A THEOREM ABOUT LIMITS WITH APPLICATION TO MACLAURIN'S POLYNOMIAL OF CERTAIN FUNCTIONS

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ABSTRACT. In this article we consider a theorem for the limit of certain functions at $x = 0$. We shall prove that in some cases, the existence of the first derivative at $x = 0$ and the existence of a "functional equation" let us able to infer the existence of some derivatives of higher order.

1. THE THEOREM

Theorem 1. *Let be $f : \mathbb{R} \to \mathbb{R}$ a continuous function such that*

- **(a):** *There exist $k > 1$, $a \in \mathbb{R}$, $h \in \mathbb{Z}^+$ such that*

$$f(kx) = kf(x) + a[f(x)]^h \quad \forall x \in \mathbb{R}.$$

- **(b):** $f(0) = 0$.
- **(c):** *There exists*

$$\lim_{x\to 0} \frac{f(x)}{x} = b.$$

- **(d):** *The function*

$$g(x) = \frac{bx - f(x)}{x^h}$$

is bounded in a deleted neighborhood of $x = 0$.

then

$$\exists \lim_{x\to 0} g(x) = L.$$

where

$$L = -a\left(\frac{b}{k}\right)^h \left(1 - \frac{1}{k^{h-1}}\right)^{-1}.$$

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Proof. Since $g(x)$ is bounded in a deleted neighborhood of $x = 0$, we have that

$$-\infty < \lambda = \liminf_{x\to 0} g(x) \leqslant \limsup_{x\to 0} g(x) = \Lambda < +\infty.$$

Now, by hypothesis (**a**), we can write

$$f(x) = kf\left(\frac{x}{k}\right) + a\left[f\left(\frac{x}{k}\right)\right]^h$$

for each $x \in \mathbb{R}$, so that

$$g(x) = \frac{bx - f(x)}{x^h} = \frac{k\left[\left(\frac{x}{k}\right) - f\left(\frac{x}{k}\right)\right]}{k^h\left(\frac{x}{k}\right)^h} - \frac{a}{k^h}\frac{\left[f\left(\frac{x}{k}\right)\right]^h}{\left(\frac{x}{k}\right)^h}$$

Hence

$$g(x) = g_1(x) + g_2(x).$$

where

$$g_1(x) = \frac{1}{k^{h-1}}\frac{\left[\left(\frac{x}{k}\right) - f\left(\frac{x}{k}\right)\right]}{\left(\frac{x}{k}\right)^h}.$$

and

$$g_2(x) = -\frac{a}{k^h}\frac{\left[f\left(\frac{x}{k}\right)\right]^h}{\left(\frac{x}{k}\right)^h}.$$

By hypotesis (**c**) it is

$$\liminf_{x\to 0} g_2(x) = \limsup_{x\to 0} g_2(x) = \lim_{x\to 0} g_2(x) = -a\left(\frac{b}{k}\right)^h = A.$$

Now, we remember that

$$\begin{aligned}\liminf_{x\to 0} g_1(x) + \liminf_{x\to 0} g_2(x) \leqslant \liminf_{x\to 0}\,(g_1(x) + g_2(x)) \leqslant \\ \leqslant \limsup_{x\to 0}\,(g_1(x) + g_2(x)) \leqslant \limsup_{x\to 0} g_1(x) + \limsup_{x\to 0} g_2(x).\end{aligned} \tag{1}$$

and so

$$\liminf_{x\to 0} g_1(x) + \liminf_{x\to 0} g_2(x) \leqslant \lambda \leqslant \Lambda \leqslant \limsup_{x\to 0} g_1(x) + \limsup_{x\to 0} g_2(x).$$

But

$$\liminf_{x\to 0} g_1(x) = \frac{1}{k^{h-1}}\lambda \tag{2}$$

$$\limsup_{x\to 0} g_1(x) = \frac{1}{k^{h-1}}\Lambda. \tag{3}$$

so that

$$\frac{1}{k^{h-1}}\lambda + A \leqslant \lambda \leqslant \Lambda \leqslant \frac{1}{k^{h-1}}\Lambda + A.$$

It follows that

$$A\left(1 - \frac{1}{k^{h-1}}\right)^{-1} \leqslant \lambda \leqslant \Lambda \leqslant A\left(1 - \frac{1}{k^{h-1}}\right)^{-1}.$$

and so

$$\lambda = \Lambda = L = \lim_{x\to 0} g(x).$$

$\square$

This theorem can be used, at least in principle, in order to obtain the Maclaurin polynomials of certain functions without the use of the derivative of order higher than one. Indeed since

$$g(x) = \frac{bx - f(x)}{x^h} = L + o(1) \quad (x \to 0)$$

we have

$$f(x) = bx - Lx^h + o(x^h).$$

We get an example in order to clarify the previous statement.

2. An example

Le us consider $f(x) = \sin x$. We remember that for any $x \in \mathbb{R}$ it is well known that

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

Thus, we have that $k = 1$, $a = -4$, $h = 3$ and $b = 1$ because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Since, in this case

$$g(x) = \frac{x - \sin x}{x^3}.$$

is an even function, we can consider $x > 0$ only. In this case, since $x < \tan x$ in a deleted right neighborhood of $x = 0$, it follows that

$$g(x) = \frac{x - \sin x}{x^3} < \frac{\tan x - \sin x}{x^3}.$$

But

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x}{x} \left(\frac{1 - \cos x}{x^2} \right) = \frac{1}{2}.$$

thus $g(x)$ is bounded in a right neighborhood of $x = 0$ and in a complete deleted neighborhood also. Hence, all hypotheses are satisfied and we have that

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = 4\left(\frac{1}{3}\right)^3 \left(1 - \frac{1}{3^2}\right)^{-1} = \frac{1}{6}.$$

This means that

$$\sin x = x - \frac{1}{6}x^3 + o(x^3). \tag{4}$$

This is the Maclaurin's polynomial of **third order** for the function $\sin x$ and we obtained it **without** use the derivatives of order **higher than one**.

Note 1. *Perhaps the most important key point is that it is possible to write*

$$g(x) = c_k g\left(\frac{x}{k}\right) + F_h\left(x\right) + s_{k,h}\left(x\right). \tag{5}$$

where $x \to s_{k,h}\left(x\right)$ *is a function of* x *such that* $\lim_{x\to 0} s_{k,h}\left(x\right) = A_{k,h} \in \mathbb{R}$. *and*

$$F_h\left(x\right) = \left[\frac{f\left(\frac{x}{k}\right)}{\left(\frac{x}{k}\right)}\right]^h.$$

This is strongly related with the existence of a "functional equation" like

$$f(kx) = kf(x) + a\left[f(x)\right]^h.$$

which can be said to be of order h*. In principle, we can obtain the Maclaurin's polynomial of a given order* n *if we are able to find the suitable functional equations for* f*. We get an example just to understand.*

3. The previous example go further

For first we are going to see that there is a relation like (5). Since

$$\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x.$$

if we consider

$$g(x) = \frac{x - \frac{1}{6}x^3 - \sin x}{x^5}.$$

we can write

$$g(x) = \frac{\left(x - \frac{1}{6}x^3\right) - \left(5\sin\frac{x}{5} - 20\sin^3\frac{x}{5}\right)}{x^5} - \frac{\left(16\sin^5\frac{x}{5}\right)}{x^5}.$$

Now, from (4), we have

$$\sin^3 x = \left[\left(x - \frac{1}{6}x^3\right) + o(x^3)\right]^3$$

thus

$$\sin^3 x = \left(x^3 - \frac{1}{2}x^5\right) + o(x^5).$$

Hence

$$g(x) = \frac{\left(x - \frac{1}{6}x^3\right) - \left(5\sin\frac{x}{5} - 20\left(\frac{x}{5^3}^3 - \frac{1}{2}\frac{x}{5^5}^5\right)\right)}{x^5} + o(1) - \frac{\left(16\sin^5\frac{x}{5}\right)}{x^5}.$$

After some calculations, we have

$$g(x) = \frac{\left[\frac{x}{5} - \frac{1}{6}\left(\frac{x}{5}\right)^3 - \sin\frac{x}{5}\right]}{5^4\left(\frac{x}{5}\right)^5} - \frac{\left(16\sin^5\frac{x}{5}\right)}{5^5\left(\frac{x}{5}\right)^5} - 10\left(\frac{1}{5^5}\right) + o(1).$$

Thus, we have the relation (5) with

$$(x) = \frac{1}{5^4}g\left(\frac{x}{5}\right) - \frac{16}{5^5}\left[F(x)\right]^5 + s_{k,h}\left(x\right)$$

where

$$s_{k,h}(x) = -10\left(\frac{1}{5^5}\right) + o(1).$$

For second, we are going to prove that $g(x)$ is bounded in a deleted neighborhood of $x = 0$. As before, it is enough to consider only a deleted right neighborhood of $x = 0$. If we consider

$$k(x) = \sin x - \left(x - \frac{1}{6}x^3\right).$$

we have that $k(0) = 0$ and

$$k'(x) = 1 - \frac{1}{2}x^2 - \cos x$$

thus

$$k'(x) = \frac{\sin^2 x}{\cos x + 1} - \frac{1}{2}x^2$$

As before, from (4), we get

$$\sin^2 x = x^2 - \frac{1}{3}x^4 + o\left(x^4\right).$$

so that

$$k'(x) = \frac{x^2}{2}\left[\frac{1-\cos x}{(\cos x + 1)}\right] - \frac{1}{3}\frac{x^4}{\cos x + 1} + o\left(x^4\right)$$

It follows that

$$k'(x) = \frac{x^2}{2}\left[\frac{\frac{x^2}{2} + o\left(x^2\right)}{(\cos x + 1)}\right] - \frac{1}{3}\frac{x^4}{\cos x + 1} + o\left(x^4\right)$$

which leads to

$$k'(x) = -\frac{1}{12}\frac{x^4}{\cos x + 1} + o\left(x^4\right) \leqslant -\frac{x^4}{24} + o\left(x^4\right) < -\frac{x^4}{48}$$

Hence

$$k(x) = \int_0^x k'(t)dt < -\frac{1}{48}\int_0^x t^4 dt < 0$$

in a suitable deleted right neighborhood of $x = 0$. In the same way we have that

$$\left(x - \frac{1}{6}x^3\right) - \sin x \geqslant -x^5.$$

This means that, even in this case, we have $-\infty < \lambda \leqslant \Lambda < +\infty$ where, of course λ and Λ have the same meaning as before but they are related with the ne function $g(x)$. In this case we get

$$\frac{1}{625}\lambda - \frac{26}{5^5} \leqslant \lambda \leqslant \Lambda \leqslant \frac{1}{625}\Lambda - \frac{26}{5^5}$$

so that

$$\lim_{x\to 0} g(x) = -\frac{1}{120}$$

and thus

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5).$$

and this result has been obtained with **no derivatives of order higher than one**.

References

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