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Basis for high order divergence-free finite element spaces

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ABSTRACT

A method classically used in the lower polynomial degree for the construction of a finite element basis of the space of divergence-free functions is here extended to any polynomial degree for a bounded domain without topological restrictions. The method uses graphs associated with two differential operators: the gradient and the divergence, and selects the basis using a spanning tree of the first graph. It can be applied for the two main families of degrees of freedom, weights and moments, used to express finite element differential forms.

1. Introduction

Graph techniques, and in particular the so-called tree-cotree decomposition, are widely used in computational electromagnetics. It was first introduced in [1] (see also [2]) and since then many works have adapted and extended this technique; see, for instance, Section 5.3 of the book of Bossavit [3] and the references therein. These works are based on the graph induced by vertices and edges of the mesh and for this reason it is not easy the extension to high order finite elements. The use of the degrees of freedom introduced in [4,5], the weights, leads to a natural extension because they have a straightforward geometrical visualization as a graph. This fact suggests how to proceed when using more classical degrees of freedom, the moments. For these latter degrees of freedom, the graph structure is not geometrically evident.

In this work we focus on the construction of a basis of the space of divergence free Raviart–Thomas finite elements of any polynomial degree using tree-cotree techniques. Starting from a basis of the space of curl conforming edge elements we compute a basis of the image of the curl operator using the tree-cotree decomposition of a graph associated with the gradient operator. If the boundary of the domain is not connected, it is necessary to complete the previous set with discrete representatives of the second de Rham cohomology group basis (which are divergence-free functions that are not curls).

It is worth noting that this is not the only possible approach to construct a basis of divergence-free finite elements. In [6] (see also the references therein) the authors proceed by computing directly a basis of the kernel of the divergence operator. Moreover, certain $H(\mathrm{div})$ conforming basis includes a basis of the space of divergence free elements (see, e.g. [7,8]).

The construction presented in this paper extends to finite elements of any polynomial degree a classical technique well-known for spaces of degree one. The first results are those in [9–11]. In these contributions the computational domain is assumed to be simply connected with a connected boundary. This approach has been extended in [12,13] to general computational domains for finite elements space of degree one. We aim now at doing so for finite elements of any polynomial degree.

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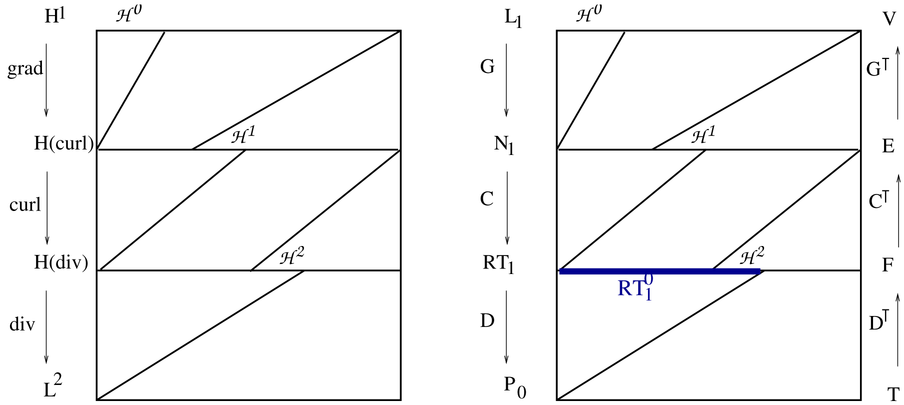

Fig. 1. The de Rham complex for the continuous spaces (left) and for Whitney differential forms (right).

We will use the Finite Element Exterior Calculus (FEEC) formalism. It unifies the notation for the different finite element spaces involved in the construction and clarifies the important role that the de Rham complex and the homology of Ω play in the construction of the basis.

Let $\mathcal{T}$ be a tetrahedral mesh of a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$. We will denote $\mathcal{P}_{r+1}^- \Lambda^k(\mathcal{T})$ the space of Whitney k-differential forms of degree $r+1$ (see e.g. [14]). They can be identified with L_{r+1}, the Lagrange finite elements of degree $r+1$, if $k=0$; with N_{r+1}, the first family of Nédélec finite elements of degree $r+1$, if $k=1$; with RT_{r+1}, the Raviart–Thomas finite elements of degree $r+1$, if $k=2$; and with P_r, the space of discontinuous piecewise polynomial functions of degree r, if $k=3$. When using the lowest order Whitney elements on a simplicial complex, $\mathcal{P}_1^- \Lambda^k(\mathcal{T})$, with $k=0,1,2,3$, the degrees of freedom are supported on the vertices (V), edges (E), faces (F) and tetrahedra (T) of the mesh respectively. It is well known (see e.g. [15]) that given an orientation to edges, faces and tetrahedra of the mesh, the matrices describing the differential operators $\mathrm{d} : \mathcal{P}_1^- \Lambda^k(\mathcal{T}) \to \mathcal{P}_1^- \Lambda^{k+1}(\mathcal{T})$ in terms of the degrees of freedom are the transposed of the matrices of the boundary operators $\partial : \mathcal{C}_{k+1}(\mathcal{T}, \mathbb{Z}) \to \mathcal{C}_k(\mathcal{T}, \mathbb{Z})$ being $\mathcal{C}_k(\mathcal{T}, \mathbb{Z})$ the group of k-chains in $\mathcal{T}$. Fig. 1 represents the de Rham's complex as in [16]. It summarizes these facts in both the continuous and the discrete case, with $\mathcal{H}^k$ denoting the cohomology groups for $k \in \{0,1,2\}$.

Since the boundary of an edge consists in two vertices, and any face belongs to the boundary of one or two tetrahedra, from the point of view of graph theory we observe that: (i) the matrix associated with the gradient is the transposed of the all-nodes incidence matrix of a directed and connected graph having a node for each vertex and an arc for each (oriented) edge of the mesh; (ii) the matrix associated with the divergence operator is an incidence matrix of a directed and connected graph having a node for each tetrahedra plus an additional node associated with the exterior of the domain, and an arc for each face. These facts have been used in different contexts as tree-cotree gauge (see [2,17–19]), construction of bases of the space of divergence-free Raviart–Thomas finite elements (see [11,20,21]) or the construction of discrete potentials (see [13,22]).

These two properties hold true also for $r > 0$ when using weights as degrees of freedom for $u \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{T})$ and a particular realizations of the moments (see e.g. [14])

$$u \ \mapsto \ \int_S \mathrm{Tr}_S(u) \wedge \eta, \quad \eta \in \mathcal{P}_{r-(\dim S-k)} \Lambda^{\dim S-k}(S),$$

being S any subsimplex of the mesh and Tr_S the trace operator on S. The key point is to use Bernstein polynomials to identify a basis of $\mathcal{P}_{r-(\dim S-k)} \Lambda^{\dim S-k}(S)$, following the approach in [23] (see also [24] where Bernstein polynomials are used to express a set of basis of $\mathcal{P}_{r+1}^- \Lambda^k(\mathcal{T})$).

If the boundary of the domain is connected, we provide a basis of RT_{r+1}^0 by selecting some elements of a cardinal basis of N_{r+1} and computing their curls. More precisely these are the elements corresponding to the moments in a *belted* tree (a spanning tree if the domain is simply connected) of the graph associated with the gradient operator.

When $r=0$, the use of a spanning tree of the graph associated with the gradient operator to identify a maximal set of linearly independent columns for the curl has been first proposed in a simply connected polyhedral domain Ω without cavities (see [11,25]) and extended to domains with an arbitrary topology in [21]. In $\mathbb{R}^2$, the kernel of the curl operator reduces to constant functions, and a basis of RT_{r+1}^0, when $r \geq 0$, can be obtained by computing the curl of a nodal basis of the space of continuous piecewise polynomial finite elements (see [26]).

Since PDEs have boundary condition we analyze how the construction has to be modified in order to take it into account. In this case we limit the analysis to simply connected computational domains.

This paper is organized as follows in Section 2 we introduce the notation concerning polynomial differential forms and the basic definitions for graphs. In Section 3 we precise the two families of degrees of freedom that will be considered in the sequel by relying on the notation for finite element exterior calculus. In Section 4 we first recall the dimension of the space RT^0_{r+1} taking into account the homology of the domain Ω. Then we construct a basis of the range of the curl operator, that is suitably completed to obtain the desired basis when the boundary of Ω is not simply connected. Section 5 contains the construction of a basis of the space of divergence free finite element functions with zero trace on $\partial\Omega$ in the case of simply connected domains and some heuristics for the general case. Few conclusions are given in Section 6.

2. Notation and basic tools

We introduce the notation concerning polynomial differential forms and some basic definitions for graphs. Note that, depending on the notion, we will be working on either the whole simplicial mesh $\mathcal{T}$ or on the single element T of $\mathcal{T}$.

2.1. Simplices and barycentric coordinates

Let j, l, m, n be integers such that $0 \le l - j \le n - m$. By $\Sigma(j : l, m : n)$ we denote the set of increasing maps from $\{j, \dots, l\}$ to $\{m, \dots, n\}$, that is

$$\Sigma(j : l, m : n) = \{\sigma : \{j, \dots, l\} \longrightarrow \{m, \dots, n\} \ : \ \sigma(j) < \sigma(j+1) < \dots < \sigma(l)\}.$$

We use multi-index notation and consider the sets

$$\mathcal{I}(d+1, r) := \{\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_d) \in \mathbb{N}^{d+1} \ : \ |\boldsymbol{\alpha}| = r\}$$

being $|\boldsymbol{\alpha}| = \sum_{i=0}^{d} \alpha_i$. The sum of multi-indexes of the same length is defined in the natural way.

Let $T \in \mathbb{R}^3$ be an 3-simplex with vertices x_0, x_1, x_2, x_3 in general position. We let $\Delta(T)$ denote all the subsimplices, or faces, of T, while $\Delta_k(T)$ is the set of subsimplices of T of dimension k, for any selected value of k between 0 and 3. For each $\sigma \in \Sigma(j : l, 0 : 3)$, we let f_σ be the (oriented) closed convex hull of the vertices $x_{\sigma(j)}, \dots, x_{\sigma(l)}$ which we henceforth denote by $f_\sigma = [x_{\sigma(j)}, \dots, x_{\sigma(l)}]$. There is a one-to-one correspondence between $\Delta_k(T)$ and $\Sigma(0 : k, 0 : 3)$.

Let $\mathcal{L}_k$ be the set of indices ℓ such that $s_\ell \in \Delta_k(\mathcal{T})$. By assigning an integer number a_ℓ to each simplex s_ℓ, we can define the k-chain $c = \sum_{\ell \in \mathcal{L}_k} a_\ell \, s_\ell$, *i.e.* a formal weighted sum of k-simplices s_ℓ in $\mathcal{T}$. We denote by

$$\mathcal{C}_k(\mathcal{T}) := \left\{ \sum_{\ell \in \mathcal{L}_k} a_\ell s_\ell \ : \ s_\ell \in \Delta_k(\mathcal{T}) \text{ and } a_\ell \in \mathbb{Z} \right\}$$

The boundary operator ∂_k takes a k-simplex s and returns the sum of all its $(k-1)$-faces f with coefficient 1 or -1 depending of whether the orientation of the $(k-1)$-face f matches or not with the orientation induced by that of the simplex s on f. The notion of boundary can be extended to a k-chain c by linearity, namely $\partial_k c = \partial_k(\sum_{\ell \in \mathcal{L}_k} a_\ell \, s_\ell) = \sum_{\ell \in \mathcal{L}_k} a_\ell \, \partial_k(s_\ell)$. Note that ∂_k is linear mapping from $\mathcal{C}_k(\mathcal{T})$ to $\mathcal{C}_{k-1}(\mathcal{T})$ and we have

$$0 \overset{\partial_0}{\longleftarrow} \mathcal{C}_0(\mathcal{T}) \overset{\partial_1}{\longleftarrow} \mathcal{C}_1(\mathcal{T}) \overset{\partial_2}{\longleftarrow} \mathcal{C}_2(\mathcal{T}) \overset{\partial_3}{\longleftarrow} \mathcal{C}_3(\mathcal{T})$$

From the property $\partial_k \partial_{k+1} = 0$, it follows that $\operatorname{Im} \partial_{k+1} \subset \operatorname{Ker} \partial_k$. The homology spaces $\mathcal{H}_k(\mathcal{T}; \mathbb{Z})$ are defined as the quotient spaces

$$\mathcal{H}_k(\mathcal{T}; \mathbb{Z}) = \operatorname{Ker} \partial_k / \operatorname{Im} \partial_{k+1}, \quad \beta_k := \dim \mathcal{H}_k(\mathcal{T}, \mathbb{Z}), \quad k = 0, 1, 2.$$

Let $\mathcal{P}_r(T)$ denote the space of polynomials in n variables of degree at most r. In the following, $\lambda_{T,0}, \lambda_{T,1}, \dots, \lambda_{T,n}$ are the barycentric coordinate functions with respect to T. Each function $\lambda_{T,i} \in \mathcal{P}_1(T)$ is determined by the equations $\lambda_{T,i}(x_j) = \delta_{i,j}$, $0 \le i, j \le n$, being $\delta_{\cdot,\cdot}$ the Kronecker's symbol. All together, the functions $\lambda_{T,i}$ form a basis of $\mathcal{P}_1(T)$, are non-negative on T, and sum to 1 identically on T. To make for the higher order $r \ge 1$, we introduce the Bernstein basis of the space $\mathcal{P}_r(T)$: it consists of all monomials of degree r in the variables $\lambda_{T,i}$. We have

$$\mathcal{P}_r(T) = \operatorname{span}\{\lambda_T^{\boldsymbol{\alpha}} \ : \ \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}, \qquad \lambda_T^{\boldsymbol{\alpha}} := \lambda_{T,0}^{\alpha_0} \lambda_{T,1}^{\alpha_1} \cdots \lambda_{T,n}^{\alpha_n}.$$

Whenever a fixed simplex T is understood, we may simplify the notation by writing

$$\lambda_i \equiv \lambda_{T,i}, \quad \lambda^{\boldsymbol{\alpha}} \equiv \lambda_T^{\boldsymbol{\alpha}}.$$

2.2. Polynomial differential forms

We denote by $\Lambda^k(T)$ the space of differential k-forms over T with smooth bounded coefficients. For $k = 0$, the set $\Lambda^0(T) = C^\infty(T)$ is the space of smooth functions over T with uniformly bounded derivatives of all orders. Furthermore, $\Lambda^k(T) \neq \{0\}$ for $0 \le k \le n$. We recall the exterior product $\omega \wedge \eta \in \Lambda^{k+l}(T)$ for $\omega \in \Lambda^k(T)$ and $\eta \in \Lambda^l(T)$. Let $\mathrm{d} : \Lambda^k(T) \to \Lambda^{k+1}(T)$ denote the exterior derivative operator.

We write $\mathrm{d}\lambda_0, \mathrm{d}\lambda_1, \dots, \mathrm{d}\lambda_n \in \Lambda^1(T)$ for the exterior derivatives of the barycentric coordinate functions. Clearly

$$\mathrm{d}\lambda_0 + \mathrm{d}\lambda_1 + \cdots + \mathrm{d}\lambda_n = 0,$$

on T since $\sum_{i=0}^n \lambda_i = 1$. If $\sigma \in \Sigma(j : l, m : n)$, we set $\mathrm{d}\lambda_\sigma := \mathrm{d}\lambda_{\sigma(j)} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(l)}$.

For $k > 0$ any element ω of $\Lambda^k(T)$ can be written as

$$\omega = \sum_{\sigma \in \Sigma(0:k-1,1:n)} a_\sigma \mathrm{d}\lambda_\sigma,$$

where $a_\sigma \in C^\infty(T)$. Taking $a_\sigma \in \mathcal{P}_r(T)$ we obtain the space $\mathcal{P}_r\Lambda^k(T)$ of polynomial differential k-forms of polynomial degree at most r. Moreover $\mathcal{P}_r\Lambda^0(T)$ coincides with $\mathcal{P}_r(T)$.

For $k > 0$,

$$\mathcal{P}_0\Lambda^k(T) = \mathrm{span}\{\mathrm{d}\lambda_\sigma \; : \; \sigma \in \Sigma(0 : k-1, 1 : n)\}.$$

Furthermore, if $0 < k < n$, we can write

$$\mathcal{P}_r\Lambda^k(T) = \mathrm{span}\{\lambda^{\boldsymbol{\alpha}}\mathrm{d}\lambda_\sigma \; : \; \sigma \in \Sigma(0 : k-1, 1 : n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}.$$

The set

$$\mathcal{BP}_r\Lambda^k(T) := \{\lambda^{\boldsymbol{\alpha}}\,\mathrm{d}\lambda_\sigma \; : \; \sigma \in \Sigma(0 : k-1, 1 : n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\} \tag{1}$$

is a basis of $\mathcal{P}_r\Lambda^k(T)$.

For $k = 0$

$$\mathcal{BP}_r\Lambda^0(T) := \{\lambda^{\boldsymbol{\alpha}} \; : \; \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$$

is a basis of $\mathcal{P}_r\Lambda^0(T)$ while for $k = n$

$$\mathcal{BP}_r\Lambda^n(T) := \{\lambda^{\boldsymbol{\alpha}}\mathrm{d}\lambda_1 \wedge \cdots \wedge \mathrm{d}\lambda_n \; : \; \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$$

is a basis of $\mathcal{P}_r\Lambda^n(T)$.

A particular set of polynomial differential k-forms of polynomial degree 1 are the Whitney's differential forms. They are associated with the k-simplices f of T. If $k = n$ then $f = T$ and the Whitney's differential form w_T is the volume form, of polynomial degree 0.

Definition 1. Let $k \geq 0$ and $f \in \Delta_k(T)$. The Whitney's differential form w_f associated with the subsimplex f is defined as follows:

- if $k = 0$ then f is a vertex of T, namely, $f = [x_i]$ for $i = 0, \dots, n$, and $w_f = w_{[x_i]} = \lambda_i$;
- if $k > 0$ then $f = f_\sigma$ for a $\sigma \in \Sigma(0 : k, 0 : n)$ and

$$w_{f_\sigma} = \sum_{i=0}^k (-1)^i \lambda_{\sigma(i)} \mathrm{d}w_{f_\sigma \setminus [x_{\sigma(i)}]}$$

being $f_\sigma \setminus [x_{\sigma(i)}] \in \Delta_{k-1}(T)$ the oriented $(k-1)$-face of T with the vertices of f_σ except $x_{\sigma(i)}$.

We can write $f_\sigma \setminus [x_{\sigma(i)}] = [x_{\sigma(0)}, \dots, \widehat{x_{\sigma(i)}}, \dots, x_{\sigma(k)}]$, where the widehat means that the underlying term is omitted from the list.

For each $\sigma \in \Sigma(0 : k, 0 : n)$ it holds that

$$\mathrm{d}w_{f_\sigma} = (k+1)!\,\mathrm{d}\lambda_\sigma = (k+1)!\,\mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)}.$$

In fact, when $k = 1$ one has $w_{f_\sigma} = \lambda_{\sigma(0)}\mathrm{d}\lambda_{\sigma(1)} - \lambda_{\sigma(1)}\mathrm{d}\lambda_{\sigma(0)}$ and

$$\mathrm{d}w_{f_\sigma} = \mathrm{d}\lambda_{\sigma(0)} \wedge \mathrm{d}\lambda_{\sigma(1)} - \mathrm{d}\lambda_{\sigma(1)} \wedge \mathrm{d}\lambda_{\sigma(0)} = 2\,\mathrm{d}\lambda_{\sigma(0)} \wedge \mathrm{d}\lambda_{\sigma(1)}.$$

For $k > 1$, by induction we obtain

$$\begin{aligned}\mathrm{d}w_{f_\sigma} &= \mathrm{d}\left(\sum_{i=0}^k (-1)^i \lambda_{\sigma(i)} \mathrm{d}w_{f_\sigma \setminus [x_{\sigma(i)}]}\right) = \sum_{i=0}^k (-1)^i \mathrm{d}\lambda_{\sigma(i)} \wedge \mathrm{d}w_{f_\sigma \setminus [x_{\sigma(i)}]} \\ &= \sum_{i=0}^k (-1)^i \mathrm{d}\lambda_{\sigma(i)} \wedge \left(k!\,\mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\sigma(i)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)}\right) \\ &= k!\,\sum_{i=0}^k \mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)} = k!\,(k+1)\,\mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)}.\end{aligned}$$

Then

$$w_{f_\sigma} = \sum_{i=0}^k (-1)^i \lambda_{\sigma(i)} \mathrm{d}w_{f_\sigma \setminus [x_{\sigma(i)}]} = k!\,\sum_{i=0}^k (-1)^i \lambda_{\sigma(i)}\,\mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\sigma(i)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)}.$$

In finite element exterior calculus, the space of Whitney's differential k-forms on T is denoted by

$$\mathcal{P}_1^- \Lambda^k(T) := \mathrm{span}\{w_f \ : \ f \in \Delta_k(T)\}.$$

Since there is a one to one correspondence between $\Delta_k(T)$ and $\Sigma(0:k,0:n)$ we can also write

$$\mathcal{P}_1^- \Lambda^k(T) := \mathrm{span}\{w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k,0:n)\}.$$

Definition 2. Whitney's differential k-forms of polynomial degree $r+1$ are the elements of the space

$$\mathcal{P}_{r+1}^- \Lambda^k(T) := \mathrm{span}\{\lambda^{\boldsymbol{\alpha}} w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k,0:n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1,r)\}.$$

For $k>0$, the space $\mathcal{P}_{r+1}^- \Lambda^k(T) \subsetneq \mathcal{P}_{r+1} \Lambda^k(T)$.

For $k=0$

$$\mathcal{P}_{r+1}^- \Lambda^0(T) = \mathrm{span}\{\lambda^{\boldsymbol{\alpha}} \lambda_i \ : \ i \in \{0,\dots,n\} \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1,r)\}$$

$$= \mathrm{span}\{\lambda^{\widetilde{\boldsymbol{\alpha}}} \ : \ \widetilde{\boldsymbol{\alpha}} \in \mathcal{I}(n+1,r+1)\} = \mathcal{P}_{r+1}\Lambda^0(T).$$

For $k=n$

$$\mathcal{P}_{r+1}^- \Lambda^n(T) = \mathrm{span}\{\lambda^{\boldsymbol{\alpha}} \mathrm{d}\lambda_1 \wedge \cdots \wedge \mathrm{d}\lambda_n \ : \ \boldsymbol{\alpha} \in \mathcal{I}(n+1,r)\} = \mathcal{P}_r \Lambda^n(T).$$

Remark 3. It is worth noting that, in the n-simplex T with vertices $x_0, x_1, \dots, x_n$, the elements belonging to the set

$$\{\lambda^{\boldsymbol{\alpha}} w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k,0:n), \ \boldsymbol{\alpha} \in \mathcal{I}(n+1,r)\}$$

are not linearly independent. As an example, for $n=2$, if $k=1$, and $r=1$, it can be verified that

$$\lambda_0 w_{[x_1,x_2]} - \lambda_1 w_{[x_0,x_2]} + \lambda_2 w_{[x_0,x_1]} = 0. \tag{2}$$

Given $\sigma \in \Sigma(0:k,0:n)$ we set

$$\mathcal{I}_\sigma(n+1,r) := \{\boldsymbol{\alpha} \in \mathcal{I}(n+1,r) \ : \ \alpha_i = 0 \ \forall\, i < \sigma(0)\}.$$

When $k=0$ then f_σ is a vertex of T, namely, $f_\sigma = [x_j]$ being $\sigma(0)=j$. In this case, to be clearer, we will sometimes use the notation $\mathcal{I}_{[x_j]}(n+1,r)$ instead of $\mathcal{I}_\sigma(n+1,r)$.

A basis of $\mathcal{P}_{r+1}^- \Lambda^k(T)$ is

$$\mathcal{BP}_{r+1}^- \Lambda^k(T) = \{\lambda^{\boldsymbol{\alpha}} w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k,0:n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1,r)\}.$$

For $n=2$, $k=1$ and $r=1$, the 8 elements of $\mathcal{BP}_2^- \Lambda^1(T)$, with $T=[x_0,x_1,x_2]$, are

$$\begin{aligned} \lambda_i\, w_{[x_0,x_1]} &= \lambda_i\,(\,\lambda_0 \mathrm{d}\lambda_1 - \lambda_1 \mathrm{d}\lambda_0\,), & i &= 0,1,2,\\ \lambda_i\, w_{[x_0,x_2]} &= \lambda_i\,(\,\lambda_0 \mathrm{d}\lambda_2 - \lambda_2 \mathrm{d}\lambda_0\,), & i &= 0,1,2,\\ \lambda_i\, w_{[x_1,x_2]} &= \lambda_i\,(\,\lambda_1 \mathrm{d}\lambda_2 - \lambda_2 \mathrm{d}\lambda_1\,), & i &= 1,2\,. \end{aligned}$$

The condition $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(3,1)$ prevents $\lambda_0 w_{[x_1,x_2]}$ from being in the set $\mathcal{BP}_2^- \Lambda^1(T)$.

2.3. Graphs

We now introduce some basic definitions and results of graph theory that will be used in the sequel (they can be found, for instance, in [27]).

A graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$ consists of two sets: a finite set $\mathcal{N} = \{\mathfrak{n}_i\}_{i=1}^n$ of nodes and a finite set $\mathcal{A} = \{\mathfrak{a}_j\}_{j=1}^m$ of arcs. Each arc is identified with a pair of nodes. The two end nodes defining an arc need not be distinct. If the arc $\mathfrak{a}_j$ has the two end points equal to the same node $\mathfrak{n}_i$ then it is called a self-loop at node $\mathfrak{n}_i$. If the arcs of $\mathcal{M}$ are identified with ordered pairs of nodes, then $\mathcal{M}$ is called a directed or an oriented graph. Otherwise $\mathcal{M}$ is called an undirected or a non-oriented graph. The following definitions concern both directed and undirected graphs.

A walk is a finite alternating sequence of nodes and arcs $\mathfrak{n}_{i_0}, \mathfrak{a}_{j_1}, \mathfrak{n}_{i_1}, \mathfrak{a}_{j_2}, \mathfrak{n}_{i_2}, \dots, \mathfrak{n}_{i_{K-1}}, \mathfrak{a}_{j_K}, \mathfrak{n}_{i_K}$, such that, for $k \in \{1,\dots,K\}$, the arc $\mathfrak{a}_{j_k}$ is identified with the pair of nodes $\mathfrak{n}_{i_{k-1}}, \mathfrak{n}_{i_k}$. This walk is usually called a $\mathfrak{n}_{i_0} - \mathfrak{n}_{i_K}$ walk with $\mathfrak{n}_{i_0}$ and $\mathfrak{n}_{i_K}$ referred to as the end or terminal nodes of this walk. A walk is open if its end nodes, $\mathfrak{n}_{i_0}$, $\mathfrak{n}_{i_K}$ are distinct; otherwise it is closed. A walk is a trail if all its arcs are distinct. An open trail is a path if all its nodes are distinct. A closed trail is a circuit if all its nodes except the end nodes are distinct. A graph is said to be acyclic if it has no circuits.

Two nodes $\mathfrak{n}_i$, $\mathfrak{n}_{i'}$ are said to be connected in a graph $\mathcal{M}$ if there exists a $\mathfrak{n}_i - \mathfrak{n}_{i'}$ path in $\mathcal{M}$. A graph $\mathcal{M}$ is connected if there exists a path between every pair of nodes in $\mathcal{M}$.

Finally we recall the definition of a spanning tree in a graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$.

Definition 4. A tree in a graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$ is a connected acyclic subgraph of $\mathcal{M}$. A spanning tree $\mathcal{S}$ is a tree in $\mathcal{M}$ containing all its nodes.

It is worth noting that if $\mathcal{S}$ is a spanning tree of $\mathcal{M} = (\mathcal{N}, \mathcal{A})$, then $\mathcal{S} = (\mathcal{N}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathcal{A}$. Moreover $\mathcal{B}$ has exactly $n-1$ arcs. If $\mathcal{M}$ is not connected, then it has not spanning trees. We recall also the definition of the all-nodes incidence matrix of a directed graph.

Definition 5. The all-nodes incidence matrix $M^e \in \mathbb{Z}^{n\times m}$ of a directed graph $\mathcal{M} = (\mathcal{N}, \mathcal{A})$, with n nodes $\mathcal{N} = \{\mathfrak{n}_i\}_{i=1}^n$, m arcs $\mathcal{A} = \{\mathfrak{a}_j\}_{j=1}^m$ and with no self-loop, is the matrix with entries

$$[M^e]_{i,j} = \begin{cases} 1 & \text{if } \mathfrak{a}_j \text{ is incident on } \mathfrak{n}_i \text{ and oriented away from it,} \\ -1 & \text{if } \mathfrak{a}_j \text{ is incident on } \mathfrak{n}_i \text{ and oriented toward it,} \\ 0 & \text{if } \mathfrak{a}_j \text{ is not incident on } \mathfrak{n}_i\,. \end{cases}$$

3. Weights and moments

We recall the two families of degrees of freedom that will be considered in the sequel by relying on the FEEC form.

3.1. Small simplices and weights

The concepts of small simplices and weights for polynomial differential forms in $\mathcal{P}_{r+1}^- \Lambda^k(T)$, were born in [4,5], for any order k and any polynomial degree $r \geq 0$, to solve the difficulty raised in [28]: "*The main problem with such forms is the interpretation of DoFs*" in geometrical terms. We recall these concepts here below with a notation adapted to the isomorphism we want to state between these new DoFs, the weights, and the classical ones, moments, introduced in [14,29].

In the n-simplex T with vertices $x_0, x_1, \dots, x_n$ the principal lattice of order $r+1$ ($r \geq 0$) in T is the set of points defined by their barycentric coordinates with respect to the vertices of T as follows

$$S_{r+1}(T) = \left\{ x \in T \;:\; \lambda_i(x) \in \left\{ 0, \frac{1}{r+1}, \dots, \frac{r}{r+1}, 1 \right\} \text{ for each } i \in \{0, \dots, n\} \right\}.$$

To each multi-index $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$ we associate an affine function, $\tau_{\boldsymbol{\alpha}} : T \longrightarrow T$, such that $\tau_{\boldsymbol{\alpha}}(\lambda_i(x)) = \frac{\lambda_i(x)+\alpha_i}{r+1}$. If f_σ is a face of T then

$$\tau_{\boldsymbol{\alpha}}(f_\sigma) := \{\tau_{\boldsymbol{\alpha}}(x) \;:\; x \in f_\sigma\}.$$

Definition 6. The small k-simplexes of order r in T are the elements of the set

$$\begin{aligned} S_r^k(T) &= \{\tau_{\boldsymbol{\alpha}}(f_\sigma) \;:\; f_\sigma \in \Delta_k(T) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\} \\ &= \{\tau_{\boldsymbol{\alpha}}(f_\sigma) \;:\; \sigma \in \Sigma(0:k, 0:n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}. \end{aligned}$$

For $k > 0$, they are $1/(r+1)$-homothetic to k-faces of T, with vertices in $S_{r+1}(T)$. For $k = 0$, we have $S_r^0(T) = S_{r+1}(T)$.

For $k > 0$ there is a one-to-one correspondence between the elements of $S_r^k(T)$ and the couples $(\sigma, \boldsymbol{\alpha})$ with $\sigma \in \Sigma(0:k, 0:n)$ and $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$. In fact, if $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathcal{I}(n+1, r)$ and $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ then $\tau_{\boldsymbol{\alpha}}(T) \cap \tau_{\boldsymbol{\alpha}'}(T)$ is either empty or an element of $S_r^0(T)$.

The weight of $\omega \in \Lambda^k(T)$ on a k-simplex s contained in T is denoted by $\int_s \omega$. If $k = 0$, for $\omega \in C^\infty(T)$ and $s \in T$ we have $\int_s \omega = \omega(s)$.

In particular we are interested in the following set of weights.

Definition 7. Let $\omega \in \Lambda^k(T)$, $\sigma \in \Sigma(0:k, 0:n)$ and $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$.

$$W_{\sigma,\boldsymbol{\alpha}}(\omega) := \int_{\tau_{\boldsymbol{\alpha}}(f_\sigma)} \omega. \tag{3}$$

The weights of Definition 7 are determinant in $\mathcal{P}_{r+1}^- \Lambda^k(T)$, namely, if $\omega \in \mathcal{P}_{r+1}^- \Lambda^k(T)$ and $\int_s \omega = 0$ for all $s \in S_r^k(T)$ then $\omega = 0$ (see [30] for a proof). However, for $0 < k < n$, the cardinality of the set of weights $\{W_{\sigma,\boldsymbol{\alpha}}(\omega) \;:\; \sigma \in \Sigma(0:k, 0:n),\ \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$ is greater than the dimension of $\mathcal{P}_{r+1}^- \Lambda^k(T)$. Hence in the sequel we often consider the following set of weights:

$$W^k := \{W_{\sigma,\boldsymbol{\alpha}}(\omega) \;:\; \sigma \in \Sigma(0:k, 0:n),\ \boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)\}. \tag{4}$$

It is worth noting that W^k is determinant (see [31]) and its cardinality coincides with the dimension of $\mathcal{P}_{r+1}^- \Lambda^k(T)$.

Remark 8. Only one of the three representations for the small node shared by the three gray small triangles in Fig. 2 verifies the condition $\alpha_i = 0$ for all $i < \sigma(0)$ required to support a weight of the set defined in (4). The first representation fails the condition (in the up-left gray small triangle, $\alpha_0 \neq 0$ with $0 < \sigma(0) = 1$), the second satisfies it (in the up-right gray small triangle, $\alpha_1 \neq 0$ with $1 > \sigma(0) = 0$), the third fails too (in the bottom-center gray small triangle, $\alpha_0 \neq 0$ with $0 < \sigma(0) = 2$).

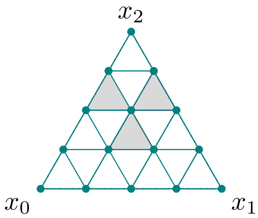

Fig. 2. Points of the principal lattice for $\mathcal{P}_4^- \Lambda^0(T)$, where T is a 2-simplex. The node with barycentric coordinates $\left(\frac{1}{4}, \frac{1}{4}, \frac{2}{4}\right)$ in T is shared by the three grey small triangles and it has different representations, as small node. Indeed, this point can be $\tau_{\boldsymbol{\alpha}}(f_\sigma)$ with, $\boldsymbol{\alpha} = (1,0,2)$, $f_\sigma = x_1$ in the top-left gray small triangle, $\boldsymbol{\alpha} = (0,1,2)$, $f_\sigma = x_0$ in the top-right gray small triangle, and $\boldsymbol{\alpha} = (1,1,1)$, $f_\sigma = x_2$ in the bottom-center gray small triangle, respectively.

3.2. Moments associated with a basis of polynomial differential forms

Let ω be a differential k-form defined on $T \subset \mathbb{R}^n$. For each d-face f_ζ of T, with $\zeta \in \Sigma(0:d, 0:n)$ and $k \le d \le n$, the moments of ω in f_ζ of degree $r-(d-k)$ are

$$M_{\zeta,\eta}(\omega) := \int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \omega \wedge \eta, \qquad \forall \eta \in \mathcal{P}_{r-(d-k)} \Lambda^{d-k}(f_\zeta), \tag{5}$$

where Tr_{f_ζ} is the trace operator on f_ζ.

It is well known that these moments are determinant in $\mathcal{P}_{r+1}^- \Lambda^k(T)$. Taking η in a basis of each space $\mathcal{P}_{r-(d-k)} \Lambda^{d-k}(f_\zeta)$, one obtains a determinant set of moments with cardinality equal to the dimension of $\mathcal{P}_{r+1}^- \Lambda^k(T)$ (see [14,30], for two different proofs).

The goal of the present work is to point out an isomorphism between moments and weights which is consistent in a sense specified in the next sections with the exterior derivative operator. To do that, we will consider a particular basis of the space $\mathcal{P}_{r-(d-k)} \Lambda^{d-k}(f_\zeta)$ in (5).

- If $d = k$ we adopt the Bernstein's basis of the space $\mathcal{P}_r(f_\zeta)$, namely

  $$\mathcal{BP}_r \Lambda^0(f_\zeta) = \{\lambda_{f_\zeta}^{\boldsymbol{\beta}} \;:\; \boldsymbol{\beta} \in \mathcal{I}(d+1, r)\},$$

  where $\lambda_{f_\zeta}^{\boldsymbol{\beta}} = \lambda_{f_\zeta,0}^{\beta_0} \cdots \lambda_{f_\zeta,d}^{\beta_d} = \lambda_{T,\zeta(0)}^{\beta_0} \cdots \lambda_{T,\zeta(d)}^{\beta_d}$.
- If $d > k$ we rely on the basis indicated in (1), namely,

  $$\mathcal{BP}_{r-(d-k)} \Lambda^{d-k}(f_\zeta) = \{\lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho \;:\; \rho \in \Sigma(0:d-(k+1), 1:d), \\ \boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k))\}.$$

  Here

  $$\begin{aligned}(\mathrm{d}\lambda_{f_\zeta})_\rho &= \mathrm{d}\lambda_{f_\zeta,\rho(0)} \wedge \cdots \wedge \mathrm{d}\lambda_{f_\zeta,\rho(d-(k+1))} \\ &= \mathrm{d}\lambda_{T,\zeta(\rho(0))} \wedge \cdots \wedge \mathrm{d}\lambda_{T,\zeta(\rho(d-(k+1)))}.\end{aligned}$$

With these choices of basis we obtain the following moments for $\omega \in \Lambda^k(T)$:
for each $\zeta \in \Sigma(0:k, 0:n)$, and $\boldsymbol{\beta} \in \mathcal{I}(k+1, r)$

$$M_{\zeta,\emptyset,\boldsymbol{\beta}}(\omega) := \int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \omega \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}}; \tag{6}$$

for each $d > k$, $\zeta \in \Sigma(0:d, 0:n)$, $\rho \in \Sigma(0:d-(k+1), 1:d)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k))$

$$M_{\zeta,\rho,\boldsymbol{\beta}}(\omega) := \int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \omega \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho. \tag{7}$$

We use the notation "$\rho = \emptyset$" when $d = k$ since $\Sigma(0:d-(k+1), 1:d)$ has not been defined for $d = k$. We thus have the following set of moments for $\omega \in \mathcal{P}_{r+1}^- \Lambda^k(T)$:

$$\begin{aligned} M^k := \{M_{\zeta,\rho,\boldsymbol{\beta}}(\omega) \;:\; &\zeta \in \Sigma(0:d, 0:n),\ \rho \in \Sigma(0:d-(k+1), 1:d), \\ &\text{and } \boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k)) \text{ with } k \le d \le n\}. \end{aligned} \tag{8}$$

Remark 9. If $\omega \in \Lambda^0(T)$, then

- when $d = k = 0$, then $\zeta \in \Sigma(0:0, 0:n)$, so $f_\zeta = [x_j]$ for some $j \in \{0, \ldots, n\}$; moreover $\mathcal{I}(1, r)$ has a unique element, hence $\boldsymbol{\beta} = (r)$ and we have

  $$M_{\zeta,\emptyset,\boldsymbol{\beta}}(\omega) = \omega(x_j),$$

- when $d > 0$, then $\zeta \in \Sigma(0:d, 0:n)$, $\rho \in \Sigma(0:d-1, 1:d)$ and $\beta \in \mathcal{I}(d+1, r-d)$. It is worth noting that $\Sigma(0:d-1,1:d)$ has a unique element and $(\mathrm{d}\lambda_{f_\zeta})_\rho = \mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}$, namely

$$M_{\zeta,\rho,\beta}(\omega) = \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\omega \wedge \lambda_{f_\zeta}^{\beta}(\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}).$$

4. A basis of the space of divergence-free finite elements

4.1. The dimension

The de Rham diagram in terms of functional spaces and corresponding conforming finite element spaces reads

$$\begin{array}{ccccccc} 0 \hookrightarrow H^1(\Omega) & \longrightarrow & H(\mathrm{curl};\Omega) & \longrightarrow & H(\mathrm{div};\Omega) & \longrightarrow & L^2(\Omega) \\ & \mathrm{grad} & & \mathrm{curl} & & \mathrm{div} & \\ 0 \hookrightarrow L_{r+1}(\mathcal{T}) & \longrightarrow & N_{r+1}(\mathcal{T}) & \longrightarrow & RT_{r+1}(\mathcal{T}) & \longrightarrow & P_r(\mathcal{T}) \end{array}$$

where $\hookrightarrow$ is the inclusion. Our aim is to construct a basis in $RT_{r+1}(\mathcal{T})$ of $\mathrm{Ker(div)}$.

In terms of forms, the de Rham diagram reads

$$\begin{array}{ccccccccc} 0 & \hookrightarrow & \Lambda^0(\Omega) & \longrightarrow & \Lambda^1(\Omega) & \longrightarrow & \Lambda^2(\Omega) & \longrightarrow & \Lambda^3(\Omega) \\ & & & \mathrm{d}_0 & & \mathrm{d}_1 & & \mathrm{d}_2 & \\ 0 & \hookrightarrow & \mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T}) & \longrightarrow & \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T}) & \longrightarrow & \mathcal{P}_{r+1}^-\Lambda^2(\mathcal{T}) & \longrightarrow & \mathcal{P}_{r+1}^-\Lambda^3(\mathcal{T}) \end{array}$$

The differential operators grad, curl, and div correspond to d_0, d_1, and d_2 respectively. The spaces $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ for $k \in \{0,1,2,3\}$ are spaces of "trimmed" polynomial differential k-forms.

The cohomology spaces $\mathcal{H}^k(\Omega)$ are defined as

$$\mathcal{H}^k(\Omega) = \mathrm{Ker}\,\mathrm{d}_k/\mathrm{Im}\,\mathrm{d}_{k-1}, \quad \beta_k := \dim \mathcal{H}^k(\Omega), \quad k = 1,2; \quad \beta_0 = \dim(\mathrm{Ker}\,\mathrm{d}_0).$$

If Ω is connected, then $\beta_0 = 1$. In $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ one obtains

$$\dim\left(\mathrm{Ker}\,\mathrm{d}_k\right) = \beta_k + \dim\left(\mathrm{Im}\,\mathrm{d}_{k-1}\right).$$

By the rank theorem, we have

$$\dim\left(\mathrm{Im}\,\mathrm{d}_{k-1}\right) = \dim\left(\mathcal{P}_{r+1}^-\Lambda^{k-1}(\mathcal{T})\right) - \dim\left(\mathrm{Ker}\,\mathrm{d}_{k-1}\right).$$

By relying on recursivity,

$$\begin{aligned} \dim\left(\mathrm{Ker}\,\mathrm{d}_1\right) &= \beta_1 + \dim\left(\mathrm{Im}\,\mathrm{d}_0\right) \\ &= \beta_1 + \dim\left(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})\right) - \dim\left(\mathrm{Ker}\,\mathrm{d}_0\right) \\ &= \beta_1 + \dim\left(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})\right) - \beta_0. \end{aligned}$$

It thus holds

$$\dim\left(\mathrm{Ker}\,\mathrm{d}_2\right) = \dim\left(\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})\right) - \dim\left(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})\right) + \beta_2 - \beta_1 + \beta_0.$$

We are interested in the case $k = 2$ that in terms of functional spaces reads

$$\begin{aligned} \dim(\mathrm{Ker\,div}) &= \beta_2 + \dim(\mathrm{Im\,curl}) \\ &= \dim\left(N_{r+1}(\mathcal{T})\right) - \dim\left(L_{r+1}(\mathcal{T})\right) + \beta_2 - \beta_1 + \beta_0. \end{aligned}$$

We will start by constructing a basis of $\mathrm{Im(curl)}$.

4.2. Construction of a basis of $RT_{r+1}(\mathcal{T}) \cap \mathrm{Im(curl)}$

A classical approach for the low order finite element spaces ($r = 0$) uses the graph defined by vertices and edges of the mesh. It is based in the construction of a spanning tree, if $\beta_1 = 0$ (see, [11]) or a belted tree, if $\beta_1 \neq 0$, of this graph (see, [12,13]). In the sequel we will study how to extend this approach to high order finite element spaces ($r > 0$).

If $\beta_1 = 0$, the extension is natural once one has a set $\{DoF_j^k\}_{j\in\mathcal{J}_k}$, $k \in \{0,1\}$ of unisolvent degrees of freedom in $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ with the following property:

Property 1. *For each $j \in \mathcal{J}_1$ there exist exactly two elements $ini(j)$, $end(j) \in \mathcal{J}_0$ such that*

$$DoF_j^1(\mathrm{d}_0\omega) = DoF_{end(j)}^0(\omega) - DoF_{ini(j)}^0(\omega), \quad \forall\, \omega \in \mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T}).$$

Let $\{\omega_j^k\}_{j\in\mathcal{J}_k}$ be the cardinal basis of $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ for a set of degrees of freedom verifying Property 1. For the construction of such a basis see, *e.g.*, [32].

If $\phi \in \mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})$ then $\phi = \sum_{i\in\mathcal{J}_0} DoF_i^0(\phi)\,\omega_i^0$ and $\mathrm{d}_0\phi \in \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})$. We thus have

$$\mathrm{d}_0\phi = \sum_{j\in\mathcal{J}_1} DoF_j^1(\mathrm{d}_0\phi)\,\omega_j^1 = \sum_{j\in\mathcal{J}_1}\left(DoF^0_{end(j)}(\phi) - DoF^0_{ini(j)}(\phi)\right)\omega_j^1.$$

This means that the matrix of the operator $\mathrm{d}_0 : \mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T}) \longrightarrow \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})$, when using these bases, is the transpose of the all nodes incidence matrix of an oriented graph $\mathcal{M}_{r+1} = (\mathcal{N}_{r+1}, \mathcal{A}_{r+1})$ where each degree of freedom DoF_i^0 of $(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T}))^*$ corresponds with a node of the graph and each degree of freedom DoF_j^1 of $(\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T}))^*$ corresponds with an arc of the graph. We thus have, $\mathrm{card}(\mathcal{N}_{r+1}) = \dim(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T}))$ and $\mathrm{card}(\mathcal{A}_{r+1}) = \dim(\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T}))$. The arc DoF_j^1 goes from the node $DoF^0_{ini(j)}$ to the node $DoF^0_{end(j)}$ if and only if $DoF_j^1(\mathrm{d}\omega) = DoF^0_{end(j)}(\omega) - DoF^0_{ini(j)}(\omega)$ for any $\omega \in \mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})$.

We are interested in two different families of degrees of freedom that satisfy Property 1: weights, $\{W_j^k\}_{j\in\mathcal{J}_k}$, and moments, $\{M_j^k\}_{j\in\mathcal{J}_k}$.

In the low order case ($r = 0$) weights and moments coincide and it is natural to identify the graph $\mathcal{M}_1$ with the graph defined by vertices and edges of the mesh. Indeed, natural degrees of freedom for $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ are the values on the vertices of the mesh for $k = 0$ and the circulation along the (oriented) edges of the mesh for $k = 1$.

In the high order case ($r > 0$) the situation is similar when using weights. In fact, it is natural to identify the graph $\mathcal{M}_{r+1}$ with the graph defined by the small vertices and those small edges chosen to obtain a unisolvent set of degrees of freedom. Also in this case there is a one to one correspondence between weights and geometrical (small) objects.

In the case of moments the geometrical realization of the graph $\mathcal{M}_{r+1}$ is more abstract because the degrees of freedom can be associated with geometrical objects of any dimension. For this reason it is useful to rely on the canonical isomorphism between weights and moments described in [33] that preserves the matrix of the d_0 operator. In other words, with this isomorphism it turns out that the graphs of weights and moments coincide.

We recall that the graph $\mathcal{M}_1$ allows to organize the set of indices in such a way that it is easy to identify the degrees of freedom that correspond with the kernel of the operator d_1. In the following we will show that the graph $\mathcal{M}_{r+1}$ can be used similarly for $r > 0$.

Let us consider a spanning tree $\mathcal{S}_{r+1} = (\mathcal{N}_{r+1,S}, \mathcal{A}_{r+1,S})$ of this graph $\mathcal{M}_{r+1}$. Then we have that $\mathrm{card}(\mathcal{A}_{r+1,S}) = \mathrm{card}(\mathcal{N}_{r+1,S}) - 1 = \dim\left(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})\right) - 1$; the arcs in $\mathcal{A}_{r+1,C} = \mathcal{A}_{r+1} \setminus \mathcal{A}_{r+1,S}$ belong to the cotree and $\mathrm{card}(\mathcal{A}_{r+1,C}) = \mathrm{card}(\mathcal{A}_{r+1}) - \mathrm{card}(\mathcal{A}_{r+1,S}) = \dim\left(\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})\right) - \dim\left(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})\right) + 1$. A spanning tree of the graph $\mathcal{M}_{r+1}$ can be constructed as explained in [34] for weights. Then, its construction can be done also for moments, by relying on the canonical isomorphism defined in [33].

Proposition 1. *Let $\{DoF_j^1\}_{j\in\mathcal{J}_1}$ be a set of unisolvent dofs in $N_{r+1}(\mathcal{T})$ and $\{DoF_j^0\}_{j\in\mathcal{J}_1}$ a set of unisolvent dofs in $L_{r+1}(\mathcal{T})$ verifying Property 1. If the associated graph $\mathcal{M}_{r+1} = (\mathcal{N}_{r+1}, \mathcal{A}_{r+1})$ is connected, let $\mathcal{S}_{r+1} = (\mathcal{N}_{r+1}, \mathcal{A}_{r+1,S})$ be a spanning tree of $\mathcal{M}_{r+1}$. Let $\mathcal{J}_1^S := \{j \in \mathcal{J}_1 \ : \ DoF_j^1 \in \mathcal{A}_{r+1,S}\}$ and $\mathcal{J}_1^C = \mathcal{J}_1 \setminus \mathcal{J}_1^S$. Let $\{\omega_j\}_{j\in\mathcal{J}_1}$ be the cardinal basis for the set $\{DoF_j^1\}_{j\in\mathcal{J}_1}$ of degrees of freedom of $N_{r+1}(\mathcal{T})$. If $\beta_1 = 0$ then the set $\{\mathrm{curl}\,\omega_j\}_{j\in\mathcal{J}_1^C}$ is linearly independent. If, in addition, $\beta_2 = 0$ then it is a basis of $RT_{r+1}(\mathcal{T}) \cap H(\mathrm{div}^0;\Omega)$.*

Proof. First we will prove that if $\sum_{j\in\mathcal{J}_1^C} c_j \mathrm{curl}\,\omega_j = 0$ and $\beta_1 = 0$ then the coefficient $c_j = 0$ for all $j \in \mathcal{J}_1^C$. We have

$$0 = \sum_{j\in\mathcal{J}_1^C} c_j \mathrm{curl}\,\omega_j = \mathrm{curl}\left(\sum_{j\in\mathcal{J}_1^C} c_j\omega_j\right) \overset{\beta_1=0}{\Longrightarrow} \sum_{j\in\mathcal{J}_1^C} c_j\omega_j = \mathrm{grad}\,\phi \tag{9}$$

for some $\phi \in L_{r+1}(\mathcal{T})$. To conclude the proof, we will show that in fact $\mathrm{grad}\,\phi = 0$; since $\{\omega_j\}_{j\in\mathcal{J}_1}$ is a basis of $N_{r+1}(\mathcal{T})$ this imply that $c_j = 0$ for all $j \in \mathcal{J}_1^C$.

Since $\{\omega_j\}_{j\in\mathcal{J}_1}$ is the cardinal basis of $N_{r+1}(\mathcal{T})$ for the set of degrees of freedom $\{DoF_j^1\}_{j\in\mathcal{J}_1}$, then $DoF^1_{j'}(\omega_j) = \delta_{j,j'}$ for all $j, j' \in \mathcal{J}_1$. Moreover, $\mathcal{J}_1^S \cap \mathcal{J}_1^C = \emptyset$, hence we have

$$DoF^1_{j'}(\mathrm{grad}\,\phi) = DoF^1_{j'}\left(\sum_{j\in\mathcal{J}_1^C} c_j\omega_j\right) = \sum_{j\in\mathcal{J}_1^C} c_j DoF^1_{j'}(\omega_j) = \sum_{j\in\mathcal{J}_1^C} c_j\delta_{j,j'} = 0 \quad \text{for each } j' \in \mathcal{J}_1^S.$$

Using Property 1 it follows that

$$0 = DoF^1_{j'}(\mathrm{grad}\,\phi) = DoF^0_{end(j')}(\phi) - DoF^0_{ini(j')}(\phi) \quad \text{for each } j' \in \mathcal{J}_1^S.$$

Since $\mathcal{S}$ is a spanning tree of $\mathcal{M}$ which is connected, it follows that there exists $c \in \mathbb{R}$ such that $DoF_i^0(\phi) = c$ for all $i \in \mathcal{J}_0$. Then in fact $DoF_j^1(\mathrm{grad}\,\phi) = DoF^0_{end(j)}(\phi) - DoF^0_{ini(j)}(\phi) = 0$ for all $j \in \mathcal{J}_1$ and $\mathrm{grad}\,\phi = 0$ because $\{DoF_j^1\}_{j\in\mathcal{J}_1}$ is a set of unisolvent dofs in $N_{r+1}(\mathcal{T})$.

Therefore $DoF_j^1(\mathrm{grad}\,\phi) = 0$ for all $j \in \mathcal{J}_1$ and then $\mathrm{grad}\,\phi = 0$ because this set of degrees of freedom is unisolvent. Recalling that $\mathrm{grad}\,\phi = \sum_{j\in\mathcal{J}_1^C} c_j\omega_j$ it follows that $c_j = 0$ for all $j \in \mathcal{J}_1^C$ because $\{\omega_j\}_{j\in\mathcal{J}_1}$ is a basis of $\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})$. □

If $\beta_1 \neq 0$, namely, if Ω is not simply connected, there exist curl free functions that are not gradients and the implication in (9) is not true.

A polynomial differential form $\omega \in \mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})$ is exact, namely, $\omega = \mathrm{d}_0\varphi$ for some $\varphi \in \mathcal{P}_{r+1}^{-}\Lambda^0(\mathcal{T})$, if and only if $\mathrm{d}_1\omega = 0$ and $\oint_\gamma \omega = 0$ for any 1-chain $\gamma = \sum_{e\in E} a_e e$ with $a_e \in \mathbb{Z}$ of oriented edges of the mesh $\mathcal{T}$ such that $\partial_1\gamma = \sum_{e\in E} a_e \partial_1 e = 0$.

In terms of weights, we get

$$\oint_\gamma \omega = \sum_{e\in \mathrm{supp}(\gamma)} a_e \int_e \omega = \sum_{e\in \mathrm{supp}(\gamma)} a_e \sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} \int_{\tau_{\boldsymbol{\alpha}}(e)} \omega = \sum_{e\in \mathrm{supp}(\gamma)} a_e \sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} W_{e,\boldsymbol{\alpha}}(\omega)$$

being $\mathrm{supp}(\gamma) = \{e \in E \ : \ a_e \neq 0\}$.

In terms of moments, we first recall that

$$1 = \left(\sum_{i=0}^{n} \lambda_i\right)^r = \sum_{\boldsymbol{\alpha}\in\mathcal{I}(n+1,r)} \frac{r!}{\boldsymbol{\alpha}!}\lambda^{\boldsymbol{\alpha}} \qquad \boldsymbol{\alpha}! := \alpha_0!\cdots\alpha_n!.$$

Then

$$\begin{aligned}\oint_\gamma \omega &= \sum_{e\in \mathrm{supp}(\gamma)} a_e \int_e \omega = \sum_{e\in \mathrm{supp}(\gamma)} a_e \sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} \int_e \omega \wedge \frac{r!}{\boldsymbol{\alpha}!}\lambda^{\boldsymbol{\alpha}} \\ &= r! \sum_{e\in \mathrm{supp}(\gamma)} a_e \sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} \int_e \omega \wedge \frac{1}{\boldsymbol{\alpha}!}\lambda^{\boldsymbol{\alpha}} = r! \sum_{e\in \mathrm{supp}(\gamma)} a_e \sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} M_{e,\emptyset,\boldsymbol{\alpha}}(\omega)\end{aligned}$$

where the last term is the sum of moments (5) for $\omega \in \mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})$ in $f_\zeta = e$, with $d = k = 1$ and $\eta = \frac{1}{\boldsymbol{\alpha}!}\lambda^{\boldsymbol{\alpha}}$.

Our aim is to extend to the high order case the notion of belted tree (see e.g. [12,18,35]).

We start by recalling the definition of belted tree for the graph $\mathcal{M}_1$ given by vertices and edges of the mesh $\mathcal{T}$.

To this end we assume to know a set of β_1 polygonal loops in $\mathcal{T}$, $\{\sigma_l\}_{l=1}^{\beta_1}$, mutually disjoint and without self-intersection, representing a basis of $\mathcal{H}_1(\mathcal{T};\mathbb{Z})$. Each loop σ_l can be written as $\sigma_l = \sum_{e\in E}\sigma_{l,e} e$ with $\sigma_{l,e} \in \{-1,0,1\}$. For each $l = \{1,\dots,\beta_1\}$, select one edge e_l^* belonging to σ_l. The set $\cup_{l=1}^{\beta_1}\left(\mathrm{supp}(\sigma_l)\setminus\{e_l^*\}\right)$ is therefore a tree and it is possible to construct a spanning tree $\mathcal{S}_1 = (\mathcal{N}_1, \mathcal{A}_{1,S})$ of the graph $\mathcal{M}_1 = (\mathcal{N}_1,\mathcal{A}_1)$ such that all the edges of each $\sigma_l \setminus \{e_l^*\}$ belong to this spanning tree, while the edges $\{e_l^*\}_{l=1}^{\beta_1}$ belong to the cotree. The subgraph $\mathcal{B}_1 = (\mathcal{N}_1, \mathcal{A}_{1,S}\cup\{e_l^*\}_{l=1}^{\beta_1})$ is called belted tree of the graph $\mathcal{M}_1$.

We are doing an abuse of notation since the arcs in the graph $\mathcal{M}_1$ are in fact degrees of freedom. So, instead of e we should write $W_e(\omega)$ or $M_e(\omega)$, being $W_e(\omega) = M_e(\omega) = \int_e \omega$ and the loop in $\mathcal{M}_1$ should be in reality $\sigma_l = \sum_{e\in E}\sigma_{l,e}W_e(\cdot) = \sum_{e\in E}\sigma_{l,e}M_e(\cdot)$.

When we are in the graph $\mathcal{M}_{r+1}$ the corresponding loops read $\mathcal{G}_l^W = \sum_{e\in E}\sigma_{l,e}\sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} W_{e,\boldsymbol{\alpha}}(\cdot)$ or $\mathcal{G}_l^M = \sum_{e\in E}\sigma_{l,e}\sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} M_{e,\emptyset,\boldsymbol{\alpha}}(\cdot)$ (we use $\mathcal{G}_l$ when it is not necessary to specify the type of degrees of freedom). The coefficients $\sigma_{l,e}$ are those of the geometrical loop σ_l while the multi-indices $\boldsymbol{\alpha}$ take care of the higher polynomial degree. Moreover when using weights one has $\mathcal{G}_l^W(\omega) = \sum_{e\in E}\sigma_{l,e}\sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} W_{e,\boldsymbol{\alpha}}(\omega) = \oint_{\sigma_l}\omega$ and when using moments $\mathcal{G}_l^M(\omega) = \sum_{e\in E}\sigma_{l,e}\sum_{\boldsymbol{\alpha}\in\mathcal{I}(2,r)} M_{e,\emptyset,\boldsymbol{\alpha}}(\omega) = \frac{1}{r!}\oint_{\sigma_l}\omega$.

The definition of a *belted tree* of the graph $\mathcal{M}_{r+1}$ for any $r \geq 0$ is a straightforward extension of that done in the geometrical graph (the one defined by the vertices and edges of the mesh).

The arcs of $\mathcal{G}_l$ are identified by a couple $\epsilon = (e,\alpha)$ with $e \in E$ and $\alpha \in \mathcal{I}(2,r)$ and the support of $\mathcal{G}_l$ is $\mathrm{supp}(\mathcal{G}_l) := \{\epsilon = (e,\alpha) \ : \ \sigma_{e,l} \neq 0\}$. For each $l = \{1,\dots,\beta_1\}$, select one arc (for weights, one small edge) $\epsilon_l^* \in \mathrm{supp}(\mathcal{G}_l)$. The set $\cup_{l=1}^{\beta_1}\left(\mathrm{supp}(\mathcal{G}_l)\setminus\{\epsilon_l^*\}\right)$ is therefore a tree and it is possible to construct a spanning tree $\mathcal{S}_{r+1} = (\mathcal{N}_{r+1},\mathcal{A}_{r+1,S})$ of the graph $\mathcal{M}_{r+1} = (\mathcal{N}_{r+1},\mathcal{A}_{r+1})$ such that all the arcs of each $\mathrm{supp}(\mathcal{G}_l)\setminus\{\epsilon_l^*\}$ belong to this spanning tree, while the arcs $\{\epsilon_l^*\}_{l=1}^{\beta_1}$ belong to the cotree. The subgraph $\mathcal{B}_{r+1} = (\mathcal{N}_{r+1},\mathcal{A}_{r+1,S}\cup\{\epsilon_l^*\}_{l=1}^{\beta_1})$ is called belted tree of the graph $\mathcal{M}_{r+1}$. It is worth noting that $\mathrm{supp}(\mathcal{G}_l) \subset \mathcal{A}_{r+1,S}\cup\{\epsilon_l^*\}_{l=1}^{\beta_1}$ for all $l \in \{1,\dots,l\}$.

Let $\mathcal{J}_1^B := \{j \in \mathcal{J}_1 \ : \ DoF_j^1 \in \mathcal{A}_{r+1,S}\cup\{\epsilon_l^*\}\}$ and $\mathcal{J}_1^{CB} = \mathcal{J}_1 \setminus \mathcal{J}_1^B$. If $\{\omega_j\}_{j\in\mathcal{J}_1}$ is the cardinal basis for the set $\{DoF_j^1\}_{j\in\mathcal{J}_1}$ then $DoF_i^1(\omega_j) = 0$ if $i \in \mathcal{J}_1^B$ and $j \in \mathcal{J}_1^{CB}$. Hence, $\mathcal{G}_l(\omega_j) = 0$ for all $j \in \mathcal{J}_1^{CB}$.

Moreover we recall that $\mathcal{A}_{r+1,BC} = \mathcal{A}_{r+1}\setminus(\mathcal{A}_{r+1,S}\cup\{\epsilon_l^*\}_{l=1}^{\beta_1})$ and thus $\mathrm{card}(\mathcal{A}_{r+1,BC}) = \mathrm{card}(\mathcal{A}_{r+1}) - \mathrm{card}(\mathcal{A}_{r+1,S}) - \beta_1 = \dim(\mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})) - \dim(\mathcal{P}_{r+1}^{-}\Lambda^0(\mathcal{T})) + 1 - \beta_1$.

Proposition 2. *Let $\{\omega_j\}_{j\in\mathcal{J}_1}$ be the cardinal basis of $N_{r+1}(\mathcal{T})$ for either weights $\{W_j^1\}_{j\in\mathcal{J}_1}$ or moments $\{M_j^1\}_{j\in\mathcal{J}_1}$. The set $\{curl\,\omega_j\}_{j\in\mathcal{J}_1^{CB}}$ is linearly independent and its cardinality coincides with the dimension of $RT_{r+1}(\mathcal{T})\cap\mathrm{Im}(curl)$, thus it is a basis for the latter space. If $\beta_2 = 0$ then it is a basis of $RT_{r+1}(\mathcal{T})\cap H(div^0;\Omega)$.*

Proof. We will proceed as in the proof of Proposition 1.

If $\sum_{j\in\mathcal{J}_1^{CB}} c_j \mathrm{curl}\,\omega_j = 0$ then clearly $\mathrm{curl}\left(\sum_{j\in\mathcal{J}_1^{CB}} c_j\,\omega_j\right) = 0$ and, by Poincaré duality, $\oint_\gamma\left(\sum_{j\in\mathcal{J}_1^{CB}} c_j\,\omega_j\right) = 0$ for all γ that is the boundary of a 2-chain of $\mathcal{T}$.

Moreover from the definition of $\mathcal{J}_1^{CB}$ and taking again into account that ω_j is the cardinal basis for the degrees of freedom, $\{DoF_j^1\}_{j\in\mathcal{J}^1}$, it is clear that for $l \in \{1,\dots,\beta_1\}$. For weights we have

$$\oint_{\sigma_l}\sum_{j\in\mathcal{J}_1^{CB}} c_j\,\omega_j = \sum_{j\in\mathcal{J}_1^{CB}} c_j \oint_{\sigma_l}\omega_j = \sum_{j\in\mathcal{J}_1^{CB}} c_j \mathcal{G}_l^W(\omega_j) = 0$$

and similarly for moments we obtain

$$\oint_{\sigma_l} \sum_{j\in\mathcal{J}_1^{CB}} c_j\,\omega_j = \sum_{j\in\mathcal{J}_1^{CB}} c_j \oint_{\sigma_l} \omega_j = r! \sum_{j\in\mathcal{J}_1^{CB}} c_j \mathcal{G}_l^M(\omega_j) = 0.$$

In conclusion $\oint_\gamma \sum_{j\in\mathcal{J}_1^{CB}} c_j\,\omega_j = 0$ for any 1-chain γ of $\mathcal{T}$ with $\partial\gamma = 0$. Then Poincaré duality yields $\sum_{j\in\mathcal{J}_1^{CB}} c_j\,\omega_j = \operatorname{grad}\phi$ for some $\phi \in L_{r+1}(\mathcal{T})$. Then, the proof ends by the same mathematical steps done to prove Proposition 1. □

If $\beta_2 = 0$, then $\operatorname{card}(\mathcal{A}_{CB})$ is the dimension of $RT_{r+1}(\mathcal{T})\cap H(\operatorname{div}^0;\Omega)$ and the "recipe" to construct a basis of $RT_{r+1}(\mathcal{T})\cap H(\operatorname{div}^0;\Omega)$ is given in Algorithm 1.

Algorithm 1 (case $\beta_2 = 0$)

1. Select a basis $\{v_j\}_{j\in\mathcal{J}_1}$ of $N_{r+1}(\mathcal{T})$.
2. Select a set $\{DoF_j^1\}_{j\in\mathcal{J}_1}$ of unisolvent dofs in $N_{r+1}(\mathcal{T})$ and a set $\{DoF_j^0\}_{j\in\mathcal{J}_1}$ of unisolvent dofs in $L_{r+1}(\mathcal{T})$ verifying Property 1.
3. Construct the cardinal basis $\{\omega_j\}_{j\in\mathcal{J}_1}$ for the set $\{DoF_j^1\}_{j\in\mathcal{J}_1}$ in terms of the basis $\{v_j\}_{j\in\mathcal{J}_1}$.
4. Construct the associated graph $\mathcal{M}_{r+1} = (\mathcal{N}_{r+1}, \mathcal{A}_{r+1})$.
5. Construct a belted spanning tree $\mathcal{B}_{r+1} = (\mathcal{N}_{r+1}, \mathcal{A}_{r+1,B})$.
6. Compute $\operatorname{curl}\omega_j$, for each index j associated with an arc not in $\mathcal{A}_{r+1,B}$.

Note that if $\beta_1 = 0$ then $\mathcal{B}_{r+1}$ is in fact a spanning tree of the graph $\mathcal{M}_{r+1}$.

The complexity of Algorithm 1 is given by the cost of the inversion, at point 3., of the Vandermonde matrix, to compute the cardinal basis, and the construction, at point 5., of the belted spanning tree. The fact of working with barycentric coordinates makes any computation independent from the current tetrahedron. Therefore, concerning the Vandermonde matrix, its inversion is done only once since this matrix has the same entries in any tetrahedron of the mesh. For the construction of the belted spanning tree, the high order enrichment inside each element does not depend on the element, it is done by the same procedure in all the tetrahedra. An analysis of this construction can be found in [34].

4.3. Making for a basis of $RT_{r+1}(\mathcal{T}) \cap H(\operatorname{div}^0;\Omega)$ when $\beta_2 \neq 0$

If $\beta_2 \neq 0$ then the space of divergence-free Raviart–Thomas finite elements that are not the curl of Nédélec finite elements is non trivial and has dimension β_2. So we have to add, for each $n = 1,\dots,\beta_2$, one solution $\mathbf{z}_{h,n} \in RT_{r+1}(\mathcal{T})$ of

$$\begin{cases} \operatorname{div}\mathbf{z}_{h,n} = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_\ell} \mathbf{z}_{h,n}\cdot\mathbf{n}_\Omega = \delta_{n,\ell} & \ell = 1\dots,\beta_2\,, \end{cases} \tag{10}$$

where $(\partial\Omega)_\ell$, for $\ell \in \{0,1,\dots,\beta_2\}$ are the connected components of $\partial\Omega$ being $(\partial\Omega)_0$ the external one. Each one of these problems has solution and it is unique up to a curl.

For any choice of $\mathbf{z}_{h,n}$ the set $\{\mathbf{z}_{h,n}\}_{n=1}^{\beta_2}$ is linearly independent. In fact, if $\sum_{n=1}^{\beta_2} c_n\mathbf{z}_{h,n} = 0$ then for any $\ell \in \{1,\dots,\beta_2\}$ one has $0 = \int_{(\partial\Omega)_\ell} \left(\sum_{n=1}^{\beta_2} c_n\mathbf{z}_{h,n}\right)\cdot\mathbf{n}_\Omega = \sum_{n=1}^{\beta_2} c_n \int_{(\partial\Omega)_\ell} \mathbf{z}_{h,n}\cdot\mathbf{n}_\Omega = \sum_{n=1}^{\beta_2} c_n\delta_{n,\ell} = c_\ell$.

Proposition 3. *Let $\{\omega_j\}_{j\in\mathcal{J}_1}$ be the cardinal basis of $N_{r+1}(\mathcal{T})$ for either weights $\{W_j^1\}_{j\in\mathcal{J}_1}$ or moments $\{M_j^1\}_{j\in\mathcal{J}_1}$. For each $n \in \{1,\dots,\beta_2\}$, let $\mathbf{z}_{h,n}$ be a solution of (10). The set $\{\operatorname{curl}\omega_j\}_{j\in\mathcal{J}_1^{CB}} \cup \{\mathbf{z}_{h,n}\}_{n=1}^{\beta_2}$ is a basis of $RT_{r+1}(\mathcal{T}) \cap H(\operatorname{div}^0;\Omega)$.*

Proof. First we notice that the cardinality of the set $\{\operatorname{curl}\omega_j\}_{j\in\mathcal{J}_1^{CB}} \cup \{\mathbf{z}_{h,n}\}_{n=1}^{\beta_2}$ is equal to $\dim(\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})) - \dim(\mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})) + 1 - \beta_1 + \beta_2$.

It remains to prove that the set $\{\operatorname{curl}\omega_j\}_{j\in\mathcal{J}_1^{CB}} \cup \{\mathbf{z}_{h,n}\}_{n=1}^{\beta_2}$ is linearly independent. If $\sum_{j\in\mathcal{J}_1^{CB}} \hat{c}_j \operatorname{curl}\omega_j + \sum_{n=1}^{\beta_2} c_n\mathbf{z}_{h,n} = 0$ then, by Stokes' theorem

$$0 = \int_{(\partial\Omega)_\ell} \left(\sum_{j\in\mathcal{J}_1^{CB}} \hat{c}_j \operatorname{curl}\omega_j + \sum_{n=1}^{\beta_2} c_n\mathbf{z}_{h,n}\right)\cdot\mathbf{n}_\Omega = \int_{(\partial\Omega)_\ell} \left(\sum_{n=1}^{\beta_2} c_n\mathbf{z}_{h,n}\right)\cdot\mathbf{n}_\Omega = c_\ell$$

for all $\ell \in \{1,\dots,\beta_2\}$. Thus $\sum_{j\in\mathcal{J}_1^{CB}} \hat{c}_j \operatorname{curl}\omega_j = 0$. In Proposition 2 we have proved that the set $\{\operatorname{curl}\omega_j\}_{j\in\mathcal{J}_1^{CB}}$ is linearly independent so, also the coefficients $\hat{c}_j$ are equal to zero. □

It is possible to consider solutions of Problem (10) in $RT_1(\mathcal{T}) \subset RT_{r+1}(\mathcal{T})$. This case has been studied in [13] Section 5 where it is proposed a very efficient algorithm for the computation of the solution. The algorithm uses the dual mesh of $\mathcal{T}$. It is an elimination procedure that follows the arcs of a spanning tree of the graph defined by the dual vertices (the barycenters of the elements) and the dual edges (one associated to each face of the mesh). This algorithm has been extended to the case $r > 0$ in [6] relying on the use of the weights as degrees of freedom for both $RT_{r+1}(\mathcal{T})$ and $P_r(\mathcal{T})$. It follows the arcs of a spanning tree of the graph with nodes

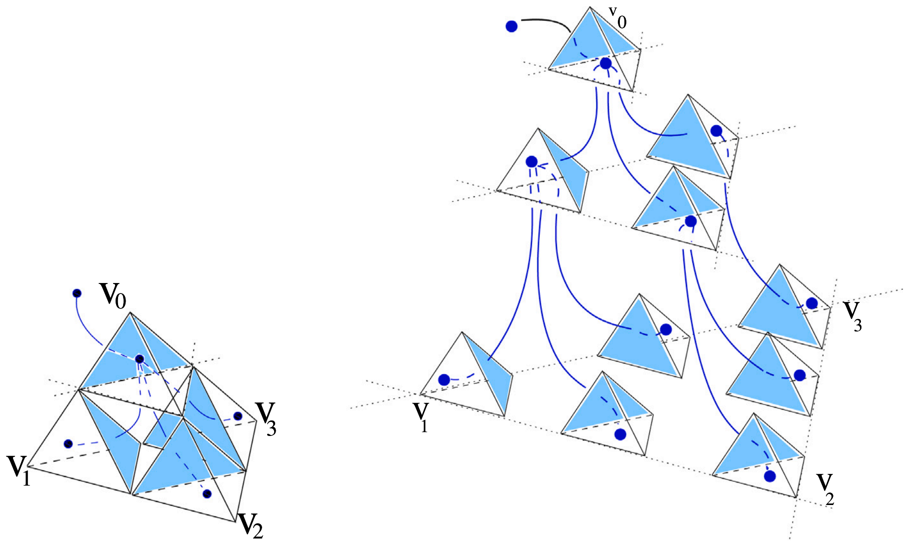

Fig. 3. Example of spanning tree in the (dual) graph, namely a selection of acyclic paths made of arcs, visiting all the nodes of the (dual) graph ($r = 1$, left and $r = 2$, right).
Source: Taken from [36].

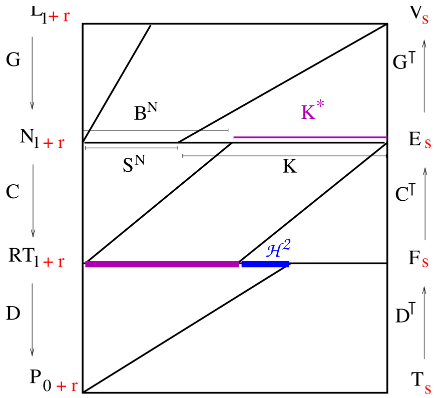

Fig. 4. A graphical summary of the structure of the basis of RT^0_{r+1} and its construction (in the case $\beta_2(\Omega) = 0$). Here, $K^* = \mathring{J}_{1,CB}$ and $K = \mathcal{J}_{1,CB}$, with or without the footindex B (standing for belted), depending on $\beta_1(\Omega)$.

the barycenters of the small elements and arcs associated to the small faces chosen to obtain a unisolvent set of degrees of freedom for $RT_{r+1}(\mathcal{T})$ (see Fig. 3).

In Fig. 4, we summarizes the situation. Due to the property that $\mathrm{d}_1\mathrm{d}_0 = 0$, we cannot construct a divergence-free basis of RT^0_{r+1} starting from the curl of a basis of N_{r+1} because they are not linear independent. However, the spanning (eventually belted) tree for the gradient of function L_{r+1} allows to identify the set (associated with the corresponding co-tree) of indices $K^* = \mathring{J}_{1,CB}$ of columns in $G^\top$ that will provide a part of this basis, once we apply on them the curl operator (see Proposition 4). If $\beta_2(\Omega) \neq 0$, the basis has to be completed *by hands*, by adding the generators of $\mathcal{H}^2$ (see Proposition 6), namely the solution of problem (10), for each $l = 1, \dots, \beta_2(\Omega) + 1$.

5. On the construction of a basis of $RT_{r+1} \cap H_0(\mathrm{div}^0; \Omega)$

In the following we denote $H_0(\mathrm{div}^0; \Omega)$ the space of divergence free fields $\mathbf{z}$ such that $\mathrm{Tr}_{\partial\Omega}\mathbf{z} = \mathbf{z} \cdot \mathbf{n}_\Omega = 0$. Similarly we set $H_0(\mathrm{curl}^0; \Omega)$ the space of curl free fields $\mathbf{u}$ such that $\mathrm{Tr}_{\partial\Omega}\mathbf{u} = \mathbf{n}_\Omega \times \mathbf{u} \times \mathbf{n}_\Omega = \mathbf{0}$.

If $\mathbf{z} \in RT_{r+1} \cap H(\mathrm{div}^0; \Omega)$ and $\mathrm{Tr}_{\partial\Omega}\mathbf{z} = 0$ then $\mathbf{z} = \mathrm{curl}\,\mathbf{u}$ for some $\mathbf{u} \in H_*(\mathrm{curl}; \Omega) = \{\mathbf{u} \in H(\mathrm{curl}; \Omega) \,:\, \mathrm{curl}\,\mathbf{u} \cdot \mathbf{n}_\Omega = 0 \text{ in } \partial\Omega\} = \{\mathbf{u} \in H(\mathrm{curl}; \Omega) \,:\, \mathrm{curl}_\tau(\mathbf{n}_\Omega \times \mathbf{u} \times \mathbf{n}_\Omega) = 0 \text{ on } \partial\Omega\}$, being $\mathrm{curl}_\tau(\mathbf{n}_\Omega \times \mathbf{u} \times \mathbf{n}_\Omega)$ the tangential curl of the trace of $\mathbf{u}$. So our aim is to construct a basis of $\mathrm{curl}(N_{r+1} \cap H_*(\mathrm{curl}; \Omega))$. Clearly $H_0(\mathrm{curl}; \Omega) \subset H_*(\mathrm{curl}; \Omega)$. First we will construct a basis of $\mathrm{curl}(N_{r+1} \cap H_0(\mathrm{curl}; \Omega))$ and then we will complete it to obtain the desired basis.

In the following, for both Lagrange and Nédélec elements we distinguish boundary and internal degrees of freedom. The boundary degrees of freedom are "supported" by subsimplices completely contained on $\partial\Omega$. The internal degrees of freedom are those that are not boundary degree of freedom. We set $\mathring{\mathcal{J}}_1 = \mathcal{J}_1 \setminus \mathcal{J}_1^\partial$ where $\mathcal{J}_1^\partial$ denotes the set of indices corresponding to Nédélec internal degrees of freedom. Similarly $\mathring{\mathcal{J}}_0 = \mathcal{J}_0 \setminus \mathcal{J}_0^\partial$ with $\mathcal{J}_0^\partial$ the set of indices corresponding to Lagrange internal degrees of freedom.

By the rank theorem

$$\dim\left(\mathrm{curl}(N_{r+1}\cap H_0(\mathrm{curl};\Omega))\right) = \dim(N_{r+1}\cap H_0(\mathrm{curl};\Omega)) - \dim(N_{r+1}\cap H_0(\mathrm{curl}^0;\Omega)).$$

We recall that $N_{r+1}\cap H_0(\mathrm{curl}^0;\Omega) = \mathrm{grad}(L_{r+1}\cap H^1_*(\Omega))$ being

$$H^1_*(\Omega) = \left\{\phi \in H^1(\Omega)\ :\ \phi_{|(\partial\Omega)_n} \text{is constant } \forall\, n \in \{1,\dots,\beta_2(\Omega)\}\right\}.$$

The dimension of $L_{r+1}\cap H^1_*(\Omega)$ is equal to the dimension of $L_{r+1}\cap H^1_0(\Omega)$ plus the number of connected components of $\partial\Omega$ namely, $\beta_2(\Omega)+1$. Moreover $\dim\left(\mathrm{grad}(L_{r+1}\cap H^1_*(\Omega))\right) = \dim(L_{r+1}\cap H^1_*(\Omega)) - 1$ (with the -1 since constants have zero gradient), hence

$$\dim\left(\mathrm{curl}(N_{r+1}\cap H_0(\mathrm{curl};\Omega))\right) = \dim(N_{r+1}\cap H_0(\mathrm{curl};\Omega)) - \dim(L_{r+1}\cap H^1_0(\Omega)) - \beta_2(\Omega).$$

We obtain

$$\dim\left(\mathrm{curl}(N_{r+1}\cap H_0(\mathrm{curl};\Omega))\right) = \mathrm{card}\mathring{\mathcal{J}}_1 - \mathrm{card}\mathring{\mathcal{J}}_0 - \beta_2(\Omega).$$

To construct a basis of $RT_{r+1}\cap\mathrm{curl}(H_0(\mathrm{curl};\Omega)) = \mathrm{curl}(N_{r+1}\cap H_0(\mathrm{curl};\Omega))$ the idea is to contract to a single node the nodes of the graph $\mathcal{M}_{r+1}$ that correspond to Lagrange degrees of freedom supported in the same connected component of the $\partial\Omega$. The incidence matrix of the new graph $\mathring{\mathcal{M}}_{r+1} = (\mathring{\mathcal{N}}_{r+1}, \mathring{\mathcal{A}}_{r+1})$ is computed by replacing all the rows of the incidence matrix of $\mathcal{M}_{r+1}$ related to Lagrange degrees of freedom supported on a connected component of $\partial\Omega$ with a single row equal to their sum, and removing the columns with all the entries equal to zero that are those of the contracted arcs. The number of arcs in $\mathring{\mathcal{M}}_{r+1}$ is equal to the number of internal Nédélec degrees of freedom, namely $\mathrm{card}(\mathring{\mathcal{J}}_1)$ while the number of nodes in $\mathring{\mathcal{M}}_{r+1}$ is equal to the number of internal Lagrange degrees of freedom plus the number of connected components of $\partial\Omega$, namely $\mathrm{card}(\mathring{\mathcal{J}}_0)+\beta_2(\Omega)+1$.

Let $\mathring{\mathcal{S}}_{r+1} = (\mathring{\mathcal{N}}_{r+1,S}, \mathring{\mathcal{A}}_{r+1,S})$ be a spanning tree of $\mathring{\mathcal{M}}_{r+1}$. We denote

$$\mathring{\mathcal{J}}_{1,S} := \{j \in \mathring{\mathcal{J}}_1\ :\ DoF^1_j \in \mathring{\mathcal{A}}_{r+1,S}\}$$

and $\mathring{\mathcal{J}}_{1,C} = \mathring{\mathcal{J}}_1 \setminus \mathring{\mathcal{J}}_{1,S}$. We recall that, being $\mathring{\mathcal{S}}_{r+1}$ a spanning tree of $\mathring{\mathcal{M}}_{r+1}$, it holds that $\mathrm{card}(\mathring{\mathcal{J}}_{1,S}) = \mathrm{card}(\mathring{\mathcal{N}}_{r+1,S})-1 = \mathrm{card}(\mathring{\mathcal{J}}_0)+\beta_2(\Omega)$.

Proposition 4. *Let $\{\omega_j\}_{j\in\mathcal{J}_1}$ be the cardinal basis of $N_{r+1}(\mathcal{T})$ for either weights $\{W^1_j\}_{j\in\mathcal{J}_1}$ or moments $\{M^1_j\}_{j\in\mathcal{J}_1}$. The set $\{\mathrm{curl}\,\omega_j\}_{j\in\mathring{\mathcal{J}}_{1,C}}$ is a basis of $\mathrm{curl}(N_{r+1}\cap H_0(\mathrm{curl};\Omega))$.*

Proof. First we notice that

$$\begin{aligned}\mathrm{card}(\mathring{\mathcal{J}}_{1,C}) &= \mathrm{card}\mathring{\mathcal{J}}_1 - \mathrm{card}(\mathring{\mathcal{J}}_{1,S})\\ &= \mathrm{card}\mathring{\mathcal{J}}_1 - (\mathrm{card}\mathring{\mathcal{J}}_0 + \beta_2(\Omega)) = \dim\left(\mathrm{curl}(N_{r+1}\cap H_0(\mathrm{curl};\Omega))\right).\end{aligned}$$

Then we prove that the set $\{\mathrm{curl}\omega_j\}_{j\in\mathring{\mathcal{J}}_{1,C}}$ is linearly independent, namely, if $\sum_{j\in\mathring{\mathcal{J}}_{1,C}} c_j \mathrm{curl}\,\omega_j = 0$ then the coefficient $c_j = 0$ for all $j \in \mathring{\mathcal{J}}_{1,C}$. The arcs of the graph $\mathring{\mathcal{M}}_{r+1}$ are the internal degrees of freedom of N_{r+1} hence we have that $\mathrm{Tr}_{\partial\Omega}\left(\sum_{j\in\mathring{\mathcal{J}}_{1,C}} c_j\omega_j\right) = 0$. Then, if

$$0 = \sum_{j\in\mathring{\mathcal{J}}_{1,C}} c_j \mathrm{curl}\,\omega_j = \mathrm{curl}\left(\sum_{j\in\mathring{\mathcal{J}}_{1,C}} c_j\omega_j\right)$$

it holds that $\sum_{j\in\mathring{\mathcal{J}}_{1,C}} c_j\omega_j = \mathrm{grad}\,\phi$ for some $\phi \in L_{r+1}(\mathcal{T})$. Now the proof follows as that for Proposition 1. □

If $\beta_1(\Omega) = 0$ then $H_*(\mathrm{curl};\Omega) = H_0(\mathrm{curl};\Omega)$. We thus have the following.

Corollary 10. *If $\beta_1(\Omega) = 0$, the set $\{\mathrm{curl}\,\omega_j\}_{j\in\mathring{\mathcal{J}}_{1,C}}$ is a basis of $RT_{r+1}\cap H_0(\mathrm{div}^0;\Omega)$.*

If $\beta_1(\Omega) \neq 0$ this basis has to be completed with $\beta_1(\Omega)$ elements that are the curl of functions that are in $N_{r+1}\cap(H_*(\mathrm{curl};\Omega)\setminus H_0(\mathrm{curl};\Omega))$. We do not know an efficient algorithm based on graphs to construct these functions. Methods using other strategies can be found in [37–39] or [40]. For the sake of completeness in the sequel we provide some insights on these functions. It is worth noting that these finite element functions could be chosen of polynomial degree lower than $r+1$, in particular of polynomial degree one.

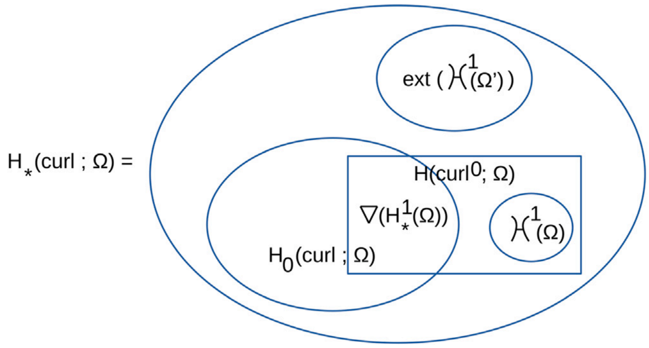

Fig. 5. A visualization of the subspaces we consider in $H_*(\mathrm{curl};\Omega)$. All the functions in the rectangle are in $H(\mathrm{curl}^0;\Omega)) = \nabla H^1(\Omega) \oplus \mathcal{H}^1(\Omega)$. Here, $\nabla H^1_*(\Omega) \subset \nabla H^1(\Omega)$, moreover $\nabla H^1_*(\Omega) \subset H_0(\mathrm{curl};\Omega) \cap H(\mathrm{curl}^0;\Omega)$ since the elements of $\nabla H^1_*(\Omega)$ are gradients, with a constant value on each connected component of $\partial\Omega$.

The first step to understand the nature of these functions is to characterize the subspace of functions in $H_*(\mathrm{curl};\Omega)$ that are not in $H_0(\mathrm{curl};\Omega)$ (see Fig. 5). It is clear that $H(\mathrm{curl}^0;\Omega) \subset H_*(\mathrm{curl};\Omega)$ but the functions of $H(\mathrm{curl}^0;\Omega)$ do not contribute to the construction of a basis of $\mathrm{curl}(N_{r+1} \cap H_*(\mathrm{curl};\Omega))$. So we are interested in functions of $H_*(\mathrm{curl};\Omega) \setminus [H_0(\mathrm{curl};\Omega) \cup H(\mathrm{curl}^0;\Omega)]$.

For the sake of simplicity in the following we assume $\beta_2(\Omega) = 0$.

Let D be a hexahedron in $\mathbb{R}^3$ such that Ω is strongly contained in D. Let us denote $\Omega' = D \setminus \Omega$ (the domain Ω' is connected when $\beta_2(\Omega) = 0$). If $\beta_1(\Omega) \neq 0$ there exist 1-chains lying on $\partial\Omega$ such that they are the boundary of 2-chains of $\mathcal{T}$ but they are not the boundary of any surface contained in Ω'. The maximum number of 1-chains with this property, that are homologically independent on $\partial\Omega$, is equal to $\beta_1(\Omega)$. For the construction see [40,41]. For each 1-chain γ of this type, there exist functions $\boldsymbol{\rho} \in H(\mathrm{curl}^0;\Omega')$ such that $\oint_\gamma \boldsymbol{\rho} = 1$ and $\mathrm{Tr}_{\partial\Omega}(\boldsymbol{\rho}) \in \mathrm{Tr}_{\partial\Omega}\left(N_{r+1}(\mathcal{T})\right)$ (see e.g. [40]). It is worth noting that the trace on $\partial\Omega$ of $\boldsymbol{\rho} \in H(\mathrm{curl}^0;\Omega')$ is not zero since $\oint_\gamma \boldsymbol{\rho} = 1$ and γ lies on $\partial\Omega$.

It is possible to construct a finite element extension of this trace in $N_{r+1}(\mathcal{T})$, namely a function $\widetilde{\boldsymbol{\rho}} \in N_{r+1}(\mathcal{T})$ such that $\mathrm{Tr}_{\partial\Omega}(\widetilde{\boldsymbol{\rho}}) = \mathrm{Tr}_{\partial\Omega}(\boldsymbol{\rho})$. Clearly $\widetilde{\boldsymbol{\rho}} \in H_*(\mathrm{curl};\Omega)$ because $\mathrm{curl}\,\widetilde{\boldsymbol{\rho}} \cdot \mathbf{n}_{\partial\Omega} = \mathrm{curl}_\tau(Tr_{\partial\Omega}(\widetilde{\boldsymbol{\rho}})) = \mathrm{curl}_\tau(Tr_{\partial\Omega}(\boldsymbol{\rho})) = \mathrm{curl}\,\boldsymbol{\rho} \cdot \mathbf{n}_{\partial\Omega} = 0$. However $\widetilde{\boldsymbol{\rho}} \notin H_0(\mathrm{curl};\Omega)$ since $\mathrm{Tr}_{\partial\Omega}(\boldsymbol{\rho}) \neq 0$, and $\mathrm{curl}\widetilde{\boldsymbol{\rho}} \neq 0$ because otherwise the function

$$\mathbf{R} = \begin{cases} \widetilde{\boldsymbol{\rho}} & \text{in } \Omega \\ \boldsymbol{\rho} & \text{in } \Omega' \end{cases}$$

would belong to $H(\mathrm{curl}^0, D)$ hence it would be a gradient. This is not possible because $\oint_\gamma \boldsymbol{\rho} \neq 0$. Hence $\widetilde{\boldsymbol{\rho}} \in N_{r+1}(\mathcal{T}) \cap \left[H_*(\mathrm{curl};\Omega) \setminus \left(H_0(\mathrm{curl};\Omega) \cup H(\mathrm{curl}^0;\Omega)\right)\right]$.

This construction has to be done for each element of a set $\{\gamma_l\}_{l=1}^{\beta_1(\Omega)}$ of 1-cycles homologically independent on $\partial\Omega$ and homologically trivial in Ω. We will denote $\boldsymbol{\rho}_{\gamma_l}$ the function on $H(\mathrm{curl}^0;\Omega')$ with $\oint_{\gamma_n} \boldsymbol{\rho}_{\gamma_l} = 1\delta_{n,l}$. The following heuristic resume this construction:

Heuristic reasoning to build a basis of the space $\mathrm{curl}(N_{r+1}(\mathcal{T}) \cap H_*(\mathrm{curl};\Omega))$.

1. Construct a basis $\{\mathrm{curl}\,\boldsymbol{\omega}_j\}_{j\in\mathring{\mathcal{J}}_{1,C}}$ of $\mathrm{curl}(N_{r+1}(\mathcal{T}) \cap H_0(\mathrm{curl};\Omega))$ using a cotree of the graph $\mathring{\mathcal{M}}_{r+1}$.
2. Compute a set $\{\gamma_l\}_{l=1}^{\beta_1(\Omega)}$ of 1-cycles homologically independent on $\partial\Omega$ and homologically trivial in Ω. These cycles generate $\mathcal{H}_1(\Omega')$.
3. For each $l \in \{1,\dots,\beta_1(\Omega)\}$
   (i) compute the traces $\mathrm{Tr}_{\partial\Omega}\,\boldsymbol{\rho}_{\gamma_l}$;
   (ii) compute an extension $\widetilde{\boldsymbol{\rho}_{\gamma_l}} := \mathrm{ext}(\mathrm{Tr}\,\boldsymbol{\rho}_{\gamma_l})$ to $N_{r+1}(\mathcal{T})$ of the traces in (i).
4. The set $\{\mathrm{curl}\,\boldsymbol{\omega}_j\}_{j\in\mathring{\mathcal{J}}_{1,C}} \cup \{\mathrm{curl}\,\widetilde{\boldsymbol{\rho}_{\gamma_l}}\}_{l=1}^{\beta_1(\Omega)}$ is a basis of $\mathrm{curl}(N_{r+1}(\mathcal{T}) \cap H_*(\mathrm{curl};\Omega))$.

6. Conclusions

We have constructed a basis of the finite element space $RT_{r+1} \cap H(\mathrm{div}^0;\Omega)$ using the tree-cotree technique that is well-known when $r = 0$. We have extended the technique to the case $r > 0$ without any restriction on the homology of the computational domain. The algorithm can be applied to the two families of degrees of freedom used in this framework: weights and moments.

The key point in the extension of the graph techniques to high order finite elements is the visualization of the graph associated with the degrees of freedom that the use of weights provides.

We have also considered the case of functions with zero trace, namely the space $RT_{r+1} \cap H_0(\mathrm{div}^0; \Omega)$. Also in this case we propose an algorithm based on the tree-cotree decomposition of a suitable graph to construct a basis when $\beta_1(\Omega) = 0$. When $\beta_1 \neq 0$ to have a basis it is necessary to complete the set obtained using the tree-cotree decomposition with β_1 functions associated to 1-cycles on $\partial\Omega$ that do not bound any surface contained in $\mathbb{R}^3 \setminus \Omega$, the complementary of Ω.

In the future we will work to design an efficient algorithm to compute a basis of $RT_{r+1} \cap H_0(\mathrm{div}^0; \Omega)$ without any restriction on the homology of the computational domain.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A

For the sake of completeness, we report here, in terms of the new notation, the proof of Proposition 2.1, presented in [42], that the graph $\mathcal{M}_{r+1}$ is connected. This is a fundamental property for all the propositions stated in the previous sections.

Proposition 5. *Let us set $d_N = \dim N_{r+1}$ and $d_L = \dim L_{r+1}$. If we denote $G \in \mathbb{R}^{d_N \times d_L}$ the matrix that computes, from the moments of φ_h, the moments of gradφ_h then $G^\top$ is the all-nodes incidence matrix of a directed graph $\mathcal{M}^G$ with a node for each Lagrange moment and an arc for each Nédélec moment: $\mathcal{M}^G = (M^L, M^N)$. This graph is connected.*

Proof. We have proven in [33] that the matrix $G \in \mathbb{R}^{d_N \times d_L}$ has two elements different from zero, one equal 1 and one equal -1, on each row hence $G^\top$ is the all-nodes incidence matrix of a directed graph $\mathcal{M}^G = (M^L, M^N)$. To prove that it is connected we decompose it in edge, face and tetrahedron subgraphs (see Fig. 6).

For all $e \in E$, $\mathcal{G}_e = (\mathcal{N}_e, \mathcal{A}_e)$ denotes the subgraph of $\mathcal{M}^G = (M^L, M^N)$ with nodes $\mathcal{N}_e = \{M^L_{e,\emptyset,\boldsymbol{\beta}'} \,:\, \boldsymbol{\beta}' \in \mathcal{I}(2, r-1)\}$ and arcs $\mathcal{A}_e = \{M^N_{e,\emptyset,\boldsymbol{\alpha}} \,:\, \boldsymbol{\alpha} \in \mathcal{I}(2, r)$ with $\alpha_0 \neq 0 \neq \alpha_1\}$. It is easy to check that all the nodes of $\mathcal{N}_e$ are connected with the node $M^L_{e,\emptyset,(r-1,0)}$. In fact if $\boldsymbol{\beta}' = (\beta'_0, \beta'_1)$ with $\beta'_1 \neq 0$ then the arc $M^N_{e,\emptyset,\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = (\beta'_0 + 1, \beta'_1)$ belongs to $\mathcal{A}_e$ and connects the node $M^L_{e,\emptyset,(\beta'_0,\beta'_1)}$ with the node $M^L_{e,\emptyset,(\beta'_0+1,\beta'_1-1)}$. Hence it is possible to construct a path with β'_1 arcs connecting $M^L_{e,\emptyset,(\beta'_0,\beta'_1)}$ with $M^L_{e,\emptyset,(\beta'_0+\beta'_1,0)} = M^L_{e,\emptyset,(r-1,0)}$.

Analogously, for all $f_\zeta = f \in F$, $\mathcal{G}_f = (\mathcal{N}_f, \mathcal{A}_f)$ denotes the subgraph of $\mathcal{M}^G = (M^L, M^N)$, with nodes $\mathcal{N}_f = \{M^L_{\zeta,\rho,\boldsymbol{\gamma}'} \,:\, \boldsymbol{\gamma}' \in \mathcal{I}(3, r-2),\ \zeta \in \Sigma(0 : d, 0 : n),\ \rho \in \Sigma(0 : d-2, 1 : d)\}$ and arcs $\mathcal{A}_f = \cup_{i=1}^2 \{M^N_{\zeta,\rho,\boldsymbol{\beta}} \,:\, \boldsymbol{\beta} \in \mathcal{I}(3, r-1)$ with $\beta_l \neq 0$ if $l \neq i$, $\zeta \in \Sigma(0 : d, 0 : n)$, $\rho \in \Sigma(0 : d-2, 1 : d)\}$. If $\gamma'_2 \neq 0$, taking $\boldsymbol{\beta}$ such that $\boldsymbol{\gamma}' = \boldsymbol{\beta} - e_0$, the arc $M^N_{\zeta,\rho,\boldsymbol{\beta}} \in \mathcal{A}_f$, with $\beta_1 \neq 0$ connects the node $M^L_{\zeta,\rho,(\gamma'_0,\gamma'_1,\gamma'_2)}$ with the node $M^L_{\zeta,\rho,(\gamma'_0+1,\gamma'_1,\gamma'_2-1)}$. On the other hand if $\gamma'_1 \neq 0$, taking $\boldsymbol{\beta}$ such that $\boldsymbol{\gamma}' = \boldsymbol{\beta} - e_0$, the arc $M^N_{\zeta,\rho,\boldsymbol{\beta}} \in \mathcal{A}_f$ with $\beta_2 \neq 0$, connects the node $M^L_{\zeta,\rho,(\gamma'_0,\gamma'_1,\gamma'_2)}$ with the node $M^L_{\zeta,\rho,(\gamma'_0+1,\gamma'_1-1,\gamma'_2)}$. Hence, if $\gamma'_1 + \gamma'_2 \neq 0$, it is possible to construct a path with $\gamma'_1 + \gamma'_2$ arcs connecting $M^L_{\zeta,\rho,(\gamma'_0,\gamma'_1,\gamma'_2)}$ with $M^L_{\zeta,\rho,(\gamma'_0+\gamma'_1+\gamma'_2,0,0)}$.

Finally for all $f_\zeta = t \in T$, $\mathcal{G}_t = (\mathcal{N}_t, \mathcal{A}_t)$ denotes the subgraph of $\mathcal{M}^G = (M^L, M^N)$, with nodes $\mathcal{N}_t = \{M^L_{\zeta,\rho,\boldsymbol{\delta}'} \,:\, \boldsymbol{\delta}' \in \mathcal{I}(4, r-3),\ \zeta \in (0 : n, 0 : n),\ \rho \in \Sigma(0 : n-2, 1 : n)\}$ and arcs $\mathcal{A}_t = \cup_{1\leq i<j\leq 3}\{M^N_{\zeta,\rho,\boldsymbol{\gamma}} \,:\, \boldsymbol{\gamma} \in \mathcal{I}(4, r-2)$ with $\gamma_l \neq 0$ if $l \notin \{i, j\}\}$. Proceeding as for edges and faces it is easy to check that $\mathcal{G}_t$ is connected.

We consider also the subgraph $\mathcal{G}^*_e = (\mathcal{N}^*_e, \mathcal{A}^*_e)$ with nodes $\mathcal{N}^*_e = \mathcal{N}_e \cup_{i=0}^1 M^L_{\mathbf{v}_{m_e(i)},\emptyset,\boldsymbol{\alpha}'}$ and arcs $\mathcal{A}_e = \{M^N_{e,\emptyset,\boldsymbol{\alpha}} \,:\, \boldsymbol{\alpha} \in \mathcal{I}(2, r)\}$. We can connect $M^L_{\mathbf{v}_{m_e(0)},\emptyset,\boldsymbol{\alpha}'}$ with a node of $\mathcal{G}_e$ while the second one $M^L_{\mathbf{v}_{m_e(1)},\emptyset,\boldsymbol{\alpha}'}$ connects with a node of $\mathcal{G}_e$ hence $\mathcal{G}^*_e$ is connected. For each $e \in E$, $\mathcal{G}^*_e$ is a path connecting the Lagrange moments associated with the vertices of the mesh in $\Delta_0(e)$. Hence if Ω is connected the graph $\cup_{e\in E}\mathcal{G}^*_e$ is connected.

To conclude the proof we notice that we have an arc of $\mathcal{M}^G = (M^L, M^N)$ that connects a node of $\mathcal{G}_f$ with a node of $\mathcal{G}_{f-[v_{m_f(2)}]}$ and an arc that connects a node of $\mathcal{G}_t$ with a node of $\mathcal{G}_{t-[v_{m_t(3)}]}$. Since each node of $\mathcal{M}^G$ belongs to a subgraph of the type $\mathcal{G}^*_e$ or $\mathcal{G}_f$ or $\mathcal{G}_t$ this prove that $\mathcal{M}^G$ is connected. □

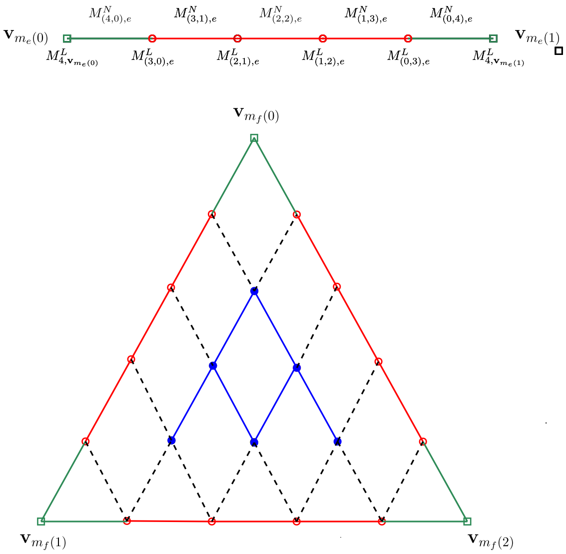

Fig. 6. On the top an example of edge subgraph $\mathcal{G}_e^*$. On the bottom, in blue, an example of face subgraph $\mathcal{G}_f$; in red and green the three subgraphs of the edges on the boundary of f, in black, other arcs of the graph that are not on any face or edge subgraph.

References

[1] Albanese R, Rubinacci G. Integral formulation for 3D eddy-current computation using edge elements. IEE Proc A: Phys Sci Meas Instrum Manag Educ Rev 1988;135(A(7)):457–62.
[2] Albanese R, Rubinacci G. Magnetostatic field computations in terms of two-component vector potentials. Int J Numer Meth Engnrg 1990;29:515–32.
[3] Bossavit A. Computational electromagnetism. In: Electromagnetism. San Diego, CA: Academic Press Inc; 1998, p. xx+352, Variational formulations, complementarity, edge elements.
[4] Rapetti F, Bossavit A. Geometrical localisation of the degrees of freedom for Whitney elements of higher order. IET Sci Meas Technol 2007;1:63–6.
[5] Rapetti F, Bossavit A. Whitney forms of higher degree. SIAM J Numer Anal 2009;47(3):2369–86.
[6] Alonso Rodríguez A, Camaño J, De Los Santos E, Rapetti F. A graph approach for the construction of high order divergence-free Raviart-Thomas finite elements. Calcolo 2018;55.
[7] Ledger PD, Zaglmayr S. *hp*-finite element simulation of three-dimensional eddy current problems on multiply connected domains. Comput Methods Appl Mech Engrg 2010;199(49–52):3386–401.
[8] Chen L, Huang X. Finite elements for div- and divdiv-conforming symmetric tensors in arbitrary dimension. SIAM J Numer Anal 2022;60(4):1932–61.
[9] Hecht F. Construction d'une base de fonctions P_1 non conforme à divergence nulle dans $\mathbf{R}^3$. RAIRO Anal Numér 1981;15(2):119–50.
[10] Dubois F. Discrete vector potential representation of a divergence-free vector field in three-dimensional domains: numerical analysis of a model problem. SIAM J Numer Anal 1990;27(5):1103–41.
[11] Scheichl R. Decoupling three-dimensional mixed problems using divergence-free finite elements. SIAM J Sci Comput 2002;23(5):1752–76.
[12] Rapetti F, Dubois F, Bossavit A. Discrete vector potentials for nonsimply connected three-dimensional domains. SIAM J Numer Anal 2003;41(4):1505–27.
[13] Alonso Rodríguez A, Valli A. Finite element potentials. Appl Numer Math 2015;95:2–14.
[14] Arnold DN, Falk RS, Winther R. Finite element exterior calculus, homological techniques, and applications. Acta Numer 2006;15:1–155.
[15] Bossavit A. Computational electromagnetism. San Diego: Academic Press Inc; 1998.
[16] Bossavit A. Magnetostatic problems in multiply connected regions: some properties of the curl operator. Phys Sci, Meas Instrum, Manag Educat - Rev, IEE Proc A 1988;135:179–87.
[17] Albanese R, Rubinacci G. Finite element methods for the solution of 3D eddy current problems. Adv Imag Electron Phys 1998;102:1–86.
[18] Ren Z, Razek A. Boundary edge elements and spanning tree technique in three-dimensional electromagnetic field computation. Internat J Numer Methods Engrg 1993;36:2877–93.
[19] Manges JB, Cendes ZJ. Tree-cotree decompositions for first-order complete tangential vector finite elements. Internat J Numer Methods Engrg 1997;40:1667–85.
[20] Alotto P, Perugia I. Mixed finite element methods and tree-cotree implicit condensation. Calcolo 1999;36(4):233–48.
[21] Alonso Rodríguez A, Camaño J, Ghiloni R, Valli A. Graphs, spanning trees and divergence-free finite elements in domains of general topology. IMA J Numer Anal 2017;37(4):1986–2003.
[22] Webb J, Forghani B. A single scalar potential method for 3D magnetostatics using edge elements. IEEE Trans Magn 1989;25(5):4126–8.
[23] Ainsworth M, Andriamaro G, Davydov O. Bernstein-Bézier finite elements of arbitrary order and optimal assembly procedures. SIAM J Sci Comput 2011;33(6):3087–109.
[24] Ainsworth M, Fu G. Bernstein–Bézier bases for tetrahedral finite elements. Comput Methods Appl Mech Engrg 2018;340:178–201.
[25] Scheichl R. Iterative solution of saddle point problems using divergence-free finite elements with applications to groundwater flow. Ann Arbor, MI: ProQuest LLC; 2000, p. 185, Thesis (Ph.D.)–University of Bath (United Kingdom).
[26] Wang J, Wang Y, Ye X. A robust numerical method for Stokes equations based on divergence-free $H(\mathrm{div})$ finite element methods. SIAM J Sci Comput 2009;31(4):2784–802.
[27] Thulasiraman K, Swamy M. Graphs: theory and algorithms. New York: A Wiley-Interscience Publication, John Wiley & Sons, Inc; 1992, p. xvi+460.
[28] Bossavit A. Generating Whitney forms of polynomial degree one and higher. IEEE Trans Magn 2002;38:341–4.
[29] Nédélec J-C. Mixed finite elements in $\mathbf{R}^3$. Numer Math 1980;35:315–41.
[30] Christiansen S, Rapetti F. On high order finite element spaces of differential forms. Math Comp 2016;85(298):517–48.

[31] Alonso Rodríguez A, Bruni Bruno L, Rapetti F. Towards nonuniform distributions of unisolvent weights for high-order Whitney edge elements. Calcolo 2022;59.
[32] Bonazzoli M, Rapetti F. High-order finite elements in numerical electromagnetism: degrees of freedom and generators in duality. Numer Algorithms 2017;74(1):111–36.
[33] Alonso Rodríguez A, Camaño J, De Los Santos E, Rapetti F. Weights for moments' geometrical localization: a canonical isomorphism. 2023, preprint.
[34] De Los Santos E, Alonso Rodríguez A, Rapetti F. Construction of a spanning tree for high-order edge elements. Int J Numer Model: Electr Netw, Dev Fields 2022;36(3).
[35] Kettunen L, Forsman K, Bossavit A. Formulation of the eddy current problem in multiply connected regions in terms of h. Internat J Numer Methods Engrg 1998;41:935–54.
[36] Alonso Rodríguez A, Bruni Bruno L, Rapetti F. Small trees for high order whitney elements. In: Spectral and high order methods for PDEs. LNCSE, vol. 134, Springer-Verlag; 2020, p. 587–97.
[37] Bermúdez A, Rodríguez R, Salgado P. A finite element method with Lagrange multipliers for low-frequency harmonic Maxwell equations. SIAM J Numer Anal 2002;40(5):1823–49.
[38] Lara E, Rodríguez R, Venegas P. Spectral approximation of the curl operator in multiply connected domains. Discrete Contin Dyn Syst Ser S 2016;9(1):235–53.
[39] Alonso Rodríguez A, Camaño J. A graph-based algorithm for the approximation of the spectrum of the curl operator. SIAM J Sci Comput 2023;45(1):A147–69.
[40] Alonso Rodríguez A, Bertolazzi E, Ghiloni R, Valli A. Construction of a finite element basis of the first de Rham cohomology group and numerical solution of 3D magnetostatic problems. SIAM J Numer Anal 2013;51(4):2380–402.
[41] Hiptmair R, Ostrowski J. Generators of $H_1(\Gamma_h, \mathbb{Z})$ for triangulated surfaces: construction and classification. SIAM J Comput 2002;31(5):1405–23.
[42] De Los Santos E. Divergence-free finite elements in general topological domains and applications (Thesis (Ph.D.)), Center for Research in Mathematical Engineering, University of Concepción (Chile); 2019, p. 100.