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Mathematical Logic

An infinitary propositional probability logic

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Abstract
We introduce a logic for a class of probabilistic Kripke structures that we call type structures, as they are inspired by Harsanyi type spaces. The latter structures are used in theoretical economics and game theory. A strong completeness theorem for an associated infinitary propositional logic with probabilistic operators was proved by Meier. By simplifying Meier's proof, we prove that our logic is strongly complete with respect to the class of type structures. In order to do that, we define a canonical model (in the sense of modal logics), which turns out to be a terminal object in a suitable category. Furthermore, we extend some standard model-theoretic constructions to type structures and we prove analogues of first-order results for those constructions.

Keywords Probability logic · Infinitary logic · Completeness

Mathematics Subject Classification Primary 03B48; Secondary 03C75 · 03C20 · 03B60

1 Introduction

In the first part of this work we provide a different and, in our opinion, simpler proof of a completeness result originally proved in [13]. Such result concerns an infinitary propositional probability logic that arises from the formalization of certain structures known as *Harsanyi type spaces*. The latter were originally introduced and partly formalized in [7] to model beliefs arising from the interactions within a set of agents and to deal with hierarchies of beliefs. Later, the study of type spaces from a logical perspective began. In this paper we are concerned with the latter.

Among the works that investigate logical systems inspired by Harsanyi type spaces, we mention [13, 21]. In this paper we build on [13]. In turn, major sources of inspiration of [13] are [8] (where a weak completeness result is established for a finitary language);

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[9] (where a universal type space is constructed in purely measure-theoretic terms, thus removing the topological assumptions needed in a previous construction proposed in [14]) and also [3] (where the relation between the semantic and the syntactic approach in game theory and economics is discussed).

We refer the reader to the exhaustive introduction and bibliography in [13, 21] for a thorough discussion on the import of Harsanyi type spaces in theoretical economics and game theory, as well as for an historical account on the development of the subject. It is also worth mentioning [15], where the authors propose a categorical approach.

Harsanyi type spaces can be formalized as probabilistic variants of Kripke structures. We deal with structures which are weaker than Harsanyi type spaces. We call them *type structures*.

We stress that, with the only exception of Sect. 5 (where we take a short categorical detour), we are concerned exclusively with the logical and model-theoretic aspects of type structures. In short, we have a class of structures (the type structures) and we introduce a probability logic for that class, with the aim of proving a completeness result. Then we investigate model-theoretic properties of the proposed logic. We focus almost exclusively on the algebraic/probabilistic content of type structures. As already mentioned, our structures and their associated deductive system are weaker than those in [13]. Hence, one cannot reasonably expect then to fully model key phenomena in economic or game theory like interaction of agents, agents introspection or hierarchies of beliefs (but, regarding introspection, see Remark 14). At this stage, our logic should be regarded as a probability logic. We believe that it can be extended to encompass the above mentioned phenomena. Such an extension goes beyond the scope this paper.

Going back to type structures, intuitively we may say that, in a type structure, a family of probability measures models the beliefs of a single agent at various states of the world. Differently from [13], who allows for multiple agents, we use a one-sorted language, as is done in [21].

Admittedly, the current framework may look quite restrictive to readers whose main interests are in theoretical economics or in game theory. On the other hand, the motivations behind this paper are of logic nature, as we stated above.

We point out that, in [21], Zhou refrains from using an infinitary language. Nevertheless he must allow for an infinitary deduction rule to mimic the crucial Archimedean property (see Sect. 3 below). Eventually, Zhou gets a weak completeness theorem. Namely, he shows that the set of logical theorems is exactly the set of logically valid formulas in his propositional probability logic.

In [13], Meier gets strong completeness (namely, the equivalence between provability and logical consequence) by working with an infinitary propositional probability logic. In the first part of this work, we take inspiration from [13] and we show that a suitable choice of an infinitary language allows for a different and, in our opinion, simpler proof of a completeness result.

Among the differences with [13], we mention: 1. a different formulation of the property of continuity at the empty set of a probability measure; 2. the use of very basic results in probability theory (for instance, we do not need to invoke Carathéodory's Extension Theorem at any point); 3. the straightforward proofs of completeness (Theorem 13 below) and of the existence of a terminal object within the category of type structures (Sect. 5).

The reader who is not interested in a comparison with the first part of [13] will find this paper basically self-contained.

We begin the second part of this work with some comments on the above mentioned category of type structures, which is the one usually studied in the literature. Such category apparently lacks some natural constructions. For this reason, we focus on algebraic rather than categorical properties of type structures. More precisely, we naturally extend products and ultraproducts of first-order structures to arbitrary families of type structures. Among others, we prove an Upward Löwenheim-Skolem-like theorem and, under an admittedly very strong set-theoretic assumption, a Łoś-like theorem. Such results hint that further model-theoretic investigations of type structures might be of interest.

By taking inspiration from the literature, we extend the inverse limit construction from systems of probability spaces to certain systems of type structures. Borrowing from the literature, we also deal the direct limit of systems of probability spaces, by pointing out the difficulties in extending such construction to systems of type structures.

We do not know whether the above mentioned constructions have economic or game-theoretic meaning.

Eventually, we point out that most logics are compositional, but probability is not. The general framework that we introduce in the first part of this work seems necessary to ensure a fruitful interplay between the logic and the probability components of the system.

2 Formulas and structures

By recursion on $n \in \omega$ we define $\beth_0 = \aleph_0$; $\beth_{n+1} = 2^{\beth_n}$ and we let $\nu = (\beth_2)^+$, the successor cardinal of $\beth_2$. Such choice of ν will be motivated in the following.

First of all we introduce the formulas of our logic, to be named $L_{\nu P}$. We do it in a rather informal way: all the definitions below can be easily formalized.

The alphabet of $L_{\nu P}$ contains denumerably many propositional variables q_i, $i \in \mathbb{N}$, and the symbol $\perp$ for "falsum" (these also play the role of atomic formulas); the usual finitary propositional connectives; the infinitary connectives $\bigvee$ and $\bigwedge$; the probability quantifiers $P^{\geq r}$, $r \in \mathbb{Q} \cap [0, 1]$; the auxiliary symbols (and).

We denote by V the set of propositional variables.

The set $\mathbb{P}$ of *probability formulas* is the least set S with the following properties: S contains the atomic formulas; S is closed under application of the finitary propositional connectives and under conjunctions and disjunctions of *countable* sequences of formulas; for all $\varphi \in S$ and all $r \in \mathbb{Q} \cap [0, 1]$, the string $P^{\geq r}\varphi$ is in S.

The restriction on the cardinality of conjunctions and disjunctions in $\mathbb{P}$ is a technical requirement which is necessary to make sure that the satisfiability relation is well-defined. See below. Each formula in $\mathbb{P}$ is a countable string of symbols from a countable alphabet. It follows that $|\mathbb{P}| = \beth_1$.

Notice that, differently from the set $\mathcal{L}_0$ of finitary formulas in [13], we allow for infinitary formulas in $\mathbb{P}$. This will simplify axioms, deduction rules and also most of the proofs of the results in [13]. It should also be noted that a disavantage of allowing for infinitary formulas in $\mathbb{P}$ is that an easily computable upper bound ($\beth_2$) for the cardinality of the canonical structure defined in Sect. 4 is higher than Meier's ($\beth_1$–when set I of agents is countable).

As is customary in set theory, throughout this paper initial letters of the greek alphabet ($\alpha, \beta, \gamma, \dots$) denote ordinals.

The set $\mathbb{F}$ of formulas is the least set S that contains $\mathbb{P}$ and is closed under application of the finitary connectives and under the following formation rule: for all $\alpha < \nu$ and all sequences $A \in S^\alpha$ the strings $\bigwedge A_0 \dots A_\beta \dots$ and $\bigvee A_0 \dots A_\beta \dots$, $\beta < \alpha$, are in S. For notational simplicity, the latter will be denoted by $\bigwedge_{\beta<\alpha} A_\beta$ and $\bigvee_{\beta<\alpha} A_\beta$ respectively. We also write $\bigwedge \Gamma$ for the conjunction of the formulas in set Γ, assuming that some well-ordering on Γ has been fixed. Similarly with $\bigvee \Gamma$.

Regularity of ν and the cardinality limitation in the definition of $\mathbb{F}$ imply that each formula in $\mathbb{F}$ is a string of length $< \nu$.

We will use the terms $\mathbb{P}$-formula and $\mathbb{F}$-formula, with the obvious meaning. When writing just "formula", we mean "$\mathbb{F}$-formula".

In order to reduce the use of brackets, we assume that the propositional connectives (finitary or infinitary) satisfy the same binding priority as in propositional logic, with the probability quantifiers having the highest one.

From now on, letters r, s, t always denote rationals in the closed real unit interval. When we write $P^{\geq r}\varphi$, it is understood that $\varphi \in \mathbb{P}$.

We write $P^{>r}\varphi$ as abbreviation for $\neg P^{\geq 1-r}\neg\varphi$. For better readability, we will also use the expressions $P^{\leq r}\varphi$ and $P^{<r}\varphi$, which can be easily defined in terms of the primitive probability quantifier: they stand for $\neg P^{>r}\varphi$ and $\neg P^{\geq r}\varphi$ respectively.

Next, we introduce the structures that we will deal with and their semantics. We take inspiration from the formalization of the Harsanyi type spaces first explicitly and completely given in [14].

Let $(\Omega, \mathcal{F})$ be a measurable space. On the family $P(\Omega, \mathcal{F})$ of probability measures on $(\Omega, \mathcal{F})$ we consider the σ-algebra $\sigma_{\Omega,\mathcal{F}}$ generated by the sets of the form

$$\{\mu \in P(\Omega, \mathcal{F}) : \mu(A) \geq r\}, \quad \text{with } A \in \mathcal{F} \text{ and } r \in \mathbb{Q} \cap [0, 1]\,.$$

Definition 1 A type structure is a 4-tuple $(\Omega, \mathcal{F}, T, v)$, where

(1) $(\Omega, \mathcal{F})$ is a measurable space;
(2) $T : \Omega \to P(\Omega, \mathcal{F})$ is a measurable function;
(3) $v : V \cup \{\bot\} \to \mathcal{F}$ is a function such that $v(\bot) = \emptyset$.

With reference to the previous definition, the triple $(\Omega, \mathcal{F}, T)$ is called a *type space*. Intuitively, for every $q \in V \cup \{\bot\}$, $v(q)$ is the set of $\omega \in \Omega$ at which q is true. The latter is required to be a measurable subset of Ω.

If there is no ambiguity, we identify a type structure $(\Omega, \mathcal{F}, T, v)$ with its support Ω.

Let $(\Omega, \mathcal{F}, T, v)$ be a type structure. We recursively define when a formula φ is true at $\omega \in \Omega$ (notation: $\Omega, \omega \models \varphi$) as follows:

(1) $\Omega, \omega \not\models \bot$;
(2) $\Omega, \omega \models q$ if and only if $\omega \in v(q)$, for $q \in V$;
(3) the finitary and infinitary propositional cases relative to $\mathbb{P}$-formulas are the same as in propositional logic;
(4) $\Omega, \omega \models P^{\geq r}\psi$ if and only if $T(\omega)(\{\omega' \in \Omega : \Omega, \omega' \models \psi\}) \geq r$, for a $\mathbb{P}$-formula ψ;
(5) the cases relative to the formulas in $\mathbb{F} \setminus \mathbb{P}$ are the same as in propositional logic.

We let $\varphi^\Omega := \{w \in \Omega : \Omega, \omega \models \varphi\}$.

Remark 2 One easily shows by induction on $\varphi \in \mathbb{P}$ that $\varphi^\Omega \in \mathcal{F}$. Here it is crucial that $\mathbb{P}$ is closed under conjunctions and disjunctions of countable cardinality only.

Furthermore, when φ of the form $P^{\geq r}\psi$, notice that

$$\varphi^\Omega = T^{-1}(\{\mu \in P(\Omega, \mathcal{F}) : \mu(\psi^\Omega) \geq r\}).$$

Then measurability of T and the inductive assumption $\psi^\Omega \in \mathcal{F}$ yield $\varphi^\Omega \in \mathcal{F}$. Therefore the satisfiability relation is well-defined.

A formula φ is satisfiable (valid) in Ω if $\varphi^\Omega \neq \emptyset$ ($\varphi^\Omega = \Omega$). Let $\Delta \subseteq \mathbb{F}$. We let $\Delta^\Omega = \bigcap_{\gamma \in \Delta} \gamma^\Omega$, with the convention that $\emptyset^\Omega = \Omega$.

We adopt a "local" notion of logical/semantic consequence. We say that $\varphi \in \mathbb{F}$ is a logical consequence of Δ (notation: $\Delta \models \varphi$) if

$$\Delta^\Omega \subseteq \varphi^\Omega, \quad \text{for all type structures } \Omega.$$

We write $\models \varphi$ for $\emptyset \models \varphi$. A formula φ is valid if $\models \varphi$.

3 Axioms, rules and soundness

As in [11, Chapter V], we denote by $\mathfrak{P}(L_\nu)$ the infinitary propositional logic (with denumerably many propositional variables) in which conjunctions and disjunctions over sequences of formulas of length less than ν are allowed. In the following we will use, often without explicit mention, [11, Theorem 5.5.4] on the strong completeness of $\mathfrak{P}(L_\nu)$.

The following set of axioms schemas for $L_{\nu P}$ is by no means intended to be minimal (this is certainly not the case with A0 below). Rather, we aim at an intuitive set of axioms. The axiomatization below is inspired by those in [12] and [13].

A0. All the $\mathbb{F}$-instances of valid formulas of $\mathfrak{P}(L_\nu)$
P1. $P^{\geq 0}\bot$
P2. $P^{\geq r}\varphi \to P^{>s}\varphi$ $\quad (s < r)$
P3. $P^{>r}\varphi \to P^{\geq r}\varphi$
P4. $P^{\leq r}\varphi \wedge P^{\leq s}\psi \to P^{\leq \min(r+s,1)}(\varphi \vee \psi)$
P5. $P^{\geq 1}(\neg(\varphi \wedge \psi)) \wedge P^{\geq r}\varphi \wedge P^{\geq s}\psi \to P^{\geq r+s}(\varphi \vee \psi)$ $\quad (r + s \leq 1)$
P6. $P^{\geq 1}(\varphi \to \psi) \to (P^{\geq r}\varphi \to P^{\geq r}\psi)$

P7. $\bigwedge_{r<s} P^{\geq r}\varphi \to P^{\geq s}\varphi$

P8. $\bigwedge_{0<n\in\mathbb{N}} \bigvee_{k\in\mathbb{N}} P^{\leq 1/n}(\bigwedge_{m\leq k}\varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}}\varphi_m)$, for all $\{\varphi_m\}_{m\in\mathbb{N}} \subseteq \mathbb{P}$

Axioms P4 and P5 state finite additivity of probability. Axioms P6 and P7 state monotonicity of probability and the Archimedean property respectively.

Remark 3 Axiom P8 formalizes the property of continuity (from above) at the empty set (the latter is often referred to as continuity at zero). Namely, P8 is intended to syntactically capture the property that, for a finite measure μ, whenever $\{A_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of measurable sets such that $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$, then $\lim_{k\to\infty}\mu(A_k) = 0$.

In order to clarify the content of P8, we argue semantically. We fix a type structure Ω and $\{\varphi_m\}_{m\in\mathbb{N}} \subseteq \mathbb{P}$. By Remark 2, each φ_m^Ω is a measurable set. Hence so is $A_k = \left(\bigwedge_{m\leq k}\varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}}\varphi_m\right)^\Omega$, for all $k \in \mathbb{N}$. Let $\omega \in \Omega$ be arbitrarily chosen. By definition of truth of a formula, we have that formula $P^{\leq 1/n}(\bigwedge_{m\leq k}\varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}}\varphi_m)$ is true at ω if and only if $T(\omega)(A_k) \leq 1/n$. Therefore truth of P8 at ω amounts to requiring that the following holds:

$(*)$ $\quad$ *for each* $0 < n \in \mathbb{N}$ *there is some* $k \in \mathbb{N}$ *such that* $T(\omega)(A_k) \leq 1/n$.

Finally, we note that the A_k's defined above form a decreasing sequence and that $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$. Since, for a finite measure, property of σ-additivity is equivalent to finite additivity plus continuity at the empty set (see, for instance, [2, §1.2]) and since $T(\omega)$ is a probability measure, statement (*) is necessarily true at ω.

We stress that the previous argument shows that P8 is a valid axiom.

Notice that, differently from [13], we formulate continuity at the empty set by means of an axiom, rather than with an inference rule.

The inference rules of $L_{\nu P}$ are the following. Albeit our formal proofs will be Hilbert style, we formulate the rules in natural deduction style for sake of better understanding.

MP (Modus Ponens)

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

C (Conjunction) For all cardinals $\kappa \leq \nu$ and all $\{\varphi_\alpha\}_{\alpha<\kappa} \subseteq \mathbb{F}$,

$$\frac{\varphi_0 \ \ldots\ \varphi_\alpha \ldots\ (\alpha < \kappa)}{\bigwedge_{\alpha<\kappa}\varphi_\alpha}$$

N (Necessitation)

$$\frac{\vdash \varphi}{\vdash P^{\geq 1}\varphi} \qquad (\varphi \in \mathbb{P})$$

Rules MP and C are exactly those in [11, §5.1]. Rule N is reminiscent of the necessitation rule in modal logic (hence the name).

Notice that, same as in modal logic and differently from MP and C-rule, N-rule can be applied only when φ is provable from logical axioms and deduction rules only (i.e. when φ is a logical theorem, according to the terminology introduced below).

As we remarked above, differently from [13], we do not need an infinitary rule expressing the property of continuity at the empty set.

The notion of (Hilbert style) proof from a given set of assumptions is formulated as usual (see the beginning of [11, Chapter 5]). We stress that a proof is a sequence of fewer than ν formulas.

We use the standard notation $\Gamma \vdash \varphi$ to say that there exists a proof of φ from the set Γ of assumptions. As is usual, $\vdash \varphi$ stands for $\emptyset \vdash \varphi$. Formulas φ satisfying the latter are called logical theorems.

In order to become familiar with the deduction system, we invite the reader to verify that $\vdash P^{\leq 1}\varphi$, for every $\mathbb{P}$-formula φ. Notice also that $\vdash P^{>1}(\varphi) \leftrightarrow \bot$.

Furthermore, if φ, ψ are $\mathbb{P}$-formulas such that $\vdash \varphi \leftrightarrow \psi$ then, for all r, $\vdash P^{\geq r}\varphi \leftrightarrow P^{\geq r}\psi$ follows from N rule and axiom P6. In the following we will often use the latter fact without explicit mention.

A set $\Gamma \subseteq \mathbb{F}$ is consistent if $\Gamma \nvdash \bot$. A maximal consistent set is a set which is maximal with respect to inclusion among consistent sets.

Let Γ be maximal consistent. It is straightforward to verify that the following hold: Γ is closed under provability; for every formula φ exactly one between φ and $\neg\varphi$ is in Γ; $(\bigwedge_{\alpha<\kappa} \varphi_\alpha) \in \Gamma$ if and only if, for all $\alpha < \kappa$, $\varphi_\alpha \in \Gamma$; $(\bigvee_{\alpha<\kappa} \varphi_\alpha) \in \Gamma$ if and only if there exists $\alpha < \kappa$ such that $\varphi_\alpha \in \Gamma$; $(\alpha \to \beta) \in \Gamma$ if and only if $(\alpha \in \Gamma \Rightarrow \beta \in \Gamma)$. The existence of (maximal) consistent sets will be a consequence of the Soundness Theorem below.

Theorem 4 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathbb{F}$. *Then*

$$\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \models \varphi.$$

Proof By induction on a proof of φ from Γ, after verifying that axioms are valid and rules preserve validity. As for validity of P8, see Remark 3 above. $\square$

As a corollary of the Soundness Theorem, we get that every satisfiable set of sentences is consistent. Furthermore, for all type structures Ω and all $\omega \in \Omega$, the set $\{\varphi \in \mathbb{F} : \Omega, \omega \models \varphi\}$ is maximal consistent.

Same as in infinitary propositional logic, it is immediate to exhibit an inconsistent set which can be obtained as union of a chain of consistent sets. Hence the usual application of Zorn's Lemma for proving that every consistent set extends to some maximal consistent one cannot be performed in the current framework.

4 Completeness

We provide a construction which vaguely resembles the canonical model construction in various modal logics.

We denote by Ω the family of sets ω of $\mathbb{P}$-formulas with the following properties:

(1) ω is consistent;
(2) every formula in ω is an atomic or a negated atomic formula or is of the form $P^{\geq r}\varphi$ or $\neg P^{\geq r}\varphi$ for some r and some $\varphi \in \mathbb{P}$;
(3) for each propositional variable q, one between q and $\neg q$ is in ω;
(4) for every formula φ of the form $P^{\geq r}\psi$, one between φ and $\neg\varphi$ is in ω;

It follows from Theorem 4 that $\Omega \neq \emptyset$.

Remark 5 Notice that every $\omega \in \Omega$ has cardinality $\beth_1$ and that $|\Omega| \leq \beth_2$. In the following we will form conjunctions of $|\Omega|$-many formulas (see Proposition 8 below). Hence the choice $\nu = (\beth_2)^+$.

We denote by $\omega_\vdash$ the closure under provability of $\omega \in \Omega$.

Remark 6 A straightforward proof by induction on formulas shows that, for every $\omega \in \Omega$ and every $\mathbb{F}$-formula ψ, the set $\omega_\vdash$ contains exactly one between ψ and $\neg\psi$. It follows that $\omega_\vdash$ is maximal consistent. Furthermore, there is a one-to-one correspondence between Ω and the family of maximal consistent sets of $\mathbb{F}$-formulas.

Next, we define a type structure based on Ω. Let $\varphi \in \mathbb{P}$ and $\omega \in \Omega$. We let

$$\varphi^\omega = \sup\{r : P^{\geq r}\varphi \in \omega\}, \quad \varphi_\omega = \inf\{s : P^{\leq s}\varphi \in \omega\}.$$

Proposition 7 *Let $\varphi \in \mathbb{P}$. For all $\omega \in \Omega$ it holds that $\varphi^\omega = \varphi_\omega$.*

Proof Let us assume that $\varphi^\omega < \varphi_\omega$. Let $r \in \mathbb{Q}$ be such that $\varphi^\omega < r < \varphi_\omega$. Then $P^{\leq r}\varphi \notin \omega$. From the latter follows that $P^{>r}\varphi \in \omega$. By axiom P3 we get $P^{\geq r}\varphi \in \omega$. On the other hand, from $\varphi^\omega < r$ we get $P^{\geq r}\varphi \notin \omega$: a contradiction. Hence $\varphi^\omega \geq \varphi_\omega$.

Eventually, let us assume $\varphi_\omega < \varphi^\omega$. Then there exist $r, s \in \mathbb{Q}$ such that $\varphi_\omega < r < s < \varphi^\omega$ and both $P^{\leq r}\varphi$, $P^{\geq s}\varphi$ are in ω. By axiom P2 we get from the latter $P^{>r}\varphi \in \omega$, thus contradicting the consistency of ω.

Therefore $\varphi^\omega = \varphi_\omega$. □

For every $\mathbb{F}$-formula η, we let

$$[\eta] = \{\omega \in \Omega : \ \eta \in \omega_\vdash\},$$

where $\omega_\vdash$ is the deductive closure of ω (see Remark 6). Let $\Gamma \subseteq \mathbb{F}$. We let $[\Gamma] = \bigcap_{\gamma\in\Gamma}[\gamma]$. Notice that if $|\Gamma| < \nu$ then $[\Gamma] = \left[\bigwedge \Gamma\right]$.

Same as in [13, Proposition 2], but with different and possibly simpler proofs, we get the following:

Proposition 8

(1) $\vdash \bigvee_{\omega\in\Omega}(\bigwedge \omega)$.
(2) For every $\mathbb{F}$-formula η,

$$\vdash \eta \leftrightarrow \bigvee_{\omega\in[\eta]} \left(\bigwedge \omega\right).$$

Proof

(1) Let R be an injective map which, restricted to V, is the identity and maps every formula of the form $P^{\geq r}\varphi$ into a new propositional variable $q_{\varphi,r}$. Application of R transforms every $\omega \in \Omega$ into a set $R(\omega)$ of propositional variables or negated propositional variables of the infinitary propositional logic $\mathfrak{P}(L_\nu)$ whose set, let us call it V_1, of propositional variables has cardinality $\beth_1$ and contains V.
Let $\mathcal{T}$ be the set of truth valuations of $\mathfrak{P}(L_\nu)$. Notice that $|\mathcal{T}| = \beth_2$, for every $v \in \mathcal{T}$ is uniquely determined by its restriction to the set V_1. For every $v \in \mathcal{T}$ and every propositional variable q we let

$$q^{v(q)} = \begin{cases} q & \text{if } v(q) = 1; \\ \neg q & \text{otherwise} \end{cases}$$

Since $\models \bigvee_{v \in \mathcal{T}} \bigwedge_{q \in V_1} q^{v(q)}$, by [11, Theorem 5.4], we have that $\bigvee_{v \in \mathcal{T}} \bigwedge_{q \in V_1} q^{v(q)}$ is a theorem of $\mathfrak{P}(L_\nu)$. As $L_{\nu P}$ extends $\mathfrak{P}(L_\nu)$, we get in the former logic

$$\vdash \bigvee_{v \in \mathcal{T}} \bigwedge_{q \in V_1} R^{-1}(q^{v(q)})$$

(with a slight notational abuse).
After recalling that $\vdash \psi \vee \bot \leftrightarrow \psi$, we remove from the formula above all the disjuncts which are inconsistent formulas of $L_{\nu P}$ and we conclude that

$$\vdash \bigvee_{\omega \in \Omega} \left(\bigwedge \omega \right).$$

(2) From (1), we get $\vdash \psi \leftrightarrow \psi \wedge \bigvee_{\omega \in \Omega} (\bigwedge \omega)$. Hence $\vdash \psi \leftrightarrow \bigvee_{\omega \in \Omega} (\psi \wedge \bigwedge \omega)$. Recalling Remark 6 and applying same argument used at the end of (1), we get the required result. $\square$

Let

$$\mathcal{F} = \{[\varphi] : \varphi \in \mathbb{P}\}.$$

It is straightforward to verify that $\mathcal{F}$ is a σ-algebra. In particular, $[\varphi]^C = [\neg\varphi]$ and $\bigcap_{n \in \mathbb{N}} [\varphi_n] = \left[\bigwedge_{n \in \mathbb{N}} \varphi_n \right]$.
We define

$$\begin{array}{rcl} v : V \cup \{\bot\} & \longrightarrow & \mathcal{F} \\ q & \longmapsto & [q] \end{array}$$

Clearly, $v(\bot) = \emptyset$.
Let $\omega \in \Omega$. We define

$$\begin{array}{rcl} T(\omega) : \mathcal{F} & \longrightarrow & [0, 1] \\ [\varphi] & \longmapsto & \varphi^\omega \end{array}$$

First of all we notice that $T(\omega)$ is well-defined. For if $[\varphi] = [\psi]$, from Proposition 8(2) we get $\vdash \varphi \leftrightarrow \psi$ and, by N rule and axiom P6, we conclude that $\varphi^{\omega} = \psi^{\omega}$.

Also, $T(\omega)([\neg\bot]) = 1$ follows from $\vdash \neg\bot$ and from N rule.

In the following we often omit brackets and we write $T(\omega)[\varphi]$ for $T(\omega)([\varphi])$.

Next, we verify that T is a measurable function. Let $\varphi \in \mathbb{P}$. As is well known, it suffices to prove that, for all $r \in \mathbb{Q}\cap[0,1]$, the set $T^{-1}(\{\mu \in P(\Omega, \mathcal{F}) : \mu([\varphi]) \geq r\})$ is measurable. We have:

$$\begin{aligned} T^{-1}(\{\mu \in P(\Omega, \mathcal{F}) : \mu([\varphi]) \geq r\}) &= \{\omega \in \Omega : T(\omega)[\varphi] \geq r\} \\ &= \bigcap \left\{\left[P^{\geq s}\varphi\right] : s \in \mathbb{Q} \cap [0,1] \text{ and } s < r\right\} \end{aligned}$$

and the latter intersection is in $\mathcal{F}$. Actually, by axiom P7 (Archimedean property), the intersection above is equal to $\left[P^{\geq r}\varphi\right]$, but we point out that P7 is not really necessary here.

We show that $T(w)$ is finitely additive. First of all, it is worth noticing that $[\varphi] \cup [\psi] = [\varphi \vee \psi]$. We begin by proving that

$$T(\omega)([\varphi] \cup [\psi]) \leq T(\omega)[\varphi] + T(\omega)[\psi]$$

holds under no additional assumption. We treat the nontrivial case when $T(\omega)[\varphi] + T(\omega)[\psi] < 1$.

By recalling Proposition 7, we have

$$\begin{aligned} T(\omega)[\varphi] + T(\omega)[\psi] &= \inf\{s : P^{\leq s}\varphi \in \omega\} + \inf\{t : P^{\leq t}\psi \in \omega\} \\ &= \inf\{s + t : P^{\leq s}\varphi \in \omega \text{ and } P^{\leq t}\varphi \in \omega\} \\ &= \inf\{s + t \leq 1 : P^{\leq s}\varphi \in \omega \text{ and } P^{\leq t}\varphi \in \omega\}. \end{aligned}$$

(Notice that the set in the last line is nonempty because of the assumption $T(\omega)[\varphi] + T(\omega)[\psi] < 1$.)

By axiom P4 we get

$$\{s + t \leq 1 : P^{\leq s}\varphi \in \omega \text{ and } P^{\leq t}\varphi \in \omega\} \subseteq \{r : P^{\leq r}(\varphi \vee \psi) \in \omega\}.$$

Therefore

$$\inf\{s + t \leq 1 : P^{\leq s}\varphi \in \omega \text{ and } P^{\leq t}\varphi \in \omega\} \geq \inf\{r : P^{\leq r}(\varphi \vee \psi) \in \omega\}.$$

Hence $T(\omega)[\varphi] + T(\omega)[\psi] \geq T(\omega)[\varphi \vee \psi]$ and the latter is equal to $T(\omega)([\varphi] \cup [\psi])$.

Next, we assume that $\varphi, \psi \in \mathbb{P}$ satisfy $[\varphi] \cap [\psi] = \emptyset$ and we prove $T(\omega)[\varphi] + T(\omega)[\psi] \leq T(\omega)([\varphi] \cup [\psi])$.

From Proposition 8(2) we get $\vdash \varphi \wedge \psi \leftrightarrow \bot$ (we follow the standard convention that an empty disjunction stands for the formula $\bot$). Application of N rule yields $\vdash P^{\geq 1}(\neg(\varphi \wedge \psi))$. We deal with the nontrivial case when $T(\omega)([\varphi] \cup [\psi]) < 1$.

$$T(\omega)[\varphi] + T(\omega)[\psi] = \sup\{s : P^{\geq s}\varphi \in \omega\} + \sup\{t : P^{\geq t}\psi \in \omega\}$$

$$= \sup\{s+t : P^{\geq s}\varphi \in \omega \text{ and } P^{\geq t}\psi \in \omega\}.$$

By axiom P5 and by the assumption $T(\omega)([\varphi] \cup [\psi]) < 1$, we get that $\{s+t : P^{\geq s}\varphi \in \omega \text{ and } P^{\geq t}\psi \in \omega\} \subseteq [0, 1[$. So, again by P5,

$$\{s+t : P^{\geq s}\varphi \in \omega \text{ and } P^{\geq t}\psi \in \omega\} \subseteq \{r : P^{\geq r}(\varphi \vee \psi) \in \omega\}.$$

Therefore

$$\begin{aligned}\sup\{s+t : P^{\geq s}\varphi \in \omega \text{ and } P^{\geq t}\psi \in \omega\} &\leq \sup\{r : P^{\geq r}(\varphi \vee \psi) \in \omega\}\\ &= T(\omega)([\varphi] \cup [\psi]),\end{aligned}$$

as required.

Next, we prove that, for all $\omega \in \Omega$, $T(\omega)$ is continuous at the empty set. This is the point where we use axiom P8. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathbb{P}$. We show that

$$\lim_{n\to\infty} T(\omega)\left(\left[\bigwedge_{m\leq n} \varphi_m\right] \cap \left[\neg \bigwedge_{m\in\mathbb{N}} \varphi_m\right]\right) = 0.$$

Being an instance of axiom P8, formula

$$\bigwedge_{0<n\in\mathbb{N}} \bigvee_{k\in\mathbb{N}} P^{\leq 1/n}\left(\bigwedge_{m\leq k} \varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}} \varphi_m\right)$$

is in the deductive closure $\omega_\vdash$ of ω. It follows that, for all $0 < n \in \mathbb{N}$, the $\mathbb{P}$-formula

$$\psi_n : \qquad \bigvee_{k\in\mathbb{N}} P^{\leq 1/n}\left(\bigwedge_{m\leq k} \varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}} \varphi_m\right)$$

is in $\omega_\vdash$.

If for some $0 < n$ and all k, the formula $P^{\leq 1/n}(\bigwedge_{m\leq k} \varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}} \varphi_m)$ is not in ω, then, for all k, the formula $\neg P^{\leq 1/n}(\bigwedge_{m\leq k} \varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}} \varphi_m)$ is in ω.

Hence

$$\eta_n : \qquad \bigwedge_{k\in\mathbb{N}} \neg P^{\leq 1/n}\left(\bigwedge_{m\leq k} \varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}} \varphi_m\right)$$

is in $\omega_\vdash$. Since ψ_n is provably equivalent to the negation of η_n, the consistency of $\omega_\vdash$ would be contradicted.

Therefore, for every $0 < n$ there exists k_n such that formula

$$P^{\leq 1/n}\left(\bigwedge_{m\leq k_n}\varphi_m \wedge \neg \bigwedge_{m\in\mathbb{N}}\varphi_m\right)$$

is in ω. It follows that

$$T(\omega)\left(\left[\bigwedge_{m\leq k_n}\varphi_m\right]\cap\left[\neg\bigwedge_{m\in\mathbb{N}}\varphi_m\right]\right) = T(\omega)\left[\bigwedge_{m\leq k_n}\varphi_m \wedge \neg\bigwedge_{m\in\mathbb{N}}\varphi_m\right] \leq 1/n.$$

By N rule and by P6, we immediately get that, for all $0 < n$ and $k_n < k$,

$$T(\omega)\left(\left[\bigwedge_{m\leq k}\varphi_m\right]\cap\left[\neg\bigwedge_{m\in\mathbb{N}}\varphi_m\right]\right) \leq 1/n.$$

Hence the conclusion.

Finally, we let

$$\begin{aligned} v : V \cup \{\bot\} &\longrightarrow \mathcal{F} \\ q &\longmapsto [q] \end{aligned}$$

We summarise the results above in the following:

Proposition 9 *For every $\omega \in \Omega$ the triple $(\omega, \mathcal{F}, T(\omega))$ is a probability space. Furthermore, $(\Omega, \mathcal{F}, T, v)$ forms a type structure.*

Proof For every $\omega \in \Omega$ the measure $T(\omega)$ is finite, finitely additive and continuous at the empty set. As we already remarked just after introducing the axioms, it follows that $T(\omega)$ is σ-additive.

As for the second part, we have shown above that map T is measurable and map v satisfies the required properties. □

We call *canonical structure* the type structure of Proposition 9.

Proposition 10 *With reference to the canonical structure, it holds that, for every $\mathbb{F}$-formula ψ,*

$$\psi^\Omega = [\psi].$$

Proof By induction on formulas. The cases relative to $\mathbb{P}$-formulas are straightforward, possibly with the exception when ψ of the form $P^{\geq r}\varphi$. We deal with the latter case in some detail. Let us assume that the statement of the theorem holds for $\varphi \in \mathbb{P}$. Then, for every $\omega \in \Omega$,

$$\begin{aligned} \Omega, \omega \models P^{\geq r}\varphi \Leftrightarrow \quad & T(\omega)(\varphi^\Omega) \geq r \quad \Leftrightarrow T(\omega)[\varphi] \geq r \Leftrightarrow \\ \Leftrightarrow \sup\{s : P^{\geq s}\varphi \in \omega\} \geq r \Leftrightarrow \quad & P^{\geq r}\varphi \in \omega, \end{aligned}$$

where the left-to-right implication in the bottom right equivalence holds by axiom (P7). Then the statement holds for ψ.

Cases relative to formulas in $\mathbb{F} \setminus \mathbb{P}$ are trivial. □

Notice that, throughout the paper, we use axiom P7 in the previous proof only.

Corollary 11 *Let Γ be a consistent set of $\mathbb{F}$-formulas. Then Γ is satisfiable in the canonical structure.*

Proof By contraposition. Let us assume that Γ is unsatisfiable in the canonical structure. Then, for each $\omega \in \Omega$ there exists $\gamma_\omega \in \Gamma$ (depending on ω) such that $\Omega, \omega \models \neg\gamma_\omega$.

The following argument applies to each $\omega \in \Omega$: by applying Proposition 10 to $\neg\gamma_\omega$ we get $\neg\gamma_\omega \in \omega_\vdash$. Hence $\omega_\vdash \cup \{\gamma_\omega\}$ is inconsistent. Since $\omega_\vdash$ is the closure under provability of ω, it follows that $\omega \cup \{\gamma_\omega\}$ is inconsistent, hence so is the set $\{\bigwedge \omega, \gamma_\omega\}$. The latter is equivalent to $\gamma_\omega \vdash \neg(\bigwedge \omega)$.

Therefore we get $\{\gamma_\omega : \omega \in \Omega\} \vdash \bigwedge_{\omega\in\Omega} \neg(\bigwedge \omega)$. A fortiori, $\Gamma \vdash \bigwedge_{\omega\in\Omega} \neg(\bigwedge \omega)$ and , by the infinitary de Morgan laws, $\Gamma \vdash \neg \bigvee_{\omega\in\Omega}(\bigwedge \omega)$. Finally, it follows from Proposition 8(1) that Γ is inconsistent. □

Corollary 12 *Every consistent set of $\mathbb{F}$-formulas extends to some maximal consistent set.*

Proof Let Γ be consistent. By Corollary 11 there exists $\omega \in \Omega$ such that $\Omega, \omega \models \Gamma$. The set $\{\varphi \in \mathbb{F} : \Omega, \omega \models \varphi\}$ is maximal consistent and extends Γ. □

Notice that in Corollaries 11 and 12 there is no limitation on the cardinality of the set of $\mathbb{F}$-formulas. The same holds for the strong completeness theorem below. Differently from [13, Theorem 1], the proof of strong completeness is straightforward.

Theorem 13 *Let $\Gamma \cup \{\psi\}$ be a set of $\mathbb{F}$-formulas. Then*

$$\Gamma \models \psi \;\Rightarrow\; \Gamma \vdash \psi.$$

Proof Suppose $\Gamma \nvdash \psi$. Then $\Gamma \cup \{\neg\psi\}$ is consistent, hence satisfiable by Corollary 11. Therefore $\Gamma \not\models \psi$. □

Eventually, we summarize the technical reasons for choosing $\nu = (\beth_2)^+$. As remarked in Sect. 2, so doing we limit the cardinality of $\mathbb{P}$-formulas to $\beth_1$. It follows that the type structure Ω defined at the very beginning of this section has cardinality $\leq \beth_2$. See Remark 5. In the latter remark, we also explain why we need to go up to $(\beth_2)^+$. Furthermore, in the background, there is another reason for dealing a regular cardinal (as ν is): regularity of ν is a necessary assumption in the proof of strong completeness of logic $\mathfrak{P}(L_\nu)$ which is mentioned at the beginning of Sect. 2). Such completeness result is needed in the proof of key Proposition 8.

Remark 14 The following axiom schemas axiomatize the phenomenon of agent introspection that Harsanyi type spaces want capture (in this regard, see [13, Definition 8]):

I1 $\quad P^{\geq r}\varphi \to P^{\geq 1}P^{\geq r}\varphi, \qquad$ for $\varphi \in \mathbb{P}$
I2 $\quad \neg P^{\geq r}\varphi \to P^{\geq 1}(\neg P^{\geq r})\varphi, \qquad$ for $\varphi \in \mathbb{P}$

Clearly, expanding our set of axioms amounts to restricting the class of type structures to those satisfying the additional axioms.

Quite interestingly, if include I1 and I2 on our list of axioms, the above defined canonical structure validates those two schemas. We provide the details for I1, the same argument applies to I2. To begin with, we notice that, by Proposition 10 and by definition of $[\psi]$, for each formulas ψ, it holds that

$$(\sharp) \qquad \{\nu \in \Omega : \ \Omega, \nu \models \psi\} = \{\nu \in \Omega : \ \psi \in \nu_{\vdash}\}.$$

Let $\omega \in \Omega$ and let us assume that $\Omega, \omega \models P^{\geq r}\varphi$. It follows from $(\sharp)$ that $P^{\geq r}\varphi \in \omega_{\vdash}$. Being I1 an axiom, it belongs to $\omega_{\vdash}$ as well. Hence $\omega_{\vdash} \vdash P^{\geq 1}P^{\geq r}\varphi$ and, by closure under provability of $\omega_{\vdash}$, $(P^{\geq 1}P^{\geq r}\varphi) \in \omega_{\vdash}$. From $(\sharp)$, we finally get $\Omega, \omega \models P^{\geq 1}P^{\geq r}\varphi$.

Being $\omega \in \Omega$ arbitrarily chosen, we conclude that the canonical structure validates schema I1.

Hence, even if it may not satisfy the characteristic property of a Harsanyi type space (see [13, Definition 8]), the above defined canonical structure validates the instances of introspection that are formalized by schemas I1 and I2.

5 A short categorial detour

In the following we recall the notion of morphism in the category of type structures given in [13]. Such definition is a natural extension of the notion of morphism in the category, to be denoted by $\mathcal{P}$, of probability spaces and measure preserving maps. More precisely, a morphism in $\mathcal{P}$ between the probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ is a measurable map $f : \Omega_1 \to \Omega_2$ with the property that

$$(*) \qquad \text{for all } E \in \mathcal{F}_2, \quad P_1(f^{-1}(E)) = P_2(E).$$

The following is taken from [13].

Definition 15 Let $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ and $(\Omega_2, \mathcal{F}_2, T_2, v_2)$ be type structures. A map $f : \Omega_1 \to \Omega_2$ is a morphism if

(1) f is a measurable function;
(2) for all $\omega_1 \in \Omega_1$ and all $q \in V$,

$$\omega_1 \in v_1(q) \Leftrightarrow f(\omega_1) \in v_2(q);$$

(3) for all $\omega_1 \in \Omega_1$ and all $E \in \mathcal{F}_2$,

$$T_2(f(\omega_1))(E) = T_1(\omega_1)(f^{-1}(E)).$$

Notice that all conditions imposed on f in Definition 15, with the exception of (2), refer to the inverse image map $f^{-1}: P(\Omega_2) \to P(\Omega_1)$.

Let us denote by id_{Ω_1} the identity morphism on a type structure $(\Omega_1, \mathcal{F}_1, T_1, v_1)$.

Let $(\Omega_1, \mathcal{F}_1, T_1, v_1)$, $(\Omega_2, \mathcal{F}_2, T_2, v_2)$ and f be as in Definition 15. By a straightforward induction on formulas one shows that

$$(**) \qquad \text{for all } \varphi \in \mathbb{F} \text{ and all } \omega_1 \in \Omega_1, \quad \Omega_1, \omega_1 \models \varphi \;\Leftrightarrow\; \Omega_2, f(\omega_1) \models \varphi.$$

Set-theoretic composition of morphisms yields a morphism and the identity map is a morphism. Therefore type structures and morphisms form a category that we denote by TS.

A morphism f from the type structure $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ to the type structure $(\Omega_2, \mathcal{F}_2, T_2, v_2)$ is an isomorphism if there exists a morphism $g: \Omega_2 \to \Omega_1$ such that $f \circ g = \mathrm{id}_{\Omega_2}$ and $g \circ f = \mathrm{id}_{\Omega_1}$. In particular, an isomorphism of type structures is a bijection between the underlying spaces of events, but this condition alone does not suffice in general.

Next we show that the canonical structure is a terminal object in TS. In our opinion, our proof is simpler than Meier's in [13].

Theorem 16 *The canonical structure $(\Omega, \mathcal{F}, T, v)$ defined before Proposition 9 is a terminal object in the category TS.*

Proof Let $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ be an arbitrary type structure. We claim that the map

$$\begin{aligned} f: \Omega_1 &\longrightarrow \qquad\qquad \Omega \\ \omega_1 &\longmapsto \{\varphi \in \mathbb{P} : \Omega_1, \omega_1 \models \varphi\} \end{aligned}$$

is a morphism. Let $\varphi \in \mathbb{P}$. From Proposition 10 we get

$$\text{for all } \omega_1 \in \Omega_1, \quad \Omega, f(\omega_1) \models \varphi \Leftrightarrow \varphi \in f(\omega_1) \Leftrightarrow \Omega_1, \omega_1 \models \varphi,$$

namely

$$(\circ) \qquad f^{-1}(\varphi^{\Omega}) = \varphi^{\Omega_1}.$$

In particular, for all $\omega_1 \in \Omega_1$ and all $q \in V$,

$$\omega_1 \in v_1(q) \Leftrightarrow \Omega_1, \omega_1 \models q \Leftrightarrow \Omega, f(\omega_1) \models q \Leftrightarrow f(\omega_1) \in v(q).$$

Moreover, $f^{-1}([\varphi]) = f^{-1}(\varphi^{\Omega}) = \varphi^{\Omega_1}$ and the latter is in $\mathcal{F}_1$ (see Remark 2). Hence f is measurable.

It remains to verify that (3) of Definition 15 is satisfied by f. Let $\varphi \in \mathbb{P}$ and $\omega_1 \in \Omega_1$. From $(\circ)$ above we get, for all $r \in \mathbb{Q} \cap [0, 1]$,

$$\begin{aligned} T(f(\omega_1))([\varphi]) \geq r \Leftrightarrow \Omega, f(\omega_1) \models P^{\geq r}\varphi \Leftrightarrow \Omega_1, \omega_1 \models P^{\geq r}\varphi \Leftrightarrow \\ \Leftrightarrow T_1(\omega_1)(\varphi^{\Omega_1}) \geq r \Leftrightarrow T_1(\omega_1)(f^{-1}(\varphi^{\Omega})) \geq r \end{aligned}$$

$$\Leftrightarrow T_1(\omega_1)(f^{-1}([\varphi])) \geq r.$$

Hence $T(f(\omega_1))([\varphi]) = T_1(\omega_1)(f^{-1}([\varphi]))$, as required.

Furthermore, let $g : (\Omega_1, \mathcal{F}_1, T_1, v_1) \to (\Omega, \mathcal{F}, T, v)$ be a morphism and let $\omega_1 \in \Omega_1$. From $(**)$ above we get

$$\text{for all } \varphi \in \mathbb{F}, \quad \Omega, f(\omega_1) \models \varphi \Leftrightarrow \Omega, g(\omega_1) \models \varphi.$$

From Proposition 10 it follows that

$$f(\omega_1)_{\vdash} = g(\omega_1)_{\vdash}$$

and so $f(\omega_1) = g(\omega_1)$, by Remark 6.

Therefore f is the only morphism from $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ to $(\Omega, \mathcal{F}, T, v)$.

We conclude that $(\Omega, \mathcal{F}, T, v)$ is a terminal object in the category TS. □

Notice that morphism f defined in the proof of Theorem 16 identifies any two points in Ω_1 that satisfy the same formulas. This admittedly strong property reflects the economic and game-theoretic intuition behind type indistinguishability.

6 A model-theoretic perspective

We begin with some considerations which may lead to an investigation of type structures from an algebraic perspective.

We notice that (3) of Definition 15 is the counterpart of $(*)$ in the category $\mathcal{P}$ of probability spaces (for the latter, see Sect. 5). It is well-known that $\mathcal{P}$ does not have products in categorical sense, simply because, in general, projections do not preserve probability. Also, category $\mathcal{P}$ has a very restrictive notion of subobject: a natural candidate for a subobject of a probability space turns out to be *isomorphic mod 0* to the original space (see, for instance, [5] for the latter notion). We explain what we mean: let $(\Omega, \mathcal{F}, P)$ be a probability space and let $B \in \mathcal{F}$. We let

$$\mathcal{F}_B = \{A \cap B : A \in \mathcal{F}\}.$$

Assuming $P(B) \neq 0$, we let

$$P_B(A \cap B) = P(A/B),$$

where $P(A/B)$ is the conditional probability of A given B. Probability space $(B, \mathcal{F}_B, P_B)$ is a natural candidate for being a subobject of $(X, \mathcal{F}, P)$. This requires the inclusion map $\iota : B \to X$ being a (mono)morphism. Measurability of ι is straightforward. Condition $(*)$ above becomes

$$P_B(A \cap B) = P(A), \quad \text{for all } A \in \mathcal{F}.$$

In particular, we get $P(B) = 1$ and so the spaces $(B, \mathcal{F}_B, P_B)$ and $(\Omega, \mathcal{F}, P)$ are isomorphic mod 0. As pointed out by [5], in ergodic theory the latter is the standard notion of isomorphism between probability spaces. Thus, in $\mathcal{P}$, the notion of subobject is extremely restrictive.

For this reason, one may want to drop condition $(*)$ in the definition of morphism, but then two probability spaces $(\Omega, \mathcal{F}, P_1)$ and $(\Omega, \mathcal{F}, P_2)$, with $P_1 \neq P_2$, would be indistinguishable.

Similar considerations apply to the category TS.

From property $(**)$ above we see that a morphism of type structures behaves "locally" like an isomorphism, even if "globally" it is not injective, in general. A natural question is whether there is a weaker notion of morphism with related notions of embedding, type substructure, etc…Even if existence of a universal type structure possibly fails, the resulting framework might be of interest. In this regard, we make a parallel with the category of sets where each singleton is a terminal object, but such property is not as relevant as other set-theoretic properties.

In light of the previous considerations, we suggest shifting the focus from categorical to algebraic properties of type structures. We first introduce a natural notion of type substructure.

6.1 Substructures

Let $(\Omega, \mathcal{F}, T, v)$ be a type structure and let $\Omega_1 \in \mathcal{F}$ be such that, for all $\omega_1, \omega_2 \in \Omega_1$, $T(\omega_1)(\Omega_1) = T(\omega_2)(\Omega_2) \neq 0$. Let s be the common value. We let $\mathcal{F}_1$ be the σ-algebra $\{A \cap \Omega_1 : A \in \mathcal{F}\}$ on Ω_1 and, for $\omega \in \Omega_1$, we define $T_1(\omega) \in \Delta(\Omega_1, \mathcal{F}_1)$ as follows:

$$T_1(\omega)(A \cap \Omega_1) = s^{-1}T(\omega)(A \cap \Omega_1), \quad A \in \mathcal{F}.$$

For all $q \in V$, we let $v_1(q) = v(q) \cap \Omega_1$ and $v_1(\bot) = \emptyset$.

We claim that $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ is a type structure. It needs only to be verified that T_1 is a measurable map. Let $r \in [0, 1]$ and $A \in \mathcal{F}_1$. We have:

$$T_1^{-1}(\{\mu \in \Delta(\Omega_1, \mathcal{F}_1) : \mu(A) \geq r\}) = \Omega_1 \cap \{\omega \in \Omega : T(\omega)(A) \geq sr\} \in \mathcal{F}_1.$$

Hence T_1 is measurable.

6.2 Products

Despite category TS does not have products in categorical sense, we can also form the product of an arbitrary family of type structures. For simplicity, we deal with the product of two structures $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ and $(\Omega_2, \mathcal{F}_2, T_2, v_2)$. The product of finitely many type structures can be defined by induction on natural numbers. Later we will deal with infinite products.

We denote the product as follows: $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, T_1 \times T_2, v_1 \times v_2)$, where

(1) $\Omega_1 \times \Omega_2$ is the cartesian product of the two sets;

(2) $\mathcal{F}_1 \times \mathcal{F}_2$ is the product σ-algebra, namely the σ-algebra generated by the rectangles $B \times C$, with $B \in \mathcal{F}_1$ and $C \in \mathcal{F}_2$;
(3) for all $\omega_1 \in \Omega_1$ and all $\omega_2 \in \Omega_2$, $(T_1 \times T_2)(\omega_1 \times \omega_2)$ is the product probability of $T_1(\omega_1)$ and $T_2(\omega_2)$;
(4) for all $q \in V$, $(v_1 \times v_2)(q)$ is the cartesian product of $v_1(q)$ and $v_2(q)$ and $(v_1 \times v_2)(\perp) = \emptyset$.

We only need to prove that $T_1 \times T_2$ is a measurable map. It suffices to verify that, for all $B \in \mathcal{F}_1$, $C \in \mathcal{F}_2$ and $r \in \mathbb{Q} \cap]0, 1]$, $(T_1 \times T_2)^{-1}(\{\mu \in P(\Omega_1 \times \Omega_2) : \mu(B \times C) > r\}) \in \mathcal{F}_1 \times \mathcal{F}_2$. Case $r = 0$ follows by taking a countable union: $r > 0 \Leftrightarrow (r > 1/n$, for some $n > 0)$.

For B, C, r as above, we have:

$$\begin{aligned}
&(T_1 \times T_2)^{-1}(\{\mu \in P(\Omega_1 \times \Omega_2) : \mu(B \times C) > r\}) \\
&\quad = \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : T_1(\omega_1)(B) \cdot T_2(\omega_2)(C) > r\} \\
&\quad = \bigcup_{\{s,t \in \mathbb{Q} \cap [0,1] : st > r\}} (\{\omega_1 \in \Omega_1 : T_1(\omega_1)(B) \geq s\} \\
&\qquad \times \{\omega_2 \in \Omega_2 : T_2(\omega_2)(C) \geq t\}),
\end{aligned}$$

and the latter countable union is in $\mathcal{F}_1 \times \mathcal{F}_2$.

As for the product $(\Omega, \mathcal{F}, T, v)$ of a countable family $\{(\Omega_i, \mathcal{F}_i, T_i, v_i)\}_{i \in \mathbb{N}}$ of type structures, we proceed as in the construction of a countably infinite product of probability spaces. See, for instance, [6, §38]. In particular, for all $(\omega_i) \in \Omega$ we define $T((\omega_i))$ as the unique probability measure S such that, for each generator $\prod_{i \in \mathbb{N}} A_i$ of $\mathcal{F}$,

$$S((\omega_i))\left(\prod_{i \in \mathbb{N}} A_i\right) = \prod_{i \in \mathbb{N}} T_i(\omega_i)(A_i).$$

Measurability of T follows as in the finite product case after recalling that, for all generators $\prod_{i \in \mathbb{N}} A_i$ of $\mathcal{F}$, $A_i \neq \Omega_i$ for finitely many i's only.

Regarding the generalization to uncountable products, see Exercise (2) at the end of [6, §38].

6.3 Ultraproducts

We develop a notion of ultraproduct of type structures. For the pertinent set-theoretic and topological notions we refer the reader to [1].

The notation and conventions previously introduced are in force.

Let U be an ultrafilter on a set I.

Let $\{(\Omega_i, \mathcal{F}_i, T_i, v_i)\}_{i \in I}$ be a family of type structures and let Ω be the set-theoretic ultraproduct of the Ω_i's with respect to U. We write ω_U for the equivalence class with respect to U of the sequence $\omega = (\omega_i)_{i \in I} \in \prod_{i \in \mathbb{N}} \Omega_i$. Let $(E_i)_{i \in I}$ be a sequence such that, for all $i \in I$, $E_i \subseteq \Omega_i$. We let

$$(E_i)_U = \{\omega_U \in \Omega_U : \{i \in I : \omega_i \in E_i\} \in U\}.$$

Notice that $(E_i)_U$ is just a convenient notation for the set-theoretic ultraproduct $(\prod_{i \in I} E_i)/U$.

It is easy to verify that the family

$$(\circ) \qquad \mathcal{F}_\bullet = \{(E_i)_U : \text{for all } i \in \mathbb{N},\ E_i \in \mathcal{F}_i\}$$

is a boolean algebra on Ω. We denote by $\mathcal{F}$ be the σ-algebra generated by $\mathcal{F}_\bullet$ and by $\Delta(\Omega, \mathcal{F}_\bullet)$ the family of pre-measures on $(\Omega, \mathcal{F}_\bullet)$.

We define a map $T' : \prod_i \Omega_i \to \Delta(\Omega, \mathcal{F}_\bullet)$ as follows: for $(E_i)_U \in \mathcal{F}_\bullet$ we let

$$T'(\omega)((E_i)_U) = \lim_U T_i(\omega_i)(E_i),$$

where $\lim_U$ stands for the limit with respect to the ultrafilter U of the bounded sequence $(T_i(\omega_i)(E_i))_{i \in I}$ (see [1]). By [1, Proposition 1.3, Ch.2], for all $\omega \in \prod \Omega_i$, map $T'(\omega)$ is well-defined and σ-additive, hence it can be uniquely extended to a probability measure $T''(\omega)$ on $\mathcal{F}$.

Furthermore, if $\omega, \psi \in \prod_{i \in \mathbb{N}} \Omega_i$ are such that $\omega_U = \psi_U$, by the filter properties we get $T''(\omega) = T''(\psi)$. Therefore T'' induces a well-defined map

$$\begin{aligned} T :\ \Omega &\to \Delta(\Omega, \mathcal{F}) \\ \omega_U &\mapsto T''(\omega) \end{aligned}$$

We turn $\Delta(\Omega, \mathcal{F})$ into a measurable space as described before Definition 1 and we prove the following:

Proposition 17 *The map T defined above is measurable.*

Proof It suffices to show that, for all $E = (E_i)_U \in \mathcal{F}_\bullet$ and all $r \in [0, 1] \cap \mathbb{Q}$, the set

$$T^{-1}(\{\mu \in \Delta(\Omega, \mathcal{F}) : \mu(E) \geq r\}) = \left\{\omega_U \in \Omega : \lim_U T_i(\omega_i)(E_i) \geq r\right\}$$

is in $\mathcal{F}$. First of all we notice that the following are equivalent for $\omega_U = (\omega_i)_U$:

(1) $\lim_U T_i(\omega_i)(E_i) \geq r$;
(2) for all $0 < k \in \mathbb{N}$, $\{i \in \mathbb{N} : T_i(\omega_i)(E_i) \geq r - 1/k\} \in U$;
(3) for all $0 < k \in \mathbb{N}$, $\omega_U \in (F_{k,i})_U$, where

$$F_{k,i} = \{u \in \Omega_i : T_i(u)(E_i) \geq r - 1/k\};$$

(4) $\omega_U \in \bigcap_{0<k \in \mathbb{N}} (F_{k,i})_U$.

Hence

$$\{\omega_U \in \Omega : \lim_U T_i(\omega_i)(E_i) \geq r\} = \bigcap_{0<k\in\mathbb{N}} (F_{k,i})_U.$$

For all $0 < k \in \mathbb{N}$ and all $i \in I$, by measurability of T_i we have $(F_{k,i})_U \in \mathcal{F}_\bullet$. Therefore

$$T^{-1}(\{\mu \in \Delta(\Omega, \mathcal{F}) : \mu(E) \geq r\}) = \bigcap_{0<k\in\mathbb{N}} (F_{k,i})_U,$$

and the latter intersection is in $\mathcal{F}$. □

Finally, we define

$$\begin{aligned} v : V &\to \quad \mathcal{F} \\ q &\mapsto (v_i(q))_U \end{aligned}$$

Summing up: the 4-tuple $(\Omega, \mathcal{F}, T, v)$ defined above is a type structure. We call it the ultraproduct of the family $\{(\Omega_i, \mathcal{F}_i, T_i, v_i)\}_{i\in I}$ with respect to the ultrafilter U.

Next we investigate whether a Łoś-like theorem holds for the ultraproduct of type structures. We recall that an ultrafiter U on a set I is ω_1-complete if any countable intersection of elements from U belongs to U. Principal ultrafilters are ω_1-complete, but the existence of a nonprincipal ω_1-complete ultrafilter is equivalent to the existence of a measurable cardinal (see [4, Theorem 1.11, Ch.6]). The latter is unprovable in ZFC. Actually, it cannot even be proved that the existence of a measurable cardinal is consistent with ZFC. See [10].

In classical first-order logic, ultraproducts with respect to principal ultrafilters are trivial, in a sense made precise by [4, Corollary 2.3, Ch.6]. An analogous result holds in the framework of type structures, as we show next.

Let Ω be the set theoretic ultraproduct of the family of sets $\{\Omega_i\}_{i\in I}$ with respect to the principal ultrafilter U generated by $\{k\}$, for some $k \in I$. An easy verification shows that the map

$$h : \Omega_k \to \Omega$$

defined by $h(\eta) = (\omega_i)_U$, where $\omega_k = \eta$ and $\omega_i \in \Omega_i$, for all $i \neq k$, is a well-defined bijection. In particular, $h(\eta)$ does not depend on the choice of the coordinates $\omega_i \in \Omega_i$, for $i \neq k$.

Proposition 18 *Let $(\Omega, \mathcal{F}, T, v)$ be the ultraproduct of the family $\{(\Omega_i, \mathcal{F}_i, T_i, v_i)\}_{i\in I}$ of type structures with respect to the principal ultrafilter U on I generated by $\{k\}$, for some $k \in I$. Let $h : \Omega_k \to \Omega$ be the bijection defined above. The following hold:*

(1) for all $(E_i)_U, (X_i)_U$ in $\prod_{i\in I} \mathcal{F}_i$,

$$(E_i)_U = (X_i)_U \;\Leftrightarrow\; E_k = X_k;$$

(2) the boolean algebra $\mathcal{F}_\bullet$ defined in ($\circ$) *is a σ-algebra, hence $\mathcal{F}_\bullet = \mathcal{F}$;*
(3) map h is bijective and induces a σ-complete isomorphism of σ-algebras $h_\times : \mathcal{F}_k \to \mathcal{F}_\bullet$;
(4) for all $\omega_k \in \Omega_k$ and all $E_k \in \mathcal{F}_k$, $T_k(\omega_k)(E_k) = T(h(\omega_k))(h(E_k))$.
(5) map h is an isomorphism of type structures.

Proof

(1) Straightforward.
(2) Let $\{(E_i^n)_U : n \in \mathbb{N}\} \subseteq \mathcal{F}_\bullet$. Then $\bigcap_{n\in\mathbb{N}}(E_i^n)_U = (X_i)_U$, where $X_k = \bigcap_{n\in\mathbb{N}} E_k^n$ and, for $i \neq k$, $X_i = \Omega_i$ (equivalently, by (1), any $X_i \in \mathcal{F}_i$ would do).
(3) For $E \in \mathcal{F}_k$, we let $h_\times(E) = (E_i)_U$, where $E_k = E$ and, for $i \neq k$, $E_i = \Omega_i$. Notice that $h_\times(E) = \{h(\eta) : \eta \in E\}$. It follows from the proof of (2) that $h_\times$ respects countable intersections.
The rest of the proof that $h_\times$ is a morphism of σ-algebras is straightforward. Moreover, map $g : \mathcal{F}_\bullet \to \mathcal{F}_k$ defined by $g((E_i)_U) = E_k$ is a morphism, which is the inverse of h.
(4) Since U is generated by $\{k\}$, for all bounded sequences (r_i) of reals it holds that $\lim_U r_i = r_k$. The conclusion follows by definition of T.
(5) Conditions (1) and (2) in Definition 15 are easily verified. Condition (3) in the same definition follows from (4). Finally, the set-theoretic inverse h^{-1} of h is also the inverse of morphism h. $\square$

Thus, in analogy with [4, Corollary 2.3, Ch.6], we see that, under the assumptions of Proposition 18, the ultraproduct $(\Omega, \mathcal{F}, T, v)$ is an isomorphic copy of the fiber $(\Omega_k, \mathcal{F}_k, T_k, v_k)$.

Finally, we prove the following Łoś-like theorem:

Theorem 19 *Let $\{(\Omega_i, \mathcal{F}_i, T_i, v_i)\}_{i\in I}$ be a family of type structures and let U be an ω_1-complete ultrafilter on I. Let $(\Omega, \mathcal{F}, T, v)$ be the corresponding ultraproduct.*
Then, for all $\mathbb{F}$-formulas φ and all $\omega \in \prod_{i\in I}\Omega_i$,

$$\Omega, \omega_U \models \varphi \;\Leftrightarrow\; \{i \in I : \Omega_i, \omega_i \models \varphi\} \in U.$$

Proof By induction. We begin with $\mathbb{P}$-formulas. If φ is $\perp$, the equivalence is trivial. If φ is the variable q, we have

$$\Omega, \omega_U \models q \Leftrightarrow \omega_U \in v(q) \Leftrightarrow \omega_U \in (v_i(q))_U \Leftrightarrow \{i \in I : \omega_i \in v_i(q)\} \in U \Leftrightarrow \\ \Leftrightarrow \{i \in I : \Omega_i, \omega_i \models q\} \in U\}.$$

Cases relative to negation and to countable conjunctions are straightforward (countably infinite conjunctions are taken care by the assumption of ω_1-completeness). If φ is of the form $P^{\geq r}\eta$, for some $\eta \in \mathbb{P}$, the inductive assumption on η is equivalent to the following:

$$\{\psi_U \in \Omega : \Omega, \psi_U \models \eta\} = (\{\psi \in \Omega_i : \Omega_i, \psi \models \eta\})_U.$$

The following are equivalent:

(1) $\Omega, \omega_U \models P^{\geq r}\eta$
(2) $T(\omega_U)((\{\psi \in \Omega_i : \Omega_i, \psi \models \eta\})_U) \geq r$
(3) $\lim_U T(\omega_i)(\{\psi \in \Omega_i : \Omega_i, \psi \models \eta) \geq r$
(4) for all $0 < k \in \mathbb{N}$, $\{i \in I : T(\omega_i)(\{\psi \in \Omega_i : \Omega_i, \psi \models \eta\}) \geq r - 1/k\} \in U$
(5) $(\bigcap_{0<k\in\mathbb{N}}\{i \in I : T(\omega_i)(\{\psi \in \Omega_i : \Omega_i, \psi \models \eta\} \geq r - 1/k\}) \in U$
(6) $\{i \in I : T(\omega_i)(\{\psi \in \Omega_i : \Omega_i, \psi \models \eta\}) \geq r\} \in U$
(7) $\{i \in I : \Omega_i, \omega_i \models P^{\geq r}\eta\} \in U$,

where the equivalence of (4) and (5) holds by ω_1-completeness of U.

We are left with the formulas in $\mathbb{F}$. The only nontrivial case is that of an infinite conjunction of formulas (recall that each conjunction involves less than ν-many formulas). We just notice that, if U is principal, then it is κ-complete for any cardinal κ, namely an arbitrary intersection of elements from U lives in U. If U is nonprincipal, then $|I|$ is a measurable cardinal. Clearly, $|I| > \nu$ and so the intersection over an arbitrary family of cardinality $< \nu$ of elements from U lives in U. □

Admittedly, the assumption of existence of a nonprincipal ω_1-complete ultrafilter in Theorem 19 is very strong. It might be interesting to investigate whether weaker set-theoretic assumptions do ensure validity of that theorem.

6.4 Upward Löwenheim-Skolem theorem

We prove the following:

Proposition 20 *Let $(\Omega, \mathcal{F}, T, v)$ be a type structure and let $\kappa > |\Omega|$ be a cardinal. Then there exists a probability structure $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ such that:*

(1) $\Omega \subset \Omega_1$ and $\mathcal{F} \subset \mathcal{F}_1$;
(2) $|\Omega_1| = \kappa$;
(3) for all $\omega \in \Omega$ and all $\varphi \in \mathbb{F}$,

$$\Omega, \omega \models \varphi \;\Leftrightarrow\; \Omega_1, \omega \models \varphi.$$

Proof We pick a set X of cardinality κ disjoint from Ω and we let $\Omega_1 = \Omega \cup X$; $v_1 = v$. Let $\mathcal{F}_1$ be the σ-algebra generated by $\mathcal{F} \cup \{X\}$. We fix $\bar{\omega} \in \Omega$ and we define $T_1 : \Omega_1 \to \Delta(\Omega_1, \mathcal{F}_1)$ as follows:

– for all $\omega \in \Omega$ and all $E \in \mathcal{F}_1$ we let $T_1(\omega)(E) = T(\omega)(E \cap \Omega)$;
– for all $\omega \in X$, we let $T_1(\omega) = T_1(\bar{\omega})$.

It is straightforward to verify that each $T_1(\omega)$ is a probability measure on $(\Omega_1, \mathcal{F}_1)$. To complete the proof that $(\Omega_1, \mathcal{F}_1, T_1, v_1)$ is a type structure it remains to prove that T_1 is a measurable map. It suffices to show that, for all $E \in \mathcal{F} \cup \{X\}$ and all $r \in [0, 1] \cap \mathbb{Q}$, the set

$$A = T_1^{-1}(\{\mu \in \Delta(\Omega_1, \mathcal{F}_1) : \mu(E) \geq r\})$$

is in $\mathcal{F}_1$.

If $E \in \mathcal{F}$, then

$$\begin{aligned} A &= \{\omega \in \Omega_1 : T_1(\omega)(E) \geq r\} \\ &= \{\omega \in \Omega : T(\omega)(E) \geq r\} \cup \{\omega \in X : T(\bar{\omega})(E) \geq r\}. \end{aligned}$$

Therefore $\{\omega \in \Omega : T(\omega)(E) \geq r\} \in \mathcal{F}$ and $\{\omega \in X : T(\bar{\omega})(E) \geq r\}$ is either $\emptyset$ or X. In any case $A \in \mathcal{F}_1$.

If $E = X$ then $A = \Omega_1$ for $r = 0$ and $A = \emptyset$ for $r > 0$.

It remains to prove that (3) holds. We proceed by induction on φ. Since the other cases are trivial, we assume that φ is of the form $P^{\geq r}\psi$ and we work under the inductive assumption $\psi^{\Omega_1} \cap \Omega = \psi^{\Omega}$, for all $\omega \in \Omega$.

Let $\omega \in \Omega$. Then

$$\begin{aligned} \Omega_1, \omega \models \varphi \Leftrightarrow T_1(\omega)(\psi^{\Omega_1}) \geq r \Leftrightarrow T(\omega)(\psi^{\Omega_1} \cap \Omega) \geq r \Leftrightarrow T(\omega)(\psi^{\Omega}) \geq r \\ \Leftrightarrow \Omega, \omega \models \varphi. \end{aligned} \qquad \square$$

We leave as an open problem whether a Downward Löwenheim-Skolem theorem hold for type structures.

In the next two sections we provide partial results relative to the existence of inverse and direct limits of systems of type structures. We begin each section by dealing with limits of systems of probability spaces. Surprisingly for this author, their construction seems to require very specific assumptions.

6.5 Inverse limits

Inverse limits of systems of probability spaces are known to exist only under suitable topological assumptions. See, for instance, the Introduction in [17]. In the latter work, the author introduces a purely measure-theoretic condition, called ϵ-completeness, that suffices for existence and uniqueness of inverse limits when the index set is the set of natural numbers. We exploit ϵ-completeness to extend the inverse limit construction to inverse families of type structures.

We suitably extend the framework in [17]. For coherence, we stick to the notation used so far. Let

$$(\Omega_i, (f_{ij}))_{i \leq j, i, j \in \mathbb{N}}$$

be an inverse system of sets and maps. Namely, for all $i \leq j \leq k$ in $\mathbb{N}$, the following hold:

- $f_{ij} : \Omega_j \to \Omega_i$ is a function (note the order);
- $f_{ii} = \mathrm{id}_{\Omega_i}$;
- $f_{ij} \circ f_{jk} = f_{ik}$.

The inverse limit $(\Omega, (p_i : \Omega \to \Omega_i)_{i \in \mathbb{N}})$ of the system above is defined as follows:

$$\Omega = \left\{ \omega \in \prod_{i \in \mathbb{N}} \Omega_i : \omega_i = f_{ij}(\omega_j), \text{ for all } i \leq j \in \mathbb{N} \right\}$$

and the p_i's are the projection maps.

Next, we consider an inverse system $((\Omega_i, \mathcal{F}_i, T_i, v_i), (f_{ij}))_{i\leq j, i, j\in\mathbb{N}}$ of type structures. By this we mean that, for all $i \leq j \leq k$,

(1) $(\Omega_i, \mathcal{F}_i, T_i, v_i)$ is a type structure;
(2) f_{ij} is a measurable function;
(3) for all $\omega \in \Omega_j$, $T_i(f_{ij}(\omega)) = T_j(\omega) \circ f_{ij}^{-1}$, where f_{ij}^{-1} denotes the restriction to $\mathcal{F}_i$ of the pre-image function associated to f_{ij}.

(Compare with [17, Definition 2.2].)

Since we are mostly concerned with the measure-theoretic features of the inverse limit construction, we forget about the assignments of values to the propositional variables $v_i : V \cup \{\perp\} \to \mathcal{F}_i$, $i \in \mathbb{N}$. Thus we deal with the inverse system $((\Omega_i, \mathcal{F}_i, T_i), (f_{ij}))_{i\leq j, i, j\in\mathbb{N}}$. We call each $(\Omega_i, \mathcal{F}_i, T_i)$ a *type pre-structure*. We will see that the v_i's play no role in the construction below.

It can be shown that $\mathcal{F}_\bullet = \bigcup_{i\in\mathbb{N}} p_i^{-1}(\mathcal{F}_i)$ is a Boolean algebra. Indeed, if $A \in \mathcal{F}_i$, we have $\Omega \setminus p_i^{-1}(A) = p_i^{-1}(\Omega_i \setminus A) \in \mathcal{F}_\bullet$. Moreover, if $i \leq j$, $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$, we first notice that $p_i^{-1}(A) = p_j^{-1}(f_{ij}^{-1}(A))$. Hence $p_i^{-1}(A) \cap p_j^{-1}(B) = p_j^{-1}(f_{ij}^{-1}(A)) \cap B) \in \mathcal{F}_\bullet$.

We let $\mathcal{F}$ be the σ-algebra generated by $\mathcal{F}_\bullet$. Clearly, $\mathcal{F}$ is the coarsest σ-algebra on Ω for which all p_i's are measurable.

For all $i \in \mathbb{N}$ and all $\omega \in \Omega_i$, let $T_i(\omega)^* : P(\Omega_i) \to [0, 1]$ be the outer measure defined by

$$T_i(\omega)^*(A) = \inf\{T_i(\omega)(B) : A \subseteq B \in \mathcal{F}_i\}.$$

We give the following, inspired by [17, Definition 3.1]:

Definition 21 The inverse system $((\Omega_i, \mathcal{F}_i, T_i), (f_{ij}))_{i\leq j, i, j\in\mathbb{N}}$ of type pre-structures is ϵ-complete if, for all $\omega \in \Omega$, all $i \leq j$ and all $A \subseteq \Omega_i$,

$$T_j(p_j(\omega))^*(f_{ij}^{-1}(A)) = T_i(p_i(\omega))^*(A).$$

Hence, ϵ-completeness of $((\Omega_i, \mathcal{F}_i, T_i), (f_{ij}))_{i\leq j, i, j\in\mathbb{N}}$ is equivalent to the condition that, for all $\omega \in \Omega$, the inverse system of probability spaces

$$((\Omega_i, \mathcal{F}_i, T_i(p_i(\omega)), (f_{ij}))_{i\leq j, i, j\in\mathbb{N}},$$

is ϵ-complete according to [17, Definition 3.1].

From now on, we assume that $((\Omega_i, \mathcal{F}_i, T_i), (f_{ij}))_{i\leq j, i, j\in\mathbb{N}}$ is an ϵ-complete inverse system of type pre-structures.

From [17, Theorem 3.2] we get that, for all $\omega \in \Omega$, there exists a unique probability measure $T(\omega)$ on $(\Omega, \mathcal{F})$ such that, for all $i \in \mathbb{N}$,

$$T(\omega) \circ p_i^{-1} = T_i(p_i(\omega)).$$

See condition (ii) in [17, Definition 2.4].

We claim that map

$$\begin{aligned} T : \Omega &\to \Delta(\Omega, \mathcal{F}) \\ \omega &\mapsto T(\omega) \end{aligned}$$

is measurable. By [16, Lemma 2.3], we only prove that, for all $A \in \mathcal{F}_\bullet$ and all $r \in [0, 1]$, $T^{-1}(\{\mu \in \Delta(\Omega, \mathcal{F}) : \mu(A) \geq r\})$ is measurable. The latter set is equal to $\{\omega \in \Omega : T(\omega)(A) \geq r\}$. We recall the definition of $T(\omega)$ from [17]:

$$T(\omega)(A) = T_n(p_n(\omega))(\hat{A}),$$

for some $n \in \mathbb{N}$ and some $\hat{A} \in \mathcal{F}_n$ such that $p_n^{-1}(\hat{A}) = A$. (It follows from [17, Theorem 3.2] that $T(\omega)$ is well-defined.) We fix n and $\hat{A}$ as above. By measurability of T_n, we have:

$$\{\eta \in \Omega_n : T_n(\eta)(\hat{A}) \geq r\} = T_n^{-1}(\{\mu \in \Delta(\Omega_n, \mathcal{F}_n) : \mu(\hat{A}) \geq r\}) \in \mathcal{F}_n.$$

Hence

$$p_n^{-1}(\{\eta \in \Omega_n : T_n(\eta)(\hat{A}) \geq r\}) \in p_n^{-1}(\mathcal{F}_n) \subseteq \mathcal{F}.$$

Since $\{\omega \in \Omega : T(\omega)(A) \geq r\} = p_n^{-1}(\{\eta \in \Omega_n : T_n(\eta)(\hat{A}) \geq r\})$, we conclude that T is measurable.

We call $(\Omega, \mathcal{F}, T)$ the inverse limit of the ϵ-complete inverse system

$$((\Omega_i, \mathcal{F}_i, T_i), (f_{ij}))_{i \leq j, i, j \in \mathbb{N}}$$

of type pre-structures.

Eventually, we may also want to define an assignment of values v to the propositional variables so to get a type structure. A natural choice is to let, for $q \in V$,

$$v(q) = \bigcap_{i \in \mathbb{N}} p_i^{-1}(v_i(q)).$$

Notice that $v(q) \in \mathcal{F}$, as required by the definition of type structure.

6.6 Direct limits

As inverse limits, direct limits of systems probability spaces are known to exist under suitable (mostly topological) assumptions. See, for instance, [18, Chapter III]. As for their existence under purely measure-theoretic assumptions, we only know the partial results obtained in [20]. Actually, [20] contains significant additional assumptions, not just "remarks and alterations", of results previously claimed by the same author in [19]. A major problem already lies with the definition of the measurable space underlying the direct limit. For such reason, in [20], the author is forced to introduce the notions of pseudo σ-algebra and pseudo probability space.

In this section, we first define the direct limit of a system of probability spaces. We do it under stronger assumptions than those in [20]. The additional assumptions are quite natural and allow to get rid of the above mentioned pseudo-notions. Eventually, we discuss the extension of the direct limit to systems of type structures.

It seems quite awkward to refer the reader directly to [20], which, in turns, makes systematic reference to [19]. For this reason, we set up the framework quite in detail.

Let $(I, \leq)$ be a directed set and let $((\Omega_i, (f_{ij}))_{i\leq j, i,j\in I}$ be an injective direct system of sets, namely, for all $i \leq j \leq k$,

- Ω_i is a nonempty set;
- $f_{ii} = \mathrm{id}_{\Omega_i}$;
- $f_{ij} : \Omega_i \to \Omega_j$ is an injective function;
- $f_{jk} \circ f_{ij} = f_{ik}$.

When writing ω_i, we implicitly assume that $\omega_i \in \Omega_i$.

The direct limit $(\Omega, (f_i)_{i\in I})$ of the above system is defined as follows: Ω is the quotient set of $\bigcup_{i\in I}(\Omega_i \times \{i\})$ with respect to the equivalence relation $\sim$ defined by

$$(\omega_i, i) \sim (\omega_j, j) \;\Leftrightarrow \text{there exists } i, j \leq k \text{ such that } f_{ik}(\omega_i) = f_{jk}(\omega_j).$$

We denote by $\omega^\sim$ the equivalence class of $\omega \in \bigcup_{i\in I} \Omega_i$.

Each $f_i : \Omega_i \to \Omega$ is a map defined by: $f_i(\omega_i) = (\omega_i, i)^\sim$. It can be easily verified that the f_i's are injective. Furthermore,

$$\text{for all } i \leq j, \quad f_i = f_j \circ f_{ij} \qquad (\star)$$

Next, we give the following:

Definition 22 An injective direct system of probability spaces

$$((\Omega_i, \mathcal{F}_i, \mu_i), (f_{ij}))_{i\leq j, i,j\in I}$$

is a structure such that $((\Omega_i, (f_{ij}))_{i\leq j, i,j\in I}$ is an injective direct system of sets and, for all $i \leq j \leq k$ and all $A \in \mathcal{F}_i$,

(1) $(\Omega_i, \mathcal{F}_i, \mu_i)$ is a probability space;
(2) $f_{ij}(A) \in \mathcal{F}_j$ (for short: $f_{ij}(\mathcal{F}_i) \subseteq \mathcal{F}_j$);
(3) $\mu_i(A) = \mu_j(f_{ij}(A))$ (for short: $\mu_i = \mu_j \circ f_{ij}$);

Definition 22 captures the setting in [20]. We point out the strength of condition (3). For instance, it implies that, whenever $i \leq j$,

$$\mu_j(\Omega_j \setminus f_{ij}(\Omega_i)) = 0.$$

Albeit rather strong, the previous assumptions do not suffice to define a σ-algebra on Ω (as wrongly claimed in [19]). Hence the notion of pseudo probability space and the statement of [20, Theorem 3].

From now on, we further assume that

– I is countable;
– for all $i \leq j$, f_{ij} is a measurable function.

The latter is a quite natural assumption on maps between probability spaces.

Then we endow Ω with a structure of probability space.

We let $f_i(\mathcal{F}_i) = \{f_i(A) : A \in \mathcal{F}_i\}$ and $R = \bigcup_{i \in I} f_i(\mathcal{F}_i)$. We claim that R is a ring of subsets of Ω. Clearly, $\emptyset \in R$. Let $X = f_i(A)$, $Y = f_j(B)$, for some $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ and let $i, j \leq k$. By ($\star$) above, we have:

$$X \cup Y = f_k(f_{ik}(A)) \cup f_k(f_{jk}(B)) = f_k(f_{ik}(A) \cup f_{jk}(B)) \in f_k(\mathcal{F}_k) \subseteq R.$$

Similarly,

$$X \setminus Y = f_k(f_{ik}(A)) \setminus f_k(f_{jk}(B)) = f_k(f_{ik}(A)) \setminus f_k(f_{jk}(B)) \in f_k(\mathcal{F}_k) \subseteq R,$$

where the rightmost equality holds by injectivity of f_k.

We define $\mu : R \to [0, 1]$ as follows: for $A \in \mathcal{F}_i$, we let

$$\mu(f_i(A)) = \mu_i(A).$$

First, we show that μ is well-defined. Let us assume that $f_i(A) = f_j(B)$. Let $i, j \leq k$. Then $f_k(f_{ik}(A)) = f_i(A) = f_j(B) = f_k(f_{jk}(B))$ and, by injectivity of f_k, we get $f_{ik}(A) = f_{jk}(B)$. Hence, by Definition 22(3), we get

$$\mu_i(A) = \mu_k(f_{ik}(A)) = \mu_k(f_{jk}(B)) = \mu_j(B).$$

We notice that,

$$\text{for all } i \in I, \quad \mu_i = \mu \circ f_i \qquad (\star\star)$$

Second, we prove that μ is additive. Let $f_i(A) \in \mathcal{F}_i$ and $f_j(B) \in \mathcal{F}_j$ be disjoint. Let $i, j \leq k$. The following hold:

$$\begin{aligned}\mu(f_i(A) \cup f_j(B)) &= \mu(f_k(f_{ik}(A)) \cup f_k(f_{jk}(B)) \\ &= \mu(f_k(f_{ik}(A) \cup f_{jk}(B))) = \mu_k(f_{ik}(A) \cup f_{jk}(B))\end{aligned}$$

$$= \mu_k(f_{ik}(A)) + \mu_k(f_{jk}(B)) = \mu_i(A) + \mu_j(B)$$
$$= \mu(f_i(A)) + \mu(f_j(B)).$$

We need a preliminary result before proving that μ is actually a pre-measure on R.

Lemma 23 *Under the assumptions above, let $i, j \in I$ and let $C \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ be such that $f_i(C) \subseteq f_j(B)$. Then there exists $D \in \mathcal{F}_j$ such that $f_i(C) = f_j(D)$.*

Proof Let $D = \{\omega_j \in B$: there exists $\omega_i \in C$ with $(\omega_i, i)^\sim = (\omega_j, j)^\sim\}$. Clearly, $f_i(C) = f_j(D)$. It remains to prove that $D \in \mathcal{F}_j$. Let $i, j \leq k \in I$. From $f_i(C) = f_j(D)$, we get $f_k(f_{ik}(C)) = f_k(f_{jk}(D))$. Hence $f_{ik}(C) = f_{jk}(D)$, by injectivity of f_k. Therefore $f_{jk}^{-1}(f_{ik}(C)) = D$. Moreover, $f_{ik}(C) \in \mathcal{F}_k$, by Definition 22(2). Hence $D = f_{jk}^{-1}(f_{ik}(C)) \in \mathcal{F}_j$, by measurability of f_{jk}. $\square$

Let $\{f_{i_n}(A_n) : n \in \mathbb{N}\} \subseteq R$ be a family of pairwise disjoint sets such that $\bigcup_{n\in\mathbb{N}} f_{i_n}(A_n) = f_k(A)$, for some $A \in \mathcal{F}_k$. By Lemma 23, for each $n \in \mathbb{N}$ there exists $B_n \in \mathcal{F}_k$ such that $f_{i_n}(A_n) = f_k(B_n)$. Notice that the B_n's are pairwise disjoint. We have:

$$\mu\left(\bigcup_{n\in\mathbb{N}} f_{i_n}(A_n)\right) = \mu\left(\bigcup_{n\in\mathbb{N}} f_k(B_n)\right) = \mu\left(f_k\left(\bigcup_{n\in\mathbb{N}} B_n\right)\right) = \mu_k\left(\bigcup_{n\in\mathbb{N}} B_n\right)$$
$$= \sum_{n\in\mathbb{N}} \mu_k(B_n) = \sum_{n\in\mathbb{N}} \mu(f_k(B_n))$$
$$= \sum_{n\in\mathbb{N}} \mu(f_{i_n}(A_n)),$$

where the second-last equality holds by $(\star\star)$.

Hence μ is a pre-measure on R. By the Carathéodory's extension theorem, μ extends to a measure, that we denote by the same name, on the σ-algebra $\mathcal{F}$ generated by R. We claim that μ is a probability measure.

First of all, since $\Omega = \bigcup_{i\in I} f_i(\Omega_i)$ and I is countable, Ω belongs to $\mathcal{F}$. Hence $\mathcal{F}$ is actually a σ-algebra on Ω. Therefore, in order to prove that μ is a probability measure on $\mathcal{F}$, it suffices to verify that the outer measure μ^* associated to μ satisfies the condition $\mu^*(\Omega) \leq 1$.

The latter follows by fixing an enumeration $\{i_n\}_{n\in\mathbb{N}}$ of I and by noticing that Ω is the disjoint union of the family

$$\{f_{i_n}(\Omega_{i_n}) \setminus \bigcup_{0\leq j<n} f_{i_j}(\Omega_{i_j}) : n \in \mathbb{N}\} \subseteq R.$$

Moreover, $\mu(f_{i_0}(\Omega_{i_0})) = 1$ and, for all $0 < n$,

$$\mu(f_{i_n}(\Omega_{i_n}) \setminus \bigcup_{0\leq j<n} f_{i_j}(\Omega_{i_j})) = 0.$$

Hence $\mu^*(\Omega) \leq 1$.

We call $((\Omega, \mathcal{F}, \mu), (f_i)_{i\in I})$ the direct limit of $((\Omega_i, \mathcal{F}_i, \mu_i), (f_{ij}))_{i\leq j, i,j\in I}$.

Eventually, we briefly comment on the extension of the direct limit construction to type pre-structures. Hence, as in Sect. 6.5, we forget about the assignments of values to propositional variables. The notation introduced in the first part of this section is in force.

We fix a direct system $((\Omega_i, \mathcal{F}_i, T_i), (f_{ij}))_{i\leq j, i,j\in I}$ of type pre-structures, where $(\Omega_i, (f_{ij}))_{i\leq j, i,j\in I}$ is a countable injective direct system. We form the set-theoretic direct limit $(\Omega, (f_i)_{i\in I})$ and, as above, we denote by $\mathcal{F}$ the σ-algebra generated by $\bigcup_{i\in I} f_i(\mathcal{F}_i)$.

For all $\omega \in \Omega$, let

$$I_\omega = \{i \in I : \text{there exists } \omega_i \in \Omega_i \text{ such that } (\omega_i, i)^\sim = \omega\}.$$

The set I_ω is a directed subset of I.

We may consider the direct subsystem $((\Omega_i, \mathcal{F}_i, T_i(\omega_i), (f_{ij}))_{i\leq j, i,j\in I_\omega}$ of probability spaces. Assuming that the latter satisfies conditions (2) and (3) in Definition 22 and that the f_{ij}'s are measurable, we denote its direct limit by $((\Omega_\omega, \mathcal{F}_\omega, \mu_\omega), (f_i^{(\omega)})_{i\in I_\omega}))$. Therefore Ω_ω is the quotient set of $\bigcup_{i\in I_\omega}(\Omega_i \times \{i\})$ with respect to the restriction $\sim_\omega$ of $\sim$ to $\bigcup_{i\in I_\omega}(\Omega_i \times \{i\})$. We define the map $T : \Omega \to \bigcup_{\omega\in\Omega} \Delta(\Omega_\omega, \mathcal{F}_\omega)$ by letting

$$T(\omega) = \mu_\omega \quad \text{for all } \omega \in \Omega.$$

We notice that the structure $(\Omega, \mathcal{F}, T)$ is not a type pre-structure, simply because each $T(\omega)$ is a probability measure on $\mathcal{F}_\omega$, not on $\mathcal{F}$. Apart from the very special case when, for all $\omega \in \Omega$, $I_\omega = I$ (the latter implies that $(\Omega, \mathcal{F}) = (\Omega_\omega, \mathcal{F}_\omega)$), there seems to be no intuitive way of obtaining a direct limit pre-structure.

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