrak{g}_1 \to \mathfrak{g}_1$ given by

$$\langle J_Z(X), Y\rangle = \langle Z, [X, Y]\rangle \quad \text{for any } X, Y \in \mathfrak{g}_1 \tag{15.6.1}$$

is orthogonal whenever $|Z| = 1$. Notice that $\mathbb{H}^n$ is of H-type for all $n \geqslant 1$.

Proof of Theorem 15.6.3. Let $T_1, \dots, T_{m_2}$ be an orthonormal basis of $\mathfrak{g}_2$ and let $X \in \mathfrak{g}_1$. By [54, Proposition 18.1.8], for any $X \in \mathfrak{g}_1$, it holds that $X, J_{T_1}(X), \dots, J_{T_{m_2}}(X)$ is an orthonormal subfamily of $\mathfrak{g}_1$, hence yielding that $m_1 \geq m_2 + 1$. Fix $V \subseteq \mathfrak{g}_1$ as in Definition 15.6.1 and let $v \in \mathfrak{g}_1 \cap V^\perp$ be such that $|v| = 1$. We now distinguish two cases.

Case 1. Let us assume that $m_1 > m_2 + 1$. In view of (15.6.1) and [54, Proposition 18.1.8], $J_{T_1}(v), \dots, J_{T_{m_2}}(v)$ is hence an orthonormal subfamily of V. Moreover, again owing to the fact that $m_1 > m_2 + 1$, there exists $w \in V$ which is orthogonal to $J_{T_1}(v), \dots, J_{T_{m_2}}(v)$ and satisfies $|w| = 1$. Again by [54, Proposition 18.1.8], we get

$$\left\langle v, J_{T_j}(w)\right\rangle = -\left\langle w, J_{T_j}(v)\right\rangle = 0$$

for any $j = 1, \dots, m_2$, which implies that $J_{T_j}(w) \in V$ for any $j = 1, \dots, m_2$. Since $[w, J_{T_j}(w)] = T_j$ for each $j = 1, \dots, m_2$ by (15.6.1), we conclude that $[V, V] = \mathfrak{g}_2$, as desired.

Case 2. Now assume that $m_1 = m_2 + 1$. We can assume that $m_1 > 2$, since otherwise $\mathbb{G}$ is isomorphic to $\mathbb{H}^1$. We recall that $\mathbb{G}$ is of H-type if and only if, for any $X \in \mathfrak{g}_1$ with $|X| = 1$, the map $\mathrm{ad}_X = [X, \cdot]$ is a surjective isometry from $\ker(\mathrm{ad}_X)^\perp \cap \mathfrak{g}_1$ to $\mathfrak{g}_2$ (cf. [181, 93]). Since $m_1 = m_2 + 1$, we infer that $\ker(\mathrm{ad}_X)^\perp \cap \mathfrak{g}_1 = X^\perp \cap \mathfrak{g}_1$. Let $X \in V$ be such that $|X| = 1$. By the previous considerations, $\dim(\mathrm{ad}_X(V \cap X^\perp)) = m_2 - 1$. Let $T \in \mathfrak{g}_2 \cap \mathrm{ad}_X(V \cap X^\perp)^\perp$ be such that $|T| = 1$. Since $[X, J_T(X)] = T$ and ad_X is injective, we infer that, up to a sign, $v = J_T(X)$. Since $m_1 > 2$, and hence $\dim(V) > 1$, let $Y \in V$ be such that $|Y| = 1$ and $\langle X, Y\rangle = 0$. By [181], we infer that

$$\langle J_T(Y), v\rangle = \langle J_T(Y), J_T(X)\rangle = -\left\langle Y, J_T^2(X)\right\rangle = \langle Y, X\rangle = 0,$$

and so $J_T(Y) \in V$. Since $[Y, J_T(Y)] = T$, we get $[V, V] = \mathfrak{g}_2$, concluding the proof. $\square$

We point out that the class of plentiful groups is broader than that of H-type groups.

Example 15.6.4. Consider the stratified Lie algebra $\mathfrak{g}_{7,5,2}$ of dimension 7, rank 5 and step 2, with only non-trivial commutation relations given by

$$[X_1, X_2] = [X_3, X_4] = T_1, \quad [X_1, X_5] = [X_2, X_3] = T_2$$

(for a construction, cf. [187, (27B)]). Let $\mathbb{G}_{7,5,2}$ be its associated Carnot group. In view of [54, Proposition 18.1.5], $\mathbb{G}_{7,5,2}$ is not of H-type. We claim that $\mathbb{G}_{7,5,2}$ is plentiful. To this aim, let us fix $V \subseteq \mathfrak{g}_1$ as in Definition 15.6.1 and let $v \in \mathfrak{g}_1 \cap V^\perp$ be such that $|v| = 1$. We let $v = \sum_{j=1}^5 a_j X_j$, where $a_j = \langle v, X_j \rangle$. We now observe that $W_j = X_j - a_j v \in V$ for $j = 1, \dots, 5$ are such that

$$[W_1, W_4] + [W_2, W_3] = \alpha T_2, \quad \text{with } \alpha = \left(a_1^2 + \left(a_4^2 - a_4 a_5 + a_5^2\right)\right) \geqslant 0, \tag{15.6.2}$$

and

$$\begin{aligned} [W_1, W_2] &= \left(1 - a_1^2 - a_2^2\right) T_1 + (a_1 a_3 - a_2 a_5)\, T_2, \\ [W_1, W_5] &= -a_2 a_5\, T_1 + \left(1 - a_1^2 - a_5^2\right) T_2, \\ [W_3, W_4] &= \left(1 - a_3^2 - a_4^2\right) T_1 + a_2 a_4\, T_2. \end{aligned} \tag{15.6.3}$$

We now distinguish two cases, depending on whether $\alpha = 0$ or $\alpha > 0$ in (15.6.2). If $\alpha = 0$, then $a_1 = a_4 = a_5 = 0$. Due to (15.6.3), we get

$$[W_1, W_5] = T_2,$$

$$[W_1, W_2] = a_3^2\, T_1$$

and

$$[W_3, W_4] = a_2^2\, T_1,$$

proving the claim, since either $a_2 \neq 0$ or $a_3 \neq 0$. If $\alpha > 0$ instead, then $T_2 \in [V, V]$ by (15.6.2). Therefore, by (15.6.3), we get

$$(1 - a_3^2 - a_4^2) T_1 \in V$$

and

$$(1 - a_1^2 - a_2^2) T_1 \in V.$$

If $a_3^2 + a_4^2 \neq 1$, then $T_1 \in V$. If $a_3^2 + a_4^2 = 1$ instead, then $a_1 = a_2 = 0$ and so $T_1 \in V$, proving the claim.

Our interest for plentiful groups is encoded in the following result, which is a sort of localized version of [Lemma 3.6][141]. This is essential in the proof of Theorem 15.6.6 below, where we prove that plentiful groups do not admit non-flat surfaces with zero excess.

Lemma 15.6.5. *Let $\mathbb{G}$ be a plentiful Carnot group. Let $\Omega \subseteq \mathbb{G}$ be a non-empty domain and let $Z_1, \dots, Z_{m_1}$ be an orthonormal basis of $\mathfrak{g}_1$. If $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ is such that $Z_1 f \geqslant 0$ and*

$Z_i f = 0$ *for* $i = 2, \dots, m_1$ *in* Ω*, then the level sets of* f *in* Ω *coincide with left translations of* $\{p \in \mathbb{G} : \langle p, Z_1(0) \rangle = 0\}$.

Proof. We can assume $f \in C^\infty(\mathbb{G})$, since the general case can be recovered by approximation. Clearly, $Z_i Z_j f = 0$ for all $i, j = 2, \dots, m_1$ and thus, since $\mathbb{G}$ is plentiful, $Tf = 0$ in Ω for any $T \in \mathfrak{g}_2$. Since the left-invariant distribution $\mathscr{D}$ generated by the vector fields $\mathfrak{g} \setminus \operatorname{span}\{Z_1\}$ is involutive, $\mathbb{G}$ is foliated by smooth $(n-1)$-dimensional manifolds tangent to $\mathscr{D}$ which, in Ω, coincide with the level sets of f. Since $Z_1, \dots, Z_{m_1}$ are orthonormal and left-invariant, each leaf of the foliation coincides with the leaf passing through $0 \in \mathbb{G}$, that is, $\{p \in \mathbb{G} : \langle p, Z_1(0) \rangle = 0\}$, up to a left translation. □

The following crucial result extends [221, Proposition 3.6] to plentiful groups. We notice that Theorem 15.6.6 below can be achieved as [221, Proposition 3.6] by a straightforward adaptation of [Lemma 3.5][221]. However, we prove Theorem 15.6.6 via a different and plainer argument, somewhat reminiscent of the proof of [Claim 3.7][141], by exploiting Lemma 15.6.5.

Theorem 15.6.6 (Locally constant normal)**.** *Let* $\mathbb{G}$ *be a plentiful Carnot group. Let* $E \subseteq \mathbb{G}$ *be a set with finite* $\mathbb{G}$*-perimeter in* $B_{d_\infty}(p, r)$*, for* $p \in \partial E$ *and* $r > 0$*. If* $\nu_{\mathbb{G}}(q) = \nu$ *for* $P_{\mathbb{G}}(E, \cdot)$*-a.e.* $q \in B_{d_\infty}(p, r)$*, then*

$$E \cap B_{d_\infty}(p, r) = \left\{q \in B_{d_\infty}(p, r) : \mathscr{h}(q) > \mathscr{h}(p)\right\}$$

up to $\mathcal{L}^n$*-negligible sets.*

Proof. We can clearly assume that $p = 0$ up to a translation. Take $\zeta \in \mathbb{R}^{m_1}$ and consider the left-invariant differential operator $L_\zeta = \sum_{j=1}^{m_1} \zeta_j X_j$ and the test horizontal vector field $\phi = \zeta \psi \in C^1_c(B_{d_\infty}(0, r); \mathbb{R}^{m_1})$ for some arbitrary $\psi \in C^1_c(B_{d_\infty}(0, r); \mathbb{R})$. By assumption, we can compute

$$\int_E L_\zeta \psi \, \mathrm{d}\mathcal{L}^n = \int_E \operatorname{div}_{\mathbb{G}} \phi \, \mathrm{d}\mathcal{L}^n = -\int_{B_{d_\infty}(0,r)} \langle \phi, \nu_{\mathbb{G}} \rangle = -\int_{B_{d_\infty}(0,r)} \psi \, \langle \zeta, \nu \rangle \, dP_{\mathbb{G}}(E, \cdot),$$

yielding that $L_\zeta \mathbf{1}_E = 0$ if $\langle \zeta, \nu \rangle = 0$ and $L_\zeta \mathbf{1}_E \geqslant 0$ if $\zeta = \nu$ in $B_{d_\infty}(0, r)$. By Lemma 15.6.5,

$$E \cap B_{d_\infty}(0, r) = \tau_q \left(\left\{ \tilde{q} \in \mathbb{G} : \mathscr{h}(\tilde{q}) > 0 \right\} \right) \cap B_{d_\infty}(0, r) \quad \text{for some } q \in \mathbb{G}.$$

To conclude, we just need to show that $\mathscr{h}(q) = 0$, as this yields

$$\tau_q \left(\left\{ \tilde{q} \in \mathbb{G} : \mathscr{h}(\tilde{q}) > 0 \right\} \right) = \left\{ \tilde{q} \in \mathbb{G} : \mathscr{h}(\tilde{q}) > 0 \right\}.$$

Indeed, if $\mathscr{h}(q) > 0$, then $B_{d_\infty}(0, \rho) \cap \tau_q \left(\left\{ \tilde{q} \in \mathbb{G} : \mathscr{h}(\tilde{q}) > 0 \right\} \right) = \emptyset$ for some $\rho \in (0, r)$, yielding

$$|B_{d_\infty}(0, \rho) \cap E| = \left| B_{d_\infty}(0, \rho) \cap \tau_q \left(\left\{ \tilde{q} \in \mathbb{G} : \mathscr{h}(\tilde{q}) > 0 \right\} \right) \right| = 0,$$

against the assumption that $0 \in \partial E$, recall (15.4.1). The case $\mathscr{h}(q) < 0$ can be similarly addressed by considering E^c in place of E. The proof is complete. □

15.7 Intrinsic cones, Lipschitz graphs and area formula

Throughout this section, we assume that $(\mathbb{G}, \cdot)$ is a Carnot group of step 2 as in Section 15.2. For a general introduction about the topics of this section, we refer to [144]. Moreover, here and for the rest of the chapter, we fix an horizontal direction ν and we choose an adapted basis $\nu = X_1, X_2, \dots, X_{m_1}, T_1, \dots, T_{m_2}$ of $\mathfrak{g}$ as in Section 15.3. In the induced exponential coordinates, we write $p = (x, t)$ for any $p \in \mathbb{G}$.

15.7.1 Intrinsic cones

The following definition rephrases [221, Definition 4.3] and [144, Definition 9].

Definition 15.7.1 (Intrinsic cones)**.** *The open X_1-*cone *with vertex $0 \in \mathbb{G}$ and aperture $\alpha \in (0, +\infty]$ is the set*

$$C(0, \alpha) = \{p \in \mathbb{G} : \|\pi_{\mathbb{W}}(p)\|_\infty < \alpha \|\pi_{\mathbb{V}}(p)\|_\infty\}.$$

The corresponding negative *and* positive cones *are*

$$C^{\pm}(0, \alpha) = \{p = (x, t) \in \mathbb{G} : \|\pi_{\mathbb{W}}(p)\|_\infty < \alpha \|\pi_{\mathbb{V}}(p)\|_\infty,\ x_1 \gtrless 0\}.$$

Consequently, we let $C(p, \alpha) = p \cdot C(0, \alpha)$ and $C^{\pm}(p, \alpha) = p \cdot C^{\pm}(0, \alpha)$ for $p \in \mathbb{G}$.

In view of (15.3.2), intrinsic cones as in Definition 15.7.1 are closed under the action of intrinsic dilations. Note that, given $p = (x, t) \in \mathbb{G}$ and $\alpha \geqslant 0$, $\|\pi_{\mathbb{W}}(p)\|_\infty \leqslant \alpha \|\pi_{\mathbb{V}}(p)\|_\infty$ rewrites as

$$\max\Big\{\big|x^\perp\big|, \varepsilon_2 \big|t - \tfrac{1}{2}\big\langle \mathsf{B} x^\perp, x^\parallel \big\rangle\big|^{1/2}\Big\} \leqslant \alpha |x_1|. \tag{15.7.1}$$

The following result collects some elementary properties of cones in Carnot groups of step 2, generalizing [221]*Lem. 4.5. We briefly detail its proof for the ease of the reader.

Lemma 15.7.2 (Properties of cones)**.** *The following hold:*

(i) $\bigcup_{s < s_0} C^+(p \cdot s\nu, \alpha) = \mathbb{G}$ *for all $\alpha > 0$, $p \in \mathbb{G}$ and $s_0 \in \mathbb{R}$;*

(ii) $C^-(0, \alpha) \subseteq \iota\left(C^+\left(0, \alpha + \varepsilon_2 \sqrt{\alpha \mathcal{M}}\right)\right)$ *for all $\alpha > 0$;*

(iii) $C^{\pm}(p, \beta) \subseteq C^{\pm}(0, \gamma)$ *for all $p \in C^{\pm}(0, \alpha)$, with $\alpha, \beta \geqslant 0$ and*

$$\gamma = \max\Big\{\alpha, \beta, \tfrac{\varepsilon_2}{2}\sqrt{(\alpha\beta + 2\beta)\,\mathcal{M}}\Big\},$$

where $\mathcal{M} > 0$ is the constant in (15.2.2).

Proof. We prove each statement separately.

Proof of (i). Assume $p = 0$ and note that, in virtue of (15.2.3) and (15.7.1), we can compute

$$\begin{aligned} C^+(s\nu,\alpha) &= s\nu\cdot C^+(0,\alpha) \\ &= s\nu\cdot\Big\{(x,t)\in\mathbb{G} : \max\Big\{\big|x^\perp\big|, \varepsilon_2\Big|t-\tfrac12\Big\langle \mathsf{B}x^\perp, x^\parallel\Big\rangle\Big|^{1/2}\Big\} < \alpha x_1\Big\} \\ &= \Big\{(x,t)\in\mathbb{G} : \max\Big\{\big|x^\perp\big|, \varepsilon_2\Big|t-\tfrac12\Big\langle \mathsf{B}x^\perp, x^\parallel - 2s\nu\Big\rangle\Big|^{1/2}\Big\} < \alpha(x_1-s)\Big\}. \end{aligned}$$

Hence (i) for $p=0$ follows from the fact that, for any $(x,t)\in\mathbb{G}$, there is $\sigma\in\mathbb{R}$ such that

$$\varepsilon_2\Big|t-\tfrac12\Big\langle \mathsf{B}x^\perp, x^\parallel-2s\nu\Big\rangle\Big|^{1/2} < \alpha(x_1-s) \quad \text{for all } s<\sigma.$$

By left translation, (i) holds for any $p\in\mathbb{G}$.

Proof of (ii). For any $\beta>0$ we have that

$$\iota\Big(C^+(0,\beta)\Big) = \Big\{(x,t)\in\mathbb{G} : \max\Big\{\big|x^\perp\big|, \varepsilon_2\Big|t+\tfrac12\Big\langle \mathsf{B}x^\perp, x^\parallel\Big\rangle\Big|^{1/2}\Big\} < -\beta x_1\Big\}.$$

Hence, if $(x,t)\in C^-(0,\alpha)$, then $\Big|\Big\langle \mathsf{B}x^\perp, x^\parallel\Big\rangle\Big| \leqslant \mathcal{M}\big|x^\perp\big|\big|x^\parallel\big| < \alpha\mathcal{M}\big|x^\parallel\big|^2$ and so

$$\varepsilon_2\Big|t+\tfrac12\Big\langle \mathsf{B}x^\perp, x^\parallel\Big\rangle\Big|^{1/2} \leqslant \varepsilon_2\Big|t-\tfrac12\Big\langle \mathsf{B}x^\perp, x^\parallel\Big\rangle\Big|^{1/2} + \varepsilon_2\Big|\Big\langle \mathsf{B}x^\perp, x^\parallel\Big\rangle\Big|^{1/2} < -\Big(\alpha+\varepsilon_2\sqrt{\alpha\mathcal{M}}\Big)\,x_1,$$

proving (ii).

Proof of (iii). If $p=(x,t)\in C^+(0,\alpha)$, then

$$\max\Big\{\big|x^\perp\big|, \varepsilon_2\Big|t-\tfrac12\Big\langle \mathsf{B}x^\perp, x^\parallel\Big\rangle\Big|^{1/2}\Big\} \leqslant \alpha x_1. \tag{15.7.2}$$

Moreover, if $q\in C^+(p,\beta)$, then $q = p * w$ with $w=(\xi,\tau)\in\mathbb{G}$ such that

$$\max\Big\{\big|\xi^\perp\big|, \varepsilon_2\Big|\tau-\tfrac12\Big\langle \mathsf{B}\xi^\perp, \xi^\parallel\Big\rangle\Big|^{1/2}\Big\} \leqslant \beta\xi_1. \tag{15.7.3}$$

Now, since $q=(x,t)\cdot(\xi,\tau) = \Big(x+\xi, t+\tau+\tfrac12\langle \mathsf{B}x,\xi\rangle\Big)$, we can write

$$\|\pi_{\mathbb{W}}(q)\|_\infty = \max\Big\{\big|x^\perp+\xi^\perp\big|, \varepsilon_2\Big|t+\tau+\tfrac12\langle \mathsf{B}x,\xi\rangle - \tfrac12\Big\langle \mathsf{B}(x^\perp+\xi^\perp), x^\parallel+\xi^\parallel\Big\rangle^{1/2}\Big|\Big\}.$$

Since $\Big\langle \mathsf{B}x^\parallel, \xi^\parallel\Big\rangle = 0$, by (15.2.3) we easily see that

$$\begin{aligned} \Big|\langle \mathsf{B}x,\xi\rangle - \Big\langle \mathsf{B}x^\perp,\xi^\parallel\Big\rangle - \Big\langle \mathsf{B}\xi^\perp, x^\parallel\Big\rangle\Big| &= \Big|\Big\langle \mathsf{B}x^\perp,\xi^\perp\Big\rangle + 2\Big\langle \mathsf{B}x^\parallel,\xi^\perp\Big\rangle\Big| \\ &\leqslant \mathcal{M}\left(\big|x^\perp\big|\big|\xi^\perp\big| + 2\big|x^\parallel\big|\big|\xi^\perp\big|\right) \\ &\leqslant \mathcal{M}\left(\alpha\beta+2\beta\right)\big|x^\parallel\big|\big|\xi^\parallel\big|. \end{aligned} \tag{15.7.4}$$

Therefore, by the triangle inequality, (15.7.2), (15.7.3) and (15.7.4) yield that

$$\begin{aligned}\varepsilon_2\Big|t+\tau+\tfrac{1}{2}\langle \mathsf{B}x,\xi\rangle-\tfrac{1}{2}\Big\langle \mathsf{B}(x^\perp+\xi^\perp),x^\parallel+\xi^\parallel\Big\rangle\Big|^{1/2} &\leqslant \varepsilon_2\Big|t-\tfrac{1}{2}\Big\langle \mathsf{B}x^\perp,x^\parallel\Big\rangle\Big|^{1/2}\\ &+\varepsilon_2\Big|\tau-\tfrac{1}{2}\Big\langle \mathsf{B}\xi^\perp,\xi^\parallel\Big\rangle\Big|^{1/2}+\tfrac{\varepsilon_2}{2}\Big|\langle \mathsf{B}x,\xi\rangle-\Big\langle \mathsf{B}x^\perp,\xi^\parallel\Big\rangle-\Big\langle \mathsf{B}\xi^\perp,x^\parallel\Big\rangle\Big|^{1/2}\\ &\leqslant \alpha x_1+\beta\xi_1+\tfrac{\varepsilon_2}{2}\sqrt{\mathcal{M}\,(\alpha\beta+2\beta)}\,x_1^{1/2}\xi_1^{1/2},\end{aligned}$$

immediately implying that $q \in C^+(0,\gamma)$. The case of negative cones is similar. □

15.7.2 Intrinsic Lipschitz graphs and functions

As already mentioned in the introduction, the right notion of Lipschitz graphs to deal with in this setting is that of *intrinsic Lipschitz graph*. In this section we propose some general definitions, while a more detailed discussion is postponed until Chapter 16, where we derive further properties in the Heisenberg group framework. The following definition rephrases [221, Definition 4.6] and [144, Definition 11].

Definition 15.7.3 (Intrinsic Lipschitz graph and function). *The* intrinsic graph *of* $\varphi\colon A\to\mathbb{R}$ *over the non-empty set* $A\subseteq\mathbb{W}$ *is*

$$\operatorname{graph}_\nu(\varphi;A)=\{\Phi(w):w\in A\}=\{w\cdot\varphi(w):w\in A\}\subseteq\mathbb{G},$$

where $\Phi\colon A\to\mathbb{G}$, $\Phi(w)=w\cdot\varphi(w)$ *for* $w\in A$, *is the* graph map. *We say that* φ *is* intrinsic Lipschitz *on* A *with* intrinsic Lipschitz constant $L\in[0,+\infty)$, *and we write* $\varphi\in\operatorname{Lip}_{\mathbb{W}}(A)$ *and* $L=\operatorname{Lip}_{\mathbb{W}}(\varphi;A)$, *if, for* $L>0$,

$$\operatorname{graph}_\nu(\varphi;A)\cap C(p,1/L)=\emptyset\quad \textit{for all } p\in\operatorname{graph}_\nu(\varphi;A),$$

and φ *constant on* A *for* $L=0$. *Equivalently, for all* $p,q\in\operatorname{graph}_\nu(\varphi;A)$, *it holds that*

$$|\varphi(\pi_{\mathbb{W}}(p))-\varphi(\pi_{\mathbb{W}}(q))|\leqslant L\|\pi_{\mathbb{W}}(q^{-1}\cdot p)\|_\infty.$$

We use the shorthand $\operatorname{Lip}_{\mathbb{W}}(\varphi)=\operatorname{Lip}_{\mathbb{W}}(\varphi;\mathbb{W})$.

As established in [144, Proposition 3.8], intrinsic Lipschitz functions are continuous (in fact, $\frac{1}{2}$-Hölder continuous, since $\mathbb{G}$ is a Carnot group of step 2). The following result, which generalizes [221, Proposition 4.8], is a particular instance of [144, Theorem 4.1] and [277, Theorem 1.5]. The key point here is to provide an explicit bound on the intrinsic Lipschitz constant of the intrinsic Lipschitz extension.

Theorem 15.7.4 (Intrinsic Lipschitz extension). *There is* $c=c(\varepsilon_2,\mathcal{M})>0$ *with the following property. If* $\varphi\in\operatorname{Lip}_{\mathbb{W}}(A)$ *for some* $\emptyset\neq A\subseteq\mathbb{W}$, *with* $L=\operatorname{Lip}_{\mathbb{W}}(\varphi;A)$, *then there exists* $\psi\in\operatorname{Lip}_{\mathbb{W}}(\mathbb{W})$ *such that* $\psi(w)=\varphi(w)$ *for all* $w\in A$, $\|\psi\|_{L^\infty(\mathbb{W})}=\|\varphi\|_{L^\infty(A)}$ *and*

$$\operatorname{Lip}_{\mathbb{W}}(\psi)\leqslant c\,\max\Big\{L,L^4\Big\}.$$

Here $\mathcal{M} > 0$ *is the constant in* (15.2.2).

Proof. Assume $L > 0$ to avoid trivialities, let $\alpha = 1/L$, and define the open set

$$E = \bigcup_{w \in A} C^+(\Phi(w), \alpha) \neq \emptyset.$$

Setting $\beta = \frac{\alpha^2}{\alpha+2} \frac{4}{4+\mathcal{M}\varepsilon_2^2}$, by Lemma 15.7.2(iii) we get that, if $q \in E$, then $C^+(q, \beta) \subseteq E$. By an elementary continuity argument, the latter inclusion also holds for any $q \in \partial E$, the topological boundary of E. Consequently, if $p, q \in \partial E$, then $p \notin C^+(q, \beta)$. As in the proof of [221, Proposition 4.8], we thus get that $\psi \colon \mathbb{W} \to \mathbb{R}$, given by

$$\psi(w) = s_w \nu, \quad \text{where } s_w = \min\Big\{\inf\{s \in \mathbb{R} : w \cdot s\nu \in E\}, \|\varphi\|_{L^\infty(A)}\Big\} \text{ for } w \in \mathbb{W},$$

is well defined and such that $\psi(w) = \varphi(w)$ for all $w \in A$, $\mathrm{graph}_\nu(\psi; \mathbb{W}) \subseteq \partial E$ and $\|\psi\|_{L^\infty(\mathbb{W})} = \|\varphi\|_{L^\infty(A)}$. Finally, given $p, q \in \mathrm{graph}_\nu(\psi; \mathbb{W})$, arguing as in the proof of [221, Proposition 4.8] and in virtue of Lemma 15.7.2(ii), we get that, if $p \notin C^+(q, \beta)$, then $q \notin C^-(p, \gamma)$, where $\gamma > 0$ is chosen such that $\beta = \gamma + \varepsilon_2 \sqrt{\gamma \mathcal{M}}$, that is, $\gamma = \frac{1}{4}\left(\sqrt{\varepsilon_2^2 \mathcal{M} + 4\beta} - \varepsilon_2 \sqrt{\mathcal{M}}\right)^2$. In particular, $\psi \in \mathrm{Lip}_{\mathbb{W}}(\mathbb{W})$ with $\mathrm{Lip}_{\mathbb{W}}(\psi) = 1/\gamma$, and a simple computation yields that $\mathrm{Lip}_{\mathbb{W}}(\psi) \leqslant c \max\{L, L^4\}$ with $c = c(\varepsilon_2, \mathcal{M}) > 0$, concluding the proof. □

15.7.3 Intrinsic gradient

Intrinsic graphs are naturally associated with an intrinsic notion of gradient, which determines in a sense their intrinsic differentiability properties. Again, a finer discussion is postponed until Chapter 16. The following definition rephrases [18, Definition 3.1].

Definition 15.7.5 (φ-gradient)**.** *Let* $A \subseteq \mathbb{W}$ *be a non-empty open set and* $\varphi \in C(A)$*. The* φ-gradient of $f \in C^\infty(\mathbb{W})$ *is* $W^\varphi f = (W_1^\varphi f, \dots W_{m_1-1}^\varphi f) \colon A \to \mathbb{R}^{m_1-1}$*, where*

$$W_i^\varphi f(w) = X_{i+1}(f \circ \pi_{\mathbb{W}})(\Phi(w))$$

for all $w \in A$ *and each* $i = 1, \dots, m_1 - 1$.

We can hence give the following definition (cf. the first lines of the proof of [18, Proposition 4.10] and [117, Definition 3.2].

Definition 15.7.6 (Intrinsic gradient)**.** *Let* $A \subseteq \mathbb{W}$ *be a non-empty open set. The* intrinsic gradient *of* $\varphi \in C(A)$ *is the distribution* $W^\varphi \varphi = (W^\varphi \varphi_1, \dots, W^\varphi \varphi_{m_1})$ *acting as*

$$\langle \nabla_i^\varphi \varphi, \vartheta \rangle = \int_A \varphi\, (\nabla_i^\varphi)^* \vartheta \, \mathrm{d}\mathcal{L}^{n-1} \quad \textit{for any } \vartheta \in C_c^1(A),$$

where $(\nabla_i^\varphi)^*$ *is the formal adjoint of* W_i^φ*, for each* $i = 1, \dots, m_1$.

The following result, which is an immediate consequence of [117, Proposition 5.3], generalizes [91, Proposition 4.4] to any Carnot group of step 2.

Theorem 15.7.7 (Bound on the intrinsic gradient)**.** *Let $A \subseteq \mathbb{W}$ be a non-empty open set. If $\varphi \in \mathrm{Lip}_{\mathbb{W}}(A)$, then $W^\varphi\varphi \in L^\infty(A;\mathbb{R}^{m_1-1})$, with $\|W^\varphi\varphi\|_{L^\infty(A)} \leqslant C_L$, for some $C_L > 0$ depending on $L = \mathrm{Lip}_{\mathbb{W}}(\varphi; A)$ only.*

15.7.4 Intrinsic area formula

The following result follows from [117, Lemma 5.2] and [117, Theorem 5.7] (also cf. [18, Proposition 4.10] for more regular functions).

Theorem 15.7.8 (Intrinsic area formula)**.** *Let $A \subseteq \mathbb{W}$ be an non-empty open set. The* intrinsic epigraph of $\varphi \in \mathrm{Lip}_{\mathbb{W}}(A)$ over A*,*

$$E_{\varphi,A} = \{\exp(sX_1) : w \in A,\ s > \varphi(w)\} \subseteq \mathbb{G},$$

has locally finite $\mathbb{G}$-perimeter in $A \cdot \mathbb{R}$, its inner horizontal normal is given by

$$\nu_{E_{\varphi,A}}(w \cdot \varphi(w)) = \left(\frac{1}{\sqrt{1+|W^\varphi\varphi(w)|^2}}, \frac{-W^\varphi\varphi(w)}{\sqrt{1+|W^\varphi\varphi(w)|^2}}\right) \quad \textit{for } \mathcal{L}^{n-1}\textit{-a.e. } w \in A,$$

and its $\mathbb{G}$-perimeter satisfies the intrinsic area formula

$$P_{\mathbb{G}}(E_{\varphi,A}; A' \cdot \mathbb{R}) = \int_{A'} \sqrt{1+|W^\varphi\varphi(w)|^2}\, \mathrm{d}\mathcal{L}^{n-1}(w) \quad \textit{for any } A' \subseteq A. \tag{15.7.5}$$

It is worth noticing that, via well-known standard arguments, the area formula (15.7.5) can be generalized as

$$\int_{\partial E_{\varphi,A} \cap A' \cdot \mathbb{R}} g(p)\, dP_{\mathbb{G}}(E,\cdot)(p) = \int_{A'} g(\Phi(w))\sqrt{1+|W^\varphi\varphi(w)|^2}\, \mathrm{d}\mathcal{L}^{n-1}(w)$$

whenever $g \colon \partial E_{\varphi,A} \to \mathbb{R}$ is a Borel function.

15.8 Intrinsic Lipschitz approximation

Throughout this section, we assume that $(\mathbb{G}, \cdot)$ is a plentiful group as in Definition 15.6.1. Our approach adapts some ideas of [221, 228, 227] to the present more general setting. The following result corresponds to [228, Lemma 3.3], which was stated in the setting of the Heisenberg groups $\mathbb{H}^n$, $n \geqslant 2$ (also cf. [202, Lemma 22.10] for the Euclidean case). The very same result holds for any plentiful group, with identical proof, thanks to Theorem 15.6.6.

Lemma 15.8.1 (Small-excess position)**.** *For any $s \in (0,1)$, $\Lambda \in [0,+\infty)$ and $r \in (0,+\infty]$ with $\Lambda r_0 \leqslant 1$, there exists $\omega(s,\Lambda,r_0) > 0$ with the following property. If $E \subseteq \mathbb{G}$ is a (Λ, r_0)-minimizer of the $\mathbb{G}$-perimeter in C_2, with $0 \in \partial E$ and $\mathbf{e}(2) \leqslant \omega(s,\Lambda,r_0)$, then*

$$\sup\Big\{|\mathfrak{h}(p)| : p \in C_1 \cap \partial E\Big\} \leqslant s,$$

$$\mathscr{L}^{n-1}\left(\left\{p \in E \cap C_1 : \mathscr{f}(p) > s\right\}\right) = 0,$$
$$\mathscr{L}^{n-1}\left(\left\{p \in C_1 \setminus E : \mathscr{f}(p) < -s\right\}\right) = 0.$$

We are now finally ready to state and prove our main result, which generalizes [221, Theorem 5.1] and, only partially, [227, Theorem 3.1] to the setting of plentiful groups. Its proof revisits that of [227, Theorem 3.1], closely following the usual approach in the Euclidean setting (cf. [202, Theorem 23.7]).

Theorem 15.8.2 (Intrinsic Lipschitz approximation)**.** *For any $L \in (0,1)$, $\Lambda \in [0,+\infty)$ and $r_0 \in (0,+\infty]$, with $\Lambda r_0 \leqslant 1$, there exist $\varepsilon, C > 0$, depending on L, Λ and r_0 only, with the following property. If $E \subseteq \mathbb{G}$ is a (Λ, r_0)-minimizer of the $\mathbb{G}$-perimeter in C_{324} with $\mathbf{e}(324) \leqslant \varepsilon$ and $0 \in \partial E$, then, letting*

$$M = C_1 \cap \partial E, \quad M_0 = \left\{q \in M : \sup_{0<r<16} \mathbf{e}(q,r) \leqslant \varepsilon\right\},$$

there exists an intrinsic Lipschitz function $\varphi\colon \mathbb{W} \to \mathbb{R}$ such that

$$\sup_{\mathbb{W}} |\varphi| \leqslant L, \quad \mathrm{Lip}_{\mathbb{W}}(\varphi) \leqslant c(\varepsilon_2, \mathcal{M})\, L, \tag{15.8.1}$$

$$M_0 \subseteq M \cap \Gamma, \quad \Gamma = \mathrm{graph}_\nu(\varphi; D_1), \tag{15.8.2}$$

$$\mathcal{S}^{Q-1}_\infty(M \Delta \Gamma) \leqslant C\, \mathbf{e}(324), \tag{15.8.3}$$

$$\int_{D_1} |W^\varphi \varphi|^2 \,\mathrm{d}\mathscr{L}^{n-1} \leqslant C\, \mathbf{e}(324), \tag{15.8.4}$$

where $c(\varepsilon_2, \mathcal{M}) > 0$ is the constant given by Theorem 15.7.4.

Proof. Let $L \in (0,1)$, $\Lambda \in [0,+\infty)$ and $r_0 \in (0,+\infty]$ be fixed and let E, M and M_0 be as in the statement. With the notation of Lemma 15.8.1, we choose

$$\varepsilon = \min\left\{\frac{\omega(L,\Lambda,r_0)}{162^{Q-1}}, \omega\left(L, 8\Lambda, \frac{r_0}{8}\right)\right\}. \tag{15.8.5}$$

The proof is then divided into three steps.

Step 1: construction of φ. Since $\mathbf{e}(324) \leqslant \omega(L,\Lambda,r_0)$ by (15.8.5), by Lemma 15.8.1 we have

$$\sup\left\{|\mathscr{f}(p)| : p \in C_1 \cap \partial E\right\} \leqslant L. \tag{15.8.6}$$

Given $p \in M$ and $q \in M_0$, we have $p, q \in C_1$, so that $\lambda = d_C(p,q) < 8$ by (15.3.7). By Remark 15.4.1, the set $F = \delta_{\lambda^{-1}}(q^{-1} \cdot E)$ is a $(\lambda\Lambda, \frac{r_0}{\lambda})$-minimizer of the $\mathbb{G}$-perimeter in $C_{\frac{324}{\lambda}}(q^{-1})$ with $0 \in \partial F$. Since $C_{\frac{324}{\lambda}}(q^{-1}) \supset C_{\frac{81}{2}}(q^{-1}) \supset C_2$ for all $q \in C_1$, by the invariance properties of the excess and by definition of M_0, we infer that

$$\mathbf{e}(F,0,2) = \mathbf{e}(E,q,2\lambda) \leqslant \varepsilon. \tag{15.8.7}$$

Recalling that $\lambda < 8$, F is a $(8\Lambda, \frac{r_0}{8})$-minimizer of the $\mathbb{G}$-perimeter in $C_{\frac{324}{\lambda}}(q^{-1})$. Since $\varepsilon \leqslant \omega(L, 8\Lambda, \frac{r_0}{8})$ due to (15.8.5), by (15.8.7) and again by Lemma 15.8.1, we infer that

$$\sup\Big\{|\not{\!\!\mathfrak{f}}(v)| : v \in C_1 \cap \partial F\Big\} \leqslant L.$$

In particular, choosing $v = \delta_{\lambda^{-1}}(q^{-1} \cdot p) \in C_1 \cap \partial F$, we get that

$$|\not{\!\!\mathfrak{f}}(q^{-1} \cdot p)| \leqslant L\, d_C(p, q).$$

Since $L < 1$, the above inequality, combined with the definition in (15.3.5), yields that $d_C(p, q) = \|\pi_{\mathbb{W}}(q^{-1} \cdot p)\|_\infty$, so that

$$|\not{\!\!\mathfrak{f}}(q^{-1} \cdot p)| \leqslant L\, \|\pi_{\mathbb{W}}(q^{-1} \cdot p)\|_\infty \quad \text{for all } p \in M,\ q \in M_0. \tag{15.8.8}$$

As a consequence, the projection $\pi_{\mathbb{W}}$ is invertible on M_0, and we can thus define a function $\varphi\colon \pi_{\mathbb{W}}(M_0) \to \mathbb{R}$ by letting $\varphi(\pi_{\mathbb{W}}(p)) = \not{\!\!\mathfrak{f}}(p)$ for all $p \in M_0$. Due to (15.8.8), we get that

$$|\varphi(\pi_{\mathbb{W}}(p)) - \varphi(\pi_{\mathbb{W}}(q))| \leqslant L\, \|\pi_{\mathbb{W}}(q^{-1} \cdot p)\|_\infty \quad \text{for all } p, q \in M_0,$$

so that $\varphi \in \mathrm{Lip}_{\mathbb{W}}(\pi_{\mathbb{W}}(M_0))$ with $\mathrm{Lip}_{\mathbb{W}}(\varphi; \pi_{\mathbb{W}}(M_0)) \leqslant L < 1$, in virtue of Definition 15.7.3. Since $M_0 \subseteq M$, from (15.8.6) we also get that $|\varphi(\pi_{\mathbb{W}}(p))| \leqslant L$ for all $p \in M_0$. By Theorem 15.7.4, we can find an extension of φ to the whole $\mathbb{W}$ (for which we keep the same notation) such that $\mathrm{Lip}_{\mathbb{W}}(\varphi) \leqslant c(\varepsilon_2, \mathcal{M})\, L$ and $|\varphi(w)| \leqslant L$ for all $w \in \mathbb{W}$. By construction, we also get that $M_0 \subseteq M \cap \Gamma$, where $\Gamma = \mathrm{graph}_\nu(\varphi; D_1)$. This proves (15.8.1) and (15.8.2).

Step 2: covering argument. We now prove (15.8.3) via a covering argument. By definition of M_0, for each $q \in M \setminus M_0$ there exists $r_q \in (0, 16)$ such that

$$\int_{C_{r_q}(q) \cap \partial E} \frac{|\nu_{\mathbb{G}} - \nu|^2}{2}\, \mathrm{d}\mathcal{S}^{Q-1}_\infty > \varepsilon\, r_q^{Q-1}. \tag{15.8.9}$$

The family of balls $\Big\{B_{2r_q}(q) : q \in M \setminus M_0\Big\}$ is a covering of $M \setminus M_0$. By Vitali's Covering Lemma, there exist $q_h \in M \setminus M_0$, for $h \in \mathbb{N}$, such that the countable subfamily

$$\{B_{d_\infty}(q_h, 2r_h) : r_h = r_{q_h},\ q_h \in M \setminus M_0,\ h \in \mathbb{N}\}$$

is disjoint, and the family $\{B_{d_\infty}(q_h, 10r_h) : h \in \mathbb{N}\}$ is still a covering of $M \setminus M_0$. Therefore, by Theorem 15.4.2, we can estimate

$$\begin{aligned} \mathcal{S}^{Q-1}_\infty(M \setminus M_0) &\leqslant \sum_{h \in \mathbb{N}} \mathcal{S}^{Q-1}_\infty\Big((M \setminus M_0) \cap B_{d_\infty}(q_h, 10r_h)\Big) \\ &\leqslant \sum_{h \in \mathbb{N}} \mathcal{S}^{Q-1}_\infty(M \cap B_{d_\infty}(q_h, 10r_h)) \leqslant c \sum_{h \in \mathbb{N}} r_h^{Q-1}, \end{aligned} \tag{15.8.10}$$

where $c > 0$ is a constant that does not dependent on L, Λ or r_0. Now note that $B_{d_\infty}(q_h, 10r_h) \subseteq$

C_{324} for all $h \in \mathbb{N}$, since, in virtue of (15.3.6), any $p \in B_{d_\infty}(q_h, 10r_h)$ satisfies

$$\|p\|_C \leqslant 2\|p\|_\infty \leqslant 2d_\infty(p, q_h) + 2\|q_h\|_\infty < 20r_h + 4\|q_h\|_C < 324.$$

Moreover, since $C_{r_h}(q_h) \subseteq B_{d_\infty}(q_h, 2r_h)$ by (15.3.7), also the cylinders $\{C_{r_h}(q_h) : h \in \mathbb{N}\}$ are disjoint and contained in C_{324}. Therefore, by combining (15.8.9) with (15.8.10), we get that

$$\mathcal{S}^{Q-1}_\infty(M \setminus M_0) \leqslant \frac{c}{\varepsilon} \sum_{h\in\mathbb{N}} \int_{C_{r_h}(q_h)\cap\partial E} \frac{|\nu_{\mathbb{G}} - \nu|^2}{2} \, \mathrm{d}\mathcal{S}^{Q-1}_\infty \leqslant \frac{c}{\varepsilon} \, \mathbf{e}(324).$$

Consequently, since $M \setminus \Gamma \subseteq M \setminus M_0$, we conclude that

$$\mathcal{S}^{Q-1}_\infty(M \setminus \Gamma) \leqslant \frac{c}{\varepsilon} \, \mathbf{e}(k),$$

which is the first half of (15.8.3). To prove the second half of (15.8.3), we observe that

$$\mathbf{e}(2) \leqslant \left(\frac{324}{2}\right)^{Q-1} \mathbf{e}(324) \leqslant \omega(L, \Lambda, r_0),$$

thanks to the properties of the excess and (15.8.5). Hence, by (15.5.1) in Lemma 15.5.2,

$$\begin{aligned} \mathcal{S}^{Q-1}_\infty(\Gamma \setminus M) &= \int_{\pi_{\mathbb{W}}(\Gamma\setminus M)} \sqrt{1 + |W^\varphi\varphi|^2} \, \mathrm{d}\mathcal{L}^{n-1} \\ &\leqslant \sqrt{1 + \|W^\varphi\varphi\|^2_{L^\infty(\mathbb{W})}} \, \mathcal{L}^{n-1}(\pi_{\mathbb{W}}(\Gamma \setminus M)) \\ &\leqslant \sqrt{1 + \|W^\varphi\varphi\|^2_{L^\infty(\mathbb{W})}} \, \mathcal{S}^{Q-1}_\infty\Big(M \cap \pi_{\mathbb{W}}^{-1}(\pi_{\mathbb{W}}(\Gamma \setminus M))\Big). \end{aligned}$$

In virtue of Theorem 15.7.7, we can estimate

$$\sqrt{1 + \|W^\varphi\varphi\|^2_{L^\infty(\mathbb{W})}} \leqslant C_L,$$

where $C_L > 0$ depends on L only. Since $M \cap \pi_{\mathbb{W}}^{-1}(\pi_{\mathbb{W}}(\Gamma \setminus M)) \subseteq M \setminus \Gamma$, we get that

$$\mathcal{S}^{Q-1}_\infty(\Gamma \setminus M) \leqslant C_L \, \mathcal{S}^{Q-1}_\infty(M \setminus \Gamma) \leqslant \frac{C_L}{\varepsilon} \, \mathbf{e}(k),$$

completing the proof of (15.8.3).

Step 3: estimate on the L^2 energy. Finally, we prove (15.8.4). By Theorem 15.7.8 and [Corollary 2.6][15], for $\mathcal{S}^{Q-1}_\infty$-a.e. $p \in M \cap \Gamma$ there exists $\sigma(p) \in \{-1, 1\}$ such that

$$\nu_{\mathbb{G}}(p) = \sigma(p) \, \frac{\big(1, -W^\varphi\varphi(\pi_{\mathbb{W}}(p))\big)}{\sqrt{1 + |W^\varphi\varphi(\pi_{\mathbb{W}}(p))|^2}}.$$

Taking into account that, for $\mathcal{S}^{Q-1}_\infty$-a.e. $p \in M \cap \Gamma$,

$$\frac{|\nu_{\mathbb{G}}(p) - \nu(p)|^2}{2} = 1 - \langle \nu_{\mathbb{G}}(p), \nu(p)\rangle \geqslant \frac{1 - \langle \nu_{\mathbb{G}}(p), \nu(p)\rangle^2}{2},$$

we get that

$$\mathbf{e}(1) \geqslant \int_{M\cap\Gamma} \frac{1-\langle \nu_{\mathbb{G}}(p), \nu(p)\rangle^2}{2}\, dP_{\mathbb{G}}(E,\cdot)(p) = \frac{1}{2}\int_{M\cap\Gamma} \frac{|W^{\varphi}\varphi(\pi_{\mathbb{W}}(p))|^2}{1+|W^{\varphi}\varphi(\pi_{\mathbb{W}}(p))|^2}\, dP_{\mathbb{G}}(E,\cdot)(p)$$
$$= \frac{1}{2}\int_{\pi_{\mathbb{W}}(M\cap\Gamma)} \frac{|W^{\varphi}\varphi(w)|^2}{1+|W^{\varphi}\varphi(w)|^2}\, \mathrm{d}\mathcal{L}^{Q-1}(w).$$

By Theorem 15.7.7 and the scaling property of the excess, we get that

$$\int_{\pi_{\mathbb{W}}(M\cap\Gamma)} |W^{\varphi}\varphi|^2 \, \mathrm{d}\mathcal{L}^{Q-1} \leqslant C_L\, \mathbf{e}(324),$$

where $C_L > 0$ depends on L only. Moreover, by Theorem 15.7.8, we can estimate

$$\int_{\pi_{\mathbb{W}}(M\Delta\Gamma)} |W^{\varphi}\varphi|^2 \, \mathrm{d}\mathcal{L}^{Q-1} \leqslant \int_{M\Delta\Gamma} \frac{|W^{\varphi}\varphi(\pi_{\mathbb{W}}(p))|^2}{1+|W^{\varphi}\varphi(\pi_{\mathbb{W}}(p))|^2}\, dP_{\mathbb{G}}(E,\cdot)(p) \leqslant \mathcal{S}_{\infty}^{Q-1}(M\Delta\Gamma),$$

and (15.8.4) immediately follows from (15.8.3). The proof is complete. □

Part VI

Hypersurfaces in the Heisenberg group

Chapter 16

Hypersurfaces in the Heisenberg group: an introduction

16.1 Introduction

Recently, the study of geometric properties of hypersurfaces in sub-Riemannian structures has been the object of a considerable interest among the sub-Riemannian community. A non-exhaustive list of related works include [83, 84, 85, 107, 120, 158, 160, 164, 238, 240, 249, 264]. These issues are particularly relevant in the sub-Riemannian Heisenberg group $\mathbb{H}^n$ since, as we know, the latter constitutes a prototypical model in the setting of Carnot groups, sub-Riemannian manifolds, CR manifolds and Carnot-Carathéodory spaces. One of the key differences between the Euclidean and the Heisenberg setting is that, as pointed out in Section 3.6, the classical Federer's notion of rectifiability in metric spaces (cf. [131]) is not suitable for the Heisenberg group. To solve this issue, the authors of [140] introduced the intrinsic notion of $\mathbb{H}$-*regular hypersurface*, which is the Heisenberg counterpart of Definition 3.6.1, together with the related notion of *intrinsic rectifiability* as in Definition 3.6.3. Roughly speaking, an $\mathbb{H}$-regular hypersurface is a subset of $\mathbb{H}^n$ which can be described locally as the zero locus of a $C^1_{\mathbb{H}}$-function, i.e. a continuous function whose horizontal gradient is continuous and locally non-vanishing. A special class of $\mathbb{H}$-regular hypersurfaces is that of *non-characteristic hypersurfaces*. In this setting, given a hypersurface S of class C^1, we say that a point $p \in S$ is *characteristic* as soon as

$$\mathcal{H}_p = T_pS,$$

being $\mathcal{H}_p$ the horizontal distribution that we previously denoted by $\mathcal{G}_p$ in general Carnot groups (cf. (3.2.3)). Otherwise, we say that p is *non-characteristic*. In this last case, the *horizontal tangent space*

$$\mathcal{H}T_pS = \mathcal{H}_p \cap T_pS$$

is a $(2n-1)$-dimensional vector space. The set of characteristic points of S is denoted by S_0 and is called the *characteristic set* of S. After [140], it was clear that the importance of $\mathbb{H}$-regular hypersurfaces went beyond rectifiability. In the following chapters, we will discuss some results which are related to two fundamental open problems, i.e. the *isoperimetric problem* and the *Bernstein problem*. Loosely speaking, the isoperimetric problem consists in characterizing sets whose boundaries minimizes the horizontal perimeter under a volume constraint. Instead, by Bernstein problem we mean the characterization of those sets whose boundaries are global minimizers of the horizontal perimeter, thus without volume constraints. In the following sections we will collect some basic definitions and preliminaries to properly settle down the aforementioned problems.

16.2 The Heisenberg group

For the sake of completeness, we recall the definition of Heisenberg groups as in Example 1.2.3.

Definition 16.2.1 (Heisenberg group)**.** *The n-th Heisenberg group $(\mathbb{H}^n, \cdot)$ is $\mathbb{R}^{2n+1}$ endowed with the group law*

$$p \cdot p' = (\bar{x}, \bar{y}, t) \cdot (\bar{x}', \bar{y}', t') = (\bar{x} + \bar{x}', \bar{y} + \bar{y}', t + t' + Q((\bar{x}, \bar{y}), (\bar{x}', \bar{y}'))),$$

where

$$Q((\bar{x}, \bar{y}), (\bar{x}', \bar{y}')) = \sum_{j=1}^{n} (x_j' y_j - x_j y_j')$$

and where we denoted points $p \in \mathbb{R}^{2n+1}$ by $p = (z, t) = (\bar{x}, \bar{y}, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$.

With this operation, it is easy to check that $\mathbb{H}^n$ is a Carnot group. Indeed, let us consider the vector fields

$$X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial t} \qquad \text{and} \qquad T = \frac{\partial}{\partial t}$$

for $j = 1, \ldots, n$. An easy computation shows that $X = (X_1, \ldots, X_n, Y_1, \ldots, Y_n, T)$ is a basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{H}^n$. Let us define

$$\mathfrak{g}_1 = \operatorname{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \qquad \text{and} \qquad \mathfrak{g}_2 = \operatorname{span}\{T\}.$$

Since the only non-trivial commutation relations are

$$[X_j, Y_j] = -[Y_j, X_j] = -2T \tag{16.2.1}$$

for any $j = 1, \ldots, n$, then $(\mathfrak{g}_1, \mathfrak{g}_2)$ is a stratification of $\mathfrak{g}$. Hence $\mathbb{H}^n$ is a Carnot group of dimension $2n + 1$, step 2 and rank $2n$. In particular $\mathbb{H}^n$ has homogeneous dimension

$$Q = 2n + 2.$$

Moreover, the intrinsic dilations associated to $\mathbb{H}^n$ can be written explicitly as

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$$

for any $p = (z,t) \in \mathbb{H}^n$ and any $\lambda > 0$. We recall that the horizontal distribution (cf. (3.2.3)) of the Heisenberg group reads as

$$\mathcal{H}_p := \text{span}\{X_1|_p, \dots, X_n|_p, Y_1|_p, \dots, Y_n|_p\}$$

for any $p \in \mathbb{H}^n$. A certain relevance in Heisenberg geometry is played by the *ninety-degrees rotation* operator J, defined by

$$J\left(\sum_{j=1}^n A_j X_j + \sum_{j=1}^n A_{n+j} Y_j + A_{2n+1}\varepsilon T\right) = -\sum_{j=1}^n A_{n+j} X_j + \sum_{j=1}^n A_j Y_j \tag{16.2.2}$$

for any vector field

$$\sum_{j=1}^n A_j X_j + \sum_{j=1}^n A_{n+j} Y_j + A_{2n+1}\varepsilon T.$$

This operator, which is usually known as *CR structure*, is one of the key ingredient of pseudohermitian geometry (cf. [83, Appendix]). Apart from its importance in the establishment of appropriate variation formulas (cf. Chapter 17), we refer e.g. to [252] for its relation with *sub-Riemannian geodesic equations* and to [87] for its connection with *rigid Heisenberg motions*.

16.3 Riemannian Heisenberg groups

Being a Carnot group, $\mathbb{H}^n$ can be endowed with a special class of Riemannian structures which are well-behaved with respect to its Lie group structure. More precisely, for any $\varepsilon \neq 0$, we consider the unique left-invariant Riemannian metric (cf. Definition 3.2.9) $g_\varepsilon = \langle \cdot, \cdot \rangle_\varepsilon$ on $\mathbb{H}^n$ for which $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \varepsilon T\}$ is an orthonormal basis at every point. We may also use the compact notation $\{Z_1, \dots Z_{2n+1}\}$ to denote the previous basis, where the dependence of Z_{2n+1} on ε is omitted for the sake of notational simplicity. Let us denote by ∇^ε the Levi-Civita connection associated to the metric g_ε. We recall that ∇^ε is the unique affine connection on $\mathbb{H}^n$ which is *torsion-free*, that is

$$\nabla^\varepsilon_A B - \nabla^\varepsilon_B A = [A, B] \tag{16.3.1}$$

for any pair of C^1-vector fields A and B, and *metric*, meaning that

$$A(g_\varepsilon(B, C)) = g_\varepsilon(\nabla^\varepsilon_A B, C) + g_\varepsilon(B, \nabla^\varepsilon_A C) \tag{16.3.2}$$

for any pair of C^1-vector fields A and B (cf. [119] for some basic preliminaries about affine connections). For future convenience, we recall that, if ∇ is any affine connection, its *torsion tensor* is defined by

$$\text{Tor}_{\nabla^{\mathbb{H}}}(A, B) = \nabla_A B - \nabla_B A - [A, B]$$

for any pair of vector fields A and B of class C^1 (cf. [119]). In the following, we will need to know the explicit behavior of ∇^ε. To this aim, its action on the frame $(Z_1, \dots, Z_{2n+1})$ can be easily computed as follows.

Proposition 16.3.1. *The following relations hold.*

$$\nabla^\varepsilon_{X_i} X_j = \nabla^\varepsilon_{Y_i} Y_j = \nabla^\varepsilon_{\varepsilon T} \varepsilon T = 0 \qquad \textit{for any } i, j = 1, \ldots, n, \tag{16.3.3}$$

$$\nabla^\varepsilon_{X_i} Y_j = \nabla^\varepsilon_{Y_i} X_j = 0 \qquad \textit{for any } i, j = 1, \ldots, n,\ i \neq j, \tag{16.3.4}$$

$$\nabla^\varepsilon_{X_i} Y_i = -T \qquad \textit{and} \qquad \nabla^\varepsilon_{Y_i} X_i = T \qquad \textit{for any } i = 1, \ldots, n, \tag{16.3.5}$$

$$\nabla^\varepsilon_{X_i} \varepsilon T = \nabla^\varepsilon_{\varepsilon T} X_i = \frac{1}{\varepsilon} Y_i \qquad \textit{for any } i = 1, \ldots, n \tag{16.3.6}$$

and

$$\nabla^\varepsilon_{Y_i} \varepsilon T = \nabla^\varepsilon_{\varepsilon T} Y_i = -\frac{1}{\varepsilon} X_i \qquad \textit{for any } i = 1, \ldots, n. \tag{16.3.7}$$

Proof. Let us recall that, if A, B, C are orthogonal vector fields, then, combining the well-known *Koszul formula* (cf. [119]) with (16.3.1) and (16.3.2), we get

$$2g_\varepsilon(C, \nabla^\varepsilon_B A) = g_\varepsilon(C, [B, A]) + g_\varepsilon(B, [C, A]) + g_\varepsilon(A, [C, B]).$$

Therefore

$$g_\varepsilon\left(X_k, \nabla^\varepsilon_{X_i} X_j\right) = g_\varepsilon\left(Y_k, \nabla^\varepsilon_{X_i} X_j\right) = g_\varepsilon\left(\varepsilon T, \nabla^\varepsilon_{X_i} X_j\right) = 0$$

for any $i, j, k = 1, \ldots, n$. Reasoning similarly, (16.3.3) and (16.3.4) follow. To prove the first part of (16.3.5), notice that

$$g_\varepsilon\left(X_k, \nabla^\varepsilon_{X_i} Y_i\right) = g_\varepsilon\left(Y_k, \nabla^\varepsilon_{X_i} Y_i\right) = 0$$

for any $i, k = 1, \ldots, n$. Moreover,

$$2g_\varepsilon\left(\varepsilon T, \nabla^\varepsilon_{X_i} Y_i\right) = g_\varepsilon\left(\varepsilon T, [X_i, Y_i]\right) = -2g_\varepsilon\left(\varepsilon T, T\right) = -\frac{2}{\varepsilon}$$

for any $i = 1, \ldots, n$, and so

$$\nabla^\varepsilon_{X_i} Y_i = \sum_{k=1}^{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{X_i} Y_i, Z_k\right) Z_k = g_\varepsilon\left(\nabla^\varepsilon_{X_i} Y_i, \varepsilon T\right) \varepsilon T = -T$$

for any $i = 1, \ldots, n$. The second part of (16.3.5) follows form (16.3.1). The first equality in (16.3.6) follows from (16.3.1). Moreover

$$g_\varepsilon\left(Z_k, \nabla^\varepsilon_{X_i} \varepsilon T\right) = 0 \qquad \text{for any } k = 1, \ldots, 2n+1,\ k \neq n+i$$

and

$$2g_\varepsilon\left(Y_i, \nabla^\varepsilon_{X_i} \varepsilon T\right) = g_\varepsilon\left(\varepsilon T, [Y_i, X_i]\right) = 2g_\varepsilon\left(\varepsilon T, T\right) = \frac{2}{\varepsilon}.$$

Hence, (16.3.6) follows. Finally, (16.3.7) follows in the same way of (16.3.6). □

In view of Proposition 16.3.1, the CR structure J can be described in terms of the Levi-

Civita connection by the formula

$$J(A) = \nabla^1_A T$$

for any vector field A. As customary in Riemannian geometry, the Levi-Civita connection allows to define the *Riemann curvature tensor* associated to $(\mathbb{H}^n, g_\varepsilon)$ (cf. [119]), namely

$$R_\varepsilon(A, B, C, D) = g_\varepsilon \left(\nabla^\varepsilon_B \nabla^\varepsilon_A C - \nabla^\varepsilon_A \nabla^\varepsilon_B C + \nabla^\varepsilon_{[A,B]} C, D \right) \tag{16.3.8}$$

for vector fields A, B, C, D of class C^2. Accordingly, recalling that $(Z_1, \ldots, Z^\varepsilon_{2n+1})$ is an orthonormal frame with respect to g_ε, the *Ricci curvature* associated to the metric g_ε (cf. [119]) is defined by

$$\mathrm{Ric}_\varepsilon(A) = \sum_{j=1}^{2n+1} R_\varepsilon(A, Z_j, A, Z_j) \tag{16.3.9}$$

for any C^2-vector field $A \in T\mathbb{H}^n$. Among the other things, the importance of the Ricci curvature in Riemannian geometry can be appreciated for instance in the computation of the second variation formula for the Riemannian perimeter. Accordingly, since Chapter 17 is devoted to the interplay between Riemannian and sub-Riemannian variation formulas in the Heisenberg group, an explicit computation of the Ricci curvature is postponed until Proposition 17.2.1.

16.4 Hypersurfaces in Riemannian Heisenberg groups

16.4.1 Some properties of the Riemannian normal

Let $\varepsilon \neq 0$ be fixed. We let $S \subseteq \mathbb{H}^n$ be an embedded orientable hypersurface of class C^2 with Riemannian unit normal ν^ε. We recall that

$$\sum_{j=1}^{2n+1} (Z_i \nu^\varepsilon_j) \nu^\varepsilon_j = 0 \tag{16.4.1}$$

for any $i = 1, \ldots, 2n+1$ for any unitary extension of ν^ε, being $(\nu^\varepsilon_1, \ldots, \nu^\varepsilon_{2n+1})$ the coordinates of ν^ε related to $\{Z_1, \ldots, Z_{2n+1}\}$. Indeed, (16.4.1) follows at once taking derivatives of the equation $|\nu^\varepsilon|^2 = 1$. Moreover, there exists a particular unitary extension of ν^ε which the special Lie group structure of $\mathbb{H}^n$ provides with some additional properties. Indeed, if we denote by d^ε be the signed Riemannian distance from S, then d^ε is of class C^2 near S, and satisfies the eikonal equation

$$|\nabla^\varepsilon d^\varepsilon| = 1$$

in a neighborhood of S (cf. e.g. [157]). Therefore, ν^ε can be extended to a suitable neighborhood of S by letting

$$\nu^\varepsilon = \nabla^\varepsilon d^\varepsilon$$

. With this extension, (16.2.1) implies that

$$Z_i(\nu^\varepsilon_j) = Z_j(\nu^\varepsilon_i) \tag{16.4.2}$$

for any $i, j = 1, \dots, 2n+1$ such that either $i = 2n+1$, $j = 2n+1$ or $|j - i| \neq n$. Moreover,

$$X_i(\nu^\varepsilon_{n+i}) = Y_i(\nu^\varepsilon_i) - \frac{2\nu^\varepsilon_{2n+1}}{\varepsilon} \qquad \text{and} \qquad Y_i(\nu^\varepsilon_i) = X_i(\nu^\varepsilon_{n+i}) + \frac{2\nu^\varepsilon_{2n+1}}{\varepsilon} \tag{16.4.3}$$

for any $i = 1, \dots, n$. Hence

$$\sum_{j=1}^{2n+1} (Z_j \nu^\varepsilon_i)\nu^\varepsilon_j = -2\frac{\nu^\varepsilon_{2n+1}}{\varepsilon} J(\nu^\varepsilon)_i \tag{16.4.4}$$

for any $i = 1, \dots, 2n+1$.

16.4.2 Mean curvature and second fundamental form

We denote by h^ε and H^ε the associated *second fundamental form* and *mean curvature* of S respectively. Let us recall (cf. [270]) that, for a given $p \in S$, the second fundamental form of S at p reads as

$$h^\varepsilon_p(v, w) = g_\varepsilon(\nabla^\varepsilon_v \nu^\varepsilon, w)$$

for any $v, w \in T_pS$, while the mean curvature of S at p is

$$H^\varepsilon(p) = \sum_{i=1}^{2n} h^\varepsilon_p(e_i, e_i) = \sum_{i=1}^{2n} g_\varepsilon(\nabla^\varepsilon_{e_i}\nu^\varepsilon, e_i) = \sum_{i=1}^{2n+1} Z_i \nu^\varepsilon_i(p) \tag{16.4.5}$$

for any orthonormal basis $e_1, \dots, e_{2n}$ of T_pS, the last equality following for any unitary extension of ν^ε. As it is well-known, h^ε and H^ε are fundamental objects in the description of the extrinsic geometry of S, for instance for their role in the celebrated *Gauss-Codazzi equations* (cf. [270]), as well as for their appearance in the first and second variation formulas (cf. [270]), as we will discuss thoroughly in Chapter 17.

16.4.3 Gradient and Laplace-Beltrami operator

A customary, S can be endowed with a Riemannian structure by restricting g_ε to S, so that the latter is naturally associated with typical objects of differential calculus, such as for instance the Laplace-Beltrami operator and the Riemannian gradient. Accordingly, let us denote by $\nabla^{\varepsilon,S}$ and $\Delta^{\varepsilon,S}$ the gradient and the Laplace-Beltrami operator in $(S, g_\varepsilon|_S)$ respectively (cf. [119]). For future purposes, we provide an explicit description of these two objects.

Proposition 16.4.1. *Let $f \in C^1(S)$. Then*

$$|\nabla^{\varepsilon,S} f|^2 = |\nabla^\varepsilon f|^2 - g_\varepsilon(\nabla^\varepsilon f, \nu^\varepsilon)^2.$$

If in addition $f \in C^2(S)$, then

$$\Delta^{\varepsilon,S} f = \sum_{i,j=1}^{2n+1} g^{i,j}_\varepsilon Z_i(Z_j f) - H^\varepsilon g_\varepsilon(\nabla^\varepsilon f, \nu^\varepsilon) + \frac{2\nu^\varepsilon_{2n+1}}{\varepsilon}\langle \nabla^\varepsilon f, J(\nu^\varepsilon)\rangle,$$

where

$$g_\varepsilon^{i,j} = \delta_{i,j} - \nu_i^\varepsilon \nu_j^\varepsilon. \tag{16.4.6}$$

Proof. Fix $p \in S$ and consider a geodesic frame $e_1, \ldots, e_{2n}$ of T_pS at p (cf. [119]), i.e. a local orthonormal frame of TS such that $\nabla^{\varepsilon,S}_{e_i} e_j(p) = 0$ for any $i, j = 1, \ldots, 2n+1$, where $\nabla^{\varepsilon,S}$ is the Levi-Civita connection of $(S, g_\varepsilon|_S)$. Let us set $e_i = \sum_{k=1}^{2n+1} \alpha_k^i Z_k$ for any $i = 1, \ldots, 2n+1$. From the definition of the basis it follow the relations

$$\sum_{k=1}^{2n+1} \alpha_k^i \alpha_k^j = \delta_{i,j}, \qquad \sum_{k=1}^{2n+1} \alpha_k^i \nu_k^\varepsilon = 0 \qquad \text{and} \qquad \sum_{k=1}^{2n} \alpha_l^k \alpha_s^k = g_\varepsilon^{l,s} \tag{16.4.7}$$

for any $i, j = 1, \ldots, 2n$ and any $l, s = 1, \ldots, 2n+1$. With respect to a geodesic frame (cf. [119]) we have that $\Delta^{\varepsilon,S} f(p) = \sum_{i=1}^{2n} e_i e_i f(p)$. Hence

$$\begin{aligned}
\Delta^{\varepsilon,S} f &= \sum_{i=1}^{2n} g_\varepsilon(\nabla^\varepsilon_{e_i} e_i, \nabla^\varepsilon f) + \sum_{i=1}^{2n} g_\varepsilon(e_i, \nabla^\varepsilon_{e_i} \nabla^\varepsilon f) \\
&= \sum_{i=1}^{2n} (g_\varepsilon(\nabla^{\varepsilon,S}_{e_i} e_i, \nabla^\varepsilon f) + g_\varepsilon(\nabla^\varepsilon f, \nu^\varepsilon) g_\varepsilon(\nabla^\varepsilon_{e_i} e_i, \nu^\varepsilon)) + \sum_{i=1}^{2n} g_\varepsilon(e_i, \nabla^\varepsilon_{e_i} \nabla^\varepsilon f) \\
&= \sum_{i=1}^{2n} -H^\varepsilon g_\varepsilon(\nabla^\varepsilon f, \nu^\varepsilon) + \sum_{i=1}^{2n} g_\varepsilon(e_i, \nabla^\varepsilon_{e_i} \nabla^\varepsilon f).
\end{aligned} \tag{16.4.8}$$

On the other hand, by (16.4.7) and (16.4.6), we have

$$\sum_{i=1}^{2n} g_\varepsilon(e_i, \nabla^\varepsilon_{e_i} \nabla^\varepsilon f) = \sum_{l,s=1}^{2n+1} g_\varepsilon^{l,s} g_\varepsilon(Z_l, \nabla^\varepsilon_{Z_s} \nabla^\varepsilon f) = \sum_{l,s=1}^{2n+1} g_\varepsilon^{l,s} \Big(Z_s(Z_l f) - g_\varepsilon(\nabla^\varepsilon_{Z_s} Z_l, \nabla^\varepsilon f) \Big). \tag{16.4.9}$$

Exploiting Proposition 16.3.1, we see that

$$\begin{aligned}
-\sum_{l,s=1}^{2n+1} g_\varepsilon^{l,s} g_\varepsilon(\nabla^\varepsilon_{Z_s} Z_l, \nabla^\varepsilon f) &= \sum_{l,s=1}^{2n+1} \nu_s^\varepsilon \nu_l^\varepsilon g_\varepsilon(\nabla^\varepsilon_{Z_s} Z_l, \nabla^\varepsilon f) \\
&= \sum_{k=1}^{n} \Big(-\nu_k^\varepsilon \nu_{n+k}^\varepsilon g_\varepsilon(\nabla^\varepsilon f, T) + \frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \nu_k^\varepsilon g_\varepsilon(\nabla^\varepsilon f, Y_k) + \nu_k^\varepsilon \nu_{n+k}^\varepsilon g_\varepsilon(\nabla^\varepsilon f, T) \\
&\quad - \frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \nu_{n+k}^\varepsilon g_\varepsilon(\nabla^\varepsilon f, X_k) + \frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \nu_k^\varepsilon g_\varepsilon(\nabla^\varepsilon f, Y_k) - \frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \nu_{n+k}^\varepsilon g_\varepsilon(\nabla^\varepsilon f, X_k) \Big) \\
&= \frac{2\nu_{2n+1}^\varepsilon}{\varepsilon} \sum_{k=1}^{n} \left(-\nu_{n+k}^\varepsilon X_k f + \nu_k^\varepsilon Y_k f \right).
\end{aligned} \tag{16.4.10}$$

Therefore, the thesis follows from (16.4.8), (16.4.9) and (16.4.10). □

16.5 sub-Riemannian Heisenberg groups

Once Riemannian Heisenberg groups are available, $\mathbb{H}^n$ can be easily equipped with a left-invariant sub-Riemannian metric, which we denote either by $\langle \cdot, \cdot \rangle$ or by $g_{\mathbb{H}}$, by restricting any of the above-introduced Riemannian metrics g_ε to the horizontal distribution $\mathcal{H}$. Accordingly, it is natural to introduce another affine connection which is tailored to the sub-Riemannian setting, namely the so-called *pseudohermitian connection* $\nabla^{\mathbb{H}}$ (cf. e.g. [251]). $\nabla^{\mathbb{H}}$ can be characterized as he unique metric connection with torsion tensor given by

$$\mathrm{Tor}_{\nabla^{\mathbb{H}}}(A,B) = \nabla^{\mathbb{H}}_A B - \nabla^{\mathbb{H}}_B A - [A,B] = 2\langle J(A), B\rangle T \tag{16.5.1}$$

for any pair of vector fields A and B of class C^1. The most relevant feature of $\nabla^{\mathbb{H}}$ (cf. [250]) is the fact that

$$\nabla^{\mathbb{H}}_{Z_i} Z_j = 0 \tag{16.5.2}$$

for any $i,j = 1,\dots,2n+1$. The major consequence of (16.5.2) is that, if one defines a suitable notion of *pseudohermitian curvature tensor* basically in the same way as in (16.3.8), then the latter is identically vanishing, whence the heuristic for which $\mathbb{H}^n$ plays the same *flat* role in pseudohermitian geometry as the Euclidean space does in Riemannian geometry.

16.6 Hypersurfaces in sub-Riemannian Heisenberg groups

16.6.1 $\mathbb{H}$-regular hypersurfaces, characteristic points

For the sake of clarity, we specialize some general notions introduced in Section 3.6 to this specific setting. Let us start by recalling the notion of $\mathbb{H}$-regular hypersurface.

Definition 16.6.1 ($\mathbb{H}$-regular hypersurfaces)**.** *We say that $S \subseteq \mathbb{H}^n$ is an* $\mathbb{H}$-regular hypersurface *if, for any $p \in S$, there exists an open neighborhood U of p and a function $f \in C^1_{\mathbb{H}}(U)$ such that*

$$S \cap U = \{q \in \mathbb{H}^n \,:\, f(q) = 0\} \qquad \textit{and} \qquad \nabla^{\mathbb{H}} f \neq 0 \textit{ on } U.$$

Here and in the following, we denote by $C^k_{\mathbb{H}}$ the spaces of horizontally differentiable functions defined in Definition 1.2.1 associated with the family $X = (Z_1, \dots, Z_{2n})$. Accordingly, the horizontal gradient and the horizontal divergence as in Definition 1.1.1 and Definition 1.1.2 are denoted respectively by $\nabla^{\mathbb{H}}$ and $\mathrm{div}_{\mathbb{H}}$. As we know, $\mathbb{H}$-regular hypersurfaces are, heuristically, the *good* hypersurfaces one wishes to deal with in sub-Riemannian geometry. For instance, as already pointed out in Definition 3.6.3, they play a crucial role in the aforementioned intrinsic rectifiability. In the smooth setting, there is a relevant class of $\mathbb{H}$-regular hypersurfaces in which we will be particularly interested.

Definition 16.6.2 (Non-characteristic points and hypersurfaces)**.** *If S is a hypersurface of class C^1, we define*

$$S_0 := \{p \in S : \mathcal{H}_p = T_p S\}$$

and we call it the characteristic set *of S. A point in S_0 is called a* characteristic point*, while a point in $S \setminus S_0$ is called a* non-characteristic point*. If $S_0 = \emptyset$, S is called a* non-characteristic hypersurface*, and otherwise it is called a* characteristic hypersurface*.*

Notice that, since S is of class C^1 and $\mathcal{H}$ is a smooth distribution, then S_0 is closed in S.

At non-characteristic points, the tangent space to a hypersurface splits into a horizontal and a non-horizontal part.

Definition 16.6.3 (Horizontal tangent space)**.** *The* horizontal tangent space *of S at p is defined by*

$$\mathcal{H}T_pS := \mathcal{H}_p \cap T_pS$$

for any $p \in S$.

When $p \in S_0$, then $\dim(\mathcal{H}T_pS) = 2n$. On the contrary, when $p \in S \setminus S_0$, we have $\dim(\mathcal{H}T_pS) = 2n-1$. In the non-characteristic part of a hypersurface of class C^1, the horizontal unit normal defined in Definition 1.4.5 admits a poinwise definition. More precisely, it holds that

$$\nu^{\mathbb{H}}(p) = \frac{N^{\mathbb{H}}(p)}{|N^{\mathbb{H}}(p)|_p} \tag{16.6.1}$$

for any $p \in S \setminus S_0$, where $N^{\mathbb{H}}(p)$ is the projection of the Euclidean unit normal onto $\mathcal{H}$, that is

$$N^{\mathbb{H}}(p) := \sum_{j=1}^{n} (\langle N(p), X_j|_p\rangle_{\mathbb{R}^{2n+1}} X_j|_p + \sum_{j=1^n} \langle N(p), Y_j|_p\rangle_{\mathbb{R}^{2n+1}} Y_j|_p, \tag{16.6.2}$$

being $N(p)$ the Euclidean unit normal to S at p. As already mentioned, non-characteristic hypersurfaces are $\mathbb{H}$-regular (cf. e.g. [263]).

Proposition 16.6.4. *Let $S \subseteq \mathbb{H}^n$ be a non-characteristic hypersurface of class C^1. Then S is $\mathbb{H}$-regular.*

On the other hand, there are instances of $\mathbb{H}$-regular hypersurfaces of class C^1 which do have characteristic points. Since an explicit example require some preliminary notions that we have not introduced yet, we postpone it until Example 16.7.2

16.6.2 Some properties of the horizontal normal

Arguing as in Section 16.4.1, we discuss some basic properties of the horizontal unit normal. To this aim, let us fix an embedded, orientable hypersurface S of class C^2. Again, it is easy to check that

$$\sum_{h=1}^{2n} \nu_h^{\mathbb{H}} Z_k(\nu_h^{\mathbb{H}}) = 0 \tag{16.6.3}$$

for any $k = 1, \dots, 2n$, where by $\nu^{\mathbb{H}}$ we mean any C^2 extension of $\nu^{\mathbb{H}}|_S$ in a neighborhood of S such that $|\nu^{\mathbb{H}}| = 1$. Again, there is a particular choice of such an extension which allows to derive further relations. Indeed, if we let $d^{\mathbb{H}}$ be the signed Carnot-Carathéodory distance from S with respect to $(Z_1, \dots, Z_{2n})$, then it is well known (cf. [251]) that $d^{\mathbb{H}}$ inherits the same regularity of S in a neighborhood of any non-characteristic point $p \in S$. Moreover, since $d^{\mathbb{H}}$ satisfies the horizontal eikonal equation in a neighborhood of S, namely

$$|\nabla^{\mathbb{H}} d^{\mathbb{H}}| = 1,$$

then $\nu^{\mathbb{H}}|_S$ can be extended by letting

$$\nu^{\mathbb{H}} = \sum_{j=1}^{n} X_j d^{\mathbb{H}} X_j + \sum_{j=1}^{n} Y_j d^{\mathbb{H}} Y_j. \tag{16.6.4}$$

With this particular extension,

$$Z_k(\nu_h^{\mathbb{H}}) = Z_h(\nu_k^{\mathbb{H}}) \tag{16.6.5}$$

for any $h, k = 1, \dots, 2n$ such that $|h-k| \neq n$. Moreover,

$$X_k(\nu_{n+k}^{\mathbb{H}}) = Y_k(\nu_k^{\mathbb{H}}) - 2Td^{\mathbb{H}} \qquad \text{and} \qquad Y_k(\nu_k^{\mathbb{H}}) = X_k(\nu_{n+k}^{\mathbb{H}}) + 2Td^{\mathbb{H}} \tag{16.6.6}$$

for any $k = 1, \dots, n$. Finally, thanks to (16.6.3), (16.6.5) and (16.6.6), we see that

$$\sum_{h=1}^{2n} \nu_h^{\mathbb{H}} Z_h(\nu_k^{\mathbb{H}}) = -2Td^{\mathbb{H}} J(\nu^{\mathbb{H}})_k \tag{16.6.7}$$

for any $k = 1, \dots, 2n$. Moreover, a simple computation shows that

$$Td^{\mathbb{H}} = \frac{N_{2n+1}}{|N^{\mathbb{H}}|} \tag{16.6.8}$$

16.6.3 Horizontal mean curvature and second fundamental forms

In the current literature, different kinds of second fundamental form are available in the sub-Riemannian setting. Let us recall the main definitions.

Definition 16.6.5 (Horizontal second fundamental form (cf. [171, 106, 87])**.** *The* horizontal second fundamental form of S at $p \in S \setminus S_0$ *is the map* $h_p^{\mathbb{H}} : \mathcal{H}T_pS \times \mathcal{H}T_pS \longrightarrow \mathbb{R}$ *defined by*

$$h_p^{\mathbb{H}}(X, Y) = -\langle \nabla_X^{\mathbb{H}} Y, \nu^{\mathbb{H}} \rangle = \langle \nabla_X^{\mathbb{H}} \nu^{\mathbb{H}}, Y \rangle$$

for any $X, Y \in \mathcal{H}T_pS$.

The second equality following being $\nabla^{\mathbb{H}}$ a metric connection. We recall that its norm is defined by

$$|h_p^{\mathbb{H}}|^2 = \sum_{i,j=1}^{2n-1} h_p(e_i, e_j)^2$$

for any $p \in S$, being $e_1, \dots, e_{2n-1}$ any orthonormal basis of $\mathcal{H}T_pS$. Notice that, in view of (16.5.1), h may not be symmetric.

Definition 16.6.6 (Horizontal mean curvature)**.** *The* horizontal mean curvature $H_p^{\mathbb{H}}$ of S at $p \in S \setminus S_0$ *is defined by*

$$H_p^{\mathbb{H}} = \operatorname{trace}(h_p) = \sum_{j=1}^{2n-1} h_p^{\mathbb{H}}(e_j, e_j).$$

In analogy with the Riemannian case, the horizontal mean curvature coincides with the divergence of the horizontal normal (cf. [106]), meaning that

$$H^{\mathbb{H}}(p) = \operatorname{div}_{\mathbb{H}} \nu^{\mathbb{H}}(p)$$

for any $p \in S$. Accordingly, the following characterization of $|h^{\mathbb{H}}|^2$, whose proof is postponed until Section 16.6.4, holds.

Proposition 16.6.7. *Let p be a non-characteristic point of S. Let $\nu^{\mathbb{H}}$ be any unitary C^2 extension of $\nu^{\mathbb{H}}|_S$. Then*

$$|h^{\mathbb{H}}_p|^2 = \sum_{h,k=1}^{2n} Z_h(\nu^{\mathbb{H}}_k) Z_k(\nu^{\mathbb{H}}_h) + 4(n-1)(Td^{\mathbb{H}})^2.$$

Moreover, if $\nu^{\mathbb{H}}|_S$ is extended as in (16.6.4), then

$$|h^{\mathbb{H}}_p|^2 = \sum_{i,j=1}^{2n} \left(Z_i \nu^{\mathbb{H}}_j(p)\right)^2 - 4(Td^{\mathbb{H}})^2.$$

Beside $h^{\mathbb{H}}$, it is possible to introduce another second fundamental form. To this aim, let us denote by $\tilde{h}^{\mathbb{H}}$ the symmetric part of $h^{\mathbb{H}}$, that is

$$\tilde{h}^{\mathbb{H}}(X,Y) = \frac{h^{\mathbb{H}}(X,Y) + h^{\mathbb{H}}(Y,X)}{2}$$

For any $X, Y \in C^1(S, \mathcal{H}TS)$, where here and in the following we denote by $C^k(S, \mathcal{H}TS)$ the family of C^k-sections of $\mathbb{H}TS$. This symmetric second fundamental form has already been considered, although through different but equivalent definitions, by several authors (cf. e.g. [106, 250, 251]). The quantities $|h^{\mathbb{H}}|$ and $|\tilde{h}^{\mathbb{H}}|$ can be related in the following way.

Proposition 16.6.8. *Let p be a non-characteristic point of S. Let $\nu^{\mathbb{H}}$ be any unitary C^2 extension of $\nu^{\mathbb{H}}|_S$. Then*

$$|\tilde{h}^{\mathbb{H}}_p|^2 = \sum_{h,k=1}^{2n} Z_h(\nu^{\mathbb{H}}_k) Z_k(\nu^{\mathbb{H}}_h) + 2(n-1)\left(Td^{\mathbb{H}}(p)\right)^2.$$

Moreover,

$$|h^{\mathbb{H}}_p|^2 = |\tilde{h}^{\mathbb{H}}_p|^2 + 2(n-1)\left(Td^{\mathbb{H}}(p)\right)^2.$$

Finally, if $\nu^{\mathbb{H}}$ is extended as in (16.6.4), then

$$|\tilde{h}^{\mathbb{H}}_p|^2 = \sum_{h,k=1}^{2n} \left(Z_h \nu^{\mathbb{H}}_k(p)\right)^2 - 2(n+1)\left(Td^{\mathbb{H}}(p)\right)^2$$

16.6.4 Horizontal gradient and horizontal tangential Laplacian

According to [106], we define the *horizontal tangential derivatives*

$$\nabla_i^{\mathbb{H},S}\xi = Z_i\bar{\xi} - g_{\mathbb{H}}\left(\nabla^{\mathbb{H}},\bar{\xi}\right)\nu_i^{\mathbb{H}}$$

for any $i = 1,\dots,2n$, where ξ is a C^1 function on an open subset of S and $\bar{\xi}$ is any C^1 extension of ξ. As customary, the horizontal tangential derivatives do not depend on the chosen extension (cf. [106]). Moreover, letting

$$\nabla^{\mathbb{H},S}\xi = \sum_{i=1}^{2n}\nabla_i^{\mathbb{H},S}\xi Z_i$$

, then

$$\nabla^{\mathbb{H},S}\xi = \nabla^{\mathbb{H}}\xi - g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi,\nu^{\mathbb{H}}\right)\nu^{\mathbb{H}} \qquad \text{and} \qquad |\nabla^{\mathbb{H},S}\xi|^2 = |\nabla^{\mathbb{H}}\xi|^2 - \left(g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi,\nu^{\mathbb{H}}\right)\right)^2. \quad (16.6.9)$$

Exploiting (16.6.9), we can give the proofs of Proposition 16.6.7 and Proposition 16.6.8.

Proof of Proposition 16.6.7. First, we show that the quantity $\sum_{h,k=1}^{2n} Z_h(\nu_k^{\mathbb{H}})Z_k(\nu_h^{\mathbb{H}})$ does not depend on the chosen unitary C^2 extension of $\nu^{\mathbb{H}}|_S$. Indeed, in view of (16.6.3), we have that

$$\begin{aligned}
\sum_{h,k=1}^{2n} Z_h(\nu_k^{\mathbb{H}})Z_k(\nu_h^{\mathbb{H}}) &= \sum_{h,k=1}^{2n}\nabla_h^{\mathbb{H},S}(\nu_k^{\mathbb{H}})Z_k(\nu_h^{\mathbb{H}}) + \sum_{h,k=1}^{2n}\nu_h^{\mathbb{H}}\langle\nabla^{\mathbb{H}}\nu_k^{\mathbb{H}},\nu^{\mathbb{H}}\rangle Z_k(\nu_h^{\mathbb{H}})\\
&= \sum_{h,k=1}^{2n}\nabla_h^{\mathbb{H},S}(\nu_k^{\mathbb{H}})\nabla_k^{\mathbb{H},S}(\nu_h^{\mathbb{H}}) + \sum_{h,k=1}^{2n}\nabla_h^{\mathbb{H},S}(\nu_k^{\mathbb{H}})\nu_k^{\mathbb{H}}\langle\nabla^{\mathbb{H}}\nu_h^{\mathbb{H}},\nu^{\mathbb{H}}\rangle\\
&= \sum_{h,k=1}^{2n}\nabla_h^{\mathbb{H},S}(\nu_k^{\mathbb{H}})\nabla_k^{\mathbb{H},S}(\nu_h^{\mathbb{H}}) + \sum_{h,k=1}^{2n} Z_h(\nu_k^{\mathbb{H}})\nu_k^{\mathbb{H}}\langle\nabla^{\mathbb{H}}\nu_h^{\mathbb{H}},\nu^{\mathbb{H}}\rangle\\
&\quad - \left(\sum_{h,k=1}^{2n}\nu_h^{\mathbb{H}}\langle\nabla^{\mathbb{H}}\nu_h^{\mathbb{H}},\nu^{\mathbb{H}}\rangle\right)^2\\
&= \sum_{h,k=1}^{2n}\nabla_h^{\mathbb{H},S}(\nu_k^{\mathbb{H}})\nabla_k^{\mathbb{H},S}(\nu_h^{\mathbb{H}}).
\end{aligned}$$

The claim then follows recalling that the horizontal tangential derivatives do not depend on the chosen extension. Let us extend $\nu^{\mathbb{H}}$ as in (16.6.4). Let $e_1,\dots,e_{2n-1}$ be an orthonormal basis of $\mathcal{H}T_pS$. For any $i = 1,\dots,2n-1$, we let $a_i^1,\dots,a_i^{2n}$ be such that

$$e_i = \sum_{j=1}^{2n} a_i^j Z_j.$$

Then, by construction,

$$\sum_{k=1}^{2n} a_i^k a_j^k = \delta_{ij}, \qquad \sum_{k=1}^{2n} a_i^k\nu_k^{\mathbb{H}} = 0 \qquad \text{and} \qquad \sum_{l=1}^{2n-1} e_k^l e_k^m = \delta_{km} - \nu_k^{\mathbb{H}}\nu_m^{\mathbb{H}} \qquad (16.6.10)$$

for any $i, j = 1, \dots, 2n-1$ and any $l, m = 1, \dots, 2n$. Hence, recalling (16.6.3) and (16.6.7),

$$\begin{aligned}
|h_p^{\mathbb{H}}|^2 &= \sum_{i,j=1}^{2n-1} \sum_{h,k,l,m=1}^{2n} a_i^h Z_h(\nu_k^{\mathbb{H}}) a_j^k a_i^l Z_l(\nu_m^{\mathbb{H}}) a_j^m \\
&= \sum_{h,k,l,m=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) Z_l(\nu_m^{\mathbb{H}}) (\delta_{hl} - \nu_h^{\mathbb{H}} \nu_l^{\mathbb{H}})(\delta_{km} - \nu_k^{\mathbb{H}} \nu_m^{\mathbb{H}}) \\
&= \sum_{h,k=1}^{2n} \left(Z_h(\nu_k^{\mathbb{H}})\right)^2 - \sum_{k=1}^{2n} \left(\sum_{h=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) \nu_h^{\mathbb{H}}\right)^2 \\
&= \sum_{i,j=1}^{2n} \left(Z_i \nu_j^{\mathbb{H}}(p)\right)^2 - 4(Td^{\mathbb{H}})^2.
\end{aligned}$$

To prove the second identity, notice that

$$\begin{aligned}
\sum_{h,k=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) Z_k(\nu_h^{\mathbb{H}}) &= \sum_{h,k=1}^{2n} \left(Z_h(\nu_k^{\mathbb{H}})\right)^2 + 2Td^{\mathbb{H}} \sum_{h,k=1}^{n} X_h(\nu_{n+k}^{\mathbb{H}}) - 2Td^{\mathbb{H}} \sum_{h,k=1}^{n} Y_h(\nu_k^{\mathbb{H}}) \\
&= \sum_{h,k=1}^{2n} \left(Z_h(\nu_k^{\mathbb{H}})\right)^2 + 2Td^{\mathbb{H}} \sum_{k=1}^{n} [X_k, Y_k] d^{\mathbb{H}} \\
&= \sum_{h,k=1}^{2n} \left(Z_h(\nu_k^{\mathbb{H}})\right)^2 - 4n(Td^{\mathbb{H}})^2
\end{aligned}$$

□

Proof of Proposition 16.6.8. Let $e_1, \dots, e_{2n-1}$ be as in the proof of Proposition 16.6.7. Notice that

$$|\tilde{h}_p^{\mathbb{H}}|^2 = \tau\left(\tilde{h}_p^{\mathbb{H}} \cdot \left(\tilde{h}_p^{\mathbb{H}}\right)^T\right) = \tau\left(\frac{\left(h_p^{\mathbb{H}} + \left(h_p^{\mathbb{H}}\right)^T\right)^2}{4}\right) = \frac{1}{2}|h_p^{\mathbb{H}}|^2 + \frac{1}{2}\tau\left(\left(h_p^{\mathbb{H}}\right)^2\right).$$

Arguing as in the proof of Proposition 16.6.7,

$$\begin{aligned}
\tau\left(\left(h_p^{\mathbb{H}}\right)^2\right) &= \sum_{i,j=1}^{2n-1} \sum_{h,k,l,m=1}^{2n} a_i^h Z_h(\nu_k^{\mathbb{H}}) a_j^k a_j^l Z_l(\nu_m^{\mathbb{H}}) a_i^m \\
&= \sum_{h,k,l,m=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) Z_l(\nu_m^{\mathbb{H}}) \sum_{i=1}^{2n-1} a_i^h a_i^m \sum_{j=1}^{2n-1} a_j^k a_j^l \\
&= \sum_{h,k,l,m=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) Z_l(\nu_m^{\mathbb{H}}) (\delta_{hm} - \nu_h^{\mathbb{H}} \nu_m^{\mathbb{H}})(\delta_{kl} - \nu_k^{\mathbb{H}} \nu_l^{\mathbb{H}}) \\
&= \sum_{h,k=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) Z_k(\nu_h^{\mathbb{H}}).
\end{aligned}$$

Exploiting Proposition 16.6.7, the thesis follows. □

Accordingly, the *horizontal tangential divergence* is defined by

$$\operatorname{div}^S_{\mathbb{H}} \varphi = \sum_{h=1}^{2n} \nabla_h^{\mathbb{H},S} \varphi_h = \operatorname{div}_{\mathbb{H}} \varphi - \sum_{h=1}^{2n} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \varphi_h, \nu^{\mathbb{H}} \right) \nu_h^{\mathbb{H}}$$

for any C^1 vector field $\varphi = \sum_{h=1}^{2n} \varphi_h Z_h$, while the *horizontal tangential Laplacian*

$$\Delta^{\mathbb{H},S} f = \sum_{i=1}^{2n} \nabla_i^{\mathbb{H},S} \nabla_i^{\mathbb{H},S} f. \tag{16.6.11}$$

Although (16.6.11) is at a first glance the natural sub-Riemannian counterpart of the Laplace-Beltrami operator of Riemannian hypersurfaces, as nicely pointed out in [106], $\Delta^{\mathbb{H},S}$ is not in general self-adjoint. To this aim, the authors of [106], introduced a *modified* version of $\Delta^{\mathbb{H},S}$, the so-called *modified horizontal tangential Laplacian*

$$\hat{\Delta}^{\mathbb{H},S} f = \Delta^{\mathbb{H},S} f + 2T d^{\mathbb{H}} \left\langle \nabla^{\mathbb{H}} f, J(\nu^{\mathbb{H}}) \right\rangle. \tag{16.6.12}$$

The most relevant feature of $\hat{\Delta}^{\mathbb{H},S}$ is that, as happens to the Laplace-Beltrami operator, it is indeed self-adjoint (cf. [106, Corollary 11.4]). Moreover, we refer to Chapter 17 for some consideration about its relation with the Laplace-Beltrami operator associated to the Riemannian Heisenberg group $(\mathbb{H}^n, g_\varepsilon)$ (cf. [32] for further insights in this direction), as well as for the derivation of a suitable *horizontal Jacobi equation*. As for the Riemannian case, we conclude this section providing an explicit expression of $\Delta^{\mathbb{H},S}$ and of $\hat{\Delta}^{\mathbb{H},S}$.

Proposition 16.6.9. *It holds that*

$$\Delta^{\mathbb{H},S} f = \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H^{\mathbb{H}} \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}} \rangle$$

and

$$\hat{\Delta}^{\mathbb{H},S} f = \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H^{\mathbb{H}} \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}} \rangle + 2T d^{\mathbb{H}} \left\langle \nabla^{\mathbb{H}} f, J(\nu^{\mathbb{H}}) \right\rangle \tag{16.6.13}$$

where $g_{\mathbb{H}}^{i,j} = \delta_{i,j} - \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}}$.

Proof. Notice that

$$\begin{aligned}
\sum_{i=1}^{2n} \nabla_i^{\mathbb{H},S}\nabla_i^{\mathbb{H},S} f &= \sum_{i=1}^{2n} \nabla_i^{\mathbb{H},S}(Z_i f - \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}}\rangle \nu_i^{\mathbb{H}}) \\
&= \sum_{i=1}^{1n} Z_i Z_i f - \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}}\rangle \sum_{i=1}^{2n} Z_i \nu_i^{\mathbb{H}} - \sum_{i,j=1}^{2n} Z_i Z_j f \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}} - \sum_{i,j=1}^{2n} Z_j f Z_i \nu_j^{\mathbb{H}} \nu_i^{\mathbb{H}} \\
&\quad - \sum_{i}^{2n} \langle \nabla^{\mathbb{H}} Z_i f, \nu^{\mathbb{H}}\rangle \nu_i^{\mathbb{H}} + \sum_{i=1}^{2n} \langle \nabla^{\mathbb{H}}(\langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}}\rangle \nu_i^{\mathbb{H}}), \nu^{\mathbb{H}}\rangle \nu_i^{\mathbb{H}} \\
&= \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H^{\mathbb{H}} \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}}\rangle - \sum_{i,j=1}^{2n} Z_j f Z_i \nu_j^{\mathbb{H}} \nu_i^{\mathbb{H}} - \sum_{i,j=1}^{2n} Z_j Z_i f \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}} \\
&\quad + \sum_{i,j,k=1}^{2n} Z_j Z_k f (\nu_i^{\mathbb{H}})^2 \nu_j^{\mathbb{H}} \nu_k^{\mathbb{H}} + \sum_{i,j,k=1}^{2n} Z_k f Z_j \nu_k^{\mathbb{H}} (\nu_i^{\mathbb{H}})^2 \nu_j^{\mathbb{H}} + \sum_{i,j,k=1}^{2n} Z_k f \nu_k^{\mathbb{H}} Z_j \nu_i^{\mathbb{H}} \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}} \\
&= \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H^{\mathbb{H}} \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}}\rangle - \sum_{i,j=1}^{2n} Z_j f Z_i \nu_j^{\mathbb{H}} \nu_i^{\mathbb{H}} - \sum_{i,j=1}^{2n} Z_j Z_i f \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}} \\
&\quad + \sum_{j,k=1}^{2n} Z_j Z_k f \nu_j^{\mathbb{H}} \nu_k^{\mathbb{H}} + \sum_{j,k=1}^{2n} Z_k f Z_j \nu_k^{\mathbb{H}} \nu_j^{\mathbb{H}} \\
&= \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H^{\mathbb{H}} \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}}\rangle,
\end{aligned}$$

and so the thesis follows. □

16.6.5 The tangent pseudohermitian connection

In this section we introduce the relevant affine connection that is typically associated with sub-Riemannian hypersurfaces in Heisenberg groups. To this aim, let us fix a non-characteristic hypersurface S of class C^2. Let $\nu^{\mathbb{H}}$ be its unit horizontal normal. The *tangent pseudohermitian connection* $\nabla^{\mathbb{H},S} : C^1(S, TS) \times C^1(S, \mathcal{H}TS) \longrightarrow C^1(S, \mathcal{H}TS)$ is defined by

$$\nabla_X^{\mathbb{H},S} Y = \nabla_X Y - \langle \nabla_X Y, \nu^{\mathbb{H}}\rangle \nu^{\mathbb{H}}.$$

Proposition 16.6.10. $\nabla^{\mathbb{H},S}$ *is well-defined.*

Proof. Let $X \in C^1(S, TS)$ and $Y \in \Gamma(\mathcal{H}TS)$. We show that $\nabla_X^{\mathbb{H},S} Y \in C^1(S, \mathcal{H}TS)$. Indeed,

$$\langle \nabla_X^{\mathbb{H},S} Y, T\rangle = \langle \nabla_X Y, T\rangle = \sum_{j=1}^{2n} X(\langle Y, Z_j\rangle)\langle Z_j, T\rangle = 0.$$

Hence $\nabla_X^S Y$ is horizontal. Therefore, for any given $\varepsilon \neq 0$ and a suitable function $\beta > 0$ of class C^1,

$$\langle \nabla_X^{\mathbb{H},S} Y, \nu^{\varepsilon}\rangle = \beta^{-1} \langle \nabla_X^{\mathbb{H},S} Y, \nu^{\mathbb{H}}\rangle = 0.$$

□

Moreover, $\nabla^{\mathbb{H},S}$ is metric in the following sense.

Proposition 16.6.11. *It holds that*

$$X\langle Y,Z\rangle = \langle \nabla_X^{\mathbb{H},S} Y, Z\rangle + \langle Y, \nabla_X^{\mathbb{H},S} Z\rangle$$

for any $X \in C^1(S,TS)$ and any $Y, Z \in C^1(S, \mathcal{H}TS)$.

Proof. If $X \in C^1(S,TS)$ and $Y, Z \in C^1(S, \mathcal{H}TS)$, then

$$\begin{aligned}
X\langle Y,Z\rangle &= \langle \nabla_X^{\mathbb{H}} Y, Z\rangle + \langle Y, \nabla_X^{\mathbb{H}} Z\rangle \\
&= \langle \nabla_X^{\mathbb{H},S} Y, Z\rangle + \langle Y, \nabla_X^{\mathbb{H},S} Z\rangle + \langle \nabla_X^{\mathbb{H}} Y, \nu^{\mathbb{H}}\rangle\langle Z, \nu^{\mathbb{H}}\rangle + \langle \nabla_X^{\mathbb{H}} Z, \nu^{\mathbb{H}}\rangle\langle Y, \nu^{\mathbb{H}}\rangle \\
&= \langle \nabla_X^{\mathbb{H},S} Y, Z\rangle + \langle Y, \nabla_X^{\mathbb{H},S} Z\rangle.
\end{aligned}$$

□

To define the torsion of $\nabla^{\mathbb{H},S}$ we have to restrict the classical definition to $C^1(S, \mathcal{H}TS)$. We define $\mathrm{Tor}_{\nabla^{\mathbb{H},S}}(X,Y) : C^1(S,\mathcal{H}TS) \times C^1(S, \mathcal{H}TS) \longrightarrow C^1(S,TS)$ by

$$\mathrm{Tor}_{\nabla^{\mathbb{H},\mathrm{S}}}(X,Y) = \nabla_X^{\mathbb{H},S} Y - \nabla_Y^{\mathbb{H},S} X - [X,Y].$$

By Frobenious theorem, $\mathrm{Tor}_{\nabla^{\mathbb{H},\mathrm{S}}}$ is well-defined.

Proposition 16.6.12. *Let $X, Y \in C^1(S, \mathcal{H}TS)$. Then*

$$\langle [X,Y], \nu^{\mathbb{H}}\rangle = 2Td^{\mathbb{H}}\langle J(X), Y)\rangle.$$

In particular

$$\mathrm{Tor}_{\nabla^{\mathbb{H},\mathrm{S}}}(X,Y) = 2\langle J(X), Y\rangle \mathcal{S},$$

where

$$\mathcal{S} = T - (Td^{\mathbb{H}})\nu^{\mathbb{H}}.$$

Proof. Let $X, Y \in C^1(S, \mathcal{H}TS)$. If $X = \sum_{j=1}^{2n} X^j Z_j$ and $Y = \sum_{j=1}^{2n} Y^j Z_j$, then

$$\begin{aligned}
-\langle [X,Y], \nu^{\mathbb{H}}\rangle &= \langle \mathrm{Tor}_{\nabla^{\mathbb{H}}}(X,Y), \nu^{\mathbb{H}}\rangle + \langle \nabla_Y X - \nabla_X Y, \nu^{\mathbb{H}}\rangle \\
&= 2\langle J(X), Y\rangle\langle \nu^{\mathbb{H}}, T\rangle + \langle \nabla_X \nu^{\mathbb{H}}, Y\rangle - \langle \nabla_Y \nu^{\mathbb{H}}, X\rangle \\
&= \sum_{i,j=1}^{2n} X^i Y^j (Z_i \nu_j^{\mathbb{H}} - Z_j \nu_i^{\mathbb{H}}) \\
&= -2(Td^{\mathbb{H}}) \sum_{i=1}^{n} X^i Y^{n+i} + 2(Td^{\mathbb{H}}) \sum_{i=1}^{n} X^{n+i} Y^i \\
&= -2(Td^{\mathbb{H}})\langle J(X), Y)\rangle.
\end{aligned}$$

In particular,

$$\mathrm{Tor}_{\nabla^{\mathbb{H},\mathrm{S}}}(X,Y) = \mathrm{Tor}_{\nabla^{\mathbb{H}}}(X,Y) - \langle \mathrm{Tor}_{\nabla^{\mathbb{H}}}(X,Y), \nu^{\mathbb{H}}\rangle\nu^{\mathbb{H}} - \langle [X,Y], \nu^{\mathbb{H}}\rangle\nu^{\mathbb{H}} = 2\langle J(X), Y\rangle \mathcal{S}.$$

□

16.7 Some relevant classes of hypersurfaces

In the following sections, we introduce some particular classes of hypersurfaces, each of which is of some importance in the overall context of the study of Heisenberg hypersurfaces.

16.7.1 t-graphs

We recall that a hypersurface S is a *t-graph* whenever there exists $u : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ such that

$$\operatorname{graph}(u) := S = \{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y})) \,:\, (\bar{x}, \bar{y}) \in \mathbb{R}^{2n}\}.$$

This class of hypersurfaces is particularly relevant in the Heisenberg setting as it allows to translates many geometric problems, such as the aforementioned Bernstein and isoperimetric problems, in terms of PDEs. On the other hand, there are several reasons for which t-graphs are not always the best class of graphs that has to be considered. For instance, t-graphs associated with an even smooth function u may have characteristic points. Accordingly, for instance, the intrinsic version of the implicit function theorem fro $\mathbb{H}$-regular hypersurfaces stated in [140] relies on a better notion of graph, namely the aforementioned *intrinsic graphs* (cf. Section 15.7.2), which will be discussed in the forthcoming Section 16.7.2. Nevertheless, determining the characteristic set of a t-graph is quite simple. To this aim, if $\Omega \subseteq \mathbb{R}^{2n}$ is open, $u \in C^1(\Omega)$ and $p = (\bar{x}, \bar{y}, u(\bar{x}, \bar{y}))$, then we recall that the Euclidean unit normal to S at p reads as

$$N(p) = \frac{1}{\sqrt{1 + |Du|^2}} \left(\sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial}{\partial y_j} - \frac{\partial}{\partial t} \right),$$

so that, recalling (16.6.2),

$$N^{\mathbb{H}}(p) = \frac{1}{\sqrt{1 + |Du|^2}} \left(\sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} - y_j \right) X_j|_p + \sum_{j=1}^{n} \left(\frac{\partial u}{\partial y_j} + x_j \right) Y_j|_p \right).$$

Therefore, letting

$$\mathcal{F}(\bar{x}, \bar{y}) = (-\bar{y}, \bar{x}) \tag{16.7.1}$$

for any $(\bar{x}, \bar{y}) \in \mathbb{R}^{2n}$, we conclude that

$$S_0 = \{(\bar{x}, \bar{y}) \in \Omega \,:\, Du(\bar{x}, \bar{y}) + \mathcal{F}(\bar{x}, \bar{y}) = 0\}. \tag{16.7.2}$$

16.7.2 Intrinsic graphs

Another relevant class of hypersurfaces is the class of *intrinsic graphs*. As we know from Section 15.7.2, an intrinsic graph is the graph of a function defined on a $(2n)$-dimensional subgroup of $\mathbb{H}^n$, say $\mathbb{W}$, onto its complementary horizontal one-dimensional subgroup, say $\mathbb{V}$.

Without loss of generality, we will assume that

$$\mathbb{V} = \text{span}\{Y_1|_0\} = \{(0, \dots, 0, y_1, \dots, 0, 0) \,:\, y_1 \in \mathbb{R}\}$$

and talk about *intrinsic Y_1-graphs.* Standard computations tell us that the complementary group of $\mathbb{V}$ is just

$$\mathbb{W} = \{p \in \mathbb{H}^n \,:\, y_1 = 0\}.$$

Let us denote points $q \in \mathbb{R}^{2n}$ by $q = (\xi_1, \dots, \xi_n, \eta_2, \dots, \eta_n, \tau) = (\bar{\xi}, \tilde{\eta}, \tau)$. We wish to identify $\mathbb{R}^{2n}$ with $\{p \in \mathbb{H}^n \,:\, y_1 = 0\}$. To this aim, we introduce the immersion map $i : \mathbb{R}^{2n} \longrightarrow \mathbb{H}^n$ defined by

$$i(\bar{\xi}, \tilde{\eta}, \tau) = (\bar{\xi}, 0, \tilde{\eta}, \tau)$$

for any $(\bar{\xi}, \tilde{\eta}, \tau) \in \mathbb{R}^{2n}$. Moreover, we identify $\mathbb{R}$ with $\{(\bar{0}, y_1, \tilde{0}, 0) \,:\, y_1 \in \mathbb{R}\}$ by means of the inclusion $j : \mathbb{R} \longrightarrow \mathbb{H}^n$ defined by

$$j(y_1) = (\bar{0}, y_1, \tilde{0}, 0)$$

for any $y_1 \in \mathbb{R}$. The maps i and j are clearly smooth, injective and open.

Definition 16.7.1 (Intrinsic Y_1-graphs.)**.** *For a given open set $\Omega \subseteq \mathbb{R}^{2n}$ and a function $\varphi : \Omega \longrightarrow \mathbb{R}$, the* Y_1-graph of φ on Ω *is defined by*

$$\begin{aligned} \text{graph}_{Y_1}(\varphi, \Omega) &= \left\{ i(\bar{\xi}, \tilde{\eta}, \tau) \cdot j\left(\varphi(\bar{\xi}, \tilde{\eta}, \tau)\right) \,:\, (\bar{\xi}, \tilde{\eta}, \tau) \in \Omega \right\} \\ &= \left\{ (\bar{\xi}, \varphi(\bar{\xi}, \tilde{\eta}, \tau), \tilde{\eta}, \tau - \xi_1 \varphi(\bar{\xi}, \tilde{\eta}, \tau)) \,:\, (\bar{\xi}, \tilde{\eta}, \tau) \in \Omega \right\}. \end{aligned}$$

Moreover, we define its parametrization map $\Psi : \Omega \longrightarrow \mathbb{H}^n$ by

$$\Psi(\bar{\xi}, \tilde{\eta}, \tau) = (\bar{\xi}, \varphi(\bar{\xi}, \tilde{\eta}, \tau), \tilde{\eta}, \tau - \xi_1 \varphi(\bar{\xi}, \tilde{\eta}, \tau))$$

for any $(\bar{\xi}, \tilde{\eta}, \tau) \in \Omega$.

We introduce the *intrinsic projection map* $\Pi : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}$ by

$$\Pi(\bar{x}, \bar{y}, t) = (\bar{x}, \tilde{y}, t + x_1 y_1)$$

for any $(\bar{x}, \bar{y}, t) \in \mathbb{H}^n$. It is easy to check that

$$\Pi(\Psi(q)) = q \qquad \text{and} \qquad \Psi(\Pi(p)) = p$$

for any $q \in \Omega$ and any $p \in \text{graph}_{Y_1}(\varphi, \Omega)$. If $\varphi \in C^1(\Omega)$ and $S = \text{graph}_{Y_1}(\varphi, \Omega)$, then

$$T_{\Psi(q)} S = \text{span}\left(\left.\frac{\partial \Psi}{\partial \xi_1}\right|_q, \dots, \left.\frac{\partial \Psi}{\partial \xi_n}\right|_q, \left.\frac{\partial \Psi}{\partial \eta_2}\right|_q, \dots, \left.\frac{\partial \Psi}{\partial \eta_n}\right|_q, \left.\frac{\partial \Psi}{\partial \tau}\right|_q \right).$$

Letting $D\varphi = (\varphi_{\xi_1}, \dots \varphi_{\xi_n}, \varphi_{\eta_2}, \dots, \varphi_{\eta_n}, \varphi_\tau)$, an easy computation shows that

$$\left.\frac{\partial \Psi}{\partial \xi_1}\right|_q = X_1|_{\Psi(q)} + \varphi_{\xi_1}(q) Y_1|_{\Psi(q)} - 2\varphi(q) T|_{\Psi(q)},$$

$$\left.\frac{\partial \Psi}{\partial \xi_j}\right|_q = X_j|_{\Psi(q)} + \varphi_{\xi_j}(q) Y_1|_{\Psi(q)} - \eta_j T|_{\Psi(q)}, \qquad \left.\frac{\partial \Psi}{\partial \eta_j}\right|_q = Y_j|_{\Psi(q)} + \varphi_{\eta_j}(q) Y_1|_{\Psi(q)} + \xi_j T|_{\Psi(q)}$$

for any $j = 2, \dots, n$ and

$$\left.\frac{\partial \Psi}{\partial \tau}\right|_q = \varphi_\tau(q) Y_1|_{\Psi(q)} + T|_{\Psi(q)}.$$

Therefor $(E_1, \dots, E_n, F_2, \dots, F_n)$ constitutes a global frame of $\mathcal{H}TS$, where

$$E_1 = X_1 + W^\varphi \varphi Y_1, \quad E_j = X_j + \tilde{X}_j \varphi Y_1 \quad \text{and} \quad F_j = Y_j + \tilde{Y}_j \varphi Y_1 \tag{16.7.3}$$

for any $j = 2, \dots, n$, and where the family of vector fields $W^\varphi = (W_1^\varphi \varphi, \tilde{X}_2, \dots \tilde{X}_n, \tilde{Y}_2, \dots, \tilde{Y}_n)$ is defined by

$$W_1^\varphi = \frac{\partial}{\partial \xi_1} + 2\varphi \tilde{T}, \quad \tilde{X}_j = \frac{\partial}{\partial \xi_j} + \eta_j \tilde{T} \quad \text{and} \quad \tilde{Y}_j = \frac{\partial}{\partial \eta_j} - \xi_j \tilde{T}$$

for any $j = 2, \dots, n$, where we have set $\tilde{T} = \frac{\partial}{\partial \tau}$. The differential operator W^φ is commonly known as *intrinsic gradient* (cf. e.g. [16, 47, 48, 46] for further insights). For future convenience, we may adopt the compact notation

$$W_j^\varphi = \tilde{X}_j \qquad \text{and} \qquad W_{n+j-1}^\varphi = \tilde{Y}_j \tag{16.7.4}$$

where $j = 2, \dots, n$. Moreover, we complete the previous family to a global frame with the vector field

$$W_{2n}^\varphi = \varepsilon \tilde{T},$$

where again the dependence on $\varepsilon \neq 0$ is left implicit. In this way, W^φ can be completed to a Riemannian family of vector fields by letting

$$W^{\varphi,\varepsilon} = (W_1^\varphi, \dots, W_{2n}^\varphi).$$

Let us denote by $\left(W_j^\varphi\right)^\star$ the adjoint operator of W_j^φ with respect to $L^2(\mathbb{R}^{2n})$, that is

$$\int_{\mathbb{R}^{2n}} \left(W_j^\varphi\right)^\star \phi\, \psi\, dw = \int_{\mathbb{R}^{2n}} W_j^\varphi \psi\, \phi\, dw$$

for any $\phi, \psi \in C_c^\infty(\mathbb{R}^{2n})$. An easy computation as in [35] implies that

$$(W_1^\varphi)^\star \phi = -W_1^\varphi \phi - 2\phi \tilde{T} \varphi \qquad \text{and} \qquad \left(W_j^\varphi\right)^\star \phi = -W_j^\varphi \phi. \tag{16.7.5}$$

$j = 2, \dots, 2n$, whence

$$\left(W_j^\varphi\right)^\star W_{2n}^\varphi = -W_{2n}^\varphi W_j^\varphi.$$

On the other hand, as a direct consequence of (16.7.3), we infer that

$$\nu^{\mathbb{H}} = W^{-\frac{1}{2}} \left(W^{\varphi}\varphi X_1 + \sum_{j=2}^{n} \tilde{X}_j \varphi X_j - Y_1 + \sum_{j=2}^{n} \tilde{Y}_j \varphi Y_j \right), \tag{16.7.6}$$

where we have set

$$W = 1 + |W^{\varphi}\varphi|^2.$$

Notice that $\nu^{\mathbb{H}}$ can be smoothly extended to a vector field on the whole $\mathbb{H}^n$ by the following construction. Let us define $\tilde{\varphi} : \mathbb{H}^n \longrightarrow \mathbb{R}$ by

$$\tilde{\varphi}(p) = \varphi(\Pi(p))$$

for any $p \in \mathbb{H}^n$. Notice that

$$\tilde{\varphi}(\Psi(q)) = \varphi(q)$$

for any $q \in \mathbb{R}^{2n}$. Moreover,

$$X_1\tilde{\varphi}(\Psi(q)) = W^{\varphi}\varphi(q), \qquad X_j\tilde{\varphi}(\Psi(q)) = \tilde{X}_j\varphi(q), \qquad Y_1\tilde{\varphi} \equiv 0, \qquad Y_j\tilde{\varphi}(\Psi(q)) = \tilde{Y}_j\varphi(q)$$

for any $q \in \mathbb{R}^{2n}$ and any $j = 2, \ldots, n$. Hence $\nu^{\mathbb{H}}$ extends in the obvious way. Notice that, since

$$[\tilde{X}_j, \tilde{Y}_j] = -2\tilde{T}$$

for any $j = 2, \ldots, n$, then (Ω, d_{φ}) is a Carnot-Carathéodory space when Ω is any domain of $\mathbb{R}^{2n}$ and d_{φ} is the Carnot-Carathéodory distance induced by W^{φ}. Moreover, as the previous computations show, it is worth noting that intrinsic graphs associated with a defining function φ of class C^1 do not have characteristic points. On the other hand, it is not always the case that intrinsic graphs of class C^1 have empty characteristic set, as the following example shows.

Example 16.7.2. In the first Heisenberg group, with coordinates (x, y, t), let us consider

$$S = \left\{ \left(x, y, y|y|^{\frac{1}{3}} - xy \right) \, : \, (x, y) \in \mathbb{R}^2 \right\}.$$

Of course, S is a t-graph associated with the function $u(x, y) = y|y|^{\frac{1}{3}} - xy$. Since $u \in C^1(\mathbb{R}^2)$, then S is a hypersurface of class C^1. We claim that

$$S = \mathrm{graph}_Y(\varphi, \mathbb{R}^2), \tag{16.7.7}$$

where $\varphi(\xi, \tau) = \mathrm{sign}(\tau)|\tau|^{\frac{3}{4}}$. Indeed, let first $(x, y) \in \mathbb{R}^2$. Choosing $\xi = x$ and

$$\tau = y|y|^{\frac{1}{3}} = \mathrm{sign}(y)|y|^{\frac{4}{3}},$$

then

$$\varphi(\tau) = \mathrm{sign}(y)|y| = y,$$

whence

$$(x,y,u(x,y)) = (\xi,\varphi(\tau),\tau-\xi\varphi(\tau)) \in \mathrm{graph}_Y(\varphi,\mathbb{R}^2).$$

If conversely $(\xi,\tau)\in\mathbb{R}^2$, then the choice $x=\xi$ and $y=\varphi(\tau)$ implies that

$$(\xi,\varphi(\tau),\tau-\xi\varphi(\tau)) = (x,y,u(x,y)) \in S,$$

whence (16.7.7) follows. Observe in addition that $\varphi \in C(\mathbb{R}^2)$ and $\varphi^2 \in C^1(\mathbb{R}^2)$. Therefore, thanks to [16, Corollary 5.11], S is $\mathbb{H}$-regular. We are left to show that $S_0 \neq \emptyset$. Notice that

$$\frac{\partial u}{\partial x} = -y \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{4}{3}|y|^{\frac{1}{3}} - x$$

for any $(x,y)\in\mathbb{R}^2$. Therefore, in view of (16.7.2), we conclude that

$$S_0 = \{(x,y)\in\mathbb{R}^2 \,:\, y=0\}.$$

In Example 16.7.2, we exhibited an instance of a hypersurface of class C^1 which is the intrinsic graphs of a function which does not share the same regularity. With the next proposition, we show that degeneration cannot occur for non-characteristic hypersurfaces.

Proposition 16.7.3. *Let $\varphi \in C(\Omega)$ be such that $W^\varphi\varphi \in C(\Omega,\mathbb{R}^{2n-1})$. Assume that $\mathrm{graph}_{Y_1}(\varphi,\Omega)$ is a non-characteristic hypersurface of class $C^{k,\alpha}$, for $k\geq 1$ and $\alpha\in[0,1]$. Then $\varphi \in C^{k,\alpha}_{loc}(\Omega)$.*

Proof. Assume, for the sake of simplicity, that $k=2$ and $\alpha=0$. Let us consider the map $g:\mathbb{H}^n \longrightarrow \mathbb{H}^n$ defined by

$$g(\bar{x},\bar{y},t) = (\bar{x},\bar{y},t-x_1y_1)$$

for any $(\bar{x},\bar{y},t)\in\mathbb{H}^n$. Notice that g is smooth, bijective and and $\det(Dg)\equiv 1$. Hence g is a smooth diffeomorphism. Let us set $\hat{S} := g(S)$. Notice that $\hat{S}$ is of class C^2. It is easy to check that

$$\hat{S}\cap g(U) = \{(\bar{\xi},\varphi(\bar{\xi},\tilde{\eta},\tau),\tilde{\eta},\tau) \,:\, (\tilde{\xi},\bar{\eta},\tau)\in\Omega\}.$$

Therefore the thesis follows provided that $(\hat{N}(\hat{p}))_{n+1}\neq 0$ for any $\hat{p}\in\hat{S}\cap g(U)$, being $\hat{N}(\hat{p})$ the Euclidean normal to $\hat{S}$ at $\hat{p}$. Assume by contradiction that there exists $\hat{p}\in\hat{S}\cap g(U)$ such that $(\hat{N}(\hat{p}))_{n+1}=0$. This implies that $(\bar{0},1,\tilde{0},0)\in T_{\hat{p}}\hat{S}$. Let $p\in S$ be such that $g(p)=\hat{p}$. Noticing that

$$(dg)|_p(Y_1|_p) = (\bar{0},1,\tilde{0},0)\in T_{\hat{p}}\hat{S},$$

we infer that $Y_1|_p \in T_pS$. Since S is non-characteristic, (16.6.1) implies that $(\nu^{\mathbb{H}}(p))_{n+1}=0$. On the other hand, we know from [16, Theorem 1.2] that $(\nu^{\mathbb{H}}(p))_{n+1}\neq 0$, a contradiction.

□

16.7.3 Intrinsic cones

A set $C \subseteq \mathbb{H}^n$ is a *cone* if

$$\delta_\lambda(C) \subseteq C$$

for any $\lambda > 0$. We point out that the natural extension of this definition to an arbitrary Carnot group embeds the particular class of cones considered in Section 15.7.1. It is easy to see that, if C is a cone, then $0 \in \overline{C}$ and $\delta_\lambda(C) = C$. Moreover, the topological boundary of a cone is itself a cone.

Proposition 16.7.4. *If C is a cone, then $\delta_\lambda(\partial C) \subseteq \partial C$ for any $\lambda > 0$.*

Proof. Let $\lambda > 0$. Let $p \in C$ and assume by contradiction that $q := \delta_\lambda(p) \notin \partial C$. Then either $q \in int(C)$ or $q \in \mathbb{H}^n \setminus \overline{C}$. Assume first that $q \in \mathbb{H}^n \setminus \overline{C}$. Let $(p_h)_h \subseteq \mathrm{int}(C)$ be such that $p_h \to p$ as $h \to \infty$. Let $q_h := \delta_\lambda(p_h)$. Since dilations are continuous, then $q_h \to q$. Being C a cone, then $(q_h)_h \subseteq C$, and so $q \in \overline{C}$. This is a contradiction. If instead $q \in int(C)$, we argue in the same way, noticing that $\mathbb{H}^n \setminus C$ is a cone. □

We say that S is a *conical hypersurface* if it is both a cone and a hypersurface. Notice that, in view of the aforementioned properties, if C is a cone with boundary of class C^k, then ∂C is a conical hypersurface of class C^k. The simplest instance of non-characteristic conical hypersurfaces is given by vertical hyperplanes passing through the origin (cf. Definition 3.6.2). Another simple instance is given by the horizontal plane $\mathcal{H}_0$. In this case we already know that $(\mathcal{H}_0)_0 = \{0\}$. Finally, if u is an homogeneous quadratic polynomial, then $\mathrm{graph}(u)$ is a conical smooth hypersurface. Moreover, in this last case, S_0 may be an infinite set. As an instance, consider the graph associated to $u(\bar{x}, \bar{y}) = \sum_{j=1}^n x_j y_j$. It is easy to see that

$$T_p S = \mathrm{span}\{X_1, \dots, X_n, Y_1 + 2x_1 T, \dots, Y_n + 2x_n T\}$$

for any $p = (\bar{x}, \bar{y}, u(\bar{x}, \bar{y})) \in \mathrm{graph}(u)$. Therefore in this case

$$S_0 = \{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y})) \in \mathrm{graph}(u) \,:\, x_1 = \ldots = x_n = 0\}.$$

When a hypersurface is a cone, we can say more about the structure of S_0.

Proposition 16.7.5. *Let S be a conical hypersurface of class C^1. Then S_0 is a cone.*

Proof. Let $p \in S_0$ and $\lambda > 0$. We prove that $q := \delta_\lambda(p) \in S_0$. If $p = 0$ the thesis is trivial. Assume that $p \neq 0$. We prove that $\mathcal{H}_q = T_q S$. Since S is a cone, then $\delta_\lambda : S \longrightarrow S$ is a diffeomorphism, and consequently, recalling (20.2.3), $d\delta_\lambda|_p : \mathcal{H}T_p S \longrightarrow \mathcal{H}T_q S$ is an isomorphism. we conclude that $\dim(\mathcal{H}T_p S) = \dim(\mathcal{H}T_q S)$, which means that $q \in S_0$. □

Proposition 16.7.6. *Let S be a conical hypersurface of class C^1. Then $S_0 \subseteq H_0$. Moreover, for any $p \in S_0$ there is a horizontal half line $\gamma : [0, +\infty) \longrightarrow S_0$ such that $\gamma(0) = 0$ and $\gamma(1) = p$.*

Proof. Let $p = (\bar{x}, \bar{y}, t) \in S_0 \setminus \{0\}$, and set $\gamma(0) = 0$ and $\gamma(\lambda) := \delta_\lambda(p)$. Then γ is a smooth curve with

$$\dot{\gamma}(\lambda) = (\bar{x}, \bar{y}, 2\lambda t) = \sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{n} y_j Y_j + 2\lambda t T.$$

Moreover, thanks to Proposition 16.7.5, then $\gamma([0, +\infty)) \subseteq S_0$. Finally, since $\gamma(1) = p$, S is a cone and $p \in S_0$, then $\dot{\gamma}(1) \in T_p S = \mathcal{H}_p$, and so $t = 0$. □

The shape of conical hypersurfaces strongly depends on the size of the associated characteristic set. Exploiting [140, Theorem 4.1], it is easy to see that vertical hyperplanes passing through the origin are the only possible examples of non-characteristic conical hypersurfaces of class C^1. Instead, when $S_0 \neq 0$, conical hypersurfaces of class C^1 are t-graphs (cf. [164, Lemma 4.4] for the same result in $\mathbb{H}^1$).

Proposition 16.7.7. *Let S be a conical hypersurface of class C^1. If $S_0 \neq \emptyset$, then $S = \mathrm{graph}(u)$ for some $u \in C^1(\mathbb{R}^{2n})$.*

Proof. Since $S_0 \neq \emptyset$, then Proposition 16.7.5 implies that $0 \in S_0$. Therefore

$$\mathcal{H}_0 = T_0 S = \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial y_n} \right\}.$$

Hence there exists $r > 0$ and a function $\tilde{u} \in C^1(B_{2r}^{2n}(0))$ such that

$$S \cap B_{2r}(0) = \{(\bar{x}, \bar{y}, \tilde{u}(\bar{x}, \bar{y})) \,:\, (\bar{x}, \bar{y}) \in B_{2r}(0)\}.$$

Notice that, being S a cone, then

$$\tilde{u}(\lambda \bar{x}, \lambda \bar{y}) = \lambda^2 u(\bar{x}, \bar{y})$$

for any $(\bar{x}, \bar{y}) \in B_{2r}^{2n}(0)$ and for any $\lambda > 0$ such that $(\lambda \bar{x}, \lambda \bar{y}) \in B_{2r}(0)$. Let us define

$$u : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$$

by

$$u(\bar{x}, \bar{y}) := \left(\frac{|(\bar{x}, \bar{y})|}{r} \right)^2 \tilde{u} \left(r \frac{(\bar{x}, \bar{y})}{|(\bar{x}, \bar{y})|} \right).$$

Then $u \in C^1(\mathbb{R}^{2n})$ and $u \equiv \tilde{u}$ on $B_{2r}(0)$. Moreover, $G := \{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y})) \,:\, (\bar{x}, \bar{y}) \in \mathbb{R}^{2n}\}$ is a conical C^1-hypersurface. Let now $p = (\bar{x}, \bar{y}, t) \in S$. Then there exists $\lambda > 0$ such that $\delta_\lambda(p) \in B_{2r}(0) \cap S$. Hence

$$\lambda^2 t = \tilde{u}(\lambda \bar{x}, \lambda \bar{y}) = u(\lambda \bar{x}, \lambda \bar{y}) = \lambda^2 u(\bar{x}, \bar{y}),$$

which allows to conclude that $t = u(\bar{x}, \bar{y})$ and $p \in G$. Therefore, being both S and G conical C^1-hypersurfaces, we conclude that $S = G$. □

Moreover, when a conical hypersurface is of class C^2, then it coincides with the t-graph of a homogeneous quadratic polynomial.

Proposition 16.7.8. *Let S be a conical hypersurfaces of class C^2. Assume that $S_0 \neq \emptyset$. Then $S = \mathrm{graph}(u)$ for some homogeneous quadratic polynomial u.*

Proof. We already know from Proposition 16.7.7 that $S = \mathrm{graph}(u)$, where $u \in C^1(\mathbb{R}^{2n})$. Moreover, since S is a hypersurface of class C^2, then $u \in C^2(\mathbb{R}^{2n})$. Finally, since $0 \in S_0$, then $Du(0) = 0$. Therefore

$$u(p) = P_2(p) + o(|p|^2),$$

where P_2 is an homogeneous quadratic polynomial. We show that $u = P_2$. Let $p \in \mathbb{R}^{2n}$, and let $\alpha > 0$. Then it holds that

$$|u(p) - P_2(p)| = \frac{|u(\alpha p) - P_2(\alpha p)|}{\alpha^2} = |p|^2 \frac{o(\alpha^2 |p|^2)}{\alpha^2 |p|^2}$$

as $\alpha \to +\infty$. The thesis then follows letting $\alpha \to +\infty$. □

16.8 Perimeters in $\mathbb{H}^n$

Since the Heisenberg sub-Riemannian structure is closely related to its Riemannian one, it is natural to wonder about the relations existing between the different notions of perimeter and total variation that can be defined. Given an open set $A \subseteq \mathbb{H}^n$ and $f \in L^1(A)$, the *ε-variation* $Var_\varepsilon(f; A)$ of f in A is defined, according to our general Definition 1.4.1, as the X-variation of f with respect to the family of vector fields $X = (Z_1, \ldots, Z_{2n+1})$. More precisely, we recall that

$$Var_\varepsilon(f; A) = \sup \left\{ \int_A f \,\mathrm{div}_\varepsilon(U)\, dx : U \in C^1_c(A; T\mathbb{H}^n),\ |U|_{\varepsilon,\infty} \leqslant 1 \right\},$$

where $C^1_c(A; T\mathbb{H}^n)$ is the space of C^1 compactly supported vector fields in A,

$$|U|_{\varepsilon,\infty} = \sup_{p \in A} |U(p)|_\varepsilon$$

and div_ε is the divergence associated to g_ε, or equivalently the X-divergence as in Definition 1.1.2 associated to the family X. The space of $L^1(A)$ functions with bounded ε-variation is denoted by $BV_\varepsilon(A)$. Accordingly, given $E \subseteq \mathbb{H}^n$ a measurable set and $A \subseteq \mathbb{H}^n$ an open set, the ε-perimeter of E in A is given by

$$P_\varepsilon(E; A) = Var_\varepsilon(\chi_E; A).$$

We point out that, in this particular setting, the ε-divergence coincides with the standard Euclidean divergence. Indeed, if $U = \sum_{j=1}^{2n+1} U_j Z_j$, then

$$U = \sum_{j=1}^{2n} U_j \frac{\partial}{\partial z_j} + \left(\varepsilon U_{2n+1} + \sum_{j=1}^{n} y_j U_j - \sum_{j=1}^{n} x_j U_{n+j} \right) \frac{\partial}{\partial t},$$

so that

$$\begin{aligned}
\operatorname{div}_\varepsilon U &= \sum_{j=1}^n X_j U_j + \sum_{j=1}^n Y_j U_{n+j} + \varepsilon T U_{2n+1}\\
&= \sum_{j=1}^{2n} \frac{\partial U_j}{\partial z_j} + \varepsilon \frac{\partial U_{2n+1}}{\partial t} + \sum_{j=1}^n y_j \frac{\partial U_j}{\partial t} - \sum_{j=1}^n x_j \frac{\partial U_{n+j}}{\partial t}\\
&= \sum_{j=1}^{2n} \frac{\partial U_j}{\partial z_j} + \frac{\partial}{\partial t}\left(\varepsilon U_{2n+1} + \sum_{j=1}^n y_j U_j - \sum_{j=1}^n x_j U_{n+j}\right)\\
&= \operatorname{div} U,
\end{aligned}$$

whence we conclude that

$$\operatorname{div} U = \operatorname{div}_\varepsilon U. \tag{16.8.1}$$

Moreover, the Riemannian norm induced by g_ε turns to be equivalent to the Euclidean one as soon as we restrict to vertical cylinders with bounded base. More precisely, if $\Omega \subseteq \mathbb{R}^{2n}$ is a bounded open set, then a simple computation shows that there exist constants $C(\Omega, \varepsilon) > 0$ and $c = c(\Omega, \varepsilon) > 0$ such that

$$C|v| \leqslant |v|_\varepsilon \leqslant c|v| \tag{16.8.2}$$

for any $v \in T_p(\Omega \times \mathbb{R})$. Indeed, we assume for clarity that $n = 1$ and let $v = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial t}$. Recalling that $(a+b)^2 \leqslant 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$, we get

$$|v|_\varepsilon^2 \leqslant v_1^2 + v_2^2 + \frac{2}{\varepsilon^2} v_3^2 + \frac{4}{\varepsilon^2} y^2 v_1^2 + \frac{4}{\varepsilon^2} x^2 v_2^2 \leqslant 4\left(1 + \frac{1 + \max_{z \in \bar\Omega} |z|^2}{\varepsilon^2}\right)|v|^2,$$

On the other hand,

$$|v|^2 \leqslant v_1^2 + v_2^2 + 4x^2 v_2^2 + 4y^2 v_1^2 + 2(v_3 - y v_1 + x v_2)^2 \leqslant \max\left\{1 + 4\max_{z \in \overline{\Omega}} |z|^2, 2\varepsilon^2\right\}|v|_\varepsilon^2.$$

Notice that, in (16.8.2), c can be chosen uniformly in ε for $|\varepsilon|$ big enough, while C can be chosen uniformly in ε for $|\varepsilon|$ small enough. Accordingly, when we are in vertical cylinders as above, the Riemannian perimeter and total variation are equivalent to their Euclidean counterparts. Precisely, given an open set $A \subseteq \Omega \times \mathbb{R}$ and a vector field $U \in C_c^1(A; T\mathbb{H}^n)$ with $|U|_{\varepsilon,\infty} \leqslant 1$, it follows from (16.8.2) that $|CU| \leqslant |U|_\varepsilon \leqslant 1$ and

$$\int_A f \operatorname{div}(U) dx = \frac{1}{C}\int_A f \operatorname{div}(CU) dx \leqslant \frac{1}{C} Var(f, A)$$

for any $f \in L^1(\Omega \times \mathbb{R})$. Hence $Var_\varepsilon(f, A) \leqslant \frac{1}{C} Var(f, A)$. Similarly, $Var(f, A) \leqslant \frac{1}{c} Var_\varepsilon(f, A)$, so that

$$\frac{1}{c} Var(f, A) \leqslant Var_\varepsilon(f, A) \leqslant \frac{1}{C} Var(f, A). \tag{16.8.3}$$

Notice that (16.8.3) implies that $BV_\varepsilon(A)$ coincide, for $A \subseteq \Omega \times \mathbb{R}$, with the space of functions of Euclidean bounded variation $BV(A)$. Moreover, the perimeters P and P_ε are absolutely continuous with respect to each other. Hence, the Euclidean reduced and essential boundaries of a Caccioppoli set coincide with the ones induced by P_ε. Beside the Riemannian and the Euclidean perimeters, we can of course introduce the *horizontal perimeter* with respect to the family $(Z_1, \dots, Z_{2n})$, again according to Definition 1.4.4. More precisely, given $E \subseteq \mathbb{H}^n$ measurable and $A \subseteq \mathbb{H}^n$ open, the *horizontal perimeter* of E in A is defined by

$$P_{\mathbb{H}}(E, A) = \sup\left\{ \int_E \operatorname{div}_{\mathbb{H}} U \, dx,\, U \in C_c^1(A, \mathcal{H}),\, |U|_{1,\infty} \leqslant 1 \right\}, \tag{16.8.4}$$

where $C_c^1(A, \mathcal{H})$ is the space of C^1 compactly supported horizontal vector fields in A and $\operatorname{div}_{\mathbb{H}}$ is the X-divergence as in Definition 1.1.2 associated to the family $X = (Z_1, \dots, Z_{2n})$. Notice that, in view of (16.8.1), then

$$\operatorname{div} U = \operatorname{div}_\varepsilon U = \operatorname{div}_{\mathbb{H}} U$$

for any $\varepsilon \neq 0$ and any C^1-vector field U. For further insights, we refer the interested reader to [140], where the main properties of the horizontal perimeter in $\mathbb{H}^n$ are discussed. Although they differs from many viewpoints, we point out that both P_ε and $P_{\mathbb{H}}$ behave like the Euclidean perimeter P for vertical sets. More precisely, given a Caccioppoli set $E \subseteq \mathbb{R}^{2n}$ and an open set $A \subseteq \mathbb{H}^n$, arguing as in [76, (3.2)] and observing that the last component of the measure theoretic Euclidean unit normal to $E \times \mathbb{R}$ is zero, then

$$P_\varepsilon(E \times \mathbb{R}; A) = P_{\mathbb{H}}(E \times \mathbb{R}; A) = P(E \times \mathbb{R}, A). \tag{16.8.5}$$

By the above definitions, the following relations between horizontal and ε-perimeter hold.

Proposition 16.8.1. *Let $A \subseteq \mathbb{H}^n$ be an open bounded set and $F \subseteq \mathbb{H}^n$ be a Caccioppoli set. Then*

$$P_{\mathbb{H}}(F, A) \leqslant P_\varepsilon(F, A) \qquad \text{and} \qquad \lim_{\varepsilon \to 0} P_\varepsilon(F, A) = P_{\mathbb{H}}(F, A).$$

Moreover, if $(\varepsilon_j)_j \subseteq (0, 1)$ satisfies $\varepsilon_j \searrow 0$ as $j \to \infty$, and E and $(E_j)_j$ are measurable sets such that $\chi_{E_j} \to \chi_E$ in $L^1_{loc}(A)$, then

$$P_{\mathbb{H}}(E; A) \leqslant \liminf_{j \to \infty} P_{\varepsilon_j}(E_j; A). \tag{16.8.6}$$

Moreover, the following compactness result holds.

Proposition 16.8.2. *Let $A \subseteq \mathbb{H}^n$ be an open and $(E_k)_k$ be a sequence of finite $\mathbb{H}$-perimeter sets in A. Assume that there exists $M > 0$ such that $\sup_k P_{\mathbb{H}}(E_k, A) < M$. Then there exists a finite $\mathbb{H}$-perimeter set E in A such that $\chi_{E_k} \to \chi_E$ in $L^1_{loc}(A)$.*

Proof. Let $A' \Subset A$ open. Let $B_1, \dots, B_k$ be a covering of A' of Carnot-Carathéodory balls with respect to the Carnot-Carathéodory distance induced by $X = (Z_1, \dots, Z_{2n})$ such that

$$A' \subseteq \bigcup_{j=1}^{s} B_j \Subset A.$$

Notice that $\chi_{E_k} \in BV_{\mathbb{H}}(B_j)$ for any $j = 1, \dots, k$. Let us consider first B_1. In view of [138, Theorem 2.2.2], there exists $v_k \in C^\infty(B_1) \cap BV_{\mathbb{H}}(B_1)$ such that

$$\|v_k - \chi_{E_k}\|_{L^1(B_1)} \leqslant \frac{1}{k} \qquad \text{and} \qquad \left| P_{\mathbb{H}}(E_k, B_1) - \int_{B_1} |\nabla^{\mathbb{H}} v_k| \, dx \right| \leqslant \frac{1}{k}$$

Therefore

$$\|v_k\|_{L^1(B_1)} \leqslant |E_k \cap B_1| + 1 \qquad \text{and} \qquad \int_{B_1} |\nabla^{\mathbb{H}} v_k| \, dx \leqslant \sup_k P_{\mathbb{H}}(E_k, A) + 1,$$

so that $(v_k)_k$ is bounded in $BV_{\mathbb{H}}(B_1)$. Hence, [150] implies that there exists $v \in BV_{\mathbb{H}}(B_1)$ such that, up to a subsequence, $v_k \to v$ in $L^1(B_1)$. This fact trivially implies the existence of a set $E^1 \subseteq B_1$ such that, up to a subsequence, $\chi_{E_k} \to \chi_{E^1}$ in $L^1(B_1)$. The thesis then easily follows by a diagonal process and the lower semicontinuity of the $\mathbb{H}$-perimeter (cf Proposition 1.4.2). □

In the development of Part VI, we will be interested in explicit ways of computing the perimeter of subgraphs of t-graphs. Since a wide range of results in the sub-Riemannian framework is available in [264], we mainly focus on some further properties of the Riemannian perimeter. Given a bounded open set $\Omega \subseteq \mathbb{R}^{2n}$ and a measurable function $u : \Omega \longrightarrow [-\infty, +\infty]$, we write the subgraph of u as

$$E_u = \{(z, t) \in \Omega \times \mathbb{R} \, : \, t < u(z)\}. \tag{16.8.7}$$

A simple computation shows that, for $u \in W^{1,1}_{loc}(\Omega)$ and $\tilde{\Omega} \subseteq \Omega$ open, the perimeter of E_u in $\tilde{\Omega} \times \mathbb{R}$ can be computed as

$$\mathcal{A}_\varepsilon(u, \tilde{\Omega}) = \int_{\tilde{\Omega}} \sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2} \, dz, \tag{16.8.8}$$

where $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined as in (16.7.1). The L^1-relaxation of $\mathcal{A}_\varepsilon$ for $u \in BV(\Omega)$ is

$$\overline{\mathcal{A}}_\varepsilon(u, \tilde{\Omega}) = \inf \left\{ \liminf_{k \to \infty} \mathcal{A}_\varepsilon(u_k, \tilde{\Omega}) : \, (u_k)_k \subseteq W^{1,1}(\Omega), \, u_k \to u \text{ in } L^1(\tilde{\Omega}) \right\}.$$

We also define

$$S_\varepsilon(u, \tilde{\Omega}) = \sup \left\{ \int_{\tilde{\Omega}} (-u \operatorname{div} \tilde{g} + \langle \mathcal{F}, \tilde{g} \rangle + \varepsilon g_{2n+1}) \, dz : \, g = (\tilde{g}, g_{2n+1}) \in C^1_c(\tilde{\Omega}, \mathbb{R}^{2n+1}), \, |g| \leqslant 1 \right\}.$$

Lemma 16.8.3. *Given a bounded open set $\Omega \subseteq \mathbb{R}^{2n}$, $\tilde{\Omega} \subseteq \Omega$ open and $u \in L^1(\Omega)$, then*

$$P_\varepsilon(E_u, \tilde{\Omega} \times \mathbb{R}) = \overline{\mathcal{A}}_\varepsilon(u, \tilde{\Omega}) = S_\varepsilon(u, \tilde{\Omega}).$$

Proof. The proof follows exactly as the proof of [264, Theorem 3.2]. □

Recall that for any $u \in BV$, its distributional derivative $\tilde{D}u$ can be decomposed as the sum of the two mutually singular measures $Du\mathcal{L}^{2n} + (Du)_s$, where $Du \in L^1$ and $(Du)_s$ is singular with respect to the Lebesgue measure $\mathcal{L}^{2n}$.

Lemma 16.8.4. *Given a bounded open set $\Omega \subseteq \mathbb{R}^{2n}$, $\tilde{\Omega} \subseteq \Omega$ open and $u \in BV(\Omega)$, it holds that*

$$P_\varepsilon(E_u, \tilde{\Omega} \times \mathbb{R}) = (Du)_s(\tilde{\Omega}) + \int_{\tilde{\Omega}} \sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2}\, dz.$$

Proof. Let us define $L : C_c^1(\Omega, \mathbb{R}^{2n+1}) \longrightarrow \mathbb{R}$ by

$$L(g) = \int_A (-u \operatorname{div} \bar{g} + \langle \mathcal{F}, \bar{g} \rangle + \varepsilon g_{2n+1})\, dz,$$

where $g = (\bar{g}, g_{2n+1})$. L is clearly linear. Moreover, since by Lemma 16.8.3 $S_\varepsilon(u) < +\infty$, then L extends to a linear bounded functional on $C_c^0(A, \mathbb{R}^{2n+1})$. Therefore, by Riesz Theorem (cf. e.g. [11, Theorem 1.54]) there exists a unique $(2n+1)$-valued finite Radon measure μ such that

$$L(g) = \int_A g \cdot d\mu \qquad \text{and} \qquad S_\varepsilon(u, A) = |\mu|(A)$$

for any $g \in C_c^1(A, \mathbb{R}^{2n+1})$. By the uniqueness of such a measure it is easy to see that $\mu = (Du + \mathcal{F} d\mathcal{L}^{2n}, \varepsilon d\mathcal{L}^{2n})$, and so

$$S_\varepsilon(u, A) = |(Du + \mathcal{F} d\mathcal{L}^{2n}, \varepsilon d\mathcal{L}^{2n})|(A).$$

A trivial computation, together with Lemma 16.8.3, concludes the proof. □

Chapter 17

Variational properties of Riemannian and sub-Riemannian hypersurfaces

17.1 Introduction

In this chapter we propose a detailed exposition of some results presented in [261, 152] concerning the limiting behavior of Riemannian second variation formulas when the Riemannian structures $(\mathbb{H}^n, g_\varepsilon)$ collapse to the sub-Riemannian Heisenberg group $(\mathbb{H}^n, \langle\cdot,\cdot\rangle)$. In addition, we exploit our computations to provide a direct proof of the so-called *Jacobi equation* for the vertical component of the Riemannian unit normal ν^ε, together with its corresponding sub-Riemannian version, finally discussing the correlation between the two. Thanks to [216, 217], it is possible to provide an explicit formula for the sub-Riemannian first and second variation of the perimeter. More precisely, the following holds.

Theorem 17.1.1. *Let $S = \partial E$ be a smooth non-characteristic hypersurface, let $A \subseteq \mathbb{H}^n$ be an open set such that $A \cap S \neq \emptyset$, and let $\xi \in C_c^\infty(A)$. Then*

$$\frac{d}{dt} P_{\mathbb{H}}(E_t, A)\Big|_{t=0} = \int_S H^{\mathbb{H}} \xi \, dP_{\mathbb{H}}(E, \cdot),$$

$$\frac{d^2}{dt^2} P_{\mathbb{H}}(E_t, A)\Big|_{t=0} = \int_S \left(|\nabla^{\mathbb{H},S} \xi|^2 - \xi^2 \left(q - \left(H^{\mathbb{H}}\right)^2 \right) \right) dP_{\mathbb{H}}(E, \cdot),$$

where

$$q = \sum_{h,k=1}^{2n} Z_h(\nu_k^{\mathbb{H}}) Z_k(\nu_h^{\mathbb{H}}) + 4\langle J(\nu^{\mathbb{H}}), \nabla^{\mathbb{H}}(T d^{\mathbb{H}})\rangle + 4n(T d^{\mathbb{H}})^2 \tag{17.1.1}$$

and where by E_t we mean a smooth variation along the vector field $\xi \nu^{\mathbb{H}}$ (cf. Section 17.3).

A detailed and direct proof of Theorem 17.1.1 is provided in Section 17.3. Observe that q does not depend on the chosen unitary extension of $\nu^{\mathbb{H}}|_S$. Apart from the squared norm of the horizontal second fundamental form, it is hard to give a sub-Riemannian interpretation to the other terms that compose q. The classical Riemannian second variation formula for smooth hypersurfaces involves, together with the squared mean curvature, also the *Ricci curvature* of the Riemannian normal and the squared norm of the Riemannian second fundamental form.

Since the family of Riemannian structures $(\mathbb{H}^n, g_\varepsilon)$ approximate in a suitable sense the sub-Riemannian manifold $(\mathbb{H}^n, \langle\cdot,\cdot\rangle$ (cf. [77]), an idea could be to understand whether $|h^\varepsilon|$ and $\mathrm{Ric}_\varepsilon(\nu^\varepsilon)$ admit a limit as $\varepsilon \to 0$. However, it is well known (cf. [77]) that both these terms diverge. Quite surprisingly we show that, coupling $|h^\varepsilon|$ and $\mathrm{Ric}_\varepsilon(\nu^\varepsilon)$, we retrieve in the limit the term q. More precisely, the following result holds.

Theorem 17.1.2. *Let $S = \partial E$ be an orientable, embedded hypersurface of class C^3. Then*

$$|h^\varepsilon|^2 + \mathrm{Ric}_\varepsilon(\nu^\varepsilon) = \sum_{h,k=1}^{2n+1} Z_k(\nu_h^\varepsilon) Z_h(\nu_k^\varepsilon) + 4g_\varepsilon\left(J(\nu^\varepsilon), T\nu^\varepsilon\right) + 4n\frac{(\nu_{2n+1}^\varepsilon)^2}{\varepsilon^2}. \tag{17.1.2}$$

Moreover,

$$\lim_{\varepsilon\to 0}\left(|h^\varepsilon|^2 + \mathrm{Ric}_\varepsilon(\nu^\varepsilon)\right) = q \tag{17.1.3}$$

locally uniformly in the non-characteristic part of S, where q is as in (17.1.1).

As a corollary of Theorem 17.1.2, we are able to provide an easy proof of the Riemannian Jacobi equation (cf. [26, 27]). Roughly speaking, the latter provides an expression for the Laplacian of the last component of the Riemannian normal in terms of mean curvature, Ricci curvature and second fundamental form, and will be exploited thoroughly in Chapter 19. The precise statement reads as follows.

Theorem 17.1.3 (Riemannian Jacobi equation)**.** *Let $S \subseteq \mathbb{H}^n$ be an orientable, embedded hypersurface of class C^3. Then*

$$\Delta^{\varepsilon,S}(\nu_{2n+1}^\varepsilon) = g_\varepsilon\left(\nabla^{\varepsilon,S}H^\varepsilon, \varepsilon T\right) - \nu_{2n+1}^\varepsilon\left(\mathrm{Ric}_\varepsilon(\nu^\varepsilon,\nu^\varepsilon) + |h^\varepsilon|^2\right),$$

where we recall that $\nu_{2n+1}^\varepsilon = g_\varepsilon(\nu^\varepsilon, \varepsilon T)$.

Moreover, arguing in the very same way as in the proof of Theorem 17.1.3, the following sub–Riemannian Jacobi equation holds.

Theorem 17.1.4 (sub-Riemannian Jacobi equation)**.** *Let $S \subseteq \mathbb{H}^n$ be an orientable, embedded, non-characteristic hypersurface of class C^3. Then*

$$\Delta^{\mathbb{H},S}(Td^{\mathbb{H}}) = TH - Td^{\mathbb{H}}\langle\nabla^{\mathbb{H}}H^{\mathbb{H}}, \nu^{\mathbb{H}}\rangle - Td^{\mathbb{H}}\left(\sum_{i,j=1}^{2n} Z_i\nu_i^{\mathbb{H}} Z_i\nu_j^{\mathbb{H}} + 6\left\langle\nabla^{\mathbb{H}}\left(Td^{\mathbb{H}}\right), J(\nu^{\mathbb{H}})\right\rangle + 4n(Td^{\mathbb{H}})^2\right).$$

In particular,

$$\hat{\Delta}^{\mathbb{H},S}(Td^{\mathbb{H}}) = TH - Td^{\mathbb{H}}\langle\nabla^{\mathbb{H}}H^{\mathbb{H}}, \nu^{\mathbb{H}}\rangle - qTd^{\mathbb{H}}.$$

Finally, combining Theorem 17.1.2, Theorem 17.1.3 and Theorem 17.1.4, we recover $\hat{\Delta}^{\mathbb{H},S}Td^{\mathbb{H}}$ as limit of a suitable normalization of $\Delta^{\varepsilon,S}(\nu_{2n+1}^\varepsilon)$.

Theorem 17.1.5. *Let $S \subseteq \mathbb{H}^n$ be an orientable, embedded, non-characteristic hypersurface of class C^3. Then*

$$\lim_{\varepsilon\to 0}\frac{\Delta^{\varepsilon,S}(\nu_{2n+1}^\varepsilon)}{\varepsilon} = \hat{\Delta}^{\mathbb{H},S}(Td^{\mathbb{H}}).$$

17.2 The Riemannian second variation formula

17.2.1 Ricci curvature

In order to prove (17.1.2), we begin by providing an explicit expression for the Ricci curvature (16.3.9).

Proposition 17.2.1. *Let $A = \sum_{j=1}^{2n+1} A_j Z_j$ for some $A_1, \dots, A_{2n+1} \in C^2(\mathbb{H}^n)$. Then*

$$\mathrm{Ric}_\varepsilon(A) = -\frac{2}{\varepsilon^2} g_\varepsilon(A, A) + \frac{2n+2}{\varepsilon^2} A_{2n+1}^2.$$

Proof. Let A be as in the statement. Recalling that

$$R_\varepsilon(A, B, C, D) = -R_\varepsilon(B, A, C, D)$$

for any $B, C, D \in T\mathbb{H}^n$, then

$$\begin{aligned}
\mathrm{Ric}_\varepsilon(A) &= \sum_{j=1}^{2n+1} R_\varepsilon(A, Z_j, A, Z_j) \\
&= -\sum_{j=1}^{2n+1} R_\varepsilon(Z_j, A, A, Z_j) \\
&= -\sum_{j=1}^{2n+1} g_\varepsilon\left(\nabla^\varepsilon_A \nabla^\varepsilon_{Z_j} A - \nabla^\varepsilon_{Z_j} \nabla^\varepsilon_A A + \nabla^\varepsilon_{[Z_j, A]} A, Z_j\right) \\
&= \sum_{j=1}^{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{Z_j} \nabla^\varepsilon_A A - \nabla^\varepsilon_A \nabla^\varepsilon_{Z_j} A + \nabla^\varepsilon_{[A, Z_j]} A, Z_j\right).
\end{aligned}$$

Computation of $R_\varepsilon(A, X_j, A, X_j)$. Let us fix $j = 1, \dots, n$. Then, recalling (16.3.2), (16.3.3) and (16.3.7),

$$\begin{aligned}
g_\varepsilon&\left(\nabla^\varepsilon_{X_j} \nabla^\varepsilon_A A, X_j\right) = X_j g_\varepsilon\left(\nabla^\varepsilon_A A, X_j\right) - g_\varepsilon\left(\nabla^\varepsilon_A A, \nabla^\varepsilon_{X_j} X_j\right) \\
&= X_j\left(\sum_{h,k=1}^{2n+1} A_k g_\varepsilon\left(\nabla^\varepsilon_{Z_k} A_h Z_h, X_j\right)\right) \\
&= X_j\left(\sum_{h,k=1}^{2n+1} A_k g_\varepsilon\left(Z_k(A_h) Z_h, X_j\right) + \sum_{h,k=1}^{2n+1} A_h A_k g_\varepsilon\left(\nabla^\varepsilon_{Z_k} Z_h, X_j\right)\right) \\
&= X_j\left(\sum_{k=1}^{2n+1} A_k Z_k(A_j) + \sum_{k=1}^{2n+1} A_{n+j} A_k g_\varepsilon\left(\nabla^\varepsilon_{Z_k} Y_j, X_j\right) + \sum_{k=1}^{2n+1} A_{2n+1} A_k g_\varepsilon\left(\nabla^\varepsilon_{Z_k} \varepsilon T, X_j\right)\right) \\
&= X_j\left(\sum_{k=1}^{2n+1} A_k Z_k(A_j) + 2A_{n+j} A_{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T} Y_j, X_j\right)\right) \\
&= X_j\left(\sum_{k=1}^{2n+1} A_k Z_k(A_j) - \frac{2}{\varepsilon} A_{n+j} A_{2n+1}\right).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
-g_\varepsilon\left(\nabla^\varepsilon_A\nabla^\varepsilon_{X_j}A, X_j\right) &= -Ag_\varepsilon\left(\nabla^\varepsilon_{X_j}A, X_j\right) + g_\varepsilon\left(\nabla^\varepsilon_{X_j}A, \nabla^\varepsilon_A X_j\right)\\
&= -A\left(\sum_{k=1}^{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{X_j}A_kZ_k, X_j\right)\right) + \sum_{h,k=1}^{2n+1} A_k g_\varepsilon\left(\nabla^\varepsilon_{X_j}A_hZ_h, \nabla^\varepsilon_{Z_k}X_j\right)\\
&= -A(X_jA_j) + \sum_{h,k=1}^{2n+1} A_kX_j(A_h)g_\varepsilon\left(Z_h, \nabla^\varepsilon_{Z_k}X_j\right) + \sum_{h,k=1}^{2n+1} A_kA_hg_\varepsilon\left(\nabla^\varepsilon_{X_j}Z_h, \nabla^\varepsilon_{Z_k}X_j\right)\\
&= -A(X_jA_j) + \sum_{h=1}^{2n+1} A_{n+j}X_j(A_h)g_\varepsilon\left(Z_h, \nabla^\varepsilon_{Y_j}X_j\right) + \sum_{h=1}^{2n+1} A_{2n+1}X_j(A_h)g_\varepsilon\left(Z_h, \nabla^\varepsilon_{\varepsilon T}X_j\right)\\
&\quad + \sum_{h=1}^{2n+1} A_{n+j}A_hg_\varepsilon\left(\nabla^\varepsilon_{X_j}Z_h, \nabla^\varepsilon_{Y_j}X_j\right) + \sum_{h=1}^{2n+1} A_{2n+1}A_hg_\varepsilon\left(\nabla^\varepsilon_{X_j}Z_h, \nabla^\varepsilon_{\varepsilon T}X_j\right)\\
&= -A(X_jA_j) + \sum_{h=1}^{2n+1} A_{n+j}X_j(A_h)g_\varepsilon\left(Z_h, T\right) + \frac{1}{\varepsilon}\sum_{h=1}^{2n+1} A_{2n+1}X_j(A_h)g_\varepsilon\left(Z_h, Y_j\right)\\
&\quad + \sum_{h=1}^{2n+1} A_{n+j}A_hg_\varepsilon\left(\nabla^\varepsilon_{X_j}Z_h, T\right) + \frac{1}{\varepsilon}\sum_{h=1}^{2n+1} A_{2n+1}A_hg_\varepsilon\left(\nabla^\varepsilon_{X_j}Z_h, Y_j\right)\\
&= -A(X_jA_j) + \frac{1}{\varepsilon}A_{n+j}X_j(A_{2n+1}) + \frac{1}{\varepsilon}A_{2n+1}X_j(A_{n+j}) - \frac{1}{\varepsilon^2}A^2_{n+j} + \frac{1}{\varepsilon^2}A^2_{2n+1}.
\end{aligned}
$$

Finally, observing that

$$
[A, X_j] = \sum_{k=1}^{2n+1} A_k(Z_kX_j - X_jZ_k) - \sum_{k=1}^{2n+1} X_j(A_k)Z_k = 2A_{n+j}T - \sum_{k=1}^{2n+1} X_j(A_k)Z_k,
$$

we see that

$$
\begin{aligned}
g_\varepsilon\left(\nabla^\varepsilon_{[A,X_j]}A, X_j\right) &= \frac{2}{\varepsilon}A_{n+j}g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A, X_j\right) - \sum_{k=1}^{2n+1} X_j(A_k)g_\varepsilon\left(\nabla^\varepsilon_{Z_k}A, X_j\right)\\
&= \frac{2}{\varepsilon}\sum_{k=1}^{2n+1} A_{n+j}g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A_kZ_k, X_j\right) - \sum_{h,k=1}^{2n+1} X_j(A_k)g_\varepsilon\left(\nabla^\varepsilon_{Z_k}A_hZ_h, X_j\right)\\
&= 2A_{n+j}T(A_j) + \frac{2}{\varepsilon}\sum_{k=1}^{2n+1} A_{n+j}A_kg_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}Z_k, X_j\right)\\
&\quad - \sum_{k=1}^{2n+1} X_j(A_k)Z_k(A_j) - \sum_{h,k=1}^{2n+1} X_j(A_k)A_hg_\varepsilon\left(\nabla^\varepsilon_{Z_k}Z_h, X_j\right)\\
&= 2A_{n+j}T(A_j) - \frac{2}{\varepsilon^2}A^2_{n+j} - \sum_{k=1}^{2n+1} X_j(A_k)Z_k(A_j)\\
&\quad - \sum_{h=1}^{2n+1} X_j(A_{n+j})A_hg_\varepsilon\left(\nabla^\varepsilon_{Y_j}Z_h, X_j\right) - \sum_{h=1}^{2n+1} X_j(A_{2n+1})A_hg_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}Z_h, X_j\right)\\
&= 2A_{n+j}T(A_j) - \frac{2}{\varepsilon^2}A^2_{n+j} - \sum_{k=1}^{2n+1} X_j(A_k)Z_k(A_j)\\
&\quad - X_j(A_{n+j})A_{2n+1}g_\varepsilon\left(\nabla^\varepsilon_{Y_j}\varepsilon T, X_j\right) - X_j(A_{2n+1})A_{n+j}g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}Y_j, X_j\right)\\
&= 2A_{n+j}T(A_j) - \frac{2}{\varepsilon^2}A^2_{n+j} - \sum_{k=1}^{2n+1} X_j(A_k)Z_k(A_j) + \frac{1}{\varepsilon}X_j(A_{n+j})A_{2n+1} + \frac{1}{\varepsilon}X_j(A_{2n+1})A_{n+j}.
\end{aligned}
$$

Computation of $R_\varepsilon(A, Y_j, A, Y_j)$. Let us fix $j = 1, \dots, n$. Arguing as above,

$$\begin{aligned}
g_\varepsilon\left(\nabla^\varepsilon_{Y_j}\nabla^\varepsilon_A A, Y_j\right) &= Y_j g_\varepsilon\left(\nabla^\varepsilon_A A, Y_j\right) \\
&= Y_j\left(\sum_{h,k=1}^{2n+1} A_k g_\varepsilon\left(\nabla^\varepsilon_{Z_k} A_h Z_h, Y_j\right)\right) \\
&= Y_j\left(\sum_{h,k=1}^{2n+1} A_k Z_k(A_h) g_\varepsilon\left(Z_h, Y_j\right) + \sum_{h,k=1}^{2n+1} A_k A_h g_\varepsilon\left(\nabla^\varepsilon_{Z_k} Z_h, Y_j\right)\right) \\
&= Y_j\left(\sum_{k=1}^{2n+1} A_k Z_k(A_{n+j}) + \sum_{k=1}^{2n+1} A_k A_j g_\varepsilon\left(\nabla^\varepsilon_{Z_k} X_j, Y_j\right) + \sum_{k=1}^{2n+1} A_k A_{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{Z_k} \varepsilon T, Y_j\right)\right) \\
&= Y_j\left(\sum_{k=1}^{2n+1} A_k Z_k(A_{n+j}) + \frac{2}{\varepsilon} A_j A_{2n+1}\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
g_\varepsilon\left(\nabla^\varepsilon_A\nabla^\varepsilon_{Y_j} A, Y_j\right) &= -A g_\varepsilon\left(\nabla^\varepsilon_{Y_j} A, Y_j\right) + g_\varepsilon\left(\nabla^\varepsilon_{Y_j} A, \nabla^\varepsilon_A Y_j\right) \\
&= -A\left(\sum_{k=1}^{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{Y_j} A_k Z_k, Y_j\right)\right) + \sum_{h,k=1}^{2n} A_k g_\varepsilon\left(\nabla^\varepsilon_{Y_j} A_h Z_h, \nabla^\varepsilon_{Z_k} Y_j\right) \\
&= -A(Y_j A_{n+j}) + \sum_{h,k=1}^{2n} A_k Y_j(A_h) g_\varepsilon\left(Z_h, \nabla^\varepsilon_{Z_k} Y_j\right) + \sum_{h,k=1}^{2n} A_k A_h g_\varepsilon\left(\nabla^\varepsilon_{Y_j} Z_h, \nabla^\varepsilon_{Z_k} Y_j\right) \\
&= -A(Y_j A_{n+j}) + \sum_{h=1}^{2n} A_j Y_j(A_h) g_\varepsilon\left(Z_h, \nabla^\varepsilon_{X_j} Y_j\right) + \sum_{h=1}^{2n} A_{2n+1} Y_j(A_h) g_\varepsilon\left(Z_h, \nabla^\varepsilon_{\varepsilon T} Y_j\right) \\
&\quad + \sum_{h=1}^{2n} A_j A_h g_\varepsilon\left(\nabla^\varepsilon_{Y_j} Z_h, \nabla^\varepsilon_{X_j} Y_j\right) + \sum_{h=1}^{2n} A_{2n+1} A_h g_\varepsilon\left(\nabla^\varepsilon_{Y_j} Z_h, \nabla^\varepsilon_{\varepsilon T} Y_j\right) \\
&= -A(Y_j A_{n+j}) - \frac{1}{\varepsilon} A_j Y_j(A_{2n+1}) - \frac{1}{\varepsilon} A_{2n+1} Y_j(A_j) - \frac{1}{\varepsilon^2} A_j^2 + \frac{1}{\varepsilon^2} A_{2n+1}^2.
\end{aligned}$$

Finally, since

$$[A, Y_j] = \sum_{k=1}^{2n+1} A_k (Z_k Y_j - Y_j Z_k) - \sum_{k=1}^{2n+1} Y_j(A_k) Z_k = -2A_j T - \sum_{k=1}^{2n+1} Y_j(A_k) Z_k,$$

then

$$g_\varepsilon\left(\nabla^\varepsilon_{[A,Y_j]}A, Y_j\right) = -\frac{2}{\varepsilon}A_j g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A, Y_j\right) - \sum_{k=1}^{2n+1} Y_j(A_k) g_\varepsilon\left(\nabla^\varepsilon_{Z_k}A, Y_j\right)$$
$$= -\frac{2}{\varepsilon}\sum_{k=1}^{2n+1} A_j g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A_k Z_k, Y_j\right) - \sum_{h,k=1}^{2n+1} Y_j(A_k) g_\varepsilon\left(\nabla^\varepsilon_{Z_k}A_h Z_h, Y_j\right)$$
$$= -2A_jT(A_{n+j}) - \frac{2}{\varepsilon^2}A_j^2 - \sum_{k=1}^{2n+1} Y_j(A_k)Z_k(A_{n+j}) - \sum_{h,k=1}^{2n+1} Y_j(A_k)A_h g_\varepsilon\left(\nabla^\varepsilon_{Z_k}Z_h, Y_j\right)$$
$$= -2A_jT(A_{n+j}) - \frac{2}{\varepsilon^2}A_j^2 - \sum_{k=1}^{2n+1} Y_j(A_k)Z_k(A_{n+j})$$
$$- \sum_{h=1}^{2n+1} Y_j(A_j)A_h g_\varepsilon\left(\nabla^\varepsilon_{X_j}Z_h, Y_j\right) - \sum_{h=1}^{2n+1} Y_j(A_{2n+1})A_h g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}Z_h, Y_j\right)$$
$$= -2A_jT(A_{n+j}) - \frac{2}{\varepsilon^2}A_j^2 - \sum_{k=1}^{2n+1} Y_j(A_k)Z_k(A_{n+j}) - \frac{1}{\varepsilon}Y_j(A_j)A_{2n+1} - \frac{1}{\varepsilon}Y_j(A_{2n+1})A_j.$$

Computation of $R_\varepsilon(A, \varepsilon T, A, \varepsilon T)$. First,

$$g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}\nabla^\varepsilon_A A, \varepsilon T\right) = \varepsilon T g_\varepsilon\left(\nabla^\varepsilon_A A, \varepsilon T\right)$$
$$= \varepsilon T\left(\sum_{h,k=1}^{2n+1} A_k g_\varepsilon\left(\nabla^\varepsilon_{Z_k}A_h Z_h, \varepsilon T\right)\right)$$
$$= \varepsilon T\left(\sum_{k=1}^{2n+1} A_k Z_k(A_{2n+1}) + \sum_{h,k=1}^{2n+1} A_k A_h g_\varepsilon\left(\nabla^\varepsilon_{Z_k}Z_h, \varepsilon T\right)\right)$$
$$= \varepsilon T\left(\sum_{k=1}^{2n+1} A_k Z_k(A_{2n+1}) + \sum_{j=1}^{n} A_j A_{n+j} g_\varepsilon\left(\nabla^\varepsilon_{X_j}Y_j, \varepsilon T\right) + \sum_{j=1}^{n} A_{n+j}A_j g_\varepsilon\left(\nabla^\varepsilon_{Y_j}X_j, \varepsilon T\right)\right)$$
$$= \varepsilon T\left(\sum_{k=1}^{2n+1} A_k Z_k(A_{2n+1})\right).$$

Then,

$$-g_\varepsilon\left(\nabla^\varepsilon_A\nabla^\varepsilon_{\varepsilon T}A, \varepsilon T\right) = -A g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A, \varepsilon T\right) + g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A, \nabla^\varepsilon_A \varepsilon T\right)$$
$$= -A\left(\sum_{k=1}^{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A_k Z_k, \varepsilon T\right)\right) + \sum_{h,k=1}^{2n+1} A_k g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}A_h Z_h, \nabla^\varepsilon_{Z_k}\varepsilon T\right)$$
$$= -A(\varepsilon T A_{2n+1}) + \sum_{h,k=1}^{2n+1} A_k \varepsilon T(A_h) g_\varepsilon\left(Z_h, \nabla^\varepsilon_{Z_k}\varepsilon T\right) + \sum_{h,k=1}^{2n+1} A_k A_h g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}Z_h, \nabla^\varepsilon_{Z_k}\varepsilon T\right)$$
$$= -A(\varepsilon T A_{2n+1}) + \sum_{j=1}^{n} A_{n+j}\varepsilon T(A_j) g_\varepsilon\left(X_j, \nabla^\varepsilon_{Y_j}\varepsilon T\right) + \sum_{j=1}^{n} A_j \varepsilon T(A_{n+j}) g_\varepsilon\left(Y_j, \nabla^\varepsilon_{X_j}\varepsilon T\right)$$
$$+ \sum_{k=1}^{2n} A_k^2 g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T}Z_k, \nabla^\varepsilon_{Z_k}\varepsilon T\right)$$
$$= -A(\varepsilon T A_{2n+1}) - \sum_{j=1}^{n} A_{n+j}T(A_j) + \sum_{j=1}^{n} A_j T(A_{n+j}) + \frac{1}{\varepsilon^2}\sum_{j=1}^{n} A_j^2 + \frac{1}{\varepsilon^2}\sum_{j=1}^{n} A_{n+j}^2.$$

Finally,

$$
\begin{aligned}
g_\varepsilon\left(\nabla^\varepsilon_{[A,\varepsilon T]}A,\varepsilon T\right) &= -\sum_{h,k=1}^{2n+1}\varepsilon T(A_k)g_\varepsilon\left(\nabla^\varepsilon_{Z_k}A_hZ_h,\varepsilon T\right)\\
&= -\sum_{k=1}^{2n+1}\varepsilon T(A_k)Z_k(A_{2n+1})-\sum_{h,k=1}^{2n+1}\varepsilon T(A_k)A_hg_\varepsilon\left(\nabla^\varepsilon_{Z_k}Z_h,\varepsilon T\right)\\
&= -\sum_{k=1}^{2n+1}\varepsilon T(A_k)Z_k(A_{2n+1})-\sum_{j=1}^{n}\varepsilon T(A_j)A_{n+j}g_\varepsilon\left(\nabla^\varepsilon_{X_j}Y_j,\varepsilon T\right)-\sum_{j=1}^{n}\varepsilon T(A_{n+j})A_jg_\varepsilon\left(\nabla^\varepsilon_{Y_j}X_j,\varepsilon T\right)\\
&= -\sum_{k=1}^{2n+1}\varepsilon T(A_k)Z_k(A_{2n+1})+\sum_{j=1}^{n}T(A_j)A_{n+j}-\sum_{j=1}^{n}T(A_{n+j})A_j.
\end{aligned}
$$

Computation of $\mathrm{Ric}_\varepsilon(A)$. In view of the previous computations, we infer that

$$
\begin{aligned}
\mathrm{Ric}_\varepsilon(A) &= \sum_{h,k=1}^{2n+1}Z_h(A_kZ_k(A_h))-A(\mathrm{div}_\varepsilon(A))-\sum_{h,k=1}^{2n+1}Z_h(A_k)Z_k(A_h)-\frac{2}{\varepsilon^2}\sum_{j=1}^{n}(A_j^2+A_{n+j}^2)\\
&\quad+\frac{2}{\varepsilon}\mathrm{div}_{\mathbb{H}}(A_{2n+1}J(A))-\frac{2}{\varepsilon}g_{\mathbb{H}}(J(A),\nabla^{\mathbb{H}}A_{2n+1})-\frac{2}{\varepsilon}A_{2n+1}\,\mathrm{div}_{\mathbb{H}}(J(A))\\
&\quad+\frac{2n}{\varepsilon^2}A_{2n+1}^2-2g_{\mathbb{H}}(J(A),T(A))\\
&= \sum_{h,k=1}^{2n+1}A_kZ_hZ_k(A_h)-A(\mathrm{div}_\varepsilon(A))-\frac{2}{\varepsilon^2}\sum_{j=1}^{n}(A_j^2+A_{n+j}^2)+\frac{2n}{\varepsilon^2}A_{2n+1}^2-2g_{\mathbb{H}}(J(A),T(A))\\
&= \sum_{h,k=1}^{2n+1}A_kZ_kZ_h(A_h)-2\sum_{j=1}^{n}A_{n+j}T(A_j)+2\sum_{j=1}^{n}A_jT(A_{n+j})-A(\mathrm{div}_\varepsilon(A))\\
&\quad-\frac{2}{\varepsilon^2}\sum_{j=1}^{n}(A_j^2+A_{n+j}^2)+\frac{2n}{\varepsilon^2}A_{2n+1}^2-2g_{\mathbb{H}}(J(A),T(A))\\
&= -\frac{2}{\varepsilon^2}g_\varepsilon\left(A,A\right)+\frac{2n+2}{\varepsilon^2}A_{2n+1}^2.
\end{aligned}
$$

□

17.2.2 Norm of the second fundamental form

To conclude the proof of (17.1.2), we are left to give an explicit expression for the squared norm of the Riemannian second fundamental form.

Proposition 17.2.2. *The following holds.*

$$
|h^\varepsilon|^2=\sum_{h,k=1}^{2n+1}Z_k(\nu_h^\varepsilon)Z_h(\nu_k^\varepsilon)+4g_\varepsilon\left(J(\nu^\varepsilon),T\nu^\varepsilon\right)+(2n-2)\frac{(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2}+\frac{2}{\varepsilon^2}.
$$

Proof. Let us fix $p\in S$. Since $S\setminus S_0$ is dense in S (cf. [25]), by continuity it suffices to assume that $p\in S\setminus S_0$. First we need to choose an orthonormal basis of T_pS. To this aim, we proceed as follows. It is well known (cf. [87]) that there exist an horizontal orthonormal system $e_1, \dots, e_{n-1}, e_{n+1}, \dots, e_{2n-1}$ in T_pS such that

$$e_{n+j} = J(e_j)$$

for any $j = 1, \dots, n-1$. Observing that

$$g_\varepsilon\left(J(\nu^\varepsilon), J(\nu^\varepsilon)\right) = 1 - (\nu^\varepsilon_{2n+1})^2,$$

we complete this orthonormal system to an orthonormal basis of T_pS by letting

$$e_n = \frac{1}{\sqrt{1-(\nu^\varepsilon_{2n+1})^2}} J(\nu^\varepsilon) \qquad \text{and} \qquad e_{2n} = -\frac{\nu^\varepsilon_{2n+1}}{\sqrt{1-(\nu^\varepsilon_{2n+1})^2}} \nu^\varepsilon_{\mathbb{H}} + \sqrt{1-(\nu^\varepsilon_{2n+1})^2}\varepsilon T,$$

where $\nu^\varepsilon_{\mathbb{H}} = \nu^\varepsilon - \nu^\varepsilon_{2n+1}\varepsilon T$. Moreover, we let

$$e_i = \sum_{k=1}^{2n+1} \alpha^i_k Z_k$$

for any $i = 1, \dots, 2n$. Clearly

$$\sum_{k=1}^{2n+1} \alpha^i_k \nu^\varepsilon_k = 0 \qquad \text{and} \qquad \sum_{k=1}^{2n+1} \alpha^i_k \alpha^j_k = \delta_{ij} \tag{17.2.1}$$

for any $i, j = 1, \dots, 2n$. Moreover,

$$\begin{aligned} \sum_{i=1}^{2n} \alpha^i_h \alpha^i_k &= \sum_{i=1}^{2n} g_\varepsilon\left(e_i, Z_k\right) g_\varepsilon\left(e_i, Z_h\right) \\ & g_\varepsilon\left(Z_k, \sum_{i=1}^{2n} g_\varepsilon\left(Z_h, e_i\right) e_i\right) \\ &= g_\varepsilon\left(Z_k, Z_h - g_\varepsilon\left(Z_h, \nu^\varepsilon\right)\nu^\varepsilon\right) \\ &= \delta_{hk} - \nu^\varepsilon_h \nu^\varepsilon_k \end{aligned} \tag{17.2.2}$$

for any $h, k = 1, \dots, 2n-1$. Finally, notice that

$$\alpha^i_{2n+1} = 0 \tag{17.2.3}$$

for any $i = 1, \dots, 2n-1$. We claim that

$$g_\varepsilon\left(\nabla^\varepsilon_{e_i} e_j, \nu^\varepsilon\right) = -\sum_{h,k=1}^{2n+1} \alpha^i_k \alpha^j_h Z_k(\nu^\varepsilon_h) + \frac{\nu^\varepsilon_{2n+1}}{\varepsilon} g_\varepsilon\left(e_i, J(e_j)\right) + \frac{\alpha^i_{2n+1}}{\varepsilon} g_\varepsilon\left(\nu^\varepsilon, J(e_j)\right) + \frac{\alpha^j_{2n+1}}{\varepsilon} g_\varepsilon\left(\nu^\varepsilon, J(e_i)\right) \tag{17.2.4}$$

for any $i, j = 1, \ldots, 2n$. Indeed, in view of Proposition 16.3.1 and (17.2.1), it holds that

$$\begin{aligned}
g_\varepsilon\left(\nabla^\varepsilon_{e_i} e_j, \nu^\varepsilon\right) &= \sum_{h,k=1}^{2n+1} \alpha^i_k g_\varepsilon\left(\nabla^\varepsilon_{Z_k} \alpha^j_h Z_h, \nu^\varepsilon\right) \\
&= \sum_{h,k=1}^{2n+1} \alpha^i_k Z_k(\alpha^j_h)\nu^\varepsilon_h + \sum_{h,k=1}^{2n+1} \alpha^i_k \alpha^j_h g_\varepsilon\left(\nabla^\varepsilon_{Z_k} Z_h, \nu^\varepsilon\right) \\
&= \sum_{h,k=1}^{2n+1} \alpha^i_k Z_k(\alpha^j_h \nu^\varepsilon_h) - \sum_{h,k=1}^{2n+1} \alpha^i_k \alpha^j_h Z_k(\nu^\varepsilon_h) + \sum_{k=1}^{n}\sum_{h=1}^{2n+1} \alpha^i_k \alpha^j_h g_\varepsilon\left(\nabla^\varepsilon_{X_k} Z_h, \nu^\varepsilon\right) \\
&\quad + \sum_{k=1}^{n}\sum_{h=1}^{2n+1} \alpha^i_{n+k} \alpha^j_h g_\varepsilon\left(\nabla^\varepsilon_{Y_k} Z_h, \nu^\varepsilon\right) + \sum_{h=1}^{2n+1} \alpha^i_{2n+1}\alpha^j_h g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T} Z_h, \nu^\varepsilon\right) \\
&= -\sum_{h,k=1}^{2n+1} \alpha^i_k \alpha^j_h Z_k(\nu^\varepsilon_h) + \sum_{k=1}^{n} \alpha^i_k \alpha^j_{n+k} g_\varepsilon\left(\nabla^\varepsilon_{X_k} Y_k, \nu^\varepsilon\right) + \sum_{k=1}^{n} \alpha^i_k \alpha^j_{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{X_k} \varepsilon T, \nu^\varepsilon\right) \\
&\quad + \sum_{k=1}^{n} \alpha^i_{n+k}\alpha^j_k g_\varepsilon\left(\nabla^\varepsilon_{Y_k} X_k, \nu^\varepsilon\right) + \sum_{k=1}^{n} \alpha^i_{n+k}\alpha^j_{2n+1} g_\varepsilon\left(\nabla^\varepsilon_{Y_k} \varepsilon T, \nu^\varepsilon\right) \\
&\quad + \sum_{k=1}^{n} \alpha^i_{2n+1}\alpha^j_k g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T} X_k, \nu^\varepsilon\right) + \sum_{k=1}^{n} \alpha^i_{2n+1}\alpha^j_{n+k} g_\varepsilon\left(\nabla^\varepsilon_{\varepsilon T} Y_k, \nu^\varepsilon\right) \\
&= -\sum_{h,k=1}^{2n+1} \alpha^i_k \alpha^j_h Z_k(\nu^\varepsilon_h) - \frac{\nu^\varepsilon_{2n+1}}{\varepsilon}\sum_{k=1}^{n} \alpha^i_k\alpha^j_{n+k} + \frac{\alpha^j_{2n+1}}{\varepsilon}\sum_{k=1}^{n}\alpha^i_k \nu^\varepsilon_{n+k} + \frac{\nu^\varepsilon_{2n+1}}{\varepsilon}\sum_{k=1}^{n}\alpha^i_{n+k}\alpha^j_k \\
&\quad - \frac{\alpha^j_{2n+1}}{\varepsilon}\sum_{k=1}^{n}\alpha^i_{n+k}\nu^\varepsilon_k + \frac{\alpha^i_{2n+1}}{\varepsilon}\sum_{k=1}^{n}\alpha^j_k\nu^\varepsilon_{n+k} - \frac{\alpha^i_{2n+1}}{\varepsilon}\sum_{k=1}^{n}\alpha^j_{n+k}\nu^\varepsilon_k
\end{aligned}$$

for any $i, j = 1, \ldots, 2n$, from which (17.2.4) follows. In view of the choice of $e_1, \ldots, e_{2n}$ we have that

$$g_\varepsilon\left(\nabla^\varepsilon_{e_i} e_{n+i}, \nu^\varepsilon\right) = -\sum_{h,k=1}^{2n+1} \alpha^i_k \alpha^{n+i}_h Z_k(\nu^\varepsilon_h) - \frac{\nu^\varepsilon_{2n+1}}{\varepsilon}$$

and

$$g_\varepsilon\left(\nabla^\varepsilon_{e_{n+i}} e_i, \nu^\varepsilon\right) = -\sum_{h,k=1}^{2n+1} \alpha^{n+i}_k \alpha^i_h Z_k(\nu^\varepsilon_h) + \frac{\nu^\varepsilon_{2n+1}}{\varepsilon}$$

for any $i = 1, \ldots, 2n-1$. Moreover, notice that

$$\begin{aligned}
g_\varepsilon\left(\nabla^\varepsilon_{e_n} e_{2n}, \nu^\varepsilon\right) &= -\sum_{h,k=1}^{2n+1} \alpha^n_k \alpha^{2n}_h Z_k(\nu^\varepsilon_h) + \frac{\nu^\varepsilon_{2n+1}}{\varepsilon} g_\varepsilon\left(e_n, J(e_{2n})\right) + \frac{\alpha^{2n}_{2n+1}}{\varepsilon} g_\varepsilon\left(\nu^\varepsilon, J(e_n)\right) \\
&\quad - \sum_{h,k=1}^{2n+1} \alpha^n_k \alpha^{2n}_h Z_k(\nu^\varepsilon_h) - \frac{(\nu^\varepsilon_{2n+1})^2}{\varepsilon(1-(\nu^\varepsilon_{2n+1})^2)} g_\varepsilon\left(J(\nu^\varepsilon), (J(\nu^\varepsilon)\right) + \frac{1}{\varepsilon} g_\varepsilon\left(\nu^\varepsilon, J(J(\nu^\varepsilon))\right) \\
&= -\sum_{h,k=1}^{2n+1} \alpha^n_k \alpha^{2n}_h Z_k(\nu^\varepsilon_h) - \frac{(\nu^\varepsilon_{2n+1})^2}{\varepsilon} - \frac{1}{\varepsilon}\left(1 - (\nu^\varepsilon_{2n+1})^2\right) \\
&= -\sum_{h,k=1}^{2n+1} \alpha^n_k \alpha^{2n}_h Z_k(\nu^\varepsilon_h) - \frac{1}{\varepsilon},
\end{aligned}$$

and

$$
\begin{aligned}
g_\varepsilon\left(\nabla^\varepsilon_{e_{2n}} e_n, \nu^\varepsilon\right) &= -\sum_{h,k=1}^{2n+1} \alpha_k^{2n}\alpha_h^{n} Z_k(\nu_h^\varepsilon) + \frac{\nu^\varepsilon_{2n+1}}{\varepsilon} g_\varepsilon\left(e_{2n}, J(e_n)\right) + \frac{\alpha^{2n}_{2n+1}}{\varepsilon} g_\varepsilon\left(\nu^\varepsilon, J(e_n)\right) \\
&= -\sum_{h,k=1}^{2n+1} \alpha_k^{2n}\alpha_h^{n} Z_k(\nu_h^\varepsilon) + \frac{(\nu^\varepsilon_{2n+1})^2}{\varepsilon} - \frac{1}{\varepsilon}\left(1 - (\nu^\varepsilon_{2n+1})^2\right) \\
&= -\sum_{h,k=1}^{2n+1} \alpha_k^{2n}\alpha_h^{n} Z_k(\nu_h^\varepsilon) - \frac{1}{\varepsilon} + \frac{2(\nu^\varepsilon_{2n+1})^2}{\varepsilon}.
\end{aligned}
$$

Finally,

$$
g_\varepsilon\left(\nabla^\varepsilon_{e_i} e_j, \nu^\varepsilon\right) = -\sum_{h,k=1}^{2n+1} \alpha_k^{i}\alpha_h^{j} Z_k(\nu_h^\varepsilon)
$$

for any $i, j = 1, \dots, 2n$ such that $|i-j| \neq n$. Let us define

$$
b_{i,n+i} = -\frac{\nu^\varepsilon_{2n+1}}{\varepsilon}, \qquad b_{n,2n} = -\frac{1}{\varepsilon}, \qquad b_{n+i,i} = \frac{\nu^\varepsilon_{2n+1}}{\varepsilon} \qquad \text{and} \qquad b_{2n,n} = -\frac{1}{\varepsilon} + \frac{2(\nu^\varepsilon_{2n+1})^2}{\varepsilon}
$$

for any $i = 1, \dots, n-1$, and $b_{i,j} = 0$ for any $i, j = 1, \dots, 2n$ with $|i-j| \neq n$. Moreover, set

$$
a_{i,j} = \sum_{h,k=1}^{2n+1} \alpha_k^{i}\alpha_h^{j} Z_k(\nu_h^\varepsilon)
$$

for any $i, j = 1, \dots, 2n$. Then we have that

$$
\begin{aligned}
|h_p^\varepsilon|^2 &= \sum_{i,j=1}^{2n} g_\varepsilon\left(\nabla^\varepsilon_{e_i} e_j, \nu^\varepsilon\right)^2 \\
&= \sum_{i,j=1}^{2n} g_\varepsilon\left(\nabla^\varepsilon_{e_i} e_j, \nu^\varepsilon\right) g_\varepsilon\left(\nabla^\varepsilon_{e_j} e_i, \nu^\varepsilon\right) \\
&= \sum_{i,j=1}^{2n} (-a_{i,j} + b_{i,j})(-a_{j,i} + b_{j,i}) \\
&= \sum_{i,j=1}^{2n} a_{i,j}a_{j,i} - \sum_{i,j=1}^{2n} a_{i,j}b_{j,i} - \sum_{i,j=1}^{2n} a_{j,i}b_{i,j} + \sum_{i,j=1}^{2n} b_{i,j}b_{j,i} \\
&= \underbrace{\sum_{i,j=1}^{2n} a_{i,j}a_{j,i}}_{A} \underbrace{-2\sum_{i=1}^{n} a_{i,n+i}b_{n+i,i} - 2\sum_{i}^{n} a_{n+i,i}b_{i,n+i}}_{B} \underbrace{+2\sum_{i=1}^{n} b_{i,n+i}b_{n+i,i}}_{C}.
\end{aligned}
$$

First, by (16.4.1) and (17.2.2),

$$
\begin{aligned}
A &= \sum_{i,j=1}^{2n} \sum_{h,k,l,m=1}^{2n+1} \alpha_k^{i}\alpha_m^{i}\alpha_h^{j}\alpha_l^{j} Z_k(\nu_h^\varepsilon) Z_l(\nu_m^\varepsilon) \\
&= \sum_{h,k,l,m=1}^{2n+1} (\delta_{km} - \nu_k^\varepsilon\nu_m^\varepsilon)(\delta_{hl} - \nu_h^\varepsilon\nu_l^\varepsilon) Z_k(\nu_h^\varepsilon) Z_l(\nu_m^\varepsilon) \\
&= \sum_{h,k=1}^{2n+1} Z_k(\nu_h^\varepsilon) Z_h(\nu_k^\varepsilon).
\end{aligned}
$$

Moreover,

$$C = -2(n-1)\frac{(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2} + \frac{2}{\varepsilon}\left(\frac{1}{\varepsilon} - \frac{2(\nu^\varepsilon_{2n+1})^2}{\varepsilon}\right) = -2(n+1)\frac{(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2} + \frac{2}{\varepsilon^2}.$$

Finally,

$$B = \underbrace{\frac{2\nu^\varepsilon_{2n+1}}{\varepsilon}\left(\sum_{i=1}^{n-1} a_{n+i,i} - \sum_{i=1}^{n-1} a_{i,n+i}\right)}_{I} + \underbrace{\frac{2}{\varepsilon}a_{2n,n} + \left(\frac{2}{\varepsilon} - \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon}\right)a_{n,2n}}_{II}.$$

Notice that, in view of (16.4.1), (16.4.2) and (16.4.3),

$$\begin{aligned}
II &= \frac{2}{\varepsilon}\sum_{h,k=1}^{2n+1} \alpha^{2n}_k \alpha^n_h Z_k(\nu^\varepsilon_h) + \left(\frac{2}{\varepsilon} - \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon}\right)\sum_{h,k=1}^{2n+1} \alpha^n_k \alpha^{2n}_h Z_k(\nu^\varepsilon_h) \\
&= \frac{2}{\varepsilon}\sum_{k=1}^{n}\sum_{h=1}^{2n} \alpha^{2n}_k \alpha^n_h X_k(\nu^\varepsilon_h) + \frac{2}{\varepsilon}\sum_{k=1}^{n}\sum_{h=1}^{2n} \alpha^{2n}_{n+k}\alpha^n_h Y_k(\nu^\varepsilon_h) + \frac{2}{\varepsilon}\sum_{h=1}^{2n} \alpha^{2n}_{2n+1}\alpha^n_h Z_{2n+1}(\nu^\varepsilon_h) \\
&\quad + \left(\frac{2}{\varepsilon} - \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon}\right)\sum_{k=1}^{2n+1}\sum_{h=1}^{2n} \alpha^n_h \alpha^{2n}_k Z_h(\nu^\varepsilon_k) \\
&= \frac{2}{\varepsilon}\sum_{k=1}^{n}\sum_{h=1}^{2n} \alpha^{2n}_k \alpha^n_h Z_h(\nu^\varepsilon_k) - \frac{4\nu^\varepsilon_{2n+1}}{\varepsilon^2}\sum_{k=1}^{n} \alpha^{2n}_k \alpha^n_{n+k} + \frac{2}{\varepsilon}\sum_{k=1}^{n}\sum_{h=1}^{2n} \alpha^{2n}_{n+k}\alpha^n_h Z_h(\nu^\varepsilon_{n+k}) + \frac{4\nu^\varepsilon_{2n+1}}{\varepsilon^2}\alpha^{2n}_{n+k}\alpha^n_k \\
&\quad + \frac{2}{\varepsilon}\sum_{h=1}^{2n} \alpha^{2n}_{2n+1}\alpha^n_h Z_h(\nu^\varepsilon_{2n+1}) + \left(\frac{2}{\varepsilon} - \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon}\right)\sum_{k=1}^{2n+1}\sum_{h=1}^{2n} \alpha^n_h \alpha^{2n}_k Z_h(\nu^\varepsilon_k) \\
&= \frac{4(1-(\nu^\varepsilon_{2n+1})^2)}{\varepsilon}\sum_{k=1}^{2n+1}\sum_{h=1}^{2n} \alpha^n_h \alpha^{2n}_k Z_h(\nu^\varepsilon_k) + \frac{4\nu^\varepsilon_{2n+1}}{\varepsilon^2} g_\varepsilon\left(e_{2n}, J(e_n)\right) \\
&= -\frac{4\nu^\varepsilon_{2n+1}}{\varepsilon}\sum_{h,k=1}^{2n} J(\nu^\varepsilon)_h \nu^\varepsilon_k Z_h(\nu^\varepsilon_k) + \frac{4(1-(\nu^\varepsilon_{2n+1})^2)}{\varepsilon}\sum_{h=1}^{2n} J(\nu^\varepsilon)_h Z_h(\nu^\varepsilon_{2n+1}) + \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2} \\
&= \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon}\sum_{h=1}^{2n} J(\nu^\varepsilon)_h Z_h(\nu^\varepsilon_{2n+1}) + \frac{4(1-(\nu^\varepsilon_{2n+1})^2)}{\varepsilon}\sum_{h=1}^{2n} J(\nu^\varepsilon)_h Z_h(\nu^\varepsilon_{2n+1}) + \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2} \\
&= 4g_\varepsilon\left(J(\nu^\varepsilon), T\nu^\varepsilon\right) + \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
I &= \frac{2\nu^\varepsilon_{2n+1}}{\varepsilon}\sum_{i=1}^{n-1}\left(\sum_{h,k=1}^{2n} \alpha^{n+i}_k \alpha^i_h Z_k(\nu^\varepsilon_h) - \sum_{h,k=1}^{2n} \alpha^{n+i}_k \alpha^i_h Z_h(\nu^\varepsilon_k)\right) \\
&= \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2}\sum_{i=1}^{n-1}\left(-\sum_{k=1}^{n} \alpha^{n+i}_k \alpha^i_{n+k} + \sum_{k=1}^{n} \alpha^{n+i}_{n+k}\alpha^i_k\right) \\
&= \frac{4(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2}\sum_{i=1}^{n-1} g_\varepsilon\left(e_{n+i}, J(e_i)\right) \\
&= 4(n-1)\frac{(\nu^\varepsilon_{2n+1})^2}{\varepsilon^2}.
\end{aligned}$$

Putting all the pieces together, the thesis follows. □

In view of Proposition 17.2.1 and Proposition 17.2.2, we are ready to prove Theorem 17.1.2.

Proof of Theorem 17.1.2. First, (17.1.2) follows combining Proposition 17.2.1 and Proposition 17.2.2. To conclude the proof, it suffices to observe that

$$\nu^\varepsilon = \frac{\nu^{\mathbb{H}}}{\sqrt{1+\varepsilon^2(Td^{\mathbb{H}})^2}} + \frac{\varepsilon T d^{\mathbb{H}}}{\sqrt{1+\varepsilon^2(Td^{\mathbb{H}})^2}}\varepsilon T.$$

Therefore, noticing that

$$\lim_{\varepsilon\to 0}\nu_j^\varepsilon = \nu_j^{\mathbb{H}} \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\frac{\nu_{2n+1}^\varepsilon}{\varepsilon} = Td^{\mathbb{H}}$$

locally uniformly for any $j = 1, \ldots, 2n$, and moreover

$$\lim_{\varepsilon\to 0} Z_i\nu_j^\varepsilon = Z_i\nu_j^{\mathbb{H}} \qquad \text{and} \qquad \lim_{\varepsilon\to 0} Z_{2n+1}\nu_k^\varepsilon = \lim_{\varepsilon\to 0} Z_k\nu_{2n+1}^\varepsilon = 0$$

locally uniformly for any $i, j = 1, \ldots, 2n$ and $k = 1, \ldots, 2n+1$, (17.1.3) follows in view of (17.1.2). □

17.3 The sub-Riemannian second variation formula

The aim of this section is to propose a proof of Theorem 17.1.1 which is tailored to the Heisenberg group. From now on we fix a non-characteristic hypersurface $S = \partial E$ of class C^3. Our approach follows essentially the Euclidean one of [163, Chapter 10]. In order to derive first and second variation formulas for the sub-Riemannian perimeter of E, we begin with the following sub-Riemannian change of variables formula.

Proposition 17.3.1 (Change of $\mathbb{H}$-perimeter formula for sets of Euclidean finite perimeter)**.** *Let $\Omega \subseteq \mathbb{H}^n$ be an open set, let E be an Euclidean Caccioppoli set in Ω and let $F : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ be a diffeomorphism. Suppose that $A \Subset \Omega$ is an open set. Then*

$$P_{\mathbb{H}}(F(E), F(A)) = \int_A |H \cdot N|\, dP(E, \cdot), \tag{17.3.1}$$

where H is the $2n \times (2n+1)$ matrix defined by

$$H(p) := |\det DF(p)|\, \mathcal{C}(F(p))(DF(p)^{-1})^T$$

for any $p \in \mathbb{H}^n$.

Proof. Let $\phi := F^{-1}$, $g \in C_c^1(F(A); \mathbb{R}^{2n})$, $g_* := g \circ \phi$ and first assume that $f \in C^1(\Omega)$. Let

$f_* := f \circ \phi$. Then

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \langle g_*, \nabla^{\mathbb{H}} f_* \rangle \, d\mathcal{L}^{2n+1} &= \int_{\mathbb{R}^{2n+1}} \langle (g \circ \phi)(p), \mathcal{C}(p) \cdot D(f \circ \phi)^T(p) \rangle \, d\mathcal{L}^{2n+1}(p) \\ &= \int_{\mathbb{R}^{2n+1}} \langle (g(\phi(p)), \mathcal{C}(F(\phi(p))) \cdot D\phi(F(\phi(p)))^T \cdot Df(\phi(p))^T \, d\mathcal{L}^{2n+1}(p) \\ &= \int_{\mathbb{R}^{2n+1}} \langle g(p), \mathcal{C}(F(p)) \cdot D\phi(F(p))^T \cdot Df(p)^T \rangle \det DF(p) \, d\mathcal{L}^{2n+1}(p) \\ &= \int_{\mathbb{R}^{2n+1}} \langle g(p), H(p) \cdot Df(p)^T \rangle \, d\mathcal{L}^{2n+1}(p).\end{aligned} \tag{17.3.2}$$

We know (cf. [138, 150]) that there exists a sequence $(f_j)_j \subset C^1(\Omega)$ such that

$$f_j \to \chi_E \text{ in } L^1(A) \qquad \text{and } f_{j*} \to \chi_{F(E)} \text{ in } L^1(A_*) \tag{17.3.3}$$

as $j \to \infty$. From (17.3.2) with $f \equiv f_j$, it follows that

$$-\int_{\mathbb{R}^{2n+1}} \operatorname{div}_{\mathbb{H}} g_* \, f_{j*} \, d\mathcal{L}^{2n+1} = \int_{\mathbb{R}^{2n+1}} \langle g_*, \nabla^{\mathbb{H}} f_{j*} \rangle \, d\mathcal{L}^{2n+1} = -\int_{\mathbb{R}^{2n+1}} \operatorname{div}(H^T g) f_j \, d\mathcal{L}^{2n+1}.$$

By (17.3.3), we can pass to the limit in the previous identity as $j \to \infty$ and get

$$\int_{\mathbb{R}^{2n+1}} \operatorname{div}_{\mathbb{H}} g_* \, \chi_{F(E)} \, d\mathcal{L}^{2n+1} = \int_{\mathbb{R}^{2n+1}} \operatorname{div}(H^T \cdot g) \chi_E \, d\mathcal{L}^{2n+1}.$$

From the previous identity we get that

$$\int_{\mathbb{R}^{2n+1}} \langle g_*, \nu^{\mathbb{H}}_{F(E)} \rangle \, dP_{\mathbb{H}}(F(E), \cdot) = \int_{\mathbb{R}^{2n+1}} \langle H^T \cdot g, N_E \rangle_{\mathbb{R}^{2n+1}} \, dP(E, \cdot) = \int_{\mathbb{R}^{2n+1}} \langle g, H \cdot N_E \rangle \, dP(E, \cdot). \tag{17.3.4}$$

Arguing *verbatim* as in the proof of [163, Lemma 10.1], (17.3.1) follows. □

Let $(F_t)_{t \in (0,1)}$ be a one-parameter family of diffeomorphisms of $\mathbb{R}^{2n+1}$ such that F_0 is the identity map. Set $\phi_t := F_t^{-1}$. Suppose also that there exists a fixed compact set $K \subseteq \mathbb{R}^{2n+1}$ such that F_t is the identity map outside K for each $t \in [0, 1]$. If $K \subseteq A$, with A bounded open set, then $F_t(A) = A$. Denote by $H_t(p) \equiv H(t, p)$ the $2n \times (2n + 1)$- matrix defined by

$$H_t(p) := \det(DF_t)(p) \, C(F_t(p)) \, (DF_t^{-1}(p))^T$$

for any $(t, p) \in [0, 1] \times \mathbb{R}^{2n+1}$, and let

$$\dot{H}_0(p) := \frac{d}{dt} H_t(p)\Big|_{t=0} \qquad \text{and} \qquad \ddot{H}_0(p) := \frac{d^2}{dt^2} H_t(p)\Big|_{t=0}$$

for any $p \in \mathbb{R}^{2n+1}$.

Theorem 17.3.2. *Let us set $E_t := F_t(E)$ for any $t \in [0, 1]$ and let $A \subseteq \mathbb{R}^{2n+1}$ be an open set.*

Assume that $\partial E \cap A \neq 0$. Then

$$\frac{d}{dt} P_{\mathbb{H}}(E_t, A)\Big|_{t=0} = \int_{A\cap\partial E} \left\langle \dot{H}_0 N_E, \nu^{\mathbb{H}} \right\rangle \, d\mathcal{H}^{2n} \tag{17.3.5}$$

and

$$\frac{d^2}{dt^2} P_{\mathbb{H}}(E_t, A)\Big|_{t=0} = \int_{A\cap\partial E} \left(\left\langle \ddot{H}_0 N_E, \nu^{\mathbb{H}} \right\rangle + \frac{\left|\dot{H}_0 N_E\right|^2 - \left\langle \dot{H}_0 N_E, \nu^{\mathbb{H}} \right\rangle^2}{|N_E^{\mathbb{H}}|} \right) d\mathcal{H}^{2n}. \tag{17.3.6}$$

Proof. First notice that, by (17.3.2),

$$P_{\mathbb{H}}(E_t, A) = \int_A |H_t \cdot N_E| \, d\,|\partial E| = \int_{A\cap\partial E} |H_t \cdot N_E| \, d\,\mathcal{H}^{2n} = \int_{A\cap\partial E} D(t,p) \, d\,\mathcal{H}^{2n}(p) \tag{17.3.7}$$

for any $t \in [0,1]$, where

$$D(t,p) := |H_t(p) \cdot N_E(p)|$$

for any $p \in \partial E \cap A$ and any $t \in [0,1]$. Without loss of generality, we can assume that there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that

$$D(t,p) \geqslant \varepsilon_0 \text{ for each } (t,p) \in [0,\delta_0] \times (K \cap \partial E)\,. \tag{17.3.8}$$

Otherwise, by contradiction, there exist two sequences $(t_h)_h \subset [0,1]$ and $(p_h)_h \subset K \cap \partial E$ with

$$0 \leqslant t_h \leqslant \frac{1}{h} \text{ and } D(t_h, p_h) \leqslant \frac{1}{h} \text{ for each } h\,.$$

Up to a subsequnce, there exists $p_0 \in K \cap \partial E$ such that $p_h \to p_0$ as $h \to \infty$. By the continuity of D, it follows that

$$|\mathcal{C}(p_0) \cdot N_E(p_0)| = D(0, p_0) = \lim_{h\to\infty} D(t_h, p_h) = 0\,.$$

Thus p_0 would be a characteristic point of $A \cap \partial E$, a contradiction with our assumptions. By (17.3.8), the function

$$[0,\delta_0] \ni t \mapsto D(t,p) := |H(t,p) N_E(p)| \text{ is regular, for each } p \in A \cap \partial E\,, \tag{17.3.9}$$

and

$$\frac{\partial}{\partial t} D(t,p) = \frac{\langle \frac{d}{dt} H_t(p) N_E(p), H_t(p) N_E(p)\rangle_{\mathbb{R}^{2n}}}{\sqrt{\langle H_t(p) N_E(p), H_t(p) N_E(p)\rangle_{\mathbb{R}^{2n}}}} = \frac{\langle \frac{d}{dt} H_t(p) N_E(p), H_t(p) N_E(p)\rangle_{\mathbb{R}^{2n}}}{D(t,p)} \tag{17.3.10}$$

for each $(t,p) \in [0,\delta_0] \times (A \cap \partial E)$, and

$$\left|\frac{\partial}{\partial t} D(t,p)\right| \leqslant \sup_{(t,p)\in[0,1]\times K} \left|\frac{d}{dt} H_t(p)\right| < \infty \text{ for each } (t,p) \in [0,1] \times (A \cap \partial E)\,. \tag{17.3.11}$$

From (17.3.10) and (17.3.11), since $\mathcal{H}^{2n}(A\cap\partial E)<\infty$, we can apply the derivation under the integral sign in (17.3.7) and we get that

$$\frac{d}{dt}P_{\mathbb{H}}(E_t,A)=\int_{A\cap\partial E}\frac{\partial}{\partial t}D(t,p)\,d\mathcal{H}^{2n}(p)=\int_{A\cap\partial E}\frac{\langle\frac{d}{dt}H_t(p)N_E(p),H_t(p)N_E(p)\rangle_{\mathbb{R}^{2n}}}{\sqrt{\langle H_t(p)N_E(p),H_t(p)N_E(p)\rangle_{\mathbb{R}^{2n}}}}\,d\mathcal{H}^{2n}(p) \tag{17.3.12}$$

for each $t\in[0,\delta_0]$. Choosing $t=0$ in (17.3.12) we get (17.3.5). By (17.3.8) and (17.3.9), we can perform the second derivative of D with respect to t and we get

$$\frac{\partial^2}{\partial t^2}D(t,p)=\frac{\left(\langle\frac{d^2}{dt^2}H_tN_E,H_tN_E\rangle+|\frac{d}{dt}H_tN_E|^2\right)\sqrt{\langle H_tN_E,H_tN_E\rangle}}{\langle H_tN_E,H_tN_E\rangle}-\frac{\left(\langle\frac{d}{dt}H_tN_E,H_tN_E\rangle\right)^2}{\left(\langle H_tN_E,H_tN_E\rangle\right)^{3/2}}$$

for each $(t,p)\in[0,\delta_0]\times(A\cap\partial E)$ Arguing as above, we can apply again the derivation under the integral sign in (17.3.12) and get that

$$\frac{d^2}{dt^2}P_{\mathbb{H}}(E_t,A)=\int_{A\cap\partial E}\frac{\partial^2}{\partial t^2}D(t,p)\,d\mathcal{H}^{2n}(p)\,d\mathcal{H}^{2n}(p)\,. \tag{17.3.13}$$

Choosing $t=0$ in (17.3.13) we get (17.3.6). □

In the following we wish to apply Theorem 17.3.2 specializing the family $(F_t)_t$. More precisely, we fix an open set $A\subseteq\mathbb{R}^{2n+1}$ such that $\partial E\cap A\neq 0$. If A is sufficiently small, we already know that we can extend $\nu^{\mathbb{H}}$ to the whole A by letting $\nu^{\mathbb{H}}=\nabla^{\mathbb{H}}d^{\mathbb{H}}$. Since we want to perform normal variations, we fix a test function $\xi\in C_c^\infty(A)$ and we define

$$F_t(p)=p\cdot\exp\delta_t(\xi\nu^{\mathbb{H}})$$

for any $p\in\mathbb{H}^n$. Notice that, if t is small enough, F_t is a diffeomorphism and $F_t=I$ outside $\operatorname{supp}(\xi)$. Before going on, let us fix some notation. Being E fixed, we let $N=N_E$ and $N^{\mathbb{H}}=N_E^{\mathbb{H}}$. Moreover, we let $g=\xi\nu^{\mathbb{H}}$ and we define the *horizontal Jacobian* matrix $\nabla^{\mathbb{H}}g$ of g by letting

$$(\nabla^{\mathbb{H}}g)_{i,j}=Z_jg^i$$

for any $i,j=1,\dots,2n$. In particular, observe that

$$\nabla^{\mathbb{H}}g=\nu^{\mathbb{H}}\otimes\nabla^{\mathbb{H}}\xi+\xi\nabla^{\mathbb{H}}\nu^{\mathbb{H}}$$

The first step to apply Theorem 17.3.2 is to compute $\dot H_0$ and $\ddot H_0$. Let us tart by computing explicitly H_t.

Lemma 17.3.3. *It holds that*

$$\dot H_0=(\operatorname{div}_{\mathbb{H}}g)C+2A-P$$

and

$$\ddot{H}_0 = 2PM^T - 4AM^T - 2(\operatorname{div}_{\mathbb{H}} g)P + 4(\operatorname{div}_{\mathbb{H}} g)A + (\operatorname{div}_{\mathbb{H}} g)^2\mathcal{C} - \tau(M^2)\mathcal{C},$$

where, for any $p \in A$, $M(p)$ is the $(2n+1)\times(2n+1)$ matrix defined by

$$M_{i,j} = \frac{\partial g_i}{\partial z_j}$$

for any $i = 1, \dots, 2n$ and any $j = 1, \dots, 2n+1$, and

$$M_{2n+1,j} = J(g)_j + \sum_{k=1}^{n} y_k \frac{\partial g^k}{\partial z_j} - \sum_{k=1}^{n} x_k \frac{\partial g^{n+k}}{\partial z_j}$$

for any $j = 1, \dots, 2n+1$, $A(p)$ is the $(2n)\times(2n+1)$ matrix defined by

$$A_{i,j} = 0$$

for any $i, j = 1, \dots, 2n$ and

$$A_{i,2n+1} = -J(g)_i$$

for any $i = 1, \dots, 2n$, and $P(p)$ is the $(2n)\times(2n+1)$ matrix defined by

$$P_{i,j} = Z_i g^j$$

for any $i, j = 1, \dots, 2n$, and

$$P_{i,2n+1} = \sum_{k=1}^{2n} y_k Z_i g^k - \sum_{k=1}^{n} x_k Z_i g^{n+k}$$

for any $i = 1, \dots, 2n$.

Proof. Notice that

$$F_s(p) = \left(x_1 + sg^1, \dots, x_n + sg^n, \dots, y_1 + sg^{n+1}, \dots, y_n + sg^{2n}, t + s\sum_{k=1}^{n} y_k g^k - s\sum_{k=1}^{n} x_k g^{n+k} \right)$$

First, by a direct computation,

$$DF_s(p) = I_{2n+1} + sM(p)$$

for any $p \in A$. Therefore it is easy to check that

$$DF_s^{-1}(p) = I_{2n+1} - sM(p) + s^2M(p)^2 + o(s^2)$$

as $s \to 0$ and for any $p \in A$. Moreover, by definition of F_s and A,

$$\mathcal{C}(F_s(p)) = \mathcal{C}(p) + sA(p).$$

Finally, we know by standard linear algebra arguments that

$$\det(DF_s(p)) = \det(I_{2n+1} + sM(p)) = 1 + s\tau(M(p)) + \frac{s^2}{2}(\tau(M(p))^2 - \tau(M(p)^2)) + o(s^2)$$

as $s \to 0$ and for any $p \in A$. Therefore,

$$\begin{aligned}
H_s &= \left(1 + s\tau(M) + \frac{s^2}{2}(\tau(M)^2 - \tau(M^2)) + o(s^2)\right)(\mathcal{C} + sA)(I_{2n+1} - sM + s^2M^2 + o(s^2))^T \\
&= \left(1 + s\tau(M) + \frac{s^2}{2}(\tau(M)^2 - \tau(M^2)) + o(s^2)\right)(\mathcal{C} - s\mathcal{C}M^T + s^2\mathcal{C}(M^T)^2 + o(s^2)) \\
&\quad + \left(1 + s\tau(M) + \frac{s^2}{2}(\tau(M)^2 - \tau(M^2)) + o(s^2)\right)(sA - s^2AM^T + o(s^2)) \\
&= \mathcal{C} + s(-\mathcal{C}M^T + \tau(M)\mathcal{C} + A) \\
&\quad + \frac{s^2}{2}\left(2\mathcal{C}(M^T)^2 - 2\tau(M)\mathcal{C}M^T + \tau(M)^2\mathcal{C} - \tau(M^2)C2AM^T + 2\tau(M)A\right) + o(s^2)
\end{aligned}$$

as $s \to 0$ and for any $p \in A$. Notice that

$$-(\mathcal{C}M^T)_{i,j} = -\frac{\partial g^j}{\partial x_i} - y_i\frac{g^j}{\partial t} = -X_ig^j \quad \text{and} \quad -(\mathcal{C}M^T)_{n+i,j} - \frac{\partial g^j}{\partial y_i} + x_i\frac{g^j}{\partial t} = -Y_ig^j$$

for any $i = 1, \dots, n$ and any $j = 1, \dots, 2n$. Moreover,

$$\begin{aligned}
-(\mathcal{C}M^T)_{i,2n+1} &= -J(g)_i - \sum_{k=1}^n y_k\frac{\partial g^k}{\partial x_i} + \sum_{k=1}^n x_k\frac{\partial g^{n+k}}{\partial x_i} - y_i\sum_{k=1}^n y_k\frac{\partial g^k}{\partial t} + y_i\sum_{k=1}^n x_k\frac{\partial g^{n+k}}{\partial t} \\
&= -J(g)_i - \sum_{k=1}^n y_kX_ig^k + \sum_{k=1}^n x_kX_ig^{n+k}
\end{aligned}$$

for any $i = 1, \dots, n$. Reasoning similarly in the remaining case, we conclude that

$$-\mathcal{C}M^T = -P + A.$$

Hence, we have that

$$\dot{H}_0 = \tau(M)\mathcal{C} - P + 2A$$

and

$$\ddot{H}_0 = 2PM^T - 4AM^T - 2\tau(M)P + \tau(M)^2\mathcal{C} - \tau(M^2)\mathcal{C} + 4\tau(M)A$$

□

Lemma 17.3.4. *It holds that*

$$\langle \dot{H}_0N, \nu^{\mathbb{H}}\rangle = |N^{\mathbb{H}}|\xi H.^{\mathbb{H}} \tag{17.3.14}$$

and

$$\frac{|\dot{H}_0N|^2}{|N^{\mathbb{H}}|} = |N^{\mathbb{H}}|\left(\xi^2\left(H^{\mathbb{H}}\right)^2 + |\nabla^{\mathbb{H},S}\xi|^2 + 4\xi Td^{\mathbb{H}}g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, J(\nu^{\mathbb{H}})\right) + 4\xi^2(Td^{\mathbb{H}})^2\right) \tag{17.3.15}$$

Proof. We claim that

$$\dot{H}_0 N = \xi H^{\mathbb{H}} N^{\mathbb{H}} + g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, \nu^{\mathbb{H}}\right) N^{\mathbb{H}} - |N^{\mathbb{H}}|\nabla^{\mathbb{H}}\xi - 2\xi|N^{\mathbb{H}}|Td^{\mathbb{H}}J(\nu^{\mathbb{H}}).$$

Notice that, in view of Lemma 17.3.3,

$$\dot{H}_0 N = (\operatorname{div}_{\mathbb{H}} g)N^{\mathbb{H}} + 2AN - PN.$$

First, notice that

$$\operatorname{div}_{\mathbb{H}} g = \xi H^{\mathbb{H}} + g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, \nu^{\mathbb{H}}\right).$$

Moreover,

$$2AN = -2N_{2n+1}J(g) = -2\xi N_{2n+1}J(\nu^{\mathbb{H}}).$$

Finally,

$$(PN)_i = \sum_{k=1}^{n} Z_i g^k N_k + \sum_{k=1}^{n} Z_i g^{n+k} N_{n+k} + N_{2n+1}\sum_{k=1}^{n} y_k Z_i g^k - N_{2n+1}\sum_{k=1}^{n} x_k Z_i g^{n+k} = \sum_{k=1}^{2n} Z_i g^k N_k^{\mathbb{H}}$$

for any $i = 1, \ldots, 2n$, which implies, recalling (16.6.3), that

$$PN = (\nabla^{\mathbb{H}} g)^T N^{\mathbb{H}} = (\nabla^{\mathbb{H}}\xi \otimes \nu^{\mathbb{H}})N^{\mathbb{H}} + \xi(\nabla^{\mathbb{H}}\nu^{\mathbb{H}})^T N^{\mathbb{H}} = |N^{\mathbb{H}}|\nabla^{\mathbb{H}}\xi.$$

Putting all the pieces together, and thanks to (16.6.8), the claim follows. Notice that (17.3.14) trivially follows. Moreover, recalling (16.6.9),

$$\begin{aligned}\frac{|\dot{H}_0 N|^2}{|N^{\mathbb{H}}|} &= \xi^2\left(H^{\mathbb{H}}\right)^2|N^{\mathbb{H}}| + \left(g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, \nu^{\mathbb{H}}\right)\right)^2|N^{\mathbb{H}}| + |\nabla^{\mathbb{H}}\xi|^2|N^{\mathbb{H}}| + 4\xi^2(Td^{\mathbb{H}})^2|N^{\mathbb{H}}| \\ &\quad - 2\left(g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, \nu^{\mathbb{H}}\right)\right)^2|N^{\mathbb{H}}| + 4\xi Td^{\mathbb{H}} g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, J(\nu^{\mathbb{H}})\right)|N^{\mathbb{H}}| \\ &= |N^{\mathbb{H}}|\left(\xi^2\left(H^{\mathbb{H}}\right)^2 + |\nabla^{\mathbb{H},S}\xi|^2 + 4\xi Td^{\mathbb{H}} g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, J(\nu^{\mathbb{H}})\right) + 4\xi^2(Td^{\mathbb{H}})^2\right)\end{aligned}$$

□

Lemma 17.3.5. *It holds that*

$$\langle \ddot{H}_0 N, \nu^{\mathbb{H}}\rangle = |N^{\mathbb{H}}|\left(\xi^2\left(H^{\mathbb{H}}\right)^2 - \xi^2\tau((\nabla^{\mathbb{H}}\nu^{\mathbb{H}})^2) - 2\xi^2 g_{\mathbb{H}}\left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}}\right) - 4\xi^2(Td^{\mathbb{H}})^2\right).$$

Proof. First, notice that

$$(PM^T)_{i,j} = \sum_{k=1}^{2n} Z_i g^k \frac{\partial g^j}{\partial z_k} + \sum_{k=1}^{n} y_k Z_i g^k \frac{\partial g^j}{\partial t} - \sum_{k=1}^{n} x_k Z_i g^{n+k}\frac{\partial g^j}{\partial t} = \sum_{k=1}^{2n} Z_i g^k Z_k g^j$$

for any $i, j = 1, \dots, 2n$. Moreover,

$$
\begin{aligned}
(PM^T)_{i,2n+1} &= \sum_{k=1}^{2n} Z_i g^k J(g)_k + \sum_{k=1}^{2n}\sum_{h=1}^{n} y_h Z_i g^k \frac{\partial g^h}{\partial z_k} - \sum_{k=1}^{2n}\sum_{h=1}^{n} x_h Z_i g^k \frac{\partial g^{n+h}}{\partial z_k} + \sum_{h,k=1}^{n} y_k y_h Z_i g^k \frac{\partial g^h}{\partial t} \\
&\quad - \sum_{h,k=1}^{n} y_k x_h Z_i g^k \frac{\partial g^{n+h}}{\partial t} - \sum_{h,k=1}^{n} x_k y_h Z_i g^{n+k} \frac{\partial g^h}{\partial t} + \sum_{h,k=1}^{n} x_k x_h Z_i g^{n+k} \frac{\partial g^{n+h}}{\partial t} \\
&= \sum_{k=1}^{2n} Z_i g^k J(g)_k + \sum_{k=1}^{2n}\sum_{h=1}^{n} y_h Z_i g^k Z_k g^h - \sum_{k=1}^{2n}\sum_{h=1}^{n} x_h Z_i g^k Z_k g^{n+h}
\end{aligned}
$$

for any $i = 1, \dots, 2n$. Therefore,

$$
\begin{aligned}
(PM^T N)_i &= \sum_{h,k=1}^{2n} Z_i g^k Z_k g^h N_h + N_{2n+1} \sum_{k=1}^{2n}\sum_{h=1}^{n} y_h Z_i g^k Z_k g^h - N_{2n+1} \sum_{k=1}^{2n}\sum_{h=1}^{n} x_h Z_i g^k Z_k g^{n+h} \\
&\quad + N_{2n+1} \sum_{k=1}^{2n} Z_i g^k J(g)_k \\
&= \sum_{h,k=1}^{2n} Z_i g^k Z_k g^h N_h^{\mathbb{H}} + |N^{\mathbb{H}}| T d^{\mathbb{H}} \sum_{k=1}^{2n} Z_i g^k J(g)_k.
\end{aligned}
$$

Hence, thanks to (16.6.3), we have that

$$
\begin{aligned}
PM^T N &= (\nabla^{\mathbb{H}} g)^T (\nabla^{\mathbb{H}} g)^T N^{\mathbb{H}} + |N^{\mathbb{H}}| T d^{\mathbb{H}} (\nabla^{\mathbb{H}} g)^T J(g) \\
&= |N^{\mathbb{H}}| \Big((\nabla^{\mathbb{H}} \xi \otimes \nu^{\mathbb{H}} + \xi (\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^T)(\nabla^{\mathbb{H}} \xi \otimes \nu^{\mathbb{H}} + \xi (\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^T) \nu^{\mathbb{H}}) \\
&\quad + \xi T d^{\mathbb{H}} (\nabla^{\mathbb{H}} \xi \otimes \nu^{\mathbb{H}} + \xi (\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^T) J(\nu^{\mathbb{H}}) \Big) \\
&= |N^{\mathbb{H}}| \Big((\nabla^{\mathbb{H}} \xi \otimes \nu^{\mathbb{H}} + \xi (\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^T) \nabla^{\mathbb{H}} \xi + \xi^2 T d^{\mathbb{H}} (\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^T J(\nu^{\mathbb{H}}) \Big) \\
&= |N^{\mathbb{H}}| \Big(g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right) \nabla^{\mathbb{H}} \xi + \xi (\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^T \nabla^{\mathbb{H}} \xi + \xi^2 T d^{\mathbb{H}} (\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^T J(\nu^{\mathbb{H}}) \Big).
\end{aligned}
$$

Therefore, recalling (16.6.7), we conclude that

$$
\begin{aligned}
\langle PM^T N, \nu^{\mathbb{H}} \rangle &= |N^{\mathbb{H}}| \left((g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 + \xi \langle \nabla^{\mathbb{H}} \xi, \left(\nabla^{\mathbb{H}} \nu^{\mathbb{H}} \right) \nu^{\mathbb{H}} \rangle + \xi^2 T d^{\mathbb{H}} \langle J(\nu^{\mathbb{H}}), \left(\nabla^{\mathbb{H}} \nu^{\mathbb{H}} \right) \nu^{\mathbb{H}} \rangle \right) \\
&= |N^{\mathbb{H}}| \left((g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 - 2\xi T d^{\mathbb{H}} \langle \nabla^{\mathbb{H}} \xi, J(\nu^{\mathbb{H}}) \rangle - 2\xi^2 (T d^{\mathbb{H}})^2 \right) \\
&= |N^{\mathbb{H}}| \left((g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 - T d^{\mathbb{H}} \langle \nabla^{\mathbb{H}} \xi^2, J(\nu^{\mathbb{H}}) \rangle - 2\xi^2 (T d^{\mathbb{H}})^2 \right).
\end{aligned}
$$

Again, since

$$A^T \nu^{\mathbb{H}} = (0, \dots, 0, -\langle J(g), \nu^{\mathbb{H}} \rangle) = 0,$$

we see that

$$\langle AM^T N, \nu^{\mathbb{H}} \rangle = N^T M A^T \nu^{\mathbb{H}} = 0.$$

Moreover, arguing as in the proof of Lemma 17.3.4,

$$\operatorname{div}_{\mathbb{H}} g \langle PN, \nu^{\mathbb{H}} \rangle = |N^{\mathbb{H}}| \left(\xi H^{\mathbb{H}} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right) + (g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 \right).$$

Arguing as above, we see that

$$\langle AN, \nu^{\mathbb{H}} \rangle = 0.$$

Moreover,

$$(\operatorname{div}_{\mathbb{H}} g)^2 \langle \mathcal{C}N, \nu^{\mathbb{H}} \rangle = |N^{\mathbb{H}}| \left(\xi^2 \left(H^{\mathbb{H}} \right)^2 + 2\xi H^{\mathbb{H}} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right) + \left(g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right) \right)^2 \right).$$

Finally, notice that

$$\begin{aligned}
\tau(M^2) &= \sum_{h,k=1}^{2n} M_{h,k} M_{k,h} + 2 \sum_{h=1}^{2n} M_{h,2n+1} M_{2n+1,h} + M_{2n+1,2n+1}^2 \\
&= \sum_{h,k=1}^{2n} \frac{\partial g^h}{\partial z_k} \frac{\partial g^k}{\partial z_h} + 2 \sum_{h=1}^{2n} J(g)_h T g^h + 2 \sum_{h,k=1}^{n} y_k \frac{\partial g^k}{\partial x_h} \frac{\partial g^h}{\partial t} - 2 \sum_{h,k=1}^{n} x_k \frac{\partial g^{n+k}}{\partial x_h} \frac{\partial g^h}{\partial t} \\
&\quad - 2 \sum_{h,k=1}^{n} x_k \frac{\partial g^{n+k}}{\partial y_h} \frac{\partial g^{n+h}}{\partial t} + 2 \sum_{h,k=1}^{n} y_k \frac{\partial g^k}{\partial y_h} \frac{\partial g^{n+h}}{\partial t} \\
&\quad + \sum_{h,k=1}^{n} x_k x_h \frac{\partial g^{n+k}}{\partial t} \frac{\partial g^{n+h}}{\partial t} - 2 \sum_{h,k=1}^{n} x_k y_h \frac{\partial g^{n+k}}{\partial t} \frac{\partial g^h}{\partial t} + \sum_{h,k=1}^{n} y_k y_h \frac{\partial g^k}{\partial t} \frac{\partial g^h}{\partial t} \\
&= \tau((\nabla^{\mathbb{H}} g)^2) + 2 g_{\mathbb{H}} \left(J(g), Tg \right) \\
&= \tau((\nu^{\mathbb{H}} \otimes \nabla^{\mathbb{H}} \xi + \xi \nabla^{\mathbb{H}} \nu^{\mathbb{H}})(\nu^{\mathbb{H}} \otimes \nabla^{\mathbb{H}} \xi + \xi \nabla^{\mathbb{H}} \nu^{\mathbb{H}})) + 2\xi^2 g_{\mathbb{H}} \left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}} \right) \\
&= \tau((\nu^{\mathbb{H}} \otimes \nabla^{\mathbb{H}} \xi)^2) + 2\xi \tau((\nu^{\mathbb{H}} \otimes \nabla^{\mathbb{H}} \xi) \nabla^{\mathbb{H}} \nu^{\mathbb{H}}) + \xi^2 \tau((\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^2) + 2\xi^2 g_{\mathbb{H}} \left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}} \right) \\
&= (g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 - 4\xi T d^{\mathbb{H}} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, J(\nu^{\mathbb{H}}) \right) + \xi^2 \tau((\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^2) + 2\xi^2 g_{\mathbb{H}} \left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}} \right) \\
&= (g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 - 2 T d^{\mathbb{H}} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi^2, J(\nu^{\mathbb{H}}) \right) + \xi^2 \tau((\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^2) + 2\xi^2 g_{\mathbb{H}} \left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}} \right).
\end{aligned}$$

Putting all the pieces together, we conclude that

$$\begin{aligned}
\langle \ddot{H}_0 N^{\mathbb{H}}, \nu^{\mathbb{H}} \rangle = |N^{\mathbb{H}}| \Big(& 2(g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 - 2 T d^{\mathbb{H}} \langle \nabla^{\mathbb{H}} \xi^2, J(\nu^{\mathbb{H}}) \rangle - 4\xi^2 (T d^{\mathbb{H}})^2 \\
& - 2\xi H^{\mathbb{H}} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right) - 2(g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 \\
& + \xi^2 \left(H^{\mathbb{H}} \right)^2 + 2\xi H^{\mathbb{H}} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right) + (g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 - (g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi, \nu^{\mathbb{H}} \right))^2 \\
& + 2 T d^{\mathbb{H}} g_{\mathbb{H}} \left(\nabla^{\mathbb{H}} \xi^2, J(\nu^{\mathbb{H}}) \right) - \xi^2 \tau((\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^2) - 2\xi^2 g_{\mathbb{H}} \left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}} \right) \Big) \\
= |N^{\mathbb{H}}| \Big(& \xi^2 \left(H^{\mathbb{H}} \right)^2 - \xi^2 \tau((\nabla^{\mathbb{H}} \nu^{\mathbb{H}})^2) - 2\xi^2 g_{\mathbb{H}} \left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}} \right) - 4\xi^2 (T d^{\mathbb{H}})^2 \Big).
\end{aligned}$$

□

Proof of Theorem 17.1.1. We are going to exploit Theorem 17.3.2. To this aim, in view of

Lemma 17.3.4 and Lemma 17.3.5, we have that

$$
\begin{aligned}
\langle \ddot{H}_0 N, \nu^{\mathbb{H}}\rangle &+ \frac{|\dot{H}_0 N|^2 - \langle \dot{H}_0 N, \nu^{\mathbb{H}}\rangle}{|N^{\mathbb{H}}|} \\
&= |N^{\mathbb{H}}|\Big(\xi^2 \left(H^{\mathbb{H}}\right)^2 - \xi^2 \tau((\nabla^{\mathbb{H}}\nu^{\mathbb{H}})^2) - 2\xi^2 g_{\mathbb{H}}\left(J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}}\right) - 4\xi^2 (Td^{\mathbb{H}})^2 = \\
&\quad + \xi^2 \left(H^{\mathbb{H}}\right)^2 + |\nabla^{\mathbb{H},S}\xi|^2 + 4\xi Td^{\mathbb{H}} g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi, J(\nu^{\mathbb{H}})\right) + 4\xi^2 (Td^{\mathbb{H}})^2 - \xi^2\left(H^{\mathbb{H}}\right)^2\Big) \\
&= |N^{\mathbb{H}}|\left(|\nabla^{\mathbb{H},S}\xi|^2 + \xi^2(\left(H^{\mathbb{H}}\right)^2 - \tau((\nabla^{\mathbb{H}}\nu^{\mathbb{H}})^2) - 2\langle J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}}\rangle) + 2Td^{\mathbb{H}}\langle \nabla^{\mathbb{H}}\xi^2, J(\nu^{\mathbb{H}})\rangle\right).
\end{aligned}
\tag{17.3.16}
$$

In order to deal with the last term in the previous expression, we exploit an horizontal integration by parts formula proved in [106]. According to the authors' notation (cf. [106, Section 2, 6 and 10]), we let

$$
c^{H,S} = 2Td^{\mathbb{H}} J(\nu^{\mathbb{H}}), \tag{17.3.17}
$$

and we consider a a vector field $\varphi \in C^1_c(S, \mathcal{H}TS)$. Then [106, Theorem 10.4] implies that

$$
\int_S \left(\operatorname{div}^S_{\mathbb{H}} \varphi + g_{\mathbb{H}}\left(c^{H,S}, \varphi\right)\right) dP_{\mathbb{H}}(E, \cdot) = \int_S H^{\mathbb{H}} g_{\mathbb{H}}\left(\varphi, \nu^{\mathbb{H}}\right) dP_{\mathbb{H}}(E, \cdot).
$$

Let us choose $\varphi = \xi^2 Td^{\mathbb{H}} J(\nu^{\mathbb{H}})$. Then, by the previous formula and (17.3.17),

$$
\int_S \operatorname{div}^S_{\mathbb{H}}(\xi^2 Td^{\mathbb{H}} J(\nu^{\mathbb{H}}))\, dP_{\mathbb{H}}(E, \cdot) = -\int_S 2\xi^2 (Td^{\mathbb{H}})^2\, dP_{\mathbb{H}}(E, \cdot). \tag{17.3.18}
$$

Now, by (16.6.7), we have that

$$
\begin{aligned}
\sum_{h=1}^{2n} \langle \nabla^{\mathbb{H}}(J(\nu^{\mathbb{H}})_h), \nu^{\mathbb{H}}\rangle \nu^{\mathbb{H}}_h &= -\sum_{h=1}^{n} \nu^{\mathbb{H}}_h \sum_{k=1}^{2n} Z_k(\nu^{\mathbb{H}}_{n+h})\nu^{\mathbb{H}}_k + \sum_{h=1}^{n} \nu^{\mathbb{H}}_{n+h} \sum_{k=1}^{2n} Z_k(\nu^{\mathbb{H}}_h)\nu^{\mathbb{H}}_k \\
&= 2Td^{\mathbb{H}} \sum_{h=1}^{n} \nu^{\mathbb{H}}_h J(\nu^{\mathbb{H}})_{n+h} - 2Td^{\mathbb{H}} \sum_{h=1}^{n} \nu^{\mathbb{H}}_{n+h} J(\nu^{\mathbb{H}})_h \\
&= 2Td^{\mathbb{H}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
2g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi^2, Td^{\mathbb{H}} J(\nu^{\mathbb{H}})\right) &= 2\operatorname{div}_{\mathbb{H}}(\xi^2 Td^{\mathbb{H}} J(\nu^{\mathbb{H}})) - 2\xi^2 \operatorname{div}_{\mathbb{H}}(Td^{\mathbb{H}} J(\nu^{\mathbb{H}})) \\
&= 2\operatorname{div}^S_{\mathbb{H}}(\xi^2 Td^{\mathbb{H}} J(\nu^{\mathbb{H}})) + 2\sum_{h=1}^{2n} \langle \nabla^{\mathbb{H}}(\xi^2 Td^{\mathbb{H}} J(\nu^{\mathbb{H}})_h), \nu^{\mathbb{H}}\rangle \nu^{\mathbb{H}}_h \\
&\quad - 2\xi^2 Td^{\mathbb{H}} \operatorname{div}_{\mathbb{H}} J(\nu^{\mathbb{H}}) - 2\xi^2 \langle J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}}\rangle \\
&= 2\operatorname{div}^S_{\mathbb{H}}(\xi^2 Td^{\mathbb{H}} J(\nu^{\mathbb{H}})) + 2\xi^2 Td^{\mathbb{H}} \sum_{h=1}^{2n} \langle \nabla^{\mathbb{H}}(J(\nu^{\mathbb{H}})_h), \nu^{\mathbb{H}}\rangle \nu^{\mathbb{H}}_h \\
&\quad - 4n\xi^2 (Td^{\mathbb{H}})^2 - 2\xi^2 \langle J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}}\rangle \\
&= 2\operatorname{div}^S_{\mathbb{H}}(\xi^2 Td^{\mathbb{H}} J(\nu^{\mathbb{H}})) - 4(n-1)\xi^2 (Td^{\mathbb{H}})^2 - 2\xi^2 \langle J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}}\rangle.
\end{aligned}
$$

This computation, together with (17.3.18), implies that

$$\int_S 2g_{\mathbb{H}}\left(\nabla^{\mathbb{H}}\xi^2, Td^{\mathbb{H}}J(\nu^{\mathbb{H}})\right)\, dP_{\mathbb{H}}(E,\cdot) = \int_S \left(-4n\xi^2(Td^{\mathbb{H}})^2 - 2\xi^2\langle J(\nu^{\mathbb{H}}), T\nu^{\mathbb{H}}\rangle\right)\, dP_{\mathbb{H}}(E,\cdot). \tag{17.3.19}$$

Combining (17.3.6) with (17.3.16) and (17.3.19), the thesis follows. □

17.4 The Riemannian Jacobi equation

Exploiting (17.1.2), we can prove Theorem 17.1.3 as follows.

Proof of Theorem 17.1.3. Since $\Delta^{\varepsilon,S}\nu^\varepsilon_{2n+1}$ depends only on $\nu^\varepsilon_{2n+1}|_S$, we extend $\nu^\varepsilon|_S$ letting $\nu^\varepsilon = \nabla^\varepsilon d^\varepsilon$, in particular $\varepsilon T d^\varepsilon = \nu^\varepsilon_{2n+1}$. Moreover, in view of (16.4.5), we extend H^ε to a neighborhood of S by letting

$$H^\varepsilon(p) = \sum_{i=1}^{2n+1} Z_i\nu^\varepsilon_i(p). \tag{17.4.1}$$

Using Proposition 16.4.1, $TZ_i = Z_iT$, (16.4.1), (16.4.6) and (17.4.1), and recalling that

$$g_\varepsilon(\nabla^{\varepsilon,S}H^\varepsilon, Z_{2n+1}) = Z_{2n+1}(H^\varepsilon) - \nu^\varepsilon_{2n+1}g_\varepsilon(\nabla^\varepsilon H^\varepsilon, \nu^\varepsilon),$$

it holds that

$$\begin{aligned}
\Delta^{\varepsilon,S}\nu^\varepsilon_{2n+1} - \frac{2\nu^\varepsilon_{2n+1}}{\varepsilon}\langle\nabla^\varepsilon(\varepsilon Td^\varepsilon), J\nu^\varepsilon\rangle &= \sum_{i,j=1}^{2n+1} g_\varepsilon^{i,j}Z_i(Z_j(\varepsilon Td^\varepsilon)) - H^\varepsilon g_\varepsilon(\nabla^\varepsilon(\varepsilon Td^\varepsilon), \nu^\varepsilon)\\
&= \sum_{i,j=1}^{2n+1} g_\varepsilon^{i,j}Z_{2n+1}(Z_i\nu^\varepsilon_j) - H^\varepsilon\sum_{j=1}^{2n+1}(Z_{2n+1}\nu^\varepsilon_j)\nu^\varepsilon_j\\
&= Z_{2n+1}\left(\sum_{i=1}^{2n+1} Z_i\nu^\varepsilon_i\right) - \sum_{i,j=1}^{2n+1} Z_{2n+1}(Z_i\nu^\varepsilon_j)\nu^\varepsilon_i\nu^\varepsilon_j\\
&= g_\varepsilon(\nabla^{\varepsilon,S}H^\varepsilon, Z_{2n+1}) + \nu^\varepsilon_{2n+1}g_\varepsilon(\nabla^\varepsilon H^\varepsilon, \nu^\varepsilon) - \sum_{i,j=1}^{2n+1} Z_{2n+1}(Z_i\nu^\varepsilon_j)\nu^\varepsilon_i\nu^\varepsilon_j.
\end{aligned} \tag{17.4.2}$$

By (16.4.1) and (16.4.4), we have

$$\begin{aligned}
\sum_{i,j=1}^{2n+1} Z_{2n+1}(Z_i\nu^\varepsilon_j)\nu^\varepsilon_i\nu^\varepsilon_j &= \sum_{i,j=1}^{2n+1}\left(\nu^\varepsilon_i Z_{2n+1}\left(Z_i\nu^\varepsilon_j\nu^\varepsilon_j\right) - (Z_{2n+1}\nu^\varepsilon_j)\left(Z_i\nu^\varepsilon_j\nu^\varepsilon_i\right)\right)\\
&= \frac{2\nu^\varepsilon_{2n+1}}{\varepsilon}\sum_{j=1}^{2n+1} Z_{2n+1}(Z_jd^\varepsilon)J(\nu^\varepsilon)_j\\
&= \frac{2\nu^\varepsilon_{2n+1}}{\varepsilon}\langle\nabla^\varepsilon(\varepsilon Td^\varepsilon), J(\nu^\varepsilon)\rangle.
\end{aligned} \tag{17.4.3}$$

Inserting (17.4.3) in (17.4.2), we get

$$\Delta^{\varepsilon,S}\nu^\varepsilon_{2n+1} = g_\varepsilon(\nabla^{\varepsilon,S}H^\varepsilon, Z_{2n+1}) + \nu^\varepsilon_{2n+1}g_\varepsilon(\nabla^\varepsilon H^\varepsilon, \nu^\varepsilon).$$

Moreover, by (16.4.2) and (16.4.3),

$$
\begin{aligned}
g_\varepsilon(\nabla^\varepsilon H^\varepsilon, \nu^\varepsilon) &= \sum_{i,j=1}^{2n+1} Z_j(Z_i \nu_i^\varepsilon)\nu_j^\varepsilon \\
&= \sum_{i,j=1}^{2n+1} Z_i(Z_j \nu_i^\varepsilon)\nu_j^\varepsilon + 2\sum_{j=1}^{n} \Big(T(X_j d^\varepsilon)(Y_j d^\varepsilon) - T(Y_j d^\varepsilon)(X_j d^\varepsilon) \Big) \\
&= \sum_{i=1}^{2n+1} Z_i \left(\sum_{j=1}^{2n+1} Z_j \nu_i^\varepsilon \nu_j^\varepsilon \right) - \sum_{i=1}^{2n+1} Z_i \nu_j^\varepsilon Z_j \nu_i^\varepsilon - 2\left\langle \nabla \left(\frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \right), J(\nu^\varepsilon) \right\rangle,
\end{aligned}
$$

and, from (16.4.4), we get

$$
\begin{aligned}
\sum_{i=1}^{2n+1} Z_i \left(\sum_{j=1}^{2n+1} Z_j \nu_i^\varepsilon \nu_j^\varepsilon \right) &= -2\sum_{i=1}^{2n+1} Z_i \left(\frac{\nu_{2n+1}^\varepsilon}{\varepsilon} J(\nu^\varepsilon)_i \right) \\
&= -2\frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \sum_{i=1}^{2n+1} Z_i(J(\nu^\varepsilon)_i) - 2\left\langle \nabla^\varepsilon \left(\frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \right), J(\nu^\varepsilon) \right\rangle \\
&= -4n\left(\frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \right)^2 - 2\left\langle \nabla^\varepsilon \left(\frac{\nu_{2n+1}^\varepsilon}{\varepsilon} \right), J(\nu^\varepsilon) \right\rangle.
\end{aligned}
$$

The thesis then follows from (17.1.2). □

17.5 The sub-Riemannian Jacobi equation

In order to prove Theorem 17.1.4, we collect some further short preliminaries. Let δ_T be the differential operator defined by

$$
\delta_T f := \frac{T\bar{f} - Td^{\mathbb{H}} \langle \nabla^{\mathbb{H}} \bar{f}, \nu^{\mathbb{H}} \rangle}{1 + (Td^{\mathbb{H}})^2}
$$

for a given $f \in C^1(S)$ and any C^1-extension $\bar{f}$ of f in a neighborhood of S. An easy computation reveals that

$$
\delta_T f = T\bar{f} - g_1(\nabla^1 \bar{f}, \nu^1)\nu_{2n+1}^1 = g_1(\nabla^{1,S} f, T),
$$

whence $\delta_T f$ is well-defined. Moreover, the very definition of δ_T implies that

$$
Tf = Td^{\mathbb{H}} \langle \nabla^{\mathbb{H}} f, \nu^{\mathbb{H}} \rangle + (1 + (Td^{\mathbb{H}})^2)\delta_T f.
$$

Proof of Theorem 17.1.4. In view of Proposition 16.6.9, it holds that

$$\begin{aligned}
\Delta^{\mathbb{H},S}(Td^{\mathbb{H}}) &= \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j (Td^{\mathbb{H}}) - H^{\mathbb{H}} \langle \nabla^{\mathbb{H}}(Td^{\mathbb{H}}), \nu^{\mathbb{H}} \rangle \\
&= \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} T(Z_i Z_j d^{\mathbb{H}}) - H^{\mathbb{H}} \sum_{j=1}^{2n} T\nu_j^{\mathbb{H}} \nu_j^{\mathbb{H}} \\
&= T\left(\sum_{i=1}^{2n} Z_i \nu_i^{\mathbb{H}}\right) - \sum_{i,j=1}^{2n} T(Z_i \nu_j^{\mathbb{H}}) \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}} \\
&= (1 + (Td^{\mathbb{H}})^2)\delta_T H^{\mathbb{H}} + Td^{\mathbb{H}} \sum_{i,j=1}^{2n} Z_j Z_i Z_i d^{\mathbb{H}} \nu_j^{\mathbb{H}} - \sum_{i,j=1}^{2n} T(Z_i \nu_j^{\mathbb{H}}) \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}}.
\end{aligned}$$

Notice that

$$\begin{aligned}
- \sum_{i,j=1}^{2n} T(Z_i \nu_j^{\mathbb{H}}) \nu_i^{\mathbb{H}} \nu_j^{\mathbb{H}} &= - \sum_{i=1}^{2n} \nu_i^{\mathbb{H}} T\left(\sum_{j=1}^{2n} Z_i \nu_j^{\mathbb{H}} \nu_j^{\mathbb{H}}\right) + \sum_{j=1}^{2n} T(\nu_j^{\mathbb{H}}) \sum_{i=1}^{2n} Z_i \nu_j^{\mathbb{H}} \nu_i^{\mathbb{H}} \\
&= -2Td^{\mathbb{H}} \sum_{j=1}^{2n} T(Z_j d^{\mathbb{H}}) J(\nu^{\mathbb{H}})_j \\
&= -2Td^{\mathbb{H}} \langle \nabla^{\mathbb{H}}(Td^{\mathbb{H}}), J(\nu^{\mathbb{H}}) \rangle.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{i,j=1}^{2n} Z_j Z_i Z_i d^{\mathbb{H}} \nu_j^{\mathbb{H}} &= \sum_{j=1}^{n} \sum_{i=1}^{2n} X_j Z_i Z_i d^{\mathbb{H}} X_j d^{\mathbb{H}} + \sum_{j=1}^{n} \sum_{i=1}^{2n} Y_j Z_i Z_i d^{\mathbb{H}} Y_j d^{\mathbb{H}} \\
&= \sum_{i,j=1}^{2n} Z_i Z_j Z_i d^{\mathbb{H}} Z_j d^{\mathbb{H}} - 2\sum_{j=1}^{n} TY_j d^{\mathbb{H}} X_j d^{\mathbb{H}} + 2\sum_{j=1}^{n} TX_j d^{\mathbb{H}} Y_j d^{\mathbb{H}} \\
&= \sum_{i=1}^{2n} Z_i \left(\sum_{j=1}^{2n} Z_j Z_i d^{\mathbb{H}} Z_j d^{\mathbb{H}}\right) - \sum_{i=1}^{2n} Z_i \nu_j^{\mathbb{H}} Z_j \nu_i^{\mathbb{H}} - 2\langle \nabla^{\mathbb{H}}(Td^{\mathbb{H}}), J(\nu^{\mathbb{H}}) \rangle \\
&= -2\sum_{i=1}^{2n} Z_i (Td^{\mathbb{H}} J(\nu^{\mathbb{H}})_i) - \sum_{i=1}^{2n} Z_i \nu_j^{\mathbb{H}} Z_j \nu_i^{\mathbb{H}} - 2\langle \nabla^{\mathbb{H}}(Td^{\mathbb{H}}), J(\nu^{\mathbb{H}}) \rangle \\
&= -2Td^{\mathbb{H}} \sum_{i=1}^{2n} Z_i (J(\nu^{\mathbb{H}})_i) - \sum_{i=1}^{2n} Z_i \nu_j^{\mathbb{H}} Z_j \nu_i^{\mathbb{H}} - 4\langle \nabla^{\mathbb{H}}(Td^{\mathbb{H}}), J(\nu^{\mathbb{H}}) \rangle \\
&= -4n(Td^{\mathbb{H}})^2 - \sum_{i=1}^{2n} Z_i \nu_j^{\mathbb{H}} Z_j \nu_i^{\mathbb{H}} - 4\langle \nabla^{\mathbb{H}}(Td^{\mathbb{H}}), J(\nu^{\mathbb{H}}) \rangle.
\end{aligned}$$

The thesis then follows. □

Chapter 18

t-graphs of prescribed mean curvature: the Dirichlet problem

18.1 Introduction

We refer to [158] as main reference for this chapter. The Plateau problem has been a fundamental issue in geometry since the pioneering works of Douglas (cf. [121]) and Radó (cf. [247]). The Euclidean prescribed mean curvature equation of the graph of a function $u \in C^2(\Omega)$ over a bounded domain $\Omega \subseteq \mathbb{R}^n$ with $H \in C(\Omega)$ reads as

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H. \tag{18.1.1}$$

Notice that, as usual, we are taking the mean curvature as the (not averaged) sum of the principal curvatures. When H is constant and $\partial\Omega$ is of class C^2, Serrin (cf. [265]) characterized the existence of solutions to the Dirichlet problem for *any* boundary datum $\varphi \in C^2(\partial\Omega)$ by the condition

$$|H| \leqslant H_{\partial\Omega}(z_0) \tag{18.1.2}$$

for any $z_0 \in \partial\Omega$, where $H_{\partial\Omega}$ is the mean curvature of the boundary of Ω. In the proof, Serrin obtained Schauder estimates for C^2 solutions first by providing height estimates for $|u|$, and then, by means of a gradient maximum principle, showing that the maximum of the gradient is attained at the boundary of Ω. In the final step, he estimated the gradient at the boundary exploiting the so-called *barriers* (cf. [157]), whose construction relies on (18.1.2). When H is not constant, an approach based on the maximum principle typically fails. Therefore, in order to deal both with non-constant sources and to allow merely continuous boundary data, it is customary to rely on suitable interior and global gradient estimates. Some references to these kind of estimates in the Euclidean space are the works of Korevaar and Simon (cf. [185]) and Wang (cf. [280]). Beyond the Euclidean framework, the Dirichlet problem for any sufficiently regular boundary datum and constant source H satisfying conditions analogous to (18.1.2) has been studied in warped products with a particular lower bound on the Ricci curvature (cf.

[271]) and in the first Heisenberg group (cf. [7]). When H is not constant, the previous results were later extended in Riemannian manifolds with a Killing vector field and a lower bound on the Ricci curvature depending on Ω (cf. [102, 101, 103]). When instead (18.1.2) fails, meaning that

$$|H| > H_{\partial\Omega}(z_0) \tag{18.1.3}$$

for some $z_0 \in \partial\Omega$, we could lose control of the norm of the gradient of a solution near the boundary, and hence of the existence of solutions. More precisely, as shown in [157], when (18.1.3) holds there always exists a boundary datum φ for which the Dirichlet problem has no solution. Nevertheless, the validity of (18.1.3) does not preclude *a priori* the existence of a *suitable* boundary datum φ for which the Dirichlet problem is solvable. As an instance, taking as domain $\Omega \subseteq \mathbb{R}^n$ the ball of radius 1 centered at 0, we can write the half sphere in $\mathbb{R}^{n+1}$ centered at 0 with radius 1 as a graph over Ω. A simple computation reveals that it satisfies (18.1.1) with $H = \frac{n}{n-1}H_{\partial\Omega}$ and with boundary datum $\varphi \equiv 0$. In particular, (18.1.3) is verified for any $z_0 \in \partial\Omega$. In this regard, when Ω has Lipschitz boundary, Giusti (cf. [162]) proved that the existence of solutions to (18.1.1) with a suitable boundary condition, not imposed *a priori*, is characterized by

$$\left|\int_{\tilde{\Omega}} H(x)\,dx\right| < P(\tilde{\Omega}) \tag{18.1.4}$$

for any set $\tilde{\Omega} \subseteq \Omega$ such that $\tilde{\Omega} \neq \emptyset$ and $\tilde{\Omega} \neq \Omega$, where $P(\tilde{\Omega})$ is the perimeter of $\tilde{\Omega}$. Moreover, [162] provides a characterization of those domains where (18.1.1) admits, up to vertical translations, a unique solution. Precisely, the previous statement is equivalent to each of the following conditions: there is no solution to (18.1.1) in any domain $\Omega \subsetneq \hat{\Omega}$; there is a solution on Ω which is vertical at every point of $\partial\Omega$; (18.1.4) holds and

$$\left|\int_{\Omega} H(x)\,dx\right| = P(\Omega). \tag{18.1.5}$$

In these cases, Ω is called an *extremal domain*. Otherwise, i.e. when (18.1.4) also holds for $\tilde{\Omega} = \Omega$, then Ω is called a *non-extremal domain*. The proof of the existence of solutions under condition (18.1.4) relies on previous results by Giaquinta (cf. [154, 153]) and Miranda (cf. [213]). In the non-extremal case the proof consists in showing the existence of BV minimizers of the penalized functional

$$\mathcal{F}(u) = \int_{\Omega} \sqrt{1+|Du|^2} + \int_{\Omega} Hu\,dx + \int_{\partial\Omega} |u-\varphi|\,d\mathcal{H}^{n-1} \tag{18.1.6}$$

for any $\varphi \in L^1(\partial\Omega)$, whose regularity is then gradually improved in several steps. The more involved extremal case, i.e. when (18.1.5) holds, follows by a compactness procedure. More precisely, in view of condition (18.1.4), every domain $\tilde{\Omega} \subsetneq \Omega$ is itself a non-extremal domain. Therefore, exploiting the existence result in the non-extremal case, together with a compactness argument based on a notion of *generalized solution* first introduced by Miranda (cf. [214]), existence in the extremal case follows. For similar results under weaker assumptions on the boundary of Ω we refer the reader to [192]. The aim of Chapter 18 and Chapter 19 is to extend the previous Euclidean considerations to the class of t-graphs in Heisenberg groups of any dimension, from both a Riemannian and a sub-Riemannian viewpoint. As we explain in detail below, in $(\mathbb{H}^n, g_\varepsilon)$ the equation of Riemannian prescribed mean curvature of a t-graph over a bounded domain $\Omega \subseteq \mathbb{R}^{2n}$ for a given source H is formally given by

$$\operatorname{div}\left(\frac{Du+\mathcal{F}}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\right) = H, \tag{ε-PMC}$$

where $\mathcal{F}$ is as in (16.7.1), while its sub-Riemannian counterpart (cf. [72, 85, 158]) reads as

$$\operatorname{div}\left(\frac{Du+\mathcal{F}}{|Du+\mathcal{F}|}\right) = H. \tag{$\mathcal{H}$-PMC}$$

We point out that (ε-PMC) and ($\mathcal{H}$-PMC), despite the apparent similarity, enjoy completely different behaviours. Indeed, while (ε-PMC) is a classical second-order elliptic equation, the possible presence of characteristic points could make ($\mathcal{H}$-PMC) both degenerate elliptic and singular (cf. [85]). Therefore, as we will see below, the sub-Riemannian prescribed mean curvature problem might be understood in a weak variational sense. In this first chapter we mainly deal with the solution of the Dirichlet problem associated to ($\mathcal{H}$-PMC). However, due to the growing interest around anisotropic geometric structures, we address this problem in an even more general context, in which the standard sub-Riemannian structure is replaced with a generic *sub-Finsler structure*. In the Heisenberg group, a sub-Finsler structure is defined by means of an asymmetric left-invariant norm $\|\cdot\|_{K_0}$ on the horizontal distribution of $\mathbb{H}^n$ associated to a convex body $K_0 \subseteq \mathbb{R}^{2n}$ containing the origin in its interior. Let us briefly introduce our approach. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded open set, $H \in L^\infty(\Omega)$, $F \in L^1(\Omega, \mathbb{R}^{2n})$ and $u \in W^{1,1}(\Omega)$. We consider the functional

$$\mathcal{I}(u) = \int_\Omega \|Du+F\|_{K_0,*}\, dz + \int_\Omega Hu\, dz, \tag{18.1.7}$$

where $\|\cdot\|_{K_0,*}$ denotes the dual norm of $\|\cdot\|_{K_0}$. In particular, when F is the vector field defined in (16.7.1), the first term in (18.1.7) coincides with the sub-Finsler area of the t-graph of u (cf. [244, 135]). Moreover, if K_0 is the Euclidean unit ball centered at the origin and $H = 0$ then (18.1.7) boils down to the classical area functional for t-graphs in Heisenberg group (cf. [83, 173] and references therein). We say that the graph of u has prescribed K_0-mean curvature H in Ω if u is a minimizer of $\mathcal{I}$. Indeed, the Euler-Lagrange equation associated to $\mathcal{I}$ out of the singular set Ω_0, i.e. the set of points where $Du+F$ vanishes, is given by

$$\operatorname{div}(\pi_{K_0}(Du+F)) = H, \tag{18.1.8}$$

where π_{K_0} is a suitable 0-homogeneous function defined in (18.2.5). Again, when K_0 is the Euclidean unit ball centered at the origin, (18.1.8) reduces to ($\mathcal{H}$-PMC). When we fix a boundary datum $\varphi \in W^{1,1}(\Omega)$, a solution to the *Dirichlet problem* for the prescribed K_0-mean curvature equation is a minimizer u of $\mathcal{I}$ such that $u-\varphi$ belongs to the Sobolev space $W^{1,1}_0(\Omega)$.

Our main result is Theorem 18.9.1, where we prove, under suitable regularity assumptions on the data, that there exists a Lipschitz solution to the Dirichlet problem for the prescribed K_0-mean curvature equation when H is *constant*, it satisfies

$$|H| < H_{K_0,\partial\Omega}(z_0) \tag{18.1.9}$$

for each $z_0 = (x_0, y_0) \in \partial\Omega$ and

$$\left| \int_\Omega Hv \, dz \right| \leqslant (1-\delta) \int_\Omega \|Dv\|_{K_0,*} \, dz \tag{18.1.10}$$

for each non-negative function $v \in C_c^\infty(\Omega)$ and a suitable $\delta = \delta(K_0, \Omega, H) \in (0, 1]$. Here $H_{K_0,\partial\Omega}$ denotes the Finsler mean curvature of the boundary $\partial\Omega \subseteq \mathbb{R}^{2n}$. Notice that the mean curvature of the graph of u is computed with respect to the downward pointing unit normal and the Finsler mean curvature of $\partial\Omega$ is computed with respect to the inner unit normal. The upper bound (18.1.9) of H in terms of the Finsler mean curvature of the boundary is the sub-optimal Finsler analogous of (18.1.2). On the other hand, (18.1.10) is a standard sufficient condition for the estimates of the supremum of $|u|$ (cf. [153] or [157]). It is worth noting that, in the Euclidean setting (cf. e.g. [162]), the weaker condition

$$\left| \int_\Omega Hv \, dz \right| \leqslant \int_\Omega \|Dv\|_{K_0,*} \, dz \tag{18.1.11}$$

for each $v \in C_c^\infty(\Omega)$, which is the functional analog of (18.1.4), is actually a necessary condition for the existence of a solution to the Euclidean prescribed mean curvature equation. Remarkably, as we will show in Section Section 18.8 and Section 18.10, there are particular settings in which Theorem 18.9.1 continues to hold even without imposing (18.1.10), such as the *first sub-Finsler* Heisenberg group $\mathbb{H}^1$ (cf. Theorem 18.9.2) and *any sub-Riemannian* Heisenberg group $\mathbb{H}^n$ (cf. Theorem 18.10.2). As already mentioned, the Dirichlet problem for constant mean curvature in the first Riemannian Heisenberg group has been studied in [7] under the same condition on the mean curvature. It is worth mentioning that this is the first time that the existence of Lipschitz solutions to the sub-Finsler Dirichlet problem has been studied when $H \neq 0$, even in the particular case in which K_0 is the unit disk centered at 0, where the sub-Finsler and the sub-Riemannian frameworks coincide. Indeed, as far as we know, the sub-Riemannian Dirichlet problem has been studied in [238, 85, 83, 82, 120, 240] only in the case of minimal surfaces under the bounded slope condition or the p-convexity assumption on Ω, and in [230] when $H \neq 0$ is small enough and in a weaker functional framework. In particular, we point out that when $n = 1$ our assumption (18.1.9) implies that $\Omega \subseteq \mathbb{R}^2$ is strictly convex (cf. Remark 18.8.9). It is easy to check that our sub-Finsler functional $\mathcal{I}$ for $H = 0$ satisfies the hypothesis of the area functional considered in [120]. Thus, assuming the bounded slope condition we directly obtain the existence of Euclidean Lipschitz minimizers for Plateau's problem. The approach of the present chapter, based on the Schauder fixed-point theory, follows the scheme developed in [85] and extends its results both to the case of prescribed constant mean curvature

$H \neq 0$ and to the sub-Finsler setting. In Theorem 18.9.1 we cannot expect better regularity than Lipschitz. Indeed, even in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ there are several examples of non-smooth area minimizers. For instance, S.D. Pauls [239] exhibited a solution of low regularity for Plateau's problem with smooth boundary datum, while in [85, 249, 164] the authors provided solutions to the *Bernstein problem* in $\mathbb{H}^1$ that are only Euclidean Lipschitz. These examples have been recently generalized to the sub-Finsler setting in [159]. We refer the interested reader to [161] for a positive result to the sub-Finsler Bernstein problem for (X, Y)-Lipschitz surfaces, which can be seen as a regularity result for global perimeter minimizers. Since equation (18.1.8) is sub-elliptic degenerate and it is singular next to the singular set, inspired by [85, 238], we first introduce a family of desingularized approximating equations given by

$$\operatorname{div}\left(\pi_{K_0}(Du+F)\frac{\|Du+F\|_*^2}{(\varepsilon^3+\|Du+F\|_*^3)^{\frac{2}{3}}}\right)=H \tag{18.1.12}$$

for each $0 < \varepsilon < 1$. A similar approximation scheme was considered in the sub-Riemannian setting in [74, 73] to study the Lipschitz regularity for non-characteristic minimal surfaces. For a detailed analysis of this approach, we refer to [72]. This family of equations can be obtained by considering a $(2n+1)$-dimensional convex body K_ε containing the origin in its interior, that converges in the Hausdorff sense to the $2n$-dimensional convex body K_0 as $\varepsilon \to 0$. The choice of the convex body K_ε is not arbitrary. Indeed, we need a specific shape in order to obtain an approximating equation well-defined in the classical sense in the singular set. It is interesting to point out that the Riemannian approximation of [85, 238, 74, 73] produces an approximation of the unit disk $D \subseteq \mathbb{R}^{2n}$ by ellipsoids in the sub-Riemannian setting, and this approximation does not work in the greater sub-Finsler generality. Indeed, if instead of (18.1.12) we were to consider the more natural equation

$$\operatorname{div}\left(\pi_{K_0}(Du+F)\frac{\|Du+F\|_*}{\sqrt{\varepsilon^2+\|Du+F\|_*^2}}\right)=H, \tag{18.1.13}$$

reminiscent of the Riemannian approximation scheme of [85] (cf. Remark 18.5.2), we would have to require certain assumptions on K_0 for (18.1.13) to be well-defined in the classical sense. We refer to Section 18.10 for a more careful analysis in this regard. On the other hand, while (18.1.12) is always well-defined, it still tends to degenerate close to the singular set, so that it could fail to be elliptic. Therefore, we need to regularize (18.1.12) by perturbing it with an Euclidean curvature term. More precisely, we consider the family of equations given by

$$\operatorname{div}\left(\pi(Du+F)\frac{\|Du+F\|_*^2}{(\varepsilon^3+\|Du+F\|_*^3)^{\frac{2}{3}}}\right)+\eta\operatorname{div}\left(\frac{Du+F}{\sqrt{1+|Du+F|^2}}\right)=H. \tag{18.1.14}$$

for any $\varepsilon \in (0,1)$ and any $\eta > 0$ sufficiently small, whose associated Finsler variational functional is given by

$$\mathcal{I}_{\varepsilon,\eta}(u) = \int_\Omega \left(\varepsilon^3 + \|Du + F\|^3_{K_0,*}\right)^{\frac{1}{3}} \, dz + \eta \int_\Omega \sqrt{1 + |Du + F|^2} \, dz + \int_\Omega Hu \, dz.$$

A direct computation (cf. Section 18.8) will show that (18.1.14) is in fact a classical, quasi-linear second-order elliptic equation. Therefore, given a boundary datum $\varphi \in C^{2,\alpha}(\bar{\Omega})$, the solvability of the Dirichlet problem associated to (18.1.12) is reduced by [157, Theorem 13.8] to *a priori* estimates in $C^1(\overline{\Omega})$ of a related family of problems. As usual the *a priori* estimates in $C^1(\overline{\Omega})$ consist of three parts: estimates of the supremum of $|u|$, boundary estimates of the gradient of u and interior estimates of the gradient of u. While the estimates of the supremum rely on assumption (18.1.10), the boundary estimates of the gradient are obtained by a barrier argument that depends on the Finsler distance from the boundary $\partial\Omega$. Due to technical reasons in the construction of the barriers we need to assume the strict inequality in (18.1.9), avoiding the optimal case when H coincides with $H_{K_0,\partial\Omega}(z_0)$ at a given point $z_0 \in \partial\Omega$. We emphasize that these results hold even if the prescribed curvature H is non-constant and Lipschitz. The only crucial step where we need H to be constant is the maximum principle for the gradient of the solution that allows us to reduce the interior estimates of the gradient to its boundary estimates. Finally, once we realize that C^1 estimates are independent of the approximation parameters ε and η, passing to the limit as $\varepsilon, \eta \to 0$ and using Arzelà-Ascoli Theorem we get the existence of a Lipschitz minimizer for the sub-Finsler Dirichlet problem.

18.2 Minkowski norms

Let us fix $d \in \mathbb{N}$, $d \geq 1$. We say that a set K is a *convex body* if it is convex, compact and has non-empty interior. We say that a convex body K is (in) $C^{k,\alpha}_+$, for $k \in \mathbb{N}$ and $\alpha \in [0,1]$, if ∂K is of class $C^{k,\alpha}$ with strictly positive principal curvatures. We follow the approach developed in [244, 257]. We say that $\|\cdot\| : \mathbb{R}^d \to [0,+\infty)$ is a *norm* if it verifies:

1. $\|v\| = 0 \Leftrightarrow v = 0$,
2. $\|sv\| = s\|v\|$ for any $s > 0$,
3. $\|v + u\| \leqslant \|v\| + \|u\|$

for any $u, v \in \mathbb{R}^d$. We stress the fact that we are not assuming the symmetry property $\|-v\| = \|v\|$. It is well known that any norm is equivalent to the Euclidean norm $|\cdot|$, that is, given a norm $\|\cdot\|$ in $\mathbb{R}^d$ there exist constants $0 < c < C$ such that

$$c|\cdot| \leqslant \|\cdot\| \leqslant C|\cdot|. \tag{18.2.1}$$

Associated to a given a norm $\|\cdot\|$ we have the set $F = \{u \in \mathbb{R}^d : ||u|| \leqslant 1\}$, which, thanks to (18.2.1) and the properties of $\|\cdot\|$, is compact, convex and includes 0 in its interior. Reciprocally, given a convex body K with $0 \in \text{int}(K)$, the function

$$||u||_K = \inf\{\lambda \geqslant 0 : u \in \lambda K\}$$

defines a norm so that $K = \{u \in \mathbb{R}^d : ||u||_K \leqslant 1\}$. In the following we let

$$B_K(v, r) := \{w \in \mathbb{R}^d \ : \ \|w - v\|_K \leqslant r\}$$

for any $v \in \mathbb{R}^d$ and $r > 0$. It is easy to check that $\|v\|_K = \| - v\|_{-K}$ for any $v \in \mathbb{R}^d$, so that

$$B_{-K}(v, r) := \{w \in \mathbb{R}^d \ : \ \|v - w\|_K \leqslant r\} \tag{18.2.2}$$

for any $v \in \mathbb{R}^d$ and $r > 0$. Given a norm $\| \cdot \|$ and a scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^d$, we consider the dual norm $\| \cdot \|_*$ of $\| \cdot \|$ with respect to $\langle \cdot, \cdot \rangle$, defined by

$$\|u\|_* = \sup_{\|v\| \leqslant 1} \langle u, v\rangle. \tag{18.2.3}$$

The dual norm is the support function of the unit ball F with respect to the scalar product $\langle \cdot, \cdot \rangle$. Moreover, thanks to the above definitions the following Cauchy-Schwarz formula holds:

$$\langle u, v\rangle \leqslant \|u\|_* \|v\| \tag{18.2.4}$$

for any $u, v \in \mathbb{R}^d$. If in addition we assume K to be strictly convex and $u \neq 0$, then the compactness and strict convexity of K guarantee the existence of a unique vector $\pi_K(u)$ in ∂K where the supremum in (18.2.3) is attained, i.e.

$$\|u\|_{K,*} = \langle u, \pi_K(u)\rangle. \tag{18.2.5}$$

It is easy to see that π_K is a positively 0-homogeneous map, i.e. $\pi_K(\lambda u) = \pi_K(u)$ for any $\lambda > 0$ and $u \in \mathbb{R}^d \setminus \{0\}$, and that $\|\pi_K(u)\|_K = 1$ for any $u \in \mathbb{R}^d \setminus \{0\}$. Moreover, if we assume that K is C^2_+, then $\pi_K|_{\mathbb{S}^{d-1}} : \mathbb{S}^{d-1} \longrightarrow \partial K$ is a C^1 diffeomorphism whose inverse is the Gauss map $\mathcal{N}_K$ of ∂K with respect to the outer unit normal. In particular,

$$D\pi_K(u) \text{ is positive definite} \tag{18.2.6}$$

for any $u \in \mathbb{R}^d \setminus \{0\}$. Furthermore, we have that the norms $\| \cdot \|_K$ and $\| \cdot \|_{K,*}$ belong to $C^{k,\alpha}(\mathbb{R}^d \setminus \{0\})$ if and only if ∂K is $C^{k,\alpha}$ for $k \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant 1$. For further details cf. [257, Section 2.5]. The relation between the dual norm and the map π_K is given by

$$D\|u\|_{K,*} = \pi_K(u). \tag{18.2.7}$$

Indeed, for any $u \in \mathbb{R}^d \setminus \{0\}$

$$D\|u\|_{K,*} = D\langle u, \pi_K(u)\rangle = \pi_K(u) + u \cdot D\pi_K(u) = \pi_K(u),$$

where the last equality follows from the fact that 0-homogeneous functions are radial.

18.3 Finsler geometry of hypersurfaces in the Euclidean space

Let $K \subseteq \mathbb{R}^d$ be a convex body in C^2_+, $0 \in \operatorname{int} K$ and $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega = \Sigma$ of class C^2. Let N be the inner unit normal to Σ. Then the derivative map

$$(W_{K,\Sigma})_p = -d_p(\pi_K \circ N) : T_p\Sigma \to T_{\pi_K(N(p))}\partial K,$$

being π_K as in (18.2.5), is called the K-*Weingarten map*. Let $\gamma \subseteq \partial K$ be a differentiable curve with $\gamma(0) = \pi_K(N(p))$ and $\gamma'(0) \in T_{\pi_K(N(p))}\partial K$. By definition of π_K, the function

$$f(t) = \langle \gamma(t), N(p)\rangle$$

has a maximum at 0 and therefore $\langle \gamma'(0), N(p)\rangle = f'(0) = 0$, which gives $T_{\pi_K(N(p))}\partial K = T_{N(p)}\mathbb{S}^{d-1}$. Moreover it is well known that $(dN)_q$ is an endomorphism of $T_q\Sigma$ and therefore $(W_{K,\Sigma})_p$ is an endomorphism of $T_p\Sigma$. We define the K-*mean curvature* of Σ as

$$H_{K,\Sigma} = \operatorname{Trace}(W_{K,\Sigma}) = -\operatorname{div}_\Sigma(\pi_K \circ N),$$

where $\operatorname{div}_\Sigma$ is the divergence in the tangent directions to Σ. We remark that $W_{K,\Sigma}$ is neither necessarily self-adjoint nor symmetric. Let us check that $W_{K,\Sigma}$ is anyway diagonalizable. Indeed, given a parametrization X of Σ, dN has a symmetric matrix representation S in the basis $B = \{\partial_{x_1}X, \dots, \partial_{x_{d-1}}X\}$. On the other hand, $\pi_K = \mathcal{N}_K^{-1}$ and, since K is in C^2_+, the matrix A which represents $d(\mathcal{N}_K^{-1})$ with respect to B is positive definite. Therefore, there exists an invertible matrix P such that $A = P^TP$. Notice that the matrices P^TPS and PSP^T have the same spectrum, and equal to the spectrum of $W_{K,\Sigma}$. Since S is symmetric we can apply Sylvester's criterion to obtain that all the eigenvalues of PSP^T are real. The eigenvalues of $W_{K,\Sigma}$ are called K-*principal curvatures* and the eigenvectors of $W_{K,\Sigma}$ are called K-*principal directions*.

18.3.1 Finsler distance from the boundary and the eikonal equation

In this and the following section we want to rely on some results by [196, 195], and so we assume that K is in C^∞_+, i.e. ∂K is of class C^∞ with strictly positive principal curvatures. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega = \Sigma$ of class $C^{2,\alpha}$, for $0 < \alpha \leqslant 1$, and inner unit normal N. We shall adapt Theorem 4.26 in [229] and the remarks at the end of Section 4.5 in

[229] to prove existence of a tubular neighborhood of Σ and compute the K-mean curvature of parallel hypersurfaces. The *interior signed K-distance* to Σ is the function $d_{K,\Sigma} : \mathbb{R}^d \to \mathbb{R}$ given by

$$d_{K,\Sigma}(p) = \begin{cases} \min\{\|p-q\|_K : q \in \Sigma\} & \text{if } p \in \Omega \\ -\min\{\|p-q\|_K : q \in \Sigma\} & \text{if } p \notin \Omega. \end{cases}$$

Consider the map $F : \Sigma \times \mathbb{R} \to \mathbb{R}^d$ given by

$$F(q,t) = q + t(\pi_K \circ N)(q).$$

For any $v \in T_q\Sigma$, we have $(dF)_{(q,t)}(v,0) = v + td(\pi_K \circ N)(v)$ and $(dF)_{(q,t)}(0,1) = (\pi_K \circ N)(q)$. Since K contains the origin,

$$\langle \pi_K(N), N\rangle > 0$$

and dF is invertible at $t = 0$. Thus F is locally a diffeomorphism and, being Σ a compact hypersurface, F is a diffeomorphism in a domain $\Sigma \times (-\delta, \delta)$. The set $F(\Sigma \times (-\delta, \delta))$ is called a *tubular neighborhood* of Σ. Notice that if $p = F(q,t)$, then

$$p - q = t(\pi_K \circ N)(q) \tag{18.3.1}$$

and, taking the K-norm, we obtain that $d_{K,\Sigma}(p) = t$. We know (cf. [196]) that, under our assumptions, there exists $\bar{\delta} > 0$ such that

$$d_{K,\Sigma} \in C^{2,\alpha}(\overline{F(\Sigma \times (-\delta,\delta))}).$$

for any $\delta < \bar{\delta}$. Given $|t| < \delta$, we let

$$\Sigma_t = \{p \in \mathbb{R}^d : p = F(q,t) \text{ for some } q \in \partial\Sigma\}. \tag{18.3.2}$$

Proposition 18.3.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega = \Sigma$ of class C^2 and let $F(\Sigma \times (-\delta,\delta))$ be a tubular neighborhood of Σ. The K-mean curvature of Σ_t at $p \in \Sigma_t$ is given by*

$$H_{K,\Sigma_t}(p) = \sum_{i=1}^{d-1} \frac{\kappa_i(q)}{1 - t\kappa_i(q)}, \tag{18.3.3}$$

where $q \in \Sigma$ satisfies $p = F(q,t)$ and $\kappa_1(q), \ldots, \kappa_{d-1}(q)$ are the K-principal curvatures of Σ at q.

Proof. Let $\{e_1, \ldots, e_{d-1}\}$ be a basis of K-principal directions of Σ. Then $(dF)_{(q,t)}(e_i, 0) = (1 - t\kappa_i)e_i$. Therefore a basis of principal directions in Σ_t is $\{\frac{e_1}{1-t\kappa_1}, \ldots, \frac{e_{d-1}}{1-t\kappa_{d-1}}\}$. Since we have

$$-d(\pi_K \circ N)_q \left(\frac{e_i}{1 - t\kappa_i}\right) = \frac{\kappa_i}{1 - t\kappa_i} e_i$$

for each $i = 1, \ldots, d-1$ we get the conclusion. □

Remark 18.3.2. From (18.3.3), we obtain that the K-mean curvature is increasing in t. In particular, given $q \in \Sigma$ and $p = F(q,t)$ for $t > 0$, it holds that

$$H_{K,\Sigma_t}(p) \geq H_{K,\Sigma}(q). \tag{18.3.4}$$

The following eikonal equation can be deduced using classical arguments. We include the proof for the sake of completeness.

Proposition 18.3.3. *It holds that*

$$\|Dd_{K,\Sigma}(p)\|_{K,*} = 1 \tag{18.3.5}$$

for any p where $d_{K,\partial\Omega}$ is differentiable.

Proof. It is clear that, for any p, p' in $\mathbb{R}^d$, we have

$$d_{K,\Sigma}(p') \leqslant \|p' - p\|_K + d_{K,\Sigma}(p).$$

Taking $p' = p + tv$ where $t > 0$, we get

$$d_{K,\Sigma}(p + tv) - d_{K,\Sigma}(p) \leqslant \|tv\|_K.$$

Therefore,

$$\langle v, Dd_{K,\Sigma}(p)\rangle \leqslant \|v\|_K. \tag{18.3.6}$$

Taking $v = \pi_K(Dd_{K,\Sigma}(p))$ in (18.3.6), we obtain

$$\|Dd_{K,\Sigma}(p)\|_{K,*} \leqslant 1.$$

On the other hand, let $\gamma(t) = F(q_0, t)$. By (18.3.1) we have that

$$d_{K,\Sigma}(\gamma(t)) = t.$$

Taking derivatives in the previous equation, we obtain

$$\langle \gamma'(t), Dd_{K,\Sigma}(\gamma(t))\rangle = 1.$$

Since $\gamma'(t) = (\pi_K \circ N)(q_0)$, we get that $\|\gamma'(t)\|_K = 1$. Using (18.2.4), we get

$$\|Dd_{K,\Sigma}(\gamma(t))\|_{K,*} \geq 1. \qquad \square$$

Given a tubular neighborhood $\mathcal{O}$ of $\partial\Omega$ and $p = F(q,t) \in \Omega$, we denote by $N_t(p)$ the inner unit normal to Σ_t at p. Let us explicitly compute $\operatorname{div}(\pi_K \circ N_t)(p)$. Let us recall that, to the 0-homogeneity of π_K, we get that

$$q \cdot D\pi_K(q) = 0$$

for any $q \in \mathbb{R}^d$. In particular, taking $q = N_t$, we obtain

$$N_t \cdot D(\pi_K \circ N_t) = N_t \cdot D\pi_K(N_t) \cdot DN_t = 0,$$

which implies that

$$-\operatorname{div}(\pi_K \circ N_t)(p) = -\operatorname{div}_\Sigma(\pi_K \circ N_t)(p) = H_{K,\Sigma_t}(p) \geq H_{K,\partial\Omega}(q). \tag{18.3.7}$$

With the next result, we better understand the relationship between the Finsler mean curvature of Σ, the Euclidean curvature of Σ and the Euclidean principal curvatures of K.

Proposition 18.3.4. *Let K be a convex body in C^2_+, $0 \in \operatorname{int} K$. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with $\partial\Omega = \Sigma$ of class C^2 and let N_q be the inner unit normal to Σ at q. Then we have*

$$H_{K,\Sigma}(q) = -\sum_{i=1}^{d-1} \frac{\langle D_{e_i} N_q, e_i\rangle}{k_i^K(\pi_K(N_q))} \tag{18.3.8}$$

where k_i^K are the Euclidean principal curvatures of ∂K and $e_1, \dots, e_{d-1}$ is an orthonormal basis of Euclidean principal directions of ∂K.

Proof. We shall drop the subscript for π_K. Let q in Σ and $e_1, \dots, e_{d-1}$ be an orthonormal basis of $\mathbb{R}^{d-1} = T_{\pi(N_q)}\partial K$ such that

$$(d\mathcal{N}_K)_{\pi(N_q)} e_i = k_i^K(\pi(N_q)) e_i.$$

By hypothesis, $k_i^K > 0$ for $i = 1, \dots, d-1$. Here $\mathcal{N}_K$ denotes the Gauss map of ∂K. Then we have

$$H_{K,\Sigma}(q) = -\operatorname{div}_\Sigma(\pi(N_q)) = -\sum_{i=1}^{d-1} \langle D_{e_i}\pi(N_q), e_i\rangle,$$

where D is the Levi-Civita connection in $\mathbb{R}^d$. We claim that $D_{e_i}\pi(N_q) = d\pi(D_{e_i}N_q)$. Indeed, let $\gamma : (\epsilon, \epsilon) \to \Sigma$ such that $\gamma(0) = q$ and $\dot\gamma(0) = e_i$ for $i = 1, \dots, d-1$. Then we have

$$\begin{aligned} D_{e_i}\pi(N_q) &= \frac{D}{ds}\Big|_{s=0} \pi(N_{\gamma(s)}) = \sum_{j=1}^d \frac{d}{ds}\Big|_{s=0} \pi_j(N_{\gamma(s)}) \frac{\partial}{\partial x_j} \\ &= \sum_{j=1}^d \nabla \pi_j(N_q) \frac{D}{ds}\Big|_{s=0} N_{\gamma(s)} \frac{\partial}{\partial x_j} = (d\pi)_{N_q} D_{e_i} N_q. \end{aligned}$$

Moreover, since $d\pi$ is a symmetric matrix we gain

$$H_{K,\Sigma}(q) = -\sum_{i=1}^{d-1} \langle (d\pi)_{N_q} D_{e_i} N_q, e_i\rangle = -\sum_{i=1}^{d-1} \langle D_{e_i} N_q, (d\pi)_{N_q} e_i\rangle. \tag{18.3.9}$$

Since $\pi = \mathcal{N}_K^{-1}$ we obtain $d\pi = (d\mathcal{N}_K)^{-1}$ and

$$e_i = d\mathcal{N}_K^{-1} d\mathcal{N}_K(e_i) = d\mathcal{N}_K^{-1}(k_i^K(\pi(N_q))e_i) = k_i^K(\pi(N_q))d\pi(e_i),$$

by linearity. Therefore, we have $d\pi(e_i) = (k_i^K(\pi(N_q)))^{-1}e_i$. Hence, plugging this last equality in (18.3.9) we gain (18.3.8). □

18.3.2 The ridge of the Finsler distance

In the previous section we obtained some regularity and geometric properties of $d_{K,\partial\Omega}$ in a tubular neighborhood of $\partial\Omega$. We shall see that some of these properties persist outside a tubular neighborhood. We fix a convex body $K \in C^\infty_+$ and a bounded domain $\Omega \subseteq \mathbb{R}^d$ with $C^{2,1}$ boundary. For any $p \in \Omega$, we let $D(p) := \{q \in \partial\Omega : d_{K,\partial\Omega}(p) = \|p - q\|_K\}$. Since $d_{K,\partial\Omega}$ is continuous, then clearly $D(p) \neq \emptyset$ for any $p \in \Omega$. Accordingly, we define the set

$$\Omega_1 := \{p \in \Omega : D(p) \text{ is a singleton}\}, \tag{18.3.10}$$

and we define the *ridge* of Ω by

$$R := \Omega \setminus \operatorname{int}\Omega_1.$$

We know, again thanks to [196], that, under our assumptions on K and Ω,

$$d_{K,\partial\Omega} \in C^{2,1}(\operatorname{int}\Omega_1 \cup \partial\Omega). \tag{18.3.11}$$

Moreover, in [195, Corollary 1.6] it is proved that the Hausdorff dimension of R is at most $d-1$. This fact implies that R has empty interior, so that

$$\partial(\operatorname{int}\Omega_1) = \partial\Omega \cup R. \tag{18.3.12}$$

The following result is inspired partially by [130, Lemma 3.4].

Proposition 18.3.5. *Let $p \in \Omega$, let $q \in D(p)$ and let*

$$(p, q) := \{tp + (1 - t)q : t \in (0, 1)\}.$$

Then $(p, q) \subseteq \operatorname{int}\Omega_1$ and

$$D(\gamma) = \{q\} \tag{18.3.13}$$

for any $\gamma \in (p, q)$.

Proof. Let p, q be as in the statement, and fix $\gamma \in (p, q)$. We already know that $D(\gamma) \neq \emptyset$. On the other hand, assume that there exists $q' \neq q$ such that $q' \in D(\gamma)$. Let us notice that p, q, q' cannot lie on the same line. Indeed, if by contradiction this was the case, then the only possibility is that p is a convex combination of γ and q'. But then the strict convexity of K

would imply that

$$\|\gamma - q'\|_K \leqslant \|\gamma - q\|_K < \|p - q\|_K \leqslant \|p - q'\|_K < \|\gamma - q'\|_K,$$

which is absurd. This in particular implies that p, γ, q' do not lie on the same line. Therefore, thanks again to the strict convexity of K, we get that

$$\|p - q'\|_K < \|p - \gamma\|_K + \|\gamma - q'\|_K \leqslant \|p - \gamma\|_K + \|\gamma - q\|_K = \|p - q\|_K,$$

a contradiction to $q \in D(p)$. Hence (18.3.13) is proved. Assume by contradiction that $\gamma \in R$. By Corollary 4.11 in [195], any point of the form $q + \lambda(\gamma - q)$ with $\lambda > 1$ has a point in $\partial\Omega$ closer than q. On the other hand, taking w the midpoint of p and γ, then by (18.3.13) it holds that $D(w) = \{q\}$, which is impossible. □

Let us take a point $p \in \operatorname{int}\Omega_1$, and let $q \in D(p)$. Thanks to Proposition 18.3.5, we know that

$$d_{K,\partial\Omega}(z) = \|z - q\|_K$$

for any z in (p, q). Recalling that $(p, q) \subseteq \operatorname{int}\Omega_1$, together with (18.3.11), and Proposition 18.3.3 it is easy to see that $Dd_{K,\partial\Omega}(z) \neq 0$. Thus, at least locally, the level set $\Sigma_{d_{K,\partial\Omega}(p)}$ is a well-defined C^2 hypersurface. Reasoning as in Section 18.3.1 we conclude that

$$-\operatorname{div}(\pi_K \circ N_{d_K,\partial\Omega})(p) \geq H_{K_0,\partial\Omega}(q) \tag{18.3.14}$$

for any $p \in \operatorname{int}\Omega_1$, where $q \in D(p)$.

18.4 Sub-Finsler norms and perimeter

Let $K_0 \subseteq \mathcal{H}_0 \equiv \mathbb{R}^{2n}$ be a convex body in C^2_+, $0 \in \operatorname{int} K_0$ and let $\|\cdot\|_{K_0}$ be the associated norm in $\mathbb{R}^{2n}$. In the following we shall write $\|\cdot\|, \|\cdot\|_*$ and π instead of $\|\cdot\|_{K_0}, \|\cdot\|_{K_0,*}$ and π_{K_0} respectively. For any $p \in \mathbb{H}^n$, we define a left-invariant norm $\|\cdot\|_p$ on $\mathcal{H}_p$ by means of the equality

$$\|v\|_p = \|d\tau_p^{-1}(v)\| \quad v \in \mathcal{H}_p,$$

where $d\tau_p$ denotes the differential of τ_p. In particular, for a horizontal vector field $\sum_{i=1}^n f_i X_i + g_i Y_i$ its norm at a point $p \in \mathbb{H}^n$ is given by

$$\left\| \sum_{i=1}^n f_i(p) X_i(0) + g_i(p) Y_i(0) \right\| = \|(f(p), g(p))\|,$$

where $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$. Similarly, we extend the dual norm $\|\cdot\|_*$ and the projection π to each fiber of the horizontal bundle. When $\|\cdot\|$ is C^l with $l \geq 2$, all norms $\|\cdot\|_p$ are C^l. Given a horizontal vector field U of class C^1, we define $\pi(U)$ as the C^1 horizontal vector field satisfying

$$\|U\|_* = \langle U, \pi(U)\rangle.$$

Proceeding as in [244, Section 2.3], it is easy to see that the projection satisfies

$$\pi\left(\sum_i^n f_i X_i + g_i Y_i\right) = \mathcal{N}_{K_0}^{-1}\left(\frac{(f,g)}{\sqrt{|f|^2+|g|^2}}\right),$$

where $|f|^2 = \langle f, f\rangle$.

Definition 18.4.1. *Given a measurable set $E \subseteq \mathbb{H}^n$ we say that E has finite horizontal K_0-perimeter if*

$$P_{K_0,\mathcal{H}}(E) = \sup\left\{\int_E \operatorname{div}(U)\ d\mathcal{L}^{2n+1},\, U \in C_c^1(\mathbb{H}^n,\mathcal{H}), \|U\|_{K_0,\infty} \leqslant 1\right\} < +\infty,$$

where $\|U\|_{K_0,\infty} = \sup_{p\in\mathbb{H}^n} \|U_p\|_p$.

Remark 18.4.2. The perimeter associated to the Euclidean norm $|\cdot|$ is the sub-Riemannian perimeter defined in (16.8.4). A set has finite perimeter for a given norm if and only if it has finite perimeter for the standard sub-Riemannian perimeter. Hence all known results in the standard case apply to the sub-Finsler perimeter. Moreover, if E has C^1 boundary ∂E, then

$$P_{K_0,\mathcal{H}}(E) = \int_{\partial E} \|N_h\|_* d\sigma =: A_{K_0,\mathcal{H}}(\partial E),$$

where N_h is the projection on the horizontal distribution $\mathcal{H}$ of the Riemannian normal N with respect to the metric g and $d\sigma$ is the Riemannian measure of ∂E (cf. [244, Section 2.4] fur further details when $n = 1$).

As a significant example, we consider a bounded open set $\Omega \subseteq \mathbb{R}^{2n}$ and a C^1 function $u : \Omega \to \mathbb{R}$. Let $\mathrm{Gr}(u) = \{(\bar{x},\bar{y},t) \in \mathbb{H}^n : u(\bar{x},\bar{y}) - t = 0\}$ be the graph of u. Then we have

$$N_h = \frac{\sum_{i=1}^n (u_{x_i} - y)X_i + (u_{y_i} + x)Y_i}{\sqrt{1+|Du+F|^2}} \quad \text{and} \quad d\sigma = \sqrt{1+|Du+\mathcal{F}|^2}\, dz,$$

where $\mathcal{F}$ is as in (16.7.1). Therefore we get

$$A_{K_0,\mathcal{H}}(\mathrm{Gr}(u)) = \int_\Omega \|Du+\mathcal{F}\|_*\, dz.$$

18.5 The sub-Finsler prescribed mean curvature equation

Inspired by the previous computation and the sub-Riemannian problem studied by [85], we consider the following problem. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded open set and let $F \in L^1(\Omega,\mathbb{R}^{2n})$,

$\varphi \in W^{1,1}(\Omega)$ and $H \in L^\infty(\Omega)$. Then we set

$$\mathcal{I}(u) = \int_\Omega \|Du + F\|_* \, dz + \int_\Omega Hu \, dz \tag{18.5.1}$$

for each $u \in W^{1,1}(\Omega)$ such that $u - \varphi \in W_0^{1,1}(\Omega)$. We say that $u \in W^{1,1}(\Omega)$ is a *minimizer* for $\mathcal{I}$ if

$$\mathcal{I}(u) \leqslant \mathcal{I}(v)$$

for all $v \in W^{1,1}(\Omega)$ such that $v - \varphi \in W_0^{1,1}(\Omega)$. In [85, Section 3] the authors investigate the first variation of the functional $\mathcal{I}$ when $\| \cdot \|_{K_0,*}$ is the Euclidean norm $| \cdot |$, taking into account the bad beaviour of the singular set

$$\Omega_0 = \{(\bar{x}, \bar{y}) \in \Omega \ : \ (Du + F)(\bar{x}, \bar{y}) = 0\}. \tag{18.5.2}$$

In the next result we derive the Euler-Lagrange equation associated to $\mathcal{I}$ for C^2 minimizers.

Proposition 18.5.1. *Let K_0 be a C^2_+ convex body such that $0 \in \operatorname{int}(K_0)$. Let $u \in C^2(\Omega)$ be a minimizer for $\mathcal{I}$ defined in (18.5.1). Assume that $F \in C^1(\Omega, \mathbb{R}^{2n})$. Let Ω_0 be the singular set defined in (18.5.2). Then u satisfies*

$$\operatorname{div}(\pi(Du + F)) = H \qquad \text{in } \Omega \setminus \Omega_0. \tag{18.5.3}$$

Proof. Given $v \in C_c^\infty(\Omega \setminus \Omega_0)$, by [244, Lemma 3.2] the first variation is given by

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} \mathcal{I}(u + sv) &= \int_{\Omega\setminus\Omega_0} \frac{d}{ds}\Big|_{s=0} \|D(u + sv) + F\|_* \, dz + \int_{\Omega\setminus\Omega_0} Hv \, dz \\
&= \int_{\Omega\setminus\Omega_0} \frac{d}{ds}\Big|_{s=0} \|Du + F + sDv\|_* \, dz + \int_{\Omega\setminus\Omega_0} Hv \, dz \\
&= \int_{\Omega\setminus\Omega_0} \langle Dv, \pi(Du + F)\rangle \, dz + \int_{\Omega\setminus\Omega_0} Hv \, dz \\
&= \int_{\Omega\setminus\Omega_0} v \, \left(H - \operatorname{div}(\pi(Du + F))\right) \, dz.
\end{aligned}$$

□

Remark 18.5.2. When K_0 is the unit disk $D_0 \subseteq \mathbb{R}^{2n}$ centered at 0 of radius 1 we have

$$\pi_{D_0}(Du + F) = \frac{Du + F}{|Du + F|}$$

and (18.5.3) is equivalent to

$$\operatorname{div}\left(\frac{Du + F}{|Du + F|}\right) = H.$$

18.6 The Finsler approximation problem

In this section we develop the Finsler approximation scheme in order to get rid of the singular nature of equation (18.5.3). To this aim, given K_0 a convex body in C^2_+ such that $0 \in \operatorname{int} K_0$

and $\varepsilon \in (0,1)$, we denote by K_ε the set

$$K_\varepsilon := \left\{ (\bar{x}, \bar{y}, t) \in \mathbb{R}^{2n+1} : \left(\frac{|t|}{\varepsilon}\right)^{\frac{3}{2}} + \|(\bar{x}, \bar{y})\|^{\frac{3}{2}} \leqslant 1 \right\}. \tag{18.6.1}$$

Notice that $K_\varepsilon \subseteq \mathbb{R}^{2n+1} \equiv T_0\mathbb{H}^n$ (here $T_0\mathbb{H}^n$ denotes the tangent space of $\mathbb{H}^n$ at $p = 0$) is a strictly convex body with $0 \in \mathrm{int}(K_\varepsilon)$. Moreover ∂K_ε is of class C^1. Indeed it is a level set of the C^1 function $g_\varepsilon(\bar{x}, \bar{y}, t) := \left(\frac{|t|}{\varepsilon}\right)^{\frac{3}{2}} + \|(\bar{x}, \bar{y})\|^{\frac{3}{2}}$, whose gradient never vanishes on ∂K_ε. Hence, the projection π_{K_ε} is well-defined and continuous. We shall write $\|\cdot\|_\varepsilon$, $\|\cdot\|_{\varepsilon,*}$ and π_ε instead of $\|\cdot\|_{K_\varepsilon}$, $\|\cdot\|_{K_\varepsilon,*}$ and π_{K_ε} respectively. The map π_ε^h is defined as the first $2n$ components of π_ε. By abuse of notation, we write $\pi_\varepsilon^h(\bar{x}, \bar{y}) = \pi_\varepsilon^h(\bar{x}, \bar{y}, -1)$ when there is no confusion.

Proposition 18.6.1. *Let K_0 be a convex body in C^2_+ such that $0 \in \mathrm{int}\, K_0$, and let $K_\varepsilon \subseteq \mathbb{R}^{2n+1}$ be the set defined in (18.6.1). Then the following assertions hold:*

(i) The map $\pi_\varepsilon^h : \mathbb{R}^{2n} \smallsetminus \{0\} \to \mathbb{R}^{2n}$ satisfies

$$\pi_\varepsilon^h(\bar{x}, \bar{y}) = \pi(\bar{x}, \bar{y}) \frac{\|(\bar{x}, \bar{y})\|_*^2}{(\varepsilon^3 + \|(\bar{x}, \bar{y})\|_*^3)^{\frac{2}{3}}}.$$

(ii) The map π_ε^h can be extended to a C^1 map in $\mathbb{R}^{2n}$ by setting $\pi_\varepsilon^h(0,0) = (0,0)$.

(iii) $\|(\bar{x}, \bar{y}, -1)\|_{K_\varepsilon,*} = (\varepsilon^3 + \|(\bar{x}, \bar{y})\|_*^3)^{\frac{1}{3}}$.

Proof. Let us prove that

$$\pi_\varepsilon(\bar{x}, \bar{y}, -1) = (\alpha\pi(\bar{x}, \bar{y}), -\varepsilon(1 - \alpha^{3/2}))^{2/3} \tag{18.6.2}$$

for some $0 < \alpha(\bar{x}, \bar{y}) < 1$. Given $(\bar{x}, \bar{y})$ in $\mathbb{R}^{2n} \setminus \{0\}$, we denote by t_0 the $(2n+1)$-th coordinate of $\pi_\varepsilon(\bar{x}, \bar{y}, -1)$ and we let $K_{t_0} \subseteq \mathbb{R}^{2n}$ be the convex set defined by

$$K_{t_0} := \{(\bar{x}', \bar{y}') \ : \ (\bar{x}', \bar{y}', t_0) \in K_\varepsilon\}.$$

Then we have

$$K_{t_0} \times \{t_0\} = \left\{ \left(\frac{|t_0|}{\varepsilon}\right)^{\frac{3}{2}} + \|(\bar{x}', \bar{y}')\|^{\frac{3}{2}} \leqslant 1 \right\} = \left\{ \|(\bar{x}', \bar{y}')\| \leqslant \left(1 - \left(\frac{|t_0|}{\varepsilon}\right)^{\frac{3}{2}}\right)^{\frac{2}{3}} \right\}.$$

Hence it follows that $\pi_{t_0} = (1 - (\frac{|t_0|}{\varepsilon})^{\frac{3}{2}})^{\frac{2}{3}}\pi$. On the other hand, since π_ε is the inverse of the Gauss map, we can see that $(\bar{x}, \bar{y}, -1)$ is normal to ∂K_ε at $\pi_\varepsilon(\bar{x}, \bar{y}, -1)$ and so $(\bar{x}, \bar{y})$ is normal to ∂K_{t_0} at $\pi_\varepsilon^h(\bar{x}, \bar{y})$, where $0 < t_0 < 1$ satisfies $\|\pi_\varepsilon^h(\bar{x}, \bar{y})\|^{\frac{3}{2}} + (\frac{|t_0|}{\varepsilon})^{\frac{3}{2}} = 1$. Since K_{t_0} is strictly convex, the projection is unique and $\pi_\varepsilon^h(\bar{x}, \bar{y}) = \pi_{t_0}(\bar{x}, \bar{y})$. Hence (18.6.2) follows. Taking the scalar product of $(\bar{x}, \bar{y}, -1)$ with the curve $\beta(s) = (s\pi(\bar{x}, \bar{y}), -\varepsilon(1 - s^{3/2})^{2/3})$, we get

$$\langle (\bar{x}, \bar{y}, -1), \beta(s) \rangle = s\|(\bar{x}, \bar{y})\|_* + \varepsilon(1 - s^{3/2})^{2/3}.$$

Notice that β is in ∂K_ε and $\beta(\alpha)$ is π_ε. Hence in $s = \alpha$ the maximum of the scalar products of $(\bar{x}, \bar{y}, -1)$ with an element of K_ε is attained. Thus we can take derivatives in $s = \alpha$, set them equal to 0 and get

$$0 = \|(\bar{x}, \bar{y})\|_* - \varepsilon \frac{\alpha^{\frac{1}{2}}}{(1 - \alpha^{3/2})^{\frac{1}{3}}}.$$

Then we obtain

$$\alpha = \frac{\|(\bar{x}, \bar{y})\|_*^2}{(\varepsilon^3 + \|(\bar{x}, \bar{y})\|_*^3)^{2/3}}$$

and we get (i). Since $\|(\bar{x}, \bar{y}, -1)\|_{K_\varepsilon,*} = \langle(\bar{x}, \bar{y}, -1), \pi_\varepsilon(\bar{x}, \bar{y}, -1)\rangle$, a straightforward computation shows (iii). Finally, (ii) follows from (i) and the 2-homogeneity of the map $\pi(\cdot)\|\cdot\|_*^2$. $\square$

Lemma 18.6.2. *Let $u, v \in T_0\mathbb{H}^n$ and $s \in \mathbb{R}$. Then we have*

$$\frac{d}{ds}\Big|_{s=0} \|u + sv\|_{\varepsilon,*} = \langle v, \pi_\varepsilon(u)\rangle. \tag{18.6.3}$$

Proof. Let $f(s) = \|u + sv\|_{\varepsilon,*}$ and $g(s) = \langle u + sv, \pi_\varepsilon(u)\rangle$. Notice that $f(s) \geqslant g(s)$ for each $s \in \mathbb{R}$, since by definition $\|u + sv\|_{\varepsilon,*} \geqslant \langle u + sv, \pi_\varepsilon(u)\rangle$ and $f(0) = \|u\|_{\varepsilon,*} = \langle u, \pi_\varepsilon(u)\rangle = g(0)$. Therefore, by a standard argument $f'(0) = g'(0)$, and the thesis follows. $\square$

Given a convex body $K_0 \subseteq \mathbb{R}^{2n}$ in C^2_+ with $0 \in \text{int}(K_0)$, and K_ε defined as in (18.6.1), we extend the reasoning of the previous section to define a left-invariant norm $\|\cdot\|_\varepsilon$ on $T\mathbb{H}^n$ by means of the equality

$$\Big\|\sum_{i=1}^n f_i X_i + g_i Y_i + hT\Big\|_{\varepsilon,p} = \|(f(p), g(p), h(p))\|_\varepsilon,$$

for any $p \in \mathbb{H}^n$ with $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$. Again, $\|\cdot\|_{\varepsilon,*}$ and π_ε can be extended to the tangent bundle in the usual way.

Definition 18.6.3. *Given a measurable set $E \subseteq \mathbb{H}^n$ we say that E has finite K_ε-perimeter if*

$$P_{K_\varepsilon}(E) = \sup\left\{\int_E \text{div}(U)\ d\mathcal{L}^{2n+1},\ U \in C^1_c(\mathbb{H}^n, T\mathbb{H}^n),\ \|U\|_{K_\varepsilon,\infty} \leqslant 1\right\} < +\infty,$$

where $\|U\|_{K_\varepsilon,\infty} = \sup_{p\in\mathbb{H}^n} \|U_p\|_\varepsilon$.

Remark 18.6.4. If E has C^1 boundary ∂E, then

$$P_{K_\varepsilon}(E) = \int_{\partial E} \|N\|_{\varepsilon,*} d\sigma = A_\varepsilon(\partial E),$$

where N is the Riemannian normal with respect to the metric g and $d\sigma$ is the Riemannian measure of ∂E.

18.7 The Finsler prescribed mean curvature equation

We are ready to derive the Finsler prescribed mean curvature equation, essentially in the same way as in the previous section. To this aim, let $\Omega \subseteq \{t = 0\}$ be a bounded open set and $u : \Omega \to \mathbb{R}$ be a C^2 function. Then we have

$$N = \frac{\sum_{i=1}^{n}(u_{x_i} - y)X_i + (u_{y_i} + x)Y_i - T}{\sqrt{1 + |Du + \mathcal{F}|^2}} \quad \text{and} \quad d\sigma = \sqrt{1 + |Du + \mathcal{F}|^2}\, dz.$$

Therefore we get

$$A_{K_\varepsilon}(\mathrm{Gr}(u)) = \int_\Omega \|(Du + \mathcal{F}, -1)\|_{\varepsilon,*}\, dz.$$

Hence, inspired by this computation and thanks to Proposition 18.6.1, given $F \in L^1(\Omega, \mathbb{R}^{2n})$, $\varphi \in W^{1,1}(\Omega)$ and $H \in L^\infty(\Omega)$, we define the approximating Finsler functional $\mathcal{I}_\varepsilon$ by

$$\mathcal{I}_\varepsilon(u) = \int_\Omega \left(\varepsilon^3 + \|(Du + F)\|_*^3\right)^{\frac{1}{3}} dz + \int_\Omega Hu\, dz, \tag{18.7.1}$$

for any $u \in W^{1,1}(\Omega)$ such that $u - \varphi \in W_0^{1,1}(\Omega)$. Arguing as in the previous section, and thanks to Lemma 18.6.2, we are able to deduce the Euler-Lagrange equation associated to (18.7.1). Indeed, given $v \in C_c^\infty(\Omega)$, by Lemma 18.6.2, the first variation is given by:

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} \mathcal{I}_\varepsilon(u + sv) &= \int_\Omega \frac{d}{ds}\Big|_{s=0} \|(D(u + sv) + F, -1)\|_{\varepsilon,*}\, dz + \int_\Omega Hv\, dz \\
&= \int_\Omega \frac{d}{ds}\Big|_{s=0} \|(Du + F, -1) + s(Dv, 0)\|_{\varepsilon,*}\, dz + \int_\Omega Hv\, dz \\
&= \int_\Omega \langle (Dv, 0), \pi_\varepsilon((Du + F, -1))\rangle\, dz + \int_\Omega Hv\, dz \\
&= \int_\Omega \langle Dv, \pi_\varepsilon^h(Du + F)\rangle\, dz + \int_\Omega Hv\, dz \\
&= \int_\Omega v(H - \mathrm{div}(\pi_\varepsilon^h(Du + F)))\, dz.
\end{aligned}$$

Then the Finsler prescribed mean curvature equation for the graph of u is given by

$$\mathrm{div}(\pi_\varepsilon^h(Du + F)) = H \quad \text{in } \Omega. \tag{18.7.2}$$

As already pointed out in the introduction, (18.7.2) is only degenerate elliptic in the singular set (cf. the computations of Section 18.8). Therefore, in the next section, we will perturb (18.7.2) as in (18.1.14) in order to apply the aforementioned classical Schauder fixed-point theory for elliptic equations.

18.8 A priori estimates

In this section we want to find classical solutions to the regularized Finsler approximating Dirichlet problem associated to (18.1.14), that is

$$\begin{cases} \operatorname{div}\left(\pi_\varepsilon^h(Du+F)\right) + \eta \operatorname{div}\left(\frac{Du+F}{\sqrt{1+|Du+F|^2}}\right) = H & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega, \end{cases} \tag{18.8.1}$$

where $\varepsilon, \eta \in (0,1)$, $\Omega \subseteq \mathbb{R}^{2n}$ is a bounded domain with $C^{2,\alpha}$ boundary for $0 < \alpha < 1$, K_0 is a convex body in $C_+^{2,\alpha}$ with $0 \in \operatorname{int} K_0$, $H \in Lip(\overline{\Omega})$, $F = (F_1, \dots, F_{2n}) \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^{2n})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. To this aim, let us fix some notation. It is easy to see that the map $G : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}^{2n}$ defined by $G(p) = \pi(p)\|p\|_*^2$ can be extended to a 2-homogeneous and C^1 map setting $G(0) = 0$. Moreover, for any $i = 1, \dots, 2n$

$$D_i(\|\cdot\|_*^3) = 3G_i(\cdot),$$

where $G = (G_1, \dots, G_{2n})$. Thanks to Proposition 18.6.1, we can write the first equation of (18.8.1) in the form

$$\operatorname{div}\left(\pi(Du+F)\frac{\|Du+F\|_*^2}{(\varepsilon^3 + \|Du+F\|_*^3)^{\frac{2}{3}}}\right) + \eta \operatorname{div}\left(\frac{Du+F}{\sqrt{1+|Du+F|^2}}\right) = H. \tag{18.8.2}$$

An easy computation yields

$$\begin{aligned} \frac{1}{(\varepsilon^3 + \|Du+F\|_*^3)^{\frac{5}{3}}} &\Big((\varepsilon^3 + \|Du+F\|_*^3) \operatorname{div}(G(Du+F)) \\ &- 2G(Du+F)(D^2u + DF)G(Du+F)^T\Big) \\ &+ \frac{\eta}{(1+|Du+F|^2)^{\frac{3}{2}}} \Big((1+|Du+F|^2) \operatorname{div}\big(Du+F\big) \\ &- (Du+F)(D^2u+DF)(Du+F)^T\Big) = H. \end{aligned}$$

Therefore, we can write (18.8.2) in the familiar form

$$\sum_{i,j=1}^{2n} A_{i,j}^{\varepsilon,\eta}(z, Du; F) D_{i,j}u + B^{\varepsilon,\eta}(z, Du; F) = H,$$

where the coefficients $A_{i,j}^{\varepsilon,\eta}$ and $B^{\varepsilon,\eta}$ are defined by

$$\begin{aligned} A_{i,j}^{\varepsilon,\eta}(z,p;F) := &\frac{1}{(\varepsilon^3 + \|p+F\|_*^3)^{\frac{2}{3}}} D_j G_i(p+F) - \frac{2}{(\varepsilon^3 + \|p+F\|_*^3)^{\frac{5}{3}}} G_i(p+F)G_j(p+F) \\ &+ \frac{\eta}{\sqrt{1+|p+F|^2}}\delta_{ij} - \frac{\eta}{(1+|p+F|^2)^{\frac{3}{2}}}(p_i+F_i)(p_j+F_j) \end{aligned} \tag{18.8.3}$$

and

$$
\begin{aligned}
B^{\varepsilon,\eta}(z,p;F) :=& \frac{1}{(\varepsilon^3+\|p+F\|_*^3)^{\frac{2}{3}}}\sum_{i,j=1}^{2n} D_jG_i(p+F)D_iF_j \\
&- \frac{2}{(\varepsilon^3+\|p+F\|_*^3)^{\frac{5}{3}}}G(p+F)\,DF\,G(p+F)^T \\
&+ \frac{\eta}{\sqrt{1+|p+F|^2}}\operatorname{div}F - \frac{\eta}{(1+|p+F|^2)^{\frac{3}{2}}}(p+F)DF(p+F)^T
\end{aligned}
$$

for any $z \in \Omega$ and $p = (p_1, \dots, p_{2n}) \in \mathbb{R}^{2n}$. Therefore (18.8.2) is a second-order quasi-linear equation. Moreover, thanks to the computations of the previous section and *(iii)* in Proposition 18.6.1, we know that (18.8.2) is the Euler-Lagrange equation associated to the functional

$$
u \mapsto \int_\Omega \left(\varepsilon^3 + \|Du+F\|_*^3\right)^{\frac{1}{3}} + \eta\sqrt{1+|Du+F|^2} + uH\,dz.
$$

Notice that the matrix $A^{\varepsilon,\eta}$ is symmetric. Moreover, observing that

$$
D_j(G_i(p)) = \begin{cases} 2\|p\|_*\pi_j(p)\pi_i(p) + \|p\|_*^2 D_i\pi_j(p) & \text{if } p \neq 0 \\ 0 & \text{if } p = 0, \end{cases} \tag{18.8.4}
$$

we infer that (18.8.2) is an elliptic equation. Indeed, assume first that $p+F=0$. Then, by (18.8.3) and (18.8.4)

$$
\sum_{i,j=1}^{2n} A_{i,j}^{\varepsilon,\eta}(z,p;F)\xi_i\xi_j = \eta|\xi|^2
$$

for any $\xi \in \mathbb{R}^{2n}$. From the other hand, when $p+F \neq 0$, (18.2.6), (18.8.4) and the Cauchy-Schwarz inequality imply that

$$
\begin{aligned}
\sum_{i,j=1}^{2n} A_{i,j}^{\varepsilon,\eta}(z,p;F)\xi_i\xi_j =& \sum_{i,j=1}^{2n} \frac{2\|p+F\|_*\pi_i(p+F)\pi_j(p+F)\xi_i\xi_j + \|p+F\|_*^2 D_i\pi_j(p+F)\xi_i\xi_j}{(\varepsilon^3+\|p+F\|_*^3)^{\frac{2}{3}}} \\
&- \sum_{i,j=1}^{2n} \frac{2\|p+F\|_*^4\pi_i(p+F)\pi_j(p+F)\xi_i\xi_j}{(\varepsilon^3+\|p+F\|_*^3)^{\frac{5}{3}}} + \eta\frac{(1+|p+F|^2)|\xi|^2 - \langle p+F,\xi\rangle^2}{(1+|p+F|^2)^{\frac{3}{2}}} \\
&\geq \frac{\|p+F\|_*^2}{(\varepsilon^3+\|p+F\|_*^3)^{\frac{2}{3}}}\left(\xi\,D\pi(p+F)\,\xi^T\right) + \eta\frac{|\xi|^2}{(1+|p+F|^2)^{\frac{3}{2}}} \\
&> \eta\frac{|\xi|^2}{(1+|p+F|^2)^{\frac{3}{2}}}
\end{aligned} \tag{18.8.5}
$$

for any $\xi \in \mathbb{R}^{2n}$, so that we conclude that

$$
\sum_{i,j=1}^{2n} A_{i,j}^{\varepsilon,\eta}(z,p;F)\xi_i\xi_j \geq \frac{\eta}{(1+|p+F|^2)^{\frac{3}{2}}}|\xi|^2 \tag{18.8.6}
$$

for any $z \in \overline{\Omega}$ and any $p, \xi \in \mathbb{R}^{2n}$. We remark that, by (18.8.5), equation (18.7.2) is elliptic outside the singular set. In view of (18.8.6), we are in position to apply the classical theory for quasi-linear elliptic equations of [157]. In particular, we wish to rely on the following fundamental result, which is a direct consequence of [157, Theorem 13.8] and subsequent remarks.

Proposition 18.8.1. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with $C^{2,\alpha}$ boundary, for some $0 < \alpha < 1$, and let $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Let us assume that $A^{\varepsilon,\eta}_{i,j}(\cdot, \cdot; \sigma F), B^{\varepsilon,\eta}(\cdot, \cdot; \sigma F) \in C^{\alpha}(\overline{\Omega} \times \mathbb{R}^{2n})$ for any $\sigma \in [0, 1]$, and that the maps*

$$\sigma \mapsto A^{\varepsilon,\eta}_{i,j}(\cdot, \cdot; \sigma F), \quad \sigma \mapsto B^{\varepsilon,\eta}(\cdot, \cdot; \sigma F)$$

are continuous as maps from $[0, 1]$ to $C^{\alpha}(\overline{\Omega} \times \mathbb{R}^{2n})$. If there exists a constant $M > 0$ such that, for any $\sigma \in [0, 1]$, any solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the problem

$$\begin{cases} \operatorname{div}(\pi^h_\varepsilon(Du + \sigma F)) + \eta \operatorname{div}\left(\frac{Du+\sigma F}{\sqrt{1+|Du+\sigma F|^2}}\right) = \sigma H & \text{in } \Omega \\ u = \sigma\varphi & \text{in } \partial\Omega \end{cases} \tag{18.8.7}$$

satisfies

$$\|u\|_{C^1(\overline{\Omega})} \leqslant M,$$

then

$$\begin{cases} \operatorname{div}(\pi^h_\varepsilon(Du + F)) + \eta \operatorname{div}\left(\frac{Du+F}{\sqrt{1+|Du+F|^2}}\right) = H & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases} \tag{18.8.8}$$

admits a solution in $C^{2,\alpha}(\overline{\Omega})$.

Remark 18.8.2. Notice that the constant $M > 0$ in Proposition 18.8.1 depends *a priori* on $\varepsilon, \eta \in (0, 1)$ and may blow up as $\varepsilon, \eta \to 0$. However, in the sequel (cf. Proposition 18.8.6, Proposition 18.8.7 and Proposition 18.8.8) we will show that the estimates for the C^1 norm of solutions to (18.8.7) can be made uniform in $\varepsilon \in (0, 1)$ and $\eta \in (0, \eta_0)$ for a sufficiently small constant $\eta_0 \in (0, 1)$. That would provide a constant $M > 0$ *a posteriori* independent of ε and η, thus allowing to pass to the limit as $\varepsilon, \eta \to 0$ (cf. Theorem 18.9.1).

We shall need also the following weak maximum principle stated in [157, Theorem 8.1].

Theorem 18.8.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. Let L be the uniformly elliptic linear operator*

$$Lw = \operatorname{div}(a_{i,j} D_j w) + c_i D_i w$$

where the coefficients $a_{i,j}$ and c_i are bounded measurable functions on Ω. Let $w \in W^{1,2}(\Omega)$ satisfy $Lw \geq 0$ in Ω in distributional sense. Then

$$\sup_{\Omega} w \leqslant \sup_{\partial\Omega} w^+,$$

where the value of $w^+ = \max\{0, w\}$ *in* $\partial\Omega$ *is understood in the sense of traces.*

First of all we need to guarantee the requested regularity for the coefficients of the equation.

Lemma 18.8.4. *Let* K_0 *be a convex body in* $C^{2,\alpha}_+$ *with* $0 \in \operatorname{int} K_0$*. Let* $F \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^{2n})$*. Then there exists* $0 < \beta < 1$ *such that* $A^{\varepsilon,\eta}_{i,j}(\cdot, \cdot; \sigma F), B^{\varepsilon,\eta}(\cdot, \cdot; \sigma F) \in C^\beta(\overline{\Omega} \times \mathbb{R}^{2n})$ *for any* $\sigma \in [0, 1]$*. Moreover, the maps*

$$\sigma \mapsto A^{\varepsilon,\eta}_{i,j}(\cdot, \cdot; \sigma F), \quad \sigma \mapsto B^{\varepsilon,\eta}(\cdot, \cdot; \sigma F)$$

are continuous as maps from $[0, 1]$ *to* $C^\beta(\overline{\Omega} \times \mathbb{R}^{2n})$*.*

Proof. The second statement easily follows from the definition of the coefficients. Let us prove the first statement. It is clear, thanks to our assumptions on K_0 and F, that $A^{\varepsilon,\eta}_{i,j}(\cdot, \cdot, \sigma F)$ and $B^{\varepsilon,\eta}(\cdot, \cdot, \sigma F)$ belong to $C^0(\overline{\Omega} \times \mathbb{R}^{2n})$ for any $\sigma \in [0, 1]$. Moreover, in view of (18.8.4), $D_j G_i$ is $C^\alpha(\mathbb{R}^{2n} \setminus 0)$ for any $i, j = 1, \ldots, 2n$, since ∂K_0 is $C^{2,\alpha}$. Finally, we get

$$\lim_{p \to 0} \frac{|D_j G_i|(p)}{|p|^\alpha} = 0.$$

Indeed, we have

$$\begin{aligned}
\frac{|D_j G_i|(p)}{|p|^\alpha} &= 2 \frac{\|p\|_*}{|p|^\alpha} |\pi_j(p)\pi_i(p) + \|p\|_*^2 D_i \pi_j(p)| \\
&\leqslant 2 \frac{\|p\|_*^\alpha}{|p|^\alpha} \|p\|_*^{1-\alpha} \left(|\pi_j(p)\pi_i(p)| + \|p\|_* |D_i \pi_j(p)| \right) \\
&\leqslant C \|p\|_*^{1-\alpha} \to 0
\end{aligned}$$

as $p \to 0$, since $\frac{\|p\|_*^\alpha}{|p|^\alpha}$ is bounded and the last factor in the previous inequality is 0-homogeneous, thus in particular bounded. Then $D_j G_i$ belongs to $C^\alpha(\mathbb{R}^{2n})$. Since $A^{\varepsilon,\eta}_{i,j}$ and $B^{\varepsilon,\eta}$ are obtained as composition, sum and product of Hölder functions, the conclusion follows. □

Therefore we are in position to apply Proposition 18.8.1. First of all we want to obtain estimates for the C^0 norm of solutions to (18.8.7). In order to do this, inspired by [154], we assume that there exists $\delta = \delta(K_0, \Omega, H) \in (0, 1]$ such that

$$\left| \int_\Omega H v dz \right| \leqslant (1 - \delta) \int_\Omega \|Dv\|_* dz \tag{18.8.9}$$

for any non-negative function $v \in C^\infty_c(\Omega)$. To justify this assumption, assume that we have a function $u \in C^2(\Omega)$ which solves (18.8.1). Then, multiplying (18.8.1) by a test function

$v \in C_c^\infty(\Omega)$, integrating over Ω and letting $\eta \to 0$, by Proposition 18.6.1 we get that

$$
\begin{aligned}
\left|\int_\Omega Hv\,dz\right| &\leqslant \left|\int_\Omega v\operatorname{div}(\pi_\varepsilon^h(Du+F))\,dz\right| + \eta\left|\int_\Omega v\operatorname{div}\left(\frac{Du+\sigma F}{\sqrt{1+|Du+\sigma F|^2}}\right)dz\right| \\
&\leqslant \int_\Omega |\langle \pi_\varepsilon^h(Du+F), Dv\rangle|\,dz + \eta\int_\Omega \left|\left\langle Dv, \frac{Du+\sigma F}{\sqrt{1+|Du+\sigma F|^2}}\right\rangle\right| dz \\
&\leqslant \int_\Omega \|Dv\|_* dz + \eta\int_\Omega |Dv|\,dz \\
&\to \int_\Omega \|Dv\|_* dz.
\end{aligned}
\tag{18.8.10}
$$

Notice that, as already pointed out in the introduction, (18.8.9) is slightly stronger than (18.8.10). We begin by proving a technical lemma.

Lemma 18.8.5. *Let $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$. Then*

$$
\langle p, \pi_\varepsilon^h(p+\sigma F)\rangle \geq \|p\|_* - 1 - \|F\|_* - \|-F\|_* \tag{18.8.11}
$$

for any $p \in \mathbb{R}^{2n}$ and $z \in \overline{\Omega}$.

Proof. Let us fix $z \in \overline{\Omega}$ and $p \in \mathbb{R}^{2n}$. If $p = 0$ or $p + \sigma F = 0$, then the assertion is trivial. Therefore, assume $p, p+\sigma F \neq 0$. It is clear, recalling Proposition 18.6.1 and using the Cauchy -Schwarz formula (18.2.4), that

$$
\begin{aligned}
\langle p, \pi_\varepsilon^h(p+\sigma F)\rangle &= \langle p+\sigma F, \pi_\varepsilon^h(p+\sigma F)\rangle - \langle \sigma F, \pi_\varepsilon^h(p+\sigma F)\rangle \\
&\geq \frac{\|p+\sigma F\|_*^3}{(\varepsilon^3 + \|p+\sigma F\|_*^3)^{\frac{2}{3}}} - \left(\frac{\|p+\sigma F\|_*^3}{\varepsilon^3 + \|p+\sigma F\|_*^3}\right)^{\frac{2}{3}} \|\sigma F\|_* \\
&\geq \frac{\|p+\sigma F\|_*^3}{(\varepsilon^3 + \|p+\sigma F\|_*^3)^{\frac{2}{3}}} - \|F\|_*.
\end{aligned}
$$

Hence, noticing that

$$
\|p+\sigma F\|_* \geq \|p\|_* - \|-\sigma F\|_* \geq \|p\|_* - \|-F\|_*
$$

by the triangle inequality, it suffices to prove that

$$
\frac{\|p+\sigma F\|_*^3}{(\varepsilon^3 + \|p+\sigma F\|_*^3)^{\frac{2}{3}}} \geq \|p+\sigma F\|_* - 1. \tag{18.8.12}
$$

When $\|p+\sigma F\|_* \leqslant 1$ (18.8.12) is trivial. Therefore let us assume $\|p+\sigma F\|_* > 1$. Notice that (18.8.12) is equivalent to

$$
\|p+\sigma F\|_*^{\frac{9}{2}} \geq (\|p+\sigma F\|_* - 1)^{\frac{3}{2}}(\varepsilon^3 + \|p+\sigma F\|_*^3).
$$

Since $a^p - b^p \geq (a-b)^p$ when $0 < b < a$ and $p > 1$, it is enough to check that

$$\begin{aligned}\|p+\sigma F\|_*^{9/2} &\geq (\|p+\sigma F\|_*^{3/2} - 1)(\varepsilon^3 + \|p+\sigma F\|_*^3) \\ &= \varepsilon^3 \|p+\sigma F\|_*^{3/2} + \|p+\sigma F\|_*^{9/2} - \varepsilon^3 - \|p+\sigma F\|_*^3,\end{aligned}$$

which is clearly true since $\|p+\sigma F\|_* > 1$ and $\varepsilon < 1$. □

Proposition 18.8.6. *Let $\alpha \in (0,1)$ and K_0 be a convex body in $C_+^{2,\alpha}$ with $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded open set, $\varphi \in C^2(\overline{\Omega})$, $H \in L^\infty(\Omega)$ and $F \in C^0(\overline{\Omega}, \mathbb{R}^{2n})$. If condition (18.8.9) is satisfied then there exist a constant $\eta_0 = \eta_0(K_0, \delta) \in (0,1)$ and a constant $C_1 = C_1(n, K_0, \Omega, \varphi, F, \delta) > 0$, independent of $\sigma \in [0,1]$, $\varepsilon \in (0,1)$ and $\eta \in (0, \eta_0)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to (18.8.7) with $\eta \in (0, \eta_0)$ it holds that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1.$$

Proof. Let us notice that (18.8.11), the equivalence between $\|\cdot\|_*$ and the Euclidean norm and the boundedness of F allow to find constants $a_0, a_2 > 0$, independent of $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$, such that

$$\langle p, \pi_\varepsilon^h(p + \sigma F)\rangle \geq a_0|p| - a_2$$

for any $z \in \Omega$ and $p \in \mathbb{R}^{2n}$. This fact, together with the boundedness of H, suggests to rely on [157, Lemma 10.8] to limit ourselves to estimate $\|u\|_{L^1(\Omega)}$. Indeed, it is not difficult to show that [157, Lemma 10.8] remains true when condition (10.23) of [157] allows a positive coefficient multiplying $|p|$. Moreover, its proof can be easily adapted to achieve estimates from above of $\sup_\Omega -u$ in terms of $\|u^-\|_{L^1(\Omega)}$ for any solution of $Qu = 0$ where Q is defined in (10.5) of [157]. In the end it suffices to estimate $\|u^+\|_{L^1(\Omega)}$ and $\|u^-\|_{L^1(\Omega)}$. We only estimate $\|u^+\|_{L^1(\Omega)}$, being the other case analogous. Moreover, up to replacing u by $u - \|\varphi\|_{L^\infty(\partial\Omega)}$, we can assume that $u \leqslant 0$ in $\partial\Omega$. Let us set $v = u^+$. Then it is clear that $v \in W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega)$, and moreover Dv exists in the classical sense for almost every $z \in \Omega$. Therefore, since u is in particular a weak solution to

$$\operatorname{div}(\pi_\varepsilon^h(Du + \sigma F)) + \eta \operatorname{div}\left(\frac{Du + \sigma F}{\sqrt{1 + |Du + \sigma F|^2}}\right) = \sigma H,$$

it follows that

$$\int_\Omega \langle Dv, \pi_\varepsilon^h(Du + \sigma F)\rangle + \eta \left\langle Dv, \frac{Du + \sigma F}{\sqrt{1 + |Du + \sigma F|^2}}\right\rangle dz = -\int_\Omega v\sigma H dz. \tag{18.8.13}$$

We claim that

$$\langle Dv, \pi_\varepsilon^h(Du + \sigma F)\rangle \geq \|Dv\|_* - 1 - \|F\|_* - \|-F\|_* \tag{18.8.14}$$

holds in any point where Dv exists in the classical sense. Indeed, in such points Dv is either 0 or Du. In the first case (18.8.14) is trivial, while in the second case it follows from Lemma 18.8.5. It is well known that, since $v \geq 0$ and $v \in W_0^{1,1}(\Omega)$, there exists a sequence of non-negative functions $(v_k)_k \subseteq C_c^\infty(\Omega)$ converging to v strongly in $W_0^{1,1}(\Omega)$. Moreover, thanks to (18.8.9) it holds that

$$\left|\int_\Omega H v_k dz\right| \leqslant (1-\delta)\int_\Omega \|Dv_k\|_*\, dz.$$

Hence, passing to the limit in the previous equation, and recalling that $\|\cdot\|_*$ is equivalent to the Euclidean norm, we conclude that (18.8.9) holds for v. Combining this information with (18.8.13) and (18.8.14) we get that

$$\begin{aligned}
0 &= \int_\Omega -\langle Dv, \pi_\varepsilon^h(Du+\sigma F)\rangle - \eta\left\langle Dv, \frac{Du+\sigma F}{\sqrt{1+|Du+\sigma F|^2}}\right\rangle dz - \int_\Omega v\sigma H\, dz \\
&\leqslant \int_\Omega -\|Dv\|_* + 1 + \|F\|_* + \|-F\|_* + \eta|Dv|\, dz + \left|\int_\Omega vH\, dz\right| \\
&\leqslant \int_\Omega -\|Dv\|_* + 1 + \|F\|_* + \|-F\|_* + C\eta\|Dv\|_* + (1-\delta)\|Dv\|_*\, dz \\
&= \int_\Omega 1 + \|F\|_* + \|-F\|_* + (C\eta - \delta)\|Dv\|_*\, dz,
\end{aligned}$$

where $C = C(K_0)$ is a positive constant as in (18.2.1). Hence, choosing $\eta_0 \in (0,1)$ such that $\delta - C\eta_0 > 0$, we conclude that

$$(\delta - C\eta_0)\int_\Omega \|Dv\|_*\, dz \leqslant (\delta - C\eta)\int_\Omega \|Dv\|_*\, dz \leqslant \int_\Omega 1 + \|F\|_* + \|-F\|_*\, dz$$

for any $\eta \in (0,\eta_0)$. Thanks to Poincaré's inequality and the equivalence between $\|\cdot\|_*$ and the Euclidean norm, we conclude that there exists a constant c_1, independent of $\sigma \in [0,1]$, $\varepsilon \in (0,1)$ and $\eta \in (0,\eta_0)$, such that

$$\int_\Omega u^+\, dz \leqslant c_1.$$

Since in the same way we can achieve an estimate for u^-, the thesis follows. □

The next step is to achieve gradient estimates, again in the C^0 norm, for solutions to (18.8.7). As customary in this framework, we want to reduce ourselves to boundary gradient estimates via a suitable maximum principle. To this aim, arguing as in [85], we need to assume the existence of scalar functions $f_1, \dots, f_{2n} \in C^1(\overline{\Omega})$ such that

$$D_k F_i = D_i f_k \quad \text{for any} \quad i,k = 1,\dots,2n. \tag{18.8.15}$$

We stress that interior gradient estimates usually depend on the bounds of the coefficients and the ellipticity nature of the equation (cf. e.g. [157, Chapter 15]). Consequently, since by (18.8.6) the ellipticity constant tends to vanish as $\eta \to 0$, the right way to achieve estimates which are uniform in $\epsilon, \eta \in (0,1)$ is to rely on a suitable maximum principle argument. Indeed, thanks to (18.8.15), the following maximum principle, which is the Finsler counterpart of [85, Proposition 4.3], holds.

Proposition 18.8.7. *Let K_0 be a convex body in $C_+^{2,\alpha}$ for $0<\alpha<1$ with $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain. Let $F \in C^1(\Omega, \mathbb{R}^{2n})$ be such that (18.8.15) holds. Let H be a* *constant. Let $u \in C^2(\overline{\Omega})$ be a solution to (18.8.7). Then*

$$\|Du\|_{L^\infty(\Omega)} \leqslant \|Du\|_{L^\infty(\partial\Omega)} + 2\|f\|_{L^\infty(\Omega)}, \tag{18.8.16}$$

where $f = (f_1, \dots, f_{2n})$ is as in (18.8.15).

Proof. Fix $\sigma \in [0,1]$, $\varepsilon \in (0,1)$ and $\eta \in (0,1)$. Let $v \in C_c^2(\Omega)$ and fix $k \in \{1, \dots, 2n\}$. Then, multiplying (18.8.7) by $D_k v$, using Proposition 18.6.1, integrating over Ω, integrating by parts and exploiting the properties of F, it holds that

$$\begin{aligned}
0 &= \int_\Omega \left(\operatorname{div} \left(\pi(Du + \sigma F) \frac{\|Du + \sigma F\|_*^2}{(\varepsilon^3 + \|Du + \sigma F\|_*^3)^{\frac{2}{3}}} + \eta \frac{Du + \sigma F}{\sqrt{1 + |Du + \sigma F|^2}} \right) - \sigma H \right) D_k v \, dz \\
&= \int_\Omega \operatorname{div} \left(\pi(Du + \sigma F) \frac{\|Du + \sigma F\|_*^2}{(\varepsilon^3 + \|Du + \sigma F\|_*^3)^{\frac{2}{3}}} + \eta \frac{Du + \sigma F}{\sqrt{1 + |Du + \sigma F|^2}} \right) D_k v \, dz \\
&= -\sum_{i=1}^{2n} \int_\Omega \left(\pi_i(Du + \sigma F) \frac{\|Du + \sigma F\|_*^2}{(\varepsilon^3 + \|Du + \sigma F\|_*^3)^{\frac{2}{3}}} + \eta \frac{D_i u + \sigma F_i}{\sqrt{1 + |Du + \sigma F|^2}} \right) D_i D_k v \, dz \\
&= -\sum_{i=1}^{2n} \int_\Omega \left(\pi_i(Du + \sigma F) \frac{\|Du + \sigma F\|_*^2}{(\varepsilon^3 + \|Du + \sigma F\|_*^3)^{\frac{2}{3}}} + \eta \frac{D_i u + \sigma F_i}{\sqrt{1 + |Du + \sigma F|^2}} \right) D_k D_i v \, dz \\
&= \sum_{i=1}^{2n} \int_\Omega D_k \left(\pi_i(Du + \sigma F) \frac{\|Du + \sigma F\|_*^2}{(\varepsilon^3 + \|Du + \sigma F\|_*^3)^{\frac{2}{3}}} + \eta \frac{D_i u + \sigma F_i}{\sqrt{1 + |Du + \sigma F|^2}} \right) D_i v \, dz \\
&= \sum_{i,j=1}^{2n} \int_\Omega A_{i,j}^{\varepsilon,\eta}(z, Du; \sigma F) D_k(D_j u + \sigma F_j) D_i v \, dz \\
&= \sum_{i,j=1}^{2n} \int_\Omega A_{i,j}^{\varepsilon,\eta}(z, Du; \sigma F) D_j(D_k u + \sigma f_k) D_i v dz,
\end{aligned}$$

being $A_{i,j}^{\varepsilon,\eta}$ as in (18.8.3). Therefore we proved that

$$\sum_{i,j=1}^{2n} \int_\Omega A_{i,j}^{\varepsilon,\eta}(z, Du; \sigma F) D_j(D_k u + \sigma f_k) D_i v \, dz = 0 \tag{18.8.17}$$

for any $v \in C_c^2(\Omega)$. Arguing as in [85, Proposition 4.3] it is easy to show that (18.8.17) actually holds for any $v \in C_c^1(\Omega)$. Therefore, recalling (18.8.6), we proved that $D_k u + \sigma f_k$ is a weak solution to the linear uniformly elliptic equation

$$\operatorname{div}(a_{i,j}^{\varepsilon,\eta} D_j w) = 0,$$

where

$$a_{i,j}^{\varepsilon,\eta}(z) := A_{i,j}^{\varepsilon,\eta}(z, Du; \sigma F(z)).$$

Hence, being $a_{i,j}^{\varepsilon,\eta}(z)$ bounded in Ω, thanks to Theorem 18.8.3 with $b_i, c_i, d = 0$ we conclude that

$$\|Du + \sigma f\|_{L^\infty(\Omega)} \leqslant \|Du + \sigma f\|_{L^\infty(\partial\Omega)},$$

which in particular implies that

$$\|Du\|_{L^\infty(\Omega)} \leqslant \|Du\|_{L^\infty(\partial\Omega)} + 2\|f\|_{L^\infty(\Omega)}. \tag{18.8.18}$$

□

Finally we are left to provide boundary gradient estimates for solutions to (18.8.7). Therefore, inspired by [154], we have to impose some constraints on the values of H depending on the Finsler mean curvature of $\partial\Omega$. More precisely, we require that

$$|H|(z_0) < H_{K_0,\partial\Omega}(z_0) \tag{18.8.19}$$

for any $z_0 \in \partial\Omega$, where $H_{K_0,\partial\Omega}$ is the K_0-mean curvature as defined in Subsection 18.3. Here and in the rest of this section we assume that K_0 is a convex body in C^∞_+ such that $0 \in \operatorname{int} K_0$, since we need to apply the results of Section 18.3.1 and Section 18.3.2.

Proposition 18.8.8. *Let K_0 be a convex body in C^∞_+ with $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be an open and bounded set with $C^{2,\alpha}$ boundary, for some $0 < \alpha < 1$. Let $\varphi \in C^2(\overline{\Omega})$, $F \in C^0(\overline{\Omega}, \mathbb{R}^{2n})$ and $H \in Lip(\Omega)$ satisfying (18.8.19). Finally, assume that there exist a constant $\eta_0 = \eta_0(n, K_0, \Omega, \varphi, F, H) \in (0,1)$ and a constant $\tilde{C}_1 = \tilde{C}_1(n, K_0, \Omega, \varphi, F, H) > 0$, independent of $\sigma \in [0,1]$, $\varepsilon \in (0,1)$ and $\eta \in (0, \eta_0)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to (18.8.7) it holds that*

$$\|u\|_{L^\infty(\Omega)} \leqslant \tilde{C}_1. \tag{18.8.20}$$

Then, up to choosing a smaller $\eta_0 = \eta_0(n, K_0, \Omega, \varphi, F, H) \in (0,1)$, there exist a constant $C_2 = C_2(n, K_0, \Omega, \varphi, F, \tilde{C}_1, H) > 0$, independent of $\sigma \in [0,1]$, $\varepsilon \in (0,1)$ and $\eta \in (0, \eta_0)$, such that any solution $u \in C^2(\overline{\Omega})$ to (18.8.7) with $\eta \in (0, \eta_0)$ satisfies

$$\|Du\|_{L^\infty(\partial\Omega)} \leqslant C_2. \tag{18.8.21}$$

Proof. First of all we notice that, being $\partial\Omega$ compact and $H_{K_0,\partial\Omega}$ continuous, (18.8.19) implies the existence of a positive constant $C_3 = C_3(K_0, \Omega, H)$ such that

$$|H(z_0)| \leqslant H_{K_0,\partial\Omega}(z_0) - 3C_3 \tag{18.8.22}$$

for any $z_0 \in \partial\Omega$. In order to prove this result we use a barrier argument as in [157, Chapter 14]. Therefore, for any $z_0 \in \partial\Omega$, we have to find a neighborhood $\mathcal{N}$ of z_0 in Ω and two functions $w^+, w^- \in C^2(\mathcal{N})$, called *upper barrier* and *lower barrier* respectively, such that

$$w^+(z_0) = w^-(z_0) = \sigma\varphi(z_0),$$

$$w^-(z) \leqslant u(z) \leqslant w^+(z)$$

for any $z \in \partial\mathcal{N}$,

$$\operatorname{div}(\pi^h_\varepsilon(Dw^+ + \sigma F)) + \eta \operatorname{div}\left(\frac{Dw^+ + \sigma F}{\sqrt{1 + |Dw^+ + \sigma F|^2}}\right) < \sigma H$$

for any $z \in \mathcal{N}$ and

$$\operatorname{div}(\pi^h_\varepsilon(\nabla w^- + \sigma F)) + \eta \operatorname{div}\left(\frac{Dw^- + \sigma F}{\sqrt{1 + |Dw^- + \sigma F|^2}}\right) > \sigma H$$

for any $z \in \mathcal{N}$. In this proof we deal only with the upper barrier, being the other case analogous. In order to find an upper barrier, we consider a tubular neighborhood $\mathcal{O}$ of $\partial\Omega$ and we let $\Gamma_\mu := \{x \in \overline{\Omega} : d_{K_0,\partial\Omega}(x) < \mu\}$, where $d_{K_0,\partial\Omega}$ is the Finsler distance from the boundary, $\mu \in (0, \mu_0)$ and $\mu_0 > 0$ is small enough to ensure that $\Gamma_\mu \subseteq \Gamma_{\mu_0} \Subset \mathcal{O}$ for any $\mu \in (0, \mu_0)$. Let us denote by $H_{\Sigma_{d(z)}}(z)$ the Euclidean mean curvature of $\Sigma_{d(z)}$ at any $z \in \overline{\Gamma}_{\mu_0}$. Being $H_{\Sigma_{d(z)}}$ continuous on $\overline{\Gamma}_{\mu_0}$, there exists a constant $C_4 = C_4(\Omega, K_0) > 0$ such that

$$|H_{\Sigma_{d(z)}}(z)| \leqslant C_4 \tag{18.8.23}$$

for any $z \in \overline{\Gamma}_{\mu_0}$. We fix $\mu \in (0, \mu_0)$ and we define $w^+ : \Gamma_\mu \longrightarrow \mathbb{R}$ by $w^+(z) := kd_{K_0,\partial\Omega}(z) + \sigma\varphi(z)$, where $k > 0$ has to be chosen. First, thanks to (18.3.11), $w^+ \in C^2(\overline{\Gamma_\mu})$, and for any $z \in \Gamma_\mu$ there exists a unique $z_0 \in \partial\Omega$ such that $d_{K_0,\partial\Omega}(z) = \|z - z_0\|$. Moreover, it is clear that $w^+(z_0) = \sigma\varphi(z_0)$ for any $z_0 \in \partial\Omega$. Thanks to (18.8.20), if we choose

$$k \geq \frac{\tilde{C}_1 + \|\varphi\|_{L^\infty(\Omega)}}{\mu},$$

it follows that $w^+(z) \geq u(z)$ for any $z \in \Omega$ with $d_{K_0,\partial\Omega}(z) = \mu$, and so we conclude that $u(z) \leqslant w^+(z)$ for any $z \in \partial\Gamma_\mu$. We are left to show that w^+ is a subsolution to (18.8.7). Therefore it suffices to show that

$$(\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{\frac{5}{3}} \left(\operatorname{div}(\pi^h_\varepsilon(Dw^+ + \sigma F)) + \eta \operatorname{div}\left(\frac{Dw^+ + \sigma F}{\sqrt{1 + |Dw^+ + \sigma F|^2}}\right) - \sigma H\right) < 0$$

on Γ_μ. Taking $k > \sup_\Omega \|-F\|_*$, (18.3.5) ensures that $kDd_{K_0,\partial\Omega}(z) + \sigma F(z) \neq 0$ for any $z \in \Gamma_\mu$ and $\sigma \in [0, 1]$. Let us notice that Proposition 18.6.1 and a simple computation imply that

$$\begin{aligned}
(\varepsilon^3 + &\|Dw^+ + \sigma F\|_*^3)^{\frac{5}{3}} \operatorname{div}(\pi^h_\varepsilon(Dw^+ + \sigma F)) \\
&= (\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{\frac{5}{3}} \operatorname{div}\left(\frac{\pi(Dw^+ + \sigma F)\|Dw^+ + \sigma F\|_*^2}{(\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{\frac{2}{3}}}\right) \\
&= (\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3) \underbrace{\operatorname{div}(\pi(Dw^+ + \sigma F)\|Dw^+ + \sigma F\|_*^2)}_{A} \\
&\quad + (\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{\frac{5}{3}} \underbrace{\|Dw^+ + \sigma F\|_*^2 \langle \pi(Dw^+ + \sigma F), D\left((\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{-\frac{2}{3}}\right)\rangle}_{B}.
\end{aligned}$$

We estimate separately A and B. In the following computations we let $d := d_{K_0,\partial\Omega}$ and $R_\sigma := \sigma D\varphi + \sigma F$. We are going to exploit the fact that, thanks to the homogeneity properties of the equation, the contribution of R_σ as $k \to \infty$ is negligible. Let us notice that by (18.3.5) and (18.2.7) we get

$$\pi(Dd_{K_0,\partial\Omega}) \cdot D^2 d_{K_0,\partial\Omega} = 0. \tag{18.8.24}$$

Hence, thanks to (18.3.5), (18.8.24), the 1-homogeneity of $\|\cdot\|_*$, the 0-homogeneity of π, the -1-homogeneity of $D\pi$ and the properties of $\|\cdot\|_*$, it holds that

$$\begin{aligned}
A =& \|kDd + R_\sigma\|_*^2 \sum_{i=1}^{2n} D_i\left(\pi_i(kDd + R_\sigma)\right) + \sum_{i=1}^{2n} \pi_i(kDd + R_\sigma) D_i\left(\|kDd + R_\sigma\|_*^2\right) \\
=& \|kDd + R_\sigma\|_*^2 \sum_{i,j=1}^{2n} D_i\pi_j(kDd + R_\sigma)(kD_{ij}d + D_iR_{\sigma,j}) \\
&+ 2\|kDd + R_\sigma\|_* \pi(kDd + R_\sigma) \cdot (kD^2d + DR_\sigma) \cdot \pi(kDd + R_\sigma)^T \\
=& k^2 \left\|Dd + \frac{R_\sigma}{k}\right\|_*^2 \sum_{i,j=1}^{2n} D_i\pi_j\left(Dd + \frac{R_\sigma}{k}\right)\left(D_{ij}d + \frac{D_iR_{\sigma,j}}{k}\right) \\
&+ 2k^2 \left\|Dd + \frac{R_\sigma}{k}\right\|_* \pi\left(Dd + \frac{R_\sigma}{k}\right) \cdot \left(D^2d + \frac{DR_\sigma}{k}\right) \cdot \pi\left(Dd + \frac{R_\sigma}{k}\right)^T \\
=& k^2(1 + o(1))(\operatorname{div}(\pi(Dd)) + o(1)) + 2k^2(1 + o(1))(\pi(Dd) \cdot D^2d \cdot \pi(Dd)^T + o(1)) \\
=& k^2 \operatorname{div}(\pi(Dd)) + o(k^2),
\end{aligned}$$

which allows to infer that

$$(\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)A = k^5 \operatorname{div}(\pi(Dd)) + o(k^5)$$

as $k \to \infty$, where $o(k^2)$ is uniform with respect to $z \in \Gamma_\mu$, $\varepsilon \in (0,1)$ and $\sigma \in [0,1]$. Now, exploiting the same properties as above, we estimate B:

$$\begin{aligned}
(\varepsilon^3 + \|kDd + R_\sigma\|_*^3)^{\frac{5}{3}} B &= -2\|kDd + R_\sigma\|_*^4 \langle \pi(kDd + R_\sigma), \nabla(\|kDd + R_\sigma\|_*)\rangle \\
&= -2\|kDd + R_\sigma\|_*^4 \pi(kDd + R_\sigma) \cdot (kD^2d + DR_\sigma) \cdot \pi(kDd + R_\sigma)^T \\
&= -2k^5 \left\|Dd + \frac{R_\sigma}{k}\right\|_*^4 \pi\left(Dd + \frac{R_\sigma}{k}\right) \cdot \left(D^2d + \frac{DR_\sigma}{k}\right) \cdot \pi\left(Dd + \frac{R_\sigma}{k}\right)^T \\
&= -2k^5(1 + o(1))(\pi(Dd) \cdot D^2d \cdot \pi(Dd)^T + o(1)) \\
&= -2k^5(1 + o(1))o(1) \\
&= o(k^5).
\end{aligned}$$

as $k \to \infty$ and uniformly with respect to $\varepsilon \in (0,1)$, $\sigma \in [0,1]$ and $z \in \Gamma_\mu$. From a similar computation, it follows that

$$
\begin{aligned}
\operatorname{div}\left(\frac{Dw^+ + \sigma F}{\sqrt{1+|Dw^+ + \sigma F|^2}}\right) &= \frac{\operatorname{div}(Dd) + \frac{\operatorname{div} R_\sigma}{k}}{\sqrt{\frac{1}{k^2} + \left|Dd + \frac{R_\sigma}{k}\right|^2}} - \frac{\left(Dd + \frac{R_\sigma}{k}\right) \cdot \left(D^2 d + \frac{DR_\sigma}{k}\right) \cdot \left(Dd + \frac{R_\sigma}{k}\right)^T}{\left(\frac{1}{k^2} + \left|Dd + \frac{R_\sigma}{k}\right|^2\right)^{3/2}} \\
&= \frac{\operatorname{div}(Dd)}{|Dd|} - \frac{Dd \cdot D^2 d \cdot Dd^T}{|Dd|^3} + o(1) \\
&= \operatorname{div}\left(\frac{Dd}{|Dd|}\right) + o(1)
\end{aligned}
$$

as $k \to \infty$ and uniformly with respect to $\sigma \in [0,1]$ and $z \in \Gamma_\mu$. Finally, it is easy to see that

$$
-(\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{\frac{5}{3}} \sigma H \leqslant (\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{\frac{5}{3}} |H| = k^5 |H| + o(k^5)
$$

as $k \to \infty$ and uniformly with respect to $\varepsilon \in (0,1)$, $\sigma \in [0,1]$ and $z \in \Gamma_\mu$. In the end we get that

$$
\begin{aligned}
(\varepsilon^3 + \|Dw^+ + \sigma F\|_*^3)^{\frac{5}{3}} &\left(\operatorname{div}(\pi_\varepsilon^h(Dw^+ + \sigma F)) + \eta \operatorname{div}\left(\frac{Dw^+ + \sigma F}{\sqrt{1+|Dw^+ + \sigma F|^2}}\right) - \sigma H\right) \\
&\leqslant k^5 \left(\operatorname{div}(\pi(Dd)) + \eta \operatorname{div}\left(\frac{Dd}{|Dd|}\right) + |H|\right) + o(k^5)
\end{aligned}
$$

as $k \to \infty$ and uniformly with respect to $\varepsilon \in (0,1)$, $\sigma \in [0,1]$ and $z \in \Gamma_\mu$. Now, let $z \in \Gamma_\mu$ and let $z_0 \in \partial\Omega$ be such that $d(z) = \|z - z_0\|$. Thanks to the Lipschitz continuity of H and the equivalence between $\|\cdot\|$ and the Euclidean norm, there exists a constant $C_5 = C_5(K_0)$ such that

$$
|H|(z) = |H|(z_0) + |H|(z) - |H|(z_0) \leqslant |H|(z_0) + C_5 d(z) \leqslant |H|(z_0) + C_5 \mu. \tag{18.8.25}
$$

Hence, thanks to (18.3.3), (18.3.7), (18.8.22) and (18.8.23), we conclude that

$$
\begin{aligned}
\operatorname{div}(\pi(Dd))(z) + \eta \operatorname{div}\left(\frac{Dd}{|Dd|}\right) + |H|(z_0) + C_5\mu &= -H_{K_0,\Sigma_{d(z)}}(z) - \eta H_{\Sigma_{d(z)}}(z) + |H|(z_0) + C_5\mu \\
&\leqslant -H_{K_0,\partial\Omega}(z_0) + \eta C_4 + |H|(z_0) + C_5\mu \\
&\leqslant -C_3 < 0,
\end{aligned}
\tag{18.8.26}
$$

provided that $\mu \leqslant \frac{C_3}{C_5}$ and $\eta \leqslant \frac{C_3}{C_4}$. Hence we found an upper barrier, from which the thesis follows. □

Remark 18.8.9. Assume that $n = 1$, let $\Omega \subseteq \mathbb{R}^2$ and $K_0 \in C_+^2$ be a convex body of $\mathbb{R}^2$. If (18.8.19) holds then Ω is strictly convex. Indeed, by Proposition 18.3.4 we have

$$
0 \leqslant |H| < -\frac{\langle D_{e_1} N_{z_0}, e_1 \rangle}{k^{K_0}(\pi(N_{z_0}))} = \frac{k^{\partial\Omega}(z_0)}{k^{K_0}(\pi(N_{z_0}))},
$$

where k^{K_0} and $k^{\partial\Omega}$ are the the Euclidean geodesic curvatures of ∂K and $\partial\Omega$. Since k^{K_0} is strictly positive we obtain $k^{\partial\Omega}(z_0) > 0$, hence Ω is strictly convex.

To conclude this section, inspired by [265] we want to show that, in the particular case in which H is constant and $n = 1$, then we can exploit (18.8.19) in order to obtain uniform estimates of the function, without requiring the validity of (18.8.9). Again, in order to apply the results of Section 18.3.1 and Section 18.3.2, we assume that K_0 is a convex body in C^∞_+ such that $0 \in \operatorname{int} K_0$ and $\partial\Omega$ belongs to $C^{2,1}$.

Proposition 18.8.10. *Assume that $n = 1$. Let K_0 be a convex body in C^∞_+ with $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with $C^{2,1}$ boundary, let $\varphi \in C^2(\overline{\Omega})$ and let H be a constant which satisfies (18.8.19). There exists a constant $C_1 = C_1(K_0, \Omega, \varphi, H, F) > 0$, independent of $\sigma \in [0, 1]$, $\varepsilon \in (0, 1)$ and $\eta \in (0, 1)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to (18.8.7), it holds that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1.$$

Proof. Let k^{K_0} be the geodesic curvature of K_0. Since $K_0 \in C^\infty_+$, then in particular $k^{K_0}(p) > 0$ for any $p \in \partial K_0$. Let $C_3 = C_3(K_0, \Omega, H)$ be as in (18.8.22). Let us define the function $v : \operatorname{int} \Omega_1 \longrightarrow \mathbb{R}$ by

$$v(z) := \sup_{\partial\Omega} |\varphi| + k d_{K_0,\partial\Omega}(z) \tag{18.8.27}$$

for any $z \in \Omega_1$, where $k > 0$ has to be chosen and Ω_1 is the set defined in (18.3.10). We already know (cf. (18.3.11)) that $v \in C^2(\operatorname{int} \Omega_1)$. We repeat *verbatim* the computations of the proof of Proposition 18.8.8 up to (18.8.26), with the difference that, being H constant, we can choose $C_5 = 0$ in (18.8.25). Since $n = 1$, we exploit Proposition 18.3.4 to infer that

$$\begin{aligned}\operatorname{div}(\pi(Dd))(z) + \eta \operatorname{div}\left(\frac{Dd}{|Dd|}\right) + |H| &= -H_{K_0,\Sigma_{d(z)}}(z) - \eta H_{\Sigma_{d(z)}}(z) + |H| \\ &= -H_{K_0,\Sigma_{d(z)}}(z) - \eta k^{K_0}\left(\pi_K(N_z)\right) H_{K_0,\Sigma_{d(z)}}(z) + |H| \\ &\leqslant -H_{K_0,\partial\Omega}(z_0) + |H| \\ &\leqslant -3C_3 < 0.\end{aligned} \tag{18.8.28}$$

Hence there exists $k > 0$, independent of $\varepsilon \in (0, 1)$, $\eta \in (0, 1)$, $\sigma \in [0, 1]$ and $z \in \Omega_1$, such that v is a subsolution to (18.8.7) on $\operatorname{int} \Omega_1$. Therefore, arguing as in the proof of [157, Theorem 10.7], it follows that $w := u - v$ is a weak supersolution on $\operatorname{int} \Omega_1$ to a linear elliptic equation of the form

$$\sum_{i,j=1}^{2n} D_i(a_{i,j}(z) D_j w(z)) + \sum_{i=1}^{2n} c_i(z) D_i w(z) = 0.$$

Hence, thanks to Theorem 18.8.3 and recalling (18.3.12), it follows that

$$\sup_{\Omega_1}(u - v) \leqslant \sup_{\partial\Omega \cup R}((u - v)^+).$$

Noticing that $u - v \leqslant 0$ on $\partial\Omega$ and that $\overline{\operatorname{int}\Omega_1} = \overline{\Omega}$, we obtain that

$$u(z) - v(z) \leqslant \sup_{\Omega}(u - v) = \sup_{\Omega_1}(u - v) \leqslant \sup_{\partial\Omega_1}((u - v)^+) = \sup_{R}((u - v)^+)$$

for any $z \in \Omega$. We are left to show that $\sup_R((u - v)^+) \leqslant 0$. Indeed, assume by contradiction that $\sup_R((u - v)^+) > 0$. Since R is compact, there exists $z_0 \in R$ such that

$$u(z_0) - v(z_0) = \sup_{R}((u - v)^+) = \sup_{R}(u - v).$$

Moreover, z_0 is a maximum point for $u - v$ on $\overline{\Omega}$. Let us fix $y_0 \in \partial\Omega$ such that $d_{K_0,\partial\Omega}(z_0) = \|z_0 - y_0\|$. Then, thanks to Proposition 18.3.5, it is easy to see that

$$d_{K_0,\partial\Omega}(z) = \|z - y_0\| \tag{18.8.29}$$

for any z belonging to (y_0, z_0), the segment connecting y_0 and z_0. Let now $\nu := \frac{y_0 - z_0}{|y_0 - z_0|}$. By (18.8.29) it holds that $v(z) < v(z_0)$ for any $z \in (y_0, z_0)$, and moreover

$$D_\nu^+ v(z_0) := \lim_{h \to 0^+} \frac{v(z_0 + h\nu) - v(z_0)}{h} < 0. \tag{18.8.30}$$

Since z_0 is a maximum point of $u - v$, it holds in particular that $D_\nu^+ u(z_0) \leqslant D_\nu^+ v(z_0)$, which implies, together with (18.8.30), that $D_\nu^+ u(z_0) = D_\nu u(z_0) < 0$. This proves that $Du(z_0) \neq 0$. Since then z_0 is a regular point for u, the level set $\{z \in \Omega : u(z) = u(z_0)\}$ is locally a C^2 hypersurface. Therefore there exists a small Euclidean ball B such that B is tangent to the level set at z_0 and moreover $B \subseteq \{z \in \Omega : u(z) \geq u(z_0)\}$. Now, since by our assumptions the Finsler balls relative to $-K_0$ are uniformly convex and C^2, there exists $\varrho > 0$ and $x_0 \in \Omega$ such that

$$\overline{B_{-K_0}(x_0, \varrho)} \subseteq \{z \in \Omega : u(z) \geq u(z_0)\} \tag{18.8.31}$$

and $B_{-K_0}(x_0, \varrho)$ is tangent to B at z_0. Indeed, fix a Finsler ball tangent to B at z_0 relative to $-K_0$, say B_F. On one hand, the principal curvatures of ∂B at z_0 are fixed. On the other hand, noticing that the principal curvatures of a C^2_+ convex set admit a positive lower bound, we can dilate and translate B_F to make the curvature of B_F as big as we want to ensure that (18.8.31) holds. Notice that

$$d_{K_0,\partial\Omega}(z) \geq d_{K_0,\partial\Omega}(z_0) \tag{18.8.32}$$

for any $z \in \overline{B_{-K_0}(x_0, \varrho)}$. Indeed, if by contradiction there exists $z \in \overline{B_{-K_0}(x_0, \varrho)}$ such that $d_{K_0,\partial\Omega}(z) < d_{K_0,\partial\Omega}(z_0)$, then (18.8.31) would imply

$$u(z) - kd_{K_0,\partial\Omega}(z) \geq u(z_0) - kd_{K_0,\partial\Omega}(z) > u(z_0) - kd_{K_0,\partial\Omega}(z_0),$$

a contradiction to the maximality of z_0. Let now $w_0 \in \partial\Omega$ be such that $d_{K,\partial\Omega}(x_0) = \|x_0 - w_0\|$, and let b_0 be the unique point of intersection between $\partial B_{-K_0}(x_0, \varrho)$ and the segment joining w_0 and x_0. Then by (18.2.2), (18.8.29), (18.8.32), the choice of b_0 and the strict convexity of K_0, it holds that

$$d_{K_0,\partial\Omega}(x_0) = \|x_0 - w_0\| = \|x_0 - b_0\| + \|b_0 - w_0\| = \varrho + d_{K_0,\partial\Omega}(b_0) \geq \varrho + d_{K_0,\partial\Omega}(z_0).$$

On the other hand, (18.2.2) and the triangle inequality imply

$$d_{K_0,\partial\Omega}(x_0) \leqslant \|x_0 - y_0\| \leqslant \|x_0 - z_0\| + \|z_0 - y_0\| = \varrho + d_{K_0,\partial\Omega}(z_0).$$

Putting together the previous inequalities we get that

$$d_{K_0,\partial\Omega}(x_0) = \|x_0 - y_0\| = \|x_0 - z_0\| + \|z_0 - y_0\|, \tag{18.8.33}$$

from which in particular we conclude, exploiting again the strict convexity of K_0, that x_0 lies on (y_0, x_0). Therefore, thanks to this fact, the first equality in (18.8.33) and Proposition 18.3.5, we conclude that $z_0 \in \operatorname{int} \Omega_1$, which is a contradiction. In the end we proved that

$$\sup_{\Omega} u \leqslant \sup_{\partial\Omega} |\varphi| + k \max_{\overline{\Omega}} d_{K_0,\partial\Omega}.$$

Since the converse estimate can be obtained in a similar way, the thesis is proved. □

Remark 18.8.11. We point out that the proof of Proposition 18.8.10 does not hold for $n \geqslant 2$. Indeed, when $n \geq 2$, the Euclidean mean curvature H_{Σ_d} in equation (18.8.28) may blow down to $-\infty$ close to the ridge R even though the Finsler mean curvature H_{K_0,Σ_d} is strictly positive on Ω_1.

18.9 Existence of sub-Finsler Lipschitz minimizers

Thanks to the *a priori* estimates of the previous section, together with Proposition 18.8.1 and the uniformity of the estimates with respect to $\varepsilon \in (0,1)$ and $\eta \in (0, \eta_0)$, we are in position to pass to the limit and find a solution to the sub-Finsler prescribed mean curvature equation.

Theorem 18.9.1. *Let $K_0 \in C^\infty_+$ be a convex body such that $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with $C^{2,1}$ boundary. Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$, for $0 < \alpha < 1$, and let $F \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^{2n})$ be such that (18.8.15) is satisfied. Assume that H is a constant such that (18.8.9) and (18.8.19) hold. Then, there exists $\eta_0 \in (0,1)$ such that for any $\varepsilon \in (0,1)$ and any $\eta \in (0, \eta_0)$, there exists a function $u_{\varepsilon,\eta} \in C^{2,\alpha}(\overline{\Omega})$ which solves (18.8.1). Moreover, there exists a constant $M > 0$, independent of $\varepsilon \in (0,1)$ and $\eta \in (0, \eta_0)$, such that any solution $u_{\varepsilon,\eta}$ to (18.8.1) satisfies*

$$\sup_{\Omega} |u_{\varepsilon,\eta}| + \sup_{\Omega} |Du_{\varepsilon,\eta}| \leqslant M. \tag{18.9.1}$$

Finally, there exists a Lipschitz continuous minimizer $u_0 \in Lip(\overline{\Omega})$ for the functional $\mathcal{I}$ defined in (18.5.1) with $u_0 = \varphi$ on $\partial\Omega$.

Proof. By Proposition 18.8.6, Proposition 18.8.7 and Proposition 18.8.8, there exists a constant $M > 0$ such that, for any $\sigma \in [0,1]$, any $0 < \varepsilon < 1$ and any $\eta \in (0, \eta_0)$ with $\eta_0 > 0$ small enough, then any solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the problem (18.8.7) satisfies

$$\sup_{\Omega} |u| + \sup_{\Omega} |Du| \leqslant M.$$

Then by Proposition 18.8.1 there exists a solution $u_{\varepsilon,\eta} \in C^{2,\alpha}(\bar{\Omega})$ to

$$\begin{cases} \operatorname{div}(\pi^h_\varepsilon(Du+F)) + \eta \operatorname{div}\left(\frac{Du+\sigma F}{\sqrt{1+|Du+\sigma F|^2}}\right) = H & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega. \end{cases}$$

Again by Proposition 18.8.6, Proposition 18.8.7 and Proposition 18.8.8, we have that

$$\sup_{\Omega} |u_{\varepsilon,\eta}| + \sup_{\Omega} |Du_{\varepsilon,\eta}| \leqslant M, \tag{18.9.2}$$

where the constant $M > 0$ is uniform in $0 < \varepsilon < 1$ and $\eta \in (0, \eta_0)$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,1)$ and $\{\eta_j\}_{j\in\mathbb{N}} \subseteq (0, \eta_0)$ be sequences such that $\varepsilon_j \to 0$ and $\eta_j \to 0$ as $j \to \infty$. Since M is uniform in ε and η by (18.9.2) we gain that $\sup_\Omega |u_{\varepsilon_j,\eta_j}| \leqslant M$ and that for any $z_1, z_2 \in \Omega$

$$|u_{\varepsilon_j,\eta_j}(z_1) - u_{\varepsilon_j,\eta_j}(z_2)| \leqslant M|z_1 - z_2|. \tag{18.9.3}$$

Then, by Ascoli-Arzelà theorem there exists $u_0 \in C(\overline{\Omega})$ such that $u_{\varepsilon_j,\eta_j} \to u_0$ uniformly in $\overline{\Omega}$. It is clear that $u = \varphi$ on $\partial\Omega$. Moreover, taking the limit as $j \to 0$ in (18.9.3), we gain that

$$\sup_{z_1 \neq z_2} \frac{|u_0(z_1) - u_0(z_2)|}{|z_1 - z_2|} \leqslant M,$$

thus u_0 is Lipschitz. We claim that u_0 is a minimizer for $\mathcal{I}$ defined in (18.5.1). Indeed, we have that $\|u_{\varepsilon_j,\eta_j}\|_{W^{1,1}(\Omega)} \leqslant M|\Omega|$, $\|u_0\|_{W^{1,1}(\Omega)} \leqslant M|\Omega|$ and u_{ε_j,η_j} converge to u_0 in $L^1(\Omega)$. Moreover, the function $(p, (\bar{x}, \bar{y})) \to \|p + F(\bar{x}, \bar{y})\|_*$ is positive, continuous and convex in p. Therefore, by [231, Theorem 4.1.2], $\mathcal{I}$ is lower semicontinuous with respect to the strong L^1-topology, from which we have that

$$\mathcal{I}(u_0) \leqslant \liminf_{j\to\infty} \mathcal{I}(u_{\varepsilon_j,\eta_j}). \tag{18.9.4}$$

For each $v \in W^{1,1}(\Omega)$ such that $v - \varphi \in W^{1,1}_0(\Omega)$, it follows that

$$\begin{aligned} \mathcal{I}(u_{\varepsilon_j,\eta_j}) &= \int_\Omega \|Du_{\varepsilon_j,\eta_j} + F\|_* \, dz + \int_\Omega H u_{\varepsilon_j,\eta_j} \, dz \\ &\leqslant \int_\Omega (\varepsilon_j^3 + \|Du_{\varepsilon_j,\eta_j} + F\|_*^3)^{\frac{1}{3}} \, dz + \int_\Omega H u_{\varepsilon_j,\eta_j} \, dz + \eta_j \int_\Omega \sqrt{1 + |Du_{\varepsilon_j,\eta_j} + F|^2} \, dz \\ &\leqslant \int_\Omega (\varepsilon_j^3 + \|Dv + F\|_*^3)^{\frac{1}{3}} \, dz + \int_\Omega H v \, dz + \eta_j \int_\Omega \sqrt{1 + |Dv + F|^2} \, dz \\ &\leqslant \varepsilon_j |\Omega| + \int_\Omega \|Dv + F\|_* \, dz + \int_\Omega H v \, dz + \eta_j \int_\Omega \sqrt{1 + |Dv + F|^2} \, dz, \end{aligned} \tag{18.9.5}$$

where we have used the fact that the Dirichlet solution $u_{\varepsilon_j,\eta_j} \in C^{2\,\alpha}(\bar{\Omega})$ is a minimizer for the functional $v \to \int_\Omega (\varepsilon_j^3 + \|Dv+F\|_*^3)^{\frac{1}{3}} + \int_\Omega Hv + \eta_j \int_\Omega \sqrt{1+|Dv+F|^2}\,dz$ for each $v \in W^{1,1}(\Omega)$ s.t. $v - \varphi \in W_0^{1,1}(\Omega)$. Passing to the liminf in (18.9.5) and taking into account (18.9.4), we obtain $\mathcal{I}(u_0) \leqslant \mathcal{I}(v)$ for each $v \in W^{1,1}(\Omega)$ s.t. $v - \varphi \in W_0^{1,1}(\Omega)$. □

We now apply the same argument of the previous proof in $\mathbb{H}^1$, using the height estimate provided by Proposition 18.8.10 instead of the one given in Proposition 18.8.6 to avoid condition (18.8.9), to obtain the following sharp result in the first Heisenberg group.

Theorem 18.9.2. *Let $n = 1$ and $K_0 \in C_+^\infty$ be a convex body such that $0 \in \operatorname{int} K_0$. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with $C^{2,1}$ boundary. Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$, for $0 < \alpha < 1$, and let $F \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^2)$ be such that (18.8.15) is satisfied. Assume that H is a constant such that (18.8.19) holds. Then, there exists $\eta_0 \in (0,1)$ such that for any $\varepsilon \in (0,1)$ and any $\eta \in (0,\eta_0)$, there exists a function $u_{\varepsilon,\eta} \in C^{2,\alpha}(\overline{\Omega})$ which solves (18.8.1). Moreover, there exists a constant $M > 0$, independent of $\varepsilon \in (0,1)$ and $\eta \in (0,\eta_0)$, such that any solution $u_{\varepsilon,\eta}$ to (18.8.1) satisfies*

$$\sup_\Omega |u_{\varepsilon,\eta}| + \sup_\Omega |Du_{\varepsilon,\eta}| \leqslant M. \tag{18.9.6}$$

Finally, there exists a Lipschitz continuous minimizer $u_0 \in Lip(\overline{\Omega})$ for the functional $\mathcal{I}$ defined in (18.5.1) with $u_0 = \varphi$ on $\partial\Omega$.

To conclude this section, according to [85] we point out that the Dirichlet problem for the prescribed K_0-mean curvature equation can be equivalently stated by means of a weak formulation which takes into account the presence of the singular set. Indeed, given a bounded domain $\Omega \subseteq \mathbb{R}^{2n}$, $\varphi \in W^{1,1}(\Omega)$, $H \in L^\infty(\Omega)$ and $F \in L^1(\Omega)$, we say that $u \in W^{1,1}(\Omega)$ is a *weak solution* to the Dirichlet problem for the prescribed K_0-mean curvature equation if $u - \varphi \in W_0^{1,1}(\Omega)$ and

$$\int_{\Omega_0} \|\nabla\phi\|_*\,dz + \int_{\Omega\setminus\Omega_0} \langle \pi(Du+F), \nabla\phi\rangle\,dz + \int_\Omega H\phi\,dz \geq 0 \tag{18.9.7}$$

for any $\phi \in W_0^{1,1}(\Omega)$, where we recall that $\Omega_0 = \{Du + F = 0\}$. The equivalence between the two formulations is proved in [85] for the sub-Riemannian setting and can be carried out for the sub-Finsler setting with slight modifications.

Remark 18.9.3. A deeper look to [196, 195] suggests that it should be possible to prove that the aforementioned results still hold only assuming that K_0 is a convex body in $C_+^{2,\alpha}$ with $0 \in \operatorname{int} K_0$, for some $0 < \alpha < 1$. Accordingly, it is reasonable that in Theorem 18.9.1 the regularity of ∂K_0 can be weakened to $C^{2,\alpha}$, for some $0 < \alpha < 1$.

18.10 A sharp existence result of Lipschitz minimizers in the sub-Riemannian setting

As pointed out in the introduction, a Finsler approximation scheme for (18.1.8) cannot be arbitrarily chosen, since one needs to guarantee classical regularity of the resulting equations. Nevertheless, for a particular class of Finsler metrics, it is possible to choose a more natural approximation scheme. More precisely, let us consider the one-parameter family of differential equations defined formally by

$$\operatorname{div}\left(\pi_{K_0}(Du+F)\frac{\|Du+F\|_*}{\sqrt{\varepsilon^2+\|Du+F\|_*^2}}\right)=H. \tag{18.10.1}$$

We point out that, when K_0 is the Euclidean unit ball centered at the origin, (18.10.1) reduces to the well-known elliptic approximating equation considered for instance in [85] (cf. Remark 18.5.2). In order to give to equation (18.10.1) a pointwise meaning, we must impose *a priori* that the function $\tilde{G}(p) := \|p\|_*\pi_{K_0}(p)$, which is C^1 outside the origin, admits a C^1 extension to the whole $\mathbb{R}^{2n}$. This regularity hypothesis turns out to be equivalent to the fact that the left-invariant sub-Finsler structure induced by K_0 comes from an underlying left-invariant sub-Riemannian metric on the distribution $\mathcal{H}$ (cf. [281]), or equivalently that K_0 is an ellipsoid centered at 0. More precisely, it is easy to check that, if $\tilde{G} \in C^1(\mathbb{R}^{2n}, \mathbb{R}^{2n})$, then $D\tilde{G}$ is necessarily a constant, symmetric and positive definite matrix, and moreover

$$\|p\|_* = \sqrt{p\cdot D\tilde{G}\cdot p^T} \qquad \text{and} \qquad \pi_K(p) = \frac{D\tilde{G}\cdot p^T}{\|p\|_*} \tag{18.10.2}$$

for any $p \in \mathbb{R}^{2n}$. When (18.10.2) holds, a direct computation shows that (18.10.1) is a well-defined, quasi-linear elliptic equation, so that in this setting a Euclidean regularization term as in (18.8.2) is no longer needed. In order to solve the Dirichlet problem associated to (18.10.1) it is then possible to replicate almost word-by-word the computations of Section 18.8, with the advantage that the absence of the Euclidean curvature term makes the process easier. The main benefit of this new approximation is that, due to the absence of the Euclidean curvature term, a result analogous to Proposition 18.8.10 actually holds for any $n \geq 1$. We include the proof for the sake of completeness.

Proposition 18.10.1. *Assume that $K_0 \in C^\infty_+$ induces a left-invariant sub-Riemannian metric on $\mathbb{H}^n$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with $C^{2,1}$ boundary, let $\varphi \in C^2(\overline{\Omega})$ and let H be a constant which satisfies (18.8.19). There exists a constant $C_1 = C_1(K_0,\Omega,\varphi,H,F) > 0$, independent of $\sigma \in [0,1]$ and $\varepsilon \in (0,1)$, such that, for any solution $u \in C^2(\overline{\Omega})$ to*

$$\begin{cases}\operatorname{div}\left(\pi_{K_0}(Du+\sigma F)\frac{\|Du+\sigma F\|_*}{\sqrt{\varepsilon^2+\|Du+\sigma F\|_*^2}}\right)=\sigma H & \text{in } \Omega\\ u=\sigma\varphi & \text{in } \partial\Omega\end{cases}$$

it holds that

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1.$$

Proof. Let $C_3 = C_3(K_0, \Omega, H)$ be as in (18.8.22). Let us define the function $v : \operatorname{int}\Omega_1 \longrightarrow \mathbb{R}$ as in (18.8.27), that is

$$v(z) := \sup_{\partial\Omega} |\varphi| + k d_{K_0,\partial\Omega}(z)$$

for any $z \in \Omega_1$, where $k > 0$ has to be chosen and Ω_1 is the set defined in (18.3.10). Again we know (cf. (18.3.11)) that $v \in C^2(\operatorname{int}\Omega_1)$. We repeat again, with minor modifications, the computations of the proof of Proposition 18.8.8 up to (18.8.26). As in the proof of Proposition 18.8.10, being H constant, we can choose $C_5 = 0$ in (18.8.25). Moreover, since $\eta = 0$, the analogous of (18.8.26) becomes

$$\operatorname{div}(\pi(Dd))(z) + |H|(z_0) = -H_{K_0,\Sigma_{d(z)}}(z) + |H|(z_0) \leqslant -H_{K_0,\partial\Omega}(z_0) + |H|(z_0) \leqslant -3C_3 < 0.$$

Hence there exists $k > 0$, independent of $\varepsilon \in (0,1)$, $\sigma \in [0,1]$ and $z \in \Omega_1$, such that v is a subsolution to (18.10.1) on $\operatorname{int}\Omega_1$. The thesis then follows *verbatim* as in the proof of Proposition 18.8.10. □

Therefore, in the sub-Riemannian setting, we can exploit Proposition 18.10.1 to avoid condition (18.8.9), so that the following sharper analogous to Theorem 18.9.1 holds.

Theorem 18.10.2. *Assume that $K_0 \in C^\infty_+$ induces a left-invariant sub-Riemannian metric on $\mathbb{H}^n$. Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with $C^{2,1}$ boundary. Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$, for $0 < \alpha < 1$, and let $F \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^{2n})$ be such that (18.8.15) is satisfied. Assume that H is a constant such that (18.8.19) holds. Then, for any $\varepsilon \in (0,1)$, there exists a function $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ which solves the Dirichlet problem associated to (18.10.1) with boundary datum φ. Moreover, there exists a constant $M > 0$, independent of $\varepsilon \in (0,1)$, such that any solution u_ε to (18.10.1) satisfies*

$$\sup_{\Omega} |u_\varepsilon| + \sup_{\Omega} |Du_\varepsilon| \leqslant M.$$

Finally, there exists a Lipschitz continuous minimizer $u_0 \in Lip(\overline{\Omega})$ for the functional $\mathcal{I}$ defined in (18.5.1) with $u_0 = \varphi$ on $\partial\Omega$.

Chapter 19

t-graphs of prescribed mean curvature: avoiding Dirichlet conditions

19.1 Introduction

We refer to [245] as main reference for this chapter. Although the Dirichlet problem, as previously discussed, has been widely studied beyond the Euclidean framework, as far as the author is aware no results in the spirit of [162] are available in the Riemannian setting. The aim of this chapter, consequently, is to lay the groundwork for the study of hypersurfaces with prescribed mean curvature outside the Euclidean setting and overcoming conditions inspired by (18.1.2). More precisely, we consider again t-graphs in both the Riemannian and the sub-Riemannian Heisenberg group. Our main achievement, in the spirit of [162], is an existence and regularity result for solutions to (ε-PMC), both in the non-extremal and in the extremal case. More precisely, our Riemannian existence statement reads as follows. Throughout this chapter, unless otherwise specified, we denote by Ω a bounded domain in $\mathbb{R}^{2n}$ with Lipschitz boundary.

Theorem 19.1.1. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with Lipschitz boundary, and let $H \in \operatorname{Lip}(\Omega) \cap C^{1,\gamma}_{loc}(\Omega)$ for some $\gamma \in (0,1)$. Then (18.1.4) holds if and only if there exists $u \in C^2(\Omega)$ which is a classical solution to (ε-PMC) on Ω. Moreover, if $H \in C^{k,\gamma}_{loc}(\Omega)$ for some $k \in \mathbb{N}$, $k \geq 1$, then $u \in C^{k+2,\gamma}_{loc}(\Omega)$. Finally, if $H \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Our approach can be summarized in the following major points.

- First, we prove existence of BV minimizers of a suitable penalized functional, analogous to (18.1.6), in the non-extremal case (cf. Proposition 19.3.3). To improve the regularity of such minimizers, we rely on suitable variational properties of minimizers (cf. Theorem 19.3.5 and Proposition 19.3.6) to infer that minimizers are locally bounded in Ω (cf. Proposition 19.3.7).

- Our second step relies on a generalization to the anisotropic setting (cf. [266]) of some celebrated regularity results for almost-minimizers of the perimeter (cf. e.g. [113, 248, 8, 259, 274, 53]). Exploiting some results from [266], we shall see that the boundary ∂E_u of the subgraph of a minimizer u is regular outside a singular portion with small Hausdorff dimension (cf. Proposition 19.3.9). A crucial result then consists in translating these regularity properties from ∂E_u to u. More precisely, we show that u is regular outside a small set $\Omega_{u,0} \subseteq \Omega$ for which $\mathcal{H}^s(\Omega_{u,0}) = 0$ for any $s > 2n-7$ (cf Proposition 19.3.11). The proof of these results is based on a careful analysis of the prescribed mean curvature equation for intrinsic graphs.

- Next, owing to some structure properties of the Riemannian perimeter induced by the metric g_ε (cf. Section 16.8), we show that minimizers enjoy Sobolev regularity (cf. Proposition 19.3.12).

- In view of the previous steps, and exploiting a suitable existence result for the Dirichlet problem in small balls (cf. Theorem 19.1.3), we provide local Lipschitz regularity by means of an approximation procedure and a comparison principle argument (cf. Proposition 19.3.13).

- To pass from Lipschitz regularity to higher regularity, exploiting a well-established approach, we write a linear uniformly elliptic equation for the function

$$u_v(z) = \frac{u(z+v) - u(z)}{|v|},$$

so that both the De Giorgi-Nash-Moser theory for $C^{1,\alpha}$ regularity and the classical Schauder theory for higher regularity apply (cf. Theorem 19.3.8). This last step basically concludes the proof of Theorem 19.1.1 in the non-extremal case.

- Finally, the existence of classical solutions in the extremal case (cf. Section 19.3.6) follows exploiting an approximating procedure as in Chapter 18, together with a suitable compactness argument and the extension to the Riemannian setting of the Euclidean notion of generalized solution to (18.1.1).

As already mentioned, since our source H may not be constant, a crucial step in the proof of local Lipschitz regularity is the use of suitable interior gradient estimates. More precisely, we extend the proof of Korevaar and Simon (cf. [185]) to achieve the following result, which may be of independent interest.

Theorem 19.1.2 (Interior gradient estimates). *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain. Let $H \in C^1(\Omega)$. Let $u \in C^3(\Omega)$ be a solution to (ε-PMC) and let $\tilde{\Omega} \Subset \Omega$ be a domain. For any domain $\hat{\Omega} \Subset \tilde{\Omega}$ there exists a constant $C = C\left(n, \varepsilon, d(\partial\hat{\Omega}, \partial\tilde{\Omega}), \|u\|_{L^\infty(\tilde{\Omega})}, \|H\|_{C^1(\tilde{\Omega})}, \|\mathcal{F}\|_{L^\infty(\tilde{\Omega})}\right) > 0$ such that*

$$\|Du\|_{L^\infty(\hat{\Omega})} \leqslant C, \tag{19.1.1}$$

where $d(\partial\hat{\Omega}, \partial\tilde{\Omega})$ the Euclidean distance between $\partial\hat{\Omega}$ and $\partial\tilde{\Omega}$.

In the proof of Theorem 19.1.2, a crucial role is played by the Riemannian Jacobi equation (17.1.3). Remarkably, assuming the additional condition

$$|H| < H_{\partial\Omega}(z_0) \tag{19.1.2}$$

for any $z_0 \in \partial\Omega$, i.e. the sub-optimal version of (18.1.2) required in [7, 158], the proof of Theorem 19.1.2 can be adapted to provide global gradient estimates (cf. Theorem 19.2.3). Consequently, we can improve the existence results proved in [7] and in Theorem 18.10.2 for the Riemannian Dirichlet problem associated to (ε-PMC) allowing a non-constant source.

Theorem 19.1.3. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with $C^{2,\alpha}$ boundary, for some $\alpha \in (0,1)$. Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$ and let $H \in \mathrm{Lip}(\Omega)$. Assume that (18.1.4) holds and that Ω is a non-extremal domain. Assume in addition that (19.1.2) holds. Then, for any $\varepsilon \neq 0$ there exists $u_\varepsilon \in \mathrm{Lip}(\Omega) \cap C^{2,\alpha}_{loc}(\Omega)$ which solves (ε-PMC) on Ω and such that $u_\varepsilon = \varphi$ on $\partial\Omega$. If in addition $H \in C^{1,\alpha}(\overline{\Omega})$, then $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$.*

Regarding uniqueness, we shall prove that the extremal condition (18.1.5) is equivalent to the maximality of the domain and the verticality of solutions to (ε-PMC), in a weak sense, at the boundary. Moreover, we show that and any of these conditions implies uniqueness of solutions of (ε-PMC) up to vertical translations.

Theorem 19.1.4. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with Lipschitz boundary. Let $H \in \mathrm{Lip}(\Omega) \cap C^{1,\gamma}_{loc}(\Omega)$ for some $\gamma \in (0,1)$ satisfy (18.1.4). The following statements are equivalent.*

(i) If $\hat{\Omega} \subseteq \mathbb{R}^{2n}$ is any domain such that $\Omega \subsetneq \hat{\Omega}$, then there is no solution $u \in C^2(\hat{\Omega})$ to (ε-PMC) in $\hat{\Omega}$.

(ii) (18.1.5) holds.

Moreover, if $\partial\Omega$ is of class C^2, then (i) and (ii) are equivalent to the following condition.

(iii) For any $u \in C^2(\Omega)$ which solves (ε-PMC) in Ω, it holds that

$$\lim_{t\to 0^+} \left| \int_{\partial\Omega_t} \frac{\langle \nu_t, Du + \mathcal{F} \rangle}{\sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2}} \, d\mathcal{H}^{2n-1} \right| = \mathcal{H}^{2n-1}(\partial\Omega),$$

where $\Omega_t = \{z \in \Omega : \min_{w\in\partial\Omega} |z-w| > t\}$ is defined for $t > 0$ small enough and ν_t is its exterior unit normal.

Finally, if $\partial\Omega$ is of class C^2, each of the previous three conditions implies that solutions to (ε-PMC) in Ω are unique up to vertical translations.

Regarding the sub-Riemannian setting, as already discussed in Chapter 18, the problem of finding t-graphs with prescribed horizontal mean curvature can be formulated by looking for local minimizers of the functional

$$\mathcal{I}(u) = \int_\Omega |Du + \mathcal{F}| + \int_\Omega H u \, dz. \tag{19.1.3}$$

Regarding the existence of such solutions, it is customary to rely on the aforementioned Riemannian approximation. Accordingly, our Riemannian existence result can be applied to study minimization problems related to (19.1.3). More precisely, we obtain solutions in $BV_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)$ to the sub-Riemannian prescribed mean curvature equation in the following sense.

Theorem 19.1.5. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be an open and bounded set with Lipschitz boundary. Let $H \in \mathrm{Lip}(\Omega)$. Assume that (18.1.4) holds. Then, there exists $u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ such that u is an H-minimizer for $P_{\mathbb{H}}$ on $\Omega \times \mathbb{R}$ in the sense of (19.5.1). Moreover, there exist a sequence of open sets such that $\Omega_j \Subset \Omega_k \Subset \Omega$ for any $j < k$ and $\bigcup_{j=0}^\infty \Omega_j = \Omega$ and a sequence $(u_j)_j \subseteq C^\infty(\Omega_j)$, such that each u_j solves (ε-PMC) in Ω_j and moreover*

$$u_j \to u \text{ almost everywhere in } \Omega \qquad \text{and} \qquad P_{\varepsilon_j}(E_{u_j},\cdot) \rightharpoonup^* P_{\mathbb{H}}(E_u,\cdot) \text{ locally in } \Omega\times\mathbb{R},$$

where $\rightharpoonup^*$ denotes the weak-* convergence *of measures.*

We point out that Theorem 19.1.5 generalizes the existence result proved in [264] for minimal t-graphs allowing H to be different from zero. Another interesting difference with respect to [264] consists in the different approach to the existence issue. Indeed, Serra Cassano and Vittone provided existence essentially via direct methods. On the other hand, although we believe that the same strategy could have worked as well in our framework, we preferred to provide existence combining direct methods and approximation. Indeed, as nicely explained in [74], sub-Riemannian minimizers arising as limit of Riemannian minimizers are only a particular subfamily of all sub-Riemannian minimizers, and they typically enjoy better regularity properties.

19.2 Interior and global gradient estimates

Throughout this section, we fix $\varepsilon \neq 0$ and the metric g_ε. Moreover, we fix a bounded domain $\Omega \subseteq \mathbb{R}^{2n}$ and $H \in C^1(\Omega)$. We shall provide interior and global gradient estimates for C^3 solutions to (ε-PMC). Our approach follows the technique developed in [184, 185]. Given $u \in C^3(\Omega)$, we denote by ν^ε the Riemannian normal to the hypersurface $S = \mathrm{graph}(u)$, which can be globally extended to $\Omega\times\mathbb{R}$ through vertical translations by

$$\nu^\varepsilon(z,t) = -\sum_{i=1}^{2n} \frac{D_i u(z) + \mathcal{F}_i(z)}{\sqrt{\varepsilon^2 + |Du(z)+\mathcal{F}(z)|^2}} Z_j|_{(z,t)} + \frac{\varepsilon}{\sqrt{\varepsilon^2 + |Du(z)+\mathcal{F}(z)|^2}} Z_{2n+1}|_{(z,t)} \tag{19.2.1}$$

for any $(z,t) \in \Omega\times\mathbb{R}$. Moreover, given $f \in C^3(\Omega)$, we can consider f as a C^3 function on S or on $\Omega\times\mathbb{R}$ by letting $f(z,u(z)) = f(z)$ and $f(z,t) = f(z)$ respectively. In particular, it holds that $\nabla^\varepsilon f = (Df, 0)$. We begin with the following preliminary result.

Proposition 19.2.1. *Let $u \in C^3(\Omega)$ be a classical solution to (ε-PMC) for $H \in L^\infty(\Omega)\cap C^1(\Omega)$. Then*

$$\Delta^{\varepsilon,S} u \geq -\|H\|_{L^\infty(|Om)}(|\varepsilon| + \max_{\overline{\Omega}} |\mathcal{F}|) - \frac{2}{|\varepsilon|}\max_{\overline{\Omega}} |\mathcal{F}|. \tag{19.2.2}$$

Proof. Since $Z_{2n+1}u \equiv 0$, (19.2.1) implies that

$$\begin{aligned}
\operatorname{div}\left(\frac{Du+\mathcal{F}}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\right) &= \frac{\Delta u}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}} + \left\langle Du+\mathcal{F}, D\left((\varepsilon^2+|Du+\mathcal{F}|^2)^{-\frac{1}{2}}\right)\right\rangle \\
&= \frac{\Delta u}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}} - \frac{\sum_{i,j=1}^{2n} D_iD_ju(Du+\mathcal{F})_i(Du+\mathcal{F})_j}{(\varepsilon^2+|Du+\mathcal{F}|^2)^{\frac{3}{2}}} \\
&= \frac{1}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\left(\sum_{i=1}^{2n+1} Z_iZ_iu - \sum_{i,j=1}^{2n+1} Z_iZ_ju\nu_i^\varepsilon\nu_j^\varepsilon\right) \\
&= \frac{1}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\left(\sum_{i,j=1}^{2n+1} g^{i,j}Z_iZ_ju\right),
\end{aligned}$$

and hence

$$H\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2} = \sum_{i,j=1}^{2n+1} g^{i,j}Z_iZ_ju.$$

Since u solves (ε-PMC), then our choice of ν^ε in (19.2.1) implies that $H^\varepsilon = -H$. Hence, by Proposition 16.4.1,

$$\begin{aligned}
\Delta^{\varepsilon,S}u &= H\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2} - \frac{H}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\langle Du, Du+\mathcal{F}\rangle + \frac{2}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\langle Du, J(\nu^\varepsilon)\rangle \\
&= \frac{H}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\left(\varepsilon^2+|Du+\mathcal{F}|^2-\langle Du, Du+\mathcal{F}\rangle\right) + \frac{2}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\langle Du, J(\nu^\varepsilon)\rangle \\
&= \frac{H}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\left(\varepsilon^2+\langle \mathcal{F}, Du+\mathcal{F}\rangle\right) - \frac{2}{\sqrt{\varepsilon^2+|Du+\mathcal{F}|^2}}\langle \mathcal{F}, J(\nu^\varepsilon)\rangle
\end{aligned}$$

Finally, (19.2.2) follows at once from the previous computation. ☐

We are ready to prove Theorem 19.1.2.

Proof of Theorem 19.1.2. Let $\tilde{\Omega}$ and $\hat{\Omega}$ be as in the statement, and let $r \in (0, d(\partial\hat{\Omega}, \partial\tilde{\Omega}))$ be fixed. In this way, $B(z_0, r) \Subset \tilde{\Omega}$ for any $z_0 \in \hat{\Omega}$. Fix then $z_0 \in \hat{\Omega}$. We set $\gamma_1 = \|u\|_{L^\infty(\tilde{\Omega})}$ and $\gamma_2 = \|H\|_{C_1(\bar{\tilde{\Omega}})}$. Let $\varphi \in C^\infty(\overline{B(z_0, r)})$ be the paraboloid centered at z_0 such that $\varphi(z_0) = u(z_0) - 1$ and $\varphi(z) = \gamma_1$ for any $z \in \partial B(z_0, r)$, and let $\gamma_3 = \|\varphi\|_{C^2(\overline{B(z_0,r)})}$. Notice that $\gamma_3 = \gamma_3(r, \gamma_1)$. We define

$$\eta(t) = \left(e^{Kt}-1\right)e^{-(\gamma_1+\gamma_3)K}$$

for any $t \in \mathbb{R}$, where $K > 0$ is a constant to be chosen later. Notice that $\eta \in C^\infty(\mathbb{R})$ and $0 \leqslant \eta((u-\varphi)^+(z)) \leqslant 1$ for any $z \in \overline{B(z_0, r)}$. Since the function $\Phi : \overline{B(z_0, r)} \longrightarrow \mathbb{R}$ defined by

$$\Phi(z) = \frac{\varepsilon \cdot \eta((u-\varphi)^+(z))}{\nu_{2n+1}^\varepsilon(z)}$$

is continuous, we can denote by M its maximum over $\overline{B(z_0, r)}$. Moreover, by the choice of φ, it holds that $\Phi \equiv 0$ on $\partial B(z_0, r)$ and $\Phi(z_0) > 0$. Therefore the maximum M is achieved at some point $\tilde{z} \in B(z_0, r)$. Since $M \geq \Phi(z_0) > 0$, then $(u-\varphi)^+ = u - \varphi$ locally near $\tilde{z}$ and Φ is of class C^2 in a neighborhood of $\tilde{z}$. In particular,

$$\Psi(z) := \eta((u-\varphi)^+(z)) - M\frac{\nu^\varepsilon_{2n+1}(z)}{\varepsilon} \leqslant 0$$

for any $z \in \overline{B(z_0, r)}$, and

$$\Psi(\tilde{z}) = 0, \qquad \nabla^{\varepsilon,S}\Psi(\tilde{z}) = 0 \qquad \text{and} \qquad \Delta^{\varepsilon,S}\Psi(\tilde{z}) \leqslant 0. \tag{19.2.3}$$

We claim that there exists $M_0 = M_0\left(\varepsilon, \|\mathcal{F}\|_{L^\infty(\Omega)}\right)$ such that, if $f \in C^2(\Omega)$ is any solution to (ε-PMC) satisfying

$$\|f\|_{L^\infty(B(z_0,r))} \leqslant \gamma_1, \tag{19.2.4}$$

and φ, M and $\tilde{z}$ are as above, then $M \geq M_0$ implies that

$$|\nabla^{\varepsilon,S}(f-\varphi)|^2(\tilde{z}) > \frac{\varepsilon^2}{2}. \tag{19.2.5}$$

Indeed, let $(f_k)_k$ be a sequence such that

$$M_k \geq k \qquad \text{and} \qquad |\nabla^{\varepsilon,S}(f_k - \varphi_k)|^2(\tilde{z}) \leqslant \frac{\varepsilon^2}{2}, \tag{19.2.6}$$

where M_k and $\tilde{z}_k$ are defined as above. Notice that

$$\sqrt{\varepsilon^2 + |Df_k + \mathcal{F}|^2(\tilde{z}_k)} \geq \Phi(\tilde{z}_k) \geq k$$

and, being $\mathcal{F}$ bounded over $\overline{\Omega}$, $|Df_k(\tilde{z}_k)|$ diverges to $+\infty$ as $k \to +\infty$. Moreover, by Proposition 16.4.1,

$$\begin{aligned}
|\nabla^{\varepsilon,S}(f_k - \varphi_k)|^2 &= |\nabla^\varepsilon(f_k-\varphi_k)|^2 - \langle \nabla^\varepsilon(f_k - \varphi_k), \nu^\varepsilon\rangle^2 \\
&= |Df_k - D\varphi_k|^2 - \frac{\langle Df_k - D\varphi_k, Df_k + \mathcal{F}\rangle^2}{\varepsilon^2 + |Df_k + \mathcal{F}|^2} \\
&\geq \frac{\varepsilon^2 |Df_k - D\varphi_k|^2}{\varepsilon^2 + |Df_k + \mathcal{F}|^2}.
\end{aligned}$$

Hence, since $(D\varphi_k(\tilde{z}_k))_k$ bounded by (19.2.4), we get

$$\liminf_{k\to+\infty} |\nabla^{\varepsilon,S}(f_k - \varphi_k)(\tilde{z}_k)|^2 \geq \lim_{k\to+\infty} \frac{\varepsilon^2|Df_k - D\varphi_k|^2(\tilde{z}_k)}{\varepsilon^2 + |Df_k + \mathcal{F}|^2(\tilde{z}_k)} = \varepsilon^2,$$

which contradicts (19.2.6). We claim that $M \leqslant M_0$ for suitable choices of K. Indeed, for a fixed $K > 0$, assume that $M > M_0$. Hence (19.2.5) holds. Notice that $(u-\varphi)^+ = u - \varphi$ locally around $\tilde{z}$. Notice that $H^\varepsilon(z,t) = -H(z)$ for any $z \in \Omega$ since u solves (ε-PMC), and H is extended vertically on $\Omega \times \mathbb{R}$. Exploiting Theorem 17.1.3, Proposition 17.2.1, (19.2.2), (19.2.3) and (19.2.5), and recalling that u solves (ε-PMC), we infer that, at $\tilde{z}$,

$$
\begin{aligned}
\Delta^{\varepsilon,S}\Psi &= \Delta^{\varepsilon,S}(\eta(u-\varphi)) - \frac{M}{\varepsilon}\Delta^{\varepsilon,S}\nu^{\varepsilon}_{2n+1} \\
&= \eta''|\nabla^{\varepsilon,S}(u-\varphi)|^2 + \eta'\left(\Delta^{\varepsilon,S}u - \Delta^{\varepsilon,S}\varphi\right) + \frac{M}{\varepsilon}\Big(g_\varepsilon(\nabla^{\varepsilon,S}H,\varepsilon T) + \nu^{\varepsilon}_{2n+1}\left(\mathrm{Ric}_\varepsilon(\nu^\varepsilon,\nu^\varepsilon) + |h^\varepsilon|^2\right)\Big) \\
&= \eta''|\nabla^{\varepsilon,S}(u-\varphi)|^2 + \eta'\left(\Delta^{\varepsilon,S}u - \Delta^{\varepsilon,S}\varphi\right) + \frac{M\nu^{\varepsilon}_{2n+1}}{\varepsilon}\Big(-g_\varepsilon(\nabla^\varepsilon H,\nu^\varepsilon) + \mathrm{Ric}_\varepsilon(\nu^\varepsilon,\nu^\varepsilon) + |h^\varepsilon|^2\Big) \\
&= \eta''|\nabla^{\varepsilon,S}(u-\varphi)|^2 + \eta'\left(\Delta^{\varepsilon,S}u - \Delta^{\varepsilon,S}\varphi\right) + \eta\Big(-g_\varepsilon(\nabla^\varepsilon H,\nu^\varepsilon) + \mathrm{Ric}_\varepsilon(\nu^\varepsilon,\nu^\varepsilon) + |h^\varepsilon|^2\Big) \\
&\geq \frac{\varepsilon^2}{2}\eta'' - C\eta' - C\eta
\end{aligned}
\tag{19.2.7}
$$

for some $C = C(n,\varepsilon,r,\gamma_2,\gamma_3) > 0$. Hence, since $\Delta^{\varepsilon,S}\Psi(\tilde{z}) \leqslant 0$ and up to choosing a different constant $C = C(n,\varepsilon,r,\gamma_2,\gamma_3) > 0$, we conclude that

$$
\eta''((u-\varphi)(\tilde{z})) - C\eta'((u-\varphi)(\tilde{z})) - C^2\eta((u-\varphi)(\tilde{z})) \leqslant 0. \tag{19.2.8}
$$

The choice $K = 2C$ is in contradiction with (19.2.8), so that, for this choice of K, we must have $M \leqslant M_0$. Hence

$$
\eta((u-\varphi)^+(z)) - M_0\frac{\nu^{\varepsilon}_{2n+1}(z)}{\varepsilon} \leqslant 0.
$$

Since $(u-\varphi)^+(z_0) = 1$, we conclude that

$$
|Du|(z_0) \leqslant \sqrt{\varepsilon^2 + |Du+\mathcal{F}|(z_0)^2} + |\mathcal{F}|(z_0) \leqslant \frac{M_0}{\eta(1)} + |\mathcal{F}|(z_0),
$$

whence the thesis follows. □

An approach as in Theorem 19.1.2 allows to reduce global gradient estimates for solutions to (ε-PMC) to boundary gradient estimates. For future convenience, we state the result for a slightly more general class of equations.

Theorem 19.2.2 (From boundary to global gradient estimates)**.** *Let $H \in C^1(\overline{\Omega})$ and $\sigma \in [0,1]$, and let $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ be a solution to*

$$
\operatorname{div}\left(\frac{Du+\sigma\mathcal{F}}{\sqrt{\varepsilon^2 + |Du+\sigma\mathcal{F}|^2}}\right) = \sigma H(z) \tag{19.2.9}
$$

on Ω. Then there exists $C = C\left(n,\varepsilon,\|u\|_{L^\infty(\Omega)},\|H\|_{C^1(\overline{\Omega})},\|\mathcal{F}\|_{L^\infty(\Omega)}\right) > 0$, thus independent of $\sigma \in [0,1]$, such that

$$
\|Du\|_{L^\infty(\Omega)} \leqslant C\Big(\|Du\|_{L^\infty(\partial\Omega)} + 1\Big).
$$

Proof. Let us set $\gamma_1 = \|u\|_{L^\infty(\Omega)}$. Let $\eta(t) = e^{K(t-\gamma_1)}$, where $K > 0$ is a fixed constant to be chosen later. Notice that

$$
e^{-2K\gamma_1} \leqslant \eta(u(z)) \leqslant 1 \tag{19.2.10}
$$

for any $z \in \overline{\Omega}$. Let $\Phi : \bar{\Omega} \to \mathbb{R}$ be defined by

$$\Phi(z) = \frac{\varepsilon(\eta(u(z))}{\nu^{\varepsilon}_{2n+1}(z)}.$$

Since $u \in C^1(\overline{\Omega})$, then $\Phi \in C(\overline{\Omega})$, so that Φ achieves its maximum M at some point $\tilde{z} \in \overline{\Omega}$. Suppose first $\tilde{z} \in \Omega$. Arguing as in the proof of Theorem 19.1.2, and thanks to (19.2.10), there exists $M_0 = M_0\left(\varepsilon, \|\mathcal{F}\|_{L^\infty(\Omega)}\right)$ such that, if $f \in C^2(\Omega)$ is any solution to (ε-PMC) satisfying (19.2.4), and M and $\tilde{z}$ are as above, then $M \geq M_0$ implies that

$$|\nabla^{\varepsilon,S} f|^2(\tilde{z}) > \frac{\varepsilon^2}{2}. \tag{19.2.11}$$

Repeating the computations of (19.2.7), and exploiting (19.2.11), we get that, for a particular choice of $K = K(n, \varepsilon, \|\mathcal{F}\|_{L^\infty(\Omega)}, \|H\|_{C^1(\bar{\Omega})}) > 0$, it holds $M \leqslant M_0$. Hence, we get that

$$\eta(u(z))\sqrt{\varepsilon^2 + |Du + \mathcal{F}|(z)^2} \leqslant \eta(u(\tilde{z}))\sqrt{\varepsilon^2 + |Du + \mathcal{F}|(\tilde{z})^2} \leqslant M_0$$

for any $z \in \overline{\Omega}$, so that, by (19.2.10),

$$|Du|(z) \leqslant \sqrt{\varepsilon^2 + |Du + \mathcal{F}|(z)^2} + |\mathcal{F}|(z) \leqslant M_0 e^{2K\gamma_1} + \|\mathcal{F}\|_{L^\infty(\Omega)}$$

for any $z \in \overline{\Omega}$. If instead $\tilde{z} \in \partial\Omega$, then

$$\begin{aligned} |Du|(z) &\leqslant e^{2K\gamma_1}\eta(u(z))\sqrt{\varepsilon^2 + |Du + \mathcal{F}|(z)^2} + |\mathcal{F}|(z) \\ &\leqslant e^{2K\gamma_1}\eta(u(\tilde{z}))\sqrt{\varepsilon^2 + |Du + \mathcal{F}|(\tilde{z})^2} + \|\mathcal{F}\|_{L^\infty(\Omega)} \\ &\leqslant e^{2K\gamma_1}\left(|\varepsilon| + \|Du\|_{L^\infty(\partial\Omega)} + \|\mathcal{F}\|_{L^\infty(\Omega)}\right) + \|\mathcal{F}\|_{L^\infty(\Omega)} \end{aligned}$$

for any $z \in \overline{\Omega}$, whence the thesis follows. □

As a corollary of Theorem 19.2.2, we provide global gradient estimates for solutions to the Dirichlet problem associated with (ε-PMC) when (19.1.2) holds.

Theorem 19.2.3 (Global gradient estimates)**.** *Let Ω be a bounded domain with boundary of class C^2. Let $H \in C^1(\overline{\Omega})$ be such that (19.1.2) holds. Let $\varphi \in C^2(\overline{\Omega})$, $\varepsilon \neq 0$ and $\sigma \in [0,1]$. Assume that there exist a constant $\gamma_1 = \gamma_1(n, \varepsilon, \Omega, \varphi, \mathcal{F}, H) > 0$, independent of $\sigma \in [0,1]$, such that any solution $u \in C^2(\overline{\Omega})$ to*

$$\begin{cases} \operatorname{div}\left(\frac{Du + \sigma\mathcal{F}}{\sqrt{\varepsilon^2 + |Du + \sigma\mathcal{F}|^2}}\right) = \sigma H(z) & \text{in } \Omega \\ u = \sigma\varphi & \text{in } \partial\Omega \end{cases} \tag{19.2.12}$$

satisfies $\|u\|_{L^\infty(\Omega)} \leqslant \gamma_1$. Then there exists $C = C\left(n, \varepsilon, \|\varphi\|_{C^2(\overline{\Omega})}, \gamma_1, \|H\|_{C^1(\overline{\Omega})}, \|\mathcal{F}\|_{L^\infty(\Omega)}\right) > 0$, *thus independent of $\sigma \in [0,1]$, such that any $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ solution to (19.2.12) satisfies*

$$\|Du\|_{L^\infty(\Omega)} \leqslant C.$$

Proof. In view of Theorem 19.2.2 we are left to provide boundary gradient estimates. Thanks to (19.1.2), and following [158, Section 6], the latter follow *verbatim* as in [158, Proposition 4.8]. □

19.3 Existence and regularity of t-graphs

19.3.1 Existence of minimizers: the non-extremal case

Throughout this subsection we fix a bounded domain $\Omega \subseteq \mathbb{R}^{2n}$ with Lipschitz boundary, and we consider $H \in L^\infty(\Omega)$ such that (18.1.4) holds and

$$\left| \int_\Omega H dz \right| < P(\Omega). \tag{***}$$

We stress that, even without imposing boundary conditions, (18.1.4) is a necessary condition to the existence of a solution $u \in C^2(\Omega)$ to (ε-PMC) in Ω. More precisely, arguing as in the Euclidean setting, the following result holds.

Theorem 19.3.1. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with Lipschitz boundary, let $H \in \mathrm{Lip}(\Omega)$ and assume that there exists $u \in C^2(\Omega)$ which solves (ε-PMC) in Ω. Then (18.1.4) holds.*

For any $\varphi \in L^1(\partial\Omega)$, we define the functional $\mathcal{I}_\varepsilon : BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_\varepsilon(v) = P_\varepsilon(E_v, \Omega \times \mathbb{R}) + \int_\Omega H u \, dz + \int_{\partial\Omega} |v - \varphi| \, d\mathcal{H}^{2n-1}, \tag{19.3.1}$$

where E_v is defined as in (16.8.7). The following result follows as [162, Lemma 1.1] (cf. also [192, Lemma 3.2]).

Lemma 19.3.2. *Assume that (18.1.4) and (***) hold. Then there exists $\delta > 0$ such that*

$$\left| \int_{\tilde{\Omega}} H \, dz \right| \leqslant (1-\delta) P(\tilde{\Omega}) \tag{19.3.2}$$

for every measurable set $\tilde{\Omega} \subseteq \Omega$.

The proof of the following proposition reproduces the argument of [154, Theorem 1.1].

Proposition 19.3.3. *Let $H \in L^\infty(\Omega)$ and assume that (18.1.4) and (***) hold. Then $\mathcal{I}_\varepsilon$ has a minimum in $BV(\Omega)$ for every $\varphi \in L^1(\partial\Omega)$.*

Proof. Let $B \subseteq \mathbb{R}^{2n}$ be a ball containing Ω such that the Euclidean distance between $\partial\Omega$ and ∂B is positive, and extend H to B by letting $H \equiv 0$ outside Ω. Fix a function $\phi \in W_0^{1,1}(B)$

with trace φ on $\partial\Omega$. Then minimizing $\mathcal{I}_\varepsilon$ is equivalent to minimize the functional

$$\mathcal{J}_\varepsilon(v) = P_\varepsilon(E_v, \Omega \times \mathbb{R}) + Var(v, B \setminus \Omega) + \int_B Hv\, dz \tag{19.3.3}$$

in $K = \{v \in BV(B) : v = \phi \text{ in } B \setminus \Omega\}$. Indeed, given v_0 and v in K with v_0 minimum of $\mathcal{J}_\varepsilon$ and writing $u_0 = v_0|_\Omega$ and $u = v|_\Omega$, it follows from [163, Remark 2.13] that

$$\mathcal{I}_\varepsilon(u) - \mathcal{I}_\varepsilon(u_0) = \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(v_0) \geq 0.$$

For a given $v \in K$, we define $v^+, v^- \in BV(B)$ by

$$v^+(z) = \begin{cases} \max\{0, v(z)\} & \text{if } z \in \Omega, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad v^-(z) = \begin{cases} \max\{0, -v(z)\} & \text{if } z \in \Omega, \\ 0 & \text{otherwise} \end{cases}.$$

Notice that

$$E_t := \{z \in B \,:\, v^+(z) > t\} = \{z \in \Omega \,:\, v(z) > t\} \subseteq \Omega \tag{19.3.4}$$

and

$$F_t := \{z \in B \,:\, v^-(z) > t\} = \{z \in \Omega \,:\, v(z) < -t\} \subseteq \Omega \tag{19.3.5}$$

for any $t > 0$. Hence, thanks to Lemma 19.3.2, (19.3.4), (19.3.5), the *layer-cake formula* (cf. [202, Remark 13.6]) and the *Coarea formula* (cf. [202, Theorem 13.1]), it follows that

$$\begin{aligned}
\int_B H(z)v(z)\, dz &= \int_\Omega H(z)v^+(z)\, dz - \int_\Omega H(z)v^-(z)\, dz \\
&= \int_0^{+\infty} dt \int_\Omega \chi_{E_t} H(z)\, dz - \int_0^{+\infty} dt \int_\Omega \chi_{F_t} H(z)\, dz \\
&\geq -(1-\delta) \int_0^{+\infty} P(E_t)\, dt - (1-\delta) \int_0^{+\infty} P(F_t)\, dt \\
&= -(1-\delta) \int_0^{+\infty} P(E_t, B)\, dt - (1-\delta) \int_0^{+\infty} P(F_t, B)\, dt \\
&\geq -(1-\delta) Var(v^+, B) - (1-\delta) Var(v^-, B).
\end{aligned}$$

Therefore, owing again to [163, Remark 2.13],

$$\begin{aligned}
&\int_B H(z)v(z)dz\\
&\quad \geq -(1-\delta)(Var(v^+,\Omega)+Var(v^-,\Omega)) - (1-\delta)\left(\int_{\partial\Omega}|v^+|\,d\mathcal{H}^{2n-1} + \int_{\partial\Omega}|v^-|\,d\mathcal{H}^{2n-1}\right)\\
&\quad = -(1-\delta)Var(v,\Omega) - (1-\delta)\int_{\partial\Omega}|v|\,d\mathcal{H}^{2n-1}\\
&\quad \geq -(1-\delta)Var(v,\Omega) - (1-\delta)\int_{\partial\Omega}|v-\varphi|\,d\mathcal{H}^{2n-1} - (1-\delta)\int_{\partial\Omega}|\varphi|\,d\mathcal{H}^{2n-1}\\
&\quad = -(1-\delta)Var(v,\overline{\Omega}) - (1-\delta)\int_{\partial\Omega}|\varphi|\,d\mathcal{H}^{2n-1}\\
&\quad \geq -(1-\delta)Var(v,B) - \int_{\partial\Omega}|\varphi|\,d\mathcal{H}^{2n-1}
\end{aligned} \tag{19.3.6}$$

On the other hand, it holds that

$$P_\varepsilon(E_v;\Omega\times\mathbb{R}) \geq Var(v,\Omega) - \int_\Omega |\mathcal{F}|\,dz. \tag{19.3.7}$$

Indeed, let $(v_k)_k \subseteq W^{1,1}(\Omega)$ be such that $v_k \to v$ in $L^1(\Omega)$. Then, for any fixed $k\in\mathbb{N}$, (16.8.8) implies that

$$P_\varepsilon(E_{v_k};\Omega\times\mathbb{R}) = \int_\Omega \sqrt{\varepsilon^2+|Dv_k+\mathcal{F}|^2}\,dz \geq \int_\Omega |Dv_k+\mathcal{F}|\,dz \geq \int_\Omega |Dv_k|\,dz - \int_\Omega |\mathcal{F}|\,dz$$

Hence, by the lower semicontinuity of $Var(\cdot,\Omega)$ with respect to the L^1-convergence, we conclude that

$$\int_\Omega |Dv|\,dz - \int_\Omega |\mathcal{F}|\,dz \leqslant \liminf_{k\to+\infty}\left(\int_\Omega |Dv_k|\,dz - \int_\Omega |\mathcal{F}|\,dz,\right) \leqslant \liminf_{k\to+\infty} P_\varepsilon(E_{v_k};\Omega\times\mathbb{R}),$$

from which (19.3.7) follows by Lemma 16.8.3. Substituting (19.3.6) and (19.3.7) in (19.3.3), we finally obtain

$$\mathcal{J}_\varepsilon(v) \geq \delta Var(v,B) - \int_\Omega |\mathcal{F}|\,dz - \int_{\partial\Omega}|\varphi|\,d\mathcal{H}^{2n-1} \tag{19.3.8}$$

Let $\{v_k\}\subseteq K$ be a minimizing sequence for $\mathcal{J}_\varepsilon$. By (19.3.8) and Poincaré's inequality, $\{v_k\}$ is bounded in $BV(B)$, so that, up to a subsequence, there exists $v_0\in BV(B)$ such that $v_k\to v_0$ in $L^1(B)$. Noticing that K is a closed with respect to the L^1 convergence on B, then $v_0\in K$. By the lower semicontinuity of the perimeter with respect to L^1 convergence, we get that v_0 is a minimizer of $\mathcal{J}_\varepsilon$ and hence $\mathcal{I}_\varepsilon$ has a minimizer in $BV(\Omega)$. □

19.3.2 Variational properties of minimizers

Throughout this subsection, we fix a bounded domain $\Omega\subseteq\mathbb{R}^{2n}$ and $H\in L^\infty_{loc}(\Omega)$.

Definition 19.3.4. *We say that a Caccioppoli set $E\subseteq\Omega\times\mathbb{R}$ is a (local) H-*minimizer *in $\Omega\times\mathbb{R}$ if*

$$P_\varepsilon(E,A) + \int_{E\cap A} H dx \leqslant P_\varepsilon(F,A) + \int_{F\cap A} H dx \tag{19.3.9}$$

for any open set $A \Subset \Omega \times \mathbb{R}$ and any measurable set F such that $E \Delta F \Subset A$, where $H(z,t) = H(z)$.

In the following theorem we prove that subgraphs of minimizers of $\mathcal{I}_\varepsilon$ are H-minimizers (cf. [132] for an exhaustive account of this and related results in the Euclidean setting). Notice that, if $u \in BV(\Omega)$ is a minimizer for $\mathcal{I}_\varepsilon$, then

$$\int_{\tilde{\Omega}} \sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2} + \int_{\tilde{\Omega}} Hu\, dz \leqslant \int_{\tilde{\Omega}} \sqrt{\varepsilon^2 + |Dv + \mathcal{F}|^2} + \int_{\tilde{\Omega}} Hv\, dz \tag{19.3.10}$$

for any open set $\tilde{\Omega} \Subset \Omega$ and any $v \in BV_{loc}(\Omega)$ such that $\{u \neq v\} \Subset \tilde{\Omega}$. When $u \in BV_{loc}(\Omega)$ satisfies (19.3.10), we refer to it as *(local) H-minimizer*. The ambiguity with Definition 19.3.4 is motivated by the following result.

Theorem 19.3.5. *Let $u \in BV_{loc}(\Omega)$. Then u satisfies (19.3.10) if, and only if, E_u is an H-minimizer in $\Omega \times \mathbb{R}$.*

Proof. Assume first that $u \in BV_{loc}(\Omega)$ satisfies (19.3.10). Let $A \Subset \Omega \times \mathbb{R}$ be open, and let $\tilde{\Omega} \Subset \Omega$ be an open set such that $A \Subset \tilde{\Omega} \times \mathbb{R}$. let now F be a measurable set such that $F \Delta E_u \Subset A$. We can assume without loss of generality that F has finite perimeter in A. Since E_u is a subgraph, we infer that

$$\lim_{t \to +\infty} \chi_F(z,t) = 0 \qquad \text{and} \qquad \lim_{t \to -\infty} \chi_F(z,t) = 1$$

for a.e. $z \in \Omega$. Inspired by [163, Lemma 14.7], we set $w \in BV_{loc}(\Omega)$ the function

$$w(z) = \lim_{k \to \infty} w_k(z) = \lim_{k \to \infty} \left(\int_{-k}^{k} \chi_F(z,t)\, dt - k \right).$$

We claim that

$$P_\varepsilon(E_w; \tilde{\Omega} \times \mathbb{R}) \leqslant P_\varepsilon(F, \tilde{\Omega} \times \mathbb{R}) \tag{19.3.11}$$

and

$$\int_{\Omega' \times \mathbb{R}} (\chi_{E_w} - \chi_F)\, dz\, dt = 0 \tag{19.3.12}$$

for any open set $\Omega' \Subset \tilde{\Omega}$. Indeed, assume first that $\exists L > 0$ such that

$$\Omega \times (-\infty, -L) \subseteq F \subseteq \Omega \times (-\infty, L). \tag{19.3.13}$$

It is clear that $-L \leqslant w \leqslant L$ and $w(z) = -L + \int_{-L}^{L} \chi_F(z,t)\, dt$. Let $\eta : \mathbb{R} \to [0,1]$ be a smooth cut-off function between $[-L, L]$ and $[-L-1, L+1]$. From a direct computation, we get that for a.e. $z \in \Omega$, it holds

$$\int_{\mathbb{R}} \chi_F(z,t)\eta(t)\, dt = w(z) + \alpha \qquad \text{and} \qquad \int_{\mathbb{R}} \chi_F(z,t)\eta'(t)\, dt = 1, \tag{19.3.14}$$

where $\alpha = L + \int_{-L-1}^{L} \eta(t)\,dt$. Let $g = (\tilde g, g_{2n+1}) \in C_c^1(\Omega, \mathbb{R}^{2n+1})$ be such that $|g| \leqslant 1$. Let us set

$$W|_{(z,t)} = \sum_{j=1}^{2n} \eta(t) g_j(z) Z_j|_{(z,t)}.$$

Then $W \in C_c^1(\Omega \times \mathbb{R}, T\mathbb{H}^n)$ and $|W|_\varepsilon \leqslant 1$. Therefore (19.3.14) implies that

$$\begin{aligned}
P_\varepsilon(F, \Omega \times \mathbb{R}) &\geq \int_{\Omega\times\mathbb{R}} \chi_F(z,t) \operatorname{div} W(z,t)\,dt\,dz \\
&= \int_\Omega \int_{\mathbb{R}} \chi_F(z,t) \left(\eta(t) \operatorname{div} \tilde g(z) - \eta'(t)\langle \tilde g(z), \mathcal{F}(z)\rangle + \eta'(t)\varepsilon g_{2n+1}(z)\right) dt\,dz \\
&= \int_\Omega \Big(\operatorname{div} \tilde g(z) \int_{\mathbb{R}} \chi_F(z,t)\eta(t)\,dt - \langle \tilde g(z), \mathcal{F}(z)\rangle \int_{\mathbb{R}} \chi_F(z,t)\eta'(t)\,dt \\
&\quad + \varepsilon g_{2n+1} \int_{\mathbb{R}} \chi_F(z,t)\eta'(t)\,dt \Big)\,dz \\
&= \int_\Omega (w(z) + \alpha) \operatorname{div} \tilde g(z) - \langle \tilde g(z), \mathcal{F}(z)\rangle + \varepsilon g_{2n+1}(z)\,dz \\
&= \int_\Omega w(z) \operatorname{div} \tilde g(z) - \langle \tilde g(z), \mathcal{F}(z)\rangle + \varepsilon g_{2n+1}(z)\,dz,
\end{aligned}$$

where we used the fact that $\operatorname{supp}(\tilde g) \subseteq \Omega$. Hence, assuming (19.3.13), (19.3.11) holds by Lemma 16.8.3. Moreover,

$$\begin{aligned}
\int_{\tilde\Omega} H(u-w)dz &= \int_{\tilde\Omega\cap\{u\geq w\}} H(u-w)dz - \int_{\tilde\Omega\cap\{u<w\}} H(w-u)dz \\
&= \int_{\tilde\Omega\cap\{u\geq w\}} H(|E_u(z)\cap A(z)| - |E_w(z)\cap A(z)|)dz \\
&\quad - \int_{\tilde\Omega\cap\{u<w\}} H(|E_w(z)\cap A(z)| - |E_u(z)\cap A(z)|)dz \\
&= \int_{(\tilde\Omega\cap\{u\geq w\})\times\mathbb{R}} H(\chi_{E_u\cap A} - \chi_{E_w\cap A})dx - \int_{\tilde\Omega\cap\{u<w\}} H(\chi_{E_w\cap A} - \chi_{E_u\cap A})dx \\
&= \int_{\tilde\Omega\times\mathbb{R}} H(\chi_{E_u\cap A} - \chi_{E_w\cap A})dx,
\end{aligned} \tag{19.3.15}$$

where $E_v(z) = \{t \in \mathbb{R} : v(z) > t\}$ and $A(z) = \{t \in \mathbb{R} : (z,t) \in A\}$. To drop (19.3.13), one can argue exactly as in the proof of [163, Theorem 14.8]. Finally, (19.3.12) follows *verbatim* as in the proof of [213, Teorema 2.3]. In view of (19.3.11), (19.3.12) and (19.3.15), arguing as in [163, Theorem 14.9] and [214] E_u is an H-minimizer in $\Omega \times \mathbb{R}$. Being the converse implication fairly straightforward, the thesis follows. □

Clearly, since $H \in L^\infty_{loc}(\Omega)$, H-minimizer sets are *almost minimizers* in the sense of [8] for suitable *anisotropic energies*. More precisely, if E is an H-minimizer in $\Omega \times \mathbb{R}$, $\tilde\Omega \Subset \Omega$ and $\|H\|_{L^\infty(\tilde\Omega)} = H_0$, then

$$P_\varepsilon(E, A) \leqslant P_\varepsilon(F, A) + H_0|E\Delta F| \tag{19.3.16}$$

for any open set $A \Subset \tilde\Omega \times \mathbb{R}$ and any measurable set $F \subseteq \Omega \times \mathbb{R}$ such that $E\Delta F \Subset A$. For almost minimizers in the previous sense, arguing as in the Euclidean setting (cf. [202]) it is possible to derive *uniform density estimates* for both volume and perimeter, in analogy with those exploited in Chapter 15. In view of (16.8.3), the following well-known estimates follow from for instance from [266, Proposition 4.5].

Proposition 19.3.6. *For any domain $\tilde{\Omega} \Subset \Omega$, there exist $c_0 = c_0(\varepsilon) > 0$, $c_1 = c_1(\varepsilon) > 0$, $c_2 > 0$ independent of $\varepsilon \in (0, 1]$ and $r_0 > 0$ such that for any H-minimizer E, any $p \in \overline{\partial^* E} \cap (\tilde{\Omega} \times \mathbb{R})$ and any $r < r_0$, it holds*

$$c_0 r^{2n+1} \leqslant \min\{|E \cap B(p, r)|, |E \setminus B(p, r)|\} \tag{19.3.17}$$

and

$$c_1 r^{2n} \leqslant P_\varepsilon(E, B(p, r)) \leqslant c_2 r^{2n}, \tag{19.3.18}$$

where $B(p, r)$ is the Euclidean ball centered at p of radius r.

The local boundedness of minimizers is a consequence of Theorem 19.3.5 and Proposition 19.3.6, and it follows as its Euclidean counterpart (cf. [163, Theorem 14.10]).

Proposition 19.3.7. *Let $u \in BV(\Omega)$ be a minimizer of $\mathcal{I}_\varepsilon$. Then $u \in L^\infty_{loc}(\Omega)$.*

19.3.3 Higher regularity of Lipschitz continuous t-graphs

In the rest of this section, inspired by [163], we show classical regularity for minimizers of $\mathcal{I}_\varepsilon$. The main difficulty consists in obtaining Lipschitz regularity since, as soon as a minimizer is Lipschitz, standard regularity results for uniformly elliptic equations apply. More precisely, the following regularity property for Lipschitz weak solutions to (ε-PMC) holds.

Theorem 19.3.8. *Let $H \in \mathrm{Lip}_{loc}(\Omega)$ and $u \in \mathrm{Lip}_{loc}(\Omega)$ be a weak solution on Ω to (ε-PMC). Then $u \in C^{2,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0, 1)$ and is a classical solution to (ε-PMC). Moreover, if $H \in C^{k,\gamma}_{loc}(\Omega)$ for some $k \geq 1$ and $\gamma \in (0, 1)$, then $u \in C^{k+2,\gamma}_{loc}(\Omega)$. Finally, if $H \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Proof. Let $\alpha \in (0, 1)$. Let $\tilde{\Omega} \Subset \Omega$ be a bounded domain. It suffices to show that $u \in C^{2,\alpha}(\tilde{\Omega})$. For $r > 0$, set

$$\tilde{\Omega}_r = \left\{ z \in \tilde{\Omega} \, : \, \min_{w \in \partial\Omega} |z - w| > r \right\}.$$

Given $\psi \in C^\infty_c(\tilde{\Omega})$, we can assume that $\psi \in C^\infty_c(\tilde{\Omega}_r)$ for $r > 0$ small enough. Let $v \in B(0, r)$. Using ψ and $\psi(\cdot - v)$ as test functions the weak formulation of (ε-PMC), we get

$$\int_{\tilde{\Omega}} \langle A(z + v, Du(z + v)) - A(z, Du(z)), D\psi(z) \rangle \, dz = \int_{\tilde{\Omega}} (H(z) - H(z + v))\psi(z) \, dz, \tag{19.3.19}$$

where

$$A(z, \xi) = \frac{\xi + \mathcal{F}(z)}{\sqrt{\varepsilon^2 + |\xi + \mathcal{F}(z)|^2}}.$$

Fixed $z_0 \in \tilde{\Omega}_r$, the fundamental theorem of calculus implies that

$$A(z_0 + v, Du(z_0 + v)) - A(z_0, Du(z_0)) = \int_0^1 \frac{d(A(\alpha_{z_0}(s))}{ds} dt, \tag{19.3.20}$$

where $\alpha_{z_0} : [0,1] \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is given by

$$\alpha_{z_0}(t) = (z_0 + tv, Du(z_0) + t(Du(z_0+v) - Du(z_0))).$$

Writing $u_v(z) = \frac{u(z+v)-u(z)}{|v|}$, it follows from a direct computation that

$$\frac{d(A(\alpha_z(t))}{dt} = A_z(\alpha_z(t)) \cdot v + |v| A_\xi(\alpha_z(t)) \cdot Du_v(z), \tag{19.3.21}$$

where A_ξ is the matrix with entries $(A_\xi)_{ij} = \frac{\partial A_j}{\partial z_i}$ given by

$$(A_\xi)_{ij}(z,\xi) = \frac{\delta_{ij}(\varepsilon^2 + |\xi + \mathcal{F}(z)|^2) - (\xi_i + \mathcal{F}(z)_i)(\xi_j + \mathcal{F}(z)_j)}{(\varepsilon^2 + |\xi + \mathcal{F}(z)|^2)^{\frac{3}{2}}}. \tag{19.3.22}$$

We set

$$\begin{aligned} \tilde{A}(z) &= \int_0^1 A_\xi(\alpha_z(t))\, dt, \\ \tilde{B}_v(z) &= \left(\int_0^1 A_z(\alpha_z(t))\, dt \right) \cdot \frac{v}{|v|} \\ \tilde{H}_v(z) &= \frac{H(z) - H(z+v)}{|v|}. \end{aligned}$$

Inserting (19.3.20) and (19.3.21) in (19.3.19) and dividing by $|v|$, we get

$$\int_{\tilde{\Omega}} \langle \tilde{A}(z) \cdot Du_v(z), D\psi(z) \rangle\, dz = \int_{\tilde{\Omega}} \tilde{H}_v(z) \psi(z)\, dz - \int_{\tilde{\Omega}} \left\langle \tilde{B}_v(z), D\psi(z) \right\rangle\, dz.$$

In other words, u_v is a weak solution on $\tilde{\Omega}$ to the linear equation

$$\operatorname{div}\left(\tilde{A} \cdot Du_v\right) - \tilde{H}_v - \operatorname{div} \tilde{B}_v.$$

Since $H \in \mathrm{Lip}(\tilde{\Omega})$, the coefficients of $\tilde{A}$, $\tilde{B}_v$ and $\tilde{H}_v$ are uniformly bounded with respect to v. Moreover, since $u \in \mathrm{Lip}(\tilde{\Omega})$, (19.3.22) implies that $\tilde{A}$ is uniformly elliptic on $\tilde{\Omega}$. Finally, since $u \in \mathrm{Lip}(\tilde{\Omega})$, then u_v is bounded in $L^\infty(\tilde{\Omega})$ uniformly for $v \in B_r$ with $r > 0$ small enough. Therefore, using the celebrated De Giorgi-Nash-Moser method (cf. [80, Chapter 4, Theorem 2.3] and cf. the proof of the Corollary right after [80, Chapter 4, Theorem 2.2]), we conclude that, up to a smaller $\tilde{\Omega}$, u_v is bounded in $C^{0,\alpha}(\tilde{\Omega})$ uniformly for $v \in B_r$, so that $u \in C^{1,\alpha}(\tilde{\Omega})$ and $u_v \in C^{1,\alpha}(\tilde{\Omega})$. In particular, $\tilde{A}$ and $\tilde{B}_v$ are of class $C^{0,\alpha}$. Therefore, in view of [157, Theorem 8,32] and up to a smaller $\tilde{\Omega}$, u_v is bounded in $C^{1,\alpha}(\tilde{\Omega})$ uniformly for $v \in B_r$, so that $u \in C^{2,\alpha}(\tilde{\Omega})$. In particular, u is a classical solution to (ε-PMC). Let $k \geq 1$ and assume $H \in C^{k,\gamma}_{loc}(\Omega)$. Since u is a weak solution to (ε-PMC), by means of [80, Chapter 12, Theorem 1.1] we infer that, for any $j = 1, \ldots, 2n$, $g := (Du)_j$ is a weak solution to the linear equation

$$\operatorname{div}\left(\frac{\partial A}{\partial \xi}(z, Du) \cdot Dg\right) = D_j H - \operatorname{div}\left(\frac{\partial A}{\partial z_j}(z, Du)\right).$$

The thesis then follows exploiting the classical Schauder's theory (cf. [155, Theorem 5.20]). □

19.3.4 The Dirichlet problem for (ε-PMC)

As a corollary of the global gradient estimates, we extend the existence result obtained in Theorem 18.10.2 for the sub-Finsler constant mean curvature equation to the existence of solutions for the sub-Riemannian prescribed, but not necessarily constant, mean curvature equation.

Proof of Theorem 19.1.3. Assume first that $H \in C^{1,\alpha}(\overline{\Omega})$. Arguing exactly as in Chapter 18, it suffices to provide *a priori* estimates in $C^1(\overline{\Omega})$ for solutions $u \in C^{2,\alpha}(\overline{\Omega})$ to (19.2.9) with boundary datum φ which are independent of u and $\sigma \in [0,1]$. First, by means of Lemma 19.3.2, uniform estimates in $L^\infty(\Omega)$ follow as in Proposition 18.8.6. Moreover, arguing *verbatim* as in Theorem 19.3.8, any solution $u \in C^{2,\alpha}(\overline{\Omega})$ to (19.2.9) belongs to $C^3(\Omega) \cap C^1(\overline{\Omega})$. Hence, Theorem 19.2.3 provides global gradient estimates uniform with respect to $\sigma \in [0,1]$. Assume now that $H \in \mathrm{Lip}(\Omega)$, and denote by H_0 its Lipschitz constant. By McShane's extension theorem, we can suppose that $H \in \mathrm{Lip}(\mathbb{R}^{2n})$ with the same Lipschitz constant H_0. By a standard mollification argument, there exists a sequence $(H_j)_j \subseteq C^\infty(\overline{\Omega})$ such that

$$H_j \to H \text{ uniformly on } \overline{\Omega} \qquad \text{and} \qquad \|H_j\|_{C^1(\overline{\Omega})} \leqslant H_0 + \|H\|_{L^\infty(\Omega)} + 1 \tag{19.3.23}$$

for any $j \in \mathbb{N}$. Since H and Ω satisfy (18.1.4) and (***), we let δ be as in the statement of Lemma 19.3.2. Let us denote by $h(\Omega)$ the *Cheeger constant* of Ω (cf. [188]), that is

$$h(\Omega) = \inf\left\{\frac{P(A)}{|A|} \,:\, A \subseteq \Omega,\, |A| > 0\right\}.$$

Being Ω bounded and open, it is well known (cf. [188, Proposition 3.5]) that $h(\Omega) > 0$. By (19.3.23), we can assume up to a subsequence that

$$\|H - H_j\|_{L^\infty(\Omega)} \leqslant \frac{\delta h(\Omega)}{2},$$

so that

$$\left|\int_A H_j dz\right| \leqslant \left|\int_A H dz\right| + |A| \|H - H_j\|_{L^\infty(\Omega)} \leqslant \left(1 - \frac{\delta}{2}\right) P(A) \tag{19.3.24}$$

for any $A \subseteq \Omega$ such that $|A| \neq 0$. On the other hand, by (19.1.2) and (19.3.23), there exists $C = C(\Omega, \|H\|_{L^\infty(\Omega)}) > 0$ such that, up to a subsequence,

$$|H_j(z_0)| \leqslant H_{\partial\Omega}(z_0) - C \tag{19.3.25}$$

for any $z_0 \in \partial\Omega$ and any $j \in \mathbb{N}$. Combining (19.3.24) and (19.3.25), from the previous step we get a solution $u_j \in C^{2,\alpha}(\overline{\Omega})$ to (ε-PMC) with boundary datum φ and source H_j. Again by (19.3.23), (19.3.24) and (19.3.25), and following Proposition 18.8.6, Proposition 18.8.8 and Theorem 19.2.3, $(u_j)_j$ is uniformly bounded in $C^1(\overline{\Omega})$, so that, by Ascoli-Arzelà Theorem, there exists $u \in \mathrm{Lip}(\Omega)$ such that $u_j \to u$ uniformly on $\overline{\Omega}$. First, notice that $u = \varphi$ on $\partial\Omega$. Finally, a compactness argument as the forthcoming Theorem 19.5.4, coupled with Theorem 19.3.5 and Theorem 19.3.8, implies that $u \in C^{2,\alpha}_{loc}(\Omega)$ and that u is a classical solution to (ε-PMC), whence the thesis follows. □

19.3.5 Lipschitz regularity of t-graphs

Throughout this subsection, we fix a bounded domain $\Omega \subseteq \mathbb{R}^{2n}$ with Lipschitz boundary and $H \in C^{1,\gamma}_{loc}(\Omega)$ for some $\gamma \in (0,1)$. We are left to show that BV-minimizers of $\mathcal{I}_\varepsilon$ are locally Lipschitz continuous.

Proposition 19.3.9. *Let $H \in C^{1,\gamma}_{loc}(\Omega)$ and let $u \in BV(\Omega)$ be such that E_u is an H-minimizer for P_ε on $\Omega \times \mathbb{R}$. Then $\partial^* E_u$ is a $C^{3,\gamma}$ manifold and $\mathcal{H}^s(\partial E \setminus \partial^* E) = 0$ for any $s > 2n - 7$.*

Proof. Let u be as in the statement. Let $\tilde{\Omega} \Subset \Omega$ be an open set. Fix $\alpha \in (0,1)$ and let $F \Subset \tilde{\Omega} \times \mathbb{R}$ be a finite perimeter set such that $E \Delta F \Subset B(p,r) \Subset \tilde{\Omega} \times \mathbb{R}$ for some $p \in \tilde{\Omega} \times \mathbb{R}$ and $r \in (0,1)$. Taking $H_0 = \|H\|_{L^\infty(\tilde{\Omega})}$, it follows that

$$P_\varepsilon(E, B(p,r)) \leqslant P_\varepsilon(F, B(p,r)) + H_0 |E \Delta F| \leqslant P_\varepsilon(F, B(p,r)) + H_0 w_n r^{2n+\alpha}.$$

Moreover, for any Caccioppoli set F in $\Omega \times \mathbb{R}$ and any open set $A \subseteq \Omega \times \mathbb{R}$ it holds that

$$P_\varepsilon(F, A) = \int_{\partial^\star F \cap A} |\mathcal{C}_\varepsilon(p)\nu(p)|\, d\mathcal{H}^{2n}(p) = \int_{\partial^\star F \cap A} \sqrt{\langle M_\varepsilon(p)\nu(p), \nu(p)\rangle}\, d\mathcal{H}^{2n}(p),$$

where $M_\varepsilon(p) = \mathcal{C}_\varepsilon(p) \cdot \mathcal{C}_\varepsilon(p)^T$ for any $p \in \Omega \times \mathbb{R}$, $\mathcal{C}^\varepsilon$ is the coefficient matrix associated with the family $(Z_1, \ldots, Z_{2n+1})$ as in (1.1) and ν is the measure theoretic Euclidean unit normal to F. It is easy to check that M_ε is uniformly positive definite and α-Hölder continuous. Then [266, Theorem 1.1] implies that E_u is a manifold of class $C^{1,\frac{\alpha}{4}}$, and moreover $\mathcal{H}^s(\partial E \setminus \partial^* E) = 0$ for any $s > 2n - 7$. Fix $0 < \alpha < 1$ such that $\frac{\alpha}{4} \leqslant \gamma$. Since S is $C^{1,\frac{\alpha}{4}}$, for any $p \in S$ we can consider an open neighborhood U of p in the tangent hyperplane of S at p where S coincides with $\Phi_{\tilde{u}}(U) := \{q - \nu(p)\tilde{u}(q) : q \in U\}$ for a suitable function $\tilde{u}$ of class $C^{1,\frac{\alpha}{4}}$. Given $v \in W^{1,1}(U)$, we let $A(v)$ be the area of $\Phi_v(U)$ with respect to the metric g_ε. By Theorem 19.3.5, $\tilde{u}$ is a critical point of the functional

$$I(v) = A(v) + \int_U \tilde{H} v, \tag{19.3.26}$$

for a suitable $\tilde{H} \in C^{1,\frac{\alpha}{4}}_{loc}(U)$. Writing A in local coordinates, and writing the Euler-Lagrange equation associated to (19.3.26) for $\tilde{u}$, one can argue exactly as in the proof of Theorem 19.3.8 to conclude that $\tilde{u} \in C^{2,\beta}_{loc}(U)$ for any $\beta \in (0,1)$. Since $\tilde{H} \in C^{1,\gamma}_{loc}(U)$, the thesis follows as in Theorem 19.3.8. □

Let $\pi : \mathbb{H}^n \to \Omega$ be the projection on the first $2n$ components (cf. (3.2.2)) and $u \in BV(\Omega)$. We denote

$$\Omega_{u,0} = \pi(\partial E_u \setminus \partial^\star E_u). \tag{19.3.27}$$

The following standard first variation formula for intrinsic graphs follows as in [35, Section 3.1].

Lemma 19.3.10. *Let $u \in BV(\Omega)$ be such that E_u is an H-minimizer for P_ε on $\Omega \times \mathbb{R}$, and set $S = \mathrm{graph}(u)$. Assume that S is the Y_1-graph of $\varphi \in C^2(U)$, where $U \subseteq \mathbb{R}^{2n}$. Then*

$$-\sum_{j=1}^{2n} \int_U \frac{W_j^\varphi \varphi \left(W_j^\varphi\right)^\star \psi}{\sqrt{1+|W^{\varphi,\varepsilon}\varphi|^2}}\, dw = \int_U H(x_1, \dots, x_n, \varphi(w), y_2, \dots, y_n)\psi \, dw \tag{19.3.28}$$

for any $\psi \in C_c^\infty(U)$.

Proposition 19.3.11. *Let $H \in C^{1,\gamma}_{loc}(\Omega)$ and let $u \in BV(\Omega)$ be such that E_u is an H-minimizer for P_ε on $\Omega \times \mathbb{R}$. Then $u \in C^{3,\gamma}_{loc}(\Omega \setminus \Omega_{u,0})$ and $\mathcal{H}^s(\Omega_{u,0}) = 0$ for any $s > 2n-7$.*

Proof. Let $z = (\bar{x}, \bar{y}) \in \Omega \setminus \Omega_{u,0}$, where $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ and $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$. By Proposition 19.3.9, $S = \mathrm{graph}(u)$ is a hypersurface of class $C^{3,\gamma}$ near $p = (z, u(z))$, and $\mathcal{H}^s(\Omega_{u,0}) = 0$ for any $s > 2n-7$. Hence it suffices to show that $g_\varepsilon(T, \nu^\varepsilon) \neq 0$ on $\Omega \setminus \Omega_{u,0}$, where ν^ε is the outer unit normal of S. Indeed, assume by contradiction that $g_\varepsilon(T, \nu^\varepsilon(p)) = 0$. Then $\nu^\varepsilon(p) \in \mathcal{H}_p$, and so p is non-characteristic, since otherwise $\nu^\varepsilon(p) \in \mathcal{H}_p = T_pS$. Therefore, by the implicit function theorem for intrinsic graphs (cf. [140, Theorem 6.5]) S is, locally near p, a Y_1-graph with respect to a continuous function φ defined on an open neighborhood $U \subseteq \mathbb{R}^{2n}$ of $\bar{w} = (\bar{x}_1, \dots, \bar{x}_n, \bar{y}_2, \dots, \bar{y}_n, \bar{t} + \bar{x}_1\bar{y}_1)$. Moreover, by means of Proposition 16.7.3, $\varphi \in C^{3,\gamma}(U)$. Since S is a t-graph vertical at p, up to choosing a smaller U, we infer that

$$\tilde{T}\varphi(\bar{w}) = 0 \qquad \text{and} \qquad \tilde{T}\varphi(w) \geq 0 \tag{19.3.29}$$

for any $w \in U$. If $W^{\varphi,\varepsilon}$ is defined as in (16.7.4), in the following we drop the superscript for the sake of clarity. Taking $\psi = W_{2n}\psi$ in (19.3.28) and using (16.7.5) and the definition of adjoint operator, we infer that

$$\begin{aligned}
\int_U \frac{\partial H}{\partial y_1} W_{2n}\varphi\psi \, dw &= -\int_U H W_{2n}\psi \, dw \\
&= \sum_{j=1}^{2n} \int_U \frac{W_j\varphi W_j^\star\left(W_{2n}\psi\right)}{\sqrt{1+|W\varphi|^2}}\, dw \\
&= \sum_{j=1}^{2n} \int_U W_{2n}\left(\frac{W_j\varphi}{\sqrt{1+|W\varphi|^2}}\right) W_j\psi \, dw \\
&= \sum_{j,k=1}^{2n} \int_U \left\{ \frac{W_{2n}\left(W_j\varphi\right)}{\sqrt{1+|W\varphi|^2}} - \frac{W_{2n}\left(W_k\varphi\right) W_k\varphi W_j\varphi}{(1+|W\varphi|^2)^{\frac{3}{2}}} \right\} W_j\psi \, dw.
\end{aligned}$$

Setting $g = W_{2n}\varphi$, we get

$$0 = \sum_{j,k=1}^{2n} \int_U \left\{ \left(\frac{W_j^\star g}{\sqrt{1+|W\varphi|^2}} - \frac{W_k^\star g W_k\varphi W_j\varphi}{(1+|W\varphi|^2)^{\frac{3}{2}}} \right) W_j\psi + \frac{\partial H}{\partial y_1} g\psi \right\} dw,$$

that can be rewritten as

$$\int_U W\psi \cdot A \cdot (Wg)^T + \langle W\psi, B\rangle g - \frac{\partial H}{\partial y_1} g\psi \, dw = 0,$$

where $A = (A_{jk})_{jk}$ and $B = (B_j)_j$ are defined by

$$A_{jk} = \frac{(1 + |W\varphi|^2)\,\delta_{jk} - W_j\varphi W_k\varphi}{(1 + |W\varphi|^2)^{\frac{3}{2}}} \qquad \text{and} \qquad B_j = 2\tilde{T}|\varphi A_{j1}$$

for any $j, k = 1, \ldots, 2n$. Since $W\varphi$ is continuous, an easy computation shows that, up to choosing a smaller U, there exists $\alpha > 0$ such that $\xi \cdot A(w) \cdot \xi^T \geq \alpha|\xi|^2$ for any $w \in U$ and any $\xi \in \mathbb{R}^{2n}$. Moreover, as $Wf = Df \cdot (\mathcal{C}^{\varphi,\varepsilon})^T$ for any $f \in C^1(U)$, being $\mathcal{C}^{\varphi,\varepsilon}$ the coefficient matrix of the family $W^{\varphi,\varepsilon}$ as in (1.1), we infer that

$$\int_U D\psi \cdot \tilde{A} \cdot (Dg)^T + \langle D\psi, \tilde{B}\rangle g - \frac{\partial H}{\partial y_1} g\psi \, dw = 0,$$

where

$$\tilde{A} = (\mathcal{C}^{\varphi,\varepsilon})^T \cdot A \cdot \mathcal{C}^{\varphi,\varepsilon} \qquad \text{and} \qquad \tilde{B} = B \cdot \mathcal{C}^{\varphi,\varepsilon}.$$

Being $\mathcal{C}^{\varphi,\varepsilon}(w)$ invertible and continuous in w, up to restricting U there exists $\tilde{\alpha} > 0$ such that

$$\xi \cdot \tilde{A} \cdot \xi^T = \xi \cdot (\mathcal{C}^{\varphi,\varepsilon})^T \cdot A \cdot \mathcal{C}^{\varphi,\varepsilon} \cdot \xi^T = \xi \cdot (\mathcal{C}^{\varphi,\varepsilon})^T \cdot A \cdot \left(\xi \cdot (\mathcal{C}^{\varphi,\varepsilon})^T\right)^T \geq \alpha|\xi \cdot (\mathcal{C}^{\varphi,\varepsilon})^T|^2 \geq \tilde{\alpha}|\xi|^2$$

on U for any $\xi \in \mathbb{R}^{2n}$. Hence $\tilde{A}$ is uniformly elliptic on U. Therefore, recalling (19.3.29) and choosing a suitable smaller neighborhood U, we can apply a strong maximum principle as in [246, page 73, Theorem 10] to conclude that $\tilde{T}\varphi \equiv 0$ on U. In particular, $g_\varepsilon(T, \nu^\varepsilon) = 0$ in a neighborhood O of p and $\mathcal{H}^{2n-1}(V) > 0$, where $V = \pi(O)$. Arguing *verbatim* as in [163, page 169], it follows that $V \subseteq \Omega_{u,0}$, which is a contradiction with $\mathcal{H}^{2n-1}(\Omega_{u,0}) = 0$. □

Proposition 19.3.12. *Let $H \in C^{1,\gamma}_{loc}(\Omega)$ and let $u \in BV(\Omega)$ be such that E_u is an H-minimizer for P_ε on $\Omega \times \mathbb{R}$. Then $u \in W^{1,1}(\Omega)$.*

Proof. By Proposition 19.3.11, $u \in C^{3,\gamma}(\Omega \setminus \Omega_{u,0})$ and $\mathcal{H}^{2n-1}(\Omega_{u,0}) = 0$, where $\Omega_{u,0}$ is defined in (19.3.27). Let $\tilde{D}u$ be the distributional derivatives of u, and consider the decomposition $\tilde{D}u = Du\mathcal{L}^{2n} + (Du)_s$ as in Lemma 16.8.4. It is enough to show that $(Du)_s \equiv 0$. Since in particular $u \in W^{1,1}_{loc}(\Omega \setminus \Omega_{u,0})$, (16.8.8) combined with Lemma 16.8.4 implies that

$$\begin{aligned} P_\varepsilon(E_u, \tilde{\Omega} \times \mathbb{R}) &= \int_{\tilde{\Omega}} \sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2} \, dz + (Du)_s(\tilde{\Omega}) \\ &= \int_{\tilde{\Omega} \setminus \Omega_{u,0}} \sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2} \, dz + (Du)_s(\tilde{\Omega}) \\ &= P_\varepsilon(E_u, (\tilde{\Omega} \setminus \Omega_{u,0}) \times \mathbb{R}) + (Du)_s(\tilde{\Omega}), \end{aligned}$$

so that

$$(Du)_s(\tilde{\Omega}) = P_\varepsilon(E_u, (\tilde{\Omega} \cap \Omega_{u,0}) \times \mathbb{R}).$$

Arguing as in Proposition 19.3.7, there exists $L > 0$ such that $\partial^\star E_u \cap (\tilde{\Omega} \times \mathbb{R}) \subseteq \tilde{\Omega} \times [-L, L]$. Therefore, exploiting (16.8.3), there exists $\tilde{C} > 0$ such that

$$(Du)_s(\tilde{\Omega}) \leqslant \tilde{C}P(E_u, (\tilde{\Omega} \cap \Omega_{u,0}) \times \mathbb{R}) = \tilde{C}\mathcal{H}^{2n}(\partial^* E_u \cap (\tilde{\Omega} \cap \Omega_{u,0}) \times \mathbb{R}) \leqslant 2\tilde{C}L\mathcal{H}^{2n-1}(\Omega_{u,0}) = 0.$$

□

Proposition 19.3.13. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with Lipschitz continuous boundary and let $H \in C^{1,\gamma}_{loc}(\Omega)$ for some $\gamma \in (0, 1)$. Let $u \in BV(\Omega)$ be such that E_u is an H-minimizer for P_ε on $\Omega \times \mathbb{R}$. then $u \in \mathrm{Lip}_{loc}(\Omega)$.*

Proof. Fix $z_0 \in \Omega$. Let $r_0 > 0$ small enough to ensure that $B_{r_0} = B(z_0, r_0) \Subset \Omega$. We claim that there exists $r \in (r, r_0)$ such that (18.1.4), (***) and (19.1.2) hold for $B_r = B(z_0, r)$. Indeed, setting $H_0 = \|H\|_{L^\infty(B_{r_0})}$, then

$$\left|\int_A H\, dz\right| \leqslant H_0|A|$$

for any $r \in (0, r_0)$ and any measurable set $A \subseteq B_r$. Therefore, recalling that

$$h(B_r) = \frac{P(B_r)}{|B_r|} = \frac{1}{r}\frac{r_0 P(B_{r_0})}{|B_{r_0}|}$$

(cf. [188]), the claim concerning (18.1.4) and (***) follows by taking $r < \frac{r_0 P(B_{r_0})}{H_0|B_{r_0}|}$. Regarding (19.1.2), it suffices to see that $H_{\partial B_r} = \frac{2n-1}{r}$, so that $H_{\partial B_r}$ can be made arbitrarily big as r becomes small. Given $w \in W^{1,1}(B_r)$, we consider the extension of w to a function $\tilde{w} \in BV(\Omega)$ by letting $\tilde{w} \equiv u$ on $\Omega \setminus \overline{B_r}$. In particular, u and $\tilde{w}$ share the same trace on $\partial\Omega$. Using that $u \in W^{1,1}(\Omega)$ is such that E_u is an H-minimizer for P_ε on $\Omega \times \mathbb{R}$ and (16.8.5), we infer that $u|_{B_r}$ minimizes the functional

$$I(w) = \int_{B_r} \sqrt{\varepsilon^2 + |Dw + \mathcal{F}|^2}\, dz + \int_{B_r} Hw dz + \int_{\partial B_r} |u - w| d\mathcal{H}^{2n-1} \tag{19.3.30}$$

in $W^{1,1}(B_r)$. Since $\mathcal{H}^{2n-1}(\Omega_{u,0}) = 0$ by Proposition 19.3.11, we can take a sequence of open sets $(\Omega_k)_k$ such that $\Omega_{u,0} \subseteq \Omega_{k+1} \subseteq \Omega_k \subseteq \Omega$ for any $k \in \mathbb{N}$, $\Omega_{u,0} = \bigcap_{k\in\mathbb{N}} \Omega_k$ and $\mathcal{H}^{2n-1}(\Omega_k \cap \partial B_r) \to 0$ as $k \to \infty$. For any $k \in \mathbb{N}$, let $\varphi_k \in C^{2,\gamma}(\overline{B_r})$ be such that $\varphi_k \equiv u$ on $\partial B_r \setminus \Omega_k$ and

$$\sup_{\partial B_r} |\varphi_k| \leqslant 2 \sup_{\partial B_r} |u|. \tag{19.3.31}$$

We apply Theorem 19.1.3 to get a classical solution $v_k \in C^2(\overline{B_r})$ to (ε-PMC) such that $v_k \equiv \varphi_k$ on ∂B_r for any $k \in \mathbb{N}$. In particular, being $\mathcal{I}_\varepsilon$ convex in $W^{1,1}(B_r)$, we infer that v_k minimizes the functional

$$I_k(w) = \int_{B_r} \sqrt{\varepsilon^2 + |Dw + \mathcal{F}|^2}\, dz + \int_{B_r} Hw dz + \int_{\partial B_r} |\varphi_k - w| d\mathcal{H}^{2n-1} \tag{19.3.32}$$

in $W^{1,1}(B_r)$. Thanks to Lemma 19.3.2 we can apply Proposition 18.8.6 which, together with (19.3.31), implies that $(v_k)_k$ is uniformly bounded in $L^\infty(B_r)$. By Theorem 19.3.8,

$v_k \in C^{3,\gamma}_{loc}(B_r)$, and by Theorem 19.1.2 the sequence $(v_k)_k$ is locally uniformly bounded in $\mathrm{Lip}(B_r)$. By Ascoli-Arzelà Theorem, there exists $v \in \mathrm{Lip}_{loc}(B_r)$ such that, up to a subsequence, $v_k \to v$ locally uniformly on B_r. Since $(v_k)_k$ is uniformly bounded in $L^\infty(B_r)$ and by the lower semicontinuity properties of Lemma 16.8.3, we can pass to the limit in (19.3.32) with $w \equiv 0$ to infer that $v \in W^{1,1}(B_r)$. Let us check that v is a minimum of I in $W^{1,1}(B_r)$. Let $y \in \partial B_r$ be a regular point for u, k big enough so that $y \in \partial B \setminus \Omega_k$ and V a neighborhood of y in ∂B_r. Let $\varphi^\pm \in C^{2,\gamma}(\overline{B_r})$ be such that

$$\varphi^\pm = u \text{ in } V \qquad \text{and} \qquad \varphi^- \leqslant \varphi_k \leqslant \varphi^+ \text{ in } \partial B_r,$$

and let $v^\pm$ be the solutions to the Dirichlet problem in B_r with boundary datum $\varphi^\pm$. Then, using the maximum principle [157, Theorem 10.7], we get $v^- \leqslant v_k \leqslant v^+$, and $v = u$ at regular points of u, which together with $\mathcal{H}^{2n-1}(\Omega_{u,0})$ implies that u and v have the same trace on ∂B_r. Moreover, by the lower semicontinuity, the local uniform convergence of $v_h \to v$ and that $\varphi_k \to u$ in $L^1(\partial B_r)$, we get

$$I(v) \leqslant \liminf_{k\to+\infty} I_k(v_k) \leqslant \liminf_{k\to+\infty} I_k(w) = I(w)$$

for any $w \in W^{1,1}(B_r)$. Therefore, u and v are minimizers in $W^{1,1}(B_r)$ of the functional I. Recalling that $u - v \in W^{1,1}_0(B_r)$, the conclusion easily follows by the strict convexity of the functional. □

As a direct consequence of Proposition 19.3.3, Theorem 19.3.8 and Proposition 19.3.13, we have the following result.

Theorem 19.3.14. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be a bounded domain with Lipschitz continuous boundary and let $H \in C^{1,\gamma}_{loc}(\Omega) \cap L^\infty(\Omega)$ for some $\gamma \in (0,1)$. Assume that* (18.1.4) *and* (***) *holds. Then there exists $u \in C^{3,\gamma}_{loc}(\Omega) \cap W^{1,1}(\Omega)$ which minimizes $\mathcal{I}_\varepsilon$ and solves* (ε-PMC)*. Moreover, if $H \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.*

19.3.6 Existence of minimizers: the extremal case

Throughout this subsection we fix $H \in \mathrm{Lip}(\Omega) \cap C^{1,\gamma}_{loc}(\Omega)$ for some $\gamma \in (0,1)$. To deal with solutions to (ε-PMC) in the extremal case (18.1.5), we follow the approach of [162]. To this aim, we generalize the notion of H-minimizer as in (19.3.10) admitting merely measurable functions.

Definition 19.3.15. *A measurable function $u : \Omega \longrightarrow [-\infty, +\infty]$ is a* generalized H-minimizer *for P_ε on $\Omega \times \mathbb{R}$ if* (19.3.9) *holds.*

According to the notation introduced in [163], if $u : \Omega \longrightarrow [-\infty, +\infty]$ is a measurable function we set

$$N_+ = \{z \in \Omega \,:\, u(z) = +\infty\} \qquad \text{and} \qquad N_- = \{z \in \Omega \,:\, u(z) = -\infty\}. \tag{19.3.33}$$

As in the Euclidean setting, we have the following minimization property.

Proposition 19.3.16. *Let u be a generalized H-minimizer. Then*

$$P(N_\pm, A) \pm \int_{N_\pm \cap A} H\,dz \leqslant P(F, A) \pm \int_{F\cap A} H\,dz \tag{19.3.34}$$

for any open set $A \Subset \Omega$ and any measurable set $F \subseteq \Omega$ such that $N_+ \Delta F \Subset A$.

Proof. Let us check this property for N_+, being the case N_- analogous. For any $j \in \mathbb{N}$, set $u_j = u - j$. Then u_j converges almost everywhere to

$$v(z) = \begin{cases} +\infty & \text{if } z \in N_+ \\ -\infty & \text{if } z \notin N_+. \end{cases}$$

Arguing *verbatim* as in [202, Section 21.5]. (cf. Section 19.5 for the proof of a similar result), E_v is an H-minimizer as in (19.3.9). Assume by contradiction that there exists an open set $A \Subset \Omega$, a measurable set F such that $N_+\Delta F \Subset A$ and $\delta > 0$ such that

$$P(F; A) + \int_{F\cap A} H\,dz \leqslant P(N_+; A) + \int_{N_+\cap A} H\,dz - \delta.$$

Given $L > 0$, we set $A_L = A \times [-L, L]$ and $A_{2L} = A \times (-2L, 2L)$, and

$$F_L = \begin{cases} F \times \mathbb{R} & \text{in } A_L \\ N_+ \times \mathbb{R} & \text{otherwise.} \end{cases}$$

Then, using (16.8.5), we have

$$\begin{aligned} P_\varepsilon(F_L; A_{2L}) + \int_{F_L\cap A_{2L}} H\,dx \leqslant\ & 4\int_\Omega \sqrt{\varepsilon^2 + |\mathcal{F}|^2}\,dz + 2LP(N_+, A) + 2LP(F; A) \\ & + 2L\int_{N_+\cap A} H\,dz + 2L\int_{F\cap A} H\,dz \\ \leqslant\ & 4\int_\Omega \sqrt{\varepsilon^2 + |\mathcal{F}|^2}\,dz + 4LP(N_+; A) + 4L\int_{F_+\cap A} H\,dz - 2L\delta \\ =\ & P_\varepsilon(E_v; A_{2L}) + \int_{E_v\cap A_{2L}} H\,dx + 4\int_\Omega \sqrt{\varepsilon^2 + |\mathcal{F}|^2}\,dz - 2L\delta \\ <\ & P_\varepsilon(E_v; A_{2L}) + \int_{E_v\cap A_{2L}} H\,dx, \end{aligned}$$

where the last strict inequality follows provided that $L > \frac{2}{\delta}\int_\Omega \sqrt{\varepsilon^2 + |\mathcal{F}|^2}\,dz$. Being E_v an H-minimizer, a contradiction follows. ☐

Proof of Theorem 19.1.1. Let us assume (18.1.5), since otherwise the thesis follows by Theorem 19.3.14. Let $(\varepsilon_j)_j \subseteq (0,1)$ be such that $\varepsilon_j \searrow 0$ as $j \to \infty$. Let $(\Omega_j)_j$ be a sequence of open domains with Lipschitz boundary such that $\Omega_j \Subset \Omega_k \Subset \Omega$ for any $j < k$m $\bigcup_{j=0}^\infty \Omega_j = \Omega$ and $P(\Omega_j) \to P(\Omega)$ as $j \to \infty$ (cf. [256]). By hypothesis

$$\left|\int_{\Omega_j} H\right| < P(\Omega_j)$$

for any $j \in \mathbb{N}$. Hence, for any fixed boundary datum $\varphi_j \in L^1(\partial\Omega_j)$, Proposition 19.3.3 implies the existence of $u_j \in BV(\Omega_j)$ which minimizes $\mathcal{I}_\varepsilon$ on Ω_j. In view of Theorem 19.3.5, E_{u_j} is an H_j-minimizer for P_{ε_j} on $\Omega_j \times \mathbb{R}$. Arguing as in [162, 192], up to vertical translations we assume that

$$\min\{|\{z \in \Omega_j \,:\, u_j(z) \leqslant 0\}|, \{z \in \Omega_j \,:\, u_j(z) \geq 0\}|\} \geq \frac{|\Omega|}{4} \tag{19.3.35}$$

for any $j \in \mathbb{N}$. Applying again a compactness argument as in [202, Section 21.5] and Section 19.5, there exists a generalized H-minimizer u for P_ε on $\Omega \times \mathbb{R}$ as in Definition 19.3.15 such that $u_j \to u$ almost everywhere on Ω. We are left to show that $u \in L^\infty_{loc}(\Omega)$. Indeed, in this case, since E_u is a Caccioppoli set, Lemma 16.8.3 would imply that $u \in BV_{loc}(\Omega)$, whence Proposition 19.3.13 and Theorem 19.3.8 guarantee the requested regularity. To show that $u \in L^\infty_{loc}(\Omega)$, we let N_+ and N_- be as in (19.3.33). We claim that $|N_+| = |N_-| = 0$. In this case, arguing *verbatim* as in [163, Proposition 16.7], volume density estimates as in Proposition 19.3.6 allow to conclude that $u \in L^\infty_{loc}(\Omega)$. We prove that $|N_+| = 0$, being the other case analogous. To this aim, we apply Proposition 19.3.16 to infer that N_+ is a minimizer as in (19.3.34). But then, (18.1.5) allows to apply [162, Lemma 1.2], whence either $|N_+| = 0$ or $|N_-| = |\Omega|$. Being the latter possibility in contradiction with (19.3.35), we conclude that $|N_+| = 0$, from which the thesis follows. □

19.4 Essential uniqueness of solutions

Similarly to what happens in the Euclidean setting (cf. [162]), the extremal case (18.1.5) describes those maximal configurations Ω for which (ε-PMC) admits a classical solution. Moreover, in these cases, solutions are unique up to vertical translations.

Proof of Theorem 19.1.4. The equivalence between (i) and (ii), thanks to Theorem 19.1.1, follows word-by-word the proof of [162, Proposition 2.2]. To prove that (i) is equivalent to (iii) it suffices to notice that, if $u \in C^2(\Omega)$ solves (ε-PMC) in Ω, then

$$\int_{\Omega_t} H\,dz = \int_{\Omega_t} \operatorname{div}\left(\frac{Du + \mathcal{F}}{\sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2}}\right) dz = \int_{\partial\Omega_t} \frac{\langle \nu_t, Du + \mathcal{F}\rangle}{\sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2}}\, d\mathcal{H}^{2n-1},$$

so that

$$\int_\Omega H\,dz = \lim_{t \to 0^+} \int_{\partial\Omega_t} \frac{\langle \nu_t, Du + \mathcal{F}\rangle}{\sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2}}\, d\mathcal{H}^{2n-1}.$$

To show uniqueness, assume that $u, v \in C^2(\Omega)$ satisfy (ε-PMC) in Ω. Up to changing the sign of u and v, we can assume that $\int_\Omega H\,dz \geq 0$. Being (ε-PMC) invariant under vertical translations, we fix $z_0 \in \Omega$ and we assume that $v(z_0) = u(z_0)$. In order to simplify the notation, we let

$$W_\varepsilon u = \frac{Du + \mathcal{F}}{\sqrt{\varepsilon^2 + |Du + \mathcal{F}|^2}},$$

and $W_\varepsilon v$ accordingly. Notice in particular that $|W_\varepsilon u|, |W_\varepsilon v| \leqslant 1$. Notice that, for any $\varphi \in$

$\mathrm{Lip}(\Omega)$ such that $\varphi \geq 0$ on Ω, and for any sufficiently small $t \in (0,1)$, it holds that

$$\int_{\Omega_t} \langle W_\varepsilon u - W_\varepsilon v, D\varphi \rangle \, dz = \int_{\partial \Omega_t} \varphi \left(\langle \nu_t, W_\varepsilon u \rangle - \langle \nu_t, W_\varepsilon v \rangle \right) dz \geq \int_{\partial \Omega_t} \varphi \left(\langle \nu_t, W_\varepsilon u \rangle - 1 \right) dz.$$

For any $k > 0$, let $\varphi_k = \max\{0, \min\{v - u, k\}\}$. Then $\varphi_k \in \mathrm{Lip}(\Omega)$, $\varphi_k \geq 0$ on Ω and $\langle W_\varepsilon u - W_\varepsilon v, D\varphi_k \rangle \leqslant 0$ on Ω, so that

$$0 \geq \int_{\Omega_t} \langle W_\varepsilon u - W_\varepsilon v, D\varphi_k \rangle \, dz \geq -k \left(P(\Omega_t) - \int_{\partial \Omega_t} \langle \nu_t, W_\varepsilon u \rangle \, dz \right).$$

Therefore, in view of (iii), we let first $t \to 0^+$ and then $k \to \infty$ to obtain

$$\int_{\Omega} \langle W_\varepsilon u - W_\varepsilon v, D\varphi \rangle \, dz = 0, \tag{19.4.1}$$

where $\varphi := \max\{0, v - u\}$ verifies again $\langle W_\varepsilon u - W_\varepsilon v, D\varphi \rangle \leqslant 0$ on Ω. Hence (19.4.1) implies that $\langle W_\varepsilon u - W_\varepsilon v, D\varphi \rangle = 0$ on Ω. In view of the definition of φ, a simple computation shows that $D\varphi = 0$ on Ω, so that $\varphi(z) = \varphi(z_0) = 0$ for any $z \in \Omega$. Hence we conclude that $u \equiv v$ on Ω. □

19.5 Existence of sub-Riemannian minimizers via Riemannian approximation

Throughout this section, we fix a bounded domain $\Omega \subseteq \mathbb{R}^{2n}$ with Lipschitz boundary and $H \in L^\infty_{loc}(\Omega)$. We already know from Chapter 18 that (ε-PMC) arises naturally as an elliptic approximation of the *sub-Riemannian prescribed mean curvature equation* ($\mathcal{H}$-PMC). In this section we provide existence of solutions to ($\mathcal{H}$-PMC) in a broad sense by means of our previous results coupled with the aforementioned Riemannian approximation scheme (cf. e.g. [183, 238, 85, 77] for further insights). To describe our approach, assume that $u \in W^{1,1}(\Omega)$ is a weak solution to ($\mathcal{H}$-PMC). A standard variational argument, together with [264, Theorem 3.2] shows that

$$\int_{\tilde{\Omega}} |Du + \mathcal{F}| + \int_{\tilde{\Omega}} Hu \, dz \leqslant \int_{\tilde{\Omega}} |Dv + \mathcal{F}| + \int_{\tilde{\Omega}} Hv \, dz \tag{19.5.1}$$

for any open set $\tilde{\Omega} \Subset \Omega$ and any $v \in BV_{loc}(\Omega)$ such that $\{u \neq v\} \Subset \tilde{\Omega}$, where, following [264], we have set

$$\int_{\tilde{\Omega}} |Dv + \mathcal{F}| := P_{\mathbb{H}}(E_v, \tilde{\Omega} \times \mathbb{R}) \tag{19.5.2}$$

for any open set $\tilde{\Omega} \Subset \Omega$ and any $v \in BV_{loc}(\Omega)$. A function $u \in BV_{loc}(\Omega)$ satisfying (19.5.1) is called a *H-minimizer for $P_{\mathbb{H}}$*. This definition is motivated by the fact that, arguing as in the proof of Theorem 19.3.5 (cf. also [264, Theorem 3.15] and [264, Corollary 3.16]), the subgraph E_u of an H-minimizer satisfies

$$P_{\mathbb{H}}(E_u, A) + \int_{E_u \cap A} H \, dz \leqslant P_{\mathbb{H}}(F, A) + \int_{F \cap A} H \, dz \tag{19.5.3}$$

for any open set $A \Subset \Omega \times \mathbb{R}$ and any measurable set F such that $E\Delta F \Subset A$. On the other hand, a truncation argument as in [163, Theorem 14.8] implies that if E_u satisfies (19.5.3), then u is an H-minimizer for $P_{\mathbb{H}}$. We stress that, in light of [264, Theorem 1.2], the sub-Riemannian area functional in (19.5.2) is finite. Therefore, for any $\varphi \in L^1(\partial\Omega)$, we define the functional $\mathcal{I}_{\mathbb{H}} : BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_{\mathbb{H}}(v) = P_{\mathbb{H}}(E_v, \Omega \times \mathbb{R}) + \int_{\Omega} Hudz + \int_{\partial\Omega} |v - \varphi| d\mathcal{H}^{2n-1}.$$

Proposition 19.5.1. *Let $H \in L^\infty(\Omega)$ and assume that (18.1.4) and (***) hold. Then $\mathcal{I}_{\mathbb{H}}$ has a minimum in $BV(\Omega) \cap L^\infty_{loc}(\Omega)$ for every $\varphi \in L^1(\partial\Omega)$.*

Proof. Let $B \subseteq \mathbb{R}^{2n}$ be a ball containing Ω such that the Euclidean distance between $\partial\Omega$ and ∂B is positive, and extend H to B by letting $H \equiv 0$ outside Ω. As done in the proof of Proposition 19.3.3, minimizing $\mathcal{I}_{\mathbb{H}}$ is equivalent to minimize

$$\mathcal{J}_{\mathbb{H}}(v) = P_{\mathbb{H}}(E_v, \Omega \times \mathbb{R}) + Var(v, B \setminus \Omega) + \int_B Hvdz$$

in $K = \{v \in BV(B) : v = \phi \text{ in } B \setminus \Omega\}$, where ϕ is a fixed function in $W^{1,1}_0(B)$ with trace φ on $\partial\Omega$. Let $(\varepsilon_j)_j \subseteq (0,1)$ be such that $\varepsilon_j \searrow 0$ as $j \to \infty$. Let v_j be a minimizer of the functional $\mathcal{J}_{\varepsilon_j}$ defined in (19.3.3). By (19.3.8) and (16.8.3), we have

$$\delta Var(v_j, B) \leqslant \mathcal{J}_\varepsilon(v_j) + \tilde{C} \leqslant \mathcal{J}_{\varepsilon_j}(\phi) + \tilde{C} \leqslant C' P(E_\phi, \Omega \times \mathbb{R}) + Var(\phi, B \setminus \Omega) + \int_B H\phi dz + \tilde{C},$$

where $\tilde{C} = \int_\Omega |\mathcal{F}| dz + \int_{\partial\Omega} |\varphi| d\mathcal{H}^{2n-1}$ and $C' = C'(\Omega)$. Hence, arguing as in Proposition 19.3.3 $(v_j)_j$ is bounded in $BV(B)$ and it converges in $L^1(B)$ to $v_0 \in K$. By (16.8.6), v_0 is a minimizer of $\mathcal{J}_{\mathbb{H}}$, whence $\mathcal{I}_{\mathbb{H}}$ has a minimizer in $BV(\Omega)$. Finally, the same arguments of Section 19.3.2 can be carried out thanks to the sub-Riemannian density estimates (c.f. [241]) to prove that $u \in L^\infty_{loc}(\Omega)$. □

As in the Riemannian setting, in the extremal case (18.1.5) we rely again on the notion of generalized solution.

Definition 19.5.2. *A measurable function $u : \Omega \longrightarrow [-\infty, +\infty]$ is a* generalized H-minimizer *for $P_{\mathbb{H}}$ on $\Omega \times \mathbb{R}$ if (19.5.3) holds.*

If N_+ and N_- are defined as in (19.3.33), in view of (16.8.5) the following analogous to (19.3.16) holds.

Proposition 19.5.3. *Let u be a generalized H-minimizer. Then*

$$P(N_\pm, A) \pm \int_{N_\pm \cap A} H\, dz \leqslant P(F, A) \pm \int_{F \cap A} H\, dz \tag{19.5.4}$$

for any open set $A \Subset \Omega$ and any measurable set $F \subseteq \Omega$ such that $N_+\Delta F \Subset A$.

Before proving Theorem 19.1.5, we need the following compactness argument, whose proof follows the approach of [202].

Theorem 19.5.4. *Let $(\varepsilon_j)_j \subseteq (0,1)$ be such that $\varepsilon_j \searrow 0$ as $j \to \infty$. Let $(\Omega_j)_j$ be a sequence of open sets such that $\Omega_j \Subset \Omega_k \Subset \Omega$ for any $j < k$ and $\bigcup_{j=0}^{\infty} \Omega_j = \Omega$. Let $(H_j)_j \subseteq L^\infty(\Omega)$ be such that $H_j \to H$ uniformly on Ω. For any $j \in \mathbb{N}$, assume that $u_j \in BV_{loc}(\Omega_j)$ is such that E_{u_j} is an H_j-minimizer for P_{ε_j} on $\Omega_j \times \mathbb{R}$ (cf. Definition 19.3.4). Then, up to a subsequence, $(u_j)_j$ converges almost everywhere to a generalized H-minimizer for $P_{\mathbb{H}}$ on $\Omega \times \mathbb{R}$. If in addition E_u is a Caccioppoli set, then*

$$P_{\varepsilon_j}(E_{u_j}, \cdot) \rightharpoonup^* P_{\mathbb{H}}(E_u, \cdot) \tag{19.5.5}$$

locally on $\Omega \times \mathbb{R}$.

Proof. We set $E_j = E_{u_j}$ and $P_{\varepsilon_j} = P_j$ for any $j \in \mathbb{N}$. First we show that there exists an $\mathbb{H}$-Caccioppoli set E on $\Omega \times \mathbb{R}$ such that, up to a subsequence,

$$\chi_{E_j} \to \chi_E \tag{19.5.6}$$

in $L^1(A')$ for any open set $A' \Subset \Omega \times \mathbb{R}$. Since $\bar{A}'$ is compact, there exists $k \in \mathbb{N}$, $p_1, \ldots, p_k \in A'$, $r_1, \ldots, r_k > 0$ and $\tilde{\Omega} \Subset \Omega$ such that

$$\bar{A}' \Subset \bigcup_{i=1}^{k} B(p_i, r_i) \Subset \tilde{\Omega} \times \mathbb{R}.$$

Therefore, in view of Proposition 16.8.1 and (19.3.18), we have

$$P_{\mathbb{H}}\left(E_j, \bigcup_{i=1}^{k} B(p_i, r_i)\right) \leqslant P_{\varepsilon_j}\left(E_j, \bigcup_{i=1}^{k} B(p_i, r_i)\right) \leqslant \sum_{i=1}^{k} P_{\varepsilon_j}(E_j, B(p_i, r_i)) \leqslant c_2 \sum_{i=1}^{k} r_i^{2n}.$$

Hence we can apply Proposition 16.8.2 to infer the existence of a finite $\mathbb{H}$-perimeter set F in $\bar{A}'$ such that, up to a subsequence, $\chi_{E_j} \to \chi_F$ in $L^1(A')$. Taking a sequence of relative compact sets that covers $\Omega \times \mathbb{R}$ and using a standard diagonal argument, (19.5.6) follows. Thanks to [163, Lemma 16.3], $E = E_u$ for a measurable function $u : \Omega \longrightarrow [-\infty, +\infty]$, and moreover, up to a subsequence, $u_j \to u$ almost everywhere on Ω. Arguing as above,

$$\sup_{j \in \mathbb{N}} P_j(E_j, K) < \infty$$

for any $K \Subset \Omega \times \mathbb{R}$. Therefore, [11, Theorem 1.59] implies the existence of a $(2n+1)$-valued Radon measure μ on $\Omega \times \mathbb{R}$ and a scalar Radon measure λ on $\Omega \times \mathbb{R}$ such that, up to a subsequence,

$$D_j \chi_{E_j} \rightharpoonup^* \mu \qquad \text{and} \qquad P_j(E_j, \cdot) \rightharpoonup^* \lambda \tag{19.5.7}$$

locally on $\Omega \times \mathbb{R}$ as $j \to \infty$. We claim that

$$\mu = (D_{\mathbb{H}} \chi_E, 0). \tag{19.5.8}$$

Indeed, fix $K \Subset \Omega \times \mathbb{R}$ and $g \in C^1_c(K, \mathbb{R}^{2n+1})$, and set

$$V_j = \sum_{i=1}^{n} (g_i X_i + g_{n+i} Y_i) + g_{2n+1} \varepsilon_j T =: V + g_{2n+1} \varepsilon_j T.$$

Then, since $\chi_{E_j} \to \chi_E$ in $L^1_{loc}(\Omega \times \mathbb{R})$, we infer that

$$\begin{aligned}\left| \int_K g \cdot dD_j \chi_{E_j} - \int_K g \cdot d(D_{\mathbb{H}} \chi_E, 0) \right| &= \left| \int_{K \cap E_j} \operatorname{div} V_j \, dx - \int_{K \cap E} \operatorname{div} V \, dx \right| \\ &\leqslant \int_{K \cap (E \Delta E_j)} |\operatorname{div} V| \, dx + \varepsilon_j \int_K \left| \frac{\partial g_{2n+1}}{\partial t} \right| dx \\ &\leqslant |K \cap (E \Delta E_j)| \| \operatorname{div} V \|_{L^\infty(K)} + \varepsilon_j |K| \left\| \frac{\partial g_{2n+1}}{\partial t} \right\|_{L^\infty(K)} \\ &\to 0\end{aligned}$$

as $j \to \infty$. An easy approximation argument allows to extend the previous convergence to any $g \in C^0_c(K, \mathbb{R}^{2n+1})$, thus proving (19.5.8). Therefore, combining (19.5.7), (19.5.8) and [11, Proposition 1.62], we conclude that

$$P_{\mathbb{H}}(E, \cdot) \leqslant \lambda(\cdot). \tag{19.5.9}$$

Let A be an open set such that $A \Subset \Omega \times \mathbb{R}$, and let F be a Caccioppoli set on $\Omega \times \mathbb{R}$ such that $F \Delta E \Subset A$. We claim that

$$P_{\mathbb{H}}(E, A) + \int_{E \cap A} H \, dx \leqslant P_{\mathbb{H}}(F, A) + \int_{F \cap A} H \, dx. \tag{19.5.10}$$

For any $j \in \mathbb{N}$, Let A_j be an open set with Lipschitz boundary such that $E \Delta F \Subset A_0 \Subset A_j \Subset A_{j+1} \Subset A$ and $\bigcup_{j \in \mathbb{N}} A_j = A$. Up to a further subsequence, we assume that $\chi_{E_j} \to \chi_E$ almost everywhere on $\Omega \times \mathbb{R}$. This fact allows to choose $(A_j)_j$ in such a way that

$$\chi_{E_j} \to \chi_E \tag{19.5.11}$$

$\mathcal{H}^{2n}$-almost everywhere on ∂A_j. Moreover, being E_j and F are Caccioppoli sets,

$$\mathcal{H}^{2n}(\partial^* E_j \cap \partial A_j) = 0 \qquad \text{and} \qquad \mathcal{H}^{2n}(\partial^* F \cap \partial A_j) = 0 \tag{19.5.12}$$

for any $j \in \mathbb{N}$. For any $j \in \mathbb{N}$, we define $F_j := (F \cap A_j) \cup (E_j \setminus A_j)$. It is clear that F_j is a Caccioppoli set such that $F_j \Delta E_j \Subset A_j \Subset A$. Therefore in particular

$$F_j \cap (A_j \setminus A_0) = E_j \cap (A_j \setminus A_0). \tag{19.5.13}$$

Notice that, thanks to (16.8.3),(19.5.11) and (19.5.12) and arguing as in [202, Theorem 21.14],

$$\lim_{j \to \infty} P_j(F_j, \partial A_j) = 0. \tag{19.5.14}$$

We are able to prove (19.5.10). Indeed, exploiting the H-minimality of E_j for P_j, together with (19.5.13), we see that

$$\begin{aligned}P_j(E_j, A) + \int_{E_j\cap A} H_j\,dx &\leqslant P_j(F_j, A) + \int_{F_j\cap A} H_j\,dx\\&= P_j(F_j, A_j) + P_j(F_j, \partial A_j) + P_j(F_j, A\setminus \overline{A_j}) + \int_{F_j\cap A} H_j\,dx\\&= P_j(F, A_j) + P_j(F_j, \partial A_j) + P_j(E_j, A\setminus \overline{A_j}) + \int_{F_j\cap A} H_j\,dx\\&\leqslant P_j(F, A) + P_j(F_j, \partial A_j) + P_j(E_j, A\setminus \overline{A_j}) + \int_{F_j\cap A} H_j\,dx,\end{aligned}$$

which implies that

$$P_j(E_j, A_j) + \int_{E_j\cap A} H_j\,dx \leqslant P_j(F, A) + P_j(F_j, \partial A_j) + \int_{F_j\cap A} H_j\,dx.$$

Therefore, exploiting Proposition 16.8.1 together with (19.5.6) and (19.5.13), we can pass to the limit and obtain

$$\lambda(A) + \int_{E\cap A} H\,dx \leqslant P_{\mathbb{H}}(F, A) + \int_{F\cap A} H\,dx.$$

Notice that (19.5.9) allows to achieve (19.5.10). Finally, if E_u is a Caccioppoli set, then we can choose $F = E$ in the previous inequality, so that, recalling (19.5.9), (19.5.5) follows. □

Proof of Theorem 19.1.5. Being the non-extremal case already covered by (19.5.1), we can assume (18.1.5). Let $(\varepsilon_j)_j$ and $(\Omega_j)_j$ be as in the statement of Theorem 19.5.4. Assume in addition that $P(\Omega_j) \to P(\Omega)$ as $j \to \infty$. Assume that, for any $j \in \mathbb{N}$, Ω_j has Lipschitz boundary. By hypothesis

$$\left|\int_{\tilde{\Omega}_j} H\right| < P(\Omega_j)$$

for any $j \in \mathbb{N}$. Arguing as in the proof of Theorem 19.1.3, there exists a sequence $(H_j)_j \subseteq C^\infty(\overline{\Omega})$ such that

$$H_j \to H \text{ uniformly on } \Omega \qquad \text{and} \qquad \|H_j\|_{C^1(\Omega)} \leqslant H_0 + \|H\|_{L^\infty(\Omega)} + 1$$

for any $j \in \mathbb{N}$, being H_0 the Lipschitz constant of H on Ω. Arguing again as in the proof of Theorem 19.1.3, up to a subsequence each H_j satisfies the hypotheses of Proposition 19.3.3 on Ω_j. Hence, for any $j \in \mathbb{N}$, there exists $u_j \in BV(\Omega_j)$ which minimizes $\mathcal{I}_{\varepsilon_j}$, with source H_j, for any $j \in \mathbb{N}$. In view of Theorem 19.3.5, E_{u_j} is an H_j-minimizer for P_{ε_j} on $\Omega_j \times \mathbb{R}$. Arguing as in the proof of Theorem 19.1.1, Theorem 19.5.4 implies the existence of a generalized H-minimizer for $P_{\mathbb{H}}$ on Ω as in Definition 19.5.2 such that, up to vertical translations, $u_j \to u$ almost everywhere on Ω. Moreover, arguing again as in Theorem 19.1.1, in view of Proposition 19.5.3 and exploiting sub-Riemannian volume density estimates as in [241], we conclude that $u \in L^\infty_{loc}(\Omega)$. Hence, being E_u an H-Caccioppoli set, then [264, Theorem 1.2] implies that $u \in BV_{loc}(\Omega)$. In this case, E_u is a Caccioppoli set, so that (19.5.5) follows. □

Chapter 20

A characterization of horizontally totally geodesic hypersurfaces

20.1 Introduction

We refer to [242] as main reference for this chapter. In this final chapter achieve a first concrete step towards a better understanding of the so-called *Bernstein problem* in higher dimensional sub-Riemannian Heisenberg groups. Namely, we reduce the solution of the latter to validity of suitable sub-Riemannian curvature estimates. The characterization of entire minimal hypersurfaces in higher dimensional sub-Riemannian Heisenberg groups is an intriguing open problem in the framework of sub-Riemannian geometry. This issue, which is typically known as *Bernstein problem* in view of its Euclidean counterpart (cf. [163] for a complete survey of the topic), fits into the broader context of minimal hypersurfaces in sub-Riemannian structures introduced in Chapter 16. A first study of the sub-Riemannian Bernstein problem in $\mathbb{H}^1$ was carried out by [83, 252] in the class of *t-graphs* of class C^2. Precisely, in the previous set of papers the authors classified minimal t-graphs of class C^2 in the first Heisenberg group $\mathbb{H}^1$, finding examples of minimal t-graphs which are not planes. These results were generalized in [173, 107, 108] to more general embedded C^2-hypersurfaces in $\mathbb{H}^1$. Moreover, as pointed out in [249], if one consider hypersurfaces with low regularity, the examples of minimal hypersurfaces which are not hyperplanes increase considerably. However, the situation is different when considering non-characteristic hypersurfaces. In this context, a meaningful counterpart of hyperplanes in the Euclidean setting is the class of *vertical hyperplanes* introduced in Definition 3.6.2. Let us recall that a vertical hyperplane is a set S of the form

$$S = \{p \in \mathbb{H}^n \,:\, \langle(\bar{x},\bar{y}),(\bar{a},\bar{b})\rangle = c\},$$

for some $0 \neq (\bar{a},\bar{b}) \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$. An easy computation (cf. Section 20.3) shows that S is non-characteristic. Moreover, every hyperplane which is not vertical is characteristic (cf. again Section 20.3). A first result in the understanding of minimal non-characteristic hypersurfaces was achieved in [35] in the class of intrinsic graphs (cf. Section 16.7.2). Indeed, the authors showed that the only minimal intrinsic graphs defined by a C^2 function in $\mathbb{H}^1$ are vertical planes. This result was generalized in [148] to the class of non-characteristic minimal C^1-hypersurfaces of $\mathbb{H}^1$, in [234] to the class of minimal intrinsic graphs defined by an Euclidean Lipschitz function in $\mathbb{H}^1$, and in [161] to the class of (X, Y)-Lipschitz surfaces in the sub-Finsler Heisenberg group $\mathbb{H}^1$. We point out that, as shown in [226], weakening the regularity of the defining function allows to find examples of minimal hypersurfaces which are not vertical planes even in the class of intrinsic graphs. While the Bernstein problem is well understood in $\mathbb{H}^1$, very few results are known in higher dimensions. On one hand, it has been recently proved in [263] that there is no rigidity in the class of smooth t-graphs in $\mathbb{H}^n$. On the other hand, when $n \geq 5$, there are counterexamples even in the class of smooth intrinsic graphs (cf. [35, 107]). The Bernstein problem for non-characteristic hypersurfaces is still open when $n = 2, 3, 4$. In $\mathbb{H}^1$, a key step consists in understanding that the non-characteristic part $S \setminus S_0$ of an area-stationary surface S is foliated by *horizontal line* segments in the following sense (cf. [84, 148]).

Ruling Property. *Let S be an area-stationary surface of class C^1 in $\mathbb{H}^1$. Then, S is foliated by horizontal line segments with endpoints in S_0.*

Here, by horizontal line, we mean a curve which is both a horizontal curve and an Euclidean line. The importance of this ruling property became even more evident in [283], where the author showed a Bernstein theorem in the class of those minimal intrinsic graphs which present the aforementioned ruling property, thus without assuming any *a priori* regularity on the surfaces. The importance of this merely differential property can be appreciated even by a sub-Riemannian viewpoint, exploiting the sub-Riemannian second fundamental forms introduced in Section 16.6.3. In the particular case of $\mathbb{H}^1$, the vanishing of the non-symmetric form $h^{\mathbb{H}}$, which coincides both with the symmetric form $\tilde{h}^{\mathbb{H}}$ and with the horizontal mean curvature $H^{\mathbb{H}}$, is equivalent to the aforementioned ruling property. In the higher dimensional case, however, $h^{\mathbb{H}}$ and $\tilde{h}^{\mathbb{H}}$ may differ, although it is in general true that the norm of $\tilde{h}^{\mathbb{H}}$ is controlled by the norm of $h^{\mathbb{H}}$ (cf. Section 16.6.3). The aim of the present chapter is twofold. On one hand, we propose a generalization of the ruling property to higher dimensional Heisenberg groups, relating this new notion with the vanishing of the symmetric form $\tilde{h}^{\mathbb{H}}$. More precisely, we will call *horizontally totally geodesic* a hypersurface such that $\tilde{h}^{\mathbb{H}} \equiv 0$ on its non-characteristic part. We stress that hypersurfaces for which $h^{\mathbb{H}} \equiv 0$ are particular instances of horizontally totally geodesic hypersurfaces. On the other hand, it is not always the case that horizontally totally geodesic hypersurfaces satisfy $h^{\mathbb{H}} \equiv 0$ (cf. Section 16.6.3). In the Riemannian framework, this name is motivated by the fact that every geodesic of a totally geodesic hypersurface is a geodesic of the ambient manifold. This last characterization allows to deduce that, in $\mathbb{R}^n$, the only totally geodesic hypersurfaces are hyperplanes. Our second aim is to provide an analogous result in the Heisenberg group. We stress that, at least in the non-characteristic case, hypersurfaces with $h \equiv 0$ are easily vertical hyperplanes (cf. Section 16.6.3). Surprisingly, the same phenomenon continues to hold under the weaker requirement $\tilde{h}^{\mathbb{H}} \equiv 0$. The main achievements of this paper can be then summarized in the following result.

Theorem 20.1.1 (Horizontally totally geodesic hypersurfaces are hyperplanes)**.** *Let $S \subseteq \mathbb{H}^n$ be a hypersurface without boundary of class C^2. The following are equivalent:*

(*i*) *S is horizontally totally geodesic;*

(*ii*) *S is ruled.*

If in addition $n \geq 2$ and S is (topologically) closed, then (i) and (ii) hold if and only if S is a hyperplane.

In particular, in the non-characteristic setting, Theorem 20.1.1 might constitute an important tool in order to approach the resolution of the Bernstein problem in the higher dimensional case.

Corollary 20.1.2. *Let $S \subseteq \mathbb{H}^n$ be a non-characteristic hypersurface without boundary of class C^2. Assume that $n \geq 2$ and that S is (topologically) closed. If S is horizontally totally geodesic, then S is a vertical hyperplane.*

Corollary 20.1.2 allows to reduce the complexity of the problem to the estimate of the norm of the horizontal second fundamental form $\tilde{h}^{\mathbb{H}}$ associated to a minimal hypersurface. We point out that an approach based on curvature estimates for minimal hypersurfaces is already available in $\mathbb{R}^n$, in view of the celebrated paper [260]. Our approach to Theorem 20.1.1 can be summarized in the following steps.

Introduction of the higher dimensional ruling property. The starting point consists in generalizing the ruling property to the higher dimensional case, which is done in Section 20.2 in two equivalent ways (cf. Definition 20.2.1, Definition 20.2.2 and Proposition 20.2.3). After discussing the connection between the characteristic set and this new notion (cf. Proposition 20.2.4), we show that the latter is well-behaved with respect to the intrinsic geometry of $\mathbb{H}^n$. Namely, we prove that the class of ruled hypersurfaces is closed under the action of intrinsic dilations (cf. Proposition 20.2.8), and the action of the so-called *pseudohermitian transformations* (cf. Theorem 20.2.1).

Rigidity results for ruled hypersurfaces. Subsequently, we exploit the ruling property to provide rigidity results for some classes of hypersurfaces. Basically, we show that under some constraints on the size of the characteristic set, the higher dimensional ruling property is more rigid than the corresponding one in $\mathbb{H}^1$.

Theorem 20.1.3. *Let S be a hypersurface of class C^1. Assume that $n \geq 2$ and that S is closed and without boundary. Assume that S_0 is countable and that S is ruled. Then S is a hyperplane.*

This result constitutes a first remarkable difference with $\mathbb{H}^1$, where there are instance of smooth ruled non-characteristic hypersurfaces which are not planes (cf. Section 20.3).

Ruled if and only if horizontally totally geodesic. In Section 16.6.3 and Section 20.5 we build a bridge between the aforementioned result, which is only differential in spirit, with the sub-Riemannian structure of $\mathbb{H}^n$. After formally introducing and motivating the notion of horizontally totally geodesic hypersurface (cf. Definition 20.4.2), in view of Theorem 20.1.3 the main remaining obstacle to prove Theorem 20.1.1 is to show the equivalence between the latter and the ruling property.

Theorem 20.1.4. *Let S be a hypersurface without boundary of class C^2. Then S is ruled if and only if S is horizontally totally geodesic.*

This result strongly relies on a local existence and uniqueness result for a particular geodesic-type initial value problem on the hypersurface (cf. Theorem 20.5.4). Sub-Riemannian geodesics have been extensively studied in the last years (cf. e.g. [218, 190, 3, 89, 222] and references therein). Although local existence results for sub-Riemannian geodesics are available (cf. e.g. [190]), it is not always the case (cf. [252]) that sub-Riemannian geodesics satisfies the standard geodesic equation

$$\nabla^{\mathbb{H},S}_{\dot\Gamma}\dot\Gamma = 0,$$

Therefore, we devote Section 20.5 to the study of the initial value problem associated to this kind of equations on hypersurfaces. The proof of Theorem 20.5.4 can be reduced to the study of curves in domains of suitable intrinsic graphs, and its main difficulty lies in the fact that that the initial value problem that we need to consider is *a priori* overdetermined. Once Theorem 20.5.4 is achieved, we are then in position to prove Theorem 20.1.4, and so, in view of the previous considerations, to conclude the proof of Theorem 20.1.1.

We point out that, in view of Theorem 20.1.4, another striking difference with the first Heisenberg group can be appreciated. Indeed, contrarily to what happens in $\mathbb{H}^1$, it is easy to provide examples of minimal smooth hypersurfaces, at least in the characteristic setting, which are not horizontally totally geodesic.

Theorem 20.1.5. *Let $n \geq 2$ and let $S := \operatorname{graph}(u)$, where $u(\bar x, \bar y) = \frac{1}{2}x_1^2 - \frac{1}{2}y_1^2$. Then S is a minimal smooth hypersurface which is not horizontally totally geodesic.*

This set of results and considerations both provides a direct way to approach the Bernstein problem via curvature estimates, and highlights once more many interesting differences between $\mathbb{H}^1$ and higher dimensional Heisenberg groups. According to the authors' hope, it may give a burst in the grasp of such an interesting open problem as the Bernstein problem in this anisotropic setting.

20.2 Higher dimensional ruled hypersurfaces

As already mentioned, a key step in the study of minimal surfaces in $\mathbb{H}^1$ consists in showing that the non-characteristic part of an area-stationary surface is foliated by horizontal line segments. This property extends to the higher dimensional case as follows.

Definition 20.2.1 (Local ruling property)**.** *Let S be a hypersurface of class C^1. We say that S is* locally ruled at $p \in S \setminus S_0$ *if there exists a neighborhood U of p such that*

$$p \cdot \mathcal{H}T_pS \cap U \subseteq S.$$

Moreover, we say that S is locally ruled *if it is locally ruled at $p \in S \setminus S_0$ for any $p \in S \setminus S_0$.*

Beside this local definition, we propose a global one, which will be useful in the following.

Definition 20.2.2 (Global ruling property)**.** *Let S be a hypersurface of class C^1. We say that S is* ruled *if for any $p \in S \setminus S_0$, for any $v \in \mathcal{H}T_pS$ and for any $s \geq 0$, the following property holds. If s is maximal with the property that*

$$p \cdot \delta_\tau(v) \in S$$

for any $\tau \in [0, s]$, then

$$p \cdot \delta_s(v) \in S_0.$$

The previous two definitions are actually equivalent.

Proposition 20.2.3. *Let S be a hypersurface of class C^1. Then the following are equivalent.*

(i) S is locally ruled.

(ii) S is ruled.

Proof. Assume that S is ruled. Assume by contradiction that there exists $p \in S \setminus S_0$ and a sequence $(p_h)_h \subseteq p \cdot \mathcal{H}T_pS \setminus S$ converging to p as $h \to +\infty$. Then, for any $h \in \mathbb{N}$, there exists $\lambda_h > 0$ and $v_h \in \mathcal{H}T_pS$ such that $p_h = p \cdot \delta_{\lambda_h}(v_h)$. If, up to a subsequence, for any h there exists $0 < \mu_h \leqslant \lambda_h$ such that $p \cdot \delta_{\mu_h}(v_h)$ belongs to the manifold boundary of S, then, being the latter closed, so does p, a contradiction with $p \in S$. Therefore, since $p_h \notin S$ and up to a subsequence, we can assume that for any h there exists $s_h \geq 0$ maximal such that $p \cdot \delta_\tau(v_h) \in S$ for any $\tau \in [0, s_h]$. Clearly $s_h \leqslant \lambda_h$. Therefore, being S ruled, then $q_h := p \cdot \delta_{s_h}(v_h) \in S_0$. But then, by construction, $(q_h)_h$ converges to p as $h \to +\infty$, and so, being S_0 closed, we conclude that $p \in S_0$, a contradiction. On the contrary, assume that S is locally ruled. Assume by contradiction that that there exists $p \in S \setminus S_0$, $w \in \mathcal{H}T_pS$ and s maximal with the property that

$$p \cdot \delta_\tau(w) \in S \tag{20.2.1}$$

for any $\tau \in [0, s]$ and

$$p \cdot (sw, 0) \notin S_0.$$

Set $\bar{p} := p \cdot (sw, 0)$. Consider the left-invariant vector field

$$W = \sum_{j=1}^{2n} w_j Z_j.$$

Recalling that left-translations preserve the horizontal distribution, and being W left invariant, we conclude that

$$d\tau_{(sw,0)}|_p(W_p) = W|_{\bar{p}} \in \mathcal{H}_{\bar{p}}.$$

Moreover, by construction, W is clearly tangent to S at $\bar{p}$. We conclude that $w \in \mathcal{H}T_{\bar{p}}S$. Since S is locally ruled and $\bar{p} \in S \setminus S_0$, there exists $\tilde{s} > 0$ such that $\bar{p} \cdot (\tilde{s}w, 0) \in S$, which implies, recalling the definition of $\bar{p}$, that

$$\bar{p} \cdot (\tilde{s}w, 0) = p \cdot (\bar{s}w, 0) \cdot (\tilde{s}w, 0) = p \cdot ((\bar{s} + \tilde{s})w, 0) \in S,$$

a contradiction with (20.2.1). □

Proposition 20.2.4. *Let S be a ruled hypersurface of class C^1. Assume that S is (topologically) closed. Let $p \in S \setminus S_0$ and $v \in \mathcal{H}T_pS$ be such that*

$$\{p \cdot \delta_s(v, 0) \;:\; s \geq 0\} \cap S_0 = \emptyset.$$

Then

$$\{p \cdot \delta_s(v, 0) \;:\; s \geq 0\} \subseteq S.$$

In particular, if

$$p \cdot \mathcal{H}T_pS \cap S_0 = \emptyset,$$

then

$$p \cdot \mathcal{H}T_pS \subseteq S. \tag{20.2.2}$$

Proof. Let $p \in S \setminus S_0$ and $v \in \mathcal{H}T_pS$ be as in the statement, and assume by contradiction that there exists $\lambda > 0$ such that $q = p \cdot \delta_\lambda(v) \notin S$. Being S closed, there exists $s \geq 0$ maximal as in Definition 20.2.2. Then we can argue as in the proof of Proposition 20.2.3 to find $s \geq 0$ such that $p \cdot \delta_s(v) \in S_0$, which is a contradiction. The second claim clearly follows from the first one. □

Notice that, in view of Proposition 20.2.4, the notion of ruled hypersurface becomes much more simpler in the case of non-characteristic hypersurfaces. Indeed, if S is a closed, non-characteristic ruled hypersurface of class C^1 and $p \in S$, then clearly $p \cdot \mathcal{H}T_pS \cap S_0 = \emptyset$. Therefore a closed non-characteristic hypersurface of class C^1 is ruled if and only if it satisfies (20.2.2). Now let us discuss some instances of ruled hypersurfaces. We begin with the simplest non-characteristic smooth hypersurface.

Example 20.2.5 (Vertical Hyperplanes)**.** Let S be a vertical hyperplane of the form

$$S = \{p \in \mathbb{H}^n \;:\; \langle(\bar{x}, \bar{y}), (\bar{a}, \bar{b})\rangle = c\}$$

for some $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$. Without loss of generality, we assume that $a_1 \neq 0$. It is easy to see that

$$T_pS = \text{span}\{(a_2, -a_1, 0, \dots, 0), (a_3, 0, -a_1, 0, \dots, 0), \dots, (b_n, 0, \dots, 0, -a_1, 0), T\}$$

for any $p \in S$. Notice that $S_0 = \emptyset$. We show that S is ruled. Indeed, noticing that $T \in T_pS$ for any $p \in S$, it follows that

$$\mathcal{H}T_pS = \text{span}\{Z_2|_p, \dots, Z_n|_p, W_1|_p, \dots, W_n|_p\}$$

for any $p \in S$, where

$$Z_i = a_iX_1 - a_1X_i \qquad \text{and} \qquad W_j = b_jX_1 - a_1Y_j$$

for any $i = 2, \dots, n$ and $j = 1, \dots, n$. Let now $p = (\bar{x}, \bar{y}, t) \in S$, and let $w = (\bar{x}', \bar{y}', 0) \in \mathcal{H}T_pS$. Then there exists $\alpha_j, \beta_j \in \mathbb{R}$ such that

$$w = \left(\sum_{j=2}^{n} \alpha_j a_j + \sum_{j=1}^{n} \beta_j b_j, -\alpha_2 a_1, \dots, -\beta_n a_1, 0 \right).$$

We conclude noticing that

$$\langle (\bar{x}', \bar{y}'), (\bar{a}, \bar{b}) \rangle = a_1 \sum_{j=2}^{n} \alpha_j a_j + a_1 \sum_{j=1}^{n} \beta_j b_j - \sum_{j=2}^{n} \alpha_j a_1 a_j - \sum_{j=1}^{n} \beta_j a_1 b_j = 0.$$

Next we consider an instance in the characteristic case.

Example 20.2.6 (Horizontal Hyperplane)**.** Let S be the horizontal hyperplane $\mathcal{H}_0$. Notice that

$$T_pS = \text{span}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial y_n} \right\} = \text{span}\{X_1 - y_1T, \dots, X_n - y_nT, Y_1 + x_1T, \dots, Y_n + x_nT\}$$

for any $p \in S$. This in particular implies that $S_0 = \{0\}$. Therefore, let $p = (\bar{x}, \bar{y}, t) \neq 0$, and assume without loss of generality that $y_1 \neq 0$. This implies that

$$\mathcal{H}T_pS = \text{span}\{y_2X_1 - y_1X_2, \dots, y_nX_1 - y_1X_n, x_1X_1 + y_1Y_1, \dots, x_nX_1 + y_1Y_n\}.$$

Therefore, let $w = (z, 0) \in \mathcal{H}T_pS$, and let $\alpha_j, \beta_j \in \mathbb{R}$ be such that

$$z = \left(\sum_{j=2}^{n} \alpha_j y_j + \sum_{j=1}^{n} \beta_j x_j, -\alpha_2 y_1, \dots, -\alpha_n y_1, \beta_1 y_1, \dots, \beta_n y_1 \right).$$

Hence it follows that

$$Q((\bar{x}, \bar{y}), z) = y_1 \sum_{j=2}^{n} \alpha_j y_j + y_1 \sum_{j=1}^{n} \beta_j x_j - \sum_{j=2}^{n} \alpha_j y_1 y_j - \sum_{j=1}^{n} \beta_j y_1 x_j = 0.$$

With the next couple of propositions we show that the class of ruled C^1-hypersurfaces is closed under the action of left translations and intrinsic dilations.

Proposition 20.2.7. *Let S be a ruled hypersurface of class C^1. Then $\tau_q(S)$ is ruled for any $q \in \mathbb{H}^n$.*

Proof. Fix $q = (\bar{x}^q, \bar{y}^q, t) \in \mathbb{H}^n$, define $\tilde{S} := \tau_q(S)$ and, given a point $\tilde{p} \in \tilde{S} \setminus \tilde{S}_0$, let $p \in S$ be such that $\tilde{p} = \tau_q(p)$. Being $\tau_q : S \longrightarrow \tilde{S}$ a diffeomorphism, then $d\tau_q|_p : T_pS \longrightarrow T_{\tilde{p}}\tilde{S}$ is an isomorphism. Therefore we have that

$$d\tau_q|_p(T_pS) = T_{\tilde{p}}\tilde{S}.$$

Moreover, by definition of $\mathcal{H}$, it is also the case that

$$d\tau_q|_p(\mathcal{H}_p) = \mathcal{H}_{\tilde{p}}.$$

Hence we infer that

$$d\tau_q|_p(\mathcal{H}T_pS) = d\tau_q|_p(\mathcal{H}_p \cap T_pS) = d\tau_q|_p(\mathcal{H}_p) \cap d\tau_q|_p(T_pS) = \mathcal{H}_{\tilde{p}} \cap T_{\tilde{p}}\tilde{S} = \mathcal{H}T_{\tilde{p}}\tilde{S}.$$

In particular, notice that $p \in S \setminus S_0$. Let $w \in \tilde{p} \cdot \mathcal{H}T_{\tilde{p}}S$ and assume that there exists $s \geq 0$ maximal with the property that $\tilde{p} \cdot \delta_\tau(w) \in \tilde{S}$ for any $\tau \in [0, s]$. We claim that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. Let $v = (\bar{a}, \bar{b}, 0) \in \mathcal{H}T_pS$ be such that $d\tau_q|_p(v) = w$. By the left-invariance of the horizontal distribution, it follows that $w = (\bar{a}, \bar{b}, 0)$. Therefore s is maximal with the property that $p \cdot \delta_\tau(v) \in S$ for any $\tau \in [0, s]$. Hence $p \cdot \delta_s(v) \in S_0$, and so, since

$$\tilde{p} \cdot \delta_s(w) = \tilde{p} \cdot (s\bar{a}, s\bar{b}, 0) = q \cdot p \cdot (s\bar{a}, \bar{b}, 0) = q \cdot (p \cdot \delta_s(v))$$

and observing that $\tau_q(S_0) = \tilde{S}_0$, we conclude that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. □

Proposition 20.2.8. *Let S be a ruled hypersurface. Then $\delta_\lambda(S)$ is ruled for any $\lambda > 0$.*

Proof. Fix $\lambda > 0$, define $\tilde{S} := \delta_\lambda(S)$ and, given a point $\tilde{p} \in \tilde{S} \setminus \tilde{S}_0$, let $p = (\bar{x}, \bar{y}, t) \in S$ be such that $\tilde{p} = \delta_\lambda(p)$. Arguing as in the proof of Proposition 20.2.7, we get that

$$d\delta_\lambda|_p(\mathcal{H}T_pS) = \mathcal{H}T_{\tilde{p}}\tilde{S}. \tag{20.2.3}$$

Therefore, again, $p \in S \setminus S_0$. Let $w \in \tilde{p} \cdot \mathcal{H}T_{\tilde{p}}S$ and assume that there exists $s \geq 0$ maximal with the property that $\tilde{p} \cdot \delta_\tau(w) \in \tilde{S}$ for any $\tau \in [0, s]$. We claim that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. Let $v = (\bar{a}, \bar{b}, 0) \in \mathcal{H}T_pS$ be such that $d\delta_\lambda|_p(v) = w$. We claim that that $w = \delta_\lambda(v)$. Indeed, recalling that the Jacobian matrix of δ_λ is a diagonal matrix with diagonal $(\lambda, \ldots, \lambda, \lambda^2)$, then

$$\begin{aligned} w(f)(q) &= \sum_{j=1}^n a_j \frac{\partial(f \circ \delta_\lambda)}{\partial x_j}(p) + \sum_{j=1}^n b_j \frac{\partial(f \circ \delta_\lambda)}{\partial x_j}(p) + \sum_{j=1}^n (a_j y_j - b_j x_j)\, T(f \circ \delta_\lambda)(p) \\ &= \sum_{j=1}^n \lambda a_j \frac{\partial f}{\partial x_j}(\tilde{p}) + \sum_{j=1}^n \lambda b_j \frac{\partial f}{\partial x_j}(\tilde{p}) + \sum_{j=1}^n ((\lambda a_j)(\lambda y_j) - (\lambda b_j)(\lambda x_j)) Tf(\tilde{p}). \end{aligned}$$

The conclusion then follows as in the previous proof, just noticing that

$$\delta_\lambda(p \cdot \delta_\tau(v)) = \delta_\lambda(p) \cdot \delta_\lambda(\delta_\tau(v)) = \tilde{p} \cdot \delta_{\lambda\tau}(v) = \tilde{p} \cdot \delta_\tau(\delta_\lambda(v)) = \tilde{p} \cdot \delta_\tau(w)$$

for any $\tau \in \mathbb{R}$, and that $\delta_\lambda(S_0) = \tilde{S}_0$. □

In view of Proposition 20.2.7, we can enlarge the class of examples of ruled hypersurfaces.

Example 20.2.9 (Non-Vertical Hyperplanes)**.** We already know that $\mathcal{H}_0$ is a characteristic ruled smooth hypersurface. For any fixed $q = (\bar{x}_q, \bar{y}_q, t_q) \in \mathbb{H}^n$, we know from Proposition 20.2.7 that $\tau_q(\mathcal{H}_0)$ is a characteristic ruled smooth hypersurface. Moreover, an easy computation shows that

$$\tau_q(\mathcal{H}_0) = \{(\bar{x}, \bar{y}, t) \in \mathbb{H}^n \,:\, \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle + t + d = 0\},$$

where $(\bar{a}, \bar{b}) = (-\bar{y}_q, \bar{x}_q)$ and $d = -t_q$. Finally, notice that any hyperplane which is not vertical can be obtained as left-translation of the horizontal hyperplane $\mathcal{H}_0$. Hence we conclude that every hyperplane of $\mathbb{H}^n$ is ruled, and it is non-characteristic if and only if it is vertical. Finally, notice that we cannot exploit Proposition 20.2.8 to obtain more ruled hypersurfaces, since dilations of hyperplanes are hyperplanes.

To conclude this section, we show that the class of ruled hypersurfaces is closed under the action of the so-called *pseudohermitian transformations* of $\mathbb{H}^n$. To introduce this notion, we define the map $\mathcal{J} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$\mathcal{J}(\bar{x}, \bar{y}, t) := (-\bar{y}, \bar{x}, t)$$

for any $p = (\bar{x}, \bar{y}, t) \in \mathbb{H}^n$. The map $\mathcal{J}$ is a global diffeomorphism which preserves the horizontal distribution, related to the CR structure J by

$$d\mathcal{J}|_{\mathcal{H}} = J|_{\mathcal{H}}.$$

A global diffeomorphism $\varphi : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ is said to be a pseudohermitian transformation of $\mathbb{H}^n$ if it preserves the horizontal distribution and it commutes with $\mathcal{J}$, that is

$$d\varphi(\mathcal{H}) \subseteq \mathcal{H} \qquad \text{and} \qquad \varphi \circ \mathcal{J} = \mathcal{J} \circ \varphi.$$

Let us begin by considering a special subclass of pseudohermitian transformations. To this aim, let us define the map $\varphi_R : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$\varphi_R(\bar{x}, \bar{y}, t) := (R(\bar{x}, \bar{y}), t), \tag{20.2.4}$$

where R is an orthogonal matrix of the form

$$R = \begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

where A and B are real-valued $n \times n$ matrices.

Proposition 20.2.10. *Let φ_R be as in (20.2.4). Then φ_R is a pseudohermitian transformation. Moreover, it holds that*

$$d\varphi_R|_p(\bar{a}, \bar{b}, 0) = (R(\bar{a}, \bar{b}), 0)$$

for any $p \in \mathbb{H}^n$ and any $(\bar{a}, \bar{b}, 0) \in H_p$.

Proof. Let $p = (\bar{x}, \bar{y}, t)$ and $(\bar{a}, \bar{b}, 0)$ as in the statement, and let $\tilde{p} := \varphi_R(p) = (\bar{\tilde{x}}, \bar{\tilde{y}}, t)$. We first claim that

$$d\varphi_R|_p(X_j|_p) = \sum_{k=1}^{n} \left(R_{kj} X_k|_{\tilde{p}} + R_{(n+k)j} Y_k|_{\tilde{p}} \right)$$

and

$$d\varphi_R|_p(Y_j|_p) = \sum_{k=1}^{n} \left(R_{k(n+j)} X_k|_{\tilde{p}} + R_{n+k)(n+j)} Y_k|_{\tilde{p}} \right)$$

for any $j = 1, \ldots, n$. Indeed, let ψ be a C^1 function defined in a neighborhood of $\tilde{p}$. Let us recall that, since $(\bar{\tilde{x}}, \bar{\tilde{y}}) = R(\bar{x}, \bar{y})$ and R is orthogonal, then $(\bar{x}, \bar{y}) = R^T(\bar{\tilde{x}}, \bar{\tilde{y}})$, which means, recalling also the special block shape of R, that

$$-x_j = \sum_{k=1}^{n} \left(-R_{kj} \tilde{x}_k - R_{(n+k)j} \tilde{y}_k \right) = \sum_{k=1}^{n} \left(-R_{(n+k)(n+j)} \tilde{x}_k + R_{k(n+j)} \tilde{y}_k \right)$$

and

$$y_j = \sum_{k=1}^{n} \left(R_{k(n+j)} \tilde{x}_k + R_{(n+k)(n+j)} \tilde{y}_k \right) = \sum_{k=1}^{n} \left(-R_{(n+k)j} \tilde{x}_k + R_{kj} \tilde{y}_k \right).$$

for any $j = 1, \ldots, n$. Then it holds that

$$\begin{aligned}
d\varphi_R|_p(X_j|_p)(\psi)(\tilde{p}) &= X_j|_p(\psi \circ \varphi_R)(p) \\
&= \frac{\partial}{\partial x_j}(\psi \circ \varphi_R)(p) + y_j T(\psi \circ \varphi_R)(p) \\
&= \sum_{k=1}^{n} \left(R_{kj} \frac{\partial \psi}{\partial x_k}(\tilde{p}) + R_{(n+k)j} \frac{\partial \psi}{\partial y_k}(\tilde{p}) \right) + y_j T(\psi)(\tilde{p}) \\
&= \sum_{k=1}^{n} \left(R_{kj} \left(\frac{\partial \psi}{\partial x_k}(\tilde{p}) + \tilde{y}_k T(\psi(\tilde{p}) \right) + R_{(n+k)j} \left(\frac{\partial \psi}{\partial y_k}(\tilde{p}) - \tilde{x}_k T(\psi)(\tilde{p}) \right) \right) \\
&= \sum_{k=1}^{n} \left(R_{kj} X_k|_{\tilde{p}}(\psi)(\tilde{p}) + R_{(n+k)j} Y_k|_{\tilde{p}}(\psi)(\tilde{p}) \right)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
d\varphi_R|_p(Y_j|_p)(\psi)(\tilde{p}) &= Y_j|_p(\psi\circ\varphi_R)(p)\\
&= \frac{\partial}{\partial y_j}(\psi\circ\varphi_R)(p) - x_jT(\psi\circ\varphi_R)(p)\\
&= \sum_{k=1}^{n}\left(R_{k(n+j)}\frac{\partial\psi}{\partial x_k}(\tilde{p}) + R_{(n+k)(n+j)}\frac{\partial\psi}{\partial y_k}(\tilde{p})\right) - x_jT(\psi)(\tilde{p})\\
&= \sum_{k=1}^{n}\left(R_{k(n+j)}\left(\frac{\partial\psi}{\partial x_k}(\tilde{p}) + \tilde{y}_kT(\psi(\tilde{p})\right) + R_{(n+k)(n+j)}\left(\frac{\partial\psi}{\partial y_k}(\tilde{p}) - \tilde{x}_kT(\psi)(\tilde{p})\right)\right)\\
&= \sum_{k=1}^{n}\left(R_{k(n+j)}X_k|_{\tilde{p}}(\psi)(\tilde{p}) + R_{(n+k)(n+j)}Y_k|_{\tilde{p}}(\psi)(\tilde{p})\right)
\end{aligned}$$

for any $j = 1, \dots, n$. Hence we conclude that

$$\begin{aligned}
d\varphi_R|_p(\bar{a},\bar{b},0) &= \sum_{j=1}^{n}\left(a_jd\varphi_R|_p(X_j|_p) + b_jd\varphi_R|_p(Y_j|_p)\right)\\
&= \sum_{j,k=1}^{n}\left(a_j\left(R_{kj}X_k|_{\tilde{p}} + R_{(n+k)j}Y_k|_{\tilde{p}}\right) + b_j\left(R_{k(n+j)}X_k|_{\tilde{p}} + R_{n+k)(n+j)}Y_k|_{\tilde{p}}\right)\right)\\
&= \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(R_{kj}a_j + R_{k(n+j)}b_j\right)X_k|_{\tilde{p}} + \sum_{j=1}^{n}\left(R_{(n+k)j}a_j + R_{(n+k)(n+j)}b_j\right)Y_k|_{\tilde{p}}\right)\\
&= (R(\bar{a},\bar{b}),0).
\end{aligned}$$

□

As a consequence of the previous result, it is easy to see that the class of ruled hypersurfaces is closed under the action of maps of the form (20.2.4).

Proposition 20.2.11. *Let S be a ruled hypersurface. Then $\varphi_R(S)$ is ruled for any φ_R as in* (20.2.4).

Proof. The proof of this result, with the help of Proposition 20.2.10, follows as the proof of Proposition 20.2.7 and Proposition 20.2.8, noticing that $\varphi_R(S_0) = (\varphi_R(S))_0$ and that, for a given $p = (z,t) \in S \setminus S_0$, $(v,0) \in \mathcal{H}T_pS$ and $s \in \mathbb{R}$, it holds that

$$\begin{aligned}
\varphi_R(p\cdot\delta_s(v,0)) &= \varphi_R(z+sv, t+Q(z,sv))\\
&= (R(z+sv), t+sQ(z,v))\\
&= (Rz+sRv, t+sQ(Rz,Rv))\\
&= (Rz,t)\cdot(sRv,0))\\
&= \varphi_R(p)\cdot\delta_s(Rv,0).
\end{aligned}$$

□

As a corollary of Proposition 20.2.10, we can conclude our initial statement.

Theorem 20.2.1. *If S is ruled, then $\varphi(S)$ is ruled for any pseudohermitian transformation φ.*

Proof. It follows combining Proposition 20.2.7, Proposition 20.2.11 and [87, Theorem 4.1]. □

20.3 Ruled hypersurfaces with countable characteristic set

The aim of this section is to characterise ruled hypersurfaces of $\mathbb{H}^n$ with countable characteristic set, when $n \geq 2$. In the first Heisenberg group $\mathbb{H}^1$ there are examples of ruled, non-characteristic, smooth surfaces which are not vertical planes.

Example 20.3.1. As an instance, let us consider the surface S parametrized by the map $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{H}^1$ defined by

$$\varphi(t, \theta) := (t \cos\theta, t \sin\theta, \theta).$$

Notice that φ is smooth and injective. Moreover,

$$\frac{\partial\varphi}{\partial t}(t, \theta) = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} = \cos\theta X|_{\varphi(t,\theta)} + \sin\theta Y|_{\varphi(t,\theta)}$$

and

$$\frac{\partial\varphi}{\partial\theta}(t, \theta) = -t\sin\theta \frac{\partial}{\partial x} + t\cos\theta \frac{\partial}{\partial y} + T = -t\sin\theta X|_{\varphi(t,\theta)} + t\cos\theta Y|_{\varphi(t,\theta)} + (1 + t^2)T.$$

This implies that S is a smooth, non-characteristic surface, and moreover

$$\mathcal{H}T_{\varphi(t,\theta)}S = \operatorname{span}\left\{\frac{\partial\varphi}{\partial t}(t, \theta)\right\}$$

for any $(t, \theta) \in \mathbb{R}^2$. Finally, for given $t, \theta, s \in \mathbb{R}$, it holds that

$$(t\cos\theta, t\sin\theta, \theta) \cdot (s\cos\theta, s\sin\theta, 0) = ((t + s)\cos\theta, (t + s)\sin\theta, \theta) \in S,$$

and so S is ruled.

However, the situation in higher dimensional Heisenberg groups is quite different, and the ruling condition turns out to be more restrictive. Indeed, we are going to prove that the only closed, ruled hypersurfaces with countable characteristic set in $\mathbb{H}^n$, with $n \geq 2$, are hyperplanes. To this aim, we already know that vertical hyperplanes are non-characteristic and ruled, and that every non-vertical hyperplane

$$P := \left\{(\bar{x}, \bar{y}, t) \in \mathbb{H}^n \,:\, \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{n} b_j y_j + ct + d = 0\right\},$$

where clearly $c \neq 0$, is ruled and satisfies

$$P_0 = \left\{\left(\frac{b_1}{c}, \ldots, \frac{b_n}{c}, -\frac{a_1}{c}, \ldots, -\frac{a_n}{c}, -\frac{d}{c}\right)\right\}.$$

Before proving Theorem 20.1.3 we establish some preliminary results.

Proposition 20.3.2. *Let S be a hypersurface of class C^1. Assume that S is closed and without boundary. Assume that S is ruled and that S_0 is countable. Then*

$$p \cdot \mathcal{H}T_pS \subseteq S$$

for any $p \in S \setminus S_0$.

Proof. Let S and p as in the statement. Assume by contradiction that there exists $q \in p \cdot \mathcal{H}T_pS \setminus S$. Combining Proposition 20.2.4 with the fact that S_0 is countable and that S is ruled, it is easy to construct a sequence $(q_h)_h \subseteq S$ converging to q as $h \to \infty$. Being S closed, then $q \in S$, a contradiction. □

Proposition 20.3.3. *Let S be a hypersurface of class C^1. Assume that S is closed and without boundary. Assume that S is ruled and that S_0 is countable. Assume that $0 \in S \setminus S_0$. Then*

$$S \cap \mathcal{H}_0 = \mathcal{H}T_0S.$$

Proof. First, since $0 \in S \setminus S_0$ and in view of Proposition 20.3.2, then $\mathcal{H}T_0S \subseteq S \cap \mathcal{H}_0$. Assume by contradiction that there exists $q = (z_q, 0) \in (S \cap \mathcal{H}_0) \setminus \mathcal{H}T_0S$. If S is tangent to $\mathcal{H}_0$ at q, then $q \in S \setminus S_0$. Otherwise, since $\mathcal{H}T_0S$ is closed and S_0 is countable, it is possible to find another point in $(S \setminus S_0) \setminus \mathcal{H}T_0S$. In the end, we can assume that $q \in S \setminus S_0$. Again, thanks to Proposition 20.3.2, $q \cdot \mathcal{H}T_qS \subseteq S$, and so $q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0 \subseteq S \cap \mathcal{H}_0$. Note that both $\mathcal{H}T_0S$ and $q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0$ are affine subspaces of $\mathcal{H}_0$. Moreover, $\dim(\mathcal{H}T_0S) = 2n - 1$ and $\dim(q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0) \geq 2n - 2$. Therefore we conclude that

$$\dim(\mathcal{H}T_0S \cap (q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0)) \geq \dim(\mathcal{H}T_0S) + \dim(q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0) - 2n = 2n - 3 \geq 1,$$

since $n \geq 2$. Therefore $(\mathcal{H}T_0S) \cap (q \cdot \mathcal{H}T_qS \cap H_0)$ contains a one-dimensional affine subspace of H_0. In particular, being S_0 countable, there exists $p = (z_p, 0) \in \mathcal{H}T_0S \cap (q \cdot \mathcal{H}T_qS \cap \mathcal{H}_0) \cap (S \setminus S_0)$. Let $v \in \mathcal{H}T_pS$ be such that $p \cdot tv = q$ for some $t \in \mathbb{R}$, and let $\gamma_p(t) := (tz_p, 0)$. Notice that, by construction, then $\gamma_p(t) \in S$ for any $t \in \mathbb{R}$. Moreover, $\dot{\gamma}_p(1) = (z_p, 0) \in \mathcal{H}_p$, and so $w := (z_p, 0) \in \mathcal{H}T_pS$. Again, since $p \in S \setminus S_0$ and in view of Proposition 20.3.2, then $p \cdot \mathcal{H}T_pS \subseteq S$. Therefore, in particular, it holds that

$$p \cdot (\alpha v + \beta w) \in S$$

for any $\alpha, \beta \in \mathbb{R}$. Hence, if we let $\gamma_q(t) := (tz_q, 0)$, we conclude that $\gamma(t) \in S \cap \mathcal{H}_0$ for any $t \in \mathbb{R}$, and so $\dot{\gamma}_q(0) = (z_q, 0) \in T_0S$. Since clearly $(z_q, 0) \in \mathcal{H}_0$, then $q \in \mathcal{H}T_0S$, which is a contradiction. □

Proposition 20.3.4. *Let S be a hypersurface of class C^1. Assume that S is closed and without boundary. Assume that S is ruled and that S_0 is countable. Then either S is a t-graph or S is a vertical hyperplane.*

Proof. If S is a t-graph we are done. If S is not a t-graph, being S_0 countable, there exists $p \in S \setminus S_0$ such that $T|_p \in T_pS$. Up to a left-translation, recalling Proposition 20.2.7, we assume that $p = 0$. We show that S is a vertical hyperplane, dividing the proof into some steps.
Step 1. Thanks to Proposition 20.3.3, we know that there exists $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ such that

$$\mathcal{H}T_0S = \mathcal{H}_0 \cap S = \{(\bar{x}, \bar{y}, 0) \in \mathbb{H}^n \,:\, \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle = 0\}.$$

We assume without loss of generality that $a_1 \neq 0$, and we let $f(\bar{x}, \bar{y}) := \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle$. We claim that

$$\pi(p \cdot \mathcal{H}T_pS) \subseteq \pi(\mathcal{H}T_0S)$$

for any $p \in \mathcal{H}T_0S \cap (S \setminus S_0)$, where $\pi : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}$ is defined as in (3.2.2). It is easy to see that π is smooth, surjective and open. Assume by contradiction that there exists $p = (z_p, 0) \in \mathcal{H}T_0S \cap (S \setminus S_0)$ and $v = (v, 0) \in \mathcal{H}T_pS$ such that $z_p + v \notin \pi(\mathcal{H}T_0S)$. This is equivalent to say that $f(z_p + v) \neq 0$. Let us define $q := p \cdot v = (z_p + v, Q(z_p, v))$. Since $p \in S \setminus S_0$ and by Proposition 20.3.2, then $q \in S$. Moreover, $Q(z_p, v) \neq 0$, since otherwise $q \in \mathcal{H}T_0S$ and consequently $f(z_p + v) = 0$. Moreover, since $z_p \in \mathcal{H}T_0S$, then, letting $\gamma(t) := (tz_p, 0)$, it holds that $\gamma(t) \in S$ for any $t \in \mathbb{R}$, and so $(z_p, 0) \in \mathcal{H}T_pS$. Hence, since $p \cdot \mathcal{H}T_pS \subseteq S$, we conclude in particular that

$$P := \{(z_p, 0) + \alpha(z_p, 0) + \beta(v, Q(z_p, v)) \,:\, \alpha, \beta \in \mathbb{R}\} \subseteq S.$$

Notice that P is a vector subspace of $\mathbb{R}^{2n+1}$. Then in particular $0 \in P$ and $(v, Q(z_p, v)) \in T_0S$. Therefore, as $T \in T_0S$, then $(v, 0) \in T_0S$, and so, since $(v, 0) \in \mathcal{H}_0$, we conclude that $(v, 0) \in \mathcal{H}T_0S$. Then $f(v) = 0$, and so, as $p \in \mathcal{H}T_0S$, $f(z_p + v) = f(z_p) + f(v) = 0$, a contradiction.
Step 2. Let $p = (z_p, 0) \in \mathcal{H}T_0S \cap (S \setminus S_0)$. Thanks to Step 1, we know that $\pi(p \cdot \mathcal{H}T_pS) \subseteq \pi(\mathcal{H}T_0S)$. Therefore, if $v \in \mathcal{H}T_pS$, then $f(z_p + v) = 0$. Since $f(z_p) = 0$, we conclude that $f(v) = 0$, which implies that

$$\mathcal{H}T_pS = \mathcal{H}T_0S \tag{20.3.1}$$

for any $p \in \mathcal{H}T_0S$. Moreover, an easy computation shows that

$$\mathcal{H}T_0S = \operatorname{span}\{Z_2|_0, \ldots, Z_n|_0, W_1|_0, \ldots, W_n|_0\},$$

where

$$Z_i = a_iX_1 - a_1X_i \qquad \text{and} \qquad W_j = b_jX_1 - a_1Y_j \tag{20.3.2}$$

for any $i = 2, \ldots, n$ and $j = 1, \ldots, n$. Then (20.3.1) allows to conclude that

$$\mathcal{H}T_pS = \operatorname{span}\{Z_2|_p, \ldots, Z_n|_p, W_1|_p, \ldots, W_n|_p\}. \tag{20.3.3}$$

Step 3. Let us define

$$\mathcal{Z} := \{z \in \pi(\mathcal{H}T_0S) \,:\, Q(z, w) = 0 \text{ for any } w \in \pi(\mathcal{H}T_0S)\}.$$

Notice that, being Q a bilinear map, then $\mathcal{Z}$ is a vector subspace of $\pi(\mathcal{H}T_0S)$. We claim that $\dim(\mathcal{Z}) \leqslant 2n-2$. Indeed, assume by contradiction that $\dim(\mathcal{Z}) \geq 2n-1$. Then, since $\mathcal{Z} \subseteq \pi(\mathcal{H}T_0S)$ and $\dim(\pi(\mathcal{H}T_0S)) = 2n-1$, we conclude that $\mathcal{Z} = \pi(\mathcal{H}T_0S)$. We show that this leads to a contradiction. Assume first that $a_2 = \ldots = a_n = b_2 = \ldots = b_n = 0$, and set $z_1 = (0,-1,0\ldots,0)$ and $z_2 = (\bar{0},0,1,0,\ldots,0)$. Then $f(z_1) = f(z_2) = 0$ and $Q(z_1,z_2) = 1 \neq 0$, which implies that $z_1, z_2 \notin \mathcal{Z}$. If it is not the case that $a_2 = \ldots = a_n = b_2 = \ldots = b_n = 0$, then assume without loss of generality that $a_2 \neq 0$. Let $z_1 = (-a_2, a_1, 0, \ldots, 0)$ and $z_2 = (-b_1, 0, \ldots, 0, a_1, 0, \ldots, 0)$. Then $f(z_1) = f(z_2) = 0$ and $Q(z_1,z_2) = a_1a_2 \neq 0$, which implies again that $z_1, z_2 \notin \mathcal{Z}$. Therefore we conclude that $\dim(\mathcal{Z}) \leqslant 2n-2$, and so in particular

$$\overline{\pi(\mathcal{H}T_0S) \setminus \mathcal{K}} = \pi(\mathcal{H}T_0S). \tag{20.3.4}$$

Step 4. We claim that for any $q = (z_q, t_q) = (x_1^q, \ldots, x_n^q, y_1^q, \ldots, y_n^q, t_q)$ such that $z_q \in \pi(\mathcal{H}T_0S) \setminus \mathcal{Z}$ there exists $p = (z_p, 0) = (x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p, 0) \in \mathcal{H}T_0S \cap (S \setminus S_0)$ and $v \in \mathcal{H}T_pS$ such that

$$q = p \cdot v. \tag{20.3.5}$$

Indeed, let q as above, and let $p \in \mathcal{H}T_0S \cap (S \setminus S_0)$ and $v \in \mathcal{H}T_pS$ to be chosen later. In view of (20.3.3), we can express v as

$$v = \sum_{j=2}^n \alpha_j Z_j|_p + \sum_{j=1}^n \beta_j W_j|_p = \left(\sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j\right) X_1|_p - \sum_{j=2}^n \alpha_j a_1 X_j|_p - \sum_{j=1}^n \beta_j a_1 Y_j|_p.$$

for some $\alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}$. Therefore, we infer that

$$p \cdot v = \left(x_1^p + \sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j, x_2^p - \alpha_2 a_1, \ldots, y_n^p - \beta_n a_1, Q(z_p, v)\right).$$

Let us choose

$$\alpha_i = \frac{x_i^p - x_i^q}{a_1} \qquad \textit{and} \qquad \beta_j = \frac{y_j^p - y_j^q}{a_1}$$

for any $i = 2, \ldots, n$ and any $j = 1, \ldots, n$. This choice implies that $(p \cdot v)_i = x_i^q$ and $(p \cdot v)_j = y_{j-n}^q$ for any $i = 2, \ldots, n$ and any $j = n+1, \ldots, 2n$. Moreover, since $f(z_p) = f(z_q) = 0$, it holds that

$$(p \cdot v)_1 = x_1^p + \sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j = \frac{1}{a_1}\left(\sum_{j=1}^n (a_j x_j^p + b_j y_j^p) - \sum_{j=2}^m a_j x_j^q + \sum_{j=1}^n b_j y_j^q\right) = x_1^q.$$

Finally, notice that

$$\begin{aligned}
Q(z_p, v) &= \left(\sum_{j=2}^{n} \alpha_j a_j + \sum_{j=1}^{n} \beta_j b_j\right) y_1^p - \sum_{j=2}^{n} \alpha_j a_1 y_j^p + \sum_{j=1}^{n} \beta_j a_1 x_j^p \\
&= \frac{1}{a_1}\Big(\sum_{j=2}^{n} a_j x_j^p y_1^p - \sum_{j=2}^{n} a_j x_j^q y_1^p + \sum_{j=1}^{n} b_j y_j^p y_1^p - \sum_{j=1}^{n} b_j y_j^q y_1^p \\
&\quad - \sum_{j=2}^{n} a_1 x_j^p y_j^p + \sum_{j=2}^{n} a_1 x_j^q y_j^p + a_1 x_1^p y_1^p + \sum_{j=2}^{n} a_1 x_j^p y_j^p - \sum_{J=1}^{n} a_1 x_j^p y_j^q \Big) \\
&= \frac{1}{a_1}\Big(- \sum_{j=1}^{n} a_j x_j^q y_1^p - \sum_{j=1}^{n} b_j y_j^q y_1^p + \sum_{j=1}^{n} a_1 x_j^q y_j^p - \sum_{j=1}^{n} a_1 x_j^p y_j^q \Big) \\
&= Q(z_p, z_q),
\end{aligned}$$

where in the third equality we exploited the fact that $f(z_p) = 0$, while the fourth equality follows from $f(z_q) = 0$. Since we assumed $z_q \notin \mathcal{Z}$, then there exists uncountably many $w \in \pi(\mathcal{H}T_0 S)$ such that $Q(w, z_q) \neq 0$. Therefore, since S_0 is countable, it is possible to choose $w \in \pi(\mathcal{H}T_0 S)$ such that, setting

$$z_p = \frac{t_q}{Q(w, z_q)} w,$$

then $p \in (S \setminus S_0)$. We conclude that $p \in \mathcal{H}T_0 S \cap (S \setminus S_0)$ and $Q(z_p, z_q) = t_q$.

Step 5. We are now able to conclude. Indeed, thanks to (20.3.5) we infer that

$$\pi(HT_0 S \setminus \mathcal{K}) \times \mathbb{R} \subseteq S.$$

But then, being S closed and recalling (20.3.4), we conclude that

$$\pi(HT_0 S) \times \mathbb{R} = \overline{\pi(HT_0 S) \setminus \mathcal{K}} \times \mathbb{R} = \overline{\pi(HT_0 S) \times \mathbb{R}} \subseteq \overline{S} = S.$$

Therefore S contains the vertical hyperplane $\pi(HT_0 S) \times \mathbb{R}$. The thesis then follows in view of the topological assumptions on S. ☐

Proof of Theorem 20.1.3. Let S be as in the statement. If S is a vertical hyperplane, we are done. If not, in view of Proposition 20.3.4, S is a t-graph. Being S_0 countable, and recalling Proposition 20.2.7, up to a left translation we may assume that $0 \in S \setminus S_0$ and that $T|_0 \notin T_0 S$. Since $0 \in S \setminus S_0$, we infer by Proposition 20.3.3 that

$$\mathcal{H}T_0 S = \mathcal{H}_0 \cap S = \{(\bar{x}, \bar{y}, 0) \in \mathbb{H}^n \,:\, \langle (\bar{x}, \bar{y}), (\bar{a}, \bar{b}) \rangle = 0\}$$

for some $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$. Again, by Proposition 20.2.10 we may assume that $a_1 \neq 0$. Being S an entire t-graph, and since $T|_0 \notin T_0 S$ and $0 \in S \setminus S_0$, there exists $v = (z_v, t_v) \in T_0 S$ such that $f(z_v) \neq 0$ and $t_v \neq 0$. Let us set $c := -\frac{f(z_v)}{t_v}$. We claim that

$$S = \{(z, t) \in \mathbb{H}^n \,:\, f(z) + ct = 0\} =: S_c.$$

Indeed, let $p = (z_p, t_p) \in S \setminus S_0$. If $t_p = 0$, then $f(z_p) = 0$, and so $p \in S_c$. Assume then $t_p \neq 0$. Then, being S a t-graph, we infer that $f(z_p) \neq 0.$. Let $v_1, \dots, v_{2n-1}$ be a basis of $\mathcal{H}T_pS$. Since $p \in S \setminus S_0$ and thanks to Proposition 20.3.2, then $p \cdot \mathcal{H}T_pS \subseteq S$. We claim that there exists $j = 1, \dots, 2n-1$ such that $Q(z_p, v_j) \neq 0$. Indeed, assume by contradiction that $Q(z_p, v_1) = \dots = Q(z_p, v_{2n-1}) = 0$. In this case, recalling that S is ruled, it holds that

$$p \cdot \mathcal{H}T_pS = \left\{ \left(z_p + \sum_{j=1}^{2n-1} \alpha_j v_j, t_p \right) : \alpha_1, \dots, \alpha_{2n-1} \in \mathbb{R} \right\} \subseteq S. \tag{20.3.6}$$

We claim that

$$\operatorname{span}\{(v_1, 0), \dots, (v_{2n-1}, 0)\} = \operatorname{span}\{Z_2|_0, \dots, Z_n|_0, W_1|_0, \dots, W_n|_0\}, \tag{20.3.7}$$

where $Z_2, \dots, Z_n, W_1, \dots, W_n$ are defined as (20.3.2). Indeed, if it was not the case, then (20.3.6) would imply the existence of $q = (z_q, t_p) \in S$ such that $z_q \in \pi(\mathcal{H}T_0S)$. But since $t_p \neq 0$ and since $(z_q, 0) \in S$, we would contradict the fact that S is a t-graph. Notice that (20.3.7) implies that

$$Z_2|_0, \dots, Z_n|_0, W_1|_0, \dots, W_n|_0 \in H_p$$

and so, observing that

$$Z_j|_0 = a_j \frac{\partial}{\partial x_1} - a_1 \frac{\partial}{\partial x_j} = a_j X_1|_p - a_1 X_j|_p + (a_1 y_j - a_j y_1) T$$

for any $j = 2, \dots, n$ and

$$W_j|_0 = b_j \frac{\partial}{\partial x_1} - a_1 \frac{\partial}{\partial y_j} = b_j X_1|_p - a_1 Y_j|_p + (-a_1 x_j - b_j y_1) T$$

for any $j = 1, \dots, n$, we conclude that

$$z_p = \frac{y_1}{a_1}(-b_1, \dots, -b_n, a_1, \dots, a_n),$$

which implies in particular that $f(z_p) = 0$, a contradiction. In this case, it holds that $p \cdot HT_pS \cap \mathcal{H}_0 \cap S = p \cdot \mathcal{H}T_pS \cap \mathcal{H}T_0S \neq 0$. Since $n \geq 2$, a dimensional argument as in the proof of Proposition 20.3.3 implies that $\dim(p \cdot \mathcal{H}T_pS \cap \mathcal{H}_0 \cap S) \geq 1$. Therefore, being S_0 countable, there exists $q = (z_q, 0) \in (p \cdot \mathcal{H}T_pS) \cap \mathcal{H}T_0S \setminus S_0$. Let then $w \in \mathcal{H}T_pS$ be such that

$$(z_q, 0) = (z_p + w, t_p + Q(z_p w)). \tag{20.3.8}$$

Arguing as in the proof of Proposition 20.3.3, recalling that $q \in S \setminus S_0$ and Proposition 20.3.2, we see that

$$P := \{(z_q, 0) + \alpha(z_q, 0) + \beta(w, Q(z_q, w)) : \alpha, \beta \in \mathbb{R}\} \subseteq S,$$

and so we conclude as above that $(w.Q(z_q, w)) \in T_0S$. This means that there exists $\tilde{w} \in$

$\pi(\mathcal{H}T_0S)$ and $\alpha \in \mathbb{R}$ such that

$$(w.Q(z_q, w)) = (\tilde{w} + \alpha z_v, \alpha t_v).$$

Therefore, recalling (20.3.8), we get that

$$(z_p, t_p) = (z_q, 0) - (w, Q(z_p, w)) = (z_q - \tilde{w} - \alpha z_v, -\alpha t_v)\,.$$

Therefore, since $z_q, \tilde{w} \in \pi(\mathcal{H}T_0S)$ we conclude that

$$f(z_p) + ct_p = -\alpha(f(z_v) + ct_v) = 0,$$

which implies that $p \in S_c$. Therefore we proved that $S \setminus S_0 \subseteq S_c$, and so, being S_0 countable and S and S_c closed, we conclude that $S \subseteq S_c$. The thesis then follows by the topological assumptions on S. □

20.4 Horizontally totally geodesic hypersurfaces

In the first Heisenberg group $\mathbb{H}^1$, $\mathcal{H}TS$ is a one dimensional distribution generated by $J(\nu^{\mathbb{H}})$. In particular (cf. [252]), $h^{\mathbb{H}}$ completely determines the behavior of $\nabla^{\mathbb{H}}_{J(\nu^{\mathbb{H}})}J(\nu^{\mathbb{H}})$, meaning that

$$\nabla^{\mathbb{H}}_{J(\nu^{\mathbb{H}})}J(\nu^{\mathbb{H}}) = -h^{\mathbb{H}}(J(\nu^{\mathbb{H}}), J(\nu^{\mathbb{H}}))\nu^{\mathbb{H}}.$$

This consideration is a first crucial step in the study of minimal surfaces, since it allows to infer that, when $H^{\mathbb{H}} = 0$, then S is ruled by horizontal lines. A horizontal line is a horizontal curve $\Gamma : I \longrightarrow \mathbb{H}^n$ such that

$$\langle \ddot{\Gamma}(s), Z_j|_{\Gamma(s)}\rangle = 0$$

for any $s \in I$ and any $j = 1, \ldots, 2n$, where here and in the following I is a domain of $\mathbb{R}$ containing 0. Indeed the following simple characterization holds.

Proposition 20.4.1. *Let $\Gamma : I \longrightarrow \mathbb{H}^n$ be a horizontal curve. The following are equivalent.*

(i) $\nabla^{\mathbb{H}}_{\dot{\Gamma}}\dot{\Gamma} = 0$ *along* Γ.

(ii) Γ *is a horizontal line.*

Proof. Let $A = \sum_{j=1}^{2n} A_j Z_j$ be any C^2 extension of $\dot{\Gamma}$. Γ is a horizontal line if and only if $t \mapsto A_j(\Gamma(t))$ is constant on I for any $j = 1, \ldots, 2n$. Notice that

$$\nabla^{\mathbb{H}}_A A\Big|_{\Gamma(s)} = \sum_{k=1}^{2n} A(A_k)|_{\Gamma(s)} Z_k|_{\Gamma(s)} = \sum_{k=1}^{2n} \dot{\Gamma}(A_k)|_{\Gamma(s)} Z_k|_{\Gamma(s)} = \sum_{k=1}^{2n} \frac{d(A_k(\Gamma(t)))}{dt}\Big|_s Z_k|_{\Gamma(s)}$$

for any $s \in I$. The thesis then follows. □

The higher dimensional case is typically more involved, since it is not always true that

$$\nabla^{\mathbb{H}}_X X = -h^{\mathbb{H}}(X,X)\nu^{\mathbb{H}}.$$

Nevertheless, there is a particular situation in which the second fundamental form provides global information.

Definition 20.4.2. *Let S be a hypersurface of class C^2. We say that S is* horizontally totally geodesic *when*

$$h^{\mathbb{H}}(X,X) = 0 \tag{20.4.1}$$

for any $X \in C^1(S, \mathcal{H}TS)$, that is when $h^{\mathbb{H}}$ is an alternating *bilinear form.*

Notice that (20.4.1) is equivalent to require that the symmetric form $\tilde{h}^{\mathbb{H}}$ is identically vanishing. Let us point out that, in view of Proposition 16.6.7 and Proposition 16.6.8, non-characteristic hypersurfaces of class C^2 with $h^{\mathbb{H}} \equiv 0$ are trivially vertical hyperplanes, provided that $n \geq 2$. Indeed, if S is such a hypersurface, N is its Euclidean unit normal and $\nu^{\mathbb{H}}$ its horizontal unit normal, then Proposition 16.6.8 and (16.6.8) imply that $\tilde{h}^{\mathbb{H}} \equiv 0$, $N_{2n+1} \equiv 0$, $N = N(\bar{x}, \bar{y})$ and $\nu^{\mathbb{H}} = (N_1, \dots, N_{2n})$. Hence

$$0 = |\tilde{h}^{\mathbb{H}}|^2 = \sum_{i,j=1}^{2n} Z_i \nu_j^{\mathbb{H}} Z_j \nu_i^{\mathbb{H}} = \sum_{i,j=1}^{2n+1} \frac{\partial N_j}{\partial z_i} \frac{\partial N_i}{\partial z_j},$$

where the last term coincides with the squared norm of the Euclidean second fundamental form of S. Hence S is a hyperplane, which is vertical since $N_{2n+1} \equiv 0$. As already mentioned, when $n \geq 2$ it is not in general true that horizontally totally geodesic hypersurfaces satisfy $h = 0$.

Example 20.4.3. As an instance, consider in $\mathbb{H}^2$ the non-vertical hyperplane

$$S := \{(\bar{x}, \bar{y}, t) \in \mathbb{H}^2 \,:\, a_1 x_1 + a_2 x_2 + b_1 y_1 + b_2 y_2 + t + d = 0\}$$

for some $a_1, a_2, b_1, b_2, d \in \mathbb{R}$. An easy computation shows that

$$N(p) = \frac{(a_1, a_2, b_1, b_2, 1)}{\sqrt{1 + a_1^2 + a_2^2 + b_1^2 + b_2^2}} \qquad \text{and} \qquad N^{\mathbb{H}}(p) = \frac{(a_1 + y_1, a_2 + y_2, b_1 - x_1, b_2 - x_2)}{\sqrt{1 + a_1^2 + a_2^2 + b_1^2 + b_2^2}}$$

for any $p \in S$. Therefore, S has a unique characteristic point $p_0 = (b_1, b_2, -a_1, -a_2, -d)$. Far from p_0, $\nu^{\mathbb{H}}$ can be expressed by

$$\nu^{\mathbb{H}}(p) = \frac{(a_1 + y_1, a_2 + y_2, b_1 - x_1, b_2 - x_2)}{\sqrt{(a_1 + y_1)^2 + (a_2 + y_2)^2 + (b_1 - x_1)^2 + (b_2 - x_2)^2}}$$

for any $p \in S \setminus S_0$.

Recalling (16.6.8), a tedious but simple computations shows that

$$\sum_{h,k=1}^{4} Z_h(\nu_k^{\mathbb{H}})Z_k(\nu_h^{\mathbb{H}}) = -\frac{2}{(a_1+y_1)^2+(a_2+y_2)^2+(b_1-x_1)^2+(b_2-x_2)^2} = -2(Td^{\mathbb{H}})^2.$$

Hence, Proposition 16.6.8 implies that $\tilde{h}^{\mathbb{H}} \equiv 0$ on $S \setminus S_0$. Nevertheless, in view of the previous computation and Proposition 16.6.7, we conclude that

$$|h_p|^2 = \frac{2}{(a_1+y_1)^2+(a_2+y_2)^2+(b_1-x_1)^2+(b_2-x_2)^2}$$

for any $p \in S \setminus S_0$.

20.5 Local existence of geodesics on hypersurfaces

Let S be a hypersurface of class C^2. Let $p \in S \setminus S_0$ and $w \in \mathcal{H}T_pS$. We wish to find a curve $\Gamma \in C^2(I, S)$ solving the differential problem

$$\begin{cases} \Gamma \text{ is horizontal} \\ \nabla^{\mathbb{H},S}_{\dot\Gamma}\dot\Gamma = 0 \qquad \text{on } I \\ \Gamma(0) = p \\ \dot\Gamma(0) = w \end{cases} \tag{20.5.1}$$

Arguing for instance as in [173], it is not difficult to show that solutions to (20.5.1) are geodesics in the Carnot-Carathéodory space associated with the sub-Riemannian structure $(S, \langle\cdot,\cdot\rangle_S)$. First, notice that, by means of [140, Theorem 6.5] and [16, Theorem 1.2] and without loss of generality, there exists $\Omega \subseteq \mathbb{R}^{2n}$ and $\varphi \in C(\Omega)$ such that $W^\varphi\varphi \in C(\Omega, \mathbb{R}^{2n-1})$, $U = i(\Omega)\cdot j(\mathbb{R})$ is an open neighborhood of p and

$$S \cap U = \operatorname{graph}_{Y_1}(\varphi, \Omega) \cap U.$$

We refer to Section 16.7.2 for the notation. Therefore we reduce (20.5.1) to a differential problem for curves in Ω. To this aim, fix $q \in \Omega$ such that $\Psi(q) = p$, and let $\gamma(s) = (\bar\xi(s), \tilde\eta(s), \tau(s)) : I \longrightarrow \Omega$. If we lift γ to a curve $\Gamma : I \longrightarrow \mathbb{H}^n$ by letting

$$\Gamma(s) = \Psi(\gamma(s)) = (\bar\xi(s), \varphi(\gamma(s)), \tilde\eta(s)), \tau(s) - \xi_1(s)\varphi(\gamma(s)))$$

for any $s \in I$, then by construction $\Gamma(I) \subseteq S$. From now on, we fix the notation $\alpha(s) := \varphi(\gamma(s))$. To give a meaning to (20.5.1) we need that $\dot\Gamma$ is horizontal. Notice that

$$\begin{aligned}\dot\Gamma &= (\dot\xi_1, \ldots, \dot\xi_n, \dot\alpha, \dot\eta_2, \ldots, \dot\eta_n, \dot\tau - \dot\xi_1\alpha - \xi_1\dot\alpha) \\ &= \sum_{j=1}^{n} \dot\xi_j X_j + \dot\alpha Y_1 + \sum_{2=1}^{n} \dot\eta_j Y_j + \left(\dot\tau - 2\alpha\dot\xi_1 - \sum_{j=2}^{n} \eta_j\dot\xi_j + \sum_{j=2}^{n} \xi_j\dot\eta_j\right) T.\end{aligned}$$

Therefore $\dot\Gamma$ admits a C^1 extension to the whole $\mathcal{H}TS$ if and only if

$$\dot\tau = 2\alpha\dot\xi_1 + \sum_{j=2}^{n} \eta_j \dot\xi_j - \sum_{j=2}^{n} \xi_j \dot\eta_j, \tag{20.5.2}$$

that is if and only if γ is horizontal in (Ω, d_φ). Let us denote such an extension by $A = \sum_{j=1}^{2n} \psi_j Z_j$. This means that $A \in C^1(S, \mathcal{H}TS)$ and

$$\psi_j(\Gamma(s)) = \dot\Gamma_j(s)$$

for any $s \in I$ and any $j = 1, \dots, 2n$. Thanks to the aforementioned properties of $\nabla^{\mathbb{H},S}$ and recalling (16.5.2), then

$$\begin{aligned}
\nabla^{\mathbb{H},S}_{\dot\Gamma}\dot\Gamma\Big|_{\Gamma(s)} &= \nabla^{\mathbb{H}}_{\dot\Gamma}\dot\Gamma\Big|_{\Gamma(s)} - \left\langle \nabla^{\mathbb{H}}_{\dot\Gamma}\dot\Gamma\Big|_{\Gamma(s)}, \nu^{\mathbb{H}}\Big|_{\Gamma(s)} \right\rangle \nu^{\mathbb{H}}\Big|_{\Gamma(s)} \\
&= \sum_{j=1}^{2n} \langle \dot\Gamma(s), \nabla^{\mathbb{H}}\psi_j(\Gamma(s))\rangle Z_j\Big|_{\Gamma(s)} - \left(\sum_{k=1}^{2n} \langle \dot\Gamma(s), \nabla^{\mathbb{H}}\psi_j(\Gamma(s))\rangle \nu_k^{\mathbb{H}}\Big|_{\Gamma(s)} \right) \nu^{\mathbb{H}}\Big|_{\Gamma(s)} \\
&= \sum_{j=1}^{2n} \ddot\Gamma_j(s) Z_j\Big|_{\Gamma(s)} - \left(\sum_{k=1}^{2n} \ddot\Gamma_k(s) \nu_k^{\mathbb{H}}\Big|_{\Gamma(s)} \right) \nu^{\mathbb{H}}\Big|_{\Gamma(s)}
\end{aligned}$$

for any $s \in I$. Hence $\nabla^{\mathbb{H},S}_{\dot\Gamma}\dot\Gamma = 0$ if and only if

$$\ddot\Gamma_j - \nu_j^{\mathbb{H}} \langle \ddot\Gamma, \nu^{\mathbb{H}} \rangle = 0 \tag{20.5.3}$$

for any $j = 1, \dots, 2n$. We need to traduce (20.5.3) in terms of γ. To this aim, recalling (20.5.2), notice that

$$\ddot\Gamma = \sum_{j=1}^{n} \ddot\xi_j X_j + \ddot\alpha Y_1 + \sum_{2=1}^{n} \ddot\eta_j Y_j.$$

Lemma 20.5.1. *It holds that*

$$\langle \ddot\Gamma, \nu^{\mathbb{H}} \rangle = -W^{-\frac{1}{2}} \left(2\tilde{T}\varphi \dot\alpha \dot\xi_1 + \langle D^2\varphi\dot\gamma, \dot\gamma \rangle \right),$$

where in the following we let

$$W = 1 + |W^{\varphi}\varphi|^2$$

Proof. Notice that

$$\dot\alpha(s) = \langle \dot\gamma, D\varphi(\gamma(s)) \rangle \qquad \text{and} \qquad \ddot\alpha(s) = \langle \ddot\gamma(s), D\varphi(\gamma(s)) \rangle + \langle D^2\varphi(\gamma(s))\dot\gamma(s), \dot\gamma(s) \rangle \tag{20.5.4}$$

for any $s \in I$. Moreover, taking derivatives in (20.5.2), we see that

$$\ddot\tau = 2\dot\alpha\dot\xi_1 + 2\alpha\ddot\xi_1 + \sum_{j=2}^{n} \eta_j \ddot\xi_j - \sum_{j=2}^{n} \xi_j \ddot\eta_j. \tag{20.5.5}$$

Exploiting (16.7.6), (20.5.4) and (20.5.5), we see that

$$
\begin{aligned}
W^{\frac{1}{2}}\langle \ddot{\Gamma}, \nu^{\mathbb{H}}\rangle &= W_1^{\varphi}\varphi\ddot{\xi}_1 + \sum_{j=2}^{n}\tilde{X}_j\varphi\ddot{\xi}_j + \sum_{j=2}^{n}\tilde{Y}_j\varphi\ddot{\eta}_j - \ddot{\alpha} \\
&= \ddot{\xi}_1\varphi_{\xi_1} + 2\ddot{\xi}_1\alpha\varphi_\tau + \sum_{j=2}^{n}\ddot{\xi}_j\varphi_{\xi_j} + \sum_{j=2}^{n}\eta_j\ddot{\xi}_j\varphi_\tau + \sum_{j=2}^{n}\ddot{\eta}_j\varphi_{\eta_j} - \sum_{j=2}^{n}\xi_j\ddot{\eta}_j\varphi_\tau \\
&\quad - \ddot{\xi}_1\varphi_{\xi_1} - \sum_{j=2}^{n}\ddot{\xi}_j\varphi_{\xi_j} - \sum_{j=2}^{n}\ddot{\eta}_j\varphi_{\eta_j} - \ddot{\tau}\varphi_\tau - \langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle \\
&= \tilde{T}\varphi\left(2\alpha\ddot{\xi}_1 + \sum_{j=2}^{n}\eta_j\ddot{\xi}_j - \sum_{j=2}^{n}\ddot{\eta}_j\xi_j - \ddot{\tau}\right) - \langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle \\
&= -2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1 - \langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle.
\end{aligned}
$$

□

In the following, we let $M = 2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1 + \langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle$. Notice that, by Lemma 20.5.1, the term $\langle \ddot{\Gamma}, \nu^{\mathbb{H}}\rangle$ does not involve second derivatives of γ. Therefore (20.5.1) is equivalent to the following differential problem.

$$
\begin{cases}
\ddot{\xi}_1 + W^{-1}W_1^{\varphi}\varphi M = 0 & \text{on } I, \quad \xi_1(0) = \xi_1^0, \quad \dot{\xi}_1(0) = w_1 \\
\ddot{\xi}_j + W^{-1}\tilde{X}_j\varphi M = 0 & \text{on } I, \quad \xi_j(0) = \xi_j^0, \quad \dot{\xi}_j(0) = w_j \quad j = 2, \dots, n \\
\ddot{\alpha} - W^{-1}M = 0 & \text{on } I, \quad \alpha(0) = y_1, \quad \dot{\alpha}(0) = w_{n+1} \\
\ddot{\eta}_j + W^{-1}\tilde{Y}_j\varphi M = 0 & \text{on } I, \quad \eta_j(0) = \eta_j^0, \quad \dot{\eta}_j(0) = w_{n+j} \quad j = 2, \dots, n \\
\dot{\tau} = 2\alpha\dot{\xi}_1 + \sum_{j=2}^{n}\eta_j\dot{\xi}_j - \sum_{j=2}^{n}\xi_j\dot{\eta}_j & \text{on } I, \quad \tau(0) = t + \xi_1^0\varphi(q)
\end{cases}
\tag{20.5.6}
$$

A key step consist in showing that the third line of (20.5.6) is redundant.

Lemma 20.5.2. *A curve $\gamma \in C^2(I, \Omega)$ solves (20.5.6) if and only if it solves the following differential system.*

$$
\begin{cases}
\ddot{\xi}_1 + W^{-1}W_1^{\varphi}\varphi M = 0 & \textit{on } I, \quad \xi_1(0) = \xi_1^0, \quad \dot{\xi}_1(0) = w_1 \\
\ddot{\xi}_j + W^{-1}\tilde{X}_j\varphi M = 0 & \textit{on } I, \quad \xi_j(0) = \xi_j^0, \quad \dot{\xi}_j(0) = w_j \quad j = 2, \dots, n \\
\ddot{\eta}_j + W^{-1}\tilde{Y}_j\varphi M = 0 & \textit{on } I, \quad \eta_j(0) = \eta_j^0, \quad \dot{\eta}_j(0) = w_{n+j} \quad j = 2, \dots, n \\
\dot{\tau} = 2\alpha\dot{\xi}_1 + \sum_{j=2}^{n}\eta_j\dot{\xi}_j - \sum_{j=2}^{n}\xi_j\dot{\eta}_j & \textit{on } I, \quad \tau(0) = t + \xi_1^0\varphi(q)
\end{cases}
\tag{20.5.7}
$$

Proof. If $\gamma \in C^2(I, \Omega)$ solves (20.5.6), then clearly solves (20.5.7). Conversely, assume that $\gamma \in C^2(I, \Omega)$ solves (20.5.7). Since $y_1 = \varphi(q)$, then $\alpha(0) = \varphi(\gamma(0)) = \varphi(q) = y_1$. Moreover, notice that

$$
\begin{aligned}
\dot{\alpha}(0) &= \langle \dot{\gamma}(0), D\varphi(q)\rangle \\
&= \dot{\xi}_1(0)\varphi_{\xi_1}(q) + \sum_{j=2}^{n} \dot{\xi}_j(0)\varphi_{\xi_j}(q) + \sum_{j=2}^{n} \dot{\eta}_j(0)\varphi_{\eta_j}(q) + \dot{\tau}(0)\varphi_{\tau}(q) \\
&= w_1 W_1^{\varphi}\varphi(q) + \sum_{j=2}^{n} w_j \tilde{X}_j\varphi(q) + \sum_{j=2}^{n} w_{n+j}\tilde{Y}_j\varphi(q) \\
&= w_{n+1},
\end{aligned}
$$

where the last equality follows from (16.7.6) and the fact that $w \in \mathcal{H}T_pS$. Observe that, recalling (20.5.5) and exploiting all the second-order equations in (20.5.7),

$$
\begin{aligned}
\langle \ddot{\gamma}, D\varphi\rangle &= \ddot{\xi}_1\varphi_{\xi_1} + \sum_{j=2}^{n} \ddot{\xi}_j\varphi_{\xi_j} + \sum_{j=2}^{n} \ddot{\eta}_j\varphi_{\eta_j} + \ddot{\tau}\tilde{T}\varphi \\
&= \ddot{\xi}_1 W_1^{\varphi}\varphi + \sum_{j=2}^{n} \ddot{\xi}_j\tilde{X}_j\varphi + \sum_{j=2}^{n} \ddot{\eta}_j\tilde{Y}_j\varphi + 2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1 \\
&= -W^{-1}M|W^{\varphi}\varphi|^2 + 2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1.
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
\ddot{\alpha} - W^{-1}M &= W^{-1}(W\langle \ddot{\gamma}, D\varphi\rangle + W\langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle - 2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1 - \langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle) \\
&= W^{-1}(\langle \ddot{\gamma}, D\varphi\rangle + |W^{\varphi}\varphi|^2\langle \ddot{\gamma}, D\varphi\rangle + |W^{\varphi}\varphi|^2\langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle - 2\tilde{T}\varphi\dot{\alpha}\dot{\xi}_1) \\
&= W^{-1}(-W^{-1}M|W^{\varphi}\varphi|^2 + |W^{\varphi}\varphi|^2\langle \ddot{\gamma}, D\varphi\rangle + |W^{\varphi}\varphi|^2\langle D^2\varphi\dot{\gamma}, \dot{\gamma}\rangle) \\
&= \frac{|W^{\varphi}\varphi|^2}{1 + |W^{\varphi}\varphi|^2}(\ddot{\alpha} - W^{-1}M),
\end{aligned}
$$

which is equivalent to say that

$$
W^{-1}(\ddot{\alpha} - W^{-1}M) = 0.
$$

Being $W^{-1} \neq 0$, the thesis follows. □

We can summarize the previous achievements in the following statement.

Proposition 20.5.3. *The following properties hold.*

(i) If $\Gamma \in C^2(I, S)$ solves (20.5.1), then $\gamma : I \longrightarrow \Omega$ defined by

$$
\gamma(s) := \Pi(\Gamma(s))
$$

for any $s \in I$ solves (20.5.7).

(ii) If $\gamma \in C^2(\Omega)$ solves (20.5.7), then $\Gamma : I \longrightarrow \Omega$ defined by

$$
\Gamma(s) := \Psi(\gamma(s))
$$

for any $s \in I$ solves (20.5.1).

Proof. (ii) follows thanks to Lemma 20.5.2. To prove (i), notice that, if Γ is as in the statement, then $\Gamma = \Psi(\sigma)$, where $\sigma = \Pi(\Gamma)$, and so (i) easily follows. □

Theorem 20.5.4. *The initial value problem* (20.5.1) *admits a unique local solution* $\Gamma \in C^2(I, S)$.

Proof. In view of Proposition 20.5.3, it suffices to show that the initial value problem (20.5.7) admits locally a unique solution. Notice that (20.5.7) can be seen as a fist-order initial value problem by means of a standard doubling variable argument. More precisely, let us introduce the equations

$$\dot{\xi}_1 = \Xi_1, \qquad \dot{\xi}_j = \Xi_j \qquad \text{and} \qquad \dot{\eta}_j = H_j \tag{20.5.8}$$

for any $j = 2, \ldots, n$, let us define the curve $\tilde{\Gamma} : I \longrightarrow \mathbb{R}^{4n-1}$ by

$$\tilde{\Gamma} = (\xi_1, \ldots, \xi_n, \eta_2, \ldots, \eta_n, \tau, \Xi_1, \ldots, \Xi_n, H_2, \ldots, H_n),$$

and let $\tilde{q} = (x_1, \ldots, x_n, y_2, \ldots, y_n, t - x_1 y_1, w_1, \ldots, w_n, w_{n+2}, \ldots, w_{2n})$. Then (20.5.7) is equivalent to the first-order initial value problem

$$\begin{cases} \dot{\tilde{\Gamma}}(s) = F(s, \Gamma(s)) & \text{on } I \\ \tilde{\Gamma}(0) = \tilde{q} \end{cases} \tag{20.5.9}$$

where $F : I \times \mathbb{R}^{4n-1} \longrightarrow \mathbb{R}^{4n-1}$ is defined in the obvious way taking into account (20.5.7) and (20.5.8). Thanks to Proposition 16.7.3, F is of class C^1 in a neighborhood of $(0, \tilde{q})$. Hence the thesis follows by means of the classical Picard-Lindelöf Theorem (cf. e.g. [169]). □

Proof of Theorem 20.1.4. Fix $p = (\bar{x}, \bar{y}, t) \in S \setminus S_0$. Assume first that there exists an open neighborhood U of p such that $\tilde{h}^{\mathbb{H}} \equiv 0$ on U. Fix $w \in \mathcal{H}T_p S$. As before, recalling also Proposition 16.7.3, we can assume that there exists $\Omega \subseteq \mathbb{R}^{2n}$ and $\varphi \in C^2(\Omega)$ such that

$$S \cap V = \text{graph}_{Y_1}(\varphi, \Omega) \cap V,$$

where $V = i(\Omega) \cdot j(\mathbb{R})$. In view of Theorem 20.5.4, there exists a small domain $I \subseteq \mathbb{R}$ such that $0 \in I$ and a curve $\Gamma \in C^2(I, S)$ solving (20.5.1) with initial data $\Gamma(0) = p$ and $\dot{\Gamma}(0) = w$. Since $\tilde{h}^{\mathbb{H}} \equiv 0$, and recalling Proposition 20.4.1, we conclude that $\Gamma(s) := p \cdot (sw, 0)$. Hence S is locally ruled at p. Conversely, assume that S is locally ruled in a suitable neighborhood U of p. Assume also that $U \cap S_0 = \emptyset$. Fix $\bar{p} \in U$ and $w \in \mathcal{H}T_{\bar{p}} S$. Since $\Gamma(s) := \bar{p} \cdot (sw, 0)$ lies locally in S, then $h_{\bar{p}}(w, w) = 0$, and so $\tilde{h}^{\mathbb{H}}_{\bar{p}} = 0$. □

Proof of Theorem 20.1.1. The first equivalence follows from Theorem 20.1.3. If in addition S is topologically closed and $n \geq 2$, arguing as in [81, Proposition 4.1] it is easy to see that the fact that S is horizontally totally geodesic implies that S_0 is constituted by isolated points, and so it is countable. The thesis then follows by Theorem 20.1.4. □

Proof of Theorem 20.1.5. S is clearly a smooth hypersurface. Let $p \in S \setminus S_0$. It is well-known that

$$N(p) = \frac{1}{\sqrt{1 + |Du(z)|^2_{\mathbb{R}^{2n}}}}(Du(z), -1) = \frac{1}{\sqrt{1 + x_1^2 + y_1^2}}(x_1, 0, \dots, 0, -y_1, 0, \dots, 0, -1),$$

and so

$$\nu^{\mathbb{H}}(p) = \nu^{\mathbb{H}}(z) = \frac{1}{\sqrt{2(x_1 - y_1)^2 + \sum_{j=2}^n (x_j^2 + y_j^2)}}(x_1 - y_1, -y_2, \dots, -y_n, x_1 - y_1, x_2, \dots, x_n).$$

Since in this case $\nu^{\mathbb{H}}$ does not depend on t, an easy computation shows that

$$\operatorname{div}_{\mathbb{H}} \nu^{\mathbb{H}}(p) = \operatorname{div}_{\mathbb{R}^{2n}} \nu^{\mathbb{H}}(z) = 0 \tag{20.5.10}$$

for any $p \in S \setminus S_0$. Since $n \geq 2$, (20.5.10) allows us to apply [85, Corollary F] and [35, Theorem 2.3], which, together with [262, Example 5.29], imply that S is minimal. We conclude noticing that, in view of Theorem 20.1.1, S is not horizontally totally geodesic. □

20.6 Ruled intrinsic cones

In this section we study ruled hypersurfaces among the class of hypersurfaces which are invariant under intrinsic dilations, that is the class of intrinsic cones introduced in Section 16.7.3. Although all the results of this section are covered by Theorem 20.1.1 and Theorem 20.1.3, we we propose some *ad hoc* procedures that may have an independent interest. The following first characterization follows at once by Theorem 20.1.3, but we give here a different proof in the spirit of [164, Lemma 4.4].

Theorem 20.6.1. *Let S be a conical hypersurface of class C^1. Assume that $S_0 = \{0\}$. Then S is ruled if and only if S is the horizontal plane H_0.*

Proof. For sake of notational simplicity, we prove the statement when $n = 2$, being the other cases completely analogous. We already know that $\mathcal{H}_0$ is ruled. Conversely, let S be ruled, and assume by contradiction that there exists $p = (z, t) \in S$ with $t \neq 0$. Then, thanks to Proposition 16.7.6, $p \in S \setminus S_0$. Moreover, $p \cdot \mathcal{H}T_pS \cap S_0 = \emptyset$, since otherwise there would be an horizontal line joining p and 0, which contradicts the fact that horizontal lines passing through 0 lie in $\mathcal{H}_0$. Therefore, being S ruled and thanks to Proposition 20.2.4, we infer that $p \cdot \mathcal{H}T_pS \subseteq S$. It is well known (cf. e.g. [87]) that there exists an orthonormal basis u, v, w of $\mathcal{H}T_pS$ such that

$$J(u) = w \qquad \text{and} \qquad J(v) = \nu_S(p). \tag{20.6.1}$$

Let us set

$$M := \begin{bmatrix} u & v & J(u) & J(v) \end{bmatrix}^T.$$

Then, defining φ_R as in (20.2.4) and thanks to Proposition 20.2.10, we can assume that $u = X_1$, $v = X_2$ and $w = Y_1$. Let us define $\varphi : (0, +\infty) \times \mathbb{R}^3 \longrightarrow S$ by

$$\varphi(\lambda, \alpha, \beta, \gamma) := \delta_\lambda \left(p \cdot \left(\frac{\alpha}{\lambda} u + \frac{\beta}{\lambda} v + \frac{\gamma}{\lambda} w \right) \right).$$

Being S a ruled cone, the map φ is well-defined. Moreover, notice that

$$\begin{aligned}\varphi(\lambda, \alpha, \beta, \gamma) &= \delta_\lambda \left(z + \frac{\alpha u + \beta v + \gamma w}{\lambda}, t + \frac{\alpha Q(z,u) + \beta Q(z,v) + \gamma Q(z,w)}{\lambda} \right) \\ &= (\lambda z + \alpha u + \beta v + \gamma w, \lambda^2 t + \lambda\alpha Q(z,u) + \lambda\beta Q(z,v) + \lambda\gamma Q(z,w)) \\ &= (\lambda x_1 + \alpha, \lambda x_2 + \beta, \lambda y_1 + \gamma, \lambda y_2, \lambda^2 t + \lambda\alpha y_1 + \lambda\beta y_2 - \lambda\gamma x_1).\end{aligned}$$

Therefore, an easy computation shows that

$$D\varphi(\lambda, \alpha, \beta, \gamma) = \begin{bmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ y_1 & 0 & 0 & 1 \\ y_2 & 0 & 0 & 0 \\ 2\lambda t + \alpha y_1 + \beta y_2 - \gamma x_1 & \lambda y_1 & \lambda y_2 & -\lambda x_1 \end{bmatrix}$$

We claim that $y_2 \neq 0$. Otherwise, recalling that $p \cdot \mathcal{H}T_p S \subseteq S$, we would have that

$$(x_1, x_2, y_1, 0, t) \cdot (\alpha, \beta, \gamma, 0, 0) = (x_1 + \alpha, x_2 + \beta, y_1 + \gamma, 0, t + \alpha y_1 - \gamma x_1) \in S$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$. Therefore, choosing $\alpha = -x_1$, $\beta = -x_2$ and $\gamma = -y_1$, we conclude that $(0,0,0,0,t) \in S$, which is a contradiction, since $0 \in S$ and S, thanks to Proposition 16.7.7, is a t-graph. Hence $y_2 \neq 0$, and so, since $\varphi(0,0,0,0) = p$, $D\varphi$ has maximum rank in a neighborhood of $(0,0,0,0)$. In particular,

$$T_{\varphi(q)} S = \operatorname{span} \left\{ \frac{\partial \varphi}{\partial \lambda}(q), \frac{\partial \varphi}{\partial \alpha}(q), \frac{\partial \varphi}{\partial \beta}(q), \frac{\partial \varphi}{\partial \gamma}(q) \right\}$$

for any q sufficiently close to $(0,0,0,0)$. Notice that, if we define the 1-form ω by

$$\omega = dt - \sum_{j=1}^n y_j dx_j + \sum_{j=1}^n x_j dy_j,$$

then v is horizontal if and only if $\omega(v) = 0$. Fix $q = (\lambda, \alpha, \beta, \gamma)$ close to $(0,0,0,0)$. Then

$$\omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \lambda}(q) \right) = 2(\lambda t + \alpha y_1 + \beta y_2 - \gamma x_1),$$

and moreover

$$\omega|_{\varphi(q)}\left(\frac{\partial\varphi}{\partial\alpha}(q)\right)=-\gamma,\qquad \omega|_{\varphi(q)}\left(\frac{\partial\varphi}{\partial\beta}(q)\right)=0,\qquad \omega|_{\varphi(q)}\left(\frac{\partial\varphi}{\partial\gamma}(q)\right)=\alpha.$$

Therefore, if we choose $\alpha=\gamma=0$, we conclude that

$$\operatorname{span}\left\{\frac{\partial\varphi}{\partial\alpha}(q),\frac{\partial\varphi}{\partial\beta}(q),\frac{\partial\varphi}{\partial\gamma}(q)\right\}\subseteq\mathcal{H}T_{\varphi(q)}(S).$$

Moreover, since $y_2\neq 0$ we can choose $\beta=-\frac{\lambda t}{y_2}$ to conclude that

$$\frac{\partial\varphi}{\partial\lambda}(q)\in\mathcal{H}T_{\varphi(q)}S.$$

Since $\operatorname{rank}(D\varphi(q))=4$, we proved that

$$\varphi\left(\lambda,0,-\frac{\lambda t}{y_2},0\right)=\left(\lambda x_1,\lambda x_2-\frac{\lambda t}{y_2},\lambda y_1,\lambda y_2,0\right)\in S_0$$

for any $\lambda>0$ small enough. Since $y_2\neq 0$, we proved that there exists $\tilde p\neq 0$ such that $\tilde p\in S_0$. This is a contradiction with the assumption $S_0=\{0\}$. □

In the following result, which is implied by Theorem 20.1.1, we characterize ruled conical hypersurfaces S of class C^2. In this case, in view of Proposition 16.7.8, it suffices to consider graphs of quadratic polynomials.

Theorem 20.6.2. *Let $n\geq 2$ and let S be a ruled conical hypersurface of class C^2. If $S_0\neq\emptyset$, then S is the horizontal hyperplane H_0.*

Proof. For sake of notational simplicity, we assume again that $n=2$, being the other cases completely analogous. We divide the proof into some steps.

Step 1. Thanks to Proposition 16.7.8, we assume that $S=\operatorname{graph}(u)$, where

$$u(\bar x,\bar y)=ax_1^2+bx_2^2+cy_1^2+dy_2^2+ex_1x_2+fx_1y_1+gx_1y_2+hx_2y_1+mx_2y_2+py_1y_2,$$

for some $a,b,\dots,m,p\in\mathbb{R}$. Let us define $\varphi:\mathbb{R}^4\longrightarrow\operatorname{graph}(u)$ by

$$\varphi(\bar x,\bar y)=(\bar x,\bar y,u(\bar x,\bar y)).$$

Then φ is a global C^2 parametrization of S. Therefore, for any $p=(\bar x,\bar y)\in\mathbb{R}^4$, $T_{\varphi(p)}S$ is generated by

$$\frac{\partial\varphi}{\partial x_1}(p)=X_1+(2ax_1+ex_2+(f-1)y_1+gy_2)T,$$

$$\frac{\partial\varphi}{\partial x_2}(p)=X_2+(ex_1+2bx_2+hy_1+(m-1)y_2)T,$$

$$\frac{\partial\varphi}{\partial x_2}(p)=Y_1+((f+1)x_1+hx_2+2cy_1+py_2)T$$

and

$$\frac{\partial \varphi}{\partial x_2}(p) = Y_2 + (gx_1 + (m+1)x_2 + py_1 + 2dy_2)T.$$

Let us define the 4×4 real-valued matrix M by

$$M = \begin{bmatrix} 2a & e & f-1 & g \\ e & 2b & h & m-1 \\ f+1 & h & 2c & p \\ g & m+1 & p & 2d \end{bmatrix},$$

and, for any $j = 1, \dots, 4$, we let v_j be the j-th row of M. Notice that $p = (z, t) \in S$ is a characteristic point of S if and only if $M \cdot z = 0$.

Step 2. We prove that $\operatorname{rank}(M) \in \{2, 3\}$. Since we are assuming that S_0 is infinite, then $\operatorname{rank}(M) \leqslant 3$, and so in particular S_0 is a linear subspace of $\mathbb{R}^4$ with $\dim(S_0) \geq 1$. Moreover, $\operatorname{rank}(M) \neq 0$, since otherwise we would have that $S = S_0 \subseteq \mathcal{H}_0$, and so $S = S_0 = \mathcal{H}_0$, which is impossible since 0 is the only characteristic point of H_0. Moreover, we claim that $\operatorname{rank}(M) \geq 2$. Otherwise, if $\operatorname{rank}(M) = 1$, then we can assume without loss of generality that $v_1 \neq 0$ and that there exist $A, B, C \in \mathbb{R}$ such that $v_2 = Av_1$, $v_3 = Bv_1$ and $v_4 = Cv_1$. Therefore in particular $e = 2Aa$, $f = 2Ba - 1$ and $g = 2Ca$. Moreover, since $h = Be$ and $h = A(f-1)$, we infer that $0 = Be - A(f-1) = 2ABa - 2ABa + 2A = 2A$, and so $A = 0$. Moreover, since $p = Bg$ and $p = C(f-1)$, we conclude as above that $C = 0$. But this is impossible, since it would imply that $m - 1 = m + 1 = 0$. Therefore we conclude that $\operatorname{rank}(M) \in \{2, 3\}$.

Step 3. Let now $p = (z, p) \in S \setminus S_0$. Since then $M \cdot z \neq 0$, we can assume that $\langle v_1, z \rangle \neq 0$. Hence, there exists an open neighborhood $\tilde{U}$ of p such that $\langle v_1, z_q \rangle \neq 0$ for any $q = (z_q, t_q) \in \tilde{U}$. This implies in particular that $M \cdot z_q \neq 0$ for any $q \in \tilde{U}$, and so $\tilde{U} \cap S \subseteq S \setminus S_0$. Let now U be an open neighborhood of p such that $U \Subset \tilde{U}$. We are going to show that there exists an open neighborhood W of 0 such that

$$\mathcal{H}T_pS \cap W \subseteq \{(\bar{x}, \bar{y}) \in \mathbb{R}^4 \,:\, u(\bar{x}, \bar{y}) = 0\} =: G. \tag{20.6.2}$$

Let us define

$$A = \frac{\langle v_2, z \rangle}{\langle v_1, z \rangle}, \qquad B = \frac{\langle v_3, z \rangle}{\langle v_1, z \rangle}, \qquad C = \frac{\langle v_4, z \rangle}{\langle v_1, z \rangle}.$$

Recalling the computations of the first step, it is clear that

$$\mathcal{H}T_pS = \operatorname{span}\{X_2 - AX_1, Y_1 - BX_1, Y_2 - CX_1\}.$$

Therefore, being S ruled and $p \in S \setminus S_0$, it follows that

$$\begin{aligned}(x_1, x_2, y_1, y_2, u(\bar{x}, \bar{y})) &\cdot (-\alpha A, -\beta B, -\gamma C, \alpha, \beta, \gamma, 0) \\ &= (x_1 - \alpha A - \beta B - \gamma C, x_2 + \alpha, y_1 + \beta, y_2 + \gamma, \\ &\quad u(\bar{x}, \bar{y}) - \alpha A y_1 - \beta B y_1 - \gamma C y_1 + \alpha y_2 - \beta x_1 - \gamma x_2) \in S\end{aligned}$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough. Hence, noticing that

$$
\begin{aligned}
&u(x_1 - \alpha A - \beta B - \gamma C, x_2 + \alpha, y_1 + \beta, y_2 + \gamma) = \\
&\qquad ax_1^2 + a\alpha^2 A^2 + a\beta^2 B^2 + a\gamma^2 C^2 - 2a\alpha A x_1 - 2a\beta B x_1 - 2a\gamma C x_1 + 2a\alpha\beta AB \\
&\qquad + 2A\alpha\gamma AC + 2a\beta\gamma BC + bx_2^2 + 2b\alpha x_2 + b\alpha^2 + cy_1^2 + 2c\beta y_1 + c\beta^2 + dy_2^2 + 2d\gamma y_2 \\
&\qquad + d\gamma^2 + ex_1x_2 + e\alpha x_1 - e\alpha A x_2 - e\alpha^2 A - e\beta B x_2 - e\alpha\beta B - e\gamma C x_2 - e\alpha\gamma C \\
&\qquad + fx_1y_1 + f\beta x_1 - f\alpha A y_1 - f\alpha\beta A - f\beta B y_1 - f\beta^2 B - f\gamma C y_1 - f\beta\gamma C \\
&\qquad + gx_2y_2 + g\gamma x_1 - g\alpha A y_2 - g\alpha\gamma A - g\beta B y_2 - g\beta\gamma B - g\gamma C y_2 - g\gamma^2 C \\
&\qquad + hx_2y_1 + h\beta x_2 + h\alpha y_1 + h\alpha\beta + mx_2y_2 + m\gamma x_2 \\
&\qquad + m\alpha y_2 + m\alpha\beta + py_1y_2 + p\gamma y_1 + p\beta y_2 + p\beta\gamma,
\end{aligned}
$$

we infer that

$$
\begin{aligned}
&a\alpha^2 A^2 + a\beta^2 B^2 + a\gamma^2 C^2 - 2a\alpha A x_1 - 2a\beta B x_1 - 2a\gamma C x_1 + 2a\alpha\beta AB \\
&+ 2A\alpha\gamma AC + 2a\beta\gamma BC + 2b\alpha x_2 + b\alpha^2 + 2c\beta y_1 + c\beta^2 + 2d\gamma y_2 \\
&+ d\gamma^2 + e\alpha x_1 - e\alpha A x_2 - e\alpha^2 A - e\beta B x_2 - e\alpha\beta B - e\gamma C x_2 - e\alpha\gamma C \\
&+ (f+1)\beta x_1 - (f-1)\alpha A y_1 - f\alpha\beta A - (f-1)\beta B y_1 - f\beta^2 B - (f-1)\gamma C y_1 - f\beta\gamma C \\
&+ g\gamma x_1 - g\alpha A y_2 - g\alpha\gamma A - g\beta B y_2 - g\beta\gamma B - g\gamma C y_2 - g\gamma^2 C \\
&+ h\beta x_2 + h\alpha y_1 + h\alpha\beta + (m+1)\gamma x_2 \\
&+ (m-1)\alpha y_2 + m\alpha\beta + p\gamma y_1 + p\beta y_2 + p\beta\gamma = 0
\end{aligned}
$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough. Hence, recalling the definition of A, B and C, we conclude that

$$
\begin{aligned}
&+ a\alpha^2 A^2 + a\beta^2 B^2 + a\gamma^2 C^2 + 2a\alpha\beta AB + 2A\alpha\gamma AC + 2a\beta\gamma BC + b\alpha^2 \\
&+ c\beta^2 + d\gamma^2 - e\alpha^2 A - e\alpha\beta B - e\alpha\gamma C - f\alpha\beta A - f\beta^2 B - f\beta\gamma C \\
&- g\alpha\gamma A - g\beta\gamma B - g\gamma^2 C + h\alpha\beta + m\alpha\beta + p\beta\gamma = 0
\end{aligned}
$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough, which is equivalent to (20.6.2).

Step 4. Let us define

$$P_p := \operatorname{span}\{(-A, 1, 0, 0), (-B, 0, 1, 0), (-C, 0, 0, 1)\}.$$

Then (20.6.2) implies that $P_p \cap \pi(W) \subseteq G$. Moreover, it is easy to see that $N := (1, A, B, C)$ is the Euclidean normal to P_p in $\mathbb{R}^4$. Let us define $V = \pi(U)$. Since π is open, then V is an open neighborhood of z. Moreover, being S a t-graph, then $\pi|_S$ is invertible, $V = \pi(U \cap S) = \pi(U \cap (S \setminus S_0))$ and $U \cap S = \pi^{-1}(V)$. Therefore, if $\tilde{z} \in V$, we let $\tilde{z} = z_q$, where q is the unique point in $U \cap S$ such that $\pi(q) = z_q$. For any $z_q \in V$, we define

$$A_q = \frac{\langle v_2, z_q \rangle}{\langle v_1, z_q \rangle}, \qquad B_q = \frac{\langle v_3, z_q \rangle}{\langle v_1, z_Q \rangle}, \qquad C_q = \frac{\langle v_4, z_q \rangle}{\langle v_1, z_Q \rangle},$$

and we let

$$P_q := \operatorname{span}\{(-A_q, 1, 0, 0), (-B_q, 0, 1, 0), (-C_q, 0, 0, 1)\}.$$

Again, $N_q := (1, A_q, B_q, C_q)$ is the Euclidean normal to P_q in $\mathbb{R}^4$. Notice in particular that $A_p = A$, $B_p = B$, $C_p = C$ and $P_p = P$, and that, since $U \Subset \tilde{U} \subseteq S \setminus S_0$, W can be chosen in such a way that $P_q \cap \pi(W) \subseteq G$ for any $z_q \in V$. Moreover, thanks to the choice of U, A_q, B_q and C_q are smooth functions on V.

Step 5. We claim that one between A_q, B_q, C_q is not constant in any neighborhood of z. Indeed, let Z be a neighborhood of z, let $a_1, \ldots, a_4, b_1, \ldots, b_4 \in \mathbb{R}$ be such that $b_1x_1' + b_2x_2' + b_3y_1' + b_4y_2' \neq 0$ for any $(\bar{x}', \bar{y}') \in Z$, and define

$$f(\bar{x}', \bar{y}') := \frac{a_1x_1' + a_2x_2' + a_3y_1' + a_4y_2'}{b_1x_1' + b_2x_2' + b_3y_1' + b_4y_2'}.$$

If f is constant on Z, then $Df \equiv 0$ on Z. A simple computation shows that this is equivalent to

$$a_1b_2 - a_2b_1 = a_1b_3 - a_3b_1 = a_1b_4 - a_4b_2 = a_2b_3 - a_3b_2 = a_2b_4 - a_4b_2 = a_3b_4 - a_4b_3 = 0.$$

This implies that the matrix

$$\hat{M} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

has rank one. Therefore, if A_q, B_q and C_q were all constant functions on Z, then we would have that $\operatorname{rank}(\hat{M}) \leqslant 1$, which contradicts the fact that $\operatorname{rank}(\hat{M}) > 1$. Therefore without loss of generality, we assume that A_q is not constant in any neighborhood of z.

Step 6. Since A_q is not constant in any neighborhood of z, there exists $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$ such that $A \in (s_1, s_2)$ and for any $s \in (s_1, s_2)$ there exists $q_s \in U$ such that $N_{q_s} = (1, s, B_{q_s}, C_{q_s})$. This implies that $\bigcup_{s \in (s_1, s_2)} P_{q_s} \cap \pi(W)$ has non-empty interior. But then, since $P_q \cap \pi(W) \subseteq G$ for any $q \in U$, G has non-empty interior. Being u a polynomial, the only possibility is that $u \equiv 0$, and thus $S = H_0$. $\square$

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