GAUSSIAN ESTIMATES ON NETWORKS WITH APPLICATIONS TO OPTIMAL CONTROL

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Abstract. We study a class of reaction-diffusion type equations on a finite network with continuity assumptions and a kind of *non-local*, stationary Kirchhoff's conditions at the nodes. A multiplicative random Gaussian perturbation acting along the edges is also included. For such a problem we prove Gaussian estimates for the semigroup generated by the evolution operator, hence generalizing similar results previously obtained in [21]. In particular our main goal is to extend known results on Gaussian upper bounds for heat equations on networks with local boundary conditions to those with non-local ones. We conclude showing how our results can be used to apply techniques developed in [13] to solve a class of Stochastic Optimal Control Problems inspired by neurological dynamics.

1. Introduction

Our main goal is to generalize results contained in [21] to prove Gaussian upper bounds for heat equations on networks with non-local boundary conditions. Such estimates can be successfully used to study a certain Stochastic Optimal Control Problem (SOCP) stated on a finite graph where the evolution on each edge is

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perturbed by a multiplicative Gaussian noise, while continuity assumptions and a non-local Kirchhoff-type law are imposed in the nodes. Namely we will consider the following stochastic evolution equation

$$\dot{X}(t,x) = (c(x)X'(t,x))' + p(x)X(t,x) + f(t,x,X(t,x)) + g(t,x,X(t,x))\dot{W}(t,x),$$

describing the electrical potential moving along each axon of a finite network, subjected to particular non-local Robin-like boundary conditions in the nodes, of the following type:

$$\sum_{l=1}^{n-1} b_{i,l} X(t,\mathsf{v}_l) = \sum_{j=1}^{m} \Phi_{ij} c_j(\mathsf{v}_i) X_j'(t,\mathsf{v}_i).$$

In Section 4 we will obtain our main result deriving a Gaussian upper bound for the Green function of the problem. In particular we extend the analogous result obtained in [21] under the more restrictive hypothesis of local boundary conditions. Our approach differs from the one used in [21, §4] since we adopt a different functional space (and associated distance) to apply the so-called Davies' trick. This simple but essential modification allows us to overcome difficulties arising in the case of non-local conditions. This allows us to exploit results given by [13] to study a controlled stochastic process X_t^u with values in a Hilbert space $\mathcal{H}$ being a mild solution of

$$\begin{cases} dX_t^u = [AX_t^u + F(t,X_t^u)]\,dt + G(t,X_t^u)R(t,X_t^u)u(t)dt + G(t,X_t^u)dW_t \\ X_{t_0}^u = x_0 \in \mathcal{H}, \end{cases} \tag{1}$$

where $t_0 \in [0,T]$. The control process u takes values in a given subset $\mathcal{U}$ of a Hilbert space U, W is a cylindrical Wiener process in an other Hilbert space $\bar{\mathcal{H}}$, $G : [0,T] \times \mathbb{R} \to L(\bar{\mathcal{H}},\mathcal{H})$, $R : [0,T] \times \mathbb{R} \to L(U,\bar{\mathcal{H}})$, $F : [0,T] \times \mathbb{R} \to \mathcal{H}$ are suitable regular functions and A is the generator of a strongly continuous semigroup of bounded linear operators $(e^{tA})_{t\geq 0}$ in $\mathcal{H}$. Under suitable hypotheses on the operator A, which will be assured by our estimates, and on the functions F, G, R, g, ψ, ϕ, it is possible to apply Theorem 6.2 of [13] to have the existence of a unique mild solution for the Hamilton-Jacobi-Bellman problem associated to (1), hence obtaining uniqueness of solution for our SOCP.

We will relax some hypotheses in a future work in order to allow F to have polynomial growth. In particular taking $\mathcal{H} = L^2(0,1)$, $F(x) = x(1-x)(x-\xi)$, with $\xi \in (0,1)$ and A as a second order elliptic operator in $L^2(0,1)$, the above equations can be used to describe a neuronal environment controlling a FitzHugh-Nagumo system by mean of the control function u. This means e.g. that we can find the optimal dose of inhibitory/stimulating drug u acting on the voltage potential $X = X(t,x)$ perturbed by a multiplicative Gaussian noise on the axon and/or external noise acting on the nodes, see e.g. [5] for an example in the deterministic case.

2. Setting of the problem

2.1. **Structure of the network.** In this paper we are concerned with a diffusion problem on a finite connected network, therefore we will adopt the by now standard setting introduced to study network flow problems. For a detailed survey on graph theory and related issues we refer to the monograph [30] and references therein (see e.g. [15, 16, 18, 28]). Throughout the whole article we will identify the network with a directed graph $\mathbb{G}$, consisting of a vertex set $V(\mathbb{G})$ of n vertices $\mathsf{v}_1, ..., \mathsf{v}_n$ and an edge set $E(\mathbb{G})$ of m oriented edges $\mathsf{e}_1, \dots, \mathsf{e}_m$ which we assume to be normalized, i.e., $\mathsf{e}_j = [0,1]$ for $j = 1, ..., m$. Now we introduce the concepts of path and distance in order to model movements in graphs. Given two nodes $\mathsf{v}_0, \mathsf{v}_k \in V(\mathbb{G})$, a $\mathsf{v}_0, \mathsf{v}_k$-*path* is a list $\mathsf{v}_0, \mathsf{e}_1, \mathsf{v}_1, ..., \mathsf{e}_k, \mathsf{v}_k$ of vertices and edges such that for $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i. The *length* of a path is its number of edges. Consequently the *distance* from v_0 to v_k, written $d_{\mathbb{G}}(\mathsf{v}_0, \mathsf{v}_k)$ is defined as the least length of a $\mathsf{v}_0, \mathsf{v}_k$-path. In general, given a point $x \in \mathsf{e}_j$ let us define the distance from x to v_k as

$$d_{\mathbb{G}}(x, \mathsf{v}_k) := (d_{\mathbb{G}}(\mathsf{e}_j(0), \mathsf{v}_k) \wedge d_{\mathbb{G}}(\mathsf{e}_j(1), \mathsf{v}_k)) + 1,$$

because we consider the additional distance due to the edge e_j. Moreover, the *eccentricity* of a vertex v_0, written $\epsilon(\mathsf{v}_0)$, is $\max_{\mathsf{v}_i \in V(\mathbb{G})} d_{\mathbb{G}}(\mathsf{v}_0, \mathsf{v}_i)$.

The orientation of the graph is described by the so-called *incidence matrix* $\Phi = \Phi^+ - \Phi^-$, where $\Phi^+ = (\phi^+_{ij})_{n\times m}$ (*incoming incident matrix*) and $\Phi^- = (\phi^-_{ij})_{n\times m}$

(*outgoing incident matrix*) are given by

$$\phi_{ij}^{-} = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(1) \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \phi_{ij}^{+} = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(0) \\ 0, & \text{otherwise.} \end{cases}$$

Accordingly, we call the edge e_j an *incoming edge* (respectively *outgoing edge*) for v_i if $\mathsf{v}_i = \mathsf{e}_j(0)$ holds (respectively if $\mathsf{v}_i = \mathsf{e}_j(1)$ holds). Note that the arcs are parameterized contrary to the direction of the current flow. Moreover, we introduce the *diffusive matrices* $\Phi_\delta^+ = (\delta_{ij}^+)_{n\times m}$ and $\Phi_\delta^- = (\delta_{ij}^-)_{n\times m}$ defined by

$$\delta_{ij}^{+} = \begin{cases} c_j(\mathsf{v}_i), & \text{if } \phi_{ij}^{+} = 1 \text{ and } i \le n-1 \\ 0, & \text{otherwise} \end{cases} \qquad \delta_{ij}^{-} = \begin{cases} c_j(\mathsf{v}_i), & \text{if } \phi_{ij}^{-} = 1 \text{ and } i \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

where c_j are nonnegative C^1-functions describing the diffusivity per unit time along the edge e_j (see Assumptions 2.2 below). Finally we define the *degree of a vertex* as the number of edges entering or leaving the node. If we denote by $\Gamma(\mathsf{v}_i)$ the set of all the indices of the edges having an endpoint at v_i, i.e.

$$\Gamma(\mathsf{v}_i) = \{j \in \{1, \dots, m\} \ : \ \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i\}$$

then we define the degree of the vertex v_i as the cardinality of the set $\Gamma(\mathsf{v}_i)$.

2.2. **Diffusion problem.** Inspired by biological motivations we will impose on every edge e_j for $j = 1, ..., m$ a stochastic reaction-diffusion equation in divergence form of this type

$$\dot{X}_j = \left(c_j X_j'\right)' + p_j X_j + f_j(t, X_j) + g_j(t, X_j)\dot{W}_j, \tag{2}$$

where $X_j(t,\cdot)$ for $t \ge 0$ is the (real-valued) electrical potential on the edge e_j. Here $W_j(t,x)$ for $j = 1, ..., m$ are real valued space-time independent Wiener processes defined on a (fixed) filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ satisfying the usual hypotheses (see e.g. [20, Definition 1.1]). Assumptions on the real-valued functions c_j, p_j, f_j and g_j for $j = 1, ..., m$ will be specified later. As usual $\dot{X}$ denotes the derivative of X with respect to the time variable t and X' with respect to the spatial variable x. The generality of the above diffusion is motivated by discussions in the biological literature, see for example [14] or [23], where concrete biological models are treated showing that the basic cable properties are not constant throughout the dendritic tree. Note that linear elliptic operators of the form

$$AX = (cX')' + pX$$

are commonly associated with mathematical models describing internal diffusions of electrical potential in neurons. The consequent (possibly semilinear) parabolic equation is often called as *cable equation* (for further details see e.g. [29, §4.2]).

The above system of equations will be endowed with suitable boundary and initial conditions. First of all, since we are dealing with a diffusion in a network, we require a continuity assumption on every node

$$X_j(t, \mathsf{v}_i) = X_k(t, \mathsf{v}_i), \qquad t > 0,\ j, k \in \Gamma(\mathsf{v}_i),\ i = 1, ..., n \tag{3}$$

therefore, setting $X(t, \cdot) := (X_1(t, \cdot), ..., X_m(t, \cdot))$, we can denote by $X(t, \mathsf{v}_i)$ the value of the electrical potential at the vertex v_i.
Initial conditions are given for simplicity at time $t = 0$ of the form

$$X_j(0, x) = X_j^0(x), \text{ with } X_j^0 \in C([0, 1]) \qquad j = 1, ..., m. \tag{4}$$

With regard to boundary conditions, the most common (and physically reasonable) ones describe a kind of Kirchhoff's second law in the ramification points, possibly with a lower-order Robin-like term (often called δ-type interactions). In particular we permit that the flow at a junction can accumulate, dissipate or hold steady according to its value in the other nodes. Therefore we will impose, in the first $n - 1$ nodes, a stationary, non-local Kirchhoff-type law as follows

$$\sum_{l=1}^{n-1} b_{i,l} X(t, \mathsf{v}_l) = \sum_{j=1}^{m} \Phi_{ij} c_j(\mathsf{v}_i) X_j'(t, \mathsf{v}_i) \qquad \text{for } i = 1, ..., n - 1. \tag{5}$$

Instead in the n-th node we will consider an homogeneous Dirichlet condition:

$$X(t, \mathsf{v}_n) = 0. \tag{6}$$

With the previous notation, the boundary conditions (3), (5) and (6) can be reformulated as follows there exists $d^X(t) \in \mathbb{R}^n$ such that

$$(\Phi^+)^T d^X(t) = X(t,0), \quad (\Phi^-)^T d^X(t) = X(t,1) \quad \text{and}$$
$$\Phi_\delta^+ X'(t,0) - \Phi_\delta^- X'(t,1) = B d^X(t) \qquad \text{for all } t \geq 0,$$

where

$$B = \begin{pmatrix} b_{1,1} & \dots & b_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & \dots & b_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

$d^X(t)$ will be in fact

$$d^X(t) = \begin{pmatrix} X(t, \mathsf{v}_1) \\ \vdots \\ X(t, \mathsf{v}_{n-1}) \\ 0 \end{pmatrix},$$

i.e. $d^X(t)$ is the n-dimensional vector of the electrical potential values at any vertex.

Remark 2.1.

- Stationary, non-local Kirckhhoff-type boundary conditions (5) in the nodes have been inspired by [21] where the author obtains upper Gaussian estimates for the integral kernel in the local case characterized by a diagonal matrix B with negative entries. Slightly modifying the functional setting we generalize previous results to the non-local boundary conditions case, including connections between entries of B, the network geometry and diffusion data slightly modify the functional setting.
- We would like to underline that similar diffusion problems have been studied by various authors for more general boundary conditions in the nodes. Let us mention some of these works.
  In [22] is treated a class of diffusion problems on a whole network of neurons, where the ramification nodes can be either active (with excitatory time-dependent boundary conditions)

$$\dot{X}(t, \mathsf{v}_i) = -\sum_{j=1}^{m} \Phi_{ij} c_j(\mathsf{v}_i) X_j'(t, \mathsf{v}_i) + b_i X(t, \mathsf{v}_i)$$

or passive (no dynamics take place, i.e. only Kirchhoff laws are imposed). In [3, 4] a stochastic version of the previous dynamical boundary conditions is considered in the following form

$$\dot{X}(t,\mathsf{v}_i) = b_i X(t,\mathsf{v}_i) - \sum_{j\in\Gamma(\mathsf{v}_i)} \Phi_{ij}\mu_j c_j(\mathsf{v}_i) X_j'(t,\mathsf{v}_i) + \sigma_i \dot{L}(t,\mathsf{v}_i),$$

where $L(t,\mathsf{v}_i)$ represents the stochastic perturbation acting on each node due to the external surrounding.

Note that in all these cases well-posedness for the problem is proved but no Gaussian estimates have been obtained so far.

- In treating a diffusion model on a network it is usual to impose on n_0 boundary vertices (i.e. those of degree 1, according to the previous definition) homogeneous Dirichlet boundary conditions. Since no evolution takes place on these nodes we may rearrange the graph identifying all of them with a unique vertex v_n of degree n_0 like in (6).

Let us state the main assumptions we shall make to handle the problem.

Assumptions 2.2.

(1) $c_j(\cdot)$ *belongs to* $C^1([0,1])$, *for* $j=1,\dots,m$ *and* $c_j(x)>0$ *for every* $x\in[0,1]$.

(2) $f_j : [0,T]\times[0,1]\times\mathbb{R}\to\mathbb{R}$, $j=1,...,m$ *is a measurable mapping, bounded and uniformly Lipschitz continuous in the last component*

$$|f_j(t,x,u)| \le K_j \qquad |f_j(t,x,u)-f_j(t,x,v)| \le L_j|u-v|$$

for some positive constants K_j, L_j , $j=1,\dots,m$, *every* $t\in[0,T]$, $x\in[0,1]$, $u,v\in\mathbb{R}$.

(3) $g_j : [0,T]\times[0,1]\times\mathbb{R}\to\mathbb{R}$, $j=1,...,m$ *is a measurable mapping, bounded and uniformly Lipschitz continuous in the last component*

$$|g_j(t,x,u)| \le \tilde{K}_j \qquad |g_j(t,x,u)-g_j(t,x,v)| \le \tilde{L}_j|u-v|$$

for some positive constants $\tilde{K}_j, \tilde{L}_j$, $j=1,\dots,m$, *every* $t\in[0,T]$, $x\in[0,1]$, $u,v\in\mathbb{R}$.

(4) $p_j \in L^\infty(0,1)$, $j = 1, ...m$ *such that* $p_j \leq 0$.
(5) $b_{i,l} \in \mathbb{R}$ *for* $i, l = 1, ..., n-1$.

3. Well-posedness and regularity results

Being interested in generation properties for the evolution problem previously introduced, we can rewrite the deterministic linear part of equation (2) endowed with continuity assumption (3), initial condition (4), boundary conditions (5) and (6) as a system of coupled evolution boundary value problems of this type

$$\begin{cases} \dot{X}_j(t,x) = \big(c_j(x) X_j'(t,x)\big)' + p_j(x) X_j(t,x) \\ X_j(t, \mathsf{v}_i) = X_k(t, \mathsf{v}_i), \qquad j,k \in \Gamma(\mathsf{v}_i),\ i = 1, ..., n \\ \displaystyle\sum_{l=1}^{n-1} b_{i,l} X(t, \mathsf{v}_l) = \sum_{j=1}^{m} \Phi_{ij} c_j(\mathsf{v}_i) X_j'(t, \mathsf{v}_i) \qquad i = 1, ..., n-1, \\ X(t, \mathsf{v}_n) = 0, \\ X_j(0,x) = X_j^0(x) \qquad j = 1, ..., m. \end{cases} \tag{7}$$

Let us introduce a functional framework which allows to reformulate our diffusion problem (7) in abstract form.

3.1. **Abstract setting.** On the real Hilbert space $\mathcal{X}_2 = [L^2(0,1)]^m$, endowed with the natural inner product

$$\langle X, Y \rangle_{\mathcal{X}_2} := \sum_{j=1}^{m} \int_0^1 X_j(x) Y_j(x) dx \qquad X, Y \in \mathcal{X}_2,$$

we define the unbounded linear operator $(A, D(A))$ as follows

$$AX := \begin{pmatrix} (c_1 X_1')' + p_1 X_1 & & 0 \\ & \ddots & \\ 0 & & (c_m X_m')' + p_m X_m \end{pmatrix} \tag{8}$$

with domain, containing the boundary and continuity conditions, defined as follows

$$D(A) := \left\{ X \in [H^2(0,1)]^m : \begin{array}{c} \text{there exists } d^X \in \mathbb{R}^n \text{ such that} \\ (\Phi^+)^T d^X = X(0), \quad (\Phi^-)^T d^X = X(1) \quad \text{and} \\ \Phi_\delta^+ X'(0) - \Phi_\delta^- X'(1) = B d^X \end{array} \right\}. \tag{9}$$

Now we can rewrite the system (7) as an abstract Cauchy problem on the Hilbert space $\mathcal{X}_2$

$$\begin{cases} \dot{X}(t) = AX(t), & t \geq 0 \\ X(0) = X^0. \end{cases} \tag{10}$$

Where $X^0 = (X_1^0, ..., X_m^0) \in [C(0,1)]^m$ is the vector of initial data.
To discuss well posedness for (10) or equivalently generation properties for A on $\mathcal{X}_2$, it is useful to apply a variational method based on forms.

Remark 3.1. Our following results are also treated in [21] for a very similar case, hence we will use some of its contents while doing detailed proofs where differences between the two approaches occur.

First of all let us introduce the domain of the form.

Lemma 3.2. *The linear space*

$$V_0 := \left\{ X \in [H^1(0,1)]^m : \begin{array}{c} \text{there exists } d^X \in \mathbb{R}^{n-1} \times \{0\} \text{ such that} \\ (\Phi^+)^T d^X = X(0) \text{ and } (\Phi^-)^T d^X = X(1) \end{array} \right\}$$

is densely and compactly embedded in $\mathcal{X}_2$. It becomes an Hilbert space when equipped with the inner product

$$\langle X, Y \rangle_{V_0} := \sum_{j=1}^{m} \int_0^1 X_j'(x) Y_j'(x) dx, \qquad X, Y \in V_0. \tag{11}$$

The equivalence between the natural norm on V_0 (as subspace of $[H^1(0,1)]^m$) and the one defined in (11) is the same existing between the norm of $H_0^1(0,1)$ and the natural one derived from $H^1(0,1)$. In particular such equivalence is based on a Poincaré-type inequality, due to the homogeneous Dirichlet boundary condition we have put on the node v_n and to the connection of the graph. Therefore, previous argument allows us to derive some estimates which are useful to characterize continuity and coercivity of the form we will introduce later. By the fundamental theorem of calculus we have

$$|X_j(x)| \leq |X_j(0)| + \int_0^x |X_j'(y)| dy$$

for all $j = 1, ..., m$ and $x \in \mathsf{e}_j$. Moreover, iterating this inequality along the shortest x, v_n-path represented by $x, \mathsf{e}_{i_0}, \mathsf{v}_{i_1}, \mathsf{e}_{i_1}, \mathsf{v}_{i_2}, ..., \mathsf{e}_{i_n}, \mathsf{v}_n$, we obtain that

$$|X_j(x)| \leq \sum_{l=i_0}^{i_n} \int_0^1 |X_l'(y)| dy,$$

thanks to $X(\mathsf{v}_n) = 0$. Recalling that $\sharp\{i_0, ..., i_n\} = d_{\mathbb{G}}(x, \mathsf{v}_n) \leq \epsilon(\mathsf{v}_n)$ the eccentricity of v_n, it follows that

$$\sum_{j=1}^{m} \int_0^1 |X_j(x)|^2 dx \leq m 2^{\epsilon(\mathsf{v}_n)-1} \sum_{j=1}^{m} \int_0^1 |X_j'(x)|^2 dx, \tag{12}$$

and analogously we have

$$|d_i^X| \leq 2^{\frac{\epsilon(\mathsf{v}_n)-1}{2}} \left(\sum_{j=1}^{m} \int_0^1 |X_j'(x)|^2 dx \right)^{\frac{1}{2}}. \tag{13}$$

Proposition 3.3. *Consider the bilinear form $\mathfrak{a} : V_0 \times V_0 \to \mathbb{R}$ defined on the Hilbert space $\mathcal{X}_2$ by*

$$\mathfrak{a}(X, Y) := \sum_{j=1}^{m} \int_0^1 \big(c_j(x) X_j'(x) Y_j'(x) - p_j(x) X_j(x) Y_j(x) \big)\, dx + \sum_{i,l=1}^{n-1} b_{i,l} d_i^X d_l^Y$$

for all $X, Y \in V_0$. Let us set

$$b := \max_{i,l=1,...,n-1} |b_{i,l}|, \qquad p := \min_{j=1,...,m} \min_{x \in [0,1]} p_j(x),$$

$$C := \max_{j=1,..,m} \max_{x \in [0,1]} c_j(x), \qquad c := \min_{j=1,..,m} \min_{x \in [0,1]} c_j(x).$$

Then $\mathfrak{a}$ enjoys the following properties:

- *$\mathfrak{a}$ is continuous, i.e. for some $M > 0$*

$$|\mathfrak{a}(X, Y)| \leq M |X|_{V_0} |Y|_{V_0} \qquad \text{for all } X, Y \in V_0.$$

- *$\mathfrak{a}$ is symmetric, i.e.*

$$\mathfrak{a}(X, Y) = \mathfrak{a}(Y, X) \qquad \text{for all } X, Y \in V_0.$$

- *if B is positive definite, $\mathfrak{a}$ is accretive and coercive, i.e. there exists $\alpha > 0$ such that*

$$\mathfrak{a}(X, X) \geq \alpha |X|_{V_0}^2 \qquad \text{for all } X \in V_0. \tag{14}$$

Otherwise, $\mathfrak{a}$ is coercive if $c - 2^{\epsilon(\mathsf{v}_n)-1} b(n-1)^2 > 0$.

- $\mathfrak{a}$ *is closed.*

Proof. Combining (12) and (13) we obtain continuity of the form:

$$\begin{aligned}
|\mathfrak{a}(X,Y)| &\le C\sum_{j=1}^{m}\int_0^1 |X_j'(x)Y_j'(x)|dx + |p|\sum_{j=1}^{m}\int_0^1 |X_j(x)Y_j(x)|dx + b\sum_{i,l=1}^{n-1}|d_i^X||d_l^Y| \\
&\le C|X|_{V_0}|Y|_{V_0} + |p|m2^{\epsilon(\mathsf{v}_n)-1}|X|_{V_0}|Y|_{V_0} + 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2|X|_{V_0}|Y|_{V_0} \\
&= M|X|_{V_0}|Y|_{V_0} \qquad \text{for all } X,Y\in V_0
\end{aligned}$$

where $M = C + 2^{\epsilon(\mathsf{v}_n)-1}(|p|m + b(n-1)^2)$.
Let us suppose B positive definite. With regard to coercivity we have

$$\begin{aligned}
\mathfrak{a}(X,X) &\ge c\sum_{j=1}^{m}\int_0^1 |X_j'(x)|^2dx + \sum_{i,l=1}^{n-1} b_{i,l}d_i^X d_l^X \\
&\ge \alpha|X|_{V_0}^2
\end{aligned}$$

where $\alpha := \frac{1}{2}c$. If B is indefinite or negative definite

$$\begin{aligned}
\mathfrak{a}(X,X) &\ge c\sum_{j=1}^{m}\int_0^1 |X_j'(x)|^2dx - \left|\sum_{i,l=1}^{n-1} b_{i,l}d_i^X d_l^X\right| \\
&\ge c\sum_{j=1}^{m}\int_0^1 |X_j'(x)|^2dx - 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2\sum_{j=1}^{m}\int_0^1 |X_j'(x)|^2dx \\
&= \left(c - 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2\right)\sum_{j=1}^{m}\int_0^1 |X_j'(x)|^2dx \\
&\ge \alpha|X|_{V_0}^2,
\end{aligned}$$

and we obtain the coercivity property taking e.g. $\alpha := \frac{1}{2}\left(c - 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2\right) > 0$.
Symmetry for the form $\mathfrak{a}$ is a direct consequence of its definition, while its closedness follows from the completeness of V_0. □

Remark 3.4. To the best of our knowledge, no reference treats of the relation between the diffusion/convection's coefficients $c_j, b_{i,l}$, the complexity of the network in term of the eccentricity $\epsilon(\mathsf{v}_n)$ (the maximal distance from the Dirichlet condition) and the dissipativity of the system (i.e. coercivity of the associated form). As a matter of fact it seems us to be reasonable that the energy's balance of the system depends on the diffusion velocities c_j on the edges and on the convective velocities $b_{i,l}$ on the nodes. In fact, Robin boundary conditions like (5) allow accumulation or dissipation of potential in a ramification point v_i according to the signs of the convective coefficients $b_{i,l}$ for $l = 1, ..., n-1$. On the other hand, the form of the coercivity constant α implies that the more its eccentricity $\epsilon(\mathsf{v}_n)$ is large and its vertices are numerous, the more its diffusions along the edges have to be fast in order to be dissipative.

In the light of Proposition 3.3 it is possible to characterize generation and spectral properties for the associated operator. In particular (see e.g. [25, Prop. 1.51, Thm. 1.52]) one can prove, in the the general case of B indefinite, the following.

Corollary 3.5. *If $c - 2^{\epsilon(\mathsf{v}_n)-1} b(n-1)^2 > 0$, the operator associated with $\mathfrak{a}$ is densely defined, self-adjoint, sectorial and resolvent compact, hence it generates an analytic, compact, contractive and uniformly exponentially stable C_0-semigroup $(T(t))_{t \geq 0}$ on $\mathcal{X}_2$. Furthermore $(T(t))_{t \geq 0}$ extrapolates to a family of contractive semigroups $(T_p(t))_{t \geq 0}$ on $\mathcal{X}_p := [L^p(0,1)]^m$, $p \in [1, \infty]$ which are strongly continuous for $p \in [1, \infty)$ and analytic for $p \in (1, \infty)$.*

The following Lemma states that the operator associated with $\mathfrak{a}$ is actually A and then by Corollary 3.5 the abstract Cauchy problem (10) is well defined.

Lemma 3.6. *The operator associated with the form $\mathfrak{a}$ is $(A, D(A))$ defined in (8), (9).*

Since we are interested in the nonlinear stochastic problem obtained considering equations (2), (3), (4), (5) and (6), we perturb the deterministic linear system (10) by means of a nonlinear term and of a Gaussian noise in multiplicative form. We set $F : [0, T] \times \mathcal{X}_2 \to \mathcal{X}_2$ such that

$$F(t, X)(\cdot) = \Big(f_j(t, \cdot, X_j(t, \cdot)) \Big)_{j=1,\ldots,m} \qquad \text{for all } X \in \mathcal{X}_2$$

and $G : [0, T] \times \mathcal{X}_2 \to L(\mathcal{X}_2)$ such that

$$[G(t, X)Y](\cdot) = \Big(g_j(t, \cdot, X_j(t, \cdot)) Y_j(\cdot) \Big)_{j=1,\ldots,m} \qquad \text{for all } Y \in \mathcal{X}_2.$$

Finally

$$W(t) = \Big(W_j(t) \Big)_{j=1,\ldots,m}$$

is a cylindrical Wiener process taking values in $\mathcal{X}_2$. With the above notations, we can rewrite our problem in the following abstract form

$$\begin{cases} dX_t = [AX_t + F(t, X_t)]dt + G(t, X_t)dW_t, & t \geq 0 \\ X(0) = X^0. \end{cases} \tag{15}$$

Assumptions 2.2 together with the Corollary 4.4 given in the next section, ensure well posedness in mild sense for (15). In particular we have

$$\|T(t)G(t,x)\|_{HS} \leq K(t)(1 + |x|_{\mathcal{X}_2}), \quad t > 0, \ x \in \mathcal{X}_2,$$

with $K(t) \simeq t^{-\frac{1}{4}}$, see [8, Hypothesis 5.1(iii)] for details.

Theorem 3.7. *Assume that coefficients in (15) satisfy Assumptions (2.2). Then for every $p \in [2, \infty)$ there exists a unique process $X \in L^p(\Omega; C([0,T]; \mathcal{X}_2))$ that is a mild solution of the equation* (15) *(mild solution being understood in the sense of* [8]*).*

See [8, Theorem 5.3.1] for a proof.

4. Gaussian estimates

In this section we derive our main result extending that one obtained in [21] to the case of non-local boundary conditions, with respect to the existence of a Gaussian upper bound for heat equations on a network. The basic point is that the particular class of functions W constructed in [21, §4], in order to apply the so-called Davies' trick, does not permit to consider non-diagonal matrix B. This is equivalent to restrict oneself to the case of local node conditions. In fact, in defining the bilinear form a^ρ (analogous to the one defined in (17)), it appears a term which depends on the difference between the boundary values of the functions belonging to the space W. Such a difference can not be uniformly (with respect to $\rho \in \mathbb{R}$ and $\phi \in W$) dominated, hence the form a^ρ is not uniform accretive (essential property to derive a Gaussian estimate following the approach in [2]). Our approach is based on the following idea: we shall define a functional space W_{bc} that contains functions with constant values on the nodes and that defines on $\mathbb{R}^n$ a metric equivalent to that Euclidean one, still being in the framework of the Davies' trick.

Theorem 4.1. *If* $c - 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2 > 0$ *then the semigroup* $(T(t))_{t\geq 0}$ *has a Gaussian upper bound. More precisely* $(T(t))_{t\geq 0}$ *is isometric to a* C_0*-semigroup* $(\tilde{T}(t))_{t\geq 0}$ *on* $L^2(0,m)$ *and there exists a kernel* $\tilde{k}(t,\cdot,\cdot) \in L^\infty((0,m)\times(0,m))$ *satisfying*

$$|\tilde{k}(t,x,y)| \leq \beta t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{\sigma t}} e^{\eta t}$$

for Lebesgue almost all $x, y \in (0,m)$ *and all* $t > 0$ *(where* $\beta, \sigma > 0$*,* $\eta \in \mathbb{R}$ *are constants) such that*

$$[\tilde{T}(t)f](x) = \int_0^m \tilde{k}(t,x,y)f(y)dy$$

for Lebesgue almost all $x \in (0,m)$ *and for all* $t > 0$*, for all Borel measurable* $f \in L^2(0,m)$*.*

We provide here a detailed proof by suitably modifying an approach first formulated in [2] (see also [21] and [22]).

Proof. The proof is divided in three steps.
Step 1: Definition of an isometry between $[L^2(0,1)]^m$ *and* $L^2(0,m)$*.*
Let us define a one-to-one mapping U from $[L^2(0,1)]^m$ onto $L^2(0,m)$ in the following way: for given functions $L^2(0,1) \ni f_j : [0,1] \to \mathbb{R}$, $j = 1,...,m$, we define a function $L^2(0,m) \ni Uf : [0,m] \to \mathbb{R}$ by

$$Uf(x) := f_j(x-j+1) \qquad \text{if } x \in (j-1,j),$$

endowed with the natural norm in $X_2 := L^2(0,m)$ defined by

$$|Uf|_{X_2} := \left(\sum_{j=1}^m \int_0^1 |f_j(x)|^2 dx \right)^{\frac{1}{2}}.$$

By the isomorphism U we can introduce the similar semigroup $\tilde{T}(t) := UT(t)U^{-1}$ defined on X_2. This "stretching" permits us to embed the problem in the right framework for proving Gaussian estimates: that one of a C_0-semigroup defined on a Hilbert space of the form $L^2(I)$ for some bounded open set $I \subset \mathbb{R}$ (see e.g. [9, §3.2] and [1, §13.1]).
Step 2: Davies' method for Gaussian upper bounds.
Our next purpose is to derive a Gaussian upper bound for the semigroup $\tilde{T}(t)$ by uniform ultracontractivity of certain semigroups $\tilde{T}^\rho(t)$ obtained by a suitable

perturbation. Then we define a slight modification of the functional space defined in the classical Davies' trick (see e.g. [2, 21]), due to the presence of boundary conditions and continuity assumptions on the nodes. Let

$$W_{bc} = \left\{ \psi \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}) : \begin{array}{l} |\psi'|_\infty \le 1, |\psi''|_\infty \le 1, \psi(0) = 0 \text{ and} \\ \psi \text{ takes a constant value for } x = 1, ..., m \end{array} \right\},$$

where, without loss of generality, we stretch the network in such a way to obtain $\mathsf{v}_n \equiv 0$. Then

$$d(x,y) := \begin{cases} \sup\{|\psi(x) - \psi(y)| : \psi \in W_{bc}\} & \text{if } (x,y) \in \mathbb{R}^2 \setminus \{0, ..., m\} \times \{0, ..., m\}, \\ |x-y| & \text{if } (x,y) \in \{0, ..., m\} \times \{0, ..., m\} \end{cases}$$

defines a metric on $\mathbb{R}$ which is actually equivalent to the Euclidean distance (see e.g. [26, pp. 200-202] for the proof). Previous equivalence is the basic ingredient for the application of the Davies' method.
Taken $\rho \in \mathbb{R}$ and $\psi \in W_{bc}$, let us define the C_0-semigroup $\tilde{T}^\rho(t)$ on X_2 by

$$\tilde{T}^\rho(t) f = e^{-\rho\psi} \tilde{T}(t)(e^{\rho\psi} f) \qquad \text{for all } f \in L^2(0,m)$$

and for $p \in [1, \infty]$ we define $X_p := L^p(0,m)$.

Theorem 4.2 (Davies' trick)**.** *The following assertions are equivalent:*

(i) *There exist $M > 0$, $\eta \in \mathbb{R}$ such that*

$$|\tilde{T}^\rho(t)|_{\mathcal{L}(X_1, X_\infty)} \le M e^{\eta(1+\rho^2)t} \cdot t^{-\frac{1}{2}} \tag{16}$$

for all $\rho \in \mathbb{R}$, $\psi \in W_{bc}$ and $t > 0$;

(ii) *$\tilde{T}(t)$ has a Gaussian bound.*

Proof. See [1, Thm. 13.1.4]. □

In the relevant cases of semigroups generated by some differential operators associated with a sesquilinear form $\mathfrak{a}$ with domain V, ultracontractivity estimates are usually proved by showing that V is continuously embedded in some space $L^p(D)$, and this is in turn usually accomplished as application of some Sobolev inequality. However, this procedure can only be performed if D is in fact an open subspace of $\mathbb{R}^n$ with $n \ge 2$ (see e.g. [2]). If instead one is interested in 1-dimensional applications (e.g. to "stretched" networks), then a method based on Nash inequality is more suitable and has been first proved in [11].

Step 3: Nash inequality.

In order to prove the ultracontractivity estimate (16) for $\tilde{T}^\rho(t)$ we first define an isometric semigroup on $\mathcal{X}_2$ as follows

$$T^\rho(t)f := U^{-1}\tilde{T}^\rho(t)Uf = (U^{-1}e^{-\rho\psi})T(t)(U^{-1}e^{\rho\psi})f \qquad \text{for all } f \in \mathcal{X}_2$$

where

$$[(U^{-1}e^{\pm\rho\psi})f](x) = \begin{pmatrix} e^{\pm\rho\psi_1(x)}f_1(x) \\ \vdots \\ e^{\pm\rho\psi_m(x)}f_m(x) \end{pmatrix} \quad \text{and } \psi_j(\cdot) = \psi(\cdot)|_{[j-1,j]}, \text{ for } j = 1, ..., m$$

is considered as a multiplication operator on $\mathcal{X}_2$. Given $\rho \in \mathbb{R}$ and $\psi \in W_{bc}$, the semigroup $T^\rho(t)$ is generated by the operator

$$A^\rho = (U^{-1}e^{-\rho\psi})A(U^{-1}e^{\rho\psi})$$

with domain

$$D(A^\rho) = \{f \in \mathcal{X}_2 : (U^{-1}e^{\rho\psi})f \in D(A)\}.$$

Note that exploiting the boundary conditions in the definition of the functional space W_{bc}, we have that $(U^{-1}e^{\pm\rho\psi})f \in V_0$ for every $\psi \in W_{bc}$, $\rho \in \mathbb{R}$ and $f \in V_0$. Therefore a direct computation shows that the bilinear form $\mathfrak{a}^\rho : V_0 \times V_0 \to \mathbb{R}$, associated with the operator A^ρ, is defined by

$$\begin{aligned}
\mathfrak{a}^\rho(f,g) &= \mathfrak{a}((U^{-1}e^{\rho\psi})f, (U^{-1}e^{-\rho\psi})g) \\
&= \sum_{j=1}^m \int_0^1 c_j(x)f_j'(x)g_j'(x)dx \\
&\quad + \rho\sum_{j=1}^m \int_0^1 c_j(x)\psi_j'(x)(f_j(x)g_j'(x) - f_j'(x)g_j(x))dx \\
&\quad - \sum_{j=1}^m \int_0^1 (\rho^2 c_j(x)\psi_j'(x)^2 + p_j(x))f_j(x)g_j(x)dx + \sum_{i,l=1}^{n-1} b_{i,l}d_i^f d_i^g
\end{aligned} \tag{17}$$

for all $f, g \in V_0$.

Let us note that the ultracontractivity estimate (16) for $\tilde{T}^\rho(t)$ is equivalent to the following one for $e^{-\eta(1+\rho^2)t}T^\rho(t)$ thanks to isomorphism U:

there exist $M > 0$, $\eta \in \mathbb{R}$ such that

$$|e^{-\eta(1+\rho^2)t}T^\rho(t)|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_\infty)} \leq M \cdot t^{-\frac{1}{2}},$$

for all $\rho \in \mathbb{R}$, $\psi \in W_{bc}$ and $t > 0$.
Moreover, a simple computation shows that the linear operator $A^\rho - \eta(1+\rho^2)I$ generates the product semigroup $e^{-\eta(1+\rho^2)t}T^\rho(t)$ and it is associated to the bilinear form $\mathfrak{a}^\rho + \eta(1+\rho^2)$ obviously endowed with the same domains as before. Therefore the following result holds.

Lemma 4.3. *If* $c - 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2 > 0$ *then the product semigroup* $e^{-\eta(1+\rho^2)t}T^\rho(t)$ *on* $\mathcal{X}_2$ *associated with* $\mathfrak{a}^\rho + \eta(1+\rho^2)$ *satisfies the estimate*

$$|e^{-\eta(1+\rho^2)t}T^\rho(t)|_{\mathcal{L}(\mathcal{X}_2,\mathcal{X}_\infty)} \leq M \cdot t^{-\frac{1}{4}}, \qquad t > 0, \tag{18}$$

for some constant M.

Proof. By [25, Thm. 6.3] estimate (18) holds if and only if

$$|f|^6_{\mathcal{X}_2} \leq M\left[\mathfrak{a}^\rho(f,f) + \eta(1+\rho^2)|f|^2_{\mathcal{X}_2}\right] \cdot |f|^4_{\mathcal{X}_1}, \qquad \forall f \in V_0. \tag{19}$$

Using also inequalities (12), (13), the constants c defined in Proposition 3.3 and $P := \max_{j=1,..,m}\max_{x\in[0,1]} p_j(x)$, we can prove that $\mathfrak{a}^\rho + \eta(1+\rho^2)$ is coercive:
(20)

$$\begin{aligned}
\mathfrak{a}^\rho(f,f) + \eta(1+\rho^2)|f|^2_{\mathcal{X}_2} =& \sum_{j=1}^m \int_0^1 c_j(x)|f_j'(x)|^2dx \\
&- \sum_{j=1}^m \int_0^1 (\rho^2 c_j(x)\psi_j'^2(x) + p_j(x))|f_j(x)|^2dx \\
&+ \sum_{i,l=1}^{n-1} b_{i,l}d_i^f d_l^f + \eta(1+\rho^2)\sum_{j=1}^m \int_0^1 |f_j(x)|^2dx \\
\geq& \; c\sum_{j=1}^m \int_0^1 |f_j'(x)|^2dx - (\rho^2 C + P)\sum_{j=1}^m \int_0^1 |f_j(x)|^2dx \\
&- 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2\sum_{j=1}^m \int_0^1 |f_j'(x)|^2dx \\
&+ \eta(1+\rho^2)\sum_{j=1}^m \int_0^1 |f_j(x)|^2dx \\
\geq& \; \omega|f|^2_{V_0} \qquad \forall f \in V_0
\end{aligned}$$

for some $\eta \geq \frac{\rho^2 C+P}{1+\rho^2}$, where $\omega := \frac{1}{2}(c - 2^{\epsilon(\mathsf{v}_n)-1}b(n-1)^2) > 0$. Combining the following multiplicative Sobolev inequality (see e.g. [19, §1.4.8, Thm. 1])

$$|f|_{L^2(0,1)} \leq N|f|^{\frac{1}{3}}_{H^1(0,1)} \cdot |f|^{\frac{2}{3}}_{L^1(0,1)} \qquad \text{for all } f \in H^1(0,1),$$

with the coercivity estimate (20) and recalling the equivalence between norms defined in V_0 and $[H^1(0,1)]^m$, we obtain that there exists a constant which we will denote again by M such that

$$\begin{aligned} |f|^2_{\mathcal{X}^2} &\leq M|f|^{\frac{2}{3}}_{V_0} \cdot |f|^{\frac{4}{3}}_{\mathcal{X}^1} \\ &\leq M\left[\mathfrak{a}^\rho(f,f) + \eta(1+\rho^2)|f|^2_{\mathcal{X}^2}\right]^{\frac{1}{3}} \cdot |f|^{\frac{4}{3}}_{\mathcal{X}^1}, \qquad \forall f \in V_0 \end{aligned}$$

which implies (19). □

Symmetry of the form $\mathfrak{a}^\rho + \eta(1+\rho^2)$ implies that condition (19) is satisfied also by its adjoint, then the semigroup $e^{-\eta(1+\rho^2)t}\,T^\rho(t)$ is ultracontractive by duality and in particular (16) holds. Hence Theorem 4.2 implies that $\tilde{T}(t)$ admits a Gaussian bound, so that Theorem 4.1 is proved. □

Corollary 4.4. *Under Assumptions 2.2, the semigroup $(T(t))_{t\geq 0}$ generated by A is an Hilbert-Schmidt operator for every fixed $t > 0$. Moreover*

$$|T(t)|_{L_2(\mathcal{X}_2)} \leq Mt^{-\frac{1}{4}} \qquad \textit{for all } t \in (0,T].$$

Proof. Note that

$$|T(t)|_{L_2(\mathcal{X}_2)} = |\tilde{T}(t)|_{L_2(X_2)} = |\tilde{k}(t,\cdot,\cdot)|_{L^2((0,m)\times(0,m))}.$$

where $X_2 = L^2(0,m)$ and $\tilde{k}(t,\cdot,\cdot)$ is the kernel defined in Theorem 4.1. Then we have

$$\begin{aligned} |T(t)|^2_{L_2(\mathcal{X}_2)} &= \int_0^m \int_0^m |\tilde{k}(t,x,y)|^2 dx\, dy \\ &\leq \int_0^m \int_0^m \beta^2 t^{-1} e^{-2\frac{|x-y|^2}{\sigma t}} e^{2\eta t} dx\, dy \\ &\leq m\beta^2 \sqrt{\pi \frac{\sigma t}{2}}\, e^{2\eta T} \qquad \text{for all } t \in (0,T]. \end{aligned}$$

This implies that $|T(t)|_{L_2(\mathcal{X}_2)}$ is bounded by $Mt^{-1/4}$ when $t \in (0,T]$ and $M = \frac{1}{2}m\beta^2\sqrt{2\pi\sigma}\, e^{2\eta T}$ is a positive constant. □

5. Stochastic Optimal Control Problem

Throughout this section we will use results obtained in [13] where the task of SOCP is studied in the infinite dimensional Kolmogorov-equation (backward/forward) framework. We would like to underline that the following methods allow to use non smooth feedbacks once the weak control formulation stated in [12] is adopted.

Let $\mathcal{H}, \bar{\mathcal{H}}, U$ be Hilbert spaces, then, for every $t_0 \geq 0$ and $x_0 \in \mathcal{H}$, we define an Admissible Control System (ACS) as $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W_t, u\big)$, where

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space,
- $(\mathcal{F}_t)_{t\geq 0}$ is a filtration in it, satisfying the usual conditions [8],
- $\{W_t : t \geq 0\}$ is a cylindrical $\mathbb{P}$-Wiener process with values in $\bar{\mathcal{H}}$ and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$,
- $L^2_{\mathcal{P}}(\Omega \times [t_0, T]; K)$ denotes the space of equivalence classes of processes $Y \in L^2(\Omega \times [t_0, T]; K)$, admitting a predictable version $L^2_{\mathcal{P}}(\Omega \times [t_0, T]; K)$ is endowed with the norm
$$|Y|^2_{L^2_{\mathcal{P}}} = \mathbb{E}\left[\int_{t_0}^{T} |Y|^2_K \, dt\right]$$
- $C_{\mathcal{P}}\left([t_0, T]; L^2(\Omega; K)\right)$ denotes the space of $K-$valued processes Y such that $Y : [t_0, T] \to L^2(\Omega; K)$ is continuous and Y has a predictable modification, endowed with the norm
$$|Y|^2_{C_{\mathcal{P}}} = \sup_{t\in[t_0,T]} \mathbb{E}\left[|Y_t|_K\right]^2$$
- $u \in L^2_{\mathcal{P}}(\Omega \times [t_0, T]; U)$ satisfies the constraint: $u(t) \in \mathcal{U}$, $\mathbb{P}$-a.s. for a.a. $t \in [t_0, T]$, where $\mathcal{U}$ is a fixed bounded subset of U.

To each ACS there is associated the mild solution $X^u \in C_{\mathcal{P}}([t_0, T]; L^2(\Omega; \mathcal{H}))$ of the state equation

$$\begin{cases} dX^u_t &= [AX^u_t + F(t, X^u_t)]\, dt + G(t, X^u_t)R(t, X^u_t)u(t)dt + G(t, X^u_t)dW_t, \\ X^u_{t_0} &= x_0 \in \mathcal{H}\ , \end{cases} \tag{21}$$

where $t \in [t_0, T]$ and the superscript u indicates the dependence of the mild solution on the particular admissible control $u(t)$.

The structure of (21), in particular with respect to the terms G and R, is discussed in [13].

Here we treat the case of a *finite time horizon* SOCP, i.e. we want to minimize, over all ACS, a cost functional of the following general form:

$$J(t_0,x_0,u) := \mathbb{E}\left[\int_{t_0}^{T} g(s,u(s),X_s^u)ds + \phi(X_T^u)\right] \tag{22}$$

in the time interval $[t_0,T]$, where the function g, resp. ϕ, represents the *running cost function*, resp. the *terminal cost*.

The Hamiltonian function associated to the above problem is defined as follows

$$\psi_0(t,x,p) := \inf\left\{g(t,x,u) + \langle p,u\rangle : u \in \mathcal{U}\right\}\ ,\ \forall (t,x,p) \in [0,T]\times\mathcal{H}\times U\ . \tag{23}$$

For a detailed description of the assumptions on the coefficients in (21) and (22) needed to have existence and uniqueness of the optimal control, we refer the reader to [13], Hyp. (7.1). Here we limit ourselves to explicit the hypothesis on the function G since these are the ones that will be verified applying our estimates in 4.4.

Hypothesis 5.1.
The map $G : [0,T]\times\mathcal{H} \to L(\bar{\mathcal{H}},\mathcal{H})$ is such that $\forall v \in \bar{\mathcal{H}}$ the map $Gv : [0,T]\times\mathcal{H} \to \mathcal{H}$ is measurable, $e^{sA}G(t,x) \in L(\bar{\mathcal{H}},\mathcal{H})$ for every $s > 0$, $t \in [0,T]$ and $x \in \mathcal{H}$, and

$$\begin{cases} \left|e^{sA}G(t,x)\right|_{L_2(\bar{\mathcal{H}},\mathcal{H})} \le Ls^{-\gamma}(1+|x|), \\ \left|e^{sA}G(t,x) - e^{sA}G(t,y)\right|_{L_2(\bar{\mathcal{H}},\mathcal{H})} \le s^{-\gamma}|x-y|\ , \\ |G(t,x)|_{L(\bar{\mathcal{H}},\mathcal{H})} \le L(1+|x|), \end{cases} \tag{24}$$

moreover

$$e^{sA}G(t,\cdot) \in \mathcal{G}^1(\mathcal{H}, L_2(\bar{\mathcal{H}},\mathcal{H}))$$

for $s > 0, t \in [0,T]$, $x,y \in \mathcal{H}$, $L > 0$, $\gamma \in \left[0,\frac{1}{2}\right)$.

By Th. 6.2 in [13], the following Hamilton-Jacobi-Bellman equation associated to (21), (22) for $t \in [0,T], x \in \mathcal{H}$:

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathcal{L}_t[v(t,\cdot)](x) = \psi(t,x,v(t,x),G(t,x)^\star\nabla_x v(t,x)), \\ v(T,x) = \phi(x), \end{cases} \tag{25}$$

admits a unique mild solution. This implies the following (cfr. Th 7.2 in [13]):

Theorem 5.2. *For all ACS we have $J(t_0, x_0, u) \geq v(t_0, x_0)$ and the equality holds iff the following feedback law is verified by u, X^u*

$$u(s) = \Gamma(s, X^u_s, R(s, X^u_s)^\star G(s, X^u_s)\nabla_x v(s, X^u_s)) \text{ }, \mathbb{P} - a.s. for\ a.a. s \in [t_0, T] \tag{26}$$

Moreover there exists at least an ACS for which (26) holds and the associated closed loop equation

$$\begin{cases} d\bar{X}_s = A\bar{X}_s ds + G(s, \bar{X}_s)R(s, \bar{X}_s)\Gamma(s\bar{X}_s, R(s, \bar{X}_s)^\star G(s, \bar{X}_s)^\star \nabla_x v(s, \bar{X}_s))ds + \\ \qquad + F(s, \bar{X}_s)ds + G(s, \bar{X}_s)dW_s, \qquad s \in [t_0, T], \\ \bar{X}_{t_0} = x_0 \in \mathcal{H} \end{cases} \tag{27}$$

admits a solution s.t. if $\bar{u}(s) = \Gamma(s, \bar{X}_s, R(s, \bar{X}_s)^\star G(s, \bar{X}_s)^\star \nabla_x v(s, \bar{X}_s))$ then $(\bar{u}, \bar{X})$ is optimal for the SOCP.

Note that in our case $\mathcal{H} = \bar{\mathcal{H}} = \mathcal{X}_2$, moreover Assumptions 2.2, Corollary 3.5 and Lemma 3.6 imply that

- the operator A is the generator of a strongly continuous semigroup e^{tA} for $t \geq 0$, in $\mathcal{X}_2$,
- $F : [0, T] \times \mathcal{X}_2 \to \mathcal{X}_2$ is a measurable, uniformly Lipschitz map with Lipschitz constant $L := \max_{j=1,\dots,m} \left\{L_j, \tilde{L}_j\right\}$ in the last component and uniformly bounded by a positive constant $K := \max_{j=1,\dots,m} \left\{K_j, \tilde{K}_j\right\}$,
- $G : [0, T] \times \mathcal{X}_2 \to L(\mathcal{X}_2)$ is a measurable, uniformly Lipschitz map with Lipschitz constant $L := \max_{j=1,\dots,m} \left\{L_j, \tilde{L}_j\right\}$ in the last component, uniformly bounded by a positive constant $K := \max_{j=1,\dots,m} \left\{K_j, \tilde{K}_j\right\}$, $e^{sA}G(t, X) \in L(\mathcal{X}_2)$ for every $s > 0$, $t \in [0, T]$ and $X \in \mathcal{X}_2$, moreover satisfies a sublinear growth condition of this type

$$|G(t, X)|_{L(\mathcal{X}_2)} \leq (L \wedge K)(1 + |X|_{\mathcal{X}_2}) \qquad \text{for } t \in [0, T], X \in \mathcal{X}_2.$$

Finally conditions on G stated in 5.1, are veryfied by the Gaussian-type estimate 4.4 for the semigroup $(T(t))_{t\geq 0}$ generated by A which implies that the Hilbert-Schmidt norm of the semigroup e^{sA} is bounded as follows

$$|e^{sA}|_{L_2(\mathcal{X}_2)} \leq M s^{-\gamma}, \tag{28}$$

for some $\gamma \in [0, 1/2)$ and positive constant M.

6. Conclusions

Our approach to the study of a class of reaction-diffusion type equations on a finite network motivated by neurological reasons allows the generalization of previously proven results for Gaussian estimates. Namely we prove Gaussian upper bounds for heat equations semigroup on finite networks extending [21] to the non-local conditions case. This results is used to apply those obtained [13] for SOCP which requires that the Hilbert-Schmidt norm of the semigroup has a bound of the form: $|e^{tA}|_{L_2(\mathcal{X}_2)} \leq Ct^{-\gamma}$ for $\gamma \in [0, 1/2)$, for some constant $C > 0$, as $t \downarrow 0$. In [3] this is obtained for a particular heat equation. The authors treat a reaction diffusion problem on an interval coupled with stochastic dynamical boundary conditions in the extremes as an abstract evolution equation on a product Hilbert space $\mathcal{X}$ (as suggested in [6]). By an analogous of characteristic polynomial for matrix in finite dimensions (see [10, 24] for a survey on spectral properties of unbounded matrix operators), they are able to explicitly calculate the eigenvalues of the leading operator A showing that they are asymptotic to $-\pi^2 n^2$, $n \in \mathbb{N}$, i.e. $|e^{tA}|_{L_2(\mathcal{X})} \leq Ct^{-1/4}$, for some constant $C > 0$, as $t \downarrow 0$. In the case of a system of coupled evolution equations on a network, it is not easy in general to characterize so precisely the spectral properties of the operator, even when the boundary conditions are static. So an alternative way is to prove Gaussian estimates for the generated semigroup, which in fact imply a stronger condition. Note that even in the one dimensional case of an interval, the dynamics of the boundary put the abstract evolution on a product space, so even what is the correct definition of Gaussian estimates is not any longer so clear. In a future work we aim at deriving some kind of Gaussian bounds for diffusion problems on networks with stochastic dynamical boundary conditions both in one and more dimensions.

Another generalization we will be interested in concerns the possibility to take a nonlinear reaction term with polynomial growth at infinity (therefore only locally Lipschitz) in order to include some biological models in neurobiology. In particular if we take the non-linear term in (2) to be a cubic

$$F(x) = x(1-x)(x-\xi)$$

we obtain the FitzHugh-Nagumo model for diffusion of electrical potential in neurons, where $0 < \xi < 1$ represents the voltage threshold (see e.g. [27] for a survey and [5] for a deterministic optimal control problem). Perhaps methods of [7] and [17] for treating stochastic optimal control problems on Banach spaces with non-Lipschitz coefficients will be useful in this context.

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