UNIVERSITY OF TRENTO - ITALY

PH.D. IN MATHEMATICS

XVII CYCLE

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On some P.D.E.s with hysteresis

Supervisor

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Introduction

The aim of this thesis is to present some results concerning some new classes of P.D.E.s containing a continuous hysteresis operator. Even if in the most frequent applications the operators involved in our treatment will be also rate independent, our results turn to be valid for the more general class of memory operators. Throughout the text, we focus our attention on the well-posedness of our model problems, dealing, when possible, with several kind of boundary conditions; in the last chapter, we present instead a result of asymptotic behaviour.

CHAPTER 1 provides some introductory material. We briefly explain what is hysteresis and its main features by means of a simple example and immediately after we introduce the concept of hysteresis operator, pointing out its basic properties.
The great part of the chapter is then devoted to the presentation of the most common examples of hysteresis operators, together with their basic properties; we will refer to this part throughout the whole manuscript.

From CHAPTER 2 we start presenting our original results. In Chapter 2 we study a class of P.D.E.s whose model equation is represented by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \triangle \left(\frac{\partial u}{\partial t} + \overline{\mathcal{G}}(u) \right) = f \qquad \text{in } \Omega \times (0, T),$$

where Ω is an open bounded set of $\mathbb{R}^2$, $\triangle$ is the Laplace operator, $\overline{\mathcal{G}}$ is a suitable hysteresis operator and f is a given function. This equation is obtained by coupling together:
- *the Maxwell equations and the Ohm law*, which are considered under severe restrictions on the geometry of the system, from what comes out the scalar character of our model equation and the presence of the euclidean space $\mathbb{R}^2$ instead of the expected $\mathbb{R}^3$;
- *the following constitutive relation*

$$H = \overline{\mathcal{G}}(B) + \gamma \frac{\partial B}{\partial t}$$

between B and H (respectively the magnetic induction and the magnetic field, which are scalars after the assumptions we made). Here $\overline{\mathcal{G}}$ is a suitable hysteresis operator and $\gamma > 0$ is a given constant. This relation can be for example obtained by the combination in series of a ferromagnetic element with hysteresis and a conducting solenoid filled with a paramagnetic core.

The presence of $\mathbb{R}^2$ instead of the expected $\mathbb{R}^3$ and the scalar character of our model equation restrict the field of applications essentially to planar problems for scalar variables.

Moreover, even if the model is meaningful for the choice $\Omega \subset \mathbb{R}^2$, then our computations are still valid in the more general setting $\Omega \subset \mathbb{R}^N$, with $N \geq 1$.

First of all we introduce a weak formulation in Sobolev spaces for the Cauchy problem associated to the previous model equation and under suitable assumptions on the hysteresis operator we get an existence and uniqueness result for the solution of our model problem.

The proof of this result is carried on by means of a technique which is based on the contraction mapping principle. Several difficulties arise due to the choice of the unusual functional setting: in fact the problem is set within the frame of a non-classical Hilbert triplet

$$L^2(\Omega) \subset H^{-1}(\Omega) \equiv (H^{-1}(\Omega))' \subset (L^2(\Omega))'$$

with continuous and dense injections, in which the role of the pivot space is played by the Sobolev space $H^{-1}(\Omega)$ endowed with a scalar product chosen ad hoc. Examples of these changes of pivot space are very few in literature and are always employed in other contexts.

Once that the uniqueness result is proved, then we easily deduce the Lipschitz continuous dependence of the solution on the data. After that we obtain a further regularity result which is based on a classical characterization of the Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$. We conclude this chapter by proving the consistency of our results for a particular choice of the hysteresis operator $\overline{\mathcal{G}}$.

In CHAPTER 3 we study a class of parabolic P.D.E.s whose model equation is

$$\frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) + \vec{v} \cdot \nabla(u + \overline{\mathcal{F}}(u)) - \triangle u = f \qquad \text{in } \Omega \times (0,T),$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\triangle$ is the Laplace operator, $\vec{v} : \Omega \times (0,T) \to \mathbb{R}^N$ is known and f is a given function.

This class of P.D.E.s is different from the model studied in [39], Chapter IX, due to the presence of the convective term $\vec{v} \cdot \nabla(u + \overline{\mathcal{F}}(u))$. Unsaturated water flow through a porous medium leads to an equation of this type, if we include a saturation versus pressure constitutive relation with hysteresis and a term of transport, together with severe restrictions on the hydraulic conductivity which is assumed to be a constant (neglecting the dependence on the saturation).

In the first part of the chapter we deal with Dirichlet boundary conditions, in the second part we introduce other nonlinear conditions on the boundary of Ω.

Also in this case we introduce a weak formulation in Sobolev spaces for a Cauchy problem associated to the previous model equation and under suitable assumptions on the hysteresis operator $\overline{\mathcal{F}}$ we find an existence result for solutions of our model problem. The technique we use is based on approximation by implicit time discretization, a priori estimates and passage to the limit by compactness. This approximation procedure is quite convenient in the analysis of equations that include a hysteresis operator, as in any time-step we have to solve a stationary problem in which the hysteresis operator is reduced to the superposition with a nonlinear function.

As the equation considered is quasilinear, we are not able to prove a uniqueness result when $\overline{\mathcal{F}}$ is a generic hysteresis operator; moreover also the techniques based on Hilpert's inequality (see [18]), which only hold for a restricted class of operators, apparently cannot be applied in our case due to the presence of the convective term. Nevertheless we are able to prove a uniqueness result for some particular choices of the operator $\overline{\mathcal{F}}$ using a method "ad hoc", which exploits the properties of the hysteresis operator we choose and the features of our specific model equation. Also in this case we prove the consistency of our results in the particular situation when $\overline{\mathcal{F}}$ is a Preisach operator, a very important case for the applications.

We analyse moreover the dependence of the solution from the data: the theorem we prove differs from the more standard ones (see for example [39], Section IX.1) for the weaker assumptions which provide a slightly weaker thesis, enough however to pass to the limit. The idea contained in the proof is new and uses the order preserving property of the hysteresis operators involved and the uniform convergence in time of the sequence of our approximate solutions (pointwise convergence would not be enough for our purposes).

We conclude this first part of the chapter by showing another way of proving existence of solutions of our model equation; the method we use is the classical "hyperbolic regularization method:" we add the term $\varepsilon \dfrac{\partial^2 u}{\partial t^2}$ in front of our parabolic model equation, transforming it into an hyperbolic-type one; then we find a solution u_ε of this modified equation. If we let $\varepsilon \to 0$, then we recover that $u_\varepsilon \to u$ in some suitable topology, where u is a solution of our original model equation.

In the second part of the chapter we deal with the same model equation but this time we change the boundary conditions; while in our first analysis we considered Dirichlet boundary conditions, this time we have a condition of nonlinear flux on a subset $\Gamma_2 \subset \Gamma$ of the boundary of Ω, which can be for example written as

$$\nabla u \cdot \vec{\nu} = [\vec{v} \cdot \vec{\nu}]\,(u + \overline{\mathcal{F}}(u)) - g(u) \qquad \text{on } \Gamma_2,$$

where $\vec{\nu}$ denotes the unit outer normal vector to Γ and $g : \mathbb{R} \to \mathbb{R}$ is a given function; on the remaining part of the boundary we assume to have Dirichlet boundary conditions. Our aim is to find assumptions on g in order to recover an existence and uniqueness theorem for the

Cauchy problem associated to the previous model equations. Also in this situation, the right tool to prove the existence result is the time discretization scheme; however a certain amount of technical difficulties arises, mainly when dealing with some terms defined on the boundary of Ω, so that we have to use a refined interpolation argument when passing to the limit. A uniqueness result together with the Lipschitz continuous dependence on the data is instead established only for a suitable restricted class of hysteresis operators.

In CHAPTER 4 we study two systems of P.D.E.s containing a continuous hysteresis operator, more precisely we deal with

$$\begin{cases} \dfrac{\partial}{\partial t}(u+w) - \triangle u = f \\ \gamma \dfrac{\partial w}{\partial t} + w = \overline{\mathcal{F}}(u) \end{cases} \qquad \text{in } \Omega \times (0,T)$$

and with

$$\begin{cases} \dfrac{\partial}{\partial t}(u+w) - \triangle u = f \\ w = \overline{\mathcal{F}}\left(u - \gamma \dfrac{\partial w}{\partial t}\right) \end{cases} \qquad \text{in } \Omega \times (0,T),$$

where $\overline{\mathcal{F}}$ is a continuous hysteresis operator, Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\triangle$ is the Laplace operator, f is a given function and γ is a constant greater than 0. Both systems arise in the context of electromagnetic processes and are characterized by the fact that the constitutive relation $w = \overline{\mathcal{F}}(u)$ is perturbed in two different ways, respectively $\gamma\, w_t + w = \overline{\mathcal{F}}(u)$ and $w = \overline{\mathcal{F}}(u - \gamma\, w_t)$, due to the presence of the relaxation term $\gamma\, w_t$.
For the first system we get an existence result; we choose to approximate our model equation in time but the presence of this new constitutive relation leads to a certain amount of technical difficulties, above all concerning the a priori estimates which are carried on through several steps.
The second system is equivalent to the following equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t} + \frac{\partial \overline{\mathcal{G}}(w)}{\partial t} - \triangle\left(\overline{\mathcal{G}}(w) + \gamma \frac{\partial w}{\partial t}\right) = f \qquad \text{in } \Omega \times (0,T)$$

where $\overline{\mathcal{G}} = \overline{\mathcal{F}}^{-1}$ provided it exists. We choose to solve this model problem in the frame of the time discretization scheme, even if this time the stationary problem which we have to face once that our model equation is approximated, requires the choice of an adequate functional setting. That's why we choose again to work with the Hilbert triplet

$$L^2(\Omega) \subset H^{-1}(\Omega) \equiv (H^{-1}(\Omega))' \subset (L^2(\Omega))'$$

with continuous and dense injections, in which the role of the pivot space is played by the Sobolev space $H^{-1}(\Omega)$, instead of the canonical $L^2(\Omega)$, endowed with a suitable scalar product.

In both cases nothing is known concerning uniqueness of solutions; moreover it is also difficult to establish what happens to these solutions when the parameter γ goes to zero; this suggests the idea of looking for other approaches to find existence results for our model problems. The analysis of these possibilities is still work in progress.

In Chapter 5 we study instead the asymptotic behaviour for the solution of an initial and boundary value problem associated to the following model equation

$$\frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for } (x,t) \in (0,1) \times (0,+\infty),$$

where $\overline{\mathcal{F}}$ is a continuous hysteresis operator; here the interval $(0,1)$ can be replaced with any other open bounded interval of $\mathbb{R}$; in any case our treatment will take place only in one space dimension.

For the same model equation, we find in literature a known result dealing with Dirichlet boundary conditions (see [39], Section IX.4, Proposition 4.1); here we deal instead with Neumann boundary conditions.

First of all we check that there exists a unique solution for a suitable Cauchy problem related to the previous model equation, which is defined on $(0,1) \times (0,+\infty)$ and then we prove that the term $\partial_x u$ exponentially decay in $L^2(0,1)$ as $t \to \infty$. This first result can be also obtained working in several space dimensions, with some suitable modifications.

At this point, if $\overline{\mathcal{F}}$ is a Preisach operator, we prove that there exists a constant u_∞ such that $\lim_{t\to\infty} u(x,t) = u_\infty$ for all $x \in [0,1]$. This result, which is proved by means of a careful procedure (only holding in one space dimension), is important because it allows us to conclude that for small amplitude oscillations of $u(x,t)$ around u_∞, the solution does not leave the convexity domain of the Preisach operator. This in turn implies that we can differentiate our (suitably space-discretized) model equation in time, test by $\partial_t u$ and obtain, by the usual convexity argument, an exponential decay for the functions $\partial_t u$ and $\partial_x^2 u$ again in $L^2(0,1)$ as $t \to \infty$.

We only would like to remark that the value u_∞, which is defined as the limit of $u(0,t)$ as t tends to infinity, is a constant and from the computations we did it appears evident that this convergence takes place independently of x. On the other hand, it is known from the general theory of dynamical systems that if the solution asymptotically converges to something, then the limit is an equilibrium of the system, so one cannot expect the limit to depend on x because all equilibria are solutions of the Laplace equation with the homogeneous Neumann boundary conditions, hence constants. This case is quite similar to the case of the linear heat equation without hysteresis and with the homogeneous Neumann boundary conditions. Also here the total energy is conserved but only a part of the initial energy is stored in u, the other stays in the hysteresis term and there is a strong energy exchange between the two during the process. For this reason the convergence proof presents some further difficulties comparing to

the Dirichlet case or the case without hysteresis.
On the other hand, if $\overline{\mathcal{F}}$ is a Prandtl-Ishlinskiĭ operator, using the convexity of the hysteresis loops we can directly differentiate our (approximated) model equation and get at once the exponential decay in $L^2(0,1)$ of the functions $\partial_t u$ and $\partial_x^2 u$.
The results contained in this chapter have been obtained in collaboration with Prof. Pavel Krejčí.

Finally Chapter A contains some complementary results, almost always without proof, which have been used throughout the whole manuscript.

Contents

CHAPTER 1

Hysteresis operators

Hysteresis is a phenomenon that occurs in several and rather different situations: for instance in physics we find it in plasticity, in ferromagnetism, in phase transitions. Hysteresis is also encountered in engineering, in chemistry, in biology and in several other settings.
According to [39] we can distinguish two main features of hysteresis phenomena: the MEMORY EFFECT and the RATE INDEPENDENCE. Here we just want to briefly explain them on a simple example.

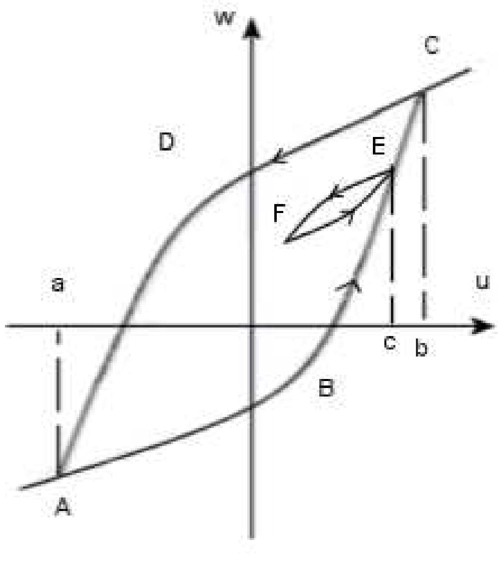

Figure 1.1: Continuous hysteresis loop.

Figure 1.1 describes the state of a system which is characterized by two scalar variables u and w depending in a continuous way on time. We will call them also *input* and *output* of the system. We have the following: if the input increases from a to b then the couple (u, w) moves along the curve ABC, on the other hand, if the input decreases from b to a then the couple input-output stays on the curve CDA. If moreover at a certain instant t such that $a < u(t) = c < b$ the input u inverts its movement, then (u, w) moves into the interior of the region bounded by the major loop $ABCDE$ in a suitable way described by the specific model, for example this can be along the curve EF as in the picture.

This means that at any instant t the value of the output $w(t)$ is not simply determined by the value $u(t)$ of the input at the same instant, but it depends also on the previous evolution of the input u. This is the *memory effect*.

On the other hand we may also require that the path of the couple $(u(t), w(t))$ is invariant with respect to any increasing time homeomorphism, that is there is no dependence on the derivative of u. This property is named *rate independence* and it is this fact that allows us to draw the characteristic pictures of hysteresis in the $(u, w)-$plane, if this did not hold we could not give a graphic representations of the hysteresis loop as the path of the couple would also depend on its velocity.

Even if hysteresis has been known and studied since the end of the eighteenth century, it was only more or less thirty-five years ago that, dealing with plasticity, a small group of Russian mathematicians introduced the concept of *hysteresis operator* and started a systematic investigation of its properties. The pioneers in this new field were certainly Krasnosel'skiĭ and Pokrovskiĭ with their important monograph [23]. From that moment onward many scientists coming also from different areas have contributed to the mathematical study of hysteresis. At this purpose we can certainly quote the recent monographes devoted to this topic, see Brokate and Sprekels [11], Krejčí [25], Mayergoyz [30] and Visintin [39], together with the references therein.

1.1. Hysteresis operators: basic properties

The evolution $u \mapsto w$ we briefly outlined before can be formalized by the introduction of the concept of hysteresis operator. This is what Krasnosel'skiĭ and Pokrovskiĭ did in 1970, dealing with a particular case. This event can be certainly regarded as the beginning of the mathematical theory of hysteresis as the concept of hysteresis operator acting among Banach spaces of time dependent functions is of basic importance in treating the mathematical aspects of several hysteresis models.

In many cases the state of the system is completely described by the couple (u, w) input-output. At any instant t, the output $w(t)$ will depend on the evolution of the input until that time t and also on the initial state of the system. So the initial value $(u(0), w(0))$ or some equivalent information must be specified. As $u(0)$ is already contained in $u_{|[0,t]}$, we say that in these cases the state of the system can be described by an operator of the following type

$$\mathcal{F}: \ \mathrm{Dom}\,(\mathcal{F}) \subset \mathcal{C}^0([0,T]) \times \mathbb{R} \to \mathcal{C}^0([0,T]) \qquad (u, w^0) \mapsto w(\cdot) := [\mathcal{F}(u, w^0)](\cdot). \tag{1.1.1}$$

This is the case, for example, of plays and stops operators (see Section 1.2).
However there are also cases in which the state of the system is not completely characterized by the couple (u, w) but there is also the presence of a variable $\eta \in X$ where X is some suitable metric space; in these situations the state of the system is described by an operator of the following type

$$\mathcal{F}: \ \mathrm{Dom}\,(\mathcal{F}) \subset \mathcal{C}^0([0,T]) \times X \to \mathcal{C}^0([0,T]) \qquad (u, \eta^0) \mapsto w(\cdot) := [\mathcal{F}(u, \eta^0)](\cdot), \tag{1.1.2}$$

where $\eta^0 \in X$ contains all the information about the initial state. This is the case for example of the Prandtl-Ishlinskiĭ operators of play or stop type or the Preisach operators (see Sections 1.3 and 1.5).
Operators of type (1.1.1) and (1.1.2) usually fulfill some properties, useful when dealing with P.D.E.s. We now make explicit some of these properties, which can be written without modifications for both operators of type (1.1.1) and (1.1.2). There are also properties which have to be presented in a different way for the two different types of operators (i.e. the *semigroup property*) but we will not quote them as we will not need them in the following.

We start with the causality property and rate independence property which respectively read

$$\begin{cases} \forall\,(u_1, w^0),\ (u_2, w^0) \in \mathrm{Dom}\,(\mathcal{F}),\ \ \forall\, t \in (0,T], \\ \text{if } u_1 = u_2 \text{ in } [0,t], \text{ then } [\mathcal{F}(u_1, w^0)](t) = [\mathcal{F}(u_2, w^0)](t), \end{cases} \tag{1.1.3}$$

$$\begin{cases} \forall\,(u, w^0) \in \mathrm{Dom}\,(\mathcal{F}),\ \ \forall\, t \in (0,T], \text{if } s : [0,T] \to [0,T] \text{ is an} \\ \text{increasing homeomorphism, then } [\mathcal{F}(u \circ s, w^0)](t) = [\mathcal{F}(u, w^0)](s(t)). \end{cases} \tag{1.1.4}$$

An operator fulfilling (1.1.3) and (1.1.4) is said to be a hysteresis operator.
In the following we will work with hysteresis operators that are continuous in the following sense

$$\begin{cases} \forall\,\{(u_n, w_n^0) \in \mathrm{Dom}\,(\mathcal{F})\}_{n\in\mathbb{N}}, \\ \text{if } u_n \to u \text{ uniformly in } [0,T] \text{ and } w_n^0 \to w^0, \\ \text{then } \mathcal{F}(u_n, w_n^0) \to \mathcal{F}(u, w^0) \text{ uniformly in } [0,T] \end{cases} \tag{1.1.5}$$

or with operators which are order preserving, that is

$$\begin{cases} \forall\,(u_1, w_1^0),\ (u_2, w_2^0) \in \ \mathrm{Dom}(\mathcal{F}),\ \ \forall\, t \in (0,T], \text{if } u_1 \le u_2 \text{ in } [0,t] \text{ and } w_1^0 \le w_2^0, \\ \text{then } \ [\mathcal{F}(u_1, w_1^0)](t) \le \ [\mathcal{F}(u_2, w_2^0)](t). \end{cases} \tag{1.1.6}$$

Moreover for an operator $\mathcal{F}$ it is also natural to require the following property, usually named PIECEWISE MONOTONICITY PRESERVATION (or briefly PIECEWISE MONOTONICITY)

$$\begin{cases} \forall\,(u,w^0)\in \mathrm{Dom}(\mathcal{F}),\ \forall\,[t_1,t_2]\subset[0,T], \\ \text{if } u \text{ is either nondecreasing or nonincreasing in } [t_1,t_2], \text{then so is } \mathcal{F}(u,w^0). \end{cases} \tag{1.1.7}$$

Another interesting property is the PIECEWISE LIPSCHITZ CONTINUITY PROPERTY

$$\begin{cases} \exists\, L_{\mathcal{F}}>0:\ \ \forall\,(v,w^0)\in \mathrm{Dom}(\mathcal{F}),\ \forall\,[t_1,t_2]\subset[0,T] \\ \text{if } v \text{ affine in } [t_1,t_2], \text{ then } |[\mathcal{F}(v,w^0)](t_2)-[\mathcal{F}(v,w^0)](t_1)|\le\ L_{\mathcal{F}}\,|v(t_2)-v(t_1)|. \end{cases} \tag{1.1.8}$$

We remark that, if $\mathcal{F}:\mathrm{Dom}(\mathcal{F})\subset\mathcal{C}^0([0,T])\times X\to\mathcal{C}^0([0,T])$ is a Lipschitz continuous hysteresis operator with Lipschitz constant $L_{\mathcal{F}}$, then it can be proved that $\mathcal{F}$ fulfills (1.1.8). This implication can be justified in the following way: let us take a function v affine in $[t_1,t_2]$ and let us consider a function $\bar{v}:[t_1,t_2]\to\mathbb{R}$ defined as follows: $\bar{v}(t)=v(t_1)$ for all $t\in[t_1,t_2]$. Then, as $\bar{v}$ is constant in $[t_1,t_2]$ then also $\mathcal{F}(\bar{v},w^0)$ is constant in $[t_1,t_2]$ due to the fact we assumed $\mathcal{F}$ to be rate independent; moreover $\mathcal{F}(v,w^0)(t_1)=\mathcal{F}(\bar{v},w^0)(t_1)$ as $\mathcal{F}$ is also a causal operator. These facts imply that

$$\begin{aligned} &|[\mathcal{F}(v,w^0)](t_1)-[\mathcal{F}(v,w^0)](t_2)|\le\ |[\mathcal{F}(v,w^0)](t_1)-[\mathcal{F}(\bar{v},w^0)](t_1)| \\ &\quad+|[\mathcal{F}(\bar{v},w^0)](t_1)-[\mathcal{F}(\bar{v},w^0)](t_2)|+|[\mathcal{F}(\bar{v},w^0)](t_2)-[\mathcal{F}(v,w^0)](t_2)| \\ &\le L_{\mathcal{F}}\,||v-\bar{v}||_{\mathcal{C}^0([0,T])}=L_{\mathcal{F}}|v(t_2)-\bar{v}(t_2)|=\ L_{\mathcal{F}}|v(t_2)-v(t_1)|. \end{aligned}$$

So property (1.1.8) is achieved.

1.2. Scalar play and stop

In this section and in the following ones, we would like to recall some important examples of hysteresis operators; our excursion is far to be deep and complete. We just outline some properties and remarks we will need in the following chapters. For more details on this topic, see the monographs we quoted in the very beginning of the chapter.

1.2.1. The play

The first simple model of hysteresis we consider is a mechanism which is known as PLAY. More precisely we have two elements, A and B which move along a horizontal line with one degree of freedom (see Figure 1.2).

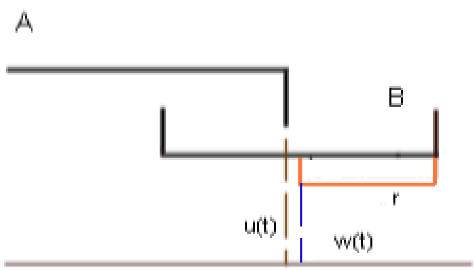

Figure 1.2: Play between two mechanical elements.

The motion of the two elements can be described as follows: the position $w(t)$ of the middle point of element B remains constant as long as the element A, represented by its end-position $u(t)$, moves in the interior region of width $2r$ which is the diameter of the element B. When u hits the boundary of the element B then w moves with the velocity $\dot{w} = \dot{u}$ which is direct outwards. The input-output behaviour is given by the hysteresis diagram which is shown in Figure 1.3.

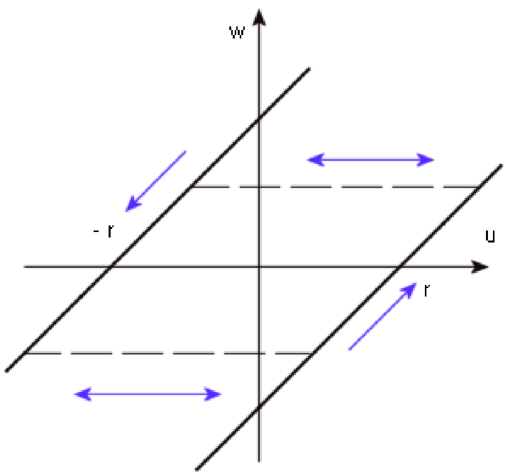

Figure 1.3: Hysteresis behaviour of the mechanical play.

On the other hand this relation $u \mapsto w$ can be also expressed by means of a hysteresis operator. In fact, for any piecewise monotone input function $u : [0, T] \to \mathbb{R}$ the output function $w(t) :=$

$\mathcal{P}_r(u)$ can be defined by induction using the following formula:

$$w(0) = \max\{u(0) - r, \min\{u(0) + r, 0\}\}$$
$$w(t) = \max\{u(t) - r, \min\{u(t) + r, w(t_{n-1})\}\} \qquad \text{for } t_{n-1} < t \le t_n,\ \ 1 \le n \le N$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ is a partition of the time interval $[0, T]$ such that the input function u is monotone on each subinterval $[t_{n-1}, t_n]$. The operator $\mathcal{P}_r$ is called PLAY OPERATOR.

1.2.2. The stop

Let us consider a device constituted by an elastic element put in series with a plastic one (see Figure 1.4).

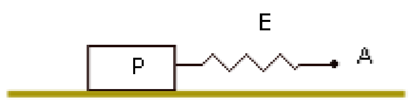

Figure 1.4: Prandtl's model of elasto-plasticity or stop.

It is simple to see that also the relation between the strain and the stress provides another example of hysteresis phenomena (see also Section 1.4). Let us denote by $r > 0$ a fixed threshold (depending on the characteristics of the bodies and on the materials in contact), which represents the yield stress. As long as the modulus of the stress w is smaller than r, the strain u and the stress w are related through the linear Hooke law. In the case when the yield value has been reached by w, then the stress remains constant under a further growth of u. The elastic behaviour is then recovered once that the strain decreases again. Figure 1.5 describes this situation in a very simplified way. Also in this case it is possible to represent the strain-stress relation by means of a hysteresis operator. More precisely, we still consider a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $[0, T]$ such that the input function u is monotone on each subintervals $[t_{n-1}, t_n]$. Then the output function $w(t) := \mathcal{S}_r(u)$ can be defined by induction using the following formula:

$$w(0) = \min\{r, \max\{-r, u(0)\}\}$$
$$w(t) = \min\{r, \max\{-r, u(t) - u(t_{n-1}) + w(t_{n-1})\}\} \qquad \text{for } t_{n-1} < t \le t_n,\ \ 1 \le n \le N.$$

The operator $\mathcal{S}_r$ is called STOP OPERATOR.

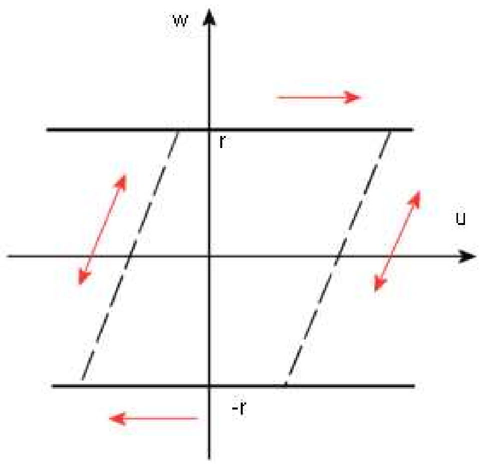

Figure 1.5: Hysteresis behaviour of the stop.

1.2.3. Play and stop operators with general initial value

In Subsections 1.2.1 and 1.2.2 we introduced the play operator $\mathcal{P}_r$ and the stop operator $\mathcal{S}_r$. More in general, for any initial value w^0 we may define for any piecewise monotone input function $u : [0,T] \to \mathbb{R}$ the output function $w(t) := \mathcal{P}_r(u, w^0)$ inductively using the following formula:

$$\begin{aligned} &w(0) = \max\{u(0) - r, \min\{u(0) + r, w^0\}\} \\ &w(t) = \max\{u(t) - r, \min\{u(t) + r, w(t_{n-1})\}\} \qquad \text{for } t_{n-1} < t \leq t_n, \ \ 1 \leq n \leq N. \end{aligned} \tag{1.2.1}$$

The following result (see [11], Example 2.2.13 and Theorem 2.3.2; see also [23], Section 2) holds

Theorem 1.2.1. *For any* $r \geq 0$, *the operator* $\mathcal{P}_r$ *can be extended to a unique Lipschitz continuous operator* $\mathcal{P}_r : \mathcal{C}^0([0,T]) \times \mathbb{R} \to \mathcal{C}^0([0,T])$ *(with Lipschitz constant 1). In addition this operator* $\mathcal{P}_r$ *is causal and rate independent in the sense of* (1.1.3) *and* (1.1.4)*, i.e. it is a hysteresis operator, and moreover it is order preserving and piecewise monotone in the sense of* (1.1.6) *and* (1.1.7)*.*

The same can be done for the stop operator; in fact, for any initial value w^0 we may define for any piecewise monotone input function $u : [0,T] \to \mathbb{R}$ the output function $w(t) := \mathcal{S}_r(u, w^0)$ inductively using the following formula:

$$\begin{aligned} &w(0) = \min\{r, \max\{-r, u(0) - w^0\}\} \\ &w(t) = \min\{r, \max\{-r, u(t) - u(t_{n-1}) + w(t_{n-1})\}\} \qquad \text{for } t_{n-1} < t \leq t_n, \ \ 1 \leq n \leq N. \end{aligned} \tag{1.2.2}$$

The following result is an immediate consequence of Theorem 1.2.1 (see [11], Proposition 2.3.4)

Proposition 1.2.2. *For any $r \geq 0$, the operator $\mathcal{S}_r$ can be extended to a unique Lipschitz continuous operator $\mathcal{S}_r : \mathcal{C}^0([0,T]) \times \mathbb{R} \to \mathcal{C}^0([0,T])$ (with Lipschitz constant 2, which is optimal). In addition it turns out that this operator $\mathcal{S}_r$ is causal and rate independent in the sense of* (1.1.3) *and* (1.1.4)*, i.e. it is a hysteresis operator, and moreover it is piecewise monotone in the sense of* (1.1.7)*.*

It is interesting to notice that the play and the stop operators are closely related, even if the corresponding models describe different situations. The simple relation is the following, which can be checked by direct computation (see also [11], Proposition 2.3.4)

$$\mathcal{P}_r(u, w^0) + \mathcal{S}_r(u, w^0) = I(u) \qquad \forall\, u \in \mathcal{C}^0([0,T]),\ \forall\, w^0 \in \mathbb{R},$$

where I is the identity operator.

1.2.4. Play and stop operators: a different approach

The play and the stop operators can be introduced also in another way. Let us come back to the classical Prandtl's model of elasto-plasticity, i.e. to the stop model. Assume that the device (an elastic and a plastic element in series) is composed by a heavy body connected to a spring which has only one degree of freedom (the horizontal line). If a longitudinal force is applied to the spring (which transmits this force to the body), the elongation u_E is proportional to the force w in the following way: $u_E = \sigma w$ where σ is a positive constant. Moreover, assuming Coulomb's friction law, as we said before there exists a threshold $r > 0$ (depending on the physical characteristics of the body and of the materials in contact) such that $\dot{u}_P = 0$ if $|w| < r$, $\dot{u}_P \geq 0$ if $w = r$ and $\dot{u}_P \leq 0$ if $w = -r$, where u_P is the displacement of the body. This is equivalent to the following variational inequality

$$|w| \leq r \qquad \frac{du_P}{dt}(w - \varphi) \geq 0 \qquad \forall\, \varphi : \ |\varphi| \leq r.$$

Summing up, if we denote by u the displacement of the point A in Figure 1.4 then $u = u_E + u_P$ and the previous considerations yield

$$|w| \leq r \qquad \left(\frac{du}{dt} - \sigma\frac{dw}{dt}\right)(w - \varphi) \geq 0 \qquad \forall\, \varphi : \ |\varphi| \leq r.$$

The analogous thing can be done for the play, we will see in Section 1.4 that the relation $u \mapsto w$ can be expressed by the following variational inequality

$$|u - w| \leq r \qquad \frac{dw}{dt}(u - w - \varphi) \geq 0 \qquad \forall\, \varphi : \ |\varphi| \leq r.$$

On the other hand it is not difficult to show (see [25], Section I.3) that the following system

$$\begin{aligned} &(i) && |x_r(t)| \leq r && \forall t \in [0,T], \\ &(ii) && (\dot{u}(t) - \dot{x}_r(t))\,(x_r(t) - \varphi) \geq 0 && \text{a.e.} \quad \forall\, \varphi \in [-r,r], \\ &(iii) && x_r(0) = x_r^0 && \end{aligned} \tag{1.2.3}$$

admits a unique solution $x_r \in W^{1,1}(0,T)$ for any given input function $u \in W^{1,1}(0,T)$ and any given initial condition $x_r^0 \in [-r,r]$. Then the stop and the play operators $\mathcal{S}_r,\ \mathcal{P}_r : [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T)$ can be introduced as solution operators of Problem (1.2.3) by the formula

$$\mathcal{S}_r(x_r^0, u) := x_r \qquad \mathcal{P}_r(x_r^0, u) := u - x_r.$$

It turns out that Theorem 1.2.1 and Proposition 1.2.2 are still valid also in this case. The set $Z := [-r,r]$ is called the *characteristic* of the operators $\mathcal{S}_r$ and $\mathcal{P}_r$. In this case it is a symmetric one-dimensional set but there are also other possibilities in which one considers tensorial extensions of the play and stop operators, or situations in which one deals with more general closed convex sets as characteristics.

Moreover, both models can be generalized in several ways, for instance if f is a strictly monotone continuous function $\mathbb{R} \to \mathbb{R}$ the relation $u \mapsto f(w)$ corresponds for both cases to diagrams similar to Figures 1.3 and 1.5. In the case of the play the hysteresis region is bounded by two parallel nonlinear exterior curves; in the case of the stop the hysteresis region is spanned by a family of parallel nonlinear curves. For more details on the topic, we refer to [39], Sections III.2 and III.3.

Finally we observe that (see [25], Section II.1, Remark 1.3) it is particularly easy to solve Problem (1.2.3) if the input is monotone in an interval $[t_1, t_2] \subset [0,T]$. What we get is nothing but formula (1.2.2) contained in Subsection 1.2.3, which provides therefore an equivalent definition for the operator $\mathcal{S}_r$.

1.2.5. Memory of the play system

Consider a system in which the output depends with hysteresis on the input. This means that the output $w(t)$ at any instant t will depend on the evolution of the input until that time t and also on the initial state of the system. Therefore, in this situation the state of the system can be described by an operator $\mathcal{F}$ of type (1.1.2).

Suppose now that $\mathcal{F} \equiv \mathcal{P}_r$, the scalar play with characteristic $[-r,r]$. Then for any given input function $u \in W^{1,1}(0,T)$ and any given initial condition $x_r^0 \in [-r,r]$ we have

$$\mathcal{P}_r(x_r^0, u)(0) := u(0) - x_r^0.$$

We notice that we can associate to any $r \in \mathbb{R}$ the corresponding value x_r^0; this suggests the idea of making the initial configuration of the play system independent of the initial conditions $\{x_r^0\}_{r\in\mathbb{R}}$ for the output function by the introduction of some suitable function of r. More precisely, following [25] Section II.2, let us consider any function $\lambda \in \Lambda$ where

$$\Lambda := \left\{ \lambda \in W^{1,\infty}(0,\infty); \ \left| \frac{d\lambda(r)}{dr} \right| \leq 1 \ \text{ a.e. in } [-r,r] \right\}.$$

We also introduce some useful subspaces of Λ, i.e.

$$\Lambda_R := \{\lambda \in \Lambda; \lambda(r) = 0 \ \text{ for } r \geq R\}, \qquad \Lambda_0 := \bigcup_{R>0} \Lambda_R. \tag{1.2.4}$$

Λ is called *configuration space* and the functions λ are called *memory configurations*.
If $Q_r : \mathbb{R} \to [-r,r]$ is the projection

$$Q_r(x) := \text{sign}\,(x) \min\{r, |x|\} = \min\{r, \max\{-r, x\}\},$$

then we set

$$x_r^0 := Q_r(u(0) - \lambda(r)).$$

This implies that the initial configuration of the play system only depends on λ and $u(0)$. The same can be done for the initial configuration $\mathcal{S}_r(x_r^0, u)(0) := x_r^0$ of the stop operator.
We introduce the following more convenient notations

$$\wp_r(\lambda, u) := \mathcal{P}_r(x_r^0, u) \qquad \mathsf{s}_r(\lambda, u) := \mathcal{S}_r(x_r^0, u), \tag{1.2.5}$$

for any $\lambda \in \Lambda$, for any $u \in \mathcal{C}^0([0,T])$ and $r > 0$, where $\mathcal{P}_r(x_r^0, u)$ and $\mathcal{S}_r(x_r^0, u)$ are then defined by induction starting from $\mathcal{P}_r(x_r^0, u)(0)$ and $\mathcal{S}_r(x_r^0, u)(0)$ according to (1.2.1) and (1.2.2).
From now on throughout this subsection we will deal only with the play operator. We then set for the sake of completeness $\wp_0(\lambda, u) = u$. It turns out that the operator $\wp_r : \Lambda \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ is Lipschitz continuous in the following sense (see [25], Section II.2, Lemma 2.3)

Lemma 1.2.3. *For every $u, v \in \mathcal{C}^0([0,T])$, every $\lambda, \mu \in \Lambda$ and $r > 0$ we have*

$$||\wp_r(\lambda, u) - \wp_r(\mu, v)||_\infty \leq \ \max\{|\lambda(r) - \mu(r)|, ||u - v||_\infty\}.$$

The introduction of the function λ plays an important role in the characterization of the memory of the play system, in the sense that, for any given λ, we can construct the play operator $\wp_r(\lambda, u)$ starting from λ and from a sequence of values (t_j, r_j) which is the so called *(reduced) memory sequence* (see [25], Section II.2 or [39], Section III.6) of any input u at a certain instant t with respect to the initial configuration λ. These values are what one simply has to know in order to evaluate the output of the play operator.

More precisely, for any $u \in \mathcal{C}^0([0,T])$, $\lambda \in \Lambda_0$, $t \in [0,T]$ we put

$$\begin{cases} \bar{r} := \max\{\min\{r \geq 0 : \; |u(\tau) - \lambda(r)| = r\}, \, \tau \in [0,t]\} \\ \bar{t} := \max\{\tau \in [0,t] : \min\{r \geq 0 : \; |u(\tau) - \lambda(r)| = r\} = \bar{r}\}, \end{cases}$$

and also

$$\begin{cases} t_0 := \bar{t}, \; r_0 := \bar{r} & \text{if} \quad u(\bar{t}) = \lambda(\bar{r}) - \bar{r}, \\ t_1 := \bar{t}, \; r_1 := \bar{r} & \text{if} \quad u(\bar{t}) = \lambda(\bar{r}) + \bar{r}; \end{cases} \tag{1.2.6}$$

at this point we continue recursively by setting

$$\begin{cases} t_{2j+1} := \max\{\tau \in [t_{2j}, t] : u(\tau) = \max\{u(\sigma) : \sigma \in [t_{2j}, t]\}\}, \; j = 1, 2, \ldots \\ t_{2j} := \max\{\tau \in [t_{2j-1}, t] : u(\tau) = \min\{u(\sigma) : \sigma \in [t_{2j-1}, t]\}\}, \; j = 1, 2, \ldots \\ r_{j+1} := \dfrac{(-1)^j}{2}(u(t_{j+1}) - u(t_j)), \; j = 1, 2, \ldots \end{cases} \tag{1.2.7}$$

and this procedure goes on until $t_{2j+1} = t$ or $t_{2j} = t$ for some j.
The (reduced) memory sequence $\{(t_j, r_j)\}$ (in symbols $RMS_\lambda(u)(t)$) can be infinite, and in such a case

$$u(t) = \lim_{j \to \infty} u(t_j), \qquad \lim_{j \to \infty} r_j = 0,$$

but it can also be finite, and in this case $t = t_n$ for some $n \in \mathbb{N}$ and we put $r_j := 0$ for $j \geq n+1$.
The important result we can state is the following (it is proved in [25], Section II.2).

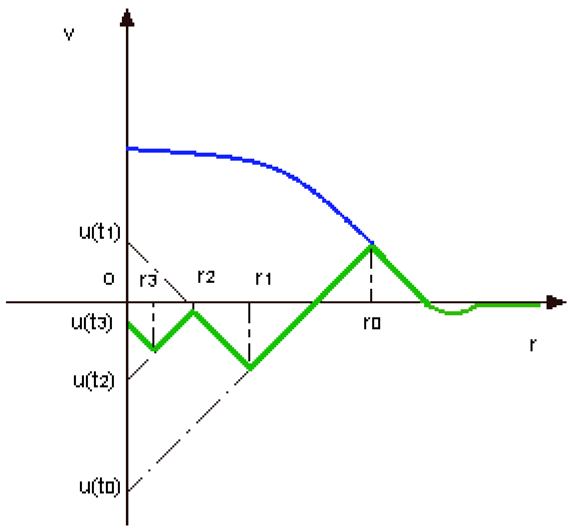

Figure 1.6: The memory structure of the play operator.

Proposition 1.2.4. *Let $u \in \mathcal{C}^0([0,T])$, $\lambda \in \Lambda_0$, $r > 0$ and $t \in [0,T]$ be given, and let $RMS_\lambda(u)(t) = \{(t_j, r_j)\}$ be the memory sequence* (1.2.6) *and* (1.2.7)*. Then we have*

$$\wp_r(\lambda, u)(t) = \begin{cases} \lambda(r) & \text{for } r \geq \bar{r} \\ u(t_j) + (-1)^j\, r & \text{for } r \in [r_{j+1}, r_j), \;\; j = 0, 1, 2, \ldots \end{cases} \tag{1.2.8}$$

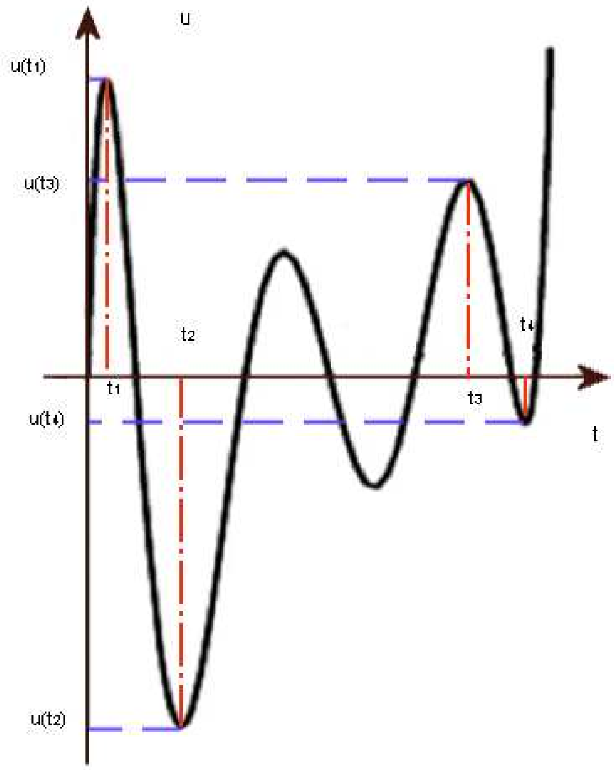

Figure 1.7: Picture illustrating the construction of the reduced memory sequence.

Figure 1.6 is explicative of this construction, the curve above represents any given memory configuration λ, the other curve represents the play operator $\wp_r(\lambda, u)$ as r varies (the two lines coincide from r_0 onward in agreement with (1.2.8)). Formula (1.2.8) shows that the increasing sequence $\{u(t_{2j})\}$ of local minima and decreasing sequence $\{u(t_{2j+1})\}$ of local maxima is precisely what the system keeps in memory. The output values are determined only by these sequences and all the rest of the input history is irrelevant. Figure 1.7 also shows the construction of the reduced memory sequence. This memory sequence is also called *reduced* because in reality for any continuous piecewise monotone function $u : [0,T] \to \mathbb{R}$ and any $t \in [0,T]$ we can also consider the finite sequence of instants $\{t_j\}$ for $j = 1, \ldots, n$ of the time interval $[0,t]$ at which the function u inverts its monotonicity. The finite sequence of the corresponding values $\{u(t_j)\}$ is usually called *complete memory sequence* (in symbols CMS) (see [25], Section II.2

or [39], Section III.6) of the function u at any instant t. This sequence determines the value at t of any hysteresis operator applied to u as any hysteresis operator is also rate independent. The fact that the RMS is mostly used in applications is due to the fact that the CMS (which can be finite or infinite for a given input function not necessarily piecewise monotone) does not necessarily exist for any continuous or even infinitely differentiable function, whereas the RMS exists for any continuous function and this entails important consequences in the study of a large class of hysteresis models.
We conclude this section with the following lemma - which can be found for example in [26], Chapter 4 - and which adds some information to the previous proposition.

Lemma 1.2.5. *Let $w \in \mathcal{C}(\mathbb{R}^+)$ (where $\mathcal{C}(\mathbb{R}^+)$ is the space of continuous functions on $\mathbb{R}^+$) and let $t \geq 0$ be given. Set*

$$w_{\max}(t) = \sup_{\tau\in[0,t]} w(\tau), \qquad w_{\min}(t) = \inf_{\tau\in[0,t]} w(\tau).$$

Then for all $\lambda \in \Lambda$ and $r > 0$ we have

$$\begin{aligned} &\wp_r[\lambda, w](\tau) \leq \max\{\lambda(r), w_{\max}(t) - r\} && \forall\, \tau \in [0,t] \\ &\wp_r[\lambda, w](\tau) \geq \min\{\lambda(r), w_{\min}(t) + r\} && \forall\, \tau \in [0,t] \\ &\wp_r[\lambda, w](t) = \lambda(r) && \text{for } r > ||m_\lambda(w(\cdot))||_{[0,t]}, \end{aligned}$$

where for $v \in \mathbb{R}$ we put $m_\lambda = \inf\{r \geq 0;\ |\lambda(r) - v| = r\}$.

1.3. Prandtl-Ishlinskiĭ operators

1.3.1. Definition and basic properties

The *Prandtl-Ishlinskiĭ operators (of play and stop type)* are more complex models than the ones just introduced. In Section 1.4 we will show a rheological construction of these models, i.e. we will present them as models of *elasto-plasticity (with strain hardening)* obtained by combining arbitrary families of stops and plays in series and in parallel. At this stage we adopt the following definition which includes both concepts. The two definitions (this one and the one contained in Section 1.4) are, of course, equivalent in the sense that they represent the same hysteresis models.

Definition 1.3.1. *Suppose that a constant $a \geq 0$ and a function $h \in BV_{\mathrm{loc}}(0,\infty)$ are given, such that $\lim_{s\to 0^+} h(s) = a$. We set*

$$\varphi(r) := \int_0^r h(s)\, ds \qquad \text{for } r > 0.$$

Then the operator $\mathcal{F}_\varphi : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ *defined by the formula*

$$\mathcal{F}_\varphi(\lambda, u) = a\,u + \int_0^\infty \wp_r(\lambda, u)\,dh(r), \qquad \lambda \in \Lambda_0, \;\; u \in \mathcal{C}^0([0,T]) \tag{1.3.1}$$

where $\wp_r$ *is the play operator* (1.2.5) *and* Λ_0 *is introduced in* (1.2.4), *is called* PRANDTL-ISHLINSKIĬ OPERATOR *generated by the function* φ *which is then called the generator of* $\mathcal{F}_\varphi$.

The Stieltjes integral in (1.3.1) is finite as $\lambda \in \Lambda_0$ and this implies, as we saw in Proposition 1.2.4, that $\wp_r(\lambda, u)$ vanishes for r sufficiently large. The distinction between Prandtl-Ishlinskiĭ operators of stop type and play type can be characterized in terms of the generator, i.e. a *convex function* φ generates an operator of play type, a *concave function* φ generates instead an operator of stop type.
Theorem 1.2.1 implies the following result.

Theorem 1.3.2. *The operator* $\mathcal{F}_\varphi$ *is causal and rate independent in the sense of* (1.1.3) *and* (1.1.4), *i.e. it is a hysteresis operator; moreover if the function* h *is nonnegative and monotone, then* $\mathcal{F}_\varphi$ *is piecewise monotone in the sense of* (1.1.7). *Finally* $\mathcal{F}_\varphi$ *is locally Lipschitz continuous in the following sense: for all* $t \geq 0$, *for all* $w_1, w_2 \in \mathcal{C}^0([0,T])$, *for all* $\lambda_1, \lambda_2 \in \Lambda_R$, *where* $R > 0$ *is given*

$$\begin{aligned} |\mathcal{F}_\varphi(\lambda_1, w_1) - \mathcal{F}_\varphi(\lambda_2, w_2)| &\leq |h(0)|\,|w_1(t) - w_2(t)| \\ &\quad + \left(\mathrm{Var}_{[0,R(t)]} h\right)\, \max\left\{||\lambda_1(r) - \lambda_2(r)||_{[0,R]}, ||w_1 - w_2||_{\mathcal{C}^0([0,t])}\right\} \end{aligned}$$

where $R(t) := \max\{R, ||w_1||_{\mathcal{C}^0([0,t])}, ||w_2||_{\mathcal{C}^0([0,t])}\}$.

1.3.2. Further properties and energy inequalities

The variational character of the Prandtl-Ishlinskiĭ operators leads to natural properties for absolutely continuous inputs. More precisely we can state the following result which can be found in [25], Section II.4.

Theorem 1.3.3. *Let* $h : [0,\infty) \to [0,\infty)$ *be a monotone function. For* $u_1, u_2 \in W^{1,1}(0,T)$, $\lambda_1, \lambda_2 \in \Lambda_0$ *and* $r > 0$ *put* $\xi_r^{(i)} := p_r(\lambda_i, u_i)$, $x_r^{(i)} := u_i - \xi_r^{(i)}$, $w_i := \mathcal{F}_\varphi(\lambda_i, u_i) = h(0)u_i + \int_0^\infty \xi_r^{(i)} dh(r)$, $i = 1, 2$, $\tilde{u} := u_1 - u_2$, $\tilde{w} := w_1 - w_2$, $\tilde{\xi}_r := \xi_r^{(1)} - \xi_r^{(2)}$, $\tilde{x} := x_r^{(1)} - x_r^{(2)}$. *Then*

$$\dot{\tilde{w}}(t)\tilde{u}(t) \geq \frac{1}{2}\frac{d}{dt}[h(0)\tilde{u}^2(t) + \int_0^\infty \tilde{\xi}_r^2(t)dh(r)] \qquad \text{a.e. if } h \text{ is nondecreasing} \tag{1.3.2}$$

$$\tilde{w}(t)\dot{\tilde{u}}(t) \geq \frac{1}{2}\frac{d}{dt}[h(\infty)\tilde{u}^2(t) - \int_0^\infty \tilde{x}_r^2(t)dh(r)] \qquad \text{a.e. if } h \text{ is nonincreasing,}$$

where, from now on (in the case these quantities exist)

$$h(0) = \lim_{s\to 0} h(s); \qquad h(\infty) := \lim_{s\to\infty} h(s). \tag{1.3.3}$$

The following result gives us explicit energy dissipation formula for the Prandtl-Ishlinskiĭ operators (1.3.1) (see [25], Section II.4, Proposition 4.6)

Proposition 1.3.4. *Let $R > 0$, $\lambda \in \Lambda_R$, $u \in W^{1,1}(0,T)$ be such that $||u||_\infty \le R$ and consider a given non-negative function $h \in BV_{\mathrm{loc}}(0,\infty)$. For $r > 0$ we put $\xi_r := \wp_r(\lambda, u)$, $x_r := u - \xi_r$. Then we have the following two cases:*

- *(Prandtl-Ishlinskiĭ operators of play type) if h is nondecreasing and*

$$w := h(0)\, u + \int_0^\infty \xi_r\, dh(r), \;\; U := \frac{1}{2} h(0)\, u^2 + \frac{1}{2}\int_0^\infty \xi_r^2\, dh(r), \;\; D := \int_0^\infty r\, \xi_r\, dh(r), \tag{1.3.4}$$

then we have

$$\dot w(t)\, u(t) - \dot U(t) = |\dot D(t)| \qquad \text{a.e. in } (0,T);$$

- *(Prandtl-Ishlinskiĭ operators of stop type) if h is nonincreasing and*

$$w := h(\infty)\, u - \int_0^\infty x_r\, dh(r), \;\; U := \frac{1}{2} h(\infty)\, u^2 - \frac{1}{2}\int_0^\infty x_r^2\, dh(r), D := -\int_0^\infty r\, \xi_r\, dh(r),$$

then we have

$$w(t)\, \dot u(t) - \dot U(t) = |\dot D(t)| \qquad \text{a.e. in } (0,T).$$

This result can be interpreted by saying that the Prandtl-Ishlinskiĭ operators of play and stop type are thermodynamically consistent. Here the quantity $\dot w(t)\, u(t)$ or the quantity $w(t)\, \dot u(t)$ is the power density, U is the internal energy density and finally $|\dot D|$ is the dissipation rate which is positive in agreement with the Second Principle of Thermodynamics.

For the last result of this subsection we restrict ourselves to the case of Prandtl-Ishlinskiĭ operators of play type, i.e. we assume that the function h is positive and nondecreasing in $(0,\infty)$.

Besides the energy inequality stated by the previous proposition, the Prandtl-Ishlinskiĭ operators of play type admit a *higher order energy inequality* which can be summarized in

$$\ddot w(t)\, \dot u(t) - \dot V(t) \ge 0 \qquad \text{in the sense of distributions}$$

where w is the same as in (1.3.4) and where

$$V(t) = \frac{1}{2} \dot w(t)\, \dot u(t) \qquad \text{a.e. in } [0,T]. \tag{1.3.5}$$

First we introduce the concept of trajectory (which can be found i.e. in [25], Section II.4).

Definition 1.3.5. *Let $\mathcal{F} : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be a rate independent operator and let $u \in \mathcal{C}^0([0,T])$ be a function which is monotone (nonincreasing or nondecreasing) in $[t_1,t_2] \subset [0,T]$, with $u(t_i) = u_i$, for $i = 1,2$. Then there exists a function $\Phi : \mathrm{Conv}\{u_1,u_2\} \to \mathbb{R}$ such that $\mathcal{F}(v)(t) = \Phi(v(t))$ for all $t \in [t_1,t_2]$ and for every function $v \in \mathcal{C}^0([0,T])$ which is monotone in $[t_1,t_2]$ and $v(t) = u(t)$ for $t \in [0,T] \setminus (t_1,t_2)$. If moreover $\mathcal{F}$ is continuous, then Φ is continuous and if $\mathcal{F}$ is locally monotone on absolutely continuous inputs, then Φ is nondecreasing and absolutely continuous. The function Φ is called* TRAJECTORY *of $\mathcal{F}$ along u in $[t_1,t_2]$.*

Now we state here the precise statement of the so called *second order energy inequality*, for more details see for example [25], Section II.4, Theorem 4.19.

Theorem 1.3.6. *Let $\mathcal{F} : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be a continuous and rate independent operator. Assume that there exist constants $R > 0$, $b_R > a_R \geq 0$ and $K_R \geq 0$ such that, for every $u \in \mathcal{C}^0([0,T])$ with $||u||_\infty \leq R$, the trajectory Φ of $\mathcal{F}$ along u in a monotonicity interval $[t_1,t_2]$ has the following properties:*
(i) Φ is absolutely continuous in $J := \mathrm{Conv}\{u(t_1),u(t_2)\}$, $a_R \leq \Phi'(v) \leq b_R$ for a.e. $v \in \mathrm{Int}\, J$;
(ii) if u is nondecreasing in $[t_1,t_2]$, then $\Phi(v) - \frac{1}{2} K_R v^2$ is convex in J;
(iii) if u is nonincreasing in $[t_1,t_2]$, then $\Phi(v) + \frac{1}{2} K_R v^2$ is concave in J.
Let $u \in W^{1,\infty}(0,T)$ be a given function such that $||u||_\infty \leq R$ and $w := \mathcal{F}(u) \in W^{2,1}(0,T)$. Then:
(a) the function $V(t)$ introduced in (1.3.5) belongs to $BV(0,T)$ and

$$\frac{a_R}{2}\,\dot{u}^2(t) \leq V(t) \leq \frac{b_R}{2}\,\dot{u}^2(t) \qquad \text{a.e.;}$$

(b)

$$\int_s^t \ddot{w}(\tau)\,\dot{u}(\tau)\,d\tau - V(t) + V(s) \geq \frac{1}{2} K_R \int_s^t |\dot{u}(\tau)|^3\,d\tau \qquad \text{for almost all } 0 < s < t < T. \quad (1.3.6)$$

The following proposition ([25], Section II.4) then allows us to verify that the previous theorem can be actually applied to Prandtl-Ishlinskiĭ operators of play type

Proposition 1.3.7. *Let $h \in BV_{\mathrm{loc}}(0,\infty)$ be a given nonnegative function and let $\mathcal{F} := \mathcal{F}_\varphi(\lambda_0,\cdot)$ be the Prandtl-Ishlinskiĭ operator (1.3.1) for some $R > 0$ and $\lambda_0 \in \Lambda_R$. Set*

$$H_-(R) := \inf\left\{ \frac{h(b)-h(a)}{b-a};\ 0 < a < b < R \right\}.$$

If $H_-(R) \geq 0$ then the hypothesis of Theorem 1.3.6 are satisfied for $K_R := \frac{1}{2} H_-(R)$.

1.4. Rheological models

Rheology is the study of mechanical constitutive properties of materials; rheology usually defines some ideal bodies which are constructed by the combination in series and in parallel of a small class of primitive elements which correspond to the main mechanical properties such as elasticity, viscosity, plasticity and so on. This primitive bodies are characterized by a constitutive relation between macroscopic strain and stress tensors ε and σ respectively. For more details on this topic we refer to [39] Chapters II and III or also [38].
From now on we confine ourselves to the one-dimensional scalar case and we present some classes of rheological models. We recall that if two or more either elementary or composed rheological models are coupled in series, they experience the same stress which is also the stress of the composed model while the strain of the global model is the sum of their strains

$$\sigma = \sigma_1 = \sigma_2 = \dots; \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 + \dots;$$

on the other hand, for the combination in parallel these properties shared by stress and strain tensors are interchanged

$$\varepsilon = \varepsilon_1 = \varepsilon_2 = \dots; \qquad \sigma = \sigma_1 + \sigma_2 + \dots.$$

If A_1 and A_2 are two either elementary or composite models we will denote by the rheological formula $A_1 - A_2$ or $A_1|A_2$ respectively the combination in series and in parallel of A_1 and A_2. We can also extend in an easy way these rules to the case when we are combining infinite elements. In this case, let $(\mathcal{P}, \mathcal{A}, \mu)$ be a measure space with μ a finite non negative measure and let $\{A_\rho\}_{\rho\in\mathcal{P}}$ be a family of rheological elements. Their combination in series and in parallel, respectively denoted by $\sum_{\rho\in\mathcal{P}} A_\rho$ and $\prod_{\rho\in\mathcal{P}} A_\rho$ correspond to the following relations

$$\begin{cases} \sigma_I = \sigma_\rho & \mu \text{ a.e. in } \mathcal{P} \\ \varepsilon_I = \displaystyle\int_{\mathcal{P}} \varepsilon_\rho \, d\mu(\rho) & \end{cases} \qquad \begin{cases} \varepsilon_I = \varepsilon_\rho & \mu \text{ a.e. in } \mathcal{P} \\ \sigma_I = \displaystyle\int_{\mathcal{P}} \sigma_\rho \, d\mu(\rho), & \end{cases}$$

where ε_I and σ_I represent the strain and the stress of the global model while ε_ρ and σ_ρ are the strain and the stress of the single rheological element.
Throughout this section we will mainly refer to two basic elements, an *elastic* and a *plastic* element, respectively denoted by E and P, which correspond to the constitutive laws of the form

$$\varepsilon = \alpha(\sigma) \qquad \text{or} \qquad \sigma = \beta(\varepsilon),$$

$$\dot{\varepsilon} \in \partial I_Z(\sigma),$$

where α and $\beta := \alpha^{-1}$ are continuous real functions and I_Z is the indicator function of a non-empty closed, convex set $Z \subset \mathbb{R}$; we assume that $\alpha(0) = 0$ and $0 \in Z$.

➜ ELASTIC AND PLASTIC ELEMENTS IN PARALLEL $E|P$. We consider the model obtained by combining *an elastic and a plastic element in parallel.* This model is usually called PLAY (see also Section 1.2 for an equivalent definition) and corresponds to the rheological law

$$\sigma \in \beta(\varepsilon) + (\partial I_Z)^{-1}(\dot{\varepsilon}) \qquad \text{or equivalently} \qquad \dot{\varepsilon} \in \partial I_Z(\sigma - \beta(\varepsilon))$$

which is also equivalent to the following variational inequality

$$\sigma - \beta(\varepsilon) \in Z, \qquad \dot{\varepsilon}[\sigma - \beta(\varepsilon) - v] \geq 0 \qquad \forall\, v \in Z;$$

we introduce also the initial condition

$$\varepsilon(0) = \varepsilon^0$$

and require that σ and ε^0 fulfill the compatibility condition $\sigma(0) - \beta(\varepsilon^0) \in Z$.
It can be easily shown that in this case the relation $\sigma \mapsto \varepsilon$ can be expressed in the form

$$\varepsilon(t) = [\mathcal{P}(\sigma, \varepsilon^0)](t) \qquad \text{in } [0,T]$$

where

$$\mathcal{P} : \mathcal{C}^0([0,T]) \times \mathbb{R} \to \mathcal{C}^0([0,T])$$

is a *causal* and *rate independent* operator (so a *hysteresis* operator) in the sense of (1.1.3) and (1.1.4). Moreover $\mathcal{P}$ is also *continuous* in the sense of (1.1.5) and *Lipschitz continuous,* i.e. there exists a constant $L_{\mathcal{P}}$ such that, for all $\sigma_1, \sigma_2 \in \mathcal{C}^0([0,T])$ and for all $\varepsilon_1^0, \varepsilon_2^0 \in \mathbb{R}$

$$||\mathcal{P}(\sigma_1, \varepsilon_1^0) - \mathcal{P}(\sigma_2, \varepsilon_2^0)||_{\mathcal{C}^0([0,T])} \leq \; L_{\mathcal{P}}\,(||\sigma_1 - \sigma_2||_{\mathcal{C}^0([0,T])} + |\varepsilon_1^0 - \varepsilon_2^0|).$$

The play operator which comes from this construction is equivalent to the play operator introduced in Subsection 1.2.4 if $\beta(\varepsilon) = \varepsilon$.

➜ ELASTIC AND PLASTIC ELEMENTS IN SERIES $E - P$. Now let us consider the model obtained by combining *an elastic and a plastic element in series.* This model is usually called PRANDTL MODEL or STOP (see also Section 1.2 for an equivalent definition) and corresponds to the rheological law

$$\dot{\varepsilon} \in \alpha(\sigma)^{\cdot} + \partial I_Z(\sigma)$$

which is equivalent to the following variational inequality

$$\sigma \in Z, \qquad [\dot{\varepsilon} - \alpha(\sigma)^{\cdot}](\sigma - v) \geq 0 \qquad \forall\, v \in Z;$$

we introduce also the following initial condition

$$\sigma(0) = \sigma^0.$$

It is not difficult to see that in this case the relation $\varepsilon \mapsto \sigma$ can be expressed in the form

$$\sigma(t) = [\mathcal{S}(\varepsilon, \sigma^0)](t) \qquad \text{in } [0, T]$$

where

$$\mathcal{S} : \mathcal{C}^0([0,T]) \times \mathbb{R} \to \mathcal{C}^0([0,T])$$

is a *causal* and *rate independent* operator (so a *hysteresis* operator) in the sense of (1.1.3) and (1.1.4). Moreover $\mathcal{S}$ is also *continuous* in the sense of (1.1.5) and *Lipschitz continuous,* i.e. there exists a constant $L_{\mathcal{S}}$ such that, for all $\varepsilon_1, \varepsilon_2 \in \mathcal{C}^0([0,T])$ and for all $\sigma_1^0, \sigma_2^0 \in \mathbb{R}$

$$||\mathcal{S}(\varepsilon_1, \sigma_1^0) - \mathcal{S}(\varepsilon_2, \sigma_2^0)||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{S}} \left(||\varepsilon_1 - \varepsilon_2||_{\mathcal{C}^0([0,T])} + |\sigma_1^0 - \sigma_2^0|\right).$$

The stop operator which comes from this construction is equivalent to the stop operator introduced in Subsection 1.2.4 if $\alpha(\sigma) = \sigma$.

➜ $\prod_{\rho\in\mathcal{P}}(E_\rho - P_\rho)$: PARALLEL COMBINATION OF ELASTIC AND PLASTIC ELEMENTS IN SERIES. Let us now assume to have a measure space $(\mathcal{P}, \mathcal{A}, \mu)$ where μ is a finite non negative Borel measure, and two families $\{E_\rho\}_{\rho\in\mathcal{P}}$ and $\{P_\rho\}_{\rho\in\mathcal{P}}$ of elastic and rigid, perfectly plastic elements fulfilling the following rheological laws

$$\varepsilon_\rho = \alpha_\rho(\sigma_\rho) \qquad \text{or} \qquad \sigma_\rho = \beta_\rho(\varepsilon_\rho)$$

$$\dot{\varepsilon_\rho} \in \partial I_{Z_\rho}(\sigma_\rho),$$

where α_ρ and $\beta_\rho = \alpha_\rho^{-1}$ are continuous and strictly monotone functions $\mathbb{R} \to \mathbb{R}$ and I_{Z_ρ} is the indicator function of a closed interval Z_ρ. At this point we consider the model obtained by combining in series a family of plays (i.e. a serial combination of a family of models each composed by an elastic element in parallel with a plastic one). This model is known as PRANDTL-ISHLINSKIĬ MODEL OF PLAY TYPE. Let us see how this model can be represented by means of a hysteresis operator. We know that for any $\rho \in \mathcal{P}$, the model obtained by combining in parallel an elastic element E_ρ and a plastic one P_ρ corresponds to the rheological law

$$\begin{cases} \dot{\varepsilon_\rho} \in \partial I_{Z_\rho}(\sigma_\rho - \beta_\rho(\varepsilon_\rho)) \\ \varepsilon_\rho(0) = \varepsilon_\rho^0 \end{cases}$$

and we said before that this system can be also represented by means of a hysteresis operator

$$\mathcal{P}_\rho : \mathcal{C}^0([0,T]) \times \mathbb{R} \to \mathcal{C}^0([0,T]) \qquad\qquad \varepsilon_\rho(t) = \mathcal{P}_\rho(\sigma_\rho, \varepsilon_\rho^0)(t).$$

Thus, the Prandtl-Ishlinskiĭ model of play type corresponds to the following system

$$\begin{cases} \dot{\varepsilon_\rho} \in \partial I_{Z_\rho}(\sigma_I - \beta_\rho(\varepsilon_\rho)) & \mu\text{−a.e. in } \mathcal{P}, \\ \varepsilon_\rho(0) = \varepsilon_\rho^0 & \mu\text{−a.e. in } \mathcal{P}, \\ \varepsilon_I = \displaystyle\int_{\mathcal{P}} \varepsilon_\rho \, d\mu(\rho), \end{cases}$$

where σ_I and ε_I are the strain and the stress of the composite model. Using the fact that for any $\rho \in \mathcal{P}$ the corresponding play model can be represented by means of a hysteresis operator, we also have

$$\varepsilon_I = \int_{\mathcal{P}} \mathcal{P}_\rho(\sigma_I, \varepsilon_\rho^0) \, d\mu_\rho =: \mathcal{I}_\mu(\sigma_I, \{\varepsilon_\rho^0\}_{\rho\in\mathcal{P}}).$$

Remark 1.4.1. Since β_ρ are increasing functions, we see that $\hat{\varepsilon}_\rho := \beta_\rho(\varepsilon_\rho)$ are the outputs of the linear play. Hence,

$$\varepsilon_I = \int_{\mathcal{P}} \beta_\rho^{-1}(\hat{\varepsilon}_\rho) \, d\mu(\rho),$$

which is essentially nothing but formula (1.5.8) with $\mathcal{P} = (0, \infty)$ and $g(\rho, v) = \mu'(\rho) \, \beta_\rho^{-1}(v)$. This construction is actually known to be equivalent to the Preisach model for regular measures μ, see [25], Section II.3, Remark 3.9.

Thus denoting by $\mathcal{M}(\mathcal{P})$ the set of measurable functions $\mathcal{P} \to \mathbb{R}$, we have that

$$\mathcal{I}_\mu : \mathcal{C}^0([0,T]) \times \mathcal{M}(\mathcal{P}) \to \mathcal{C}^0([0,T])$$

is a *causal* and *rate independent* operator (so a *hysteresis* operator) i.e. fulfilling (1.1.3) and (1.1.4) (for more details we refer to [39] Chapters II and III or also [38]). Moreover $\mathcal{I}_\mu$ is also *continuous* in the sense of (1.1.5) and finally

$$\mathcal{I}_\mu : \mathcal{C}^0([0,T]) \times L^1(\mathcal{P}) \to \mathcal{C}^0([0,T])$$

is *Lipschitz continuous,* in the sense that there exists a constant $L_{\mathcal{I}}$ such that, for all $\sigma_I^1, \sigma_I^2 \in \mathcal{C}^0([0,T])$ and for all $\varepsilon_\rho^{0\,1}, \varepsilon_\rho^{0\,2} \in L^1(\mathcal{P})$

$$||\mathcal{I}_\mu(\sigma_I^1, \varepsilon_\rho^{0\,1}) - \mathcal{I}_\mu(\sigma_I^2, \varepsilon_\rho^{0\,2})||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{I}} \left(||\sigma_I^1 - \sigma_I^2||_{\mathcal{C}^0([0,T])} + ||\varepsilon_\rho^{0\,1} - \varepsilon_\rho^{0\,2}||_{L^1(\mathcal{P})}\right).$$

This operator is equivalent to the Prandtl-Ishlinskiĭ operator introduced in (1.3.1) under suitable assumptions on the measure μ and on the function h.

➜ $\sum_{\rho\in\mathcal{P}}(E_\rho|P_\rho)$ Serial combination of elastic and plastic elements in parallel. We now consider the model obtained by combining in parallel a family of stops (i.e. a parallel combination of a family of models each composed by an elastic element in series with a plastic one). This model is known as PRANDTL-ISHLINSKIĬ MODEL OF STOP TYPE. Let us see how this model can be represented by means of a hysteresis operator. We know that for any $\rho \in \mathcal{P}$, the model obtained by combining in series an elastic element E_ρ and a plastic one P_ρ corresponds to the rheological law

$$\begin{cases} \dot{\varepsilon_\rho} \in \alpha_\rho(\sigma)^{\cdot} + \partial I_{K_\rho}(\sigma_\rho) \\ \sigma_\rho(0) = \sigma_\rho^0 \end{cases}$$

and we said before that this system can be also represented by means of a hysteresis operator

$$\mathcal{S}_\rho : \mathcal{C}^0([0,T]) \times \mathbb{R} \to \mathcal{C}^0([0,T]) \qquad \sigma_\rho(t) = \mathcal{S}_\rho(\varepsilon_\rho, \sigma_\rho^0)(t).$$

Thus, the Prandtl-Ishlinskiĭ model of stop type corresponds to the following system

$$\begin{cases} \dot{\varepsilon_I} \in \alpha_\rho(\sigma_\rho)^{\cdot} + \partial I_{K_\rho}(\sigma_\rho) & \mu\text{−a.e. in } \mathcal{P}, \\ \sigma_\rho(0) = \sigma_\rho^0 & \mu\text{−a.e. in } \mathcal{P}, \\ \sigma_I = \displaystyle\int_{\mathcal{P}} \sigma_\rho \, d\mu(\rho), \end{cases}$$

where σ_I and ε_I are the strain and the stress of the composite model. Using the fact that for any $\rho \in \mathcal{P}$ the corresponding stop model can be represented by means of a hysteresis operator, we also have

$$\sigma_I = \int_{\mathcal{P}} \mathcal{S}_\rho(\varepsilon_I, \sigma_\rho^0) \, d\mu_\rho =: \mathcal{J}_\mu(\varepsilon_I, \{\sigma_\rho^0\}_{\rho \in \mathcal{P}}).$$

Thus we have that

$$\mathcal{J}_\mu : \mathcal{C}^0([0,T]) \times \mathcal{M}(\mathcal{P}) \to \mathcal{C}^0([0,T])$$

is a *hysteresis* operator (i.e. a *causal* and *rate independent* operator in the sense of (1.1.3) and (1.1.4)) (for more details we refer to [39] Chapters II and III or also [38]). Moreover $\mathcal{J}_\mu$ is also *continuous* i.e. fulfilling (1.1.5) and finally

$$\mathcal{J}_\mu : \mathcal{C}^0([0,T]) \times L^1(\mathcal{P}) \to \mathcal{C}^0([0,T])$$

is *Lipschitz continuous,* in the sense that there exists a constant $L_{\mathcal{J}}$ such that, for all $\varepsilon_I^1, \varepsilon_I^2 \in \mathcal{C}^0([0,T])$ and for all $\sigma_\rho^{0\,1}, \sigma_\rho^{0\,2} \in L^1(\mathcal{P})$

$$||\mathcal{J}_\mu(\varepsilon_I^1, \sigma_\rho^{0\,1}) - \mathcal{J}_\mu(\varepsilon_I^2, \sigma_\rho^{0\,2})||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{J}} \, (||\varepsilon_I^1 - \varepsilon_I^2||_{\mathcal{C}^0([0,T])} + ||\sigma_\rho^{0\,1} - \sigma_\rho^{0\,2}||_{L^1(\mathcal{P})}).$$

1.5. The Preisach operator

1.5.1. Delayed relay

It is the simplest example of discontinuous hysteresis nonlinearity. It is characterized by two thresholds, say ρ_1, ρ_2 and two output values which we assume to be equal to -1 and $+1$. For any couple $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ with $\rho_1 < \rho_2$, the DELAYED RELAY OPERATOR

$$h_\rho : \mathcal{C}^0([0,T]) \times \{-1,1\} \to BV(0,T) \cap \mathcal{C}^0_r([0,T)), \tag{1.5.1}$$

(where $BV(0,T)$ is the Banach space of functions $[0,T] \to \mathbb{R}$ having finite total variation and $\mathcal{C}^0_r([0,T))$ is the linear space of functions which are continuous on the right in $[0,T)$), can be defined in the following way: for any $u \in \mathcal{C}^0([0,T])$ and any $\xi \in \{-1,+1\}$, the function $w = h_\rho(u,\xi)$ is given by

$$w(0) := \begin{cases} -1 & \text{if } u(0) \le \rho_1, \\ \xi & \text{if } \rho_1 < u(0) < \rho_2, \\ 1 & \text{if } u(0) \ge \rho_2 \end{cases}$$

and, for any $t \in (0,T]$, setting $W_t := \{\tau \in (0,t] : u(\tau) = \rho_1 \text{ or } \rho_2\}$, by

$$w(t) := \begin{cases} w(0) & \text{if } W_t = \emptyset \\ -1 & \text{if } W_t \ne \emptyset \text{ and } u(\max W_t) = \rho_1, \\ 1 & \text{if } W_t \ne \emptyset \text{ and } u(\max W_t) = \rho_2. \end{cases}$$

Thus w is uniquely defined in $[0,T]$ and has the regularity outlined in (1.5.1). It turns out that the operator h_ρ is causal, rate independent, order preserving and piecewise monotone in the sense of (1.1.3), (1.1.4), (1.1.6) and (1.1.7).
We conclude the subsection presenting an interesting connection between the relay and the system of play operators $\{\wp_r(\lambda, u)\}_{r \ge 0}$ introduced in (1.2.5).
First of all we give the following definition which will be also useful in the following

Definition 1.5.1. *The* PREISACH PLANE

$$\mathcal{P} := \{\rho = (\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2\} \tag{1.5.2}$$

is the set of thresholds of delayed relay operators h_ρ.

In the following we will often use a different system of coordinates, in order to describe $\mathcal{P}$. For example we can consider the *half-width* $\sigma_1 = \dfrac{\rho_2 - \rho_1}{2}$ and the *mean value* $\sigma_2 = \dfrac{\rho_1 + \rho_2}{2}$; in this case the condition on σ_1 and σ_2 in order to have admissible thresholds is $\sigma_1 > 0$ and so the Preisach plane can be written as

$$\mathcal{P} = \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 : \sigma_1 > 0\}. \tag{1.5.3}$$

We will also set in the following $\sigma_1 := r$ and $\sigma_2 := v$ in order to establish a connection with the notations introduced in the previous sections; in this way we obtain

$$\mathcal{P} = \{(r, v) \in \mathbb{R}^2 : r > 0\}. \tag{1.5.4}$$

In this setting we state this lemma whose proof can be found in [25], Section II.3.

Lemma 1.5.2. *Let $\lambda \in \Lambda_0$ and $u \in \mathcal{C}^0([0,T])$ be given. For any given $(r,v) \in \mathcal{P}$ we set $\xi_\lambda(r,v) := -1$ if $v \geq \lambda(r)$ and $\xi_\lambda(r,v) := +1$ if on the other hand $v < \lambda(r)$. Then for every $t \in [0,T]$ and $(r,v) \in \mathcal{P}$ with $v \neq \wp_r(\lambda, u)(t)$ we have*

$$h_{(r,v)}(u, \xi_\lambda(r,v))(t) = \begin{cases} +1 & \text{if } v < \wp_r(\lambda, u)(t) \\ -1 & \text{if } v > \wp_r(\lambda, u)(t). \end{cases}$$

1.5.2. Definition of the Preisach operator and some properties

In 1935 Preisach (see [34]) proposed a model of ferromagnetism based on an idea of Weiss and de Freudenreich [43]; in particular he introduced a geometric interpretation of this model which actually became one of its main features. This construction gained much success and now is known as the *Preisach model of ferromagnetism.* Mathematical aspects of this model were dealt with by Krasnosel'skiĭ and Pokrovskiĭ [21], [22], [23]; the model has been also studied in connection with partial differential equations by Visintin for example in [37] [39]; it is interesting to quote also the contributions of Brokate and Sprekels [10], [11] and Krejčí [24], [25]. We also refer to the monograph of Mayergoyz [30] for the discussion of many generalization of the Preisach model and also for the important problem of characterizing Preisach operators. We present the construction and the main properties of the Preisach operator following [39].

Definition 1.5.3. *The* PREISACH OPERATOR *can be defined as follows*

$$\begin{cases} \mathcal{H}_\mu : \mathcal{C}^0([0,T]) \times \mathcal{R} \to L^\infty(0,T) \cap \mathcal{C}^0_r([0,T)), \\ [\mathcal{H}_\mu(u,\xi)](t) := \displaystyle\int_{\mathcal{P}} [h_\rho(u, \xi_\rho)](t)\, d\mu(\rho) \quad \forall t \in [0,T] \end{cases} \tag{1.5.5}$$

where $\mathcal{R}$ is the family of Borel measurable functions $\mathcal{P} \to \{-1, 1\}$, ξ_ρ is the image of $\rho \in \mathcal{P}$ by the function $\xi \in \mathcal{R}$, μ is any finite (signed) Borel measure over $\mathcal{P}$ and $\mathcal{P}$ is the Preisach plane introduced in one of the equivalent ways (1.5.2), (1.5.3) *or* (1.5.4).

The Preisach model can be interpreted as the superposition of a family of delayed relays, distributed with a certain density; from a physical point of view, it can represent a circuital model obtained by setting in parallel a family of magnets whose constitutive laws $H \mapsto M$ between magnetic field H and magnetization M are given by delayed relays. The following proposition can be proved at once (see [39], Section IV.1, Theorem 1.2. and Corollary 1.3).

Proposition 1.5.4. *For any finite Borel measure μ over $\mathcal{P}$, it turns out that*
- *the operator $\mathcal{H}_\mu$ is* causal *and* rate independent *so it is a* hysteresis operator*;*
- *if $\mu \geq 0$, then $\mathcal{H}_\mu$ is* piecewise monotone *and* order preserving*;*
- *if there exists $\delta > 0$ such that $\mu(\{(\rho_1, \rho_2) \in \mathcal{P} : \rho_2 - \rho_1 \leq \delta\}) = 0$, then $\mathcal{H}_\mu$ maps $\mathcal{C}^0([0,T]) \times \mathcal{R}$ into $BV(0,T)$.*

In the next chapters we shall use the following continuity properties of the Preisach operators.

Proposition 1.5.5. *The following three conditions are equivalent*

$$(i) \qquad \mathcal{H}_\mu(\cdot, \xi) : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$$

$$(ii) \qquad |\mu|(\mathbb{R} \times \{r\}) = |\mu|(\{r\} \times \mathbb{R}) = 0 \qquad \forall\, r \in \mathbb{R}$$

$$(iii) \qquad \mathcal{H}_\mu(\cdot, \xi) : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T]) \qquad \textit{is strongly continuous, for any } \xi \in \bar{\mathcal{S}},$$

where $\bar{\mathcal{S}}$ is the family of relay configurations which can be attained by applying a continuous input to a system initially in the so-called virgin state

$$\xi^v_\rho = \begin{cases} 1 & \text{if } \rho_1 + \rho_2 < 0, \\ -1 & \text{if } \rho_1 + \rho_2 > 0. \end{cases}$$

Condition *(ii)* in Proposition 1.5.5 says that any horizontal or vertical segment contained in the Preisach plane has zero measure. This in particular holds when the measure μ is absolutely continuous with respect to the Lebesgue measure over the Preisach plane.

Proposition 1.5.6. *The following two conditions are equivalent*

$$(i) \qquad \mathcal{H}_\mu : \mathcal{C}^0([0,T]) \times \bar{\mathcal{S}} \to \mathcal{C}^0([0,T]) \qquad \textit{is uniformly strongly continuous}$$

$$(ii) \qquad |\mu|(B) = 0 \qquad \forall\, B \in \mathcal{B},$$

where $\mathcal{B} = \{B_\xi : \xi \in \bar{\mathcal{S}}\}$ are the graphs of the corresponding elements in $\bar{\mathcal{S}}$.

Proposition 1.5.7. *We get*

$$\mathcal{H}_\mu : \mathcal{C}^0([0,T]) \times \bar{\mathcal{S}} \to \mathcal{C}^0([0,T]) \qquad \textit{is Lipschitz continuous with Lipschitz constant } L$$

if and only if

$$\sup_{B\in\mathcal{B}} |\mu|(N(B,\varepsilon)) \le L\,\varepsilon \qquad \forall\,\varepsilon>0,$$

where

$$N(B,\delta) := \{(s_1, s_2+\alpha)\in\mathbb{R}^+\times\mathbb{R} : (s_1,s_2)\in B, |\alpha|\le\delta\},$$

for any $B\in\mathcal{B}$ and any $\varepsilon>0$.

1.5.3. A particular situation

Suppose now that in (1.5.5) the measure μ is absolutely continuous with respect to the two-dimensional Lebesgue measure. This means that there exists $\psi\in L^1_{\mathrm{loc}}(\mathcal{P})$ such that

$$\mathcal{H}_\mu(u,\xi)(t) := \int_0^\infty\int_{-\infty}^\infty h_{(r,v)}(u,\xi_{(r,v)})\,\psi(r,v)\,dv\,dr. \tag{1.5.6}$$

We do the following technical assumptions:

Assumptions 1.5.8.

∗ the antisymmetric part of ψ stays in $L^1(\mathcal{P})$, i.e.

$\psi_a(r,v) := \dfrac{1}{2}\,(\psi(r,v)-\psi(r,-v))\in L^1(\mathcal{P})$;

∗ the integral in (1.5.6) is considered in the sense of principal value;

∗ there exist β_0, $\beta_1\in L^1_{\mathrm{loc}}(0,\infty)$, $\beta_1(r)\ge\beta_0(r)\ge 0$ a.e., $b_0 := \displaystyle\int_0^\infty \beta_0(r)\,dr$ such that $b_0<\infty$ and $\beta_1(r)\ge\psi(r,v)\ge-\beta_0(r)$ for a.e. $(r,v)\in\mathcal{P}$.

We also put $b_1(R) := \displaystyle\int_0^R \beta_1(r)\,dr$ for $R>0$.

Now we use Lemma 1.5.2 to derive a new representation of the Preisach operator in this particular situation. Without loss of generality, we replace in (1.5.6) the density function ψ with $\dfrac{1}{2}\,\psi$. So in the assumptions of Lemma 1.5.2, we can certainly say that

$$\frac{1}{2}\int_0^\infty\int_{-\infty}^\infty h_{(r,v)}(u,\xi_{(r,v)})\,\psi(r,v)\,dv\,dr = \frac{1}{2}\int_0^\infty\left[\int_{-\infty}^{\wp_r(\lambda,u)}\psi(r,v)\,dv - \int_{\wp_r(\lambda,u)}^\infty\psi(r,v)\,dv\right]dr$$

$$=\frac{1}{2}\int_0^\infty\int_0^{\wp_r(\lambda,u)}\psi(r,v)\,dv\,dr + \frac{1}{2}\int_0^\infty\int_{-\infty}^0\psi(r,v)\,dv\,dr - \frac{1}{2}\int_0^\infty\int_{\wp_r(\lambda,u)}^\infty\psi(r,v)\,dv\,dr$$

$$=-\frac{1}{2}\int_0^\infty\int_0^\infty\psi(r,v)\,dv\,dr + \frac{1}{2}\int_0^\infty\int_0^\infty\psi(r,-v)\,dv\,dr + \int_0^\infty\int_0^{\wp_r(\lambda,u)}\psi(r,v)\,dv\,dr$$

$$=-\int_0^\infty\int_0^\infty\frac{1}{2}\,(\psi(r,v)-\psi(r,-v))\,dv\,dr + \int_0^\infty\int_0^{\wp_r(\lambda,u)}\psi(r,v)\,dv\,dr$$

$$= - \int_0^\infty \int_0^\infty \psi_a(r,v)\, dr\, dv + \int_0^\infty g(r, \wp_r(\lambda,u)(t))\, dr,$$

where we set

$$g(r,v) := \int_0^v \psi(r,z)\, dz \qquad \text{for } (r,v) \in \mathcal{P}. \tag{1.5.7}$$

This justifies the following definition (which is equivalent to the corresponding definition of the Preisach operator (1.5.5) in the particular case when Assumptions 1.5.8 hold, see also Remark 1.4.1).

Definition 1.5.9. *Let $\psi \in L^1_{\mathrm{loc}}(\mathcal{P})$ be given and let g be chosen as in* (1.5.7)*. Then the Preisach operator $\mathcal{W} : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ generated by the function g is defined by the formula*

$$\mathcal{W}[\lambda,u](t) := \int_0^\infty g(r, \wp_r[\lambda,u](t))\, dr \tag{1.5.8}$$

for any given $\lambda \in \Lambda_0$, $u \in \mathcal{C}^0([0,T])$ and $t \in [0,T]$.

Then we have the following result (see [25], Section II.3, Proposition 3.11).

Proposition 1.5.10. *Let Assumptions* 1.5.8 *be satisfied and let $R > 0$ be given. Then for every $\lambda, \mu \in \Lambda_R$ and $u, v \in \mathcal{C}^0([0,T])$ such that $||u||_{\mathcal{C}^0([0,T])}, ||v||_{\mathcal{C}^0([0,T])} \le R$, the Preisach operator* (1.5.8) *satisfies*

$$||\mathcal{W}[\lambda,u] - \mathcal{W}[\mu,v]||_{\mathcal{C}^0([0,T])} \le \int_0^R |\lambda(r) - \mu(r)|\, \beta_1(r)\, dr \;+\; b_1(R)\, ||u-v||_{\mathcal{C}^0([0,T])}.$$

Before going on we introduce the *Preisach potential energy* $\mathcal{U}$ as

$$\mathcal{U}[\lambda,u](t) := \int_0^\infty G(r, \wp_r[\lambda,u](t))\, dr, \tag{1.5.9}$$

where

$$G(r,v) := v\, g(r,v) - \int_0^v g(r,z)\, dz = \int_0^v z\, \psi(r,z)\, dz,$$

with $\psi(r,z) = \partial_z g(r,z)$. We moreover introduce the *Preisach dissipation operator* as

$$\mathcal{D}[\lambda,u](t) := \int_0^\infty r\, g(r, \wp_r[\lambda,u](t))\, dr. \tag{1.5.10}$$

If in addiction to Assumptions 1.5.8 we also add the following assumptions

Assumptions 1.5.11.

$$(i) \quad \frac{\partial \psi}{\partial v} \in L^\infty_{\mathrm{loc}}(\mathcal{P});$$

$$(ii) \quad \psi(r,v) \ge 0, \quad \text{a.e.},$$

we recover the following results (see [25], Section II.4, Proposition 4.8 and Theorem 4.3 respectively).

Proposition 1.5.12. *Let Assumptions* 1.5.8 *and* 1.5.11 *be satisfied and let* $R > 0$ *be given. Suppose moreover to have* $b \geq b_0$, $\lambda \in \Lambda_R$ *and* $u \in W^{1,1}(0,T)$ *be given such that* $||u||_{\mathcal{C}^0([0,T])} \leq R$. *Put* $w := b\,u + \mathcal{W}[\lambda, u]$. *Then*

$$(b - b_0)\,\dot{u}^2(t) \leq \dot{w}(t)\,\dot{u}(t) \leq (b + b_1(R))\dot{u}^2(t) \qquad \text{a.e.}$$

Theorem 1.5.13. *Let Assumptions* 1.5.8 *and* 1.5.11 *be satisfied and let* $R > 0$ *be given. For arbitrary* $\lambda \in \Lambda_R$ *and* $u \in W^{1,1}(0,T)$ *such that* $||u||_{\mathcal{C}^0([0,T])} \leq R$, *we put*

$$w := \mathcal{W}[\lambda, u] \qquad U := \mathcal{U}[\lambda, u] \qquad D := \mathcal{D}[\lambda, u],$$

where $\mathcal{U}$ *and* $\mathcal{D}$ *are respectively the Preisach potential energy and the Preisach dissipation operator introduced in* (1.5.9) *and* (1.5.10). *Then we have*

$$\begin{aligned} &(i) \quad U(t) \geq \frac{1}{2b_1(R)} w^2(t) \qquad \forall\, t \in [0,T] \\ &(ii) \quad \dot{w}(t)\,u(t) - \dot{U}(t) = |\dot{D}(t)| \qquad \text{a.e.} \end{aligned}$$

We go on with the presentation of another important result which will be useful later (see [25], Section II.3, Theorem 3.17, see also [12], Theorem 5.8).

Theorem 1.5.14. *Let Assumptions* 1.5.8 *be fulfilled and let* $\lambda \in \Lambda_0$, $b > b_0$ *be given. Then the operator* $b\,I + \mathcal{W}(\lambda, \cdot) : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$, *where* I *is the identity operator and* $\mathcal{W}$ *is the Preisach operator introduced in* (1.5.8), *is invertible and its inverse is Lipschitz continuous.*

We conclude this subsection with the following theorem, which illustrates the assumptions under which also the Preisach operator satisfies, in a certain region, the convexity property (see [25], Section II.4, Proposition 4.22).

Theorem 1.5.15. *Let* $\mathcal{W}$ *be the Preisach operator in* (1.5.8), *fulfilling Assumptions* 1.5.8 *and* 1.5.11. *Suppose that there exists* $\rho > 0$ *such that* $A_\rho := \operatorname{ess\,inf}\{\psi(r,v) : |v| + r \leq \rho\} > 0$. *Then there exists* $R > 0$ *such that, for every* $\lambda_0 \in \Lambda_R$ *and* $b \geq 0$, *the operator* $b\,I + \mathcal{W}(\lambda_0, \cdot)$ *satisfies the hypothesis of Theorem* 1.3.6.

1.6. Space dependent memory operators

The hysteresis operators introduced in Section 1.1 work between spaces of continuous functions (or their suitable subspaces), i.e.

$$\mathcal{F} : \mathcal{C}^0([0,T]) \times X \to \mathcal{C}^0([0,T]), \tag{1.6.1}$$

where X is a suitable metric space which contains all the information about the desired initial state. These operators are usually employed in problems in which time is the only independent variable, like in the case of O.D.E.s. When also the space variable appears, for example as in the case of P.D.E.s, (and so like the situations we will deal throughout the thesis), then operators (1.6.1) cannot be directly applied and it is necessary to extend $\mathcal{F}$ to some suitable operator $\overline{\mathcal{F}}$ acting between Fréchet spaces involving also the space variable.
Indeed, given an operator $\mathcal{F}$ as in (1.6.1), we can introduce, for any $u : \Omega \times [0,T] \to \mathbb{R}$ and any $\varepsilon^0 : \Omega \to X$ the corresponding space dependent operator $\tilde{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T]) \times X) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ as follows

$$[\tilde{\mathcal{F}}(u, \varepsilon^0)](x,t) := [\mathcal{F}(u(x,\cdot), \varepsilon^0(x))](t) \qquad \forall\, (x,t) \in \Omega \times [0,T].$$

Then, if $\mathcal{F}$ is a hysteresis operator which is Lipschitz continuous and piecewise monotone in the sense of (1.1.7), it is easy to verify (see also Proposition A.1.2) that, for any fixed $\varepsilon^0 : \Omega \to X$, the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ defined as follows

$$\overline{\mathcal{F}}(u)(x,t) := \tilde{\mathcal{F}}(u, \varepsilon^0)(x,t)$$

fulfills (2.1.11), (2.1.12), (2.1.13), (2.1.25), (3.1.2), (3.1.3) and (3.1.4), all properties which will be used in the following.
We therefore carry on our analysis dealing with operators of the following type

$$\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \tag{1.6.2}$$

where $\mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ is the space of strongly measurable functions briefly recalled in Subsection A.1.2.
We finally remark that, even if in the most common physical situations and applications one usually encounters rate independent operators (and so hysteresis operators), the results contained in this thesis are valid for the more general class of memory operators (of type (1.6.2)). This does not prevent us to keep in mind the previous extension and above all the case when $\mathcal{F}$ is rate independent, and also to check, when possible, the consistency of our theorems for the choice of a suitable hysteresis operator.

Just to make an example (and to introduce a result which will be relevant later), consider any finite (signed) Borel measure μ defined in $\mathcal{P}$, (where $\mathcal{P}$ is the Preisach plane, see (1.5.2)) and let $\mathcal{H}_\mu$ be the corresponding Preisach operator introduced in (1.5.5). Let us briefly consider how the Preisach operator acts on functions dependent on a space parameter, in view of the study of our partial differential equations containing the hysteresis term.
We therefore introduce, for any $(u,\xi) \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]) \times \bar{\mathcal{S}})$ the corresponding space dependent Preisach operator in the following way

$$[\overline{\mathcal{H}}_\mu(u,\xi)](x,t) := [\mathcal{H}_\mu(u(x,\cdot), \xi(x,\cdot))](t) \qquad \forall\, t \in [0,T], \ \text{ a.e. in } \Omega, \tag{1.6.3}$$

where $\bar{\mathcal{S}}$ was introduced in Subsection 1.5.2. Then the following result holds (for more details concerning also the proof of this proposition, see [39], Chapter IV, Proposition 3.11).

Proposition 1.6.1. *The operator* $\overline{\mathcal{H}}_\mu$ *introduced in* (1.6.3) *is a causal and rate independent operator, i.e. it is a hysteresis operator; if* $\mu \geq 0$ *then* $\overline{\mathcal{H}}_\mu$ *is piecewise monotone and order preserving. If*

$$|\mu|(\mathbb{R} \times \{r\}) = |\mu|(\{r\} \times \mathbb{R}) = 0 \qquad \forall\, r \in \mathbb{R}, \tag{1.6.4}$$

holds, then for any $\xi \in \mathcal{M}(\Omega; \bar{\mathcal{S}})$, $\overline{\mathcal{H}}_\mu(\cdot, \xi) : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ *is a continuous operator. Moreover, if*

$$\sup_{B \in \mathcal{B}} |\mu|(x, N(B, \varepsilon)) \leq L\, \varepsilon \qquad \forall\, \varepsilon > 0, \tag{1.6.5}$$

holds, (where $N(B, \varepsilon)$ *and* $\mathcal{B}$ *were introduced in Subsection* 1.5.2*), then for any* $p \in [1, +\infty]$, $\overline{\mathcal{H}}_\mu : L^p(\Omega; \mathcal{C}^0([0,T]) \times \bar{\mathcal{S}}) \to L^p(\Omega; \mathcal{C}^0([0,T]))$ *is a Lipschitz continuous operator. Finally, under the same assumption, for any* $s \in [0,1]$ *and for any* $q \in [1, +\infty]$, $\overline{\mathcal{H}}_\mu : W^{s,q}(\Omega; \mathcal{C}^0([0,T]) \times \bar{\mathcal{S}}) \to W^{s,q}(\Omega; \mathcal{C}^0([0,T]))$ *is a linearly bounded operator.*

CHAPTER 2

First class of P.D.E.s with hysteresis

The aim of this chapter is to study a class of P.D.E.s containing a continuous hysteresis operator $\overline{\mathcal{G}}$, whose model equation can be represented by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \triangle \left(\frac{\partial u}{\partial t} + \overline{\mathcal{G}}(u) \right) = f, \tag{2.0.1}$$

where $\triangle$ is the Laplace operator and f is a given function. This model equation arises in the context of electromagnetic processes. In particular it is obtained by coupling in a suitable way the Maxwell equations, the Ohm law and a constitutive relation between the magnetic field and the magnetic induction. More precisely we consider the Maxwell equations and the Ohm law defined in an open bounded set $\Omega \subset \mathbb{R}^3$; then we simplify our vectorial model by imposing severe restrictions on the geometry of the system, from what comes out the scalar character of our model equation (2.0.1).
First of all, we introduce a weak formulation in Sobolev spaces for the Cauchy problem associated to equation (2.0.1). Under suitable assumptions on the hysteresis operator $\overline{\mathcal{G}}$ we get existence and uniqueness of the solution by means of a technique based on the contraction mapping principle. Then we obtain Lipschitz continuous dependence on the data and further space-regularity results for the Cauchy problem associated to this class of P.D.E.s. Finally we discuss these results for a particular choice of the operator $\overline{\mathcal{G}}$.

2.1. First model problem

2.1.1. First physical interpretation: electromagnetic processes

Electromagnetic processes in ferromagnetic materials can be described by coupling in a suitable way the Maxwell equations with the Ohm law. Here we consider $\mathcal{D} \subset \mathbb{R}^3$ to be an electromagnetic material; we set $\mathcal{D}_T := \mathcal{D} \times (0,T)$, for a fixed $T > 0$ and we recall the Ampère, Faraday and Ohm laws (where $\nabla\times$ is the curl operator)

$$c\,\nabla \times \vec{H} = 4\pi\,\vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{in } \mathcal{D}_T \tag{2.1.1}$$

$$c\,\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{in } \mathcal{D}_T \tag{2.1.2}$$

$$\vec{J} = \sigma(\vec{E} + \vec{g}) \qquad \text{in } \mathcal{D}_T$$

where c is the speed of light in vacuum, $\vec{H}$ is the magnetic field, $\vec{J}$ is the electric current density, $\vec{D}$ is the electric displacement, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic induction, σ is the electric conductivity and finally $\vec{g}$ is an applied electromotive force. For more details about these facts see a classical text of electromagnetism, for example [20].
In a ferromagnetic material we may assume that $\vec{D} = \epsilon\vec{E}$, where ϵ is the dielectric permittivity. Applying the curl operator to (2.1.1), differentiating (2.1.2) in time and eliminating $\vec{J}, \vec{D}$ and $\vec{E}$ we then get

$$\epsilon\,\frac{\partial^2 \vec{B}}{\partial t^2} + 4\pi\sigma\,\frac{\partial \vec{B}}{\partial t} + c^2\,\nabla \times \nabla \times \vec{H} = 4\pi\,c\,\sigma\,\nabla \times \vec{g} \qquad \text{in } \mathcal{D}_T. \tag{2.1.3}$$

This equation is a bit difficult to work with as the variables involved are vectors; that's why we simplify equation (2.1.3) considering it in a particular setting. More precisely, let Ω be a domain of $\mathbb{R}^2$, we set $\Omega_T := \Omega \times (0,T)$ and assume that (using orthogonal Cartesian coordinates x, y, z) $\vec{H}$ is parallel to the z-axis and only depends on the coordinates x, y, i.e. $\vec{H} = (0, 0, H(x, y))$. Then

$$\nabla \times \nabla \times \vec{H} = (0, 0, -\triangle_{x,y} H) \qquad \left(\triangle_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \qquad \text{in } \Omega_T.$$

Dealing with a strongly anisotropic material, we can also assume that $\vec{B} = (0, 0, B(x, y))$. If moreover $\nabla \times \vec{g} := (0, 0, \tilde{f})$, the equation (2.1.3) is then reduced to an equation for scalar variables

$$\epsilon\,\frac{\partial^2 B}{\partial t^2} + 4\pi\sigma\,\frac{\partial B}{\partial t} - c^2\,\triangle_{x,y} H = 4\pi\,c\,\sigma\,\tilde{f} =: f \qquad \text{in } \Omega_T. \tag{2.1.4}$$

At this point we would like to combine equation (2.1.4) with the relation

$$H = \overline{\mathcal{G}}(B) + \gamma\,\frac{\partial B}{\partial t} \tag{2.1.5}$$

where $\overline{\mathcal{G}}$ is a suitable hysteresis operator and $\gamma > 0$ is a given constant. The relation (2.1.5) can be for example obtained by the combination in series of a ferromagnetic element with hysteresis and a conducting solenoid filled with a paramagnetic material.
In the case of a ferromagnetic element with hysteresis we may assume that $B = H + 4\pi M$ and the magnetization M depends with hysteresis on H, i.e. $M = \overline{\mathcal{W}}(H)$ where $\overline{\mathcal{W}}$ is a scalar

Preisach operator. This means that $B = H + 4\pi\overline{\mathcal{W}}(H) =: \overline{\mathcal{F}}(H)$. Provided that the inverse of the operator $\overline{\mathcal{F}}$ exists (we will discuss this fact in Subsection 2.1.7) we may then assume that

$$H = \overline{\mathcal{F}}^{-1}(B) =: \overline{\mathcal{G}}(B).$$

In the case of the conducting solenoid having a paramagnetic core, the equation

$$H = \gamma \frac{\partial B}{\partial t} \tag{2.1.6}$$

describes the so called *linear induction*; this equation can be justified as follows: a flux variation $\dfrac{\partial B}{\partial t}$ induces the magnetic field

$$\tilde{H} = -\gamma \frac{\partial B}{\partial t}$$

and this can be seen using the Faraday-Lenz and the Ampère laws (the constant $\gamma > 0$ depends on the geometry of the circuit). Hence to vary the flux, the opposite magnetic field must be applied $H = -\tilde{H}$ and this leads to equation (2.1.6). So in (2.1.5) we have the presence of a rate independent element and a rate dependent one.

If we consider the equation which results from the combination of (2.1.4) with (2.1.5) and we write it without displaying the coefficients (in order to simplify our formula layout, since they do not play any role in our developments) we get

$$\frac{\partial^2 B}{\partial t^2} + \frac{\partial B}{\partial t} - \triangle_{x,y}\left(\frac{\partial B}{\partial t} + \overline{\mathcal{G}}(B)\right) = f.$$

2.1.2. Second physical interpretation: a model of visco-elasto-plasticity

Let us consider the evolution of a continuous medium which at the beginning occupies a bounded domain $\Omega \subset \mathbb{R}^3$; let $\{x_i(t)\}_{i=1,2,3}$ be the coordinates at time t of a generic material particle with respect to a system of orthogonal Cartesian axes. Following for example [39], Sections II.2 and VII.1 (for more details see also [15] and [17]), we introduce the following quantities:

➜ displacement vector

$$u_i(x,t) := x_i(t) - x_i(0) \qquad (i = 1,2,3)$$

➜ (linearized) strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (i = 1,2,3)$$

which gives a measure of the local deformation of the body.

➜ STRESS TENSOR σ which represents the interior traction exerted on the surface due to the deformations. The tensor σ is symmetric, as a consequence of the principle of conservation of angular momentum, i.e.

$$\sigma_{ij} = \sigma_{ji} \qquad (i, j = 1, 2, 3).$$

Let h be an external force field applied to the body. In the classical continuum mechanics the dynamic equation reads

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \sigma + h, \qquad \text{i.e.} \qquad \rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{\ell=1}^{3} \frac{\partial \sigma_{i\ell}}{\partial x_\ell} + h_i \qquad \text{in } \Omega_T := \Omega \times (0, T),\ i = 1, 2, 3, \tag{2.1.7}$$

where ρ is the density.
We moreover introduce the tensors $A\sigma$ and g in the following way

$$(A\sigma)_{ij} := -\frac{1}{2} \sum_{\ell=1}^{3} \left(\frac{\partial^2 \sigma_{i\ell}}{\partial x_j \partial x_\ell} \right), \qquad g_{ij} := \frac{1}{2} \left(\frac{\partial h_i}{\partial x_j} + \frac{\partial h_j}{\partial x_i} \right) \qquad \text{in } \Omega_T,\ i, j = 1, 2, 3.$$

We consider the density ρ normalized and we deal with homogeneous Dirichlet boundary conditions; if we differentiate (2.1.7) with respect to the space variable and take the symmetric parts, we have

$$\frac{\partial^2 \varepsilon}{\partial t^2} + A\sigma = g, \qquad \text{i.e.} \qquad \frac{\partial^2 \varepsilon_{ij}}{\partial t^2} + (A\sigma)_{ij} = g_{ij} \qquad \text{in } \Omega_T,\ i, j = 1, 2, 3. \tag{2.1.8}$$

At this point, we suppose to deal with *univariate systems*; in this case the strain and the stress tensors are scalars and equation (2.1.8) becomes

$$\frac{\partial^2 \varepsilon}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} = g \qquad \text{in } \Omega \times (0, T), \tag{2.1.9}$$

where this time Ω is a bounded interval of $\mathbb{R}$.
Now we would like to couple equation (2.1.9) with a constitutive relation between ε and σ. We choose the following one

$$\sigma = \overline{\mathcal{G}}(\varepsilon) + \eta\, \dot{\varepsilon}, \tag{2.1.10}$$

where $\overline{\mathcal{G}}$ is a suitable hysteresis operator and $\eta > 0$ is a constant.
Equation (2.1.10) is a combination in series of two elements:
- a rate independent element, described by the constitutive relation $\sigma = \overline{\mathcal{G}}(\varepsilon)$. This may represent for example an elastic and a plastic element in series and therefore $\overline{\mathcal{G}}$ will be a stop operator (see for example Section 1.4);
- a rate dependent element, described by the constitutive relation $\sigma = \eta\, \dot{\varepsilon}$. This takes into

account the so-called *linear short-memory viscosity* and corresponds to the rheological model known as Newton's fluid.
Coupling (2.1.9) and (2.1.10) we have

$$\frac{\partial^2 \varepsilon}{\partial t^2} + A\frac{\partial \varepsilon}{\partial t} + A\overline{\mathcal{G}}(\varepsilon) = f \qquad \text{in } \Omega \times (0,T),$$

where $A = -\dfrac{\partial^2}{\partial x^2}$ and this equation belongs to the class of model equations studied in this chapter, in the one-dimensional case.

2.1.3. Choice of the functional setting; weak formulation of the problem

We fix an open bounded set $\Omega \subset \mathbb{R}^N$, $N \geq 1$ of Lipschitz class with boundary Γ and set $Q := \Omega \times (0,T)$.
Although the physical situations outlined in Subsections 2.1.1 and 2.1.2 say that the model equation (2.0.1) is meaningful if $\Omega \subset \mathbb{R}$ or $\Omega \subset \mathbb{R}^2$, we choose to deal here with the more general situation $\Omega \subset \mathbb{R}^N$, $N \geq 1$.
We consider the following operator

$$\overline{\mathcal{G}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$$

which is assumed to be causal, i.e.

$$\begin{cases} \forall\, v_1, v_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])),\ \ \forall\, t \in [0,T],\ \text{if } v_1 = v_2 \text{ in } [0,t], \text{ a.e. in } \Omega, \\ \text{then } [\overline{\mathcal{G}}(v_1)](\cdot,t) = [\overline{\mathcal{G}}(v_2)](\cdot,t) \text{ a.e. in } \Omega, \end{cases} \tag{2.1.11}$$

where we denoted by $\mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ the Fréchet space of (strongly) measurable functions $\Omega \to \mathcal{C}^0([0,T])$ (see Subsection A.1.2). Moreover we suppose that $\overline{\mathcal{G}}$ is Lipschitz continuous in the following sense

$$\begin{cases} \exists\, L > 0 :\ \forall\, u_1, u_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \\ ||\, [\overline{\mathcal{G}}(u_1)](x,\cdot) - [\overline{\mathcal{G}}(u_2)](x,\cdot)||_{\mathcal{C}^0([0,T])} \leq L\, ||u_1(x,\cdot) - u_2(x,\cdot)||_{\mathcal{C}^0([0,T])} \qquad \text{a.e. in } \Omega. \end{cases} \tag{2.1.12}$$

We also assume that, for a.a. $x, y \in \Omega$, for all $u \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$,

$$||\, [\overline{\mathcal{G}}(u)](x,\cdot) - [\overline{\mathcal{G}}(u)](y,\cdot)||_{\mathcal{C}^0([0,T])} \leq \tilde{L}||u(x,\cdot) - u(y,\cdot)||_{\mathcal{C}^0([0,T])}. \tag{2.1.13}$$

It is certainly not restrictive to assume that $\tilde{L} = L$.
At this point we have to discuss the setting of our model problem. The choice of the right functional spaces to work with plays a fundamental role in our results. In our case the choice we make is a bit unusual in the sense that we don't work with the classical Hilbert triplet

$$H_0^1(\Omega) \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset H^{-1}(\Omega)$$

but we change the pivot space. This argument is not new in literature; some examples can be found in [28], Chapter 2, Section 3 or in [40], Section II.6. Let us explain a bit our choice which will be relevant for the right interpretation of the weak formulation of our problem.
⇨ First of all we consider the injection of the space $L^2(\Omega)$ into the space $H^{-1}(\Omega)$. More precisely we take the map $j : L^2(\Omega) \to H^{-1}(\Omega)$ which acts in the following way

$$ {}_{H^{-1}(\Omega)}\langle j\,(f), \varphi \rangle_{H^1_0(\Omega)} := \int_\Omega f\;\varphi\,dx \qquad \forall\, f \in L^2(\Omega),\ \forall\, \varphi \in H^1_0(\Omega). \tag{2.1.14} $$

It is not difficult to see that j is a continuous and dense injection, i.e. $L^2(\Omega)$ is a linear subspace of $H^{-1}(\Omega)$ and it is dense with respect to the strong topology of $H^{-1}(\Omega)$. By Theorem A.6.1 we have that $(H^{-1}(\Omega))'$ can be identified with a linear subspace of $(L^2(\Omega))'$, i.e. $(H^{-1}(\Omega))' \subset (L^2(\Omega))'$ with continuous injection (let us call this map j^*). More precisely we identify functionals with their restrictions, i.e.

$$ {}_{(L^2(\Omega))'}\langle j^*\psi, f \rangle_{L^2(\Omega)} :={}_{(H^{-1}(\Omega))'}\ \langle \psi, j(f) \rangle_{H^{-1}(\Omega)} \qquad \forall\, \psi \in (H^{-1}(\Omega))',\ \forall\, f \in L^2(\Omega). \tag{2.1.15} $$

In the following we will avoid to write each time j, j^* when it will be clear from the context, in order to simplify the notations. So for example (2.1.15) will simply become

$$ {}_{(L^2(\Omega))'}\langle \psi, f \rangle_{L^2(\Omega)} :={}_{(H^{-1}(\Omega))'}\ \langle \psi, f \rangle_{H^{-1}(\Omega)}, \qquad \forall\, \psi \in (H^{-1}(\Omega))',\ \forall\, f \in L^2(\Omega). $$

⇨ Now we introduce the operator $A : H^1_0(\Omega) \to H^{-1}(\Omega)$ defined as follows

$$ {}_{H^{-1}(\Omega)}\ \langle Au, v\, \rangle_{H^1_0(\Omega)} := \int_\Omega \nabla u \cdot \nabla v\, dx \qquad \forall u, v \in H^1_0(\Omega); $$

so it is clear that $Au = -\triangle\, u \left(:= -\sum_{i=1}^{N} \dfrac{\partial^2 u}{\partial x_i^2} \right)$ in the sense of distributions.
In this setting we can think to invert the Laplace operator, i.e. the operator A^{-1} can be interpreted as the inverse of the operator $-\triangle$ associated with the homogeneous Dirichlet boundary conditions. More precisely, for any $v \in H^{-1}(\Omega)$,

$$ u = A^{-1}v \quad \text{if and only if} \quad \begin{cases} u \in H^1(\Omega) \\ -\triangle u = v \quad \text{in } \mathcal{D}'(\Omega) \\ \gamma_0 u = 0 \quad \text{on } \partial\Omega = \Gamma, \end{cases} \tag{2.1.16} $$

where $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ is the unique linear continuous trace operator such that

$$ \gamma_0 v = v_{|\Gamma} \qquad \forall\, v \in \mathcal{C}^\infty(\bar{\Omega}) \cap H^1(\Omega). $$

⇨ At this point we consider the space $H^{-1}(\Omega)$ endowed with the scalar product

$$ (u, v)_{H^{-1}(\Omega)} :={}_{H^{-1}(\Omega)}< u, A^{-1}v >_{H^1_0(\Omega)}\ . \tag{2.1.17} $$

It is clear, using (2.1.14), that

$$(u,v)_{H^{-1}(\Omega)} := \int_\Omega u\ A^{-1}v\,dx \qquad \forall\, u \in L^2(\Omega).$$

⇨ Finally we identify the space $H^{-1}(\Omega)$ to its dual by means of the Riesz operator $\mathcal{R} : H^{-1}(\Omega) \to (H^{-1}(\Omega))'$ which acts in the following way

$$_{(H^{-1}(\Omega))'}\langle \mathcal{R}u, v\rangle_{H^{-1}(\Omega)} := (u,v)_{H^{-1}(\Omega)} \qquad \forall\, u,v \in H^{-1}(\Omega). \tag{2.1.18}$$

Let us remark that with this identification we immediately get, (omitting from now on also the Riesz operator $\mathcal{R}$ for the sake of simplicity),

$$\begin{aligned} _{(L^2(\Omega))'}\langle \psi, f\rangle_{L^2(\Omega)} &\overset{(2.1.15)\,(2.1.18)}{=} (\psi,f)_{H^{-1}(\Omega)} = (f,\psi)_{H^{-1}(\Omega)} \overset{(2.1.17)}{=} {}_{H^{-1}(\Omega)}\langle f, A^{-1}\psi\rangle_{H_0^1(\Omega)} \\ &\overset{(2.1.14)}{=} \int_\Omega f\ A^{-1}\psi\,dx \qquad \forall\, f \in L^2(\Omega),\ \forall\,\psi \in H^{-1}(\Omega), \end{aligned}$$

where we also used the fact that the scalar product $(\cdot,\cdot)_{H^{-1}(\Omega)}$ is symmetric.
As $L^2(\Omega) \subset H^{-1}(\Omega)$ with continuous and dense injection, we then have the Hilbert triplet

$$L^2(\Omega) \subset H^{-1}(\Omega) \equiv (H^{-1}(\Omega))' \subset (L^2(\Omega))'$$

with continuous and dense injections.
From now on we set $L^2(\Omega) := V,\ H^{-1}(\Omega) := H$ and $(L^2(\Omega))' := V'$. We assume that $u^0 \in V$, $v^0 \in H$ and $f \in L^2(0,T;H)$. We want to solve the following problem

Problem 2.1.1. *To find two functions $u \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \cap H^1(0,T;V)$ and $v \in L^2(Q)$ such that $\overline{\mathcal{G}}(u) \in L^2(Q)$ and for any $\psi \in H^1(0,T;V)$ with $\psi(\cdot,T) = 0$ a.e. in Ω*

$$\begin{aligned} \int_0^T - {}_{V'}\langle v+u, \frac{\partial\psi}{\partial t}\rangle_V\,dt + \int_0^T\int_\Omega (v+\overline{\mathcal{G}}(u))\,\psi\,dx\,dt = \int_0^T {}_{V'}\langle f,\psi\rangle_V\,dt \\ + {}_{V'}\langle (v^0+u^0)(\cdot), \psi(\cdot,0)\rangle_V \end{aligned} \tag{2.1.19}$$

$$-\int_0^T\int_\Omega u\,\frac{\partial\psi}{\partial t}\,dx\,dt = \int_0^T\int_\Omega v\,\psi\,dx\,dt + \int_\Omega u^0(\cdot)\,\psi(\cdot,0)\,dx. \tag{2.1.20}$$

Interpretation. First of all we want to prove that (2.1.19) yields

$$A^{-1}\left(\frac{\partial v}{\partial t} + \frac{\partial u}{\partial t}\right) + v + \overline{\mathcal{G}}(u) = A^{-1}f \qquad \text{in } \mathcal{D}'(0,T;L^2(\Omega)), \tag{2.1.21}$$

where the space $\mathcal{D}'(0,T;X)$ with X a Banach space is briefly recalled in Subsection A.1.3 (see also the reference therein).

In fact, let us take any $\varphi \in \mathcal{D}(0,T)$ and any $\eta \in L^2(\Omega)$. We consider $\psi \in \mathcal{D}(0,T; L^2(\Omega))$ defined in the following way

$$\psi(t,x) := \varphi(t)\eta(x).$$

It is easy to see that in particular $\psi \in H^1(0,T;V)$ and $\psi(\cdot,T) = 0$ a.e. in Ω, so (2.1.19) holds for this particular choice of ψ. Taking into account that $\dfrac{\partial\varphi}{\partial t}(t) \in \mathbb{R}$ for any $t \in [0,T]$, we have

$$\int_0^T -{}_{V'}\langle v(t)+u(t), \frac{\partial\psi}{\partial t}(t)\rangle_V \, dt \overset{(2.1.15)\,(2.1.18)}{=} \int_0^T -\left(v(t)+u(t), \frac{\partial\varphi}{\partial t}(t)\,\eta \right)_{H^{-1}(\Omega)} dt$$

$$\overset{(2.1.17)}{=} \int_0^T -{}_{H^{-1}(\Omega)}\langle \eta, \frac{\partial\varphi}{\partial t}(t)\, A^{-1}(v(t)+u(t))\rangle_{H_0^1(\Omega)} \, dt \overset{(2.1.14)}{=} -\int_0^T \int_\Omega \frac{\partial\varphi}{\partial t} A^{-1}(v+u)\,\eta \, dx\, dt,$$

where we also used the fact the scalar product of $H^{-1}(\Omega)$ is symmetric and the fact that η does not depend on t. The same can be done with the term $\displaystyle\int_0^T {}_{V'}\langle f, \psi\rangle_V \, dt$ so that we obtain for all $\eta \in L^2(\Omega)$

$$\int_\Omega \left[\int_0^T \left\{ -A^{-1}(v(t)+u(t)) \frac{\partial\varphi}{\partial t}(t) + (v(t)+\overline{\mathcal{G}}(u)(t))\,\varphi(t) - A^{-1}f(t)\varphi(t) \right\} dt \right] \eta \, dx = 0;$$

let us notice that with our choice of ψ the term ${}_{V'}\langle (v^0+u^0)(\cdot), \psi(\cdot,0)\rangle_V$ vanishes as, a.e. in Ω, we have $\psi(\cdot,0) = 0$. The previous equation is equivalent to

$$\int_0^T \left\{ -A^{-1}(v(t)+u(t)) \frac{\partial\varphi}{\partial t}(t) + (v(t)+\overline{\mathcal{G}}(u)(t))\,\varphi(t) - A^{-1}f(t)\varphi(t) \right\} dt = 0 \quad \text{in } L^2(\Omega).$$

Using the remarks contained in Subsection A.1.3, we can easily get (2.1.21). Working in the same manner, the variational equation (2.1.20) is immediately equivalent to

$$\frac{\partial u}{\partial t} = v \qquad \text{in } \mathcal{D}'(0,T; L^2(\Omega))$$

and so at the end (2.1.19) and (2.1.20) yield

$$\begin{cases} A^{-1}\dfrac{\partial v}{\partial t} + A^{-1}\dfrac{\partial u}{\partial t} + v + \overline{\mathcal{G}}(u) = A^{-1} f \\[2ex] \dfrac{\partial u}{\partial t} = v \end{cases} \qquad \text{in } \mathcal{D}'(0,T; L^2(\Omega)). \tag{2.1.22}$$

At this point, by comparison we have that

$$A^{-1}\frac{\partial v}{\partial t} \in L^2(Q)$$

thus (2.1.22) holds in $L^2(\Omega)$ a.e. in $(0,T)$. Therefore $A^{-1}v \in H^1(0,T;L^2(\Omega))$ and, integrating by parts in (2.1.19) and (2.1.20), we obtain

$$A^{-1}\, u_{|t=0} = A^{-1}\, u^0 \qquad A^{-1}\, v_{|t=0} = A^{-1}\, v^0 \qquad \text{in } L^2(\Omega), \text{ in the sense of traces.} \tag{2.1.23}$$

In turn, (2.1.22) and (2.1.23) yield (2.1.19) and (2.1.20) and the two formulations are equivalent. We end this subsection by noticing that, if in addition the solution (u, v) is more regular in space, (as in Theorem 2.1.5), then (2.1.19) and (2.1.20) can be also interpreted as

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial t} - \triangle(v + \overline{\mathcal{G}}(u)) = f \\ \dfrac{\partial u}{\partial t} = v \end{cases} \qquad \text{in } L^2(0,T;V')$$

and so we come back to the original model equation from which our discussion started.

2.1.4. An existence and uniqueness result for the first model problem

Theorem 2.1.2. *(Existence and uniqueness)*
Let us assume that the operator $\overline{\mathcal{G}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ *is causal and Lipschitz continuous according to* (2.1.11) *and* (2.1.12)*. Suppose moreover that*

$$u^0 \in V, \quad v^0 \in H, \quad f \in L^2(0,T;H). \tag{2.1.24}$$

Then Problem 2.1.1 *has a unique solution*

$$u \in H^1(0,T;V), \qquad v \in L^2(Q)$$

such that

$$\overline{\mathcal{G}}(u) \in L^2(\Omega; \mathcal{C}^0([0,T])).$$

If moreover $\overline{\mathcal{G}}$ *fulfills the following* piecewise Lipschitz continuity property

$$\begin{cases} \exists\, \bar{L} > 0 : \ \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \ \forall\, [t_1, t_2] \subset [0,T] \\ \text{if } v(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega, \text{ then} \\ |[\overline{\mathcal{G}}(v)](x,t_2) - [\overline{\mathcal{G}}(v)](x,t_1)| \le \bar{L}\, |v(x,t_1) - v(x,t_2)| \qquad \text{a.e. in } \Omega, \end{cases} \tag{2.1.25}$$

then in addition we deduce that

$$\overline{\mathcal{G}}(u) \in H^1(0,T;L^2(\Omega)).$$

Proof. The proof of this theorem consists in two steps.
➜ step 1: freezing. In this first step we fix $z \in H^1(0,T;V)$ and consider the auxiliary problem obtained starting from Problem 2.1.1 and replacing $\overline{\mathcal{G}}(u)$ by $\overline{\mathcal{G}}(z)$. We study therefore the following problem

Problem 2.1.3. *If $z \in H^1(0,T;V)$ is a fixed function, we search for two functions $u \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \cap H^1(0,T;V)$ and $v \in L^2(Q)$ such that, for any $\psi \in H^1(0,T;V)$ with $\psi(\cdot,T) = 0$ a.e. in Ω*

$$\begin{aligned} \int_0^T - {}_{V'}\langle v+u, \frac{\partial \psi}{\partial t} \rangle_V \, dt + \int_0^T \int_\Omega (v + \overline{\mathcal{G}}(z))\, \psi \, dx\, dt &= \int_0^T {}_{V'}\langle f, \psi \rangle_V \, dt \\ &\quad + {}_{V'}\langle (v^0+u^0)(\cdot), \psi(\cdot,0) \rangle_V \end{aligned} \tag{2.1.26}$$

$$-\int_0^T \int_\Omega u \frac{\partial \psi}{\partial t}\, dx\, dt = \int_0^T \int_\Omega v\, \psi\, dx\, dt + \int_\Omega u^0(\cdot)\, \psi(\cdot,0)\, dx. \tag{2.1.27}$$

We can interpret this problem working as we did in the case of Problem 2.1.1, more precisely we get that (2.1.26) and (2.1.27) are equivalent to the following system

$$\begin{cases} A^{-1} \dfrac{\partial v}{\partial t} + A^{-1} \dfrac{\partial u}{\partial t} + v = A^{-1} f - \overline{\mathcal{G}}(z) = A^{-1}(f - A\,\overline{\mathcal{G}}(z)) =: A^{-1} G_z \\ \dfrac{\partial u}{\partial t} = v, \end{cases}$$

which holds in $\mathcal{D}'(0,T;L^2(\Omega))$ and the initial values

$$A^{-1}\, u_{|t=0} = A^{-1}\, u^0 \qquad A^{-1}\, v_{|t=0} = A^{-1}\, v^0 \qquad \text{in } L^2(\Omega), \text{ in the sense of traces.} \tag{2.1.28}$$

By comparison we have that the previous system actually holds in $L^2(Q)$ and then in $L^2(\Omega)$ a.e. in $(0,T)$. So in this case, in order to prove that this problem admits a unique solution (u,v), it is enough to prove that there exists a unique v such that

$$\int_\Omega \phi\, A^{-1} \frac{\partial v}{\partial t}\, dx + \int_\Omega (A^{-1} v + v)\, \phi\, dx = \int_\Omega \phi\, A^{-1} G_z\, dx \qquad \forall\, \phi \in L^2(\Omega), \text{ for a.a. } t \in [0,T], \tag{2.1.29}$$

with the initial data (2.1.28). We apply Theorem A.8.1 with the following choices: $H := H^{-1}(\Omega)$ endowed with the scalar product (2.1.17), $V := L^2(\Omega)$ and $a(t;\xi,\zeta) := \int_\Omega (A^{-1}\xi + \xi)\, \zeta\, dx$, for all $\xi, \zeta \in L^2(\Omega)$. Then condition (i) of Theorem A.8.1 is trivially verified; condition (ii) is also ok as

$$|a(t;\xi,\zeta)| = \left| \int_\Omega (A^{-1}\xi + \xi)\, \zeta\, dx \right| \le ||A^{-1}\xi||_{L^2(\Omega)}\, ||\zeta||_{L^2(\Omega)} \le c\, ||\xi||_{L^2(\Omega)}\, ||\zeta||_{L^2(\Omega)}$$

as $A^{-1} \in \mathcal{L}(L^2(\Omega); H^2(\Omega))$ (where $\mathcal{L}(L^2(\Omega); H^2(\Omega))$ is the space of linear and continuous operators from $L^2(\Omega)$ to $H^2(\Omega)$, for more details see for example [28], Chapter 1, Section 1.6 and the references therein). Moreover, as

$$a(t;\zeta,\zeta) = \int_\Omega (A^{-1}\zeta + \zeta)\, \zeta\, dx = \int_\Omega (A^{-1}\zeta)\, \zeta\, dx + ||\zeta||^2_{L^2(\Omega)} = ||\zeta||^2_{H^{-1}(\Omega)} + ||\zeta||^2_{L^2(\Omega)},$$

then also condition (iii) holds with $\alpha = 1$ and $C = -1$. We moreover choose $G \in L^2(0,T;V')$ in the following way

$$_{V'}\langle G(t), \varphi \rangle_V := \int_\Omega (A^{-1}f + \overline{\mathcal{G}}(z))\,\varphi\,dx \qquad \forall\,\varphi \in V, \ \text{ a.e. in } [0,T].$$

Then (2.1.29) holds.

➜ SECOND STEP: FIXED POINT. At this point, we define the set

$$B = \left\{ z \in H^1(0,T;V) : z(0) = u^0 \right\}.$$

For each $z \in B$ we found, in the previous step, a unique solution (u, v) of Problem 2.1.3. Thus we may introduce an operator

$$J : B \to B \qquad z \mapsto u.$$

Now we consider a couple of data $z_1, z_2 \in B$; let us define $u_1 := J(z_1)$, $u_2 := J(z_2)$. If $v_i := \dfrac{\partial u_i}{\partial t}$ for $i = 1, 2$, then

$$A^{-1}\,\frac{\partial}{\partial t}(v_1 + u_1) + v_1 + \overline{\mathcal{G}}(z_1) = A^{-1} f \qquad \text{in } L^2(Q)$$

and also

$$A^{-1}\,\frac{\partial}{\partial t}(v_2 + u_2) + v_2 + \overline{\mathcal{G}}(z_2) = A^{-1} f \qquad \text{in } L^2(Q).$$

Taking the difference between the two previous equations we have

$$A^{-1}\,\frac{\partial}{\partial t}(v_1 - v_2 + u_1 - u_2) + v_1 - v_2 + \overline{\mathcal{G}}(z_1) - \overline{\mathcal{G}}(z_2) = 0 \qquad \text{in } L^2(Q). \tag{2.1.30}$$

Let us multiply (2.1.30) by $(v_1 - v_2)$ in the scalar product of $L^2(\Omega)$; this is possible due to the fact that (2.1.30) holds in $L^2(Q)$. We have

$$\int_\Omega \left(A^{-1}\,\frac{\partial}{\partial t}(v_1 - v_2) \right) (v_1 - v_2)\,dx + \int_\Omega \left(A^{-1}\frac{\partial}{\partial t}(u_1 - u_2) \right) (v_1 - v_2)\,dx$$

$$+ \int_\Omega (v_1 - v_2 + \overline{\mathcal{G}}(z_1) - \overline{\mathcal{G}}(z_2))\,(v_1 - v_2)\,dx = \left(\frac{\partial}{\partial t}(v_1 - v_2), v_1 - v_2 \right)_H (t) + \,||v_1 - v_2||_H^2\,(t)$$

$$+ \int_\Omega (v_1 - v_2 + \overline{\mathcal{G}}(z_1) - \overline{\mathcal{G}}(z_2))\,(v_1 - v_2)\,dx = \frac{1}{2}\,\frac{d}{dt}||v_1 - v_2||_H^2(t) + ||v_1 - v_2||_H^2(t)$$

$$+ \,||v_1 - v_2||_{L^2(\Omega)}^2(t) + \int_\Omega (\overline{\mathcal{G}}(z_1) - \overline{\mathcal{G}}(z_2))\,(v_1 - v_2)\,dx = 0.$$

Now we set $D(t) = \dfrac{1}{2}||v_1 - v_2||_H^2(t)$ and $\dot{D}(t) := \dfrac{1}{2}\,\dfrac{d}{dt}\,||v_1 - v_2||_H^2(t)$. We deduce

$$\dot{D}(t) \,+\, ||v_1 - v_2||_{L^2(\Omega)}^2(t) \,\le\, ||\overline{\mathcal{G}}(z_1) - \overline{\mathcal{G}}(z_2)||_{L^2(\Omega)}^2(t) + \frac{1}{4}\,||v_1 - v_2||_{L^2(\Omega)}^2(t). \tag{2.1.31}$$

On the other hand

$$
\begin{aligned}
&\int_\Omega [\overline{\mathcal{G}}(z_1)(x,t) - \overline{\mathcal{G}}(z_2)(x,t)]^2\,dx \overset{(2.1.12)}{\leq} L^2 \int_\Omega ||z_1(x,\cdot) - z_2(x,\cdot)||^2_{\mathcal{C}^0([0,T])}\,dx \\
&\leq L^2 \int_\Omega \left(\int_0^t \left| \frac{\partial}{\partial \tau}(z_1 - z_2) \right| (x,\tau)\,d\tau \right)^2 dx \leq L^2\, t \int_0^t \int_\Omega \left| \frac{\partial}{\partial \tau}(z_1 - z_2) \right|^2 (x,\tau)\,dx\,d\tau,
\end{aligned}
\tag{2.1.32}
$$

where we used the fact that $z_1(0) = z_2(0)$, as $z_1, z_2 \in B$. We define an equivalent norm on the closed subset B of $H^1(0,T;V)$

$$
||\eta||_B := \left(||\eta(0)||^2_{L^2(\Omega)} + \int_0^T e^{-L^2\,t^2} \left|\left| \frac{\partial \eta}{\partial t} \right|\right|^2_{L^2(\Omega)} (t)\,dt \right)^{1/2} \qquad \forall\, \eta \in B \subset H^1(0,T;V).
$$

At this point we multiply (2.1.31) by $\exp(-L^2 t^2)$ and integrate in time, for $t \in (0,T)$. We use (2.1.32) and the fact that $v_1(x,0) = v_2(x,0)$ (this makes sense as, from Theorem A.8.1 we have $v \in \mathcal{C}^0(0,T;H^{-1}(\Omega))$). So we deduce

$$
\begin{aligned}
&\frac{3}{4}\,||J(z_1) - J(z_2)||^2_B \leq \int_0^T e^{-L^2\,t^2} L^2\, t \int_0^t \int_\Omega \left| \frac{\partial}{\partial \tau}(z_1 - z_2) \right|^2 (x,\tau)\,dx\,d\tau \\
&= -\frac{e^{-L^2 T^2}}{2} \int_0^T \int_\Omega \left| \frac{\partial}{\partial t}(z_1 - z_2) \right|^2 (x,t)\,dx\,dt + \frac{1}{2} \int_0^T e^{-L^2 t^2} \int_\Omega \left| \frac{\partial}{\partial t}(z_1 - z_2) \right|^2 (x,t)\,dx\,dt
\end{aligned}
$$

which in turn gives

$$
||J(z_1) - J(z_2)||^2_B \leq \frac{2}{3}\,||z_1 - z_2||^2_B.
$$

Hence J is a contraction on the closed subset B of $H^1(0,T;V)$, which yields the existence and uniqueness of solutions.

Concerning the second part of the proof, as the family of continuous, piecewise linear functions is dense in $W^{1,1}(0,T)$, then (2.1.25) entails that for all $u \in \mathcal{M}(\Omega;W^{1,1}(0,T))$, $\overline{\mathcal{G}}(u) \in \mathcal{M}(\Omega;W^{1,1}(0,T))$ and we get

$$
\left| \frac{\partial}{\partial t} \overline{\mathcal{G}}(u) \right| \leq \bar{L} \left| \frac{\partial u}{\partial t} \right| \qquad \text{a.e. in } Q. \tag{2.1.33}
$$

Thus, as we have $u \in H^1(0,T;L^2(\Omega))$, then from (2.1.33) we also have $\overline{\mathcal{G}}(u) \in H^1(0,T;L^2(\Omega))$ and this finishes the proof. □

2.1.5. Lipschitz continuous dependence on the data

Theorem 2.1.4. *(Lipschitz continuous dependence on the data)*
Assume that the operator $\overline{\mathcal{G}} : \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega;\mathcal{C}^0([0,T]))$ *is causal and Lipschitz*

continuous according to (2.1.11) *and* (2.1.12)*. Then the dependence of the solution on the data is Lipschitz continuous in the following sense. For* $i = 1, 2$*, let* u_i^0*,* v_i^0*,* f_i *fulfill* (2.1.24) *and let* (u_i, v_i) *be the corresponding unique solution of Problem* 2.1.1*. Then*

$$||u_1 - u_2||^2_{L^2(Q)} \leq 2\,||u_1^0 - u_2^0||^2_{L^2(\Omega)} + c_1\,||v_1^0 - v_2^0||^2_{H^{-1}(\Omega)} + c_2\,||A^{-1}(f_1 - f_2)||^2_{L^2(Q)},$$

where the constants c_j *for* $j = 1, 2$ *are functions of* T*.*

Proof. From our assumptions, we immediately have that the following systems

$$\begin{cases} A^{-1}\dfrac{\partial}{\partial t}(v_1 + u_1) + v_1 + \overline{\mathcal{G}}(u_1) = A^{-1} f_1 \\ \dfrac{\partial u_1}{\partial t} = v_1 \end{cases} \qquad \begin{cases} A^{-1}\dfrac{\partial}{\partial t}(v_2 + u_2) + v_2 + \overline{\mathcal{G}}(u_2) = A^{-1} f_2 \\ \dfrac{\partial u_2}{\partial t} = v_2 \end{cases}$$

hold in $L^2(Q)$. Taking the difference between the corresponding equations of the previous systems we deduce

$$\begin{cases} A^{-1}\dfrac{\partial}{\partial t}(v_1 - v_2 + u_1 - u_2) + v_1 - v_2 + \overline{\mathcal{G}}(u_1) - \overline{\mathcal{G}}(u_2) = A^{-1}(f_1 - f_2) \\ \dfrac{\partial}{\partial t}(u_1 - u_2) = v_1 - v_2. \end{cases}$$

We multiply the first equation of the previous system by $v_1 - v_2$ in the scalar product of $L^2(\Omega)$ and we work as we did in Subsection 2.1.4 to obtain (2.1.31). This time we get

$$\dot{D}(t) + \frac{5}{8}\,||v_1 - v_2||^2_{L^2(\Omega)}(t) \leq ||\overline{\mathcal{G}}(u_1) - \overline{\mathcal{G}}(u_2)||^2_{L^2(\Omega)}(t) \; + \; 2\,||A^{-1}(f_1 - f_2)||^2_{L^2(\Omega)}(t), \quad (2.1.34)$$

where we recall that $\dot{D}(t) = \dfrac{1}{2}\dfrac{d}{dt}\,||v_1 - v_2||^2_H(t)$.
Estimate (2.1.32) is still valid also here (with z_1, z_2 of course replaced by u_1, u_2). So we introduce the following equivalent norm on $L^2(Q)$

$$|||z|||_{L^2(Q)} := \left(\int_0^T e^{-L^2 t^2}\,||z||^2_{L^2(\Omega)}(t)\,dt\right)^{1/2}.$$

Therefore we multiply (2.1.34) by $\exp(-L^2 t^2)$ and integrate in time from 0 to T. We obtain

$$\frac{5}{8}\,|||v_1 - v_2|||^2_{L^2(Q)} \leq ||v_1(x,0) - v_2(x,0)||^2_H \; + \; \frac{1}{2}\,|||v_1 - v_2|||^2_{L^2(Q)} \; + \; 2\,|||A^{-1}(f_1 - f_2)|||^2_{L^2(Q)}.$$

But $||\cdot||_{L^2(Q)}$ and $|||\cdot|||_{L^2(Q)}$ are equivalent norms, so

$$||v_1 - v_2||^2_{L^2(Q)} \leq \tilde{c}_1\,||v_1(x,0) - v_2(x,0)||^2_H \; + \; \tilde{c}_2\,||A^{-1}(f_1 - f_2)||^2_{L^2(Q)},$$

with $\tilde{c}_1$ and $\tilde{c}_2$ functions of T. Now

$$||u_1 - u_2||^2_{L^2(Q)} = \int_\Omega \int_0^T |u_1(x,t) - u_2(x,t)|^2 \, dt\, dx \le 2 \int_\Omega \int_0^T |u_1(x,0) - u_2(x,0)|^2 \, dt\, dx$$

$$+ 2 \int_\Omega \int_0^T \left| \int_0^t v_1(x,t) - v_2(x,t)\, dt \right|^2 dt\, dx \le 2||u_1^0 - u_2^0||^2_{L^2(Q)} + 2\,T\,||v_1 - v_2||^2_{L^2(Q)}$$

$$\le 2\,||u_1^0 - u_2^0||^2_{L^2(Q)} + c_1||v_1^0 - v_2^0||^2_H + c_2||A^{-1}(f_1 - f_2)||^2_{L^2(Q)},$$

where c_1 and c_2 are functions of T. This finishes the proof. $\square$

2.1.6. Regularity

Theorem 2.1.5. *(Regularity)*
Let us assume that the operator $\overline{\mathcal{G}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ is causal, Lipschitz continuous and bounded in the sense of (2.1.11), (2.1.12) *and* (2.1.13). *Moreover, let $f \in L^2(0,T; H^{-1}(\Omega))$, $u^0, v^0 \in H^1(\Omega)$. Then Problem* 2.1.1 *has a unique solution*

$$u \in H^1(0,T; H^1(\Omega)), \qquad v \in L^2(0,T; H^1(\Omega)).$$

On the other hand, if $f \in L^2(0,T; H^{-1}(\Omega))$, $u^0, v^0 \in H^1_0(\Omega)$ then Problem 2.1.1 *has a unique solution*

$$u \in H^1(0,T; H^1_0(\Omega)), \qquad v \in L^2(0,T; H^1_0(\Omega)).$$

Proof. First of all, for any given function f defined on an open set $\Omega \subset \mathbb{R}^N$, we introduce its extension to zero outside Ω in the following way

$$[f]_0(x) = \begin{cases} f(x) & x \in \Omega; \\ 0 & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.1.35}$$

Obviously, if $f \in H^1(\Omega)$, then this does not imply that $[f]_0 \in H^1(\mathbb{R}^N)$. Our idea to prove this higher regularity result is to use Proposition A.2.1; so let us refer to this auxiliary result and assume the notations therein to hold.
Now let us fix any $D \subset\subset \Omega$ (this notation means that D is an open bounded set contained in Ω such that $\bar{D} \subset \Omega$); let χ_D be the characteristic function of the open set D. Let (u, v) be the unique solution of Problem 2.1.1. We take the space increments in equation

$$A^{-1} \frac{\partial}{\partial t}(v + u) + v + \overline{\mathcal{G}}(u) = A^{-1} f,$$

which holds in $L^2(Q)$, multiply it by χ_D (so that now the equation holds in $L^2(D \times (0,T))$) and then multiply the result by $(\delta_h\, v)\, \chi_D$ in the scalar product of $L^2(\Omega)$ (or equivalently in the scalar product of $L^2(D)$; the notation δ_h is introduced in Proposition A.2.1). Then we integrate in time between 0 and T. We get

$$\int_0^T \int_D \left[A^{-1} \frac{\partial}{\partial t}(\delta_h\, v + \delta_h\, u)\right] (\delta_h\, v)\, dx\, dt + \int_0^T \int_D (\delta_h\, v + \delta_h\, \overline{\mathcal{G}}(u))\, (\delta_h\, v)\, dx\, dt$$

$$= \frac{1}{2} \int_0^T \frac{\partial}{\partial t} ||\delta_h\, v||^2_{H^{-1}(D)}\, dt + \int_0^T \left[||\delta_h\, v||^2_{H^{-1}(D)} + ||\delta_h\, v||^2_{L^2(D)}\right]\, dt + \int_0^T \int_D \delta_h\, \overline{\mathcal{G}}(u)\, \delta_h\, v\, dx\, dt$$

$$= \int_0^T \int_D (A^{-1}\, \delta_h\, f)\, (\delta_h\, v)\, dx\, dt \leq \frac{1}{8} \int_0^T ||\delta_h\, v||^2_{L^2(D)}\, dt + 2 \int_0^T ||A^{-1}\, \delta_h\, f||^2_{L^2(D)}\, dt.$$

At this point, using assumption (2.1.13) and working as in (2.1.32)

$$\begin{aligned} ||\delta_h\, \overline{\mathcal{G}}(u)||_{L^2(0,T;L^2(D))} &\leq \sqrt{T}\, ||\delta_h\, \overline{\mathcal{G}}(u)||_{L^2(D;\mathcal{C}^0([0,T]))} \\ &\leq \sqrt{T}\, L\, ||\delta_h\, u||_{L^2(D;\mathcal{C}^0([0,T]))} \leq T\, L\, ||\delta_h\, v||_{L^2(0,T;L^2(D))} \end{aligned}$$

from what we deduce

$$\frac{1}{2}\, ||\delta_h\, v(x,T)||^2_{H^{-1}(D)} + ||\delta_h\, v||^2_{L^2(0,T;H^{-1}(D))}\, dt + \frac{7}{8}\, ||\delta_h\, v||^2_{L^2(0,T;L^2(D))}$$

$$\leq \frac{1}{4}\, ||\delta_h\, v||^2_{L^2(0,T;L^2(D))} + 2\, ||A^{-1} \delta_h\, f||^2_{L^2(0,T;L^2(D))} + ||\delta_h\, \overline{\mathcal{G}}(u)||^2_{L^2(0,T;L^2(D))} + \frac{1}{2}\, ||\delta_h v(x,0)||^2_{H^{-1}(D)}$$

$$\leq \left(\frac{1}{4} + T^2\, L^2\right)\, ||\delta_h\, v||^2_{L^2(0,T;L^2(D))} + 2\, ||A^{-1} \delta_h\, f||^2_{L^2(0,T;L^2(D))} + \frac{1}{2}\, ||\delta_h v(x,0)||^2_{H^{-1}(D)}.$$

By means of a technique similar to the one employed in Subsections 2.1.4 and 2.1.5 and using our assumptions on the data, we obtain

$$\frac{1}{2}\, ||\delta_h\, v(x,T)||^2_{H^{-1}(D)} + \frac{1}{8}\, ||\delta_h\, v||^2_{L^2(0,T;L^2(D))} \leq c_1\, (T)\, |h|^2,$$

where the constant c_1 depends on T but it is independent of D. Thus, using the characterization of the space $H^1(\Omega)$ we have that

$$v(x,t) \in H^1(\Omega) \qquad \text{a.e. in } [0,T];$$

on the other hand, using the Lebesgue dominated convergence theorem, we also immediately get

$$v \in L^2(0,T;H^1(\Omega)).$$

which gives us the regularity we were looking for.
Concerning the second part of the proof, we choose $h \in \mathbb{R}^N$ and notice that

$$\int_0^T \int_\Omega \left[A^{-1} \frac{\partial}{\partial t}(\delta_h\, v + \delta_h\, u)\right] (\delta_h\, v)\, dx\, dt + \int_0^T \int_\Omega (\delta_h\, v + \delta_h\, \overline{\mathcal{G}}(u))\, (\delta_h\, v)\, dx\, dt$$

$$= \int_0^T \int_\Omega (\delta_h\, A^{-1}\, f)\, (\delta_h\, v)\, dx\, dt$$

is equivalent to (recall the definition (2.1.35))

$$\int_0^T \int_{\mathbb{R}^N} \left[A^{-1} \frac{\partial}{\partial t}(\delta_h [v]_0 + \delta_h [u]_0) \right] \delta_h [v]_0 \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} (\delta_h [v]_0 + \delta_h [\overline{\mathcal{G}}(u)]_0) \, \delta_h [v]_0 \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{R}^N} (\delta_h [A^{-1} f]_0) \, \delta_h [v]_0 \, dx \, dt.$$

Therefore we develop the computations as we did before; then taking into account that Proposition A.2.1 yields

$$A^{-1} f \in H_0^1(\Omega) \Leftrightarrow [A^{-1} f]_0 \in H^1(\mathbb{R}^N),$$

we easily obtain

$$||\delta_h [u]_0||^2_{L^2(\mathbb{R}^N)} \le c\,|h|^2;$$

this tell us that $[u]_0 \in H^1(\mathbb{R}^N)$ a.e. in $[0,T]$ and thus again by Proposition A.2.1 $u \in H_0^1(\Omega)$, a.e. in $[0,T]$. This allows us to conclude also this second part of the proof. □

2.1.7. The case of the Preisach operator

Let us consider the following model equation for scalar variables which we discussed in Subsection 2.1.1

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \triangle_{x,y} \left(\frac{\partial u}{\partial t} + \overline{\mathcal{G}}(u) \right) = f \qquad \text{in } \Omega \times (0,T),$$

where $\triangle_{x,y} := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and Ω is an open bounded set of $\mathbb{R}^2$; moreover $\overline{\mathcal{G}}$ is the inverse of the operator $\overline{\mathcal{F}} := I + \overline{\mathcal{W}}$ where I is the identity operator and $\overline{\mathcal{W}}$ is a suitable extension to space dependent functions of a scalar Preisach operator.

In this subsection we want to recover the results established in the previous subsections for this particular choice of $\overline{\mathcal{G}}$.

More precisely, let us consider the setting outlined in Subsection 2.1.3; let $\mathcal{W} : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be a Preisach operator defined in (1.5.8). Then Theorem 1.5.14 assures us that under reasonable assumptions on the density function ψ (i.e. Assumptions 1.5.8) the operator

$$\mathcal{F} : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T]) \qquad \mathcal{F}(\lambda, u)(t) := u(t) + \mathcal{W}(\lambda, u)(t)$$

is invertible and its inverse $\mathcal{G}$ is Lipschitz continuous. This inverse operator $\mathcal{G}$ turns out to be also a hysteresis operator as the same holds for $\mathcal{F}$.

At this point, let us fix any initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_K) \qquad \text{for some } K > 0, \tag{2.1.36}$$

where Λ_K was introduced in (1.2.4).
From Section 1.6, it is easy to see that the operator

$$\overline{\mathcal{G}}(u)(x,t) := \tilde{\mathcal{G}}(\lambda, u)(x,t) := \mathcal{G}(\lambda(x), u(x,\cdot))(t)$$

fulfills (2.1.11), (2.1.12), (2.1.13) and (2.1.25).
So, let us assume that $u^0 \in V$, $v^0 \in H$ and $f \in L^2(0,T;H)$, where we recall that we set $V := L^2(\Omega)$, $H := H^{-1}(\Omega)$ and $V' := (L^2(\Omega))'$. We want to solve the following problem

Problem 2.1.6. *To find $u \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \cap H^1(0,T;V)$ and $v \in L^2(Q)$ such that $\overline{\mathcal{G}}(u) \in L^2(Q)$ and for any $\phi \in H^1(0,T;V)$ with $\phi(\cdot,T) = 0$ a.e. in Ω*

$$\int_0^T - {}_{V'}\langle v+u, \frac{\partial \phi}{\partial t} \rangle_V \, dt + \int_0^T \int_\Omega (v + \overline{\mathcal{G}}(u))\, \phi \, dx\, dt = \int_0^T {}_{V'}\langle f, \phi \rangle_V \, dt$$

$$+ {}_{V'}\langle (v^0 + u^0)(\cdot), \phi(\cdot,0) \rangle_V$$

$$-\int_0^T \int_\Omega u \frac{\partial \phi}{\partial t}\, dx\, dt = \int_0^T \int_\Omega v\, \phi \, dx\, dt + \int_\Omega u^0(\cdot)\, \phi(\cdot,0)\, dx.$$

Working as we did in the previous sections, we have that this problem can be interpreted as follows

$$\begin{cases} A^{-1}\dfrac{\partial v}{\partial t} + A^{-1}\dfrac{\partial u}{\partial t} + v + \overline{\mathcal{G}}(u) = A^{-1} f \\ \dfrac{\partial u}{\partial t} = v. \end{cases} \qquad \text{in } L^2(Q), \text{ a.e. in } (0,T).$$

If in addition the solution (u,v) is more regular in space, (as in Corollary 2.1.7) then this is equivalent to

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial t} - \triangle_{x,y}(v + \overline{\mathcal{G}}(u)) = f \\ \dfrac{\partial u}{\partial t} = v \end{cases} \qquad \text{in } L^2(0,T;V'), \text{ a.e. in } (0,T);$$

this is the system from which our discussion started.
The main result of this subsection is the following.

Corollary 2.1.7. *Let $\mathcal{W}(\lambda,\cdot) : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be a Preisach operator (introduced in* (1.5.8)*), for λ as in* (2.1.36) *and assume that the density function ψ fulfills Assumption* 1.5.8 *with $b_0 = 0$. Suppose moreover that*

$$u^0 \in L^2(\Omega), \;\; v^0 \in H^{-1}(\Omega), \;\; f \in L^2(0,T;H^{-1}(\Omega)).$$

Then Problem 2.1.6 *has a unique solution*

$$u \in H^1(0,T;L^2(\Omega)) \qquad v \in L^2(Q)$$

such that

$$\overline{\mathcal{G}}(u) \in H^1(0,T;L^2(\Omega)).$$

On the other hand, if

$$u^0 \in H^1(\Omega), \;\; v^0 \in H^1(\Omega), \;\; f \in L^2(0,T;H^{-1}(\Omega))$$

holds, then the unique solution of Problem 2.1.6 *is such that*

$$u \in H^1(0,T;H^1(\Omega)) \qquad v \in L^2(0,T;H^1(\Omega)),$$

while if

$$u^0 \in H^1_0(\Omega), \;\; v^0 \in H^1_0(\Omega), \;\; f \in L^2(0,T;H^{-1}(\Omega)) \tag{2.1.37}$$

holds, then the unique solution of Problem 2.1.6 *is such that*

$$u \in H^1(0,T;H^1_0(\Omega)) \qquad v \in L^2(0,T;H^1_0(\Omega)).$$

Finally the dependence of the solution on the data is Lipschitz continuous in the following sense. For $i = 1,2$ *let* u_i^0, v_i^0, f_i *fulfill* (2.1.37) *and let* (u_i, v_i) *be the corresponding unique solution of Problem* 2.1.6*. Then*

$$||u_1 - u_2||^2_{L^2(Q)} \leq 2\,||u_1^0 - u_2^0||^2_{L^2(\Omega)} + c_1\,||v_1^0 - v_2^0||^2_{H^{-1}(\Omega)} + c_2\,||A^{-1}(f_1 - f_2)||^2_{L^2(Q)},$$

where the constants c_j *for* $j = 1,2$ *depend on* T*.*

Proof. This result can be immediately proved using Theorems 2.1.2, 2.1.4 and 2.1.5 together with Theorem 1.5.14. □

CHAPTER 3

Second class of P.D.E.s with hysteresis

The aim of this chapter is to study a class of parabolic P.D.E.s containing a continuous hysteresis operator $\overline{\mathcal{F}}$; the model equation we take into consideration is the following

$$\frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) + \vec{v} \cdot \nabla(u + \overline{\mathcal{F}}(u)) - \triangle u = f \qquad \text{in } \Omega \times (0, T) \tag{3.0.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\triangle$ is the Laplace operator, $\vec{v} : \Omega \times (0, T) \to \mathbb{R}^N$ is known and f is a given function.

This class of P.D.E.s is different from the model studied for example in [39], Chapter IX, due to the presence of the convective term $\vec{v} \cdot \nabla(u + \overline{\mathcal{F}}(u))$.

First of all, we introduce a weak formulation in Sobolev spaces for a Cauchy problem associated to the previous model equation, dealing in the first part of the chapter with Dirichlet boundary conditions. Under suitable assumptions on the hysteresis operator $\overline{\mathcal{F}}$ and on the data we are able to prove an existence and uniqueness theorem. The existence result is based on approximation by implicit time discretization, a priori estimates and passage to the limit by compactness; the uniqueness result is established for a suitably restricted class of hysteresis operators. At the end we discuss our results for a particular choice of the operator $\overline{\mathcal{F}}$. We also prove some results of stable dependence on the data; finally we get an existence theorem for solutions of the Cauchy problem associated to the previous model equation by means of a hyperbolic regularization method.

In the second part of the chapter we deal with the same model equation but this time we deal with a condition of nonlinear flux on a subset $\Gamma_2 \subset \Gamma$, which can be for example written as

$$\nabla u \cdot \vec{\nu} = [\vec{v} \cdot \vec{\nu}]\,(u + \overline{\mathcal{F}}(u)) - g(u) \qquad \text{on } \Gamma_2,$$

where $\vec{\nu}$ denotes the unit outer normal vector to Γ and $g : \mathbb{R} \to \mathbb{R}$ is a given function. On Γ_1 instead (which is the remaining part of the boundary) we consider Dirichlet boundary conditions. Our aim is to find assumptions on the function g so that an existence and a

uniqueness theorem can be stated and proved for the Cauchy problem associated to the previous model equation. Also in this case the existence theorem is proved by means of a technique based on approximation by implicit time discretization, a priori estimates and passage to the limit by compactness and a uniqueness result is obtained for a suitably restricted class of hysteresis operators.
These kinds of problems have already been studied, even if in different settings, for example in the case of the Stefan problem we may quote [13], [33], [36].

3.1. First case: Dirichlet boundary conditions

3.1.1. Model problem

Let us consider an open bounded set of Lipschitz class $\Omega \subset \mathbb{R}^N$, $N \geq 1$ with boundary Γ and set $Q := \Omega \times (0,T)$. Assume that the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ is:
∗ CAUSAL, i.e.

$$\begin{cases} \forall\, v_1, v_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \ \ \forall\, t \in [0,T], \text{ if } v_1 = v_2 \text{ in } [0,t], \text{ a.e. in } \Omega, \\ \text{then } [\overline{\mathcal{F}}(v_1)](\cdot,t) = [\overline{\mathcal{F}}(v_2)](\cdot,t) \text{ a.e. in } \Omega, \end{cases} \tag{3.1.1}$$

where we denoted by $\mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ the Fréchet space of (strongly) measurable functions $\Omega \to \mathcal{C}^0([0,T])$ (see Subsection A.1.2);
∗ STRONGLY CONTINUOUS, i.e.

$$\begin{cases} \forall\, \{v_n \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))\}_{n\in\mathbb{N}}, \quad \text{if } v_n \to v \text{ uniformly in } [0,T] \\ \text{a.e. in } \Omega, \text{ then } \overline{\mathcal{F}}(v_n) \to \overline{\mathcal{F}}(v) \text{ uniformly in } [0,T], \text{ a.e. in } \Omega; \end{cases} \tag{3.1.2}$$

∗ AFFINELY BOUNDED, i.e.

$$\begin{cases} \exists\, L_{\mathcal{F}} > 0, \ \ \exists\, \tau \in L^2(\Omega) : \forall v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \\ ||[\overline{\mathcal{F}}(v)](x,\cdot)||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{F}} ||v(x,\cdot)||_{\mathcal{C}^0([0,T])} + \tau(x) \ \text{ a.e. in } \Omega; \end{cases} \tag{3.1.3}$$

∗ PIECEWISE MONOTONE i.e.

$$\begin{cases} \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \ \forall\, [t_1,t_2] \subset [0,T], \text{ if } v(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega, \\ \text{then } \{[\overline{\mathcal{F}}(v)](x,t_2) - [\overline{\mathcal{F}}(v)](x,t_1)\} \cdot [v(x,t_2) - v(x,t_1)] \geq 0 \text{ a.e. in } \Omega; \end{cases} \tag{3.1.4}$$

∗ PIECEWISE LIPSCHITZ CONTINUOUS, i.e.

$$\begin{cases} \exists\, \tilde{L}_{\mathcal{F}} > 0 : \ \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \ \ \forall\, [t_1,t_2] \subset [0,T] \\ \text{if } v(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega, \text{ then} \\ |[\overline{\mathcal{F}}(v)](x,t_2) - [\overline{\mathcal{F}}(v)](x,t_1)| \leq \tilde{L}_{\mathcal{F}}\, |v(x,t_1) - v(x,t_2)| \qquad \text{a.e. in } \Omega. \end{cases} \tag{3.1.5}$$

It is not restrictive to assume that $L_{\mathcal{F}} = \tilde{L}_{\mathcal{F}}$.
The causality property entails that $[\overline{\mathcal{F}}(v)](\cdot,0)(\in \mathcal{M}(\Omega))$ depends just on $\overline{\mathcal{F}}$ and $v(\cdot,0)$; so we can set

$$\mathcal{H}_{\overline{\mathcal{F}}}(v(\cdot,0)) := [\overline{\mathcal{F}}(v)](\cdot,0)(\in \mathcal{M}(\Omega)) \qquad \forall\, v \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])). \tag{3.1.6}$$

We set $V := H_0^1(\Omega)$, $H := L^2(\Omega)$ and $V' := H^{-1}(\Omega)$ and we consider V endowed with the norm $||u||_V := ||\nabla u||_{L^2(\Omega)}$. We then identify the space $L^2(\Omega)$ to its topological dual $(L^2(\Omega))'$; as the injection of V into $L^2(\Omega)$ is continuous and dense, $(L^2(\Omega))'$ can be identified to a subspace of V' (see Theorem A.6.1). This yields the Hilbert triplet

$$V \subset H \equiv H' \subset V'$$

with dense and continuous injections.
Now we denote by ${}_{V'}\langle\cdot,\cdot\rangle_V$ the duality pairing between V' and V and we then define the linear and continuous operator $A : V \to V'$ as follows

$${}_{V'}\langle Au, v\rangle_V := \int_\Omega \nabla u\cdot\nabla v\,dx \qquad \forall u,v\in V. \tag{3.1.7}$$

We assume that $u^0, w^0 = \mathcal{H}_{\overline{\mathcal{F}}}(u^0) \in L^2(\Omega)$ are given initial conditions; moreover, let us consider a known function

$$\vec{v} : \Omega\times(0,T) \to \mathbb{R}^N \qquad\qquad \vec{v}(x,t) := (v_1(x,t), v_2(x,t),\dots,v_N(x,t))$$

satisfying the following assumptions

$$\vec{v},\ \frac{\partial\vec{v}}{\partial t} \in L^\infty(Q)^N \qquad\qquad \nabla\cdot\vec{v} = 0 \qquad \text{a.e. in } Q. \tag{3.1.8}$$

We want to solve the following problem.

Problem 3.1.1. *Let us consider a known function $\vec{v}$ satisfying* (3.1.8)*; we search for a function $u \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \cap L^2(0,T;V)$ such that $\overline{\mathcal{F}}(u) \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T]))\cap L^2(Q)$ and for any $\psi \in H^1(0,T;L^2(\Omega))\cap L^2(0,T;V)$ with $\psi(\cdot,T) = 0$ a.e. in Ω*

$$\begin{aligned}&\int_0^T\int_\Omega -(u+\overline{\mathcal{F}}(u))\frac{\partial\psi}{\partial t}\,dx\,dt - \int_0^T\int_\Omega [\vec{v}\cdot\nabla\psi]\,(u+\overline{\mathcal{F}}(u))\,dx\,dt\\ &+\int_0^T\int_\Omega \nabla u\cdot\nabla\psi\,dx\,dt = \int_0^T {}_{V'}\langle f,\psi\rangle_V\,dt + \int_\Omega [u^0(x)+w^0(x)]\,\psi(x,0)\,dx.\end{aligned} \tag{3.1.9}$$

Interpretation. The variational equation (3.1.9) yields

$$\begin{cases} \dfrac{\partial w}{\partial t} + \vec{v}\cdot\nabla w - \triangle u = f & \text{in } \mathcal{D}'(0,T;V') \\ w = (I+\overline{\mathcal{F}})(u) \end{cases} \tag{3.1.10}$$

where we used the fact that $\nabla \cdot \vec{v} = 0$, the standard Green formulae (for more details see e.g. [9], Chapter 2, see also Section A.5) and the definition of derivatives in the sense of distributions (see Subsection A.1.3). Thus, by comparison,

$$\frac{\partial}{\partial t}[u + \overline{\mathcal{F}}(u)] \in L^2(0,T;V')$$

so $u + \overline{\mathcal{F}}(u) \in H^1(0,T;V')$ and (3.1.10) holds in V' a.e. in $(0,T)$. Hence, integrating by parts in time in (3.1.9) we get

$$[u + \overline{\mathcal{F}}(u)]_{|t=0} = u^0 + w^0 \quad \text{in } V' \tag{3.1.11}$$

in the sense of the traces. In turn (3.1.10) and (3.1.11) yield (3.1.9), and the two formulations are equivalent.

Physical interpretation.
Let $D \subset \mathbb{R}^3$ represent the region occupied by a porous medium. We consider the equation of continuity

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \vec{q} = 0 \qquad \text{in } D_T := D \times (0,T),$$

where θ is the water content of the medium and $\vec{q}$ is the flux. We have $\theta = \varphi\, s$ where $\varphi : D \to [0,1]$ is the porosity of the medium and s is the saturation.
We couple this equation with Darcy's law

$$\vec{q} = -k\,(\nabla u + \rho\, g\, \vec{z})$$

where k is the hydraulic conductivity, u is the pressure, ρ is the density of the fluid, g is the gravity acceleration and $\vec{z}$ is the upward vertical unit vector. The saturation s and the pressure u are unknown.
Therefore the system we find is the following

$$\begin{cases} \varphi\, \dfrac{\partial s}{\partial t} - \nabla \cdot k\,(\nabla u + \rho\, g\, \vec{z}) = 0 \\ s = \overline{\mathcal{F}}(u), \end{cases} \qquad \text{in } D_T \tag{3.1.12}$$

where the dependence of s upon u is formally represented by the operator $\overline{\mathcal{F}}$. Experimental results indicate the occurrence of a quite relevant hysteresis effect which has occasionally been represented by Preisach models in engineering literature.
A model of this type with saturation versus pressure constitutive relation with hysteresis has been studied for example in [2] and [3].
In the model considered in this chapter we make a strong restriction on the hydraulic conductivity k i.e. we assume that it is a constant (as it happens in the case of a saturated flow through a porous medium). In this way we neglect the dependence of k upon the saturation

s, which instead occurs in the case of unsaturated flow through a porous medium (for more details on this topic we refer for example to [6], Section 9.4).
Moreover we suppose that in (3.1.12) the derivative in time is not a Eulerian derivative but a *material* derivative, that is, with the notation introduced in Section A.11, we consider the system

$$\begin{cases} \varphi \dfrac{Ds}{Dt} - \nabla \cdot k\,(\nabla u + \rho\, g\, \vec{z}) = 0 \\ s = \overline{\mathcal{F}}(u), \end{cases} \tag{3.1.13}$$

If we want to express (3.1.13) in terms of the Eulerian derivative, recalling (A.11.5), the system we get is included in our model system (3.1.10), which may then represent a model with saturation versus pressure constitutive relation with hysteresis and with a term of transport.

3.1.2. Existence

Theorem 3.1.2. *(Existence)*
Let us assume that the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ is causal, strongly continuous, affinely bounded, piecewise monotone and piecewise Lipschitz continuous according to (3.1.1)*,* (3.1.2)*,* (3.1.3)*,* (3.1.4) *and* (3.1.5)*. Moreover let*

$$f \in L^2(Q), \;\; u^0 \in V, \;\; w^0 \in H. \tag{3.1.14}$$

Then Problem 3.1.1 *admits at least one solution*

$$u \in H^1(0,T;H) \cap L^\infty(0,T;V)$$

such that

$$\overline{\mathcal{F}}(u) \in H^1(0,T;H).$$

Proof.

① *First step: approximation.*

Let us fix $m \in \mathbb{N}$ and set $k := T/m$. As in the last part of the proof m will go to infinity, it is not restrictive to suppose that

$$k < \min\left(\frac{1}{4\|\vec{v}\|^2_{L^\infty(Q)^N}(1+L_{\mathcal{F}})^2}, \frac{1}{c_6} \right), \tag{3.1.15}$$

where the constant c_6 will be introduced in (3.1.25). This assumption will be useful later. Now for $n = 1, \dots, m$ let us consider $f^n_m(x) := f(x, nk)$, $u^0_m := u^0$ and $w^0_m := w^0$. We approximate our problem by an implicit time discretization scheme.

❒ Statement of the problem. We want to solve the following problem

Problem 3.1.3. *To find $u_m^n \in V$ for $n = 1, \dots m$, such that, if $u_m(x, \cdot)$ is the linear time interpolate of $u_m(x, nk) := u_m^n(x)$, for $n = 1, \dots, m$, a.e. in Ω and $w_m^n := [\overline{\mathcal{F}}(u_m)](x, nk)$ for $n = 1, \dots, m$, a.e. in Ω, then, for any $\psi \in V$*

$$\begin{aligned} &\frac{1}{k}\int_\Omega (u_m^n - u_m^{n-1})\,\psi\,dx + \frac{1}{k}\int_\Omega (w_m^n - w_m^{n-1})\,\psi\,dx - \int_\Omega [\vec{v}_m^{\,n} \cdot \nabla\psi]\,(u_m^n + w_m^n)\,dx \\ &+ \int_\Omega \nabla u_m^n \cdot \nabla\psi\,dx = \int_\Omega f_m^n\,\psi\,dx \end{aligned} \tag{3.1.16}$$

where we used the following notation $\vec{v}_m^{\,n}(x) := \vec{v}(x, nk)$.

For any $n \in \{1, \dots, m\}$, we suppose to know $u_m^1, \dots, u_m^{n-1} \in V$; the problem is now to determine u_m^n.

❒ introduction of the operators $\widehat{F}_m^n$ and C. For almost any $x \in \Omega$, $u_m(x, \cdot)$ is the linear time interpolate of $u_m^n(x)$, so it is affine in $[(n-1)k, nk]$; this implies that $[\overline{\mathcal{F}}(u_m)](x, nk)$ depends only on $u_m(x, \cdot)_{|[0,(n-1)k]}$, which is known and on $u_m^n(x)$, which must be determined. Hence, there exists a function $F_m^n : \mathbb{R} \times \Omega \to \mathbb{R}$ such that

$$w_m^n(x) = [\overline{\mathcal{F}}(u_m)](x, nk) := F_m^n(u_m^n(x), x) \qquad \text{a.e. in } \Omega.$$

This allows us to introduce an operator $\widehat{F}_m^n$ acting between spaces of measurable functions $\mathcal{M}(\Omega)$ in the following way $\widehat{F}_m^n(v) := F_m^n(v(\cdot), \cdot)$.
Let us outline some properties of the operator $\widehat{F}_m^n$.
First of all we want to show that (for any fixed m and $n \in \{1, \dots, m\}$)

$$[(3.1.2)] \Rightarrow [\widehat{F}_m^n : L^2(\Omega) \to L^2(\Omega) \text{ is a strongly continuous operator}].$$

This is equivalent to show that

$$[w_j \to w \text{ strongly in } L^2(\Omega)] \;\Rightarrow\; [\widehat{F}_m^n(w_j) \to \widehat{F}_m^n(w) \text{ strongly in } L^2(\Omega)].$$

So let us consider a sequence $\{w_j\}_{j\in\mathbb{N}}$ of elements of $L^2(\Omega)$ such that $w_j \to w$ strongly in $L^2(\Omega)$. We recall that $u_m^{\tilde{n}}(x)$ are given for $\tilde{n} = 0, \dots, n-1$; so we construct the two linear-time interpolates W_j and W in the following way

$$\begin{cases} W_j(x, \tilde{n}k) := u_m^{\tilde{n}}(x) \ \text{ for } \tilde{n} = 0, \dots, n-1 \\ W_j(x, nk) = w_j(x) \end{cases}$$

and

$$\begin{cases} W(x, \tilde{n}k) := u_m^{\tilde{n}}(x) \ \text{ for } \tilde{n} = 0, \dots, n-1 \\ W(x, nk) = w(x). \end{cases}$$

As $W_j \equiv W$ in $[0,(n-1)k]$ and W_j, W are affine in $[(n-1)k, nk]$, we can immediately deduce that

$$[w_j \to w \text{ strongly in } L^2(\Omega)] \;\Rightarrow [W_j \to W \text{ uniformly in } \mathcal{C}^0([0,T]), \text{ a.e. in } \Omega]$$

$$\overset{(3.1.2)}{\Rightarrow} [\overline{\mathcal{F}}(W_j) \to \overline{\mathcal{F}}(W) \text{ uniformly in } \mathcal{C}^0([0,T]), \text{ a.e. in } \Omega]$$

$$\Rightarrow [\overline{\mathcal{F}}(W_j)(x,nk) \to \overline{\mathcal{F}}(W)(x,nk) \text{ a.e in } \Omega]$$

$$\Rightarrow [\widehat{F}^n_m(w_j) = \overline{\mathcal{F}}(W_j)(\cdot,nk) \to \overline{\mathcal{F}}(W)(\cdot,nk) = \widehat{F}^n_m(w) \text{ strongly in } L^2(\Omega)],$$

where in the last line we used the Lebesgue dominated convergence theorem. This finishes the proof of the first property of the operator $\widehat{F}^n_m$.
Moreover from (3.1.3) we get that there exist two constants $C_1^{\mathcal{F}}, C_2^{\mathcal{F}}$ (actually $C_1^{\mathcal{F}} = L_{\mathcal{F}}$) such that

$$||\widehat{F}^n_m(v)||_{L^2(\Omega)} \leq C_1^{\mathcal{F}}\, ||v||_{L^2(\Omega)} + C_2^{\mathcal{F}} \qquad \forall\, v \in L^2(\Omega); \tag{3.1.17}$$

finally, using (3.1.17) and (3.1.4) we have that there exist also other two constants $C_3^{\mathcal{F}}, C_4^{\mathcal{F}} \in \mathbb{R}^+$, depending on m, n such that

$$\int_\Omega \widehat{F}^n_m(v)\, v\, dx \geq -C_3^{\mathcal{F}} ||v||_{L^2(\Omega)} - C_4^{\mathcal{F}} \qquad \forall v \in L^2(\Omega). \tag{3.1.18}$$

Moreover we introduce the operator $C : V \to V'$ acting in the following way

$${}_{V'}\langle C(\Phi), \psi\rangle_V := -\int_\Omega \vec{v}^{\,n}_m\, (\Phi + \widehat{F}^n_m(\Phi)) \cdot \nabla\psi\, dx \qquad \forall\, \psi, \Phi \in V.$$

❒ how to get a solution of problem 3.1.3. Thus we can write (3.1.16) in the following way

$$\frac{u^n_m - u^{n-1}_m}{k} + \frac{w^n_m - w^{n-1}_m}{k} + C(u^n_m) + A(u^n_m) = f^n_m \qquad \text{in } V' \tag{3.1.19}$$

which in turn yields

$$u^n_m + \widehat{F}^n_m(u^n_m) + k\, C(u^n_m) + kAu^n_m = g^n_m \qquad \text{in } V'$$

where $g^n_m := kf^n_m + w^{n-1}_m + u^{n-1}_m$, so it is a known function. For the sake of simplicity, we omit the fixed indexes m and n (so in the following we will also write $\vec{v}$ in place of $\vec{v}^{\,n}_m$); thus we get

$$u + \widehat{F}(u) + k\, C(u) + kAu = g \qquad \text{in } V'. \tag{3.1.20}$$

We claim that (3.1.20) admits at least one solution $u \in V$; we will use Theorem A.9.1.
Let $\{V_j\}_{j\in\mathbb{N}}$ be a sequence of finite dimensional subspaces invading V; for any $j \in \mathbb{N}$ let us consider the problem of finding $u_j \in V_j$ such that

$$Z(u_j) := u_j + \widehat{F}(u_j) + k\, Au_j + k\, C(u_j) = g \qquad \text{in } V'. \tag{3.1.21}$$

First of all, we want to prove that the operator $Z : V \to V'$ defined as

$$Z(w) := w + \widehat{F}(w) + k\,Aw + k\,C(w)$$

is strongly continuous and coercive.

• *Strong continuity.* It is immediate to verify that the operator $\bar{Z} : V \to V'$ defined as $\bar{Z}(w) = w + \widehat{F}(w) + k\,Aw$ is strongly continuous from V to V'. The same happens for the operator $\tilde{Z} : V \to V'$ which is defined as $\tilde{Z}(w) = k\,C(w)$; in fact if we suppose that $w_n \to w$ in V, then

$$||k\,C(w_n) - \,k\,C(w)||_{V'} = \sup_{||\psi||_V=1} \left\{ -k \int_\Omega [\vec{v} \cdot \nabla\psi]\,[(w_n - w) + (\widehat{F}(w_n) - \widehat{F}(w))]\,dx \right\}$$

$$\leq k\,||\vec{v}||_{L^\infty(Q)^N} \sup_{||\psi||_V=1} \left\{ ||\psi||_V \left[||w_n - w||_{L^2(\Omega)} + ||\widehat{F}(w_n) - \widehat{F}(w)||_{L^2(\Omega)} \right] \right\} \to 0,$$

where we used the strong continuity of $\widehat{F}$. Thus also the operator $Z : V \to V'$ is strongly continuous.

• *Coercivity.* Now the aim is to prove that Z is also coercive, that is

$$\frac{1}{||w||_V}\, {}_{V'}\langle Z(w), w\rangle_V \to \infty \qquad \text{as } ||w||_V \to \infty.$$

Recalling that we set $H := L^2(\Omega)$, we have

$${}_{V'}\langle Z(w), w\rangle_V = ||w||_H^2 + \int_\Omega \widehat{F}(w)\,w\,dx + \,k\,||\nabla w||_H^2 - k \int_\Omega \vec{v}\,(w + \widehat{F}(w)) \cdot \nabla\,w\,dx$$

$$\overset{(3.1.18)}{\geq} ||w||_H^2 - C_3^{\mathcal{F}}\,||w||_H - C_4^{\mathcal{F}} + \,k\,||\nabla w||_H^2 - k \int_\Omega ||\vec{v}||_{L^\infty(Q)^N}\,|w + \widehat{F}(w)|\,|\nabla w|\,dx$$

$$\geq ||w||_H^2 - C_3^{\mathcal{F}}\,||w||_H - C_4^{\mathcal{F}} + \,k\,||\nabla w||_H^2 - k\,||\vec{v}||_{L^\infty(Q)^N}\,||w + \widehat{F}(w)||_H\,||\nabla w||_H$$

$$\overset{(3.1.17)}{\geq} ||w||_H^2 - C_3^{\mathcal{F}}||w||_H - C_4^{\mathcal{F}} + k\,||\nabla w||_H^2 - k\,||\vec{v}||_{L^\infty(Q)^N}\,[(1 + C_1^{\mathcal{F}})||w||_H + C_2^{\mathcal{F}}]\,||\nabla w||_H$$

$$\geq ||w||_H^2 - C_3^{\mathcal{F}}||w||_H - C_4^{\mathcal{F}} + \frac{3}{4}k\,||\nabla w||_H^2 - k\,||\vec{v}||^2_{L^\infty(Q)^N}[(1 + C_1^{\mathcal{F}})||w||_H + C_2^{\mathcal{F}}]^2$$

$$\geq ||w||_H^2 - C_3^{\mathcal{F}}||w||_H - C_4^{\mathcal{F}} + \frac{3}{4}k\,||\nabla w||_H^2 - 2\,k\,||\vec{v}||^2_{L^\infty(Q)^N}(1 + C_1^{\mathcal{F}})^2\,||w||_H^2$$

$$-2\,k\,||\vec{v}||^2_{L^\infty(Q)^N}\,(C_2^{\mathcal{F}})^2 \overset{(3.1.15)}{\geq} \frac{1}{2}||w||_H^2 + \frac{3}{4}k||\nabla w||_H^2 - C_3^{\mathcal{F}}||w||_H - C_4^{\mathcal{F}} - 2\,k\,||\vec{v}||^2_{L^\infty(Q)^N}(C_2^{\mathcal{F}})^2$$

$$\geq \frac{1}{4}||w||_H^2 + \frac{3}{4}k\,||\nabla w||_H^2 - (C_3^{\mathcal{F}})^2 - C_4^{\mathcal{F}} - 2\,k\,||\vec{v}||^2_{L^\infty(Q)^N}(C_2^{\mathcal{F}})^2 \geq \frac{3}{4}k\,||w||_V^2 - \hat{c}_1$$

where we set

$$\hat{c}_1 := (C_3^{\mathcal{F}})^2 + C_4^{\mathcal{F}} + 2\,k\,||\vec{v}||^2_{L^\infty(Q)^N}(C_2^{\mathcal{F}})^2$$

so that $\hat{c}_1$ depends on k but it is independent of j. Thus summing up all the contributions we get the coercivity of the operator Z.
Hence, using Theorem A.9.1, we get the existence of at least a solution u_j of (3.1.21). If we multiply (3.1.21) by u_j and use the coercivity of the operator Z we get that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is uniformly bounded in V. In fact we have

$$\begin{aligned}\frac{3}{4}\,k\,||u_j||_V^2 &\leq {}_{V'}\langle Z(u_j), u_j\,\rangle_V \,+\, \hat{c}_1 \,=\, {}_{V'}\langle g, u_j\,\rangle_V \,+\, \hat{c}_1 \leq\, ||g||_{V'}\,||u_j||_V \,+\, \hat{c}_1\\ &\leq \frac{k}{8}\,||u_j||_V^2 \,+\, \frac{2}{k}\,||g||_{V'}^2 \,+\, \hat{c}_1;\end{aligned}$$

if we set

$$\hat{c}_2 := \frac{8}{5\,k}\left(\frac{2}{k}||g||_{V'}^2 + \hat{c}_1\right),$$

then $\hat{c}_2$ depends on k but it is independent of j and we have exactly what we would like to obtain

$$||u_j||_V^2 \leq\, \hat{c}_2.$$

Thus there exists u such that, possibly extracting a subsequence, $u_j \rightharpoonup u$ in V. By the compactness of the inclusion $V \subset L^2(\Omega)$ and by the continuity of the operator Z we may pass to the limit taking $j \to \infty$ in (3.1.21) getting (3.1.20). This allows us to conclude that Problem 3.1.3 has at least a solution.

② *Second step: a priori estimates.*

We recall that, for brevity, we set $L^2(\Omega) := H$. The idea we have is to multiply (3.1.19) by $(u_m^n - u_m^{n-1})$ in the duality ${}_{V'}\langle\cdot,\cdot\rangle_V$ (i.e. to consider (3.1.16) with the choice $\psi := u_m^n - u_m^{n-1}$) and sum for $n = 1,\dots,j$, for $j \in \{1,\dots,m\}$. It is clear that the difficulties come from the term

$$\sum_{n=1}^{j}\int_\Omega \vec{v}_m^{\,n}(u_m^n + w_m^n)\cdot\nabla(u_m^n - u_m^{n-1})\,dx,$$

as the term $\nabla(u_m^n - u_m^{n-1})$ cannot be controlled; we thus need to integrate in the time variable. More precisely, the details are the following:

$$\begin{aligned}&\sum_{n=1}^{j}{}_{V'}\langle\frac{u_m^n - u_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j}{}_{V'}\langle\frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V\\ &+\sum_{n=1}^{j}{}_{V'}\langle C(u_m^n), u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j}{}_{V'}\langle Au_m^n, u_m^n - u_m^{n-1}\rangle_V\\ \geq\,& k\sum_{n=1}^{j}\left|\left|\frac{u_m^n - u_m^{n-1}}{k}\right|\right|_H^2 + k\sum_{n=1}^{j}{}_{V'}\langle\frac{w_m^n - w_m^{n-1}}{k}, \frac{u_m^n - u_m^{n-1}}{k}\rangle_V\end{aligned}$$

$$- \sum_{n=1}^{j} \int_\Omega [\vec{v}_m^{\,n}(u_m^n + w_m^n) \cdot \nabla(u_m^n - u_m^{n-1})]\, dx + \frac{1}{2} \sum_{n=1}^{j} \int_\Omega (|\nabla u_m^n|^2 - |\nabla u_m^{n-1}|^2)\, dx$$

$$\geq k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 - \int_\Omega (u_m^j + w_m^j)\, [\vec{v}_m^{\,j} \cdot \nabla u_m^j]\, dx + \int_\Omega (u_m^0 + w_m^0)\, [\vec{v}_m^{\,0} \cdot \nabla u_m^0]\, dx$$

$$+ \sum_{n=1}^{j} \int_\Omega [(\vec{v}_m^{\,n} - \vec{v}_m^{\,n-1})(u_m^n + w_m^n)] \cdot \nabla u_m^n\, dx + k \sum_{n=1}^{j} \int_\Omega \left(\frac{u_m^n - u_m^{n-1}}{k} \right) [\vec{v}_m^{\,n} \cdot \nabla u_m^n]\, dx$$

$$+ k \sum_{n=1}^{j} \int_\Omega \left(\frac{w_m^n - w_m^{n-1}}{k} \right) [\vec{v}_m^{\,n} \cdot \nabla u_m^n]\, dx + \frac{1}{2} \int_\Omega (|\nabla u_m^j|^2 - |\nabla u_m^0|^2)\, dx,$$

where we used (3.1.4) and where $\vec{v}_m^{\,n} - \vec{v}_m^{\,n-1} = (v_{1m}^n - v_{1m}^{n-1}, v_{2m}^n - v_{2m}^{n-1}, \ldots, v_{Nm}^n - v_{Nm}^{n-1})$. On the other hand

$$\sum_{n=1}^{j} {}_{V'}\langle \frac{u_m^n - u_m^{n-1}}{k}, u_m^n - u_m^{n-1} \rangle_V + \sum_{n=1}^{j} {}_{V'}\langle \frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1} \rangle_V$$

$$+ \sum_{n=1}^{j} {}_{V'}\langle C(u_m^n), u_m^n - u_m^{n-1} \rangle_V + \sum_{n=1}^{j} {}_{V'}\langle A u_m^n, u_m^n - u_m^{n-1} \rangle_V = \sum_{n=1}^{j} {}_{V'}\langle f_m^n, u_m^n - u_m^{n-1} \rangle_V$$

$$= k \sum_{i=1}^{j} \int_\Omega f_m^n \left(\frac{u_m^n - u_m^{n-1}}{k} \right) dx \leq k \sum_{i=1}^{j} ||f_m^n||_H^2 + \frac{k}{4} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 .$$

From the previous two chains of inequalities, we deduce

$$\frac{1}{2} \int_\Omega |\nabla u_m^j|^2\, dx + \frac{3}{4}\, k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 \leq \frac{1}{2} \int_\Omega |\nabla u_m^0|^2\, dx + \int_\Omega (u_m^j + w_m^j)\, [\vec{v}_m^{\,j} \cdot \nabla u_m^j]\, dx$$

$$- \int_\Omega (u_m^0 + w_m^0)\, [\vec{v}_m^{\,0} \cdot \nabla u_m^0]\, dx - \sum_{n=1}^{j} \int_\Omega [(\vec{v}_m^{\,n} - \vec{v}_m^{\,n-1})\, (u_m^n + w_m^n)] \cdot \nabla u_m^n\, dx$$

$$- k \sum_{n=1}^{j} \int_\Omega [\vec{v}_m^{\,n} \cdot \nabla u_m^n] \left(\frac{u_m^n - u_m^{n-1}}{k} + \frac{w_m^n - w_m^{n-1}}{k} \right) dx + k \sum_{n=1}^{j} ||f_m^n||_H^2 \leq \frac{1}{2}\, ||u_m^0||_V^2$$

$$+ \int_\Omega |u_m^j|\, |\nabla u_m^j|\, |\vec{v}_m^{\,j}|\, dx + \int_\Omega |w_m^j|\, |\nabla u_m^j|\, |\vec{v}_m^{\,j}|\, dx + \int_\Omega |u_m^0|\, |\nabla u_m^0|\, |\vec{v}_m^{\,0}|\, dx$$

$$+ \int_\Omega |w_m^0|\, |\nabla u_m^0|\, |\vec{v}_m^{\,0}|\, dx + k \sum_{n=1}^{j} \int_\Omega \left| \frac{\vec{v}_m^{\,n} - \vec{v}_m^{\,n-1}}{k} \right| |u_m^n + w_m^n|\, |\nabla u_m^n|\, dx$$

$$+ k \sum_{n=1}^{j} \int_\Omega |\vec{v}_m^{\,n}|\, |\nabla u_m^n| \left(\left| \frac{u_m^n - u_m^{n-1}}{k} \right| + \left| \frac{w_m^n - w_m^{n-1}}{k} \right| \right) dx + k \sum_{n=1}^{j} ||f_m^n||_H^2.$$

This implies that

$$\frac{1}{2}\int_\Omega |\nabla u_m^j|^2\,dx + \frac{3}{4}\,k\sum_{n=1}^{j}\left|\left|\frac{u_m^n - u_m^{n-1}}{k}\right|\right|_H^2 \le \frac{1}{2}\,||u_m^0||_V^2 + ||\vec{v}||_{L^\infty(Q)^N}\int_\Omega (|u_m^j| + |w_m^j|)|\nabla u_m^j|\,dx$$

$$+\ ||\vec{v}||_{L^\infty(Q)^N}\ \int_\Omega (|u_m^0| + |w_m^0|)\,|\nabla u_m^0|\,dx + \left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\ k\sum_{n=1}^{j}\int_\Omega (|u_m^n| + |w_m^n|)\,|\nabla u_m^n|\,dx$$

$$+\ ||\vec{v}||_{L^\infty(Q)^N}\ k\sum_{n=1}^{j}\int_\Omega |\nabla u_m^n|\left(\left|\frac{u_m^n - u_m^{n-1}}{k}\right| + \left|\frac{w_m^n - w_m^{n-1}}{k}\right|\right)dx + k\sum_{n=1}^{j}||f_m^n||_H^2$$

$$\le \frac{1}{2}\,||u_m^0||_V^2 + \frac{1}{4}\int_\Omega |\nabla u_m^j|^2\,dx + 2\,||\vec{v}||^2_{L^\infty(Q)^N}\int_\Omega (|u_m^j|^2 + |w_m^j|^2)\,dx + ||\vec{v}||_{L^\infty(Q)^N}\,||\nabla u_m^0||_H^2$$

$$+\ \frac{||\vec{v}||_{L^\infty(Q)^N}}{2}\,(||u_m^0||_H^2 + ||w_m^0||_H^2)\ +\ \frac{1}{2}\left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\ k\sum_{n=1}^{j}(||u_m^n||_H^2 + ||w_m^n||_H^2)$$

$$+\left[\left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N} + ||\vec{v}||^2_{L^\infty(Q)^N}(L_{\mathcal{F}}^2 + 1)\right] k\sum_{n=1}^{j}||\nabla u_m^n||_H^2\ +\ \frac{k}{4}\sum_{n=1}^{j}\left|\left|\frac{u_m^n - u_m^{n-1}}{k}\right|\right|_H^2$$

$$+\ \frac{k}{4\,L_{\mathcal{F}}^2}\sum_{n=1}^{j}\left|\left|\frac{w_m^n - w_m^{n-1}}{k}\right|\right|_H^2 + k\sum_{n=1}^{j}||f_m^n||_H^2.$$

This is one of the most delicate point in the a priori estimates as we have now to control the terms containing hysteresis. As to the element $\sum_{n=1}^{j}\left|\left|\frac{w_m^n - w_m^{n-1}}{k}\right|\right|_H^2$, we can control it by (3.1.5) and then absorb the result in $\frac{3}{4}\,k\sum_{n=1}^{j}\left|\left|\frac{u_m^n - u_m^{n-1}}{k}\right|\right|_H^2$. We remark that the constant $\frac{1}{4\,L_{\mathcal{F}}^2}$ has been taken ad hoc for this aim. On the other hand,

$$\begin{aligned}
&2\,||\vec{v}||^2_{L^\infty(Q)^N}\int_\Omega (|u_m^j|^2 + |w_m^j|^2)\,dx\ +\ \frac{1}{2}\left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\ k\sum_{n=1}^{j}(||u_m^n||_H^2 + ||w_m^n||_H^2)\\
&= 2\,||\vec{v}||^2_{L^\infty(Q)^N}\int_\Omega (|u_m^j|^2 + |w_m^j|^2)\,dx\ +\ \frac{1}{2}\left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\int_\Omega k\sum_{n=1}^{j}(|u_m^n|^2 + |w_m^n|^2)\,dx\\
&\le \max\left(2\,||\vec{v}||^2_{L^\infty(Q)^N}, \frac{1}{2}\left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\right)\int_\Omega\left(\left[\max_{n=1,\dots,j}|u_m^n(x)|\right]^2 + \left[\max_{n=1,\dots,j}|w_m^n(x)|\right]^2\right)dx\\
&\overset{(3.1.3)}{\le} \max\left(2\,||\vec{v}||^2_{L^\infty(Q)^N}, \frac{1}{2}\left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\right) 2\,L_{\mathcal{F}}^2\int_\Omega\left[\max_{n=1,\dots,j}|u_m^n(x)|\right]^2 dx
\end{aligned}\tag{3.1.22}$$

$$+ \, 2 \, \max\left(2\,||\vec{v}||^2_{L^\infty(Q)^N}, \frac{1}{2}\,\left|\left|\frac{\partial \vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\right)\,||\tau||^2_{L^2(\Omega)},$$

where $\tau \in L^2(\Omega)$ was introduced in (3.1.3). We set

$$c_1 := \max\left(2\,||\vec{v}||^2_{L^\infty(Q)^N}, \frac{1}{2}\,\left|\left|\frac{\partial \vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\right)\,2\,L^2_{\mathcal{F}} \tag{3.1.23}$$

$$c_2 := \left|\left|\frac{\partial \vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N} + ||\vec{v}||^2_{L^\infty(Q)^N}(L^2_{\mathcal{F}}+1)$$

$$c_3 := \left(||\vec{v}||_{L^\infty(Q)^N} + \frac{1}{2}\right)\,||\nabla u^0_m||^2_H + \frac{||\vec{v}||_{L^\infty(Q)^N}}{2}\,(||u^0_m||^2_H + ||w^0_m||^2_H) \, + \, k\,\sum_{n=1}^{j}||f^n_m||^2_H$$

$$+ \, 2 \, \max\left(2\,||\vec{v}||^2_{L^\infty(Q)^N}, \frac{1}{2}\,\left|\left|\frac{\partial \vec{v}}{\partial t}\right|\right|_{L^\infty(Q)^N}\right)\,||\tau||^2_{L^2(\Omega)}.$$

Therefore we have, for any $j = 1, \ldots, m$

$$\frac{1}{4}\int_\Omega |\nabla u^j_m|^2\,dx + \frac{k}{4}\,\sum_{n=1}^{j}\left|\left|\frac{u^n_m - u^{n-1}_m}{k}\right|\right|^2_H \le \, c_1\,\int_\Omega \left[\max_{n=1,\ldots,j}|u^n_m(x)|\right]^2\,dx + c_2\,k\,\sum_{n=1}^{j}||\nabla u^n_m||^2_H + c_3.$$

At this point consider that the following holds (where c_1 is the constant introduced in (3.1.23))

$$\begin{aligned}
|u^j_m(x)|^2 =& |u^0_m(x)|^2 \, + \, \sum_{n=1}^{j}(u^n_m(x) - u^{n-1}_m(x))\,(u^n_m(x) + u^{n-1}_m(x))\\
\le& |u^0_m(x)|^2 \, + \, \left(\sum_{n=1}^{j}|u^n_m(x) - u^{n-1}_m(x)|^2\right)^{1/2}\,\left(\sum_{n=1}^{j}(|u^n_m(x)| + |u^{n-1}_m(x)|)^2\right)^{1/2}\\
\le& |u^0_m(x)|^2 \, + \, \frac{k}{8c_1}\,\sum_{n=1}^{j}\left|\frac{u^n_m(x) - u^{n-1}_m(x)}{k}\right|^2 \, + \, 2\,k\,c_1\,\sum_{n=1}^{j}\left(2|u^n_m(x)|^2 + 2|u^{n-1}_m(x)|^2\right)\\
\le& 8\,c_1\,\left(|u^0_m(x)|^2 + \, k\,\sum_{n=1}^{j}|u^n_m(x)|^2\right) \, + \, \frac{k}{8c_1}\,\sum_{n=1}^{j}\left|\frac{u^n_m(x) - u^{n-1}_m(x)}{k}\right|^2
\end{aligned}$$

which yields

$$\frac{1}{4}\int_\Omega |\nabla u^j_m|^2\,dx \, + \, \frac{k}{4}\,\sum_{n=1}^{j}\left|\left|\frac{u^n_m - u^{n-1}_m}{k}\right|\right|^2_H \le \, c_1^2\,\left[8\,||u^0_m||^2_H \, + \, 8\,k\,\sum_{n=1}^{j}||u^n_m||^2_H\right]$$

$$+ \, \frac{k}{8}\,\sum_{n=1}^{j}\left|\left|\frac{u^n_m - u^{n-1}_m}{k}\right|\right|^2_H \, + \, c_2\,k\,\sum_{n=1}^{j}||\nabla u^n_m||^2_H \, + \, c_3.$$

As we are dealing with Dirichlet boundary conditions, we can use Poincaré inequality and obtain in particular that

$$\int_\Omega |\nabla u_m^j|^2\,dx \leq c_4\,k \sum_{n=1}^{j} ||\nabla u_m^n||_H^2 + c_5 \leq c_6 \left(1 + k \sum_{n=1}^{j} ||\nabla u_m^n||_H^2\right) \tag{3.1.24}$$

where

$$c_4 := 4\,c_2 + 32\,c_1^2\,c_P^2 \qquad c_5 := 32\,c_1^2\,||u_m^0||_H^2 + 4\,c_3 \qquad c_6 := \max(c_4, c_5) \tag{3.1.25}$$

and c_P is the Poincaré constant.

We conclude at this point using a discrete version of Gronwall's inequality (see also Section A.7). For the sake of completeness we illustrate here (once for all the thesis) the details of the proof.

Assumption (3.1.15) tells us that $k < 1/c_6$. We set, for any $j = 1, \ldots, m$

$$q = 1 - c_6\,k \qquad\qquad z_j := ||\nabla u_m^j||_H^2.$$

Then equation (3.1.24) reads

$$z_j \leq c_6 \left(1 + k \sum_{n=1}^{j} z_n\right) \tag{3.1.26}$$

hence

$$q^j \sum_{n=1}^{j} z_n - q^{j-1} \sum_{n=1}^{j-1} z_n = q^{j-1} \left(q \sum_{n=1}^{j} z_n - \sum_{n=1}^{j-1} z_n\right) = q^{j-1} \left(z_j - c_6\,k \sum_{n=1}^{j} z_n\right) \overset{(3.1.26)}{\leq} c_6\,q^{j-1}.$$

Thus summing up over $j = 1, \ldots, m$, we have

$$q^m \sum_{n=1}^{m} z_n \leq c_6 \sum_{j=1}^{m} q^{j-1} \leq c_6 \left(\frac{1}{1-q}\right) = \frac{1}{k}.$$

This entails that, for any $j = 1, \ldots, m$

$$z_j \leq c_6 \left(1 + k \sum_{n=1}^{j} z_n\right) \leq c_6 \left(1 + k\,\frac{q^{-m}}{k}\right) = c_6\,(1 + q^{-m}).$$

As $\lim_{m\to\infty} q^{-m} = e^{c_6}$, this gives a bound independent of the discretization parameter.

Then, for any $j \in \{1, \ldots, m\}$, we have the following a priori estimate

$$\frac{1}{4}\,||\nabla u_m^j||_{L^2(\Omega)}^2 + \frac{k}{8} \sum_{n=1}^{j} \left|\left|\frac{u_m^n - u_m^{n-1}}{k}\right|\right|_{L^2(\Omega)}^2 \leq \text{constant (independent of } m). \tag{3.1.27}$$

③ *Third step: limit procedure*

At this point we introduce some further notation. A.e. in Ω, let $w_m(x,\cdot)$ be the linear time interpolate of $w_m(x,nk) := w_m^n(x)$ for $n = 0,\ldots,m$; moreover set $\bar{u}_m(x,t) := u_m^n(x)$ if $(n-1)k < t \leq nk$ for $n = 1,\ldots,m$ and define $\bar{w}_m$ and $\bar{f}_m$ in a similar way. We also set $\vec{v}_m(x,t) := \vec{v}_m^{\,n}(x)$ if $(n-1)k < t \leq nk$ for $n = 1,\ldots,m$. Thus (3.1.19) yields

$$\frac{\partial u_m}{\partial t} + \frac{\partial w_m}{\partial t} + C(\bar{u}_m) + A\bar{u}_m = \bar{f}_m \qquad \text{in } V', \text{ a.e. in } (0,T) \tag{3.1.28}$$

while (3.1.27) becomes

$$\begin{aligned} ||u_m||_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)} &\leq \text{ constant (independent of } m); \\ ||\bar{u}_m||_{L^\infty(0,T;V)} &\leq \text{ constant (independent of } m). \end{aligned} \tag{3.1.29}$$

The a priori estimates we found allow us to conclude that there exists u such that, possibly taking $m \to +\infty$ along a subsequence,

$$\begin{array}{lll} u_m \to u & \text{weakly star in} & H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \\ \bar{u}_m \to u & \text{weakly star in} & L^\infty(0,T;V). \end{array}$$

Moreover $H^1(0,T;L^2(\Omega)) = L^2(\Omega;H^1(0,T)) \subset L^2(\Omega;\mathcal{C}^0([0,T]))$ with continuous injection, so by (3.1.3) and (3.1.29) we get

$$||w_m||_{L^2(Q)} \leq \sqrt{T}\,||w_m||_{L^2(\Omega;\mathcal{C}^0([0,T]))} \leq \sqrt{T}\,L_{\mathcal{F}}\,||u_m||_{L^2(\Omega;\mathcal{C}^0([0,T]))} + \sqrt{T}\,||\tau||_{L^2(\Omega)} \leq c$$

with c constant independent of m; this entails that there exists w such that, possibly taking $m \to +\infty$ along a subsequence

$$w_m \to w \quad \text{weakly in } L^2(Q). \tag{3.1.30}$$

On the other hand, using again (3.1.3) it is also clear that

$$||\bar{u}_m + \bar{w}_m||_{L^2(Q)} \leq \text{ constant (independent of } m).$$

Using this fact, it is easy to verify that $C(\bar{u}_m) \in L^2(0,T;V')$ which in turn gives us

$$\left|\left|\frac{\partial}{\partial t}(u_m + w_m)\right|\right|_{L^2(0,T;V')} \leq \text{ constant (independent of } m).$$

At this point, possibly taking $m \to +\infty$ along a subsequence, we get

$$\begin{array}{lll} \dfrac{\partial}{\partial t}(u_m + w_m) \to \dfrac{\partial}{\partial t}(u + w) & \text{weakly star in} & L^2(0,T;V') \\ C(\bar{u}_m) \to C(u) & \text{weakly star in} & L^2(0,T;V'). \end{array} \tag{3.1.31}$$

Hence, taking $m \to +\infty$ in (3.1.28), we get

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} + C(u) + Au = f. \tag{3.1.32}$$

Now we have only to show that $w = \overline{\mathcal{F}}(u)$. We already remarked that the a priori estimates we found yield

$$u_m \to u \qquad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V).$$

On the other hand, by interpolation and after a suitable choice of representatives in equivalence classes, we may deduce, for any $s \in (0,1/2)$ (for more details see for example [29], Chapter 4)

$$H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \subset H^1(Q) \subset H^s(\Omega;H^{1-s}(0,T)) \subset L^2(\Omega;\mathcal{C}^0([0,T])) \tag{3.1.33}$$

where the last inclusion is also compact; so possibly extracting a subsequence, we have

$$u_m \to u \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega.$$

Using the strong continuity of the operator $\overline{\mathcal{F}}$ we deduce

$$\overline{\mathcal{F}}(u_m) \to \overline{\mathcal{F}}(u) \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega.$$

As $w_m(x,\cdot)$ is the linear time interpolate of

$$w_m(x,nk) = [\overline{\mathcal{F}}(u_m)](x,nk)$$

for $n = 1,\dots,m$, a.e. in Ω, we have

$$w_m \to \overline{\mathcal{F}}(u) \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega.$$

Therefore, by (3.1.30) we get $w = \overline{\mathcal{F}}(u)$ a.e. in Q.
By (3.1.3), the sequence $\left\{ ||w_m(\cdot,t)||^2_{\mathcal{C}^0([0,T])} \right\}$ is equiintegrable in Ω, as the same holds for u_m. Hence w_m converges strongly in $L^2(\Omega;\mathcal{C}^0([0,T]))$.
Then Problem 3.1.1 admits at least one solution $u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$. On the other hand, by (3.1.5), as $u \in H^1(0,T;L^2(\Omega)) = L^2(\Omega;H^1(0,T))$, then also $\overline{\mathcal{F}}(u) \in H^1(0,T;L^2(\Omega))$. This finishes the proof. □

3.1.3. Uniqueness

The aim of this subsection is to establish a uniqueness result for solutions of the Cauchy problem (with Dirichlet boundary conditions) related to the model equation

$$\frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) + \vec{v} \cdot \nabla(u + \overline{\mathcal{F}}(u)) - \triangle u = f \qquad \text{in } \Omega \times (0,T)$$

which is studied in this first part of the chapter. As it is expected (the equation considered is quasi-linear), we are not able to prove a uniqueness result when $\overline{\mathcal{F}}$ is a generic hysteresis operator. In the case of equation

$$\frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) - \triangle u = f \qquad \text{in } \Omega \times (0,T)$$

a partial result can be established using a suitable inequality due to Hilpert [18], which only holds for a restricted class of hysteresis operators; however in our case, apparently we cannot do the same because the properties of hysteresis operators do not allow us to get a sort of Hilpert's inequality dealing with the gradient of the operator $\overline{\mathcal{F}}$ with respect to the space variable. We are anyway able to state and prove a uniqueness result using an inequality which is typical for Prandtl-Ishlinskiĭ operators of play type.

More precisely, let us consider the setting outlined in Subsection 3.1.1. Let $\mathcal{F}_\varphi : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be a Prandtl-Ishlinskiĭ operator of play type (according to Definition 1.3.1, see also (1.3.3))

$$\mathcal{F}_\varphi(\lambda, u) = h(0)u + \int_0^\infty \wp_r(\lambda, u)\, dh(r)$$

generated by the convex function

$$\varphi(r) = \int_0^r h(s)\, ds, \qquad r > 0,$$

where $\wp_r(\lambda, u)$ is the play operator introduced in (1.2.5), Λ_0 is introduced in (1.2.4) and h is a given nondecreasing function. We fix any initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_K) \qquad \text{for some } K > 0, \tag{3.1.34}$$

where Λ_K was introduced in (1.2.4). From Section 1.6 it is easy to see that the operator

$$\overline{\mathcal{F}}_\varphi(u)(x,t) := \tilde{\mathcal{F}}_\varphi(\lambda, u)(x,t) := \mathcal{F}_\varphi(\lambda(x), u(x,\cdot))(t) := h(0)u(x,t) + \int_0^\infty \overline{\wp}_r(\lambda, u)(x,t)\, dh(r) \tag{3.1.35}$$

fulfills (3.1.1), (3.1.2), (3.1.3), (3.1.4) and (3.1.5), where

$$\overline{\wp}_r(\lambda, u)(x,t) := \wp_r(\lambda(x), u(x,\cdot))(t).$$

The uniqueness result we are able to prove is the following.

Theorem 3.1.4. *(Uniqueness)*
Consider λ as in (3.1.34) *and $\overline{\mathcal{F}}_\varphi$ as in* (3.1.35). *Moreover, suppose that h is bounded and assume the following data*

$$f \in L^2(Q), \;\; u^0 \in V, \;\; w^0 \in L^2(\Omega).$$

Then Problem $3.1.1_\varphi$ *(which is Problem* 3.1.1 *with $\overline{\mathcal{F}}$ replaced by $\overline{\mathcal{F}}_\varphi$) admits a unique solution.*

Proof. First of all Theorem 1.3.2 and Proposition A.1.2 entail that Problem $3.1.1_\varphi$ admits at least a solution $u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$. Let us now show that this solution is also unique.

The proof of this result may appear quite involved; we try to clarify it by showing first the *formal computations* in *one space dimension.*

Suppose by contradiction that u_1 and u_2 are both solutions of our model problem. This implies that

$$\frac{\partial}{\partial t}(u_1 - u_2 + \overline{\mathcal{F}}_\varphi(u_1) - \overline{\mathcal{F}}_\varphi(u_2)) + \frac{\partial}{\partial x}(u_1 - u_2 + \overline{\mathcal{F}}_\varphi(u_1) - \overline{\mathcal{F}}_\varphi(u_2)) - \frac{\partial^2}{\partial x^2}(u_1 - u_2) = 0.$$

We set

$$\tilde{F}(u_i) := u_i + \overline{\mathcal{F}}_\varphi(u_i) \qquad G := \tilde{F}^{-1} \qquad v_i := \tilde{F}(u_i) \qquad\qquad i = 1,2.$$

We then obtain

$$(v_1 - v_2)_t + (v_1 - v_2)_x - (G(v_1) - G(v_2))_{xx} = 0.$$

Integrating twice in space and setting $V_{xx} = v_1 - v_2$, we deduce

$$V_t + V_x - (G(v_1) - G(v_2)) = 0.$$

Now it is enough to test this equation by $-(v_1 - v_2)_t = -V_{xxt}$. In fact, using the properties of the operator G, this will lead to

$$\int V_{xt}^2 + \frac{d}{dt}\int V_{xx}^2 \leq \int V_{xx}\, V_{xt}$$

and at this point the term on the right side can be in part absorbed from the contribution on the left side and in part controlled by the Gronwall lemma. This is the idea which undergoes our proof. Let us now show the details, this time in several space dimensions.

Suppose by contradiction that $u_1,\, u_2 \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$ are both solutions of Problem $3.1.1_\varphi$. Then starting from (3.1.9) we may deduce, for any $\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ and for any $t \in (0,T)$

$$\int_0^t \int_\Omega \frac{\partial}{\partial t}(u_1 - u_2 + \overline{\mathcal{F}}_\varphi(u_1) - \overline{\mathcal{F}}_\varphi(u_2))\,\psi\,dx\,dt$$
$$- \int_0^t \int_\Omega [\vec{v}\cdot\nabla\psi]\,(u_1 - u_2 + \overline{\mathcal{F}}_\varphi(u_1) - \overline{\mathcal{F}}_\varphi(u_2))\,dx\,dt \;+\; \int_0^t \int_\Omega \nabla(u_1 - u_2)\,\nabla\psi\,dx\,dt = 0. \tag{3.1.36}$$

It is easy to see that the operator $I + \mathcal{F}_\varphi$, where I is the identity operator, is still a Prandtl-Ishlinskiĭ operator of play type with the function h replaced by $\tilde{h} := h + 1$; let us set

$$\tilde{\varphi}(r) := \int_0^r \tilde{h}(s)\,ds, \qquad r > 0$$

and $\overline{\mathcal{G}}_{\tilde{\varphi}} := I + \overline{\mathcal{F}}_{\varphi}$. Thus

$$\overline{\mathcal{G}}_{\tilde{\varphi}}(u_i)(x,t) := u_i(x,t) + \overline{\mathcal{F}}_{\varphi}(u_i)(x,t) \qquad i = 1,2, \text{ a.e. in } \Omega \times (0,T).$$

We also set, for brevity

$$w_i := \overline{\mathcal{G}}_{\tilde{\varphi}}(u_i) \qquad \text{for } i = 1,2.$$

At this point we fix any $t \in (0,T)$, then we set $Q_t := \Omega \times (0,t)$ and also $\tilde{w} := w_1 - w_2$. We endow the space $H^{-1}(\Omega)$ with the following scalar product which will be useful for our purposes

$$(u,v)_{H^{-1}(\Omega)} :=_{H^{-1}(\Omega)} \langle u, A^{-1}v \rangle_{H^1_0(\Omega)} \qquad \forall\, u,v \in H^{-1}(\Omega)$$

where the operator A^{-1} is introduced in (2.1.16). We remember in particular that

$$(u,v)_{H^{-1}(\Omega)} := \int_\Omega u\, A^{-1}v\, dx \qquad \forall\, u \in L^2(\Omega),\ \forall\, v \in H^{-1}(\Omega).$$

It is evident that this choice produces no contradiction with the functional setting in which we are working. This procedure - where we "invert" the Laplace operator associated to homogeneous Dirichlet boundary conditions - is the counterpart, in several dimensions, of the double integration in space we did in the formal computations in one space dimension.
Now we choose in (3.1.36) $\psi := \dfrac{\partial}{\partial t}(A^{-1}\tilde{w})$, obtaining

$$\int_0^t \int_\Omega \frac{\partial \tilde{w}}{\partial t} \frac{\partial}{\partial t}(A^{-1}\tilde{w})\, dx\, dt - \int_0^t \int_\Omega \left[\vec{v} \cdot \frac{\partial}{\partial t} \nabla (A^{-1}\tilde{w})\right] \tilde{w}\, dx\, dt + \int_0^t \int_\Omega (u_1 - u_2) \frac{\partial \tilde{w}}{\partial t}\, dx\, dt = 0. \tag{3.1.37}$$

First of all we have

$$\int_0^t \left\| \frac{\partial \tilde{w}}{\partial t} \right\|^2_{H^{-1}(\Omega)} dt = \int_0^t \int_\Omega \frac{\partial \tilde{w}}{\partial t} \frac{\partial}{\partial t}(A^{-1}\tilde{w})\, dx\, dt$$

$$= \int_0^t \int_\Omega \frac{\partial}{\partial t}[A\, A^{-1}\tilde{w}] \frac{\partial}{\partial t}(A^{-1}\tilde{w})\, dx\, dt = \int_0^t \int_\Omega \left| \frac{\partial}{\partial t} \nabla\, (A^{-1}\tilde{w}) \right|^2 dx\, dt.$$

On the other hand, using property (1.3.2), we get

$$\int_0^t \int_\Omega \frac{\partial \tilde{w}}{\partial t}(u_1 - u_2)\, dx\, dt = \int_0^t \int_\Omega \frac{\partial}{\partial t}(\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2))\,(u_1 - u_2)\, dx\, dt$$

$$\geq \frac{1}{2} \int_\Omega \tilde{h}(0)\{[(u_1 - u_2)(x,t)]^2 - [(u_1 - u_2)(x,0)]^2\}\, dx$$

$$+ \frac{1}{2} \int_\Omega \int_0^\infty \{[\overline{\wp}_r(\lambda, u_1)(x,t) - \overline{\wp}_r(\lambda, u_2)(x,t)]^2 - [\overline{\wp}_r(\lambda, u_1)(x,0) - \overline{\wp}_r(\lambda, u_2)(x,0)]^2\}\, d\tilde{h}(r)\, dx$$

$$= \frac{1}{2} \int_\Omega \left[\tilde{h}(0)[(u_1 - u_2)(x,t)]^2 + \int_0^\infty [\overline{\wp}_r(\lambda, u_1)(x,t) - \overline{\wp}_r(\lambda, u_2)(x,t)]^2\, d\tilde{h}(r)\right] dx,$$

due to the fact that $u_1(x,0) \equiv u_2(x,0)$ and consequently, using the causality property for the play operators, also $[\overline{\wp}_r(\lambda,u_1)(x,0) \equiv \overline{\wp}_r(\lambda,u_2)(x,0)]$. Thus (3.1.37) yields

$$\int_0^t \int_\Omega \left|\frac{\partial}{\partial t}\nabla(A^{-1}(w_1-w_2))\right|^2 dx\,dt + \frac{1}{2}\int_\Omega \tilde{h}(0)[(u_1-u_2)(x,t)]^2\,dx$$

$$+\frac{1}{2}\int_\Omega \int_0^\infty [\overline{\wp}_r(\lambda,u_1)(x,t) - \overline{\wp}_r(\lambda,u_2)(x,t)]^2\,d\tilde{h}(r)\,dx$$

$$\leq ||\vec{v}||_{L^\infty(Q)^N} \int_0^t \int_\Omega |\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2)| \cdot \left|\frac{\partial}{\partial t}\nabla(A^{-1}(w_1-w_2))\right| dx\,dt$$

$$\leq \frac{||\vec{v}||^2_{L^\infty(Q)^N}}{2} \int_0^t \int_\Omega |\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2)|^2\,dx\,dt + \frac{1}{2}\int_0^t \int_\Omega \left|\frac{\partial}{\partial t}\nabla(A^{-1}(w_1-w_2))\right|^2 dx\,dt$$

$$\leq \frac{||\vec{v}||^2_{L^\infty(Q)^N}}{2} \int_0^t \int_\Omega \left|\tilde{h}(0)u_1 + \int_0^\infty \overline{\wp}_r(\lambda,u_1)\,d\tilde{h}(r) - \tilde{h}(0)u_2 - \int_0^\infty \overline{\wp}_r(\lambda,u_2)\,d\tilde{h}(r)\right|^2 dx\,dt$$

$$+\frac{1}{2}\int_0^t \int_\Omega \left|\frac{\partial}{\partial t}\nabla(A^{-1}(w_1-w_2))\right|^2 dx\,dt \leq \tilde{h}(0)\,||\vec{v}||^2_{L^\infty(Q)^N} \int_0^t \int_\Omega \tilde{h}(0)[(u_1-u_2)(x,t)]^2\,dx\,dt$$

$$+\,||\vec{v}||^2_{L^\infty(Q)^N}\,(\tilde{h}(\infty)-\tilde{h}(0)) \int_0^t \int_\Omega \left[\int_0^\infty [\overline{\wp}_r(\lambda,u_1)(x,t) - \overline{\wp}_r(\lambda,u_2)(x,t)]^2\,d\tilde{h}(r)\right] dx\,dt$$

$$+\frac{1}{2}\int_0^t \int_\Omega \left|\frac{\partial}{\partial t}\nabla(A^{-1}(w_1-w_2))\right|^2 dx\,dt.$$

Now Gronwall's lemma (see Lemma A.7.1) allows us to obtain

$$u_1(x,t) \equiv u_2(x,t) \qquad \text{a.e. in } \Omega, \text{ for all } t \in (0,T)$$

and this finishes the proof. □

Remark 3.1.5. In the last lines of the previous computations we used the assumption that the function h is bounded. This is essential since it is certainly true that $\overline{\wp}_r(\lambda,u)(x,t)$ vanishes for r sufficiently large, say $r > R(x)$, but by Proposition 1.2.4 we have in general no control on $R(x)$, unless Ω is a bounded one-dimensional interval. In this case the assumption of boundedness of h can be removed as $R(x)$ is a priori bounded independently of h. The same happens in Theorem 3.2.8.

3.1.4. The case of the Preisach operator

The aim of this subsection is just to show that the results obtained so far in this chapter are valid also if $\overline{\mathcal{F}}$ is a Preisach operator, a case which is very important for the applications under some conditions on the corresponding measure μ.

Consider any finite (signed) Borel measure μ defined in $\mathcal{P}$, (where $\mathcal{P}$ is the Preisach plane, see (1.5.2)) and let $\mathcal{H}_\mu$ and $\overline{\mathcal{H}}_\mu$ be the corresponding Preisach operators introduced in (1.5.5) and (1.6.3) respectively.
Consider the setting outlined in Subsection 3.1.1 and fix any $\xi \in L^1(\Omega; L^1(\mathcal{P}))$; we deal with the following problem.

Problem 3.1.6. *Let us consider a known function $\vec{v}$ satisfying* (3.1.8)*; we search for a function $u \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \cap L^2(0,T;V)$ such that $\overline{\mathcal{H}}_\mu(u,\xi) \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \cap L^2(Q)$ and for any $\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ with $\psi(\cdot,T) = 0$ a.e. in Ω*

$$\int_0^T \int_\Omega -(u + \overline{\mathcal{H}}_\mu(u,\xi)) \frac{\partial \psi}{\partial t}\, dx\, dt - \int_0^T \int_\Omega [\vec{v} \cdot \nabla \psi]\, (u + \overline{\mathcal{H}}_\mu(u,\xi))\, dx\, dt$$
$$+ \int_0^T \int_\Omega \nabla u \cdot \nabla \psi\, dx\, dt = \int_0^T {}_{V'}\langle f, \psi \rangle_V\, dt + \int_\Omega [u^0(x) + w^0(x)]\, \psi(x,0)\, dx,$$

where u^0, $w^0 \in L^2(\Omega)$ and $f \in L^2(0,T;V')$.

Let us investigate under what assumptions on the measure μ the previous problem admits at least one solution.

Corollary 3.1.7. *(Existence)*
Let μ be a nonnegative finite Borel measure over Ω fulfilling (1.6.4) *and* (1.6.5)*.*
Moreover, we assume that (3.1.14) *holds. Then Problem* 3.1.6 *admits at least one solution*

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

such that

$$\overline{\mathcal{H}}_\mu(u,\xi) \in H^1(0,T;L^2(\Omega)).$$

Proof. The thesis follows from Proposition 1.6.1 and Theorem 3.1.2. □

Remark 3.1.8. The problem of uniqueness in this particular case is for the moment open, as inequality (1.3.2) is typical for Prandtl-Ishlinskiĭ operators and cannot be extended to more general Preisach-type models.

3.1.5. Stable dependence on the data

In this subsection we would like to state and prove a result of stable dependence on the data for solutions of Problem 3.1.1. We consider a sequence of memory operators $\overline{\mathcal{F}}_n : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ being order preserving in a sense specified below and

fulfilling the assumptions of Theorem 3.1.2; we require moreover a pointwise convergence in time, a.e. in space, of these operators to some operator $\overline{\mathcal{F}}$. Let us notice that it is new the idea of using such a weak assumption. The trick stays in exploiting, in a suitable way, the order preserving property and the uniform convergence in time of the sequence of approximate solutions u_n; this last property comes from the good regularity results for solutions of Problem 3.1.1 (the pointwise convergence in time of these solutions would be not enough for our purposes).
The result we are able to state and prove is the following.

Theorem 3.1.9. *(Stable dependence on the data)*
Let us consider a sequence of operators $\overline{\mathcal{F}}_n : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ *being causal, strongly continuous, affinely bounded, piecewise monotone and piecewise Lipschitz continuous according to* (3.1.1), (3.1.2), (3.1.3), (3.1.4) *and* (3.1.5), *(where the constant* $L_{\mathcal{F}}$ *is the same for all* $n \in \mathbb{N}$*) and being moreover* ORDER PRESERVING *in the following sense:*

$$\begin{cases} \forall\, u_1, u_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \ \ \forall\, t \in [0,T], \ \text{if } u_1 \le u_2 \text{ in } [0,t], \text{ a.e. in } \Omega, \text{ then} \\ [\overline{\mathcal{F}}_n(u_1)](x,t) \le [\overline{\mathcal{F}}_n(u_2)](x,t) \ \text{ a.e. in } \Omega. \end{cases}$$

Suppose moreover that, for all $v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$

$$\overline{\mathcal{F}}_n(v) \to \overline{\mathcal{F}}(v) \qquad \text{pointwise in } \mathcal{C}^0([0,T]), \quad \text{a.e. in } \Omega, \tag{3.1.38}$$

for some operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ *fulfilling the same assumptions as* $\overline{\mathcal{F}}_n$*. Finally let us assume that* (3.1.14) *holds for a sequence of data* $\{(u_n^0, f_n)\}_{n\in\mathbb{N}}$ *and suppose that*

$$\begin{aligned} &u_n^0 \to u^0 \qquad w_n^0 \to w^0 && \text{strongly in } L^2(\Omega) \\ &f_n \to f && \text{weakly in } L^2(Q). \end{aligned}$$

For any $n \in \mathbb{N}$, *let* u_n *be a solution of Problem* $3.1.1_n$ *(which is Problem* 3.1.1 *corresponding to* u_n^0, w_n^0, f_n, $\overline{\mathcal{F}}_n$*); then*

$$u_n \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T;V)$$

and there exists u *such that, possibly taking* $n \to \infty$ *along a subsequence,*

$$u_n \to u \qquad \text{weakly star in } H^1(0,T; L^2(\Omega)) \cap L^2(0,T;V)$$

and

$$\overline{\mathcal{F}}_n(u_n) \to \overline{\mathcal{F}}(u) \qquad \text{strongly in } L^2(Q).$$

Finally u *is a solution of Problem* 3.1.1.

Proof. First we define the operators $\overline{\mathcal{F}}^{(-)}, \overline{\mathcal{F}}^{(+)} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ as follows

$$\overline{\mathcal{F}}^{(-)}(u)(x,\cdot) := \sup\{\overline{\mathcal{F}}(v)(x,\cdot) : v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), v(x,\cdot) < u(x,\cdot)\} \qquad \text{a.e. in } \Omega$$

$$\overline{\mathcal{F}}^{(+)}(u)(x,\cdot) := \inf\{\overline{\mathcal{F}}(v)(x,\cdot) : v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), v(x,\cdot) > u(x,\cdot)\} \qquad \text{a.e. in } \Omega.$$

It is not difficult to see that

$$\forall\, u \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \qquad \overline{\mathcal{F}}^{(-)}(u)(x,\cdot) \leq \overline{\mathcal{F}}(u)(x,\cdot) \leq \overline{\mathcal{F}}^{(+)}(u)(x,\cdot), \qquad \text{a.e in } \Omega,$$

and that $\overline{\mathcal{F}}^{(-)}(u)(x,\cdot) = \overline{\mathcal{F}}(u)(x,\cdot) = \overline{\mathcal{F}}^{(+)}(u)(x,\cdot)$, a.e in Ω, if $\overline{\mathcal{F}}$ is a strongly continuous operator.

Now, using our assumptions it is clear that Problem 3.1.1$_n$ admits at least one solution u_n such that

$$||u_n||_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)} \leq \ \text{constant (independent of } n). \tag{3.1.39}$$

This entails that there exists u such that, possibly taking the limit for $n \to \infty$ along a subsequence

$$u_n \to u \qquad \text{weakly star in } H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T;V).$$

On the other hand, by interpolation and after a suitable choice of representatives, we deduce

$$H^1(0,T; L^2(\Omega)) \cap L^2(0,T;V) \subset L^2(\Omega; \mathcal{C}^0([0,T]))$$

with continuous and compact injection and this assures us that

$$u_n \to u \qquad \text{strongly in } L^2(\Omega; \mathcal{C}^0([0,T]))$$

from what we immediately get

$$u_n \to u \qquad \text{uniformly in } [0,T], \text{ a.e. in } \Omega. \tag{3.1.40}$$

At this point, as $\overline{\mathcal{F}}_n$ are order preserving, from (3.1.38) and (3.1.40), we get that, for all $\varepsilon > 0$, $\exists\, \bar{n}$ such that $\forall\, n \geq \bar{n}$ and for all $t \in [0,T]$

$$\overline{\mathcal{F}}_n(u_n)(x,t) \leq \overline{\mathcal{F}}_n(u+\varepsilon)(x,t) \to \overline{\mathcal{F}}(u+\varepsilon)(x,t) \qquad \text{a.e. in } \Omega.$$

Now we first take the superior limit as $n \to \infty$ and then the infimum with respect to ε. We deduce

$$\limsup_{n\to\infty}[\overline{\mathcal{F}}_n(u_n)](x,t) \leq \inf_{\varepsilon>0}[\overline{\mathcal{F}}(u+\varepsilon)](x,t) =: \overline{\mathcal{F}}^{(+)}(u)(x,t) \qquad \text{a.e. in } \Omega.$$

Arguing in a similar way we get

$$\overline{\mathcal{F}}_n(u_n)(x,t) \geq \overline{\mathcal{F}}_n(u-\varepsilon)(x,t) \to \overline{\mathcal{F}}(u-\varepsilon)(x,t) \qquad \text{a.e. in } \Omega,$$

from what we deduce

$$\liminf_{n\to\infty} \overline{\mathcal{F}}_n(u_n)(x,t) \geq \overline{\mathcal{F}}^{(-)}(u)(x,t) \qquad \text{a.e. in } \Omega.$$

As $\overline{\mathcal{F}}$ is assumed to be strongly continuous, we have

$$\lim_{n\to\infty} \overline{\mathcal{F}}_n(u_n)(x,t) = \overline{\mathcal{F}}(u)(x,t) \qquad \text{a.e. in } Q.$$

At this point, (3.1.3) gives us the possibility of applying the Lebesgue dominated convergence theorem both in space and in time, thus we deduce

$$\overline{\mathcal{F}}_n(u_n) \to \overline{\mathcal{F}}(u) \qquad \text{strongly in } L^2(Q).$$

This is enough in order to pass to the limit in Problem $3.1.1_n$ and get that u is a solution of Problem 3.1.1. □

3.1.6. Existence via hyperbolic regularization method

The aim of this subsection is to find an alternative existence theorem for solutions of Problem 3.1.1 using the so-called in literature *hyperbolic regularization method*. More precisely, we consider the following hyperbolic equation

$$\varepsilon\,\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) + \vec{v}\cdot\nabla(u + \overline{\mathcal{F}}(u)) - \triangle u = f \qquad \text{in } \Omega \times (0,T)$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\triangle$ is the Laplace operator, $\varepsilon > 0$, $\vec{v} : \Omega\times(0,T) \to \mathbb{R}^N$ is known, f is a given function and $\overline{\mathcal{F}}$ is a continuous hysteresis operator. This equation derives from (3.0.1) with in addition the perturbation term $\varepsilon\,\dfrac{\partial^2 u}{\partial t^2}$. First we introduce a weak formulation in Sobolev spaces for the Cauchy problem associated to this model hyperbolic equation, then under suitable assumptions on the hysteresis operator $\overline{\mathcal{F}}$ we state an existence theorem for the solutions of the same problem; finally we study the behavior of these solutions when the parameter $\varepsilon \to 0$, getting at the end the existence of a solution for Problem 3.1.1.

❒ HYPERBOLIC REGULARIZATION METHOD. Let us consider the setting outlined in Subsection 3.1.1, in particular let us consider the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega;\mathcal{C}^0([0,T]))$ fulfilling (3.1.1), (3.1.2), (3.1.3), (3.1.4) and (3.1.5). We assume that $u^0, w^0, z^0 \in L^2(\Omega)$ and $f \in L^2(0,T;V')$. We want to solve the following problem

Problem 3.1.10. *Let us consider a known function $\vec{v}$ fulfilling* (3.1.8)*; for any fixed $\varepsilon > 0$, we search for a couple of functions $u_\varepsilon \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \cap L^2(0,T;V)$ and $z_\varepsilon \in L^2(Q)$ such* *that* $\overline{\mathcal{F}}(u_\varepsilon) \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \cap L^2(Q)$ *and for any* $\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ *with* $\psi(\cdot,T) = 0$ *a.e. in* Ω

$$\int_0^T\int_\Omega -(\varepsilon\, z_\varepsilon + u_\varepsilon + \overline{\mathcal{F}}(u_\varepsilon))\frac{\partial\psi}{\partial t}\,dx\,dt - \int_0^T\int_\Omega [\vec{v}\cdot\nabla\psi]\,(u_\varepsilon + \overline{\mathcal{F}}(u_\varepsilon))\,dx\,dt$$
$$+\int_0^T\int_\Omega \nabla u_\varepsilon\cdot\nabla\psi\,dx\,dt = \int_0^T {}_{V'}\langle f,\psi\rangle_V\,dt + \int_\Omega [z^0(x)+u^0(x)+w^0(x)]\,\psi(x,0)\,dx \tag{3.1.41}$$

$$-\int_0^T\int_\Omega u_\varepsilon\,\frac{\partial\psi}{\partial t}\,dx\,dt = \int_0^T\int_\Omega z_\varepsilon\,\psi\,dx\,dt + \int_\Omega u^0(\cdot)\,\psi(\cdot,0)\,dx \tag{3.1.42}$$

Interpretation. The variational equations (3.1.41) and (3.1.42) yield

$$\begin{cases} \varepsilon\,\dfrac{\partial z_\varepsilon}{\partial t} + \dfrac{\partial}{\partial t}(u_\varepsilon + \overline{\mathcal{F}}(u_\varepsilon)) + \vec{v}\cdot\nabla(u_\varepsilon + \overline{\mathcal{F}}(u_\varepsilon)) + Au_\varepsilon = f \\ \dfrac{\partial u_\varepsilon}{\partial t} = z_\varepsilon \end{cases} \quad \text{a.e. in } \mathcal{D}'(0,T;V') \tag{3.1.43}$$

thus, by comparison,

$$\frac{\partial}{\partial t}[\varepsilon\, z_\varepsilon + u_\varepsilon + \overline{\mathcal{F}}(u_\varepsilon)] \in L^2(0,T;V')$$

so $z_\varepsilon + u_\varepsilon + \overline{\mathcal{F}}(u_\varepsilon) \in H^1(0,T;V')$ and (3.1.43) holds in V' a.e. in $(0,T)$. Hence, integrating by parts in time in (3.1.42) we get

$$[z_\varepsilon + u_\varepsilon + \overline{\mathcal{F}}(u_\varepsilon)]_{|t=0} = z^0 + u^0 + w^0 \ \text{ in } V' \tag{3.1.44}$$

in the sense of the traces. In turn (3.1.43) and (3.1.44) yield (3.1.41) and (3.1.42) and the two formulations are equivalent.

Theorem 3.1.11. *(Existence)*
Let us assume that the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega;\mathcal{C}^0([0,T]))$ *is causal, strongly continuous, affinely bounded, piecewise monotone and piecewise Lipschitz continuous according to* (3.1.1), (3.1.2), (3.1.3), (3.1.4) *and* (3.1.5). *Moreover let*

$$f \in L^2(Q),\ \ u^0 \in V,\ \ w^0 \in L^2(\Omega),\ \ z^0 \in L^2(\Omega). \tag{3.1.45}$$

Then for any $\varepsilon > 0$, *Problem* 3.1.10 *admits at least one solution*

$$u_\varepsilon \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), \qquad\qquad z_\varepsilon \in L^2(Q)$$

such that

$$\overline{\mathcal{F}}(u_\varepsilon) \in H^1(0,T;L^2(\Omega)).$$

Proof. The proof of this theorem relays on the same approximation argument used for proving Theorem 3.1.2, i.e. also this time we approximate our problem using an implicit time discretization scheme. Here we just would like to sketch the main points; therefore we fix $m \in \mathbb{N}$, set $k := T/m$ and for any $n = 1, \dots, m$ we consider $f_m^n(x) := f(x, nk)$, $u_m^0 := u^0$, $w_m^0 := w^0$ and $z_m^0 = z^0$. So we have to solve the following

Problem 3.1.12. *For any fixed $\varepsilon > 0$, we search for two functions $u_m^n \in V$ and $z_m^n \in L^2(\Omega)$ for any $n = 1, \dots m$, such that, if $u_m(x, \cdot)$ is the linear time interpolate of $u_m(x, nk) := u_m^n(x)$, for $n = 1, \dots, m$, a.e. in Ω and $w_m^n := [\overline{\mathcal{F}}(u_m)](x, nk)$ for $n = 1, \dots, m$, a.e. in Ω, then*

$$\frac{\varepsilon}{k} \int_\Omega (z_m^n - z_m^{n-1})\, \psi \, dx + \frac{1}{k} \int_\Omega (u_m^n - u_m^{n-1})\, \psi \, dx + \frac{1}{k} \int_\Omega (w_m^n - w_m^{n-1})\, \psi \, dx$$

$$- \int_\Omega [\vec{v}_m^{\,n} \cdot \nabla \psi]\, (u_m^n + w_m^n)\, dx + \int_\Omega \nabla u_m^n \cdot \nabla \psi \, dx = \int_\Omega f_m^n \, \psi \, dx \tag{3.1.46}$$

$$\frac{1}{k} \int_\Omega (u_m^n - u_m^{n-1})\, \psi \, dx = \int_\Omega z_m^n \, \psi \, dx$$

where $\vec{v}_m^{\,n}(x) := \vec{v}(x, nk)$.

By induction, let us assume that the pairs $(u_m^1, z_m^1), \dots, (u_m^{n-1}, z_m^{n-1})$ are known, after noticing that (u_m^0, z_m^0) is given; we try to determine the couple (u_m^n, z_m^n). We introduce the operators $\widehat{F}_m^n$ and C as in Subsection 3.1.2; it turns out that (3.1.17) and (3.1.18) still hold also in this case. If we rewrite (3.1.46) by means of the operator C and then separate the terms which are known from the ones we have to determine, we deduce

$$\begin{cases} (\varepsilon + k)\, u_m^n + k\, \widehat{F}_m^n(u_m^n) + k^2\, C(u_m^n) + k^2\, A u_m^n = (G_\varepsilon)_m^n \\ \dfrac{u_m^n - u_m^{n-1}}{k} = z_m^n, \end{cases}$$

where $(G_\varepsilon)_m^n$ is a known function depending on m, n, k, ε. Arguing as we did in this chapter in the proof of Theorem 3.1.2, it is not difficult to obtain that this problem admits at least a solution (u_m^n, z_m^n).
At this point we should get some a priori estimates. In order to do this, we consider system (3.1.46) with the choice $\psi := (u_m^n - u_m^{n-1})$; then we sum for $n = 1, \dots, j$, for $j \in \{1, \dots, m\}$. All the terms can be estimated like in Subsection 3.1.2, with the exception of the following one which has to be added this time

$$\varepsilon \sum_{n=1}^{j} {}_{V'}\langle \frac{z_m^n - z_m^{n-1}}{k}, u_m^n - u_m^{n-1} \rangle_V = \varepsilon \sum_{n=1}^{j} {}_{V'}\langle z_m^n - z_m^{n-1}, z_m^n \rangle_V$$

$$= \varepsilon \sum_{n=1}^{j} \int_\Omega (|z_m^n|^2 - z_m^{n-1}\, z_m^n)\, dx \geq \frac{\varepsilon}{2} \sum_{n=1}^{j} \int_\Omega (|z_m^n|^2 - |z_m^{n-1}|^2)\, dx = \frac{\varepsilon}{2} \int_\Omega (|z_m^j|^2 - |z_m^0|^2)\, dx.$$

Therefore after summing up we deduce the following a priori estimate, for any $j \in \{1, \dots, m\}$

$$\frac{\varepsilon}{2} ||z_m^j||^2_{L^2(\Omega)} + \frac{k}{8} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|^2_{L^2(\Omega)} + \frac{1}{4} ||\nabla u_m^j||^2_{L^2(\Omega)} \leq \text{ constant (independent of } m \text{ and } \varepsilon). \tag{3.1.47}$$

Passing from the discrete case to the continuous one, we deduce that the system (3.1.46) yields

$$\begin{cases} \varepsilon \dfrac{\partial z_m}{\partial t} + \dfrac{\partial u_m}{\partial t} + \dfrac{\partial w_m}{\partial t} + C(\bar{u}_m) + A\bar{u}_m = \bar{f}_m \\ \dfrac{\partial u_m}{\partial t} = \bar{z}_m, \end{cases} \tag{3.1.48}$$

where, a.e. in Ω, $w_m(x, \cdot)$ is the linear time interpolate of $w_m(x, nk) := w_m^n(x)$ for $n = 0, \dots, m$, $\bar{u}_m(x, t) := u_m^n(x)$ if $(n-1)k < t \leq nk$ for $n = 1, \dots, m$ and the quantities $\bar{w}_m$, $\bar{z}_m$, z_m and $\bar{f}_m$ are also defined in a similar way. We moreover set $\vec{v}_m(x,t) := \vec{v}_m^{\,n}(x)$ if $(n-1)k < t \leq nk$ for $n = 1, \dots, m$.
On the other hand, the a priori estimate (3.1.47) gives us

$$||u_m||_{W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)} \leq \text{ constant (independent of } m\text{, depending on } \varepsilon).$$

We notice that (3.1.47) also entails the following weaker information

$$||u_m||_{H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)} \leq \text{ constant (independent of } m \text{ and } \varepsilon),$$

that is this time we have a bound by means of a constant independent of both m and ε. This last information is what is actually enough for our purposes and in particular it entails that there exist u_ε and z_ε such that

$$u_m \to u_\varepsilon \qquad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

$$z_m \to z_\varepsilon \qquad \text{weakly in } L^2(Q).$$

At this point a standard procedure can be used in order to show that there exists also w_ε such that

$$w_m \to w_\varepsilon \qquad \text{weakly in } L^2(Q)$$

and that $w_\varepsilon = \overline{\mathcal{F}}(u_\varepsilon)$. Then the rest of the limit procedure can be done working as in the proof of Theorem 3.1.2. □

❒ PASSAGE TO THE LIMIT. Now we suppose that the parameter ε tends to 0. We have the following result

Theorem 3.1.13. *In the assumptions of Theorem* 3.1.11*, for any given* $\varepsilon > 0$*, let* $(u_\varepsilon, z_\varepsilon)$ *be a solution of Problem* 3.1.10*. Then there exists a function*

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

such that

$$u_\varepsilon \to u \qquad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T,V)$$

if $\varepsilon \to 0$. *Moreover* u *is a solution of Problem* 3.1.1.

Proof. For any $\varepsilon > 0$, Theorem 3.1.11 yields the existence of a solution $(u_\varepsilon, z_\varepsilon)$ such that in particular

$$\begin{aligned} ||u_\varepsilon||_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)} &\le \text{ constant (independent of } \varepsilon) \\ ||z_\varepsilon||_{L^2(Q)} &\le \text{ constant (independent of } \varepsilon). \end{aligned}$$

From this fact we deduce that there exist $u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$ and $z \in L^2(Q)$ such that, if $\varepsilon \to 0$ possibly along a subsequence

$$\begin{aligned} u_\varepsilon \to u &\qquad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V); \\ z_\varepsilon \to z &\qquad \text{weakly in } L^2(Q). \end{aligned}$$

On the other hand, using (3.1.33) and possibly extracting a subsequence, we have

$$u_m \to u \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega.$$

Using the strong continuity of the operator $\overline{\mathcal{F}}$ we deduce

$$\overline{\mathcal{F}}(u_m) \to \overline{\mathcal{F}}(u) \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega.$$

By (3.1.3), the sequence $\left\{ ||\overline{\mathcal{F}}(u_\varepsilon)(\cdot,t)||^2_{\mathcal{C}^0([0,T])} \right\}$ is equiintegrable in Ω (bounded by a constant independent of ε) since the same holds for u_ε. Hence $\overline{\mathcal{F}}(u_\varepsilon) \to \overline{\mathcal{F}}(u)$ strongly in $L^2(\Omega;\mathcal{C}^0([0,T]))$ and so in $L^2(Q)$.

At this point we want to pass to the limit in (3.1.43) as $\varepsilon \to 0$. We have

$$\left|\left| \varepsilon \frac{\partial z_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial \overline{\mathcal{F}}(u_\varepsilon)}{\partial t} \right|\right|_{L^2(0,T;V')} \le \text{ constant (independent of } \varepsilon).$$

Thus, as $\varepsilon \to 0$ possibly along a subsequence

$$\begin{aligned} &\varepsilon \frac{\partial z_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial \overline{\mathcal{F}}(u_\varepsilon)}{\partial t} \to \frac{\partial u}{\partial t} + \frac{\partial \overline{\mathcal{F}}(u)}{\partial t} &&\text{weakly star in } L^2(0,T;V'). \\ &C(u_\varepsilon) \to C(u) &&\text{weakly star in } L^2(0,T;V'). \end{aligned}$$

So if ε goes to zero, u and z satisfy

$$\begin{cases} \dfrac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) + \vec{v} \cdot \nabla(u + \overline{\mathcal{F}}(u)) + Au = f \\ \dfrac{\partial u}{\partial t} = z \end{cases}$$

and thus u is a solution of Problem 3.1.1. This finishes the proof. □

3.2. Second case: a nonlinear boundary condition

As we said in the introduction in this part of the chapter we deal with the same model equation

$$\frac{\partial}{\partial t}(u+\overline{\mathcal{F}}(u))+\vec{v}\cdot\nabla(u+\overline{\mathcal{F}}(u))-\triangle u=f \qquad \text{in } \Omega\times(0,T)$$

but instead of Dirichlet boundary conditions we consider this time a condition of nonlinear flux on a subset $\Gamma_2\subset\Gamma$, where Γ is the boundary of Ω; this condition can be written as

$$\nabla u\cdot\vec{\nu}=[\vec{v}\cdot\vec{\nu}]\,(u+\overline{\mathcal{F}}(u))-g(u) \qquad \text{on } \Gamma_2,$$

where $\vec{\nu}$ denotes the unit outer normal vector to Γ and $g:\mathbb{R}\to\mathbb{R}$ is a given function. On the other part of the boundary of Ω we consider Dirichlet boundary conditions. We look for assumptions on the function g in order to recover an existence and a uniqueness theorem for the Cauchy problem associated to the previous model equation.

3.2.1. Setting of the quantities and statement of the model problem

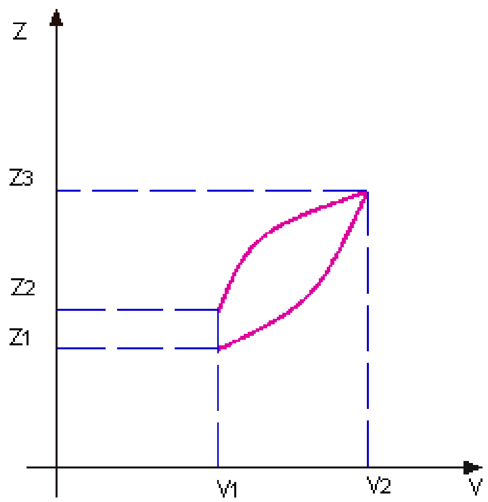

Figure 3.1: Example in which the hysteresis cycle is covered counterclockwisely.

Consider an open bounded set of Lipschitz class $\Omega\subset\mathbb{R}^N$, $N\geq 1$ with boundary Γ and set $Q:=\Omega\times(0,T)$. First of all let us assume that the operator $\overline{\mathcal{F}}:\mathcal{M}(\Omega;\mathcal{C}^0([0,T]))\to\mathcal{M}(\Omega;\mathcal{C}^0([0,T]))$ is causal, strongly continuous, affinely bounded, piecewise monotone and piecewise Lipschitz continuous according to (3.1.1), (3.1.2), (3.1.3), (3.1.4) and (3.1.5).

We assume moreover that $\overline{\mathcal{F}}$ fulfills the following two further properties. The first one is a condition on the hysteresis cycles and can be stated as follows

$$\forall\, v \in \mathcal{M}(\Omega; W^{1,1}(0,T)) \qquad \int_0^T [v(\cdot,t) - v(\cdot,0)]\, \frac{\partial}{\partial t}\overline{\mathcal{F}}(v)\, dt \,\geq\, 0 \qquad \text{a.e. in } \Omega. \tag{3.2.1}$$

Condition (3.2.1) is fulfilled if $\overline{\mathcal{F}} : v \mapsto z$ is a piecewise monotone hysteresis operator corresponding to counterclock cycles in the $(v, z)-$plane, as happens in Figure 3.1.
This property is fulfilled by generalized Prandtl-Ishlinskiĭ operators of play type and by Preisach operators corresponding to nonnegative measures; it is not satisfied by Prandtl-Ishlinskiĭ operators of stop type.

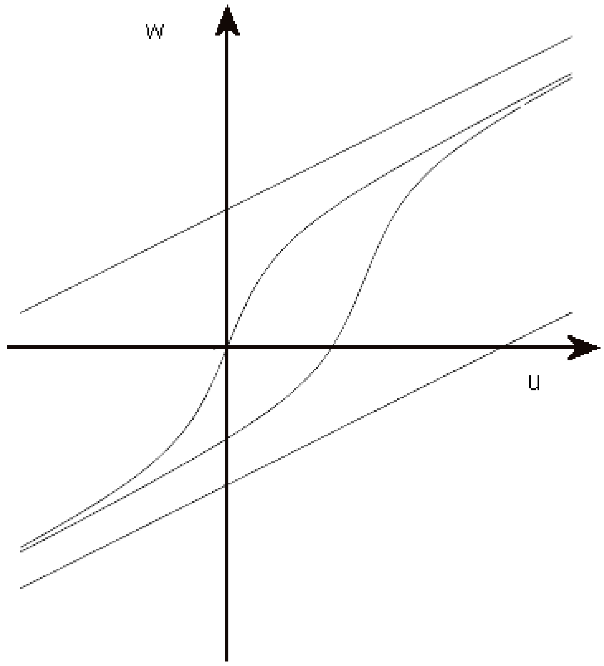

Figure 3.2: Example in which the hysteresis loops always stay in a strip.

The second property we require for $\overline{\mathcal{F}}$ is a geometrical condition on the hysteresis loops: we ask that the hysteresis cycles do not exit from a strip constituted by two parallel lines of some angular coefficient $\alpha > 0$, i.e. (see also Figure 3.2)

$$w := \overline{\mathcal{F}}(u) = \alpha\, u + \overline{\mathcal{J}}(u) \qquad \text{a.e. in } \Omega \times [0,T], \tag{3.2.2}$$

where $\overline{\mathcal{J}}$ is a memory operator fulfilling the same assumptions as $\overline{\mathcal{F}}$ and being in addition on operator of bounded range, i.e.

$$|\overline{\mathcal{J}}(u)(x,t)| \leq \beta \qquad \forall\, u \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])),\ \forall\,(x,t) \in \Omega \times [0,T]. \tag{3.2.3}$$

Finally assume that also (3.1.6) holds.
We consider the boundary Γ of the open set Ω to be divided in two parts Γ_1 and Γ_2 such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. If this condition does not hold, the local regularity of u holds only away from $\Sigma := \overline{\Gamma}_1 \cap \overline{\Gamma}_2$. We refer to [9] Chapter 3, for more details and examples on this topic.
First of all let $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ be the unique linear continuous trace operator such that

$$\gamma_0 v = v_{|\Gamma} \qquad \forall\, v \in \mathcal{C}^\infty(\bar{\Omega}) \cap H^1(\Omega). \tag{3.2.4}$$

We introduce the Hilbert space

$$V := \{v \in H^1(\Omega) : \gamma_0 v = 0 \ \text{ on } \Gamma_1\}$$

and we consider V equipped with the norm

$$||v||_V := ||v||_{H^1(\Omega)} := ||v||_{L^2(\Omega)} + ||\nabla v||_{L^2(\Omega)}. \tag{3.2.5}$$

Remark 3.2.1. In general, $||v||_V$ is not equivalent to the norm $|||v||| := ||\nabla v||_{L^2(\Omega)}$; however, when Γ_1 is such that

$$0 < |\Gamma_1| := (N-1) - \text{dimensional Hausdorff measure of } \Gamma_1$$

then Theorem A.4.1 holds and so on V we may put the equivalent norm $|||v|||$. Nevertheless we will deal with the general case and so we will consider V with the norm (3.2.5).

For the sake of simplicity we moreover set $H := L^2(\Omega)$; also in this case we deal with the Hilbert triplet $V \subset H \equiv H' \subset V'$, with dense and continuous injections. We introduce the linear and continuous operator $A : V \to V'$ as in (3.1.7); we assume that $u^0, w^0 = \mathcal{H}_{\overline{\mathcal{F}}}(u^0) \in L^2(\Omega)$ are given initial conditions; as we did in Chapter 3, we introduce a known function

$$\vec{v} : \Omega \times (0,T) \to \mathbb{R}^N \qquad\qquad \vec{v}(x,t) := (v_1(x,t), v_2(x,t), \ldots, v_N(x,t)).$$

satisfying (3.1.8). We want to solve the following problem.

Problem 3.2.2. *Let $\vec{v}$ be a given function satisfying* (3.1.8)*; consider two functions $g : \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ such that g and $f(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ have linear growth. We search for a function $u \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \cap L^2(0,T;V)$ such that $\gamma_0 u \in \mathcal{M}(\Gamma; \mathcal{C}^0([0,T])) \cap L^2(0,T;L^2(\Gamma_2))$ and for any $\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ with $\psi(\cdot,T) = 0$ a.e. in Ω*

$$\int_0^T \int_\Omega -(u + \overline{\mathcal{F}}(u)) \frac{\partial \psi}{\partial t}\, dx\, dt - \int_0^T \int_\Omega [\vec{v} \cdot \nabla \psi]\, (u + \overline{\mathcal{F}}(u))\, dx\, dt + \int_0^T \int_\Omega \nabla u \cdot \nabla \psi\, dx\, dt$$
$$+ \int_0^T \int_{\Gamma_2} g(\gamma_0 u)\, \gamma_0 \psi\, d\sigma\, dt = \int_\Omega [u^0(x) + w^0(x)]\, \psi(x,0)\, dx + \int_0^T \int_\Omega f(x,t,u(x,t))\, \psi(x,t)\, dx\, dt. \tag{3.2.6}$$

Remark 3.2.3. In order to make sense to the previous formula, it is certainly necessary that $g \circ h \in L^2(\Gamma_2)$ for any function $h \in L^2(\Gamma_2)$ and that for any $\ell \in L^2(Q)$, $f \circ \hat{\ell} \in L^2(Q)$, where $\hat{\ell} : \Omega \times [0,T] \to \Omega \times [0,T] \times \mathbb{R}$ is defined as $\hat{\ell}(x,t) = (x,t,\ell(x,t))$. A necessary and sufficient condition for this is that f and g have linear growth, that is what we assumed. The assumption contained in Theorem 3.2.4 that they are Lipschitz continuous makes this implication trivial.

Interpretation. If (2.1.13) holds, then we may use the standard Green formulae (for more details see e.g. [9], Chapter 2, see also Section A.5), the fact that $\nabla \cdot \vec{v} = 0$ a.e. in Q and the definition of derivatives in the sense of distributions (see Subsection A.1.3) to interpret the variational equation (3.2.6) as

$$\begin{cases} \dfrac{\partial w}{\partial t} + \vec{v} \cdot \nabla w - \triangle u = f(\cdot,\cdot,u(\cdot,\cdot)) \\ w = (I + \overline{\mathcal{F}})(u) \end{cases} \qquad \text{in } \mathcal{D}'(0,T;L^2(\Omega)) \tag{3.2.7}$$

with

$$\begin{cases} \gamma_0 u = 0 & \text{on } \Gamma_1 \times (0,T) \\ \gamma_A u = \gamma_0 \left[\vec{v} \cdot \vec{\nu}\right] (u + \overline{\mathcal{F}}(u)) - g(\gamma_0 u) & \text{on } \Gamma_2 \times (0,T), \end{cases} \tag{3.2.8}$$

where γ_A is the unique trace operator given by Theorem 2.27 contained in [9], Section 2.9. As by comparison

$$\frac{\partial}{\partial t}[u + \overline{\mathcal{F}}(u)] \in L^2(Q),$$

we may integrate by parts in time in (3.2.6) and get

$$[u + \overline{\mathcal{F}}(u)]_{|t=0} = u^0 + w^0 \ \text{ in } L^2(\Omega) \tag{3.2.9}$$

in the sense of the traces. In turn (3.2.7), (3.2.8) and (3.2.9) yield (3.2.6) and the two formulations are equivalent.

3.2.2. An existence result

Now we are ready to state and prove the following existence result.

Theorem 3.2.4. *(Existence)*
Let us assume that the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega;\mathcal{C}^0([0,T]))$ is causal, strongly continuous, affinely bounded, piecewise monotone and piecewise Lipschitz continuous according to (3.1.1), (3.1.2), (3.1.3), (3.1.4) *and* (3.1.5). *Assume moreover that $\overline{\mathcal{F}}$ fulfills* (3.2.1) *and* (3.2.2). *We suppose that the following assumptions hold for the function $g : \mathbb{R} \to \mathbb{R}$*

- g is a Lipschitz continuous, nondecreasing function with Lipschitz constant L_g;
- there exists $x_0 \in \mathbb{R} \setminus \{0\}$ such that $g(x_0) = 0$.

(3.2.10)

We also assume that the following holds for the function $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$

$$\begin{array}{l} \bullet \ \text{a.e. in } Q,\ f(x,t,\cdot) : \mathbb{R} \to \mathbb{R} \text{ is a Lipschitz continuous} \\ \quad \text{function with Lipschitz constant } L_f \\ \bullet \ f(\cdot,\cdot,0) \in L^2(Q) \\ \bullet \ \text{there exist two constants } C_1^f \text{ and } C_2^f \text{ such that, for any } v \in L^2(Q) \\ \quad \displaystyle\int_\Omega f(x,t,v(x,t))\,dx \geq \,-C_1^f\,||v||_{L^2(\Omega)} - \,C_2^f \qquad \text{a.e. in } (0,T). \end{array} \tag{3.2.11}$$

Finally let us consider the data

$$u^0 \in V, \;\; w^0 \in L^2(\Omega). \tag{3.2.12}$$

Then Problem 3.2.2 *admits at least one solution*

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

such that

$$\overline{\mathcal{F}}(u) \in H^1(0,T;L^2(\Omega)).$$

Remark 3.2.5. First of all the Lipschitz continuity of the function g easily implies the existence of two constants C_1^g and C_2^g possibly depending on L_g such that

$$|g(\eta)| \leq C_1^g|\eta| + C_2^g \qquad \forall\, \eta \in \mathbb{R}. \tag{3.2.13}$$

On the other hand, the Lipschitz continuity of the function f with respect to the third argument easily entails the existence of two constants C_3^f and C_4^f possibly depending on L_f such that

$$|f(x,t,\xi)| \leq C_3^f|\xi| + C_4^f \qquad \forall\, \xi \in \mathbb{R}, \;\text{ a.e. in } Q. \tag{3.2.14}$$

Moreover the fact that g is assumed to be a nondecreasing function, vanishing in some point x_0 entails the existence of two constants C_3^g and C_4^g such that for any $z \in L^2(\Gamma_2)$

$$\int_{\Gamma_2} g(z(\sigma))\, z(\sigma)\, d\sigma \geq \,-C_3^g\,||z||_{L^2(\Gamma_2)} - \,C_4^g. \tag{3.2.15}$$

This can be seen in the following way: for any $z \in L^2(\Gamma_2)$

$$(g(z(\sigma)) - g(x_0))\,(z(\sigma) - x_0) \geq \,0 \qquad \text{for a.a. } \sigma \in \Gamma_2.$$

Then

$$\int_{\Gamma_2} g(z(\sigma))z(\sigma)\,d\sigma \geq \int_{\Gamma_2} g(z(\sigma))x_0\,d\sigma \geq -\int_{\Gamma_2} |g(z(\sigma))||x_0|\,d\sigma \overset{(3.2.13)}{\geq} -\int_{\Gamma_2} (C_1^g|z(\sigma)| + C_2^g)\,|x_0|\,d\sigma$$

$$= -\int_{\Gamma_2} C_1^g|z(\sigma)|\,|x_0|\,d\sigma - \int_{\Gamma_2} C_2^g\,|x_0|\,d\sigma = -|x_0|\,C_1^g||z||_{L^2(\Gamma_2)} - C_2^g\,|x_0|\,|\Gamma_2|.$$

Here it is evident that x_0 must be different from 0, but this fact is not restrictive, in the sense that (3.2.15) can be also proved in a different way, without involving x_0. We assumed $x_0 \neq 0$ for simplicity.
Also in the case of f we notice that the coercivity property required in (3.2.11) could be easily deduced from the assumption $f(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ being nondecreasing.

Proof.

① *First step: approximation.*

Let us fix $m \in \mathbb{N}$, set $k := T/m$. As at the end of the proof we will let m go to infinity, it is not restrictive to assume

$$k < \frac{1}{8||\vec{v}||^2_{L^\infty(Q)^N}\,(1+L_{\mathcal{F}})^2}. \tag{3.2.16}$$

For any $n = 1, \dots, m$ let us consider $u^0_m := u^0$, $w^0_m := w^0$ and $f^n_m(x,\xi) = f(x, nk, \xi)$, a.e. in Ω, for any $\xi \in \mathbb{R}$. Also this time, the right tool we need is the approximation of our problem by an implicit time discretization scheme.

❒ Statement of the problem. We want to solve the following problem.

Problem 3.2.6. *To find $u^n_m \in V$ for $n = 1, \dots m$, such that, if $u_m(x,\cdot)$ is the linear time interpolate of $u_m(x,nk) := u^n_m(x)$, for $n = 1, \dots, m$, a.e. in Ω and $w^n_m := [\overline{\mathcal{F}}(u_m)](x,nk)$ for $n = 1, \dots, m$, a.e. in Ω, then for all $\psi \in V$*

$$\begin{aligned} &\frac{1}{k}\int_\Omega (u^n_m - u^{n-1}_m)\,\psi\,dx + \frac{1}{k}\int_\Omega (w^n_m - w^{n-1}_m)\,\psi\,dx - \int_\Omega [\vec{v}^{\,n}_m \cdot \nabla\psi\,]\,(u^n_m + w^n_m)\,dx \\ &+ \int_\Omega \nabla u^n_m \cdot \nabla\psi\,dx + \int_{\Gamma_2} g(\gamma_0 u^n_m)\,\gamma_0\psi\,d\sigma = \int_\Omega f^n_m(x, u^n_m(x))\,\psi(x)\,dx, \end{aligned} \tag{3.2.17}$$

where $\vec{v}^{\,n}_m(x) := \vec{v}(x, nk)$.

First of all we notice that $u^0_m := u^0$ is known; then, working by induction, for any $n \in \{1, \dots, m\}$ we suppose to know $u^1_m, \dots, u^{n-1}_m \in V$; we therefore want to determine u^n_m.

❒ Introduction of some auxiliary operators. The main idea in solving Problem 3.2.6 is to consider (3.2.17) as an abstract equation of the form

$$Z(u^n_m) = \Lambda$$

with a suitable operator $Z : V \to V'$ and a given right-hand side Λ, which will be constructed as follows.
❖ *The operator F.* We first introduce an operator F as a counterpart of the operator $\widehat{F}^n_m$ we had in the proof of Theorem 3.1.2. In particular also $F : L^2(\Omega) \to L^2(\Omega)$ is a strongly continuous operator fulfilling (3.1.17) and (3.1.18).

❖ *The operator C.* We introduce the operator $C : V \to V'$ acting in the following way

$$ {}_{V'}\langle C(\Phi), \psi \rangle_V := - \int_\Omega \vec{v}_m^{\,n} \, (\Phi + F(\Phi)) \cdot \nabla \psi \, dx \quad \forall \, \psi, \Phi \in V. \tag{3.2.18} $$

❖ *The operator G.* We also introduce the operator $G : V \to V'$ acting in the following way

$$ {}_{V'}\langle G(\Phi), \psi \rangle_V := \int_{\Gamma_2} g(\gamma_0(\Phi)(\sigma)) \, \gamma_0 \psi(\sigma) \, d\sigma \quad \forall \, \psi, \Phi \in V. $$

❖ *The operator K.* Finally we introduce the operator $K : V \to V'$ acting in the following way

$$ {}_{V'}\langle K(\Phi), \psi \rangle_V := \int_\Omega f_m^n(x, \Phi(x)) \, \psi(x) \, dx \quad \forall \, \psi, \Phi \in V. $$

❒ HOW TO GET A SOLUTION OF PROBLEM 3.2.6. At this point, we may rewrite equation (3.2.17) using the operators we just introduced. We obtain

$$ \frac{u_m^n - u_m^{n-1}}{k} + \frac{w_m^n - w_m^{n-1}}{k} + C(u_m^n) + A(u_m^n) + G(u_m^n) = K(u_m^n) \qquad \text{in } V'. \tag{3.2.19} $$

Now, if we separate the terms which are known from the ones we have to determine, the previous equation is equivalent to the following one

$$ u_m^n + F(u_m^n) + k \, C(u_m^n) + k \, A(u_m^n) + k \, G(u_m^n) = \, k \, K(u_m^n) + \Lambda \qquad \text{in } \, V' $$

where $\Lambda := w_m^{n-1} + u_m^{n-1}$, so it is a known function. For the sake of simplicity we omit the fixed indexes m and n (so in the following we will also write $\vec{v}$ in place of $\vec{v}_m^{\,n}$); thus we get

$$ u + F(u) + k \, C(u) + k \, A(u) + k \, G(u) = \, k \, K(u) + \Lambda \qquad \text{in } \, V'. \tag{3.2.20} $$

We claim that (3.2.20) admits at least one solution $u \in V$. The procedure we use is based on Theorem A.9.1.

Let $\{V_j\}_{j \in \mathbb{N}}$ be a sequence of finite dimensional subspaces invading V; for any $j \in \mathbb{N}$ let us consider the problem of finding $u_j \in V_j$ such that

$$ u_j + F(u_j) + k \, A(u_j) + \, k \, C(u_j) + \, k \, G(u_j) = \, k \, K(u_j) + \Lambda \qquad \text{in } V_j'. \tag{3.2.21} $$

We have to prove that the operator $Z : V \to V'$ defined as

$$ Z(u) := u + F(u) + k \, A(u) + k \, C(u) + k \, G(u) - k \, K(u) $$

is strongly continuous and coercive.

❖ *Strong continuity.* We recall that for brevity we set $L^2(\Omega) =: H$. We verify immediately that the operator $\bar{Z} : V \to V'$ defined as $\bar{Z}(u) = u + F(u) + \, k \, A(u)$ is strongly continuous. The same happens for the operator $\tilde{Z} : V \to V'$ which is defined as $\tilde{Z}(u) = k\,C(u) + k\,G(u) - k\,K(u)$. In fact, let $u_n \to u$ in V; we claim that $C(u_n) + G(u_n) - K(u_n) - C(u) - G(u) + K(u) \to 0$ in V'. Using the Lipschitz continuity of the functions g and f, we rapidly deduce

$$
\begin{aligned}
& ||C(u_n) + G(u_n) - K(u_n) - C(u) - G(u) + K(u)||_{V'} \\
= & \sup_{||\psi||_V = 1} {}_{V'}\langle C(u_n) - C(u) + G(u_n) - G(u) - K(u_n) + K(u), \psi \rangle_V \\
\le & \sup_{||\psi||_V = 1} \Bigg[- \int_\Omega \vec{v}\,(u_n - u + F(u_n) - F(u)) \cdot \nabla\psi \, dx + {}_{V'}\langle G(u_n) - G(u), \psi \rangle_V \\
& + {}_{V'}\langle K(u) - K(u_n), \psi \rangle_V \Bigg] \le \sup_{||\psi||_V = 1} \Bigg[||\vec{v}||_{L^\infty(Q)^N} \, (||u_n - u||_H + ||F(u_n) - F(u)||_H) \\
& \times ||\nabla\psi||_H \Bigg] + \sup_{||\psi||_V = 1} \Bigg[\bigg| \int_{\Gamma_2} (g(\gamma_0 u_n(\sigma)) - g(\gamma_0 u(\sigma)))\, \gamma_0\psi(\sigma)\, d\sigma \bigg| \\
& + \bigg| \int_\Omega (f(x, u(x)) - f(x, u_n(x)))\, \psi(x)\, dx \bigg| \Bigg] \le ||\vec{v}||_{L^\infty(Q)^N} (||u_n - u||_H + ||F(u_n) - F(u)||_H) \\
& + \sup_{||\psi||_V = 1} L_g\, ||\gamma_0 u_n - \gamma_0 u||_{L^2(\Gamma_2)}\, ||\gamma_0\psi||_{L^2(\Gamma_2)} + \sup_{||\psi||_V = 1} L_f\, ||u_n - u||_H\, ||\psi||_H \le ||\vec{v}||_{L^\infty(Q)^N} \\
& \times \Big(||u_n - u||_H + ||F(u_n) - F(u)||_H \Big) + L_g\, ||\gamma_0 u_n - \gamma_0 u||_{L^2(\Gamma_2)} + L_f\, ||u_n - u||_H.
\end{aligned}
$$

Remark 3.2.7. Notice that in the previous step, for the sake of simplicity, we omitted the fixed indexes m and n; so in the sixth line of the previous chain of inequalities f indicates actually the time-discretized f_m^n and not the original f which depends also on time.

Now, the first and the fourth term of the last sum obviously go to zero; the second one vanishes thanks to the strong continuity of the operator F; finally the continuity of the trace operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ together with the inclusion $H^{1/2}(\Gamma) \subset L^2(\Gamma)$ yield

$$
\begin{aligned}
u_n \to u \quad \text{in } V \quad &\Rightarrow \quad \gamma_0 u_n \to \gamma_0 u \quad \text{in } H^{1/2}(\Gamma) \\
\Rightarrow \quad \gamma_0 u_n \to \gamma_0 u \quad \text{in } L^2(\Gamma) \quad &\Rightarrow \quad \gamma_0 u_n \to \gamma_0 u \quad \text{in } L^2(\Gamma_2)
\end{aligned}
$$

(the last passage because $u_n, u \in V$ and so $\gamma_0 u_n = \gamma_0 u = 0$ on Γ_1). This fact allows us to conclude that also the third term in the last sum vanishes.

❖ *Coercivity.* Now the aim is to prove that Z is also coercive, that is

$$
\frac{1}{||u||_V}\, {}_{V'}\langle Z(u), u \rangle_V \to \infty \qquad \text{as } ||u||_V \to \infty.
$$

Recalling that we set $L^2(\Omega) =: H$, we have

$$
\begin{aligned}
&{}_{V'}\langle Z(u), u\rangle_V = ||u||^2_H + \int_\Omega F(u)\, u\, dx + k\, ||\nabla u||^2_H + k\, {}_{V'}\langle C(u), u\rangle_V + k\, {}_{V'}\langle G(u), u\rangle_V \\
&- k\, {}_{V'}\langle K(u), u\rangle_V \overset{(3.1.18)}{\geq} ||u||^2_H - C^{\mathcal{F}}_3 ||u||_H - C^{\mathcal{F}}_4 + k\, ||\nabla u||^2_H - k \int_\Omega \vec{v}\, (u + F(u)) \cdot \nabla u\, dx \\
&+ k \int_{\Gamma_2} g(\gamma_0 u(\sigma))\, \gamma_0 u(\sigma)\, d\sigma - k \int_\Omega f(x, u(x))\, u(x)\, dx \overset{(3.2.15)\ (3.2.11)}{\geq} ||u||^2_H + k\, ||\nabla u||^2_H \\
&- C^{\mathcal{F}}_3 ||u||_H - C^{\mathcal{F}}_4 - k\, ||\vec{v}||_{L^\infty(Q)^N} \int_\Omega |u + F(u)|\, |\nabla u|\, dx - C^g_3 ||\gamma_0 u||_{L^2(\Gamma_2)} - C^g_4 - C^f_1\, ||u||_H \\
&- C^f_2 \geq ||u||^2_H + k\, ||\nabla u||^2_H - (C^{\mathcal{F}}_3 + C^f_1)\, ||u||_H - (C^{\mathcal{F}}_4 + C^g_4 + C^f_2) - C^g_3\, ||u||_V \\
&- k\, ||\vec{v}||_{L^\infty(Q)^N} \left(||u||_H + ||F(u)||_H\right) ||\nabla u||_H \overset{(3.1.17)}{\geq} ||u||^2_H + k\, ||\nabla u||^2_H - (C^{\mathcal{F}}_3 + C^f_1)^2 \\
&- \frac{1}{4}\, ||u||^2_H - (C^{\mathcal{F}}_4 + C^g_4 + C^f_2) - k\, ||\vec{v}||_{L^\infty(Q)^N} [(1 + C^{\mathcal{F}}_1) ||u||_H + C^{\mathcal{F}}_2]\, ||\nabla u||_H - \frac{(C^g_3)^2}{k} \\
&- \frac{k}{4}\, ||u||^2_V \geq \frac{3}{4}\, ||u||^2_H + k\, ||\nabla u||^2_H - k\, ||\vec{v}||^2_{L^\infty(Q)^N} \left[(1 + C^{\mathcal{F}}_1) ||u||_H + C^{\mathcal{F}}_2\right]^2 - \frac{k}{4}\, ||\nabla u||^2_H \\
&- \frac{k}{4}\, ||u||^2_V - (C^{\mathcal{F}}_3 + C^f_1)^2 - (C^{\mathcal{F}}_4 + C^g_4 + C^f_2) - \frac{(C^g_3)^2}{k} \overset{(3.2.16)}{\geq} \frac{k}{4} ||u||^2_V + \tilde{c}
\end{aligned}
$$

where

$$\tilde{c} := k\, ||\vec{v}||^2_{L^\infty(Q)^N}\, 2\, (C^{\mathcal{F}}_2)^2 + (C^{\mathcal{F}}_3 + C^f_1)^2 + C^{\mathcal{F}}_4 + C^g_4 + C^f_2 + \frac{(C^g_3)^2}{k}.$$

This immediately yields the coercivity of the operator Z. Also in this case we refer to Remark 3.2.7 concerning the third line of the previous chain of inequalities.

Now, using Theorem A.9.1, we get the existence of at least a solution u_j of (3.2.21). If we multiply (3.2.21) by u_j and use the coercivity of the operator Z we get that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is uniformly bounded in V. Thus there exists u such that, possibly extracting a subsequence, $u_j \rightharpoonup u$ in V. By the compactness of the inclusion $V \subset L^2(\Omega)$ and by the strong continuity of the operator Z we may pass to the limit taking $j \to \infty$ in (3.2.21) getting (3.2.20).

② *Second step: a priori estimates.*

At a first sight one may think that the a priori estimates can be carried on in a similar way as we did in the first part of the chapter. The problem is that in Subsection 3.1.2 we used the Poincaré inequality in order to apply the discrete Gronwall inequality and this tool cannot be used here anymore, as we are not dealing with Dirichlet boundary conditions. What we do instead is to divide this step in two parts. In the first part we multiply equation (3.2.19) by $k\,u_m^n$ in the duality pairing ${}_{V'}\langle\cdot,\cdot\rangle_V$ and sum for $n=1,\dots,j$, for $j\in\{1,\dots,m\}$. In this manner we are able to control the term $\displaystyle\int_\Omega |u_m^j|^2\,dx$ by means of the following quantity

$$\tilde c_1\,k\sum_{n=1}^{j}\left|\left|\frac{u_m^n-u_m^{n-1}}{k}\right|\right|^2_{L^2(\Omega)}\;+\;\tilde c_2\,k\sum_{n=1}^{j}||u_m^n||^2_{L^2(\Omega)}\;+\;\tilde c_3\,k\sum_{n=1}^{j}||\nabla u_m^n||^2_{L^2(\Omega)}\;+\;\tilde c_4,$$

where the constants $\tilde c_1,\tilde c_2,\tilde c_3$ only depend on $\vec v,\alpha,L_f,L_{\mathcal F},T$ while $\tilde c_4$ also depends on the data u_m^0, w_m^0. In particular $\tilde c_1$ can be taken small at will, so that the term $\displaystyle\tilde c_1\,k\sum_{n=1}^{j}\left|\left|\frac{u_m^n-u_m^{n-1}}{k}\right|\right|^2_{L^2(\Omega)}$ will be absorbed when developing the second part of the computations. The other terms will be controlled at the end by means of a discrete version of the Gronwall lemma.
In the second part instead we multiply equation (3.2.19) by $u_m^n-u_m^{n-1}$ in the duality pairing ${}_{V'}\langle\cdot,\cdot\rangle_V$ and sum for $n=1,\dots,j$, for $j\in\{1,\dots,m\}$; this will be enough to conclude this step.

➜ first part: test by $k\,u_m^n$. As we said in the preamble, we multiply equation (3.2.19) by $k\,u_m^n$ in the duality pairing ${}_{V'}\langle\cdot,\cdot\rangle_V$ and sum for $n=1,\dots,j$, for $j\in\{1,\dots,m\}$, getting

$$\begin{aligned}
&\sum_{n=1}^{j}{}_{V'}\langle\frac{u_m^n-u_m^{n-1}}{k},k\,u_m^n\rangle_V\;+\;\sum_{n=1}^{j}{}_{V'}\langle\frac{w_m^n-w_m^{n-1}}{k},k\,u_m^n\rangle_V\;+\;\sum_{n=1}^{j}{}_{V'}\langle C(u_m^n)+A(u_m^n),k\,u_m^n\rangle_V\\
&+\sum_{n=1}^{j}{}_{V'}\langle G(u_m^n),k\,u_m^n\rangle_V\geq\frac12\sum_{n=1}^{j}\int_\Omega(|u_m^n|^2-|u_m^{n-1}|^2)\,dx+k\sum_{n=1}^{j}\int_\Omega\left(\frac{w_m^n-w_m^{n-1}}{k}\right)u_m^0\,dx\\
&+k\sum_{n=1}^{j}\int_\Omega\left(\frac{w_m^n-w_m^{n-1}}{k}\right)(u_m^n-u_m^0)\,dx\;-k\sum_{n=1}^{j}\int_\Omega[\vec v_m^{\,n}\cdot\nabla u_m^n]\,(u_m^n+w_m^n)\,dx\\
&+k\sum_{n=1}^{j}||\nabla u_m^n||^2_{L^2(\Omega)}+k\sum_{n=1}^{j}\int_{\Gamma_2}g(\gamma_0u_m^n(\sigma))\,\gamma_0u_m^n(\sigma)\,d\sigma\overset{(3.2.1)\,(3.2.15)}{\geq}\frac12\int_\Omega(|u_m^j|^2-|u_m^0|^2)\,dx\\
&+k\sum_{n=1}^{j}\int_\Omega\left(\frac{w_m^n-w_m^{n-1}}{k}\right)u_m^0\,dx-k\sum_{n=1}^{j}\int_\Omega[\vec v_m^{\,n}\cdot\nabla u_m^n]\,(u_m^n+w_m^n)\,dx\\
&+k\sum_{n=1}^{j}||\nabla u_m^n||^2_{L^2(\Omega)}\;-\;k\,C_3^g\sum_{n=1}^{j}||\gamma_0u_m^n||_{L^2(\Gamma_2)}\;-\;C_4^g\,T.
\end{aligned}$$

We also have

$$\sum_{n=1}^{j}{}_{V'}\langle\frac{u_m^n-u_m^{n-1}}{k},k\,u_m^n\rangle_V\;+\;\sum_{n=1}^{j}{}_{V'}\langle\frac{w_m^n-w_m^{n-1}}{k},k\,u_m^n\rangle_V\;+\;\sum_{n=1}^{j}{}_{V'}\langle C(u_m^n)+A(u_m^n),k\,u_m^n\rangle_V$$

$$+\sum_{n=1}^{j} {}_{V'}\langle G(u_m^n), k\, u_m^n\rangle_V = \sum_{n=1}^{j} {}_{V'}\langle K(u_m^n), k\, u_m^n\rangle_V \le k \sum_{n=1}^{j} \int_\Omega |f_m^n(x, u_m^n(x))|\,|u_m^n(x)|\,dx$$

$$\stackrel{(3.2.14)}{\le} k \sum_{n=1}^{j} \int_\Omega (C_3^f |u_m^n| + C_4^f)\,|u_m^n|\,dx \le k\,(C_3^f + 1) \sum_{n=1}^{j} ||u_m^n||^2_{L^2(\Omega)} + (C_4^f)^2.$$

On the other hand we notice that

$$\begin{aligned} |w_m^j(x)| - |w_m^0(x)| &= \sum_{n=1}^{j} (|w_m^n(x)| - |w_m^{n-1}(x)|) \le k \sum_{n=1}^{j} \left| \frac{w_m^n(x) - w_m^{n-1}(x)}{k} \right| \\ &\le \sqrt{T} \left(k \sum_{n=1}^{j} \left| \frac{w_m^n(x) - w_m^{n-1}(x)}{k} \right|^2 \right)^{1/2} \end{aligned} \tag{3.2.22}$$

thus

$$\begin{aligned} ||w_m^j||^2_{L^2(\Omega)} &\le 2\,||w_m^0||^2_{L^2(\Omega)} + 2\,T\,k \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|^2_{L^2(\Omega)} \\ &\stackrel{(3.1.5)}{\le} 2\,||w_m^0||^2_{L^2(\Omega)} + 2\,T\,L^2_{\mathcal{F}}\,k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|^2_{L^2(\Omega)}. \end{aligned} \tag{3.2.23}$$

We denote by $\tilde c_9 = 2\,||\vec v||^2_{L^\infty(Q)^N}\,(1+\alpha) + 1/4$, where α is introduced in (3.2.2). This constant $\tilde c_9$ will come out in the second part of the computations. Therefore we have

$$\frac{1}{2} \int_\Omega |u_m^j|^2\,dx + k \sum_{n=1}^{j} ||\nabla u_m^n||^2_{L^2(\Omega)} \le \frac{1}{2}\,||u_m^0||^2_{L^2(\Omega)} + k \sum_{n=1}^{j} \int_\Omega \left| \frac{w_m^n - w_m^{n-1}}{k} \right| |u_m^0|\,dx$$

$$+ k\,||\vec v||_{L^\infty(Q)^N} \sum_{n=1}^{j} \int_\Omega |u_m^n|\,|\nabla u_m^n|\,dx + k\,||\vec v||_{L^\infty(Q)^N} \sum_{n=1}^{j} \int_\Omega |w_m^n|\,|\nabla u_m^n|\,dx + \frac{k}{2} \sum_{n=1}^{j} ||u_m^n||^2_V$$

$$+ k\,(C_3^g)^2 + C_4^g\,T + k\,(C_3^f + 1) \sum_{n=1}^{j} ||u_m^n||^2_{L^2(\Omega)} + (C_4^f)^2 \le \frac{1}{2}\,||u_m^0||^2_{L^2(\Omega)}(1 + 16\,T\,L^2_{\mathcal{F}}\,\tilde c_9)$$

$$+ \frac{k}{32\,L^2_{\mathcal{F}}\,\tilde c_9} \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|^2_{L^2(\Omega)} + k\,(||\vec v||_{L^\infty(Q)^N} + 2 + C_3^f) \sum_{n=1}^{j} ||u_m^n||^2_{L^2(\Omega)}$$

$$+ k\,(||\vec v||_{L^\infty(Q)^N} + 16\,T\,L^2_{\mathcal{F}}\,\tilde c_9\,||\vec v||^2_{L^\infty(Q)^N} + 1) \sum_{n=1}^{j} ||\nabla u_m^n||^2_{L^2(\Omega)} + \frac{k}{64\,T\,L^2_{\mathcal{F}}\,\tilde c_9} \sum_{n=1}^{j} ||w_m^n||^2_{L^2(\Omega)}$$

$$+ k\,(C_3^g)^2 + C_4^g\,T + (C_4^f)^2$$

and this in particular yields (using (3.2.23) and (3.1.5))

$$\int_\Omega |u_m^j|^2\,dx \;\le\; \tilde c_1\,k\sum_{n=1}^{j}\left|\left|\frac{u_m^n-u_m^{n-1}}{k}\right|\right|^2_{L^2(\Omega)} \;+\;\tilde c_2\,k\sum_{n=1}^{j}||u_m^n||^2_{L^2(\Omega)} \;+\;\tilde c_3\,k\sum_{n=1}^{j}||\nabla u_m^n||^2_{L^2(\Omega)}\;+\;\tilde c_4 \tag{3.2.24}$$

where

$$\tilde c_1 := \frac{1}{8\,\tilde c_9} \qquad \tilde c_2 := 2(||\vec v||_{L^\infty(Q)^N}+2+C_3^f) \qquad \tilde c_3 := 2(||\vec v||_{L^\infty(Q)^N}(1+16\,T\,L_{\mathcal F}^2\,\tilde c_9))+2$$

$$\tilde c_4 := \left[||u_m^0||^2_{L^2(\Omega)}(1+16\,T\,L_{\mathcal F}^2\,\tilde c_9)+\frac{k}{16\,L_{\mathcal F}^2\,\tilde c_9\,T}\,||w_m^0||^2_{L^2(\Omega)}\;+\;2\,k\,(C_3^g)^2\;+\;2\,C_4^g\,T\;+\;2\,(C_4^f)^2\right].$$

➜ SECOND PART: TEST BY $u_m^n - u_m^{n-1}$. As announced, we multiply equation (3.2.19) by $u_m^n - u_m^{n-1}$ in the duality pairing ${}_{V'}\langle\cdot,\cdot\rangle_V$ and sum for $n=1,\dots,j$, for $j\in\{1,\dots,m\}$; recalling that for brevity we set $H := L^2(\Omega)$, we get

$$\begin{aligned}
&\sum_{n=1}^{j}{}_{V'}\langle\frac{u_m^n-u_m^{n-1}}{k},u_m^n-u_m^{n-1}\rangle_V+\sum_{n=1}^{j}{}_{V'}\langle\frac{w_m^n-w_m^{n-1}}{k},u_m^n-u_m^{n-1}\rangle_V\\
&+\sum_{n=1}^{j}{}_{V'}\langle C(u_m^n),u_m^n-u_m^{n-1}\rangle_V+\sum_{n=1}^{j}{}_{V'}\langle A(u_m^n),u_m^n-u_m^{n-1}\rangle_V\\
&+\sum_{n=1}^{j}{}_{V'}\langle G(u_m^n),u_m^n-u_m^{n-1}\rangle_V\;\ge\;k\sum_{n=1}^{j}\left|\left|\frac{u_m^n-u_m^{n-1}}{k}\right|\right|^2_H+k\sum_{n=1}^{j}{}_{V'}\langle\frac{w_m^n-w_m^{n-1}}{k},\frac{u_m^n-u_m^{n-1}}{k}\rangle_V\\
&-\sum_{n=1}^{j}\int_\Omega[\vec v_m^{\,n}(u_m^n+w_m^n)\cdot\nabla(u_m^n-u_m^{n-1})]\,dx+\frac12\sum_{n=1}^{j}\int_\Omega(|\nabla u_m^n|^2-|\nabla u_m^{n-1}|^2)\,dx\\
&+\sum_{n=1}^{j}{}_{V'}\langle G(u_m^n),u_m^n-u_m^{n-1}\rangle_V\overset{(3.1.4)}{\ge}k\sum_{n=1}^{j}\left|\left|\frac{u_m^n-u_m^{n-1}}{k}\right|\right|^2_H-\int_\Omega\vec v_m^{\,j}(u_m^j+w_m^j)\cdot\nabla u_m^j\,dx\\
&+\int_\Omega\vec v_m^{\,0}(u_m^0+w_m^0)\cdot\nabla u_m^0\,dx+\sum_{n=1}^{j}\int_\Omega[(\vec v_m^{\,n}-\vec v_m^{\,n-1})(u_m^n+w_m^n)]\cdot\nabla u_m^n\,dx\\
&+\;k\sum_{n=1}^{j}\int_\Omega\left(\frac{u_m^n-u_m^{n-1}}{k}\right)[\vec v_m^{\,n}\cdot\nabla u_m^n]\,dx+k\sum_{n=1}^{j}\int_\Omega\left(\frac{w_m^n-w_m^{n-1}}{k}\right)[\vec v_m^{\,n}\cdot\nabla u_m^n]\,dx\\
&+\frac12\int_\Omega(|\nabla u_m^j|^2-|\nabla u_m^0|^2)\,dx+\sum_{n=1}^{j}\int_{\Gamma_2}g(\gamma_0u_m^n(\sigma))\,(\gamma_0u_m^n(\sigma)-\gamma_0u_m^{n-1}(\sigma))\,d\sigma,
\end{aligned}$$

where $\vec{v}_m^{\,n} - \vec{v}_m^{\,n-1} = (v_{1m}^n - v_{1m}^{n-1}, v_{2m}^n - v_{2m}^{n-1}, \dots, v_{Nm}^n - v_{Nm}^{n-1})$. We also have

$$\sum_{n=1}^{j} {}_{V'}\langle \frac{u_m^n - u_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j} {}_{V'}\langle \frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V$$

$$+ \sum_{n=1}^{j} {}_{V'}\langle C(u_m^n), u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j} {}_{V'}\langle A(u_m^n), u_m^n - u_m^{n-1}\rangle_V$$

$$+ \sum_{n=1}^{j} {}_{V'}\langle G(u_m^n), u_m^n - u_m^{n-1}\rangle_V = \sum_{n=1}^{j} {}_{V'}\langle K(u_m^n), u_m^n - u_m^{n-1}\rangle_V$$

$$\le k \sum_{n=1}^{j} \int_\Omega |f_m^n(x, u_m^n(x))| \left| \frac{u_m^n(x) - u_m^{n-1}(x)}{k} \right| dx \overset{(3.2.14)}{\le} k \sum_{n=1}^{j} \int_\Omega (C_3^f |u_m^n| + C_4^f) \left| \frac{u_m^n - u_m^{n-1}}{k} \right| dx$$

$$\le 2\,(C_3^f)^2\, k \sum_{n=1}^{j} ||u_m^n||_H^2 + 2\,(C_4^f)^2\, T + \frac{k}{4} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 .$$

Now, let us denote with $\tilde{G} : \mathbb{R} \to \mathbb{R}$ a primitive function of g, i.e. such that $\tilde{G}'(z) = g(z)$ for all $z \in \mathbb{R}$. From the assumptions taken on g it is clear that $\tilde{G}$ is first decreasing then increasing and so it attains its minimum in $\tilde{G}(x_0)$. Thus

$$\sum_{n=1}^{j} \int_{\Gamma_2} g(\gamma_0 u_m^n(\sigma))\,(\gamma_0 u_m^n(\sigma) - \gamma_0 u_m^{n-1}(\sigma))\, d\sigma = \int_{\Gamma_2} [\tilde{G}(\gamma_0 u_m^j(\sigma)) - \tilde{G}(\gamma_0 u_m^0(\sigma))]\, d\sigma$$

$$= \int_{\Gamma_2} \tilde{G}(\gamma_0 u_m^j(\sigma))\, d\sigma - \int_{\Gamma_2} \tilde{G}(\gamma_0 u_m^0(\sigma))\, d\sigma \ge \tilde{G}(x_0)|\Gamma_2| - \int_{\Gamma_2} \tilde{G}(\gamma_0 u_m^0(\sigma))\, d\sigma.$$

On the other hand

$$\int_\Omega \left\{ (u_m^j + w_m^j)[\vec{v}_m^{\,j} \cdot \nabla u_m^j] - (u_m^0 + w_m^0)[\vec{v}_m^{\,0} \cdot \nabla u_m^0] \right\} dx - k \sum_{n=1}^{j} \int_\Omega [\vec{v}_m^{\,n} \cdot \nabla u_m^n] \left(\frac{u_m^n - u_m^{n-1}}{k} \right) dx$$

$$- \, k \sum_{n=1}^{j} \int_\Omega [\vec{v}_m^{\,n} \cdot \nabla u_m^n] \left(\frac{w_m^n - w_m^{n-1}}{k} \right) dx - \sum_{n=1}^{j} \int_\Omega [(\vec{v}_m^{\,n} - \vec{v}_m^{\,n-1}) \cdot \nabla u_m^n]\,(u_m^n + w_m^n)\, dx$$

$$\le \; ||\vec{v}||_{L^\infty(Q)^N} \int_\Omega (|u_m^j| + |w_m^j|)\,|\nabla u_m^j|\, dx \; + \; ||\vec{v}||_{L^\infty(Q)^N} \int_\Omega (|u_m^0| + |w_m^0|)\,|\nabla u_m^0|\, dx$$

$$+ \; ||\vec{v}||_{L^\infty(Q)^N}\, k \sum_{n=1}^{j} \int_\Omega |\nabla u_m^n| \left(\left| \frac{u_m^n - u_m^{n-1}}{k} \right| + \left| \frac{w_m^n - w_m^{n-1}}{k} \right| \right) dx + \left|\left| \frac{\partial \vec{v}}{\partial t} \right|\right|_{L^\infty(Q)^N}$$

$$\times\, k \int_\Omega (|u_m^n| + |w_m^n|)\,|\nabla u_m^n|\, dx \overset{(3.2.2)}{\le} \frac{1}{4} \int_\Omega |\nabla u_m^j|^2\, dx + 2\,||\vec{v}||_{L^\infty(Q)^N}^2\, (1 + 2\alpha^2) \int_\Omega |u_m^j|^2\, dx$$

$$+\ 4\,||\vec{v}||^2_{L^\infty(Q)^N}\,|\Omega|\,\beta^2\ +\ ||\vec{v}||_{L^\infty(Q)^N}\,||\nabla u^0_m||^2_H\ +\ \frac{||\vec{v}||_{L^\infty(Q)^N}}{2}\,(||u^0_m||^2_H+||w^0_m||^2_H)$$

$$+\ \frac{k}{8}\sum_{n=1}^{j}\left|\left|\frac{u^n_m-u^{n-1}_m}{k}\right|\right|^2_H\ +\ \frac{k}{8\,L^2_{\mathcal{F}}}\sum_{n=1}^{j}\left|\left|\frac{w^n_m-w^{n-1}_m}{k}\right|\right|^2_H\ +\ \frac{k}{4}\sum_{n=1}^{j}||u^n_m||^2_H$$

$$+\ k\left(||\vec{v}||^2_{L^\infty(Q)^N}\,2\,(L^2_{\mathcal{F}}+1)+\left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|^2_{L^\infty(Q)^N}\,(1+4\,L^2_{\mathcal{F}}\,T)\right)\sum_{n=1}^{j}||\nabla u^n_m||^2_H$$

$$\overset{(3.2.23)}{+}\ \frac{k}{8\,L^2_{\mathcal{F}}\,T}\,||w^0_m||^2_H\ \overset{(3.2.23)}{+}\ \frac{k}{8\,L^2_{\mathcal{F}}}\sum_{n=1}^{j}\left|\left|\frac{w^n_m-w^{n-1}_m}{k}\right|\right|^2_H.$$

Therefore summing up we find

$$\begin{aligned}\frac{3}{8}\,k\sum_{n=1}^{j}\left|\left|\frac{u^n_m-u^{n-1}_m}{k}\right|\right|^2_H\ +\ \frac{1}{4}\int_\Omega|\nabla u^j_m|^2\,dx\le\ \tilde{c}_5\int_\Omega|u^j_m|^2\,dx\ +\ \tilde{c}_6\,k\sum_{n=1}^{j}||u^n_m||^2_{L^2(\Omega)}\\ +\ \tilde{c}_7\,k\sum_{n=1}^{j}||\nabla u^n_m||^2_{L^2(\Omega)}\ +\ \tilde{c}_8,\end{aligned}\tag{3.2.25}$$

where

$$\tilde{c}_5:=\ 2\,||\vec{v}||^2_{L^\infty(Q)^N}(1+2\alpha^2);\qquad \tilde{c}_6:=\frac{1}{4}+2\,(C^f_3)^2$$

$$\tilde{c}_7:=||\vec{v}||^2_{L^\infty(Q)^N}\,2\,(L^2_{\mathcal{F}}+1)\ +\ \left|\left|\frac{\partial\vec{v}}{\partial t}\right|\right|^2_{L^\infty(Q)^N}\,(1+4\,L^2_{\mathcal{F}}\,T)$$

$$\tilde{c}_8:=\ 4\,||\vec{v}||^2_{L^\infty(Q)^N}\,|\Omega|\,\beta^2\ +\ (||\vec{v}||_{L^\infty(Q)^N}+1)\,||\nabla u^0_m||^2_H\ +\ \frac{||\vec{v}||_{L^\infty(Q)^N}}{2}\,(||u^0_m||^2_H+||w^0_m||^2_H)$$

$$+\ \frac{k}{8\,L^2_{\mathcal{F}}\,T}\,||w^0_m||^2_H\ +\ |\tilde{G}(x_0)|\,|\Gamma_2|\ +\ \int_{\Gamma_2}\tilde{G}(\gamma_0 u^0_m(\sigma))\,d\sigma\ +\ 2\,(C^f_4)^2\,T.$$

Combining (3.2.24) and (3.2.25), we have

$$\frac{1}{4}\int_\Omega|u^j_m|^2\,dx\ +\ \frac{1}{4}\int_\Omega|\nabla u^j_m|^2\,dx\ +\ \frac{3}{8}\,k\sum_{n=1}^{j}\left|\left|\frac{u^n_m-u^{n-1}_m}{k}\right|\right|^2_H\ \overset{(3.2.25)}{\le}\ \left(\tilde{c}_5+\frac{1}{4}\right)\int_\Omega|u^j_m|^2\,dx$$

$$+\,\tilde{c}_6\,k\sum_{n=1}^{j}||u^n_m||^2_H+\ \tilde{c}_7\,k\sum_{n=1}^{j}||\nabla u^n_m||^2_H\ +\ \tilde{c}_8\ \overset{(3.2.24)}{\le}\ \frac{k}{8}\sum_{n=1}^{j}\left|\left|\frac{u^n_m-u^{n-1}_m}{k}\right|\right|^2_H$$

$$+\ \left(\tilde{c}_6+\frac{\tilde{c}_2}{\tilde{c}_9}\right)\,k\sum_{n=1}^{j}||u^n_m||^2_{L^2(\Omega)}+\ \left(\tilde{c}_7+\frac{\tilde{c}_3}{\tilde{c}_9}\right)\,k\sum_{n=1}^{j}||\nabla u^n_m||^2_{L^2(\Omega)}\ +\ \tilde{c}_8+\frac{\tilde{c}_4}{\tilde{c}_9}$$

which in particular implies that, for any $j\in\{1,\dots,m\}$

$$||u^j_m||^2_V\ \le\ \bar{c}\left(1+\ k\sum_{n=1}^{j}||u^n_m||^2_V\right),$$

where

$$\bar{c} := 4 \max\left(\tilde{c}_6 + \frac{\tilde{c}_2}{\tilde{c}_9}, \tilde{c}_7 + \frac{\tilde{c}_3}{\tilde{c}_9}, \tilde{c}_8 + \frac{\tilde{c}_4}{\tilde{c}_9}\right).$$

A discrete version of the Gronwall lemma (see also the second step of the proof of Theorem 3.1.2) yields the following a priori estimate, which is valid for any $j \in \{1, \dots, m\}$

$$k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|^2_{L^2(\Omega)} + ||u_m^j||_V^2 \leq \text{ constant (independent of } m). \tag{3.2.26}$$

③ *Third step: limit procedure*

With the usual notations, we denote with $w_m(x, \cdot)$ the linear time interpolate of $w_m(x, nk) := w_m^n(x)$ for $n = 0, \dots, m$, a.e. in Ω; moreover we set $\bar{u}_m(x,t) := u_m^n(x)$ if $(n-1)k < t \leq nk$, for $n = 1, \dots, m$, a.e. in Ω and define $\bar{w}_m$ in a similar way. Thus (3.2.19) yields

$$\frac{\partial u_m}{\partial t} + \frac{\partial w_m}{\partial t} + C(\bar{u}_m) + A(\bar{u}_m) + G(\bar{u}_m) = K(\bar{u}_m) \tag{3.2.27}$$

while (3.2.26) yields

$$\begin{aligned} ||u_m||_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)} &\leq \text{ constant (independent of } m) \\ ||\bar{u}_m||_{L^\infty(0,T;V)} &\leq \text{ constant (independent of } m). \end{aligned} \tag{3.2.28}$$

The a priori estimates we found allow us to conclude that there exists u such that, possibly taking $m \to +\infty$ along a subsequence,

$$\begin{aligned} u_m \to u &\quad \text{weakly star in} \quad H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \\ \bar{u}_m \to u &\quad \text{weakly star in} \quad L^\infty(0,T;V). \end{aligned}$$

Moreover, as $H^1(0,T;L^2(\Omega)) = L^2(\Omega; H^1(0,T)) \subset L^2(\Omega; \mathcal{C}^0([0,T]))$ with continuous injection, then we may easily obtain

$$||w_m||_{L^2(Q)} \leq \sqrt{T}\, ||w_m||_{L^2(\Omega;\mathcal{C}^0([0,T]))} \overset{(3.1.3)}{\leq} \sqrt{T}\, L_{\mathcal{F}}\, ||u_m||_{L^2(\Omega;\mathcal{C}^0([0,T]))} + \sqrt{T}\, ||\tau||_{L^2(\Omega)} \leq c$$

with c constant independent of m; this allows us to conclude that there exists w such that, possibly taking $m \to +\infty$ along a subsequence

$$w_m \to w \quad \text{weakly in } L^2(Q). \tag{3.2.29}$$

Using once more (3.1.3), we may also say that

$$||\bar{u}_m + \bar{w}_m||_{L^2(Q)} \leq \text{ constant (independent of } m). \tag{3.2.30}$$

This fact easily entails that

$$C(\bar{u}_m) + A(\bar{u}_m) + G(\bar{u}_m) - K(\bar{u}_m) \in L^2(0,T;V')$$

which in turn gives us

$$\left|\left|\frac{\partial}{\partial t}(u_m + w_m)\right|\right|_{L^2(0,T;V')} \leq \text{ constant (independent of } m).$$

So we are able to pass to the limit in the first two terms of (3.2.27) obtaining

$$\frac{\partial u_m}{\partial t} + \frac{\partial w_m}{\partial t} \to \frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} \qquad \text{weakly star in } L^2(0,T;V').$$

Working as in the third step of the proof of Theorem 3.1.2 it is not difficult to prove that $w = \overline{\mathcal{F}}(u)$.

On the other hand, the a priori estimate (3.2.28) allows us to pass to the limit easily in the third, fourth and the sixth term of (3.2.27), (in the last case we also use the Lipschitz continuity of the function f with respect to the third argument) namely

$$C(\bar{u}_m) + A(\bar{u}_m) - K(\bar{u}_m) \to C(u) + A(u) - K(u) \qquad \text{weakly star in } L^2(0,T;V').$$

In order to finish the proof we have only to pass to the limit in the fifth term of (3.2.27) and this fact is not completely trivial.

By interpolation, (see [29], Chapter 4), we deduce, for any $\theta \in (1/2, 1)$

$$H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \subset H^1(Q) \subset H^\theta(\Omega; H^{1-\theta}(0,T)).$$

Taking the traces we get

$$\gamma_0 u_m \in H^{\theta-1/2}(\Gamma; H^{1-\theta}(0,T)).$$

But now, for any $\epsilon > 0$ sufficiently small we deduce

$$H^{\theta-1/2}(\Gamma; H^\theta(0,T)) \subset H^{\theta-1/2-\epsilon}(\Gamma; H^{\theta-\epsilon}(0,T)) \subset L^2(\Gamma \times (0,T)),$$

where the first inclusion is also compact. Thus, possibly taking the limit along a subsequence, we deduce

$$\gamma_0 u_m \to \gamma_0 u \qquad \text{pointwise a.e. in } \Gamma \times (0,T)$$

which gives

$$\gamma_0 \bar{u}_m \to \gamma_0 u \qquad \text{pointwise a.e. in } \Gamma \times (0,T);$$

on the other hand, the sequence $||\gamma_0 \bar{u}_m||_{L^2(\Gamma\times(0,T))}$ is dominated by the sequence $||\bar{u}_m||_{L^2(0,T;V)}$ which is controlled by a constant independent of m, thus the Lebesgue dominated convergence theorem yields that

$$\gamma_0 \bar{u}_m \to \gamma_0 u \qquad \text{strongly in } L^2(\Gamma \times (0,T)).$$

At this point we have all the elements in order to conclude the passage to the limit, since, using the arguments exposed so far, we finally obtain the desired goal, i.e.

$$G(\bar{u}_m) \to G(u) \qquad \text{weakly in } L^2(0,T;V').$$

In fact, in order to prove this, it is enough to show that, for all $\psi \in L^2(0,T;V)$,

$$\int_0^T {}_{V'}\langle G(\bar{u}_m)(t,\cdot) - G(u)(t,\cdot), \psi(t,\cdot)\rangle_V \, dt \to 0.$$

But, using the Lipschitz continuity of the function g we have

$$\begin{aligned}
&\int_0^T {}_{V'}\langle G(\bar{u}_m)(t,\cdot) - G(u)(t,\cdot), \psi(t,\cdot)\rangle_V \, dt \\
&= \int_0^T \int_{\Gamma_2} (g(\gamma_0 \bar{u}_m(\sigma,t)) - g(\gamma_0 u(\sigma,t)))\, \gamma_0\psi(\sigma,t)\, d\sigma\, dt \\
&\leq \int_0^T \int_{\Gamma_2} |g(\gamma_0 \bar{u}_m(\sigma,t)) - g(\gamma_0 u(\sigma,t))|\, |\gamma_0\psi(\sigma,t)|\, d\sigma\, dt \\
&\leq L_g \int_0^T \int_{\Gamma_2} |\gamma_0 \bar{u}_m(\sigma,t) - \gamma_0 u(\sigma,t)|\, |\gamma_0\psi(\sigma,t)|\, d\sigma\, dt \\
&\leq L_g \, ||\gamma_0 \bar{u}_m - \gamma_0 u||_{L^2(\Gamma_2)} \, ||\gamma_0 \psi||_{L^2(\Gamma_2)}
\end{aligned}$$

and this last term vanishes. The fact that $\overline{\mathcal{F}}(u) \in H^1(0,T;L^2(\Omega))$ then easily follows by (3.1.5). This is enough to conclude the proof. □

3.2.3. Uniqueness and Lipschitz continuous dependence on the data

For a particular choice of the operator $\overline{\mathcal{F}}$ we are able to state and prove a uniqueness result for solutions of Problem 3.2.2 together with the corresponding consequence of the Lipschitz continuous dependence on the data.

More precisely, let us consider the setting outlined in Subsection 3.1.1. Let $\mathcal{F}_\varphi : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be a Prandtl-Ishlinskiĭ operator of play type (according to Definition 1.3.1)

$$\mathcal{F}_\varphi(\lambda, u) = h(0)u + \int_0^\infty \wp_r(\lambda, u)\, dh(r) \tag{3.2.31}$$

generated by the convex function

$$\varphi(r) = \int_0^r h(s)\, ds, \qquad r > 0,$$

where $\wp_r(\lambda, u)$ is the play operator introduced in (1.2.5), Λ_0 is introduced in (1.2.4) and h is a given nondecreasing function. We fix any initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_K) \qquad \text{for some } K > 0, \tag{3.2.32}$$

where Λ_K was also introduced in (1.2.4). Working as in the Subsection 3.1.3, it is not difficult to see that the operator

$$\overline{\mathcal{F}}_\varphi(u)(x,t) := \tilde{\mathcal{F}}_\varphi(\lambda, u)(x,t) := \mathcal{F}_\varphi(\lambda(x), u(x,\cdot))(t) := h(0)u(x,t) + \int_0^\infty \overline{\wp}_r(\lambda, u)(x,t)\, dh(r) \tag{3.2.33}$$

fulfills (3.1.1), (3.1.2), (3.1.3), (3.1.4), (3.1.5) and (3.2.1), where

$$\overline{\wp}_r(\lambda, u)(x,t) := \wp_r(\lambda(x), u(x,\cdot))(t).$$

Moreover, if the function h is bounded, then also (3.2.2) is fulfilled by $\overline{\mathcal{F}}_\varphi$.
This time, for the sake of simplicity, we consider a given function f only dependent on x and t but independent of u, so in particular we will not need (3.2.11) anymore.
The main result of this subsection is therefore the following

Theorem 3.2.8. *Let λ be as in (3.2.32) and $\overline{\mathcal{F}}_\varphi$ as in (3.2.33); suppose that h is nondecreasing and bounded and the function $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, Lipschitz continuous with Lipschitz constant L_g and such that $g(x_0) = 0$ for some $x_0 \in \mathbb{R} \setminus \{0\}$. Assume moreover that the known function f is defined over $\Omega \times [0,T]$ with $f \in L^2(Q)$. Finally let (3.2.12) hold. Then Problem 3.2.2$_\varphi$ (which is our original model Problem 3.2.2 with $\overline{\mathcal{F}}$ replaced by $\overline{\mathcal{F}}_\varphi$ and our current choice of the data), admits a unique solution.*
Moreover with the same assumptions the dependence of the solution on the data is Lipschitz continuous in the following sense: if u_i^0, w_i^0, f_i for $i = 1, 2$ are data fulfilling

$$u_i^0 \in V, \quad w_i^0 \in L^2(\Omega), \quad f_i \in L^2(Q) \qquad \text{for } i = 1, 2 \tag{3.2.34}$$

and u_i is the corresponding unique solution of Problem 3.2.2 with $\overline{\mathcal{F}}$ replaced by $\overline{\mathcal{F}}_\varphi$, then, setting $\xi_i^0 := \overline{\wp}_r(\lambda, u_i)(0)$ we obtain for any $t \in (0,T)$

$$||u_1(t) - u_2(t)||^2_{L^2(\Omega)} \leq \bar{c}_1 \, ||u_1^0 - u_2^0||^2_{L^2(\Omega)} + \bar{c}_2 \int_\Omega \int_0^\infty |\xi_1^0 - \xi_1^0|^2 \, dh(r)\, dx + \bar{c}_3 \, ||f_1 - f_2||^2_{L^2(Q_t)},$$

where the constants $\bar{c}_i$ for $i = 1, 2, 3$ are functions of t.

Proof. First of all Theorem 1.3.2 and Proposition A.1.2 entail that Problem 3.2.2$_\varphi$ admits at least a solution $u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; V)$. Let us now show that this solution is also unique.

Suppose by contradiction that Problem $3.2.2_\varphi$ admits two solutions u_1, u_2 such that $u_1, u_2 \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$. Then, for $i = 1,2$ we have

$$u_i + \mathcal{F}_\varphi(\lambda, u_i) := (h(0)+1)u_i + \int_0^\infty \wp_r(\lambda, u_i)\, dh(r) = \tilde{h}(0)u_i + \int_0^\infty \wp_r(\lambda, u_i)\, dh(r)$$

and this means that the operator $I + \mathcal{F}_\varphi$ (where I is the identity operator) is still a Prandtl-Ishlinskiĭ operator of play type generated by the convex function

$$\tilde{\varphi}(r) := \int_0^r \tilde{h}(s)\, ds, \qquad r > 0$$

where $\tilde{h} := h + 1$; let us set $\overline{\mathcal{G}}_{\tilde{\varphi}} := I + \overline{\mathcal{F}}_\varphi$. So we certainly come to the following

$$\int_\Omega \frac{\partial}{\partial t}(\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2))\,(u_1 - u_2)\, dx - \int_\Omega [\vec{v} \cdot \nabla(u_1 - u_2)]\,(\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2))\, dx$$

$$+ \int_\Omega |\nabla(u_1 - u_2)|^2\, dx + \int_{\Gamma_2} (g(\gamma_0 u_1) - g(\gamma_0 u_2))\,\gamma_0(u_1 - u_2)\, d\sigma = 0 \qquad \text{a.e. in } [0,T].$$

At this point the term

$$\int_\Omega \frac{\partial}{\partial t}(\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2))\,(u_1 - u_2)\, dx$$

is controlled by inequality (1.3.2) in the following way

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial t}(\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2))\,(u_1 - u_2)\, dx \\ &\geq \int_\Omega \frac{1}{2}\frac{d}{dt}\left\{\tilde{h}(0)[(u_1 - u_2)(x,t)]^2 + \int_0^\infty [\overline{\wp}_r(\lambda, u_1)(x,t) - \overline{\wp}_r(\lambda, u_2)(x,t)]^2\, d\tilde{h}(r)\right\} dx \end{aligned} \tag{3.2.35}$$

and moreover

$$\int_{\Gamma_2} (g(\gamma_0 u_1) - g(\gamma_0 u_2))\,\gamma_0(u_1 - u_2)\, d\sigma \geq 0 \tag{3.2.36}$$

since the function g is nondecreasing. Summing up and integrating in time from 0 to any $t \in [0,T]$ we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^t \int_\Omega |\nabla(u_1 - u_2)|^2\, dx\, dt + \int_\Omega \frac{1}{2}\tilde{h}(0)\{[(u_1 - u_2)(x,t)]^2 - [(u_1 - u_2)(x,0)]^2\}\, dx \\ &+ \frac{1}{2}\int_\Omega \int_0^\infty \{[\overline{\wp}_r(\lambda, u_1)(x,t) - \overline{\wp}_r(\lambda, u_2)(x,t)]^2 - [\overline{\wp}_r(\lambda, u_1)(x,0) - \overline{\wp}_r(\lambda, u_2)(x,0)]^2\}\, d\tilde{h}(r)\, dx \\ &\leq \frac{\|\vec{v}\|^2_{L^\infty(Q)^N}}{2}\int_0^t \int_\Omega \left|\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2)\right|^2\, dx\, dt \leq \tilde{h}(0)\|\vec{v}\|^2_{L^\infty(Q)^N}\int_0^t \int_\Omega \tilde{h}(0)[(u_1 - u_2)(x,t)]^2\, dx\, dt \\ &+ \|\vec{v}\|^2_{L^\infty(Q)^N}\,(\tilde{h}(\infty) - \tilde{h}(0))\int_0^t \int_\Omega \left[\int_0^\infty [\overline{\wp}_r(\lambda, u_1)(x,t) - \overline{\wp}_r(\lambda, u_2)(x,t)]^2(t)\, d\tilde{h}(r)\right] dx\, dt. \end{aligned}$$

Now we can use the fact that $u_1(x,0) \equiv u_2(x,0)$, which in turns entails $[\overline{\wp}_r(\lambda, u_1)(x,0) \equiv \overline{\wp}_r(\lambda, u_2)(x,0)]$ (due the causality property for the play operators), and Gronwall's lemma to obtain

$$u_1(x,t) \equiv u_2(x,t) \qquad \text{a.e. in } \Omega, \text{ for all } t \in (0,T)$$

and this finishes the first part of the proof.

For the Lipschitz continuous dependence on the data, if u_1, u_2 are the unique solutions of Problem $3.2.2_\varphi$ corresponding to the data (3.2.34), then

$$\int_\Omega \frac{\partial}{\partial t}(\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2))\,(u_1-u_2)\,dx - \int_\Omega [\vec{v}\cdot\nabla(u_1-u_2)]\,(\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2))\,dx$$

$$+\int_\Omega |\nabla(u_1-u_2)|^2\,dx + \int_{\Gamma_2} (g(\gamma_0 u_1) - g(\gamma_0 u_2))\,\gamma_0(u_1-u_2)\,d\sigma = \int_\Omega (f_1-f_2)\,(u_1-u_2)\,dx.$$

Now, we first use (3.2.35) and (3.2.36) and then we integrate in time from 0 to any $t \in [0,T]$. We deduce that

$$\frac{1}{2}\int_0^t\int_\Omega |\nabla(u_1-u_2)|^2\,dx\,dt + \int_\Omega \frac{1}{2}\,\tilde{h}(0)\{[(u_1-u_2)(x,t)]^2 - [(u_1-u_2)(x,0)]^2\}\,dx$$

$$+\frac{1}{2}\int_\Omega\int_0^\infty \left\{[\overline{\wp}_r(\lambda,u_1)(x,t) - \overline{\wp}_r(\lambda,u_2)(x,t)]^2 - [\overline{\wp}_r(\lambda,u_1)(x,0) - \overline{\wp}_r(\lambda,u_2)(x,0)]^2\right\}\,d\tilde{h}(r)\,dx$$

$$\le \frac{||\vec{v}||^2_{L^\infty(\Omega)^N}}{2}\int_0^t\int_\Omega |\overline{\mathcal{G}}_{\tilde{\varphi}}(u_1) - \overline{\mathcal{G}}_{\tilde{\varphi}}(u_2)|^2\,dx\,dt \,+\, \frac{1}{2}\,||f_1-f_2||^2_{L^2(Q_t)} \,+\, \frac{1}{2}\,||u_1-u_2||^2_{L^2(Q_t)}$$

$$\le \left((\tilde{h}(0))^2\,||\vec{v}||^2_{L^\infty(Q)^N} + \frac{1}{2}\right)\,||u_1-u_2||_{L^2(Q_t)} \,+\, ||f_1-f_2||^2_{L^2(Q_t)}$$

$$+\; ||\vec{v}||^2_{L^\infty(Q)^N}\,(\tilde{h}(\infty) - \tilde{h}(0))\int_0^t\int_\Omega \left[\int_0^\infty [\overline{\wp}_r(\lambda,u_1)(x,t) - \overline{\wp}_r(\lambda,u_2)(x,t)]^2\,d\tilde{h}(r)\right]\,dx\,dt.$$

The Gronwall lemma then yields

$$||u_1(t)-u_2(t)||^2_{L^2(\Omega)} \le \bar{c}_1\,||u_1^0 - u_2^0||^2_{L^2(\Omega)} + \bar{c}_2\int_\Omega\int_0^\infty |\xi_1^0 - \xi_1^0|^2\,d\tilde{h}(r)\,dx + \bar{c}_3\,||f_1-f_2||^2_{L^2(Q_t)},$$

where the constants $\bar{c}_i$ for $i = 1,2,3$ are functions of t. This finishes also the second part of the proof. $\square$

CHAPTER 4

On some systems of P.D.E.s with hysteresis

In [39], Chapter IX, the following model equation

$$\frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) - \triangle u = f \qquad \text{in } \Omega \times (0, T)$$

is studied, where $\overline{\mathcal{F}}$ is a continuous hysteresis operator, Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\triangle$ is the Laplace operator and f is a given function. This equation can be rewritten as a system, i.e.

$$\begin{cases} \dfrac{\partial}{\partial t}(u + w) - \triangle u = f \\ w = \overline{\mathcal{F}}(u) \end{cases} \qquad \text{in } \Omega \times (0, T). \tag{4.0.1}$$

In the first part of this chapter we would like to study the following system

$$\begin{cases} \dfrac{\partial}{\partial t}(u + w) - \triangle u = f \\ \gamma \, \dfrac{\partial w}{\partial t} + w = \overline{\mathcal{F}}(u) \end{cases} \qquad \text{in } \Omega \times (0, T). \tag{4.0.2}$$

where γ is some positive constant. Here the constitutive relation $\gamma \, w_t + w = \overline{\mathcal{F}}(u)$ can be seen as a perturbation of $w = \overline{\mathcal{F}}(u)$ if we let the parameter γ tend to zero.
For this model system we get an existence theorem, based on approximation by implicit time discretization scheme; once that some good a priori estimates are found, we pass to the limit, with a particular care when dealing with the nonlinear hysteresis term and we get a weak solution of our problem in a sense that will be specified in the following. The presence of this new constitutive relation leads to some technical difficulties, mainly as to concern the a priori estimates, which are found through several steps.

The last part of the chapter is instead devoted to the study of the following model system

$$\begin{cases} \dfrac{\partial}{\partial t}(u+w) - \triangle u = f \\ w = \overline{\mathcal{F}}\left(u - \gamma \dfrac{\partial w}{\partial t}\right) \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{4.0.3}$$

where $\overline{\mathcal{F}}$ is a continuous hysteresis operator, Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\triangle$ is the Laplace operator, f is a given function and $\gamma > 0$ is a suitable constant. This system differs from system (4.0.1), due to another kind of perturbation which has been considered in the constitutive equation.
Provided that the inverse of the hysteresis operator $\overline{\mathcal{F}}$ exists - we will set $\overline{\mathcal{G}} := \overline{\mathcal{F}}^{-1}$ - our perturbed system is equivalent to the following equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t} + \frac{\partial \overline{\mathcal{G}}(w)}{\partial t} - \triangle \left(\overline{\mathcal{G}}(w) + \gamma \frac{\partial w}{\partial t}\right) = f \qquad \text{in } \Omega \times (0,T).$$

We approach this new problem using a method based on the approximation by implicit time discretization scheme. However some difficulties arise, mainly when dealing with the stationary problem and the choice of an adequate functional setting to work with seems actually to be necessary. We thus consider our problem in the setting of the following Hilbert triplet

$$L^2(\Omega) \subset H^{-1}(\Omega) \equiv (H^{-1}(\Omega))' \subset (L^2(\Omega))'$$

where the role of the pivot space is played by $H^{-1}(\Omega)$ instead of $L^2(\Omega)$.
Both systems (4.0.2) and (4.0.3) arise in the context of electromagnetic processes, we refer to Section 4.1 for more details on this purpose.

Remark 4.0.9. For the systems presented in this chapter we chose an approach based on the approximation of the problem by an implicit time discretization scheme. On the other hand, we would like to look for other possibilities because the approach chosen in this chapter provides no information concerning uniqueness of solutions of our model problems and the behaviour of the solutions when the parameter γ goes to zero. For example it seems to be possible to obtain an existence and uniqueness result using a method based on the contraction mapping principle. This part of the analysis is still work in progress.

Remark 4.0.10. According to the whole presentation of the thesis, also these results are obtained for the general class of memory operators $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$. This is not restrictive, as explained in Section 1.6 to which we refer for more details on the topic.

4.1. Physical interpretation of systems (4.0.2) and (4.0.3)

Let a ferromagnetic material occupy a bounded region $\mathcal{D} \subset \mathbb{R}^3$; we set $\mathcal{D}_T := \mathcal{D} \times (0,T)$, for a fixed $T > 0$. We assume that the body is homogeneous and isotropic, that it is surrounded by vacuum and that the dielectric permeability ϵ is a scalar constant. The facts contained in this section can be found in a classical text of electromagnetism, for example [20].
We denote by $\vec{g}$ a prescribed electromotive force; then Ohm's law reads

$$\vec{J} = \sigma\,(\vec{E} + \vec{g}) \qquad \text{in } \mathcal{D},$$

whereas $\vec{J} = 0$ outside $\mathcal{D}$, where σ is the electric conductivity, $\vec{J}$ is the electric current density and $\vec{E}$ is the electric field.
In the case of a ferromagnetic metal σ is very large, hence we can certainly assume

$$4\,\pi\,|\vec{J}| \gg \left|\frac{\partial \vec{D}}{\partial t}\right| \qquad \text{in } \mathcal{D}_T,$$

where $\vec{D}$ is the electric displacement, provided that the field $\vec{g}$ does not vary too rapidly.
Therefore we confine to $\mathcal{D}$ the Ampère and the Faraday laws, i.e.

$$c\,\nabla \times \vec{H} = 4\pi\,\vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{in } \mathcal{D}_T$$

$$c\,\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{in } \mathcal{D}_T,$$

where c is the speed of light in vacuum, $\vec{H}$ is the magnetic field, $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic induction. Then we neglect the displacement current $\dfrac{\partial \vec{D}}{\partial t}$ in Ampère law's; this is the so-called *eddy current approximation*.
By coupling this reduced law with Faraday's and Ohm's laws, in Gauss units we get

$$4\,\pi\,\sigma\,\frac{\partial \vec{B}}{\partial t} \,+\, c^2\,\nabla \times \nabla \times \vec{H} \,=\, 4\,\pi\,c\,\sigma\,\nabla \times \vec{g} \qquad \text{in } \mathcal{D}_T. \tag{4.1.1}$$

We also have $\vec{B} = \vec{H} + 4\,\pi\,\vec{M}$, where $\vec{M}$ is the magnetization, so we can rewrite (4.1.1) as follows

$$4\,\pi\,\sigma\,\frac{\partial}{\partial t}(\vec{H} + 4\,\pi\,\vec{M}) \,+\, c^2\,\nabla \times \nabla \times \vec{H} \,=\, 4\,\pi\,c\,\sigma\,\nabla \times \vec{g} \qquad \text{in } \mathcal{D}_T.$$

On the other hand the previous equation involves vectors; we would like to simplify this model problem by dealing with a special case.
More precisely, let Ω be a domain of $\mathbb{R}^2$, we set $\Omega_T := \Omega \times (0,T)$ and assume (using the orthogonal Cartesian coordinates x, y, z) that $\vec{H}$ is parallel to the $z-$axis and only depends on the coordinates x, y, i.e.

$$\vec{H} = (0, 0, H(x,y)).$$

Then

$$\nabla \times \nabla \times \vec{H} = (0, 0, -\triangle_{x,y} H) \qquad \left(\triangle_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{4.1.2}$$

Dealing with a strongly anisotropic material, we can also assume that

$$\vec{M} = (0, 0, M(x, y)).$$

If moreover $\nabla \times \vec{g} = (0, 0, \tilde{f})$, then equation (4.1.1) is reduced to an equation for scalar variables (we avoid from now on to write the constants π, c, σ in order to simplify our formula layout)

$$\frac{\partial}{\partial t}(H + M) - \triangle_{x,y} H = \tilde{f}. \tag{4.1.3}$$

Now we would like to introduce a constitutive relation between H and M. In this chapter we choose the following two

$$\gamma \frac{\partial M}{\partial t} + M = \overline{\mathcal{F}}(H) \tag{4.1.4}$$

and

$$M = \overline{\mathcal{F}} \left(H - \gamma \frac{\partial M}{\partial t} \right), \tag{4.1.5}$$

where $\overline{\mathcal{F}}$ is a scalar Preisach operator and $\gamma > 0$ a suitable constant. Here the constitutive relation $M = \overline{\mathcal{F}}(H)$ is perturbed in two different ways by means of the relaxation term $\gamma \frac{\partial M}{\partial t}$. In equations (4.1.4) and (4.1.5) we have therefore the presence of a rate independent element and a rate dependent one.
By coupling (4.1.3) respectively with (4.1.4) and (4.1.5), we have exactly (4.0.2) and (4.0.3).

4.2. First system of P.D.E.s

4.2.1. Weak formulation of the problem

Consider an open bounded set of Lipschitz class $\Omega \subset \mathbb{R}^N$, $N \geq 1$ and set $Q := \Omega \times (0, T)$. Assume that the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0, T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0, T]))$ is causal, strongly continuous and affinely bounded according to (3.1.1), (3.1.2) and (3.1.3).
From now on, for the sake of simplicity, we set $V := H^1_0(\Omega)$, $H := L^2(\Omega)$ and $V' := H^{-1}(\Omega)$ and we consider V endowed with the norm $||u||_V := ||\nabla u||_{L^2(\Omega)}$ (we are thus considering Dirichlet boundary conditions).
Let us introduce the operator $A : V \to V'$ defined as follows

$$_{V'}\langle Au, v \rangle_V := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall u, v \in V.$$

We then identify the space $L^2(\Omega)$ to its topological dual $(L^2(\Omega))'$; as the injection of V into $L^2(\Omega)$ is continuous and dense, $(L^2(\Omega))'$ can be identified to a subspace of V' (see Theorem A.6.1). This yields the Hilbert triplet

$$V \subset H \equiv H' \subset V'$$

with dense and continuous injections.
We assume that $u^0, w^0 \in L^2(\Omega)$ are given initial data and that $f \in L^2(0,T;V')$; moreover let γ be a fixed positive constant. We would like to solve the following problem

Problem 4.2.1. *We search for two functions $u \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \cap L^2(0,T;V)$ and $w \in L^2(Q)$ such that $\overline{\mathcal{F}}(u) \in L^2(Q)$ and for any $\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ with $\psi(\cdot,T) = 0$ a.e. in Ω*

$$\begin{aligned} \int_0^T \int_\Omega -(u+w)\frac{\partial \psi}{\partial t}\, dx\, dt \; + \int_0^T \int_\Omega \nabla u \cdot \nabla \psi \, dx\, dt \\ = \int_0^T {}_{V'}\langle f, \psi \rangle_V \, dt + \int_\Omega [u^0(x) + w^0(x)]\, \psi(x,0)\, dx. \end{aligned} \tag{4.2.1}$$

$$-\gamma \int_0^T \int_\Omega w \frac{\partial \psi}{\partial t}\, dx\, dt = \int_0^T \int_\Omega [\overline{\mathcal{F}}(u) - w]\, \psi \, dx\, dt + \int_\Omega w^0(x)\, \psi(x,0)\, dx. \tag{4.2.2}$$

Interpretation. The variational equations (4.2.1) and (4.2.2) yield

$$\begin{cases} \dfrac{\partial}{\partial t}(u+w) - \triangle u = f \\ \gamma \dfrac{\partial w}{\partial t} + w = \overline{\mathcal{F}}(u) \end{cases} \qquad \text{in } \mathcal{D}'(0,T;V') \tag{4.2.3}$$

thus, by comparison,

$$\frac{\partial}{\partial t}[u+w] \in L^2(0,T;V')$$

so $u + w \in H^1(0,T;V')$ and (4.2.3) holds in V' a.e. in $(0,T)$. Hence, integrating by parts in time in (4.2.1) we get

$$[u+w]_{|t=0} = u^0 + w^0 \;\text{ in } V' \tag{4.2.4}$$

in the sense of the traces. In turn (4.2.3) and (4.2.4) yield (4.2.1) and (4.2.2) and the two formulations are equivalent.

4.2.2. An existence result

As the proof of our existence result may appear quite involved, in order to clarify it as much as we can, we present here the formal computations which undergo our treatment.

We deal with the following system

$$
\begin{cases}
\dfrac{\partial}{\partial t}(u+w) - \triangle\, u = f & \text{(4.2.5a)}\\[2ex]
\gamma\, \dfrac{\partial w}{\partial t} + w = \overline{\mathcal{F}}(u). & \text{(4.2.5b)}
\end{cases}
$$

The first idea is to test equation (4.2.5a) by u_t with the hope of having a control on the term $||u_t||_{L^2(Q)}$ which would be certainly good; the problem is that, doing this, the term $\int_0^T \int_\Omega w_t\, u_t\, dx\, dt$ turns out to be without control. We try to avoid this difficulty by testing equation (4.2.5b) by u_t, but in this way we cannot estimate the term $||w||_{L^2(Q)}$; for this reason we conclude by a further test of equation (4.2.5b), this time by w_t. At this point the outline of the proof is clear: as we said in the introduction the basic hint which is useful in this case is the approximation of our problem by an implicit time discretization scheme. The stationary problem is solved without great difficulties; the a priori estimates are deduced according to the previous tests (between the corresponding suitably discretized equations) and finally the conclusion is achieved passing to the limit, paying attention to the nonlinear term.
The result we can state and prove is therefore the following.

Theorem 4.2.2. *(Existence)*
Let us assume that the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ is causal, strongly continuous and affinely bounded according to (3.1.1), (3.1.2) *and* (3.1.3). *Moreover let*

$$
f \in L^2(Q), \;\; u^0 \in V, \;\; w^0 \in L^2(\Omega). \tag{4.2.6}
$$

Then Problem 4.2.1 *has at least one solution*

$$
u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T;V) \qquad w \in H^1(0,T;L^2(\Omega))
$$

such that

$$
\overline{\mathcal{F}}(u) \in L^2(Q).
$$

Proof. ➜ approximation via implicit time discretization scheme.

We fix $m \in \mathbb{N}$, set $k := T/m$ and for any $n = 1, \ldots, m$ we consider $f_m^n(x) := f(x, nk)$, a.e. in Ω, $u_m^0 := u^0$ and $w_m^0 := w^0$.
We search for a couple of functions $(u_m^n, w_m^n) \in V \times L^2(\Omega)$ for $n = 1, \ldots, m$; if we consider the time dependent functions $u_m(x, \cdot)$ and $w_m(x, \cdot)$ which are introduced as the linear time interpolate of $u_m(x, nk) := u_m^n(x)$ and $w_m(x, nk) := w_m^n(x)$ respectively, a.e. in Ω, then the discrete counterpart of the variational equations (4.2.1) and (4.2.2) reads

$$
\begin{cases}
\dfrac{u_m^n - u_m^{n-1}}{k} + \dfrac{w_m^n - w_m^{n-1}}{k} + Au_m^n = f_m^n & \text{in } V',\\[2ex]
\gamma\, \dfrac{w_m^n - w_m^{n-1}}{k} + w_m^n = z_m^n & \text{in } L^2(\Omega),
\end{cases} \tag{4.2.7}
$$

where we set $z_m^n := [\overline{\mathcal{F}}(u_m)](x, nk)$ and where we recall that $\gamma > 0$.
Working by induction, after noticing that the couple $(u_m^0, w_m^0) = (u^0, w^0)$ is known, for any $n \in \{1, \dots, m\}$ we suppose to know also the couples $(u_m^1, w_m^1), \dots, (u_m^{n-1}, w_m^{n-1}) \in V \times L^2(\Omega)$; the problem is now to determine the pair (u_m^n, w_m^n).
Using this fact we may rewrite (4.2.7) distinguishing the known quantities from the terms we would like to determine. More precisely we have

$$\begin{cases} u_m^n + w_m^n + k\, A u_m^n = k\, f_m^n + u_m^{n-1} + w_m^{n-1} & \text{in } V', \\ (\gamma + k)\, w_m^n - k\, z_m^n = \gamma\, w_m^{n-1} & \text{in } L^2(\Omega). \end{cases}$$

At this point we remark that actually the quantity z_m^n can be seen as a function of u_m^n. In fact the causality property of the operator $\overline{\mathcal{F}}$ allows us to say that the quantity $z_m^n(x) = [\overline{\mathcal{F}}(u_m)](x, nk)$ just depends on the end point of the linear time interpolate u_m i.e. there exists a function F_m^n such that

$$z_m^n(x) = [\overline{\mathcal{F}}(u_m)](x, nk) := F_m^n(u_m^n(x), x) \qquad \text{a.e. in } \Omega;$$

this is due to the fact that $u_m(x, \cdot)_{|[0,(n-1)k]}$ is known. At this point we can introduce an operator $F_S : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ acting in the following way $F_S(v) := F_m^n(v(\cdot), \cdot)$; the operator F_S, despite to the notation, depends on the choice of m and n. It turns out that $F_S : L^2(\Omega) \to L^2(\Omega)$ is strongly continuous. Moreover from (3.1.3) we deduce once again (3.1.17) and (3.1.18) Therefore (4.2.7) is now equivalent to

$$\begin{cases} u_m^n + w_m^n + k\, A u_m^n = k\, f_m^n + u_m^{n-1} + w_m^{n-1} & \text{in } V', \\ (\gamma + k)\, w_m^n - k\, F_S(u_m^n) = \gamma\, w_m^{n-1} & \text{in } L^2(\Omega). \end{cases} \tag{4.2.8}$$

Now we express w_m^n in terms of u_m^n, i.e. we have

$$w_m^n := \frac{1}{k + \gamma}(\gamma\, w_m^{n-1} + k\, F_S(u_m^n))$$

and we insert this relation in the first equation of system (4.2.8), getting

$$(\gamma + k)\, u_m^n + k\, F_S(u_m^n) + (\gamma + k)\, k\, A u_m^n = k\, (\gamma + k)\, f_m^n + (\gamma + k)\, u_m^{n-1} + k\, w_m^{n-1} =: \Lambda_m^n, \tag{4.2.9}$$

and therefore Λ_m^n is a known function. If we want to simplify the notations, we can erase the fixed indexes m and n, dealing therefore with the equation

$$(\gamma + k)\, u + k\, F_S(u) + (\gamma + k)\, k\, A u = \Lambda. \tag{4.2.10}$$

This equation can be solved as we did in Chapter 3, passing through a finite dimensional setting and using Theorem A.9.1. Therefore it is easy to see that equation (4.2.9) has at least one solution u_m^n and in turn system (4.2.8) has at least one solution $(u_m^n, w_m^n) \in V \times L^2(\Omega)$; this was our goal.

➜ a priori estimates.

As we briefly explained in the outline, the strategy we follow is to multiply the first equation of system (4.2.7) by $u_m^n - u_m^{n-1}$ in the duality pairing ${}_{V'}\langle \cdot, \cdot \rangle_V$, and sum for $n = 1, \dots j$, for $j \in \{1, \dots, m\}$. We get

$$\begin{aligned} &\sum_{n=1}^{j} {}_{V'}\langle \frac{u_m^n - u_m^{n-1}}{k}, u_m^n - u_m^{n-1} \rangle_V + \sum_{n=1}^{j} {}_{V'}\langle \frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1} \rangle_V \\ &+ \sum_{n=1}^{j} {}_{V'}\langle A u_m^n, u_m^n - u_m^{n-1} \rangle_V = \sum_{n=1}^{j} {}_{V'}\langle f_m^n, u_m^n - u_m^{n-1} \rangle_V . \end{aligned} \tag{4.2.11}$$

The aim in fact is to obtain the following a priori estimates which will fit our purposes

$$k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|^2_{L^2(\Omega)} + ||\nabla u_m^j||^2_{L^2(\Omega)} \leq \text{ constant (independent of } m), \ \forall\, j \in \{1, \dots, m\}.$$

But, acting in this way, we have to control the term

$$\sum_{n=1}^{j} {}_{V'}\langle \frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1} \rangle_V = \sum_{n=1}^{j} \frac{1}{k} \int_\Omega (w_m^n - w_m^{n-1})\,(u_m^n - u_m^{n-1})\, dx. \tag{4.2.12}$$

This suggests the idea of multiplying

$$\gamma\, \frac{w_m^n - w_m^{n-1}}{k} + w_m^n = z_m^n \qquad \text{in } L^2(\Omega) \tag{4.2.13}$$

by $u_m^n - u_m^{n-1}$ in the duality pairing ${}_{V'}\langle \cdot, \cdot \rangle_V$ (or equivalently in the scalar product of $L^2(\Omega)$) in order to estimate the term (4.2.12). We then sum for $n = 1, \dots j$, for $j \in \{1, \dots, m\}$. We have

$$\frac{\gamma}{k} \sum_{n=1}^{j} \int_\Omega (w_m^n - w_m^{n-1})\,(u_m^n - u_m^{n-1})\, dx + \sum_{n=1}^{j} \int_\Omega w_m^n\,(u_m^n - u_m^{n-1})\, dx = \sum_{n=1}^{j} \int_\Omega z_m^n\,(u_m^n - u_m^{n-1})\, dx$$

and this implies (we recall that we set $H := L^2(\Omega)$)

$$\begin{aligned} &\frac{1}{k} \sum_{n=1}^{j} \int_\Omega (w_m^n - w_m^{n-1})\,(u_m^n - u_m^{n-1})\, dx \geq -\frac{k}{\gamma} \sum_{n=1}^{j} \left| \int_\Omega (z_m^n - w_m^n) \left(\frac{u_m^n - u_m^{n-1}}{k} \right) dx \right| \\ &\geq -\frac{k}{\gamma} \sum_{n=1}^{j} \int_\Omega |z_m^n - w_m^n| \left| \frac{u_m^n - u_m^{n-1}}{k} \right| dx \geq -\frac{k}{\gamma} \sum_{n=1}^{j} \int_\Omega (|z_m^n| + |w_m^n|) \left| \frac{u_m^n - u_m^{n-1}}{k} \right| dx \\ &\geq -\frac{k}{\gamma^2} \sum_{n=1}^{j} (||z_m^n||_H^2 + ||w_m^n||_H^2) - \frac{k}{4} \sum_{n=1}^{j} \int_\Omega \left| \frac{u_m^n - u_m^{n-1}}{k} \right|^2 dx. \end{aligned} \tag{4.2.14}$$

Now, while the first and the last term on the right side of the previous inequality can be absorbed when developing (4.2.11), the second one cannot be controlled; that's why a third series of computations is needed, more precisely we multiply equation (4.2.13) by $w_m^n - w_m^{n-1}$ in the scalar product of $L^2(\Omega)$ and sum for $n = 1, \dots, j$, for $j \in \{1, \dots, m\}$. Thus we obtain

$$\gamma\, k \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|_H^2 + \frac{1}{2} \int_\Omega (|w_m^j|^2 - |w_m^0|^2)\, dx \leq \gamma\, k \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|_H^2$$

$$+ \frac{1}{2} \sum_{n=1}^{j} \int_\Omega (|w_m^n|^2 - |w_m^{n-1}|^2)\, dx \leq \frac{\gamma}{k} \sum_{n=1}^{j} \int_\Omega (w_m^n - w_m^{n-1})\,(w_m^n - w_m^{n-1})\, dx$$

$$+ \sum_{n=1}^{j} \int_\Omega w_m^n\,(w_m^n - w_m^{n-1})\, dx = \sum_{n=1}^{j} \int_\Omega z_m^n\,(w_m^n - w_m^{n-1})\, dx$$

$$\leq k \sum_{n=1}^{j} \int_\Omega |z_m^n| \left| \frac{w_m^n - w_m^{n-1}}{k} \right| dx \leq \frac{k}{2\,\gamma} \sum_{n=1}^{j} ||z_m^n||_H^2 + k\,\frac{\gamma}{2} \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|_H^2$$

which entails

$$k\,\frac{\gamma}{2} \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|_H^2 + \frac{1}{2}\, ||w_m^j||_H^2 \leq ||w_m^0||_H^2 + \frac{k}{2\,\gamma} \sum_{n=1}^{j} ||z_m^n||_H^2. \tag{4.2.15}$$

This inequality can be used to obtain the following

$$k \sum_{n=1}^{j} ||w_m^n||_H^2 \leq 2 ||w_m^0||_H^2 + 2\,Tk \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|_H^2 \leq ||w_m^0||_H^2 \left(2 + \frac{4}{\gamma} T\right) + \frac{2T}{\gamma^2} k \sum_{n=1}^{j} ||z_m^n||_H^2. \tag{4.2.16}$$

If we insert (4.2.14) and (4.2.16) into (4.2.11), we rapidly obtain

$$\frac{k}{2} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 + \frac{1}{2} \int_\Omega |\nabla u_m^j|^2\, dx \leq ||\nabla u_m^0||_H^2 + (\gamma^2 + 2T)\,\frac{1}{\gamma^4}\, k \sum_{n=1}^{j} ||z_m^n||_H^2$$

$$+ ||w_m^0||_H^2 \left(2 + \frac{4}{\gamma} T\right) \frac{1}{\gamma^2} + k \sum_{n=1}^{j} ||f_m^n||_{L^2(\Omega)}^2 \leq ||\nabla u_m^0||_H^2 + (\gamma^2 + 2T) \frac{1}{\gamma^4} \int_\Omega \left[\max_{n=1,\dots,j} |z_m^n| \right]^2 dx$$

$$+ ||w_m^0||_H^2 \left(2 + \frac{4}{\gamma}\, T\right) \frac{1}{\gamma^2} + k \sum_{n=1}^{j} ||f_m^n||_H^2 \overset{(3.1.3)}{\leq} (\gamma^2 + 2T)\, \frac{1}{\gamma^4}\, 2\, L_{\mathcal{F}}^2 \int_\Omega \left[\max_{n=1,\dots,j} |u_m^n(x)| \right]^2 dx$$

$$+ (\gamma^2 + 2T)\, \frac{2}{\gamma^4}\, ||\tau||_H^2 + ||\nabla u_m^0||_H^2 + ||w_m^0||_H^2 \left(2 + \frac{4}{\gamma}\, T\right) \frac{1}{\gamma^2} + k \sum_{n=1}^{j} ||f_m^n||_H^2.$$

Considering that the following holds

$$
\begin{aligned}
|u_m^j(x)|^2 =& |u_m^0(x)|^2 + \sum_{n=1}^{j} (u_m^n(x) - u_m^{n-1}(x))\,(u_m^n(x) + u_m^{n-1}(x)) \\
\le& |u_m^0(x)|^2 + \left(\sum_{n=1}^{j} |u_m^n(x) - u_m^{n-1}(x)|^2 \right)^{1/2} \left(\sum_{n=1}^{j} (|u_m^n(x)| + |u_m^{n-1}(x)|)^2 \right)^{1/2} \\
\le& |u_m^0(x)|^2 + \frac{k}{4} \sum_{n=1}^{j} \left| \frac{u_m^n(x) - u_m^{n-1}(x)}{k} \right|^2 + k \sum_{n=1}^{j} \left(2|u_m^n(x)|^2 + 2|u_m^{n-1}(x)|^2 \right) \\
\le& 4 \left(|u_m^0(x)|^2 + k \sum_{n=1}^{j} |u_m^n(x)|^2 \right) + \frac{k}{4} \sum_{n=1}^{j} \left| \frac{u_m^n(x) - u_m^{n-1}(x)}{k} \right|^2
\end{aligned}
\tag{4.2.17}
$$

and using the fact that we have Dirichlet boundary conditions, if we denote by c_P the Poincaré constant, we may deduce, for any $j \in \{1, \ldots, m\}$

$$
\frac{k}{4} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega |\nabla u_m^j|^2 \, dx \le c_1 \, k \sum_{n=1}^{j} ||\nabla u_m^n||_{L^2(\Omega)}^2 + c_2,
$$

where

$$
c_1 := 8 \, L_{\mathcal{F}}^2 \, \frac{1}{\gamma^4} \, (\gamma^2 + 2T) \, c_P^2
$$

$$
c_2 := \frac{1}{\gamma^4} \, (\gamma^2 + 2T) \left[8 \, L_{\mathcal{F}}^2 \, ||u_m^0||_H^2 + 2 \, ||\tau||_H^2 \right] + ||\nabla u_m^0||_H^2 + ||w_m^0||_H^2 \left(\frac{2}{\gamma^2} + \frac{4}{\gamma^3} \, T \right) + k \sum_{n=1}^{j} ||f_m^n||_H^2.
$$

A discrete version of the Gronwall lemma then yields the first desired a priori estimates, for any $j \in \{1, \ldots, m\}$

$$
k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_{L^2(\Omega)}^2 + ||\nabla u_m^j||_{L^2(\Omega)}^2 \le \text{ constant (independent of } m); \tag{4.2.18}
$$

therefore, combining (4.2.15) and (4.2.18) we also deduce this second a priori estimate

$$
k \sum_{n=1}^{j} \left|\left| \frac{w_m^n - w_m^{n-1}}{k} \right|\right|_{L^2(\Omega)}^2 \le \text{ constant (independent of } m.) \tag{4.2.19}
$$

➜ PASSAGE TO THE LIMIT AND CONCLUSIONS.

At this point we introduce some notations in order to pass from the discretized problem to the continuous one. A.e. in Ω, let $w_m(x, \cdot)$ be the linear time interpolate of $w_m(x, nk) := w_m^n(x)$ for $n = 0, \ldots, m$; moreover set $\bar{u}_m(x,t) := u_m^n(x)$ if $(n-1)k < t \le nk$ for $n = 1, \ldots, m$ and define $\bar{w}_m$, $\bar{z}_m$ and $\bar{f}_m$ in a similar way. Thus (4.2.7) yields

$$\begin{cases} \dfrac{\partial u_m}{\partial t} + \dfrac{\partial w_m}{\partial t} + A\bar{u}_m = \bar{f}_m \\ \gamma \dfrac{\partial w_m}{\partial t} + \bar{w}_m = \bar{z}_m \end{cases} \tag{4.2.20}$$

while (4.2.18) and (4.2.19) become

$$\begin{aligned} ||u_m||_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)} &\leq \text{ constant (independent of } m) \\ ||\bar{u}_m||_{L^\infty(0,T;V)} &\leq \text{ constant (independent of } m) \\ ||w_m||_{H^1(0,T;L^2(\Omega))} &\leq \text{ constant (independent of } m) \\ ||\bar{w}_m||_{L^2(Q)} &\leq \text{ constant (independent of } m). \end{aligned}$$

The a priori estimates we found allow us to conclude that there exist u and w such that, possibly taking $m \to +\infty$ along a subsequence,

$$\begin{array}{lll} u_m \to u & \text{weakly star in} & H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \\ \bar{u}_m \to u & \text{weakly star in} & L^\infty(0,T;V) \\ w_m \to w & \text{weakly in} & H^1(0,T;L^2(\Omega)) \\ \bar{w}_m \to w & \text{weakly in} & L^2(Q). \end{array}$$

On the other hand, the a priori estimates we found immediately yield

$$\left|\left| \frac{\partial}{\partial t}(u_m + w_m) \right|\right|_{L^2(0,T;V')} \leq \text{ constant (independent of } m).$$

This is enough to conclude that, possibly taking $m \to +\infty$ along a subsequence

$$\frac{\partial}{\partial t}(u_m + w_m) \to \frac{\partial}{\partial t}(u + w) \quad \text{weakly in } L^2(0,T;V').$$

Hence, taking $m \to +\infty$ in the first equation of (4.2.20) we get

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} + Au = f \qquad \text{in } L^2(0,T;V'). \tag{4.2.21}$$

We only have to pass to the limit in the second equation of (4.2.20); in order to do this we notice that, by comparison

$$||\bar{z}_m||_{L^2(Q)} \leq ||\bar{w}||_{L^2(Q)} + \gamma \left|\left| \frac{\partial w_m}{\partial t} \right|\right|_{L^2(Q)} \leq \text{ constant (independent of } m),$$

thus there exists z such that, up to subsequences, $\bar{z}_m \to z$ weakly in $L^2(Q)$. This allows us to pass to the limit also in the second equation, getting

$$\gamma \frac{\partial w}{\partial t} + w = z$$

in $L^2(Q)$ and therefore in $L^2(0,T;V')$. Using the strong continuity of the operator $\overline{\mathcal{F}}$ and following the procedure established in the previous chapters based on some interpolation results, it is also quite easy to find that actually $z = \overline{\mathcal{F}}(u)$. This finishes the proof. □

Remark 4.2.3. It would be an interesting point to ask what happens if $\gamma \to 0$ in our model problem. More in detail, let us fix any $\gamma > 0$ and let (u_γ, w_γ) be a solution of Problem 4.2.1. The point is: can we obtain the existence of a pair of functions (u, w) such that $u_\gamma \to u$ and $w_\gamma \to w$ in some suitable topology and (u, w) is a solution of system (4.0.1)? At a first analysis, this doesn't seem possible as the constants obtained in the a priori estimates (4.2.18) and (4.2.19) depend on γ in the following way: $\lim_{\gamma \to 0} c(\gamma) = +\infty$. Therefore at least using the time approximation scheme as a possible approach to find existence of solutions of our model equation, this problem seems hard to be solved (see also Remark 4.0.9).

4.3. Second system of P.D.E.s

4.3.1. Functional setting and statement of the model problem

Suppose to have an open bounded set $\Omega \subset \mathbb{R}^N$, $N \geq 1$ of class $\mathcal{C}^1$ and set $Q := \Omega \times (0,T)$. Let us consider an operator $\overline{\mathcal{G}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ which is causal, Lipschitz continuous, bounded and piecewise monotone according to (2.1.11) (2.1.12), (2.1.13) and (3.1.4). We set

$$\mathcal{H}_{\overline{\mathcal{G}}}(v(\cdot,0)) := [\overline{\mathcal{G}}(v)](\cdot,0)(\in \mathcal{M}(\Omega)) \qquad \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])).$$

We describe now the functional setting we will use in the following.
The starting point of our development will be the injection of the space $L^2(\Omega)$ into the space $H^{-1}(\Omega)$. More precisely we take the map $j : L^2(\Omega) \to H^{-1}(\Omega)$ which acts in the following way

$$_{H^{-1}(\Omega)}\langle j\,(f), \varphi\rangle_{H^1_0(\Omega)} := \int_\Omega f\ \varphi\, dx \qquad \forall\, f \in L^2(\Omega),\ \forall\, \varphi \in H^1_0(\Omega). \tag{4.3.1}$$

It is not difficult to see that j is a continuous and dense injection, i.e. $L^2(\Omega)$ is a linear subspace of $H^{-1}(\Omega)$ and it is dense with respect to the strong topology of $H^{-1}(\Omega)$. By Theorem A.6.1 we have that $(H^{-1}(\Omega))'$ can be identified with a linear subspace of $(L^2(\Omega))'$, i.e. $(H^{-1}(\Omega))' \subset (L^2(\Omega))'$ with continuous injection (let us call this map j^*). More precisely we identify functionals with their restrictions in the following way

$$_{(L^2(\Omega))'}\langle j^*\psi, f\rangle_{L^2(\Omega)} :=_{(H^{-1}(\Omega))'} \langle \psi, j(f)\rangle_{H^{-1}(\Omega)} \qquad \forall\, \psi \in (H^{-1}(\Omega))',\ \forall\, f \in L^2(\Omega). \tag{4.3.2}$$

In the following we will avoid to write each time j, j^* when it will be clear from the context, in order to simplify the notations. So for example (4.3.2) will simply become

$$ {}_{(L^2(\Omega))'}\langle \psi, f\rangle_{L^2(\Omega)} :={}_{(H^{-1}(\Omega))'}\langle \psi, f\rangle_{H^{-1}(\Omega)}, \qquad \forall\, \psi \in (H^{-1}(\Omega))',\ \forall\, f \in L^2(\Omega). $$

Now we introduce the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined as follows

$$ {}_{H^{-1}(\Omega)}\langle Au, v\rangle_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall u, v \in H_0^1(\Omega); \tag{4.3.3} $$

so it is clear that $Au = -\triangle u \left(:= -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \right)$ in the sense of distributions.
At this point we remark that, for any $k > 0$, the operator

$$ (I + kA) : H_0^1(\Omega) \to H^{-1}(\Omega) $$

is an isomorphism (for example, we refer to [16], Section 6.2.2); this allows us to consider the space $H^{-1}(\Omega)$ endowed with the scalar product

$$ (u, v)_{H^{-1}(\Omega)} :={}_{H^{-1}(\Omega)}< u, (I + kA)^{-1} v >_{H_0^1(\Omega)} . \tag{4.3.4} $$

It is clear, using (4.3.1), that

$$ (u, v)_{H^{-1}(\Omega)} := \int_\Omega u\ (I + kA)^{-1} v \, dx \qquad \forall\, u \in L^2(\Omega). $$

At this point we identify the space $H^{-1}(\Omega)$ to its dual by means of the Riesz operator $\mathcal{R}$: $H^{-1}(\Omega) \to (H^{-1}(\Omega))'$ which acts in the following way

$$ {}_{(H^{-1}(\Omega))'}\langle \mathcal{R}u, v\rangle_{H^{-1}(\Omega)} := (u, v)_{H^{-1}(\Omega)} \qquad \forall\, u, v \in H^{-1}(\Omega). \tag{4.3.5} $$

Let us remark that with this identification we immediately get, (omitting from now on also the Riesz operator $\mathcal{R}$ for the sake of simplicity),

$$ \begin{aligned} {}_{(L^2(\Omega))'}\langle \psi, f\rangle_{L^2(\Omega)} &\overset{(4.3.2),(4.3.5)}{=} (\psi, f)_{H^{-1}(\Omega)} = (f, \psi)_{H^{-1}(\Omega)} \overset{(4.3.4)}{=} {}_{H^{-1}(\Omega)}\langle f, (I + kA)^{-1}\psi\rangle_{H_0^1(\Omega)} \\ &\overset{(4.3.1)}{=} \int_\Omega f\ (I + kA)^{-1}\psi \, dx \qquad \forall\, f \in L^2(\Omega),\ \forall\, \psi \in H^{-1}(\Omega), \end{aligned} \tag{4.3.6} $$

where we also used the fact that the scalar product is symmetric.
As $L^2(\Omega) \subset H^{-1}(\Omega)$ with continuous and dense injection, we then have the Hilbert triplet

$$ L^2(\Omega) \subset H^{-1}(\Omega) \equiv (H^{-1}(\Omega))' \subset (L^2(\Omega))' \tag{4.3.7} $$

with continuous and dense injections. Finally, we consider $H_0^1(\Omega)$ endowed with the norm $||u||_{H_0^1(\Omega)} := ||\nabla u||_{L^2(\Omega)}$.

We assume that $u^0, v^0, w^0 := \mathcal{H}_{\overline{\mathcal{G}}}(u^0) \in H_0^1(\Omega)$, and that $f \in L^2(0, T; H^{-1}(\Omega))$.

Problem 4.3.1. *We search for two functions* $u \in H^1(0,T;H^1_0(\Omega))$ *and* $v \in L^2(0,T;H^1_0(\Omega))$ *such that* $\overline{\mathcal{G}}(u) \in H^1(\Omega;\mathcal{C}^0([0,T]))$ *and for any* $\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ *with* $\psi(\cdot,T) = 0$ *a.e. in* Ω

$$\begin{aligned}&\int_0^T \int_\Omega -(v + u + \overline{\mathcal{G}}(u)) \frac{\partial \psi}{\partial t}\, dx\, dt + \int_0^T \int_\Omega (\nabla \overline{\mathcal{G}}(u) + \nabla v) \cdot \nabla \psi \, dx\, dt \\ &= \int_0^T {}_{H^{-1}(\Omega)}\langle f, \psi \rangle_{H^1_0(\Omega)}\, dt + \int_\Omega [v^0(x) + u^0(x) + w^0(x)]\, \psi(x,0)\, dx.\end{aligned} \tag{4.3.8}$$

$$-\int_0^T \int_\Omega u \frac{\partial \psi}{\partial t}\, dx\, dt = \int_0^T \int_\Omega v\, \psi\, dx\, dt + \int_\Omega u^0(x)\, \psi(x,0)\, dx. \tag{4.3.9}$$

Interpretation. The variational equations (4.3.8) and (4.3.9) yield

$$\begin{cases} \dfrac{\partial}{\partial t}(v + u + \overline{\mathcal{G}}(u)) - \triangle(\overline{\mathcal{G}}(u) + v) = f \\ \dfrac{\partial u}{\partial t} = v \end{cases} \qquad \text{in } \mathcal{D}'(0,T;H^{-1}(\Omega)) \tag{4.3.10}$$

thus, by comparison,

$$\frac{\partial}{\partial t}[v + u + \overline{\mathcal{G}}(u)] \in L^2(0,T;H^{-1}(\Omega))$$

so $v + u + \overline{\mathcal{G}}(u) \in H^1(0,T;H^{-1}(\Omega))$ and (4.3.10) holds in $H^{-1}(\Omega)$ a.e. in $(0,T)$. Hence, integrating by parts in time in (4.3.8) we get

$$[v + u + \overline{\mathcal{G}}(u)]_{|t=0} = v^0 + u^0 + w^0 \ \text{ in } H^{-1}(\Omega) \tag{4.3.11}$$

in the sense of traces. In turn (4.3.10) and (4.3.11) yield (4.3.8) and (4.3.9) and the two formulations are equivalent.

4.3.2. An existence result

Theorem 4.3.2. *(Existence)*
Let us consider an operator $\overline{\mathcal{G}} : \mathcal{M}(\Omega;\mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega;\mathcal{C}^0([0,T]))$ *which is causal, Lipschitz continuous, bounded and piecewise monotone according to* (2.1.11), (2.1.12), (2.1.13) *and* (3.1.4). *Moreover assume that*

$$f \in L^2(Q), \ \ u^0, w^0, v^0 \in H^1_0(\Omega). \tag{4.3.12}$$

Then Problem 4.3.1 *admits at least one solution*

$$u \in H^1(0,T;H^1_0(\Omega)) \qquad v \in L^2(0,T;H^1_0(\Omega)).$$

Proof. ➜ APPROXIMATION VIA IMPLICIT TIME DISCRETIZATION SCHEME.

We fix $m \in \mathbb{N}$ and set $k := T/m$; as at the end of the proof k is supposed to go to zero, it is not restrictive to assume

$$k < \frac{1}{\sqrt{2}\,L}, \tag{4.3.13}$$

where L is the Lipschitz constant of $\overline{\mathcal{G}}$, introduced in (2.1.12). For any $n = 1, \dots, m$, we consider $f_m^n(x) := f(x, nk)$, a.e. in Ω, $u_m^0(x) := u^0(x)$, $w_m^0(x) := w^0(x)$ and $v_m^0(x) := v^0(x)$. We would like to solve the following problem

Problem 4.3.3. *To find $u_m^n \in H_0^1(\Omega)$ and $v_m^n \in H_0^1(\Omega)$ for any $n = 1, \dots m$, such that, if $u_m(x, \cdot)$ is the linear time interpolate of $u_m(x, nk) := u_m^n(x)$, for $n = 1, \dots, m$, a.e. in Ω and $w_m^n := [\overline{\mathcal{G}}(u_m)](x, nk)$ for $n = 1, \dots, m$, a.e. in Ω, then*

$$\begin{cases} \dfrac{v_m^n - v_m^{n-1}}{k} + \dfrac{u_m^n - u_m^{n-1}}{k} + \dfrac{w_m^n - w_m^{n-1}}{k} + A\, w_m^n + A\, v_m^n = f_m^n & \text{in } H^{-1}(\Omega), \\ v_m^n := \dfrac{u_m^n - u_m^{n-1}}{k}, & \text{in } L^2(\Omega). \end{cases} \tag{4.3.14}$$

Working by induction, after noticing that the couple $(u_m^0, v_m^0) = (u^0, v^0)$ is known, for any $n \in \{1, \dots, m\}$ we suppose to know also the couples $(u_m^1, v_m^1), \dots, (u_m^{n-1}, v_m^{n-1})$; the problem is now to determine the pair (u_m^n, v_m^n).
We remark once more that the causality property of the operator $\overline{\mathcal{G}}$ allows us to say that the quantity $w_m^n(x) = [\overline{\mathcal{G}}(u_m)](x, nk)$ just depends on the end point of the linear time interpolate u_m, i.e. there exist a function $G_m^n : \mathbb{R} \times \Omega \to \mathbb{R}$ such that

$$w_m^n(x) = [\overline{\mathcal{G}}(u_m)](x, nk) := G_m^n(u_m^n(x), x) \qquad \text{a.e. in } \Omega;$$

this is due to the fact that $u_m(x, \cdot)_{|[0,(n-1)k]}$ is known. At this point we can introduce an operator $\Phi : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$ acting in the following way $\Phi(v) := G_m^n(v(\cdot), \cdot)$; the operator Φ, despite the notation, depends on the choice of m and n. It turns out that $\Phi : L^2(\Omega) \to L^2(\Omega)$ is strongly continuous and inherits the properties (2.1.12) and (2.1.13) of the operator $\overline{\mathcal{G}}$ in the sense that the following two inequalities, which will be needed later, hold

$$||\Phi(u_1) - \Phi(u_2)||_{L^2(\Omega)} \le L\, ||u_1 - u_2||_{L^2(\Omega)} \qquad \forall\, u_1, u_2 \in L^2(\Omega) \tag{4.3.15}$$

and

$$|\Phi(u)(x) - \Phi(u)(y)| \le L\, |u(x) - u(y)| \quad \forall u \in L^2(\Omega), \text{ for a.a. } x, y \in \Omega. \tag{4.3.16}$$

We would like to rewrite the discrete system we are going to solve by distinguishing the quantities which are known from the ones we have to determine. We have

$$v_m^n - v_m^{n-1} + u_m^n - u_m^{n-1} + w_m^n - w_m^{n-1} + k\, A w_m^n + k\, A v_m^n = k\, f_m^n$$

which yields

$$v_m^n + u_m^n + w_m^n + \, k\, Aw_m^n + \, k\, Av_m^n = k\, f_m^n + v_m^{n-1} + u_m^{n-1} + w_m^{n-1};$$

now, using the fact that $v_m^n = \dfrac{u_m^n - u_m^{n-1}}{k}$, we have

$$u_m^n - u_m^{n-1} + ku_m^n + kw_m^n + k^2 Aw_m^n + kAu_m^n = k^2 f_m^n + kv_m^{n-1} + kw_m^{n-1} + ku_m^{n-1} + kAu_m^{n-1}$$

which is equivalent to

$$(k+1)\, u_m^n + kw_m^n + k^2 Aw_m^n + kAu_m^n = k^2 f_m^n + kv_m^{n-1} + kw_m^{n-1} + (k+1)\, u_m^{n-1} + kAu_m^{n-1};$$

setting

$$k^2 f_m^n + kv_m^{n-1} + kw_m^{n-1} + (k+1)\, u_m^{n-1} + kAu_m^{n-1} =: G_m^n$$

it turns out that we have to find $u_m^n \in H_0^1(\Omega)$ such that

$$(k+1)\, u_m^n + kw_m^n + k^2 Aw_m^n + kAu_m^n = G_m^n \qquad \text{in } H^{-1}(\Omega).$$

Removing the fixed indexes m and n for the sake of simplicity, we have then to solve the following problem.

Problem 4.3.4. *To find $u \in H_0^1(\Omega)$ such that*

$$k\, u + [I + kA]\, (u + k\, \Phi(u)) = G \qquad \text{in } H^{-1}(\Omega). \tag{4.3.17}$$

➜ SOLVING PROBLEM 4.3.4.

This is the crucial point where we need the Hilbert triplet (4.3.7) and the functional setting we introduced in the previous pages. Before entering in the heart of the problem, we notice that (4.3.17) is equivalent to

$${}_{H^{-1}(\Omega)}\langle k\, u, \tilde{\varphi}\rangle_{H_0^1(\Omega)} + {}_{H^{-1}(\Omega)}\langle (I{+}k\, A)\, (u{+}\, k\, \Phi(u)), \tilde{\varphi}\rangle_{H_0^1(\Omega)} = {}_{H^{-1}(\Omega)}\, \langle G, \tilde{\varphi}\rangle_{H_0^1(\Omega)} \qquad \forall\, \tilde{\varphi} \in H_0^1(\Omega).$$

On the other hand, the operator $(I + k\, A) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism and therefore, for all $\tilde{\varphi} \in H_0^1(\Omega)$, there exists a unique $\varphi \in H^{-1}(\Omega)$ such that $\tilde{\varphi} = (I + k\, A)^{-1}\, \varphi$. So recalling the definition of the scalar product (4.3.4) we put on the space $H^{-1}(\Omega)$, we have that (4.3.17) is also equivalent to

$$(k\, u, \varphi)_{H^{-1}(\Omega)} \; + \; ((I + k\, A)(u + k\, \Phi(u)), \varphi)_{H^{-1}(\Omega)} \; = \; (G, \varphi)_{H^{-1}(\Omega)} \qquad \forall\, \varphi \in H^{-1}(\Omega). \tag{4.3.18}$$

We reach our goal if we solve (4.3.18) for all $\varphi \in L^2(\Omega)$. On the other hand, if (4.3.18) holds for all $\varphi \in L^2(\Omega)$, then it is equivalent to the following one

$$k \int_\Omega \varphi\, [(I + kA)^{-1}\, u]\, dx + \int_\Omega \varphi\, (u + \, k\, \Phi(u))\, dx = \int_\Omega \varphi\, [(I + kA)^{-1} G]\, dx \qquad \forall\, \varphi \in L^2(\Omega); \tag{4.3.19}$$

that's why, we are going to solve (4.3.19) in order to conclude.

❒ step 1. We fix $z \in L^2(\Omega)$ and we consider the problem of finding $u \in L^2(\Omega)$ such that the following equation is fulfilled

$$k \int_\Omega \varphi\,[(I+kA)^{-1}u]\,dx + \int_\Omega u\,\varphi\,dx = \int_\Omega ((I+kA)^{-1}G - k\,\Phi(z))\,\varphi\,dx \qquad \forall\,\varphi \in L^2(\Omega). \tag{4.3.20}$$

We set

$$a(u,\varphi) := k \int_\Omega \varphi\,(I+k\,A)^{-1}\,u\,dx + \int_\Omega u\,\varphi\,dx$$

and $\tilde{G} \in (L^2(\Omega))'$ defined in the following way

$${}_{(L^2(\Omega))'}\langle \tilde{G},\varphi\rangle_{L^2(\Omega)} := \int_\Omega ((I+k\,A)^{-1}\,G - k\,\Phi(z))\,\varphi\,dx \qquad \forall\,\varphi \in L^2(\Omega).$$

We remark that $\tilde{G}$ is known, so our aim is to find $u \in L^2(\Omega)$ such that

$$a(u,\varphi) = {}_{(L^2(\Omega))'}\langle \tilde{G},\varphi\rangle_{L^2(\Omega)} \qquad \forall\,\varphi \in L^2(\Omega).$$

Let us do some considerations concerning the form $a : L^2(\Omega)\times L^2(\Omega) \to \mathbb{R}$. It is not hard to see that this form is bilinear, symmetric and positive definite (in fact $a(u,u) := k\,||u||^2_{H^{-1}(\Omega)} + ||u||^2_{L^2(\Omega)} \geq 0$ and $a(u,u) > 0$ if $u \neq 0$). So it defines a scalar product on $L^2(\Omega)$, more precisely, setting

$$|||u|||^2_{L^2(\Omega)} := ((u,u))_{L^2(\Omega)} := a(u,u) \qquad \forall\,u \in L^2(\Omega),$$

we have that $|||\cdot|||_{L^2(\Omega)}$ is a norm in $L^2(\Omega)$ and it is equivalent to the standard norm in $L^2(\Omega)$. At this point we can conclude by the Riesz Theorem (see Theorem A.10.1) and so equation (4.3.20) admits a unique solution $u \in L^2(\Omega)$.

❒ step 2. Now we introduce the following operator

$$J : L^2(\Omega) \to L^2(\Omega)$$

which associates to any $z \in L^2(\Omega)$ the corresponding solution $u \in L^2(\Omega)$ of the equation (4.3.20). Let us consider any couple of data $z_1, z_2 \in L^2(\Omega)$ and let $u_1 = J(z_1),\ u_2 = J(z_2)$ be the corresponding solutions. We have, for all $\varphi \in L^2(\Omega)$

$$k \int_\Omega \varphi\,(I+kA)^{-1}(u_1-u_2)\,dx + \int_\Omega (u_1-u_2)\,\varphi\,dx = -k \int_\Omega (\Phi(z_1)-\Phi(z_2))\,\varphi\,dx.$$

At this point, we choose $\varphi = (u_1-u_2) \in L^2(\Omega)$ and develop the computations. We deduce

$$k\,||u_1-u_2||^2_{H^{-1}(\Omega)} + ||u_1-u_2||^2_{L^2(\Omega)} = k \int_\Omega (I+kA)^{-1}|u_1-u_2|^2\,dx + ||u_1-u_2||^2_{L^2(\Omega)}$$

$$\leq k\,||\Phi(z_1)-\Phi(z_2)||_{L^2(\Omega)}\,||u_1-u_2||_{L^2(\Omega)} \overset{(4.3.15)}{\leq} k\,L\,||z_1-z_2||_{L^2(\Omega)}\,||u_1-u_2||_{L^2(\Omega)}.$$

Therefore in particular we deduce

$$||J(z_1) - J(z_2)||_{L^2(\Omega)} = ||u_1 - u_2||_{L^2(\Omega)} \leq \, k\, L\, ||z_1 - z_2||_{L^2(\Omega)}.$$

Using (4.3.13), we easily have that the operator J has a unique fixed point, and therefore (4.3.19) admits a unique solution $u \in L^2(\Omega)$ (the fixed point of the operator J).

❒ STEP 3. In this step we show that actually equation (4.3.19) has a unique solution $u \in H^1_0(\Omega)$. First of all, for any given function f defined on an open set $\Omega \subset \mathbb{R}^N$, we introduce its extension to zero outside Ω in the following way

$$[f]_0(x) = \begin{cases} f(x) & x \in \Omega; \\ 0 & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

We recall that, if $f \in H^1(\Omega)$, then this does not implies that $[f]_0 \in H^1(\mathbb{R}^N)$. In this step we will use Proposition A.2.1, to which we refer for the notations therein. We then choose $h \in \mathbb{R}^N$; we recall that, from the previous steps, we are able to find $u \in L^2(\Omega)$ such that, for all $\varphi \in L^2(\Omega)$

$$k \int_\Omega \varphi\, [(I + kA)^{-1}\, u]\, dx + \int_\Omega (u + \, k\, \Phi(u))\, \varphi\, dx = \int_\Omega [(I + kA)^{-1}\, G]\, \varphi\, dx$$

which is of course equivalent to the following, which holds for all $\tilde{\varphi} \in L^2(\mathbb{R}^N)$,

$$k \int_{\mathbb{R}^N} \tilde{\varphi}\, [(I + kA)^{-1}\, [u]_0]\, dx + \int_{\mathbb{R}^N} ([u]_0 + \, k\, [\Phi(u)]_0)\, \tilde{\varphi}\, dx = \int_{\mathbb{R}^N} [(I + kA)^{-1}\, G]_0\, \tilde{\varphi}\, dx.$$

If we set $\tilde{G} := [(I + kA)^{-1}\, G]_0$ then from the previous equation we can certainly deduce

$$k \int_{\mathbb{R}^N} \tilde{\varphi}\, [(I + kA)^{-1}\, \delta_h[u]_0]\, dx + \int_{\mathbb{R}^N} (\delta_h[u]_0 + \, k\, \delta_h\, [\Phi(u)]_0)\, \tilde{\varphi}\, dx = \int_{\mathbb{R}^N} [\delta_h\, \tilde{G}]\, \tilde{\varphi}\, dx,$$

for all $\tilde{\varphi} \in L^2(\mathbb{R}^N)$, where the notation $\delta_h f(x) := f(x+h) - f(x)$ for all $f \in H^1(\Omega)$ is introduced in Proposition A.2.1. Now we choose $\tilde{\varphi} := \delta_h\, [u]_0 \in L^2(\mathbb{R}^N)$ in the previous equation and we get

$$k\, ||\, \delta_h[u]_0||^2_{H^{-1}(\mathbb{R}^N)} + ||\, \delta_h[u]_0||^2_{L^2(\mathbb{R}^N)} \leq \, k \int_{\mathbb{R}^N} |\, \delta_h[\Phi(u)]_0|\, |\, \delta_h[u]_0|\, dx + \int_{\mathbb{R}^N} |\, \delta_h\tilde{G}|\, |\, \delta_h[u]_0|\, dx$$

$$\leq k^2\, ||\, \delta_h[\Phi(u)]_0||^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\, ||\, \delta_h[u]_0||^2_{L^2(\mathbb{R}^N)} + \, ||\, \delta_h\tilde{G}||^2_{L^2(\mathbb{R}^N)}.$$

At this point we remark that (4.3.16) immediately yields

$$||\delta_h[\Phi(u)]_0||^2_{L^2(\mathbb{R}^N)} \leq \, L^2\, ||\delta_h[u]_0||^2_{L^2(\mathbb{R}^N)};$$

moreover $(I+kA)^{-1}G \in H^1_0(\Omega)$ so from the characterization of the space $H^1_0(\Omega)$ (more precisely from the fact that $(iv) \Rightarrow (vi)$ in Proposition A.2.1) we have that $\tilde{G} \in H^1(\mathbb{R}^N)$. This in turns implies (iii) in the case when $\Omega = \mathbb{R}^N$. We therefore deduce

$$||\delta_h[u]_0||^2_{L^2(\mathbb{R}^N)} \leq 2\,k^2\,||\delta_h[\Phi(u)]_0||^2_{L^2(\mathbb{R}^N)} + 2\,||\delta_h\tilde{G}||^2_{L^2(\mathbb{R}^N)} \leq 2\,k^2\,L^2||\delta_h[u]_0||_{L^2(\mathbb{R}^N)} + c\,|h|^2$$

and so, using (4.3.13), we get

$$||\delta_h[u]_0||^2_{L^2(\mathbb{R}^N)} \leq c\,|h|^2;$$

this tells us that $[u]_0 \in H^1(\mathbb{R}^N)$ and thus again by Proposition A.2.1, $u \in H^1_0(\Omega)$. This allows us to conclude this first part of the proof of our existence result, in particular we have deduced that equation (4.3.17) has a unique solution $u \in H^1_0(\Omega)$.

➜ A PRIORI ESTIMATES.

Let us consider the following equation

$$\frac{v^n_m - v^{n-1}_m}{k} + \frac{u^n_m - u^{n-1}_m}{k} + \frac{w^n_m - w^{n-1}_m}{k} + Aw^n_m + Av^n_m = f^n_m \qquad \text{in } H^{-1}(\Omega)$$

with the notations introduced in the previous steps; we now multiply it by $k\,v^n_m = u^n_m - u^{n-1}_m$ in the duality pairing ${}_{H^{-1}(\Omega)}\langle\cdot,\cdot\rangle_{H^1_0(\Omega)}$ then we sum for $n = 1,\dots,j$ for $j \in \{1,\dots,m\}$. In this part we only need the embedding of $L^2(\Omega)$ in $H^{-1}(\Omega)$ and in particular (4.3.1) together with the definition of the operator A (see (4.3.3)). We in particular don't need to use the following Hilbert triplet

$$H^1_0(\Omega) \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset H^{-1}(\Omega)$$

so there is no contradiction with the choice of our functional setting and in particular of our Hilbert triplet (4.3.7). Therefore we obtain

$$\begin{aligned}
&\sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle \frac{v^n_m - v^{n-1}_m}{k}, k\,v^n_m\rangle_{H^1_0(\Omega)} + \sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle \frac{u^n_m - u^{n-1}_m}{k}, u^n_m - u^{n-1}_m\rangle_{H^1_0(\Omega)}\\
&+ \sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle \frac{w^n_m - w^{n-1}_m}{k}, u^n_m - u^{n-1}_m\rangle_{H^1_0(\Omega)} + \sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle Aw^n_m, u^n_m - u^{n-1}_m\rangle_{H^1_0(\Omega)}\\
&+ \sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle Av^n_m, u^n_m - u^{n-1}_m\rangle_{H^1_0(\Omega)} = \sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle f^n_m, u^n_m - u^{n-1}_m\rangle_{H^1_0(\Omega)}.
\end{aligned}$$

We analyze all the terms one by one. First of all we have

$$\begin{aligned}
&\sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle \frac{v^n_m - v^{n-1}_m}{k}, k\,v^n_m\rangle_{H^1_0(\Omega)} = \sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle v^n_m - v^{n-1}_m, v^n_m\rangle_{H^1_0(\Omega)}\\
&\geq \frac{1}{2}\sum_{n=1}^{j}\int_\Omega (|v^n_m|^2 - |v^{n-1}_m|^2)\,dx = \frac{1}{2}\int_\Omega |v^j_m|^2\,dx - \frac{1}{2}\int_\Omega |v^0_m|^2\,dx.
\end{aligned}$$

Moreover

$$\sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle \frac{u_m^n - u_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_{H_0^1(\Omega)} = k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_{L^2(\Omega)}^2 .$$

On the other hand, using assumption (3.1.4), it is true that

$$\sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle \frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_{H_0^1(\Omega)} = k \sum_{n=1}^{j} \int_\Omega \left(\frac{w_m^n - w_m^{n-1}}{k} \right) \left(\frac{u_m^n - u_m^{n-1}}{k} \right) dx \geq 0.$$

Moreover

$$\sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle A w_m^n, u_m^n - u_m^{n-1}\rangle_{H_0^1(\Omega)} = k \sum_{n=1}^{j} \int_\Omega \nabla w_m^n \cdot \nabla v_m^n \, dx$$

$$\leq k \sum_{n=1}^{j} ||\nabla w_m^n||_{L^2(\Omega)}^2 + \frac{k}{4} \sum_{n=1}^{j} ||\nabla v_m^n||_{L^2(\Omega)}^2.$$

We also notice that

$$\sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle A v_m^n, u_m^n - u_m^{n-1}\rangle_{H_0^1(\Omega)} = k \sum_{n=1}^{j} ||\nabla v_m^n||_{L^2(\Omega)}^2$$

and that

$$\sum_{n=1}^{j} {}_{H^{-1}(\Omega)}\langle f_m^n, u_m^n - u_m^{n-1}\rangle_{H_0^1(\Omega)} = k \sum_{n=1}^{j} \int_\Omega f_m^n \left(\frac{u_m^n - u_m^{n-1}}{k} \right) dx$$

$$\leq k \sum_{n=1}^{j} ||f_m^n||_{L^2(\Omega)}^2 + \frac{k}{4} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_{L^2(\Omega)}^2 .$$

Therefore, summarizing, we can say that

$$\begin{aligned} &\frac{1}{2} \int_\Omega |v_m^j|^2 \, dx + \frac{3}{4} k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_{L^2(\Omega)}^2 + \frac{3}{4} k \sum_{n=1}^{j} ||\nabla v_m^n||_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2} \int_\Omega |v_m^0|^2 \, dx + k \sum_{n=1}^{j} ||\nabla w_m^n||_{L^2(\Omega)}^2 + k \sum_{n=1}^{j} ||f_m^n||_{L^2(\Omega)}^2. \end{aligned} \tag{4.3.21}$$

Assumption (2.1.13) implies that

$$|\nabla w_m^n| \leq \max_{k=1,\dots,n} |\nabla w_m^k| \leq L \max_{k=1,\dots,n} |\nabla u_m^k|.$$

At this point we introduce an auxiliary variable $\eta_m^n \in L^2(\Omega)$ such that, for all $n \in \{1, \dots, m\}$

$$\eta_m^0(x) := |\nabla u_m^0(x)|, \qquad \eta_m^n(x) := |\nabla u_m^0(x)| + \sum_{k=1}^{n} |\nabla u_m^k(x) - \nabla u_m^{k-1}(x)|, \qquad \text{a.e. in } \Omega.$$

It turns out that

$$\left|\frac{\eta_m^n - \eta_m^{n-1}}{k}\right| = \left|\frac{\nabla u_m^n - \nabla u_m^{n-1}}{k}\right| = |\nabla v_m^n|$$

and also that

$$\max_{k=1,\dots,n} |\nabla u_m^k| \le \eta_m^n;$$

this in turn entails that

$$|\nabla w_m^n| \le L \max_{k=1,\dots,n} |\nabla u_m^k| \le L\, \eta_m^n$$

and so, starting from (4.3.21), we may certainly deduce

$$\frac{3}{4}\, k \sum_{n=1}^{j} \left|\left|\frac{\eta_m^n - \eta_m^{n-1}}{k}\right|\right|^2_{L^2(\Omega)} \le \frac{1}{2}\, ||v_m^0||^2_{L^2(\Omega)} + L^2\, k \sum_{n=1}^{j} ||\eta_m^n||^2_{L^2(\Omega)} + k \sum_{n=1}^{j} ||f_m^n||^2_{L^2(\Omega)}.$$

On the other hand

$$||\eta_m^j||_{L^2(\Omega)} - ||\eta_m^0||_{L^2(\Omega)} = \sum_{n=1}^{j} (||\eta_m^n||_{L^2(\Omega)} - ||\eta_m^{n-1}||_{L^2(\Omega)})$$

$$\le k \sum_{n=1}^{j} \left|\left|\frac{\eta_m^n - \eta_m^{n-1}}{k}\right|\right|_{L^2(\Omega)} \le \sqrt{T} \left(k \sum_{n=1}^{j} \left|\left|\frac{\eta_m^n - \eta_m^{n-1}}{k}\right|\right|^2_{L^2(\Omega)} \right)^{1/2}$$

and this entails that

$$||\eta_m^j||^2_{L^2(\Omega)} \le C_1\, k \sum_{n=1}^{j} ||\eta_m^n||^2_{L^2(\Omega)} + C_2$$

where

$$C_1 := \frac{8}{3}\, T\, L^2 \qquad\qquad C_2 := 2\, ||\nabla u_m^0||^2_{L^2(\Omega)} + \frac{4}{3}\, T\, ||v_m^0||^2_{L^2(\Omega)} + \frac{8}{3}\, k \sum_{n=1}^{j} ||f_m^n||^2_{L^2(\Omega)}.$$

A discrete version of the Gronwall lemma then yields the following a priori estimate, for any $j \in \{1, \dots, m\}$

$$||v_m^j||^2_{L^2(\Omega)} + k \sum_{n=1}^{j} ||\nabla v_m^n||^2_{L^2(\Omega)} \le \text{ constant (independent of } m).$$

➜ PASSAGE TO THE LIMIT AND CONCLUSION.

Now it is time to pass from the discrete to the continuous setting. We have to introduce some further notations: a.e. in Ω we denote $w_m(x,\cdot)$ the linear time interpolate of $w_m^n(x, nk) := w_m^n(x)$ for $n = 0, \dots, m$; moreover we set $\bar{u}_m(x,t) := u_m^n(x)$ if $(n-1)k < t \le nk$ for $n = 1, \dots, m$. Finally we define in a similar way $v_m, \bar{w}_m, \bar{v}_m, \bar{f}_m, \bar{u}_m$. Thus (4.3.14) yields

$$\begin{cases} \dfrac{\partial v_m}{\partial t} + \dfrac{\partial u_m}{\partial t} + \dfrac{\partial w_m}{\partial t} + A\, \bar{w}_m + A\, \bar{v}_m = \bar{f}_m \\ \bar{v}_m = \dfrac{\partial u_m}{\partial t} \end{cases} \qquad \text{in } L^2(0,T; H^{-1}(\Omega)).$$

The a priori estimates we found on the other hand entail

$$||v_m||_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))} \leq \text{ constant (independent of } m)$$
$$||\bar{v}_m||_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))} \leq \text{ constant (independent of } m).$$

Therefore there exists v such that

$$v_m \to v \qquad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$
$$\bar{v}_m \to v \qquad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)).$$

From these facts we can also deduce the existence of u such that

$$u_m \to u \qquad \text{weakly star in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H_0^1(\Omega))$$

and

$$||\bar{u}_m||_{L^2(0,T;H_0^1(\Omega))} \leq \text{ constant (independent of } m).$$

Moreover we notice that

$$||\bar{w}_m||_{L^2(0,T;H_0^1(\Omega))} \overset{(2.1.13)}{\leq} c\,||\bar{u}_m||_{L^2(0,T;H_0^1(\Omega))} \leq \text{ costant (independent of } m)$$

and this allows us to obtain the existence of w such that

$$\nabla \bar{w}_m \to \nabla w \qquad \text{weakly in } L^2(Q).$$

The fact that actually $w = \overline{\mathcal{G}}(u)$ comes working as in the previous case, due to the right regularity of u_m and the good properties of the operator $\overline{\mathcal{G}}$.
This was the last thing we had to check; now we are able to conclude that the following holds

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial t} + \dfrac{\partial \overline{\mathcal{G}}(u)}{\partial t} + A\,\overline{\mathcal{G}}(u) + A\,v = f \\ v = \dfrac{\partial u}{\partial t} \end{cases} \qquad \text{in } L^2(0,T;H^{-1}(\Omega)).$$

and this finishes the proof. □

CHAPTER 5

Time-asymptotic behaviour for a class of P.D.E.s with hysteresis and Neumann boundary conditions

The aim of this chapter is to study the asymptotic behaviour of solutions of an initial and boundary value problem associated to the following parabolic model equation

$$\frac{\partial}{\partial t}(u + \overline{\mathcal{F}}(u)) - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in } (0,1) \times (0,+\infty),$$

where $(0,1)$ is the open bounded set of $\mathbb{R}$ (we choose $(0,1)$ for the sake of simplicity, but we could actually take any other open bounded interval of $\mathbb{R}$) and $\overline{\mathcal{F}}$ is a continuous hysteresis operator. Thus our development will take place in one space dimension; this because there are points in our treatment where the computations cannot be extended in the more general setting of several space dimensions.
In literature, for the same model equation, a result has been established by Krejčí (this proposition can be found in [39], Section IX.4) dealing with Dirichlet boundary conditions. Here the new fact is that we deal with Neumann boundary conditions.
We first introduce a weak formulation in Sobolev spaces for a Cauchy problem associated to the previous model equation. After checking that our model problem admits a unique solution defined in $(0,1) \times (0,+\infty)$, it is not too hard to prove an exponential decay estimate in $L^2(0,1)$ for the function $\partial_x u$. This can be done either if $\overline{\mathcal{F}}$ is a Preisach operator or if $\overline{\mathcal{F}}$ is a Prandtl-Ishlinskiĭ operator of play type. This is of course not enough for deducing an exponential decay for the solution u, as we have Neumann boundary conditions and so Poincaré inequality does not hold any more. However further improvements can be done and at this point we distinguish the two cases, i.e. the case when $\overline{\mathcal{F}}$ is a Prandtl-Ishlinskiĭ operator and the case when $\overline{\mathcal{F}}$ is a Preisach operator.

➜ In the case when $\overline{\mathcal{F}}$ is a Preisach operator, we are able to prove that the solution u converges pointwise as $t \to \infty$ to a constant u_∞, i.e.

$$\lim_{t\to\infty} u(x,t) = u_\infty \qquad \text{for all } x \in [0,1].$$

This turns to be the most delicate part of the proof of our results. This is proved by contradiction, exploiting in a very essential way the connection between the Preisach operator introduced in (1.5.8) and the play operator, in particular Lemma 1.2.5 is used, where a representation of the play starting from a given memory configuration is outlined. The essential point is that these computations can be done only in one space dimension (the generalization to several space dimensions is an open problem). This result is important by itself but it also provides very nice consequences: in fact for small amplitude oscillations of the solution around u_∞, we have that $u(x,t)$ does not leave the convexity domain of the Preisach operator (see Theorem 1.5.15), so we are allowed to differentiate our equation, test by $\partial_t u$ and we obtain, by the usual convexity argument, an exponential decay in $L^2(0,1)$ for the function $\partial_t u$ (and therefore for $\partial_x^2 u$). We remark that, in order to differentiate our equation, we should get some higher regularity of the solution of our model problem that actually we don't have. For this reason we will differentiate an approximate equation obtained by a semi-discretization in space of the model equation we deal with, passing then to the limit the corresponding estimates which turn to be uniform with respect to the space parameter. It is relevant to underline that the time-discretization scheme, which is usually employed for having the existence of solutions of parabolic equations, may fail in general as to concern asymptotic results; the difficulty of the time discrete schemes is that one needs decay estimates which are uniform with respect to the discretized parameter and this is often hard to prove.

We also want to point out that u_∞ is a constant and all the computations are done in order to show that u_∞ does not depend on x; this result is expected from the general theory of dynamical system (see also Remark 5.2.6).

➜ If $\overline{\mathcal{F}}$ is a Prandtl-Ishlinskiĭ operator, we can exploit the convexity of the hysteresis loops of $\overline{\mathcal{F}}$ and we can directly differentiate our model equation, test by $\partial_t u$ and get the further estimate which leads to an exponential decay in $L^2(0,1)$ of the functions $\partial_t u$ and $\partial_x^2 u$.

The plan of the chapter is the following: first of all we present some preliminary propositions which will be needed later; for two of them we present also the proof; in fact even if these results have been sometimes used in literature, it is hard to find explicitly their proof. After that we study the case when $\overline{\mathcal{F}}$ is a Preisach operator and in the last part of the chapter we present a theorem dealing with $\overline{\mathcal{F}}$ being a Prandtl-Ishlinskiĭ operator.

All these results have been obtained in collaboration with Prof. Pavel Krejčí.

5.1. Some preliminary results

5.1.1. First result: an auxiliary lemma

Before entering the heart of the problems, we state and prove the following preliminary lemma.

Lemma 5.1.1. *Consider a function $h : (0, \infty) \to \mathbb{R}$ such that:*

$$h \text{ is nonincreasing;} \tag{5.1.1}$$

$$h \in L^1(0, \infty). \tag{5.1.2}$$

Then we can deduce that

$$\lim_{t \to \infty} t\, h(t) = 0. \tag{5.1.3}$$

Proof. Suppose by contradiction that (5.1.3) does not hold. Then there exists $\varepsilon > 0$ such that $\forall\, M > 0,\ \exists\, t > M$ such that

$$t\, h(t) \geq \varepsilon.$$

This implies that there exists a sequence of time instants $0 < t_0 < t_1 < t_2 < \dots$, such that $\lim\limits_{n \to \infty} t_n = \infty$ and

$$t_n\, h(t_n) \geq \varepsilon \qquad\qquad \forall\, n \in \mathbb{N}. \tag{5.1.4}$$

At this point

$$c \overset{(5.1.2)}{\geq} \int_0^\infty h(t)\, dt = \sum_{n=1}^{\infty} \int_{t_{n-1}}^{t_n} h(t)\, dt \overset{(5.1.1)}{\geq} \sum_{n=1}^{\infty} h(t_n)(t_n - t_{n-1}) = \sum_{n=1}^{\infty} h(t_n)\, t_n \left(1 - \frac{t_{n-1}}{t_n}\right)$$
$$\overset{(5.1.4)}{\geq} \varepsilon \sum_{n=1}^{\infty} \left(1 - \frac{t_{n-1}}{t_n}\right).$$

Let us set

$$a_n := 1 - \frac{t_{n-1}}{t_n}.$$

From the previous analysis, it is clear that $\sum\limits_{n=1}^{\infty} a_n$ is a convergent series and therefore $a_n \to 0$ (as the general term of a convergent series goes to zero).

On the other hand, let us consider the function

$$f(x) = \begin{cases} \dfrac{\log(1+x)}{x} & x \in (-1, 0) \\ 1 & x = 0 \end{cases}$$

which is continuous in $(-1, 0]$ and positive. As $a_n \to 0$, then $f(-a_n) \to 1$ and so there exists $\bar{n}$ such that for all $n \geq \bar{n}$ for example $f(-a_n) \leq 2$. The comparison criterium for series with non-negative terms yields

$$\sum_{n=1}^{\infty} a_n\, f(-a_n) \;\leq\; 2 \sum_{n=1}^{\infty} a_n \;\leq\; c$$

and this implies that the series $\sum_{n=1}^{\infty} a_n\, f(-a_n)$ is convergent. But we have also the following

$$\sum_{n=1}^{\infty} a_n\, f(-a_n) = \sum_{n=1}^{\infty} - \left(a_n \, \frac{\log(1-a_n)}{a_n} \right) = \sum_{n=1}^{\infty} - \log(1-a_n)$$

$$= - \sum_{n=1}^{\infty} [\log(t_{n-1}) - (\log t_n)] = - \log t_0 + \lim_{n \to \infty} \log t_n = +\infty$$

and this is clearly in contradiction with what we said before. This concludes the proof. □

5.1.2. Second result: Hilpert inequality for the Preisach operator

First of all, let $\mathcal{W}[\lambda, u]$ be the Preisach operator introduced in (1.5.8), i.e.

$$\mathcal{W}[\lambda, u](t) := \int_0^{\infty} g(r, \wp_r[\lambda, u](t))\, dr \tag{5.1.5}$$

for any $\lambda \in \Lambda_0$, (Λ_0 is introduced in (1.2.4)), $u \in \mathcal{C}^0([0,T])$ and $t \in [0,T]$; we recall moreover that the generating function $g : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is chosen as in (1.5.7) and that $\wp_r$ is the play operator with threshold $r > 0$ and initial configuration λ introduced in (1.2.5).
Now we are going to state an important proposition which have a central role for the uniqueness of the solution of our model problem.

Proposition 5.1.2. *(Hilpert inequality for the Preisach operator)*
For $u, v \in W^{1,1}(0,T)$ and for a.a. $t \in [0,T]$ we have

$$\left[\frac{d}{dt} (\mathcal{W}[\lambda, u] - \mathcal{W}[\lambda, v]) \right] \mathrm{sign}(u - v) \geq \frac{d}{dt} \int_0^{\infty} |g(r, \wp_r[\lambda, u]) - g(r, \wp_r[\lambda, v])|\, dr, \tag{5.1.6}$$

where $\mathcal{W}[\lambda, u]$ is the Preisach operator as in (5.1.5) *and where we denoted with* sign *the sign function.*

Proof. We set $\xi_r(t) = \wp_r[\lambda, u](t)$ and $\eta_r(t) = \wp_r[\lambda, v](t)$. Then, using the definition of the play and the stop operators given in (1.2.3), we readly have

$$\begin{aligned} &\dot{\xi}_r(u - \xi_r - z_1) \geq 0, \\ &\dot{\eta}_r(v - \eta_r - z_2) \geq 0, \end{aligned} \qquad \text{for any } z_1, z_2 \in [-r, r], \text{ for a.a. } t \in [0,T].$$

As the function $\psi(r,z)=\partial_z g(r,z)$ is non-negative, this immediately implies that

$$\begin{aligned}&\left[\frac{d}{dt}g(r,\xi_r)\right](u-\xi_r-z_1)\geq 0,\\&\left[\frac{d}{dt}g(r,\eta_r)\right](v-\eta_r-z_2)\geq 0\end{aligned}\qquad \text{for any } z_1,z_2\in[-r,r], \text{ for a.a. } t\in[0,T].$$

We choose in the previous inequalities $z_1=v-\eta_r$ and $z_2=u-\xi_r$ and then we take the sum of the resulting inequalities. We obtain

$$\left[\frac{d}{dt}(g(r,\xi_r)-g(r,\eta_r))\right][(u-v)-(\xi_r-\eta_r)]\geq 0.$$

Moreover we can certainly notice that the previous inequality is equivalent to the following one

$$\left[\frac{d}{dt}(g(r,\xi_r)-g(r,\eta_r))\right][f(u-v)-f(\xi_r-\eta_r)]\geq 0$$

for any f nondecreasing function. We can take in particular $f(z)=\operatorname{sign}(z)$ and use the following

$$\operatorname{sign}(\xi_r-\eta_r)=\operatorname{sign}(g(r,\xi_r)-g(r,\eta_r))$$

which is an immediate consequence of the definition of g and of the fact that the function $\psi(r,z)=\partial_z g(r,z)$ is non-negative. Summarizing we obtain

$$\left[\frac{d}{dt}(g(r,\xi_r)-g(r,\eta_r))\right]\operatorname{sign}(u-v)\geq\frac{d}{dt}|g(r,\xi_r)-g(r,\eta_r)|$$

and the desired goal now follows from this inequality integrating in r. $\square$

5.1.3. A classical result

We conclude this section by recalling the following classical result for monotone functions (see for example, [25], Section II.4, Proposition 4.17).

Proposition 5.1.3. *Let $(a,b)\subset\mathbb{R}$ be a bounded interval and let $f\in L^\infty(a,b)$, $\eta\in W^{1,1}(a,b)$ be given functions, $\eta(v)\geq 0$ for all $v\in[a,b]$. Let us set*

$$f(b-):=\lim_{x\to b^-}f(x),\qquad f(a^+):=\lim_{x\to a^+}f(x).$$

(i) Assume that the function $f(v)-K\,v$ is nondecreasing for some $K\geq 0$. Then

$$\int_a^b f(v)\,\eta'(v)\,dv\;\leq\;f(b-)\,\eta(b)\;-\;f(a+)\,\eta(a)\;-\;K\int_a^b\eta(v)\,dv,$$

$$\int_a^b \frac{\eta'(v)}{f(v)}\,dv \geq \frac{\eta(b)}{f(b-)} - \frac{\eta(a)}{f(a+)} + K\int_a^b \frac{\eta(v)}{f^2(v)}\,dv,$$

provided $f(a+) > 0$.

(ii) Assume that the function $f(v) + K\,v$ *is nonincreasing for some* $K \geq 0$. *Then*

$$\int_a^b f(v)\,\eta'(v)\,dv \;\geq\; f(b-)\,\eta(b) \;-\; f(a+)\,\eta(a) \;+\; K\int_a^b \eta(v)\,dv,$$

$$\int_a^b \frac{\eta'(v)}{f(v)}\,dv \;\leq\; \frac{\eta(b)}{f(b-)} - \frac{\eta(a)}{f(a+)} \;-\; K\int_a^b \frac{\eta(v)}{f^2(v)}\,dv,$$

provided $f(b-) > 0$.

5.2. The case $\mathcal{F}$ Preisach operator

5.2.1. Main assumptions and setting of the quantities

Let Ω be an open bounded interval of $\mathbb{R}$, for the sake of simplicity, let us take $\Omega = (0,1)$. First of all we set $Q_\infty := (0,1)\times(0,\infty)$. We recall that in Section 1.2.5 we introduced the configuration space

$$\Lambda := \left\{\lambda \in W^{1,\infty}(0,\infty);\ \left|\frac{d\lambda(r)}{dr}\right| \leq 1 \ \text{ a.e. in } [-r,r]\right\}$$

and its subspaces

$$\Lambda_K := \{\lambda \in \Lambda; \lambda(r) = 0 \ \text{ for } r \geq K\}, \qquad \Lambda_0 := \bigcup_{K>0} \Lambda_K.$$

Let $\mathcal{W} : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be the Preisach operator introduced in (5.1.5), generated by the function $g : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ introduced in (1.5.7).

Let us recall that the *Preisach potential energy* $\mathcal{U}$ is defined in the following way

$$\mathcal{U}[\lambda,u](t) := \int_0^\infty G(r, \wp_r[\lambda,u](t))\,dr,$$

where

$$G(r,v) := v\,g(r,v) - \int_0^v g(r,z)\,dz = \int_0^v z\,\psi(r,z)\,dz,$$

with $\psi(r,z) = \partial_z g(r,z)$. Moreover the *Preisach dissipation operator* is introduced as

$$\mathcal{D}[\lambda,u](t) := \int_0^\infty r\,g(r,\wp_r[\lambda,u](t))\,dr.$$

Now we fix any initial memory distribution

$$\lambda \in L^2(0,1;\Lambda_K) \qquad \text{for some } K>0. \tag{5.2.1}$$

Suppose in addition that the following assumptions are satisfied

Assumptions 5.2.1.

* $g(r,0)=0$ a.e., and the function $\psi(r,z):=\partial_z g(r,z)$ is non-negative and belongs to the space $L^\infty(\mathcal{P})$, where $\mathcal{P}$ is the Preisach plane $\mathcal{P}=\{(r,v)\in\mathbb{R}^2 : r>0\}$ (see (1.5.4));
* the initial configuration λ belongs to $L^2(0,1;\Lambda_K)$, for some $K>0$;
* the antisymmetric part of ψ stays in $L^1(\mathcal{P})$, i.e.

  $\psi_a(r,v):=\dfrac{1}{2}\left(\psi(r,v)-\psi(r,-v)\right)\in L^1(\mathcal{P})$;
* there exist β_0, $\beta_1 \in L^1_{\text{loc}}(0,\infty)$, $\beta_1(r)\geq\beta_0(r)\geq 0$ a.e., $b_0:=\displaystyle\int_0^\infty \beta_0(r)\,dr$ such that $b_0<\infty$ and $\beta_1(r)\geq\psi(r,v)\geq-\beta_0(r)$ for a.e. $(r,v)\in\mathcal{P}$.

  We also put $b_1(R):=\displaystyle\int_0^R \beta_1(r)\,dr$ for $R>0$.

It is not hard to find that the operator $\overline{\mathcal{W}}:\mathcal{M}((0,1);\mathcal{C}^0([0,T]))\to\mathcal{M}((0,1);\mathcal{C}^0([0,T]))$ defined as follows

$$\overline{\mathcal{W}}(u)(x,t):=\mathcal{W}[\lambda(x),u(x,\cdot)](t)=\int_0^\infty g(r,\wp_r[\lambda(x),u(x,\cdot)](t))\,dr=:\int_0^\infty g(r,\bar{\wp}_r(u)(x,t))\,dr$$

is causal, strongly continuous, affinely bounded and piecewise monotone according to (3.1.1), (3.1.2), (3.1.3) and (3.1.4).
This can be easily verified using Proposition 1.5.4 (which is valid for any choice of the measure μ), Proposition 1.5.10, Proposition A.1.1 and Proposition A.1.2.

5.2.2. Model problem and statement of the main results

We deal with a model problem in the framework of Sobolev spaces; we recall that we are setting our problem in one-space dimension. We set $V:=H^1(0,1)$ and we identify the space $L^2(0,1)$ to its dual $(L^2(0,1))'$. As V is a dense subspace of $L^2(0,1)$, then $(L^2(0,1))'$ can be identified to a subspace of V' (see Theorem A.6.1). So we get the Hilbert triplet

$$V\subset L^2(0,1)\equiv(L^2(0,1))'\subset V'$$

with dense and continuous injections.

We define the operator $A : V \to V'$ as follows

$$ {}_{V'}\langle Au, v\rangle_V := \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx \qquad \forall\, u, v \in V $$

hence $Au = -\dfrac{\partial^2 u}{\partial x^2}$ in the sense of distributions.

We assume that $u^0 \in H^1(0,1)$ and $w^0 \in L^2(0,1)$ are given initial conditions. Moreover we notice that, due to the causality property, the operator $\overline{\mathcal{W}}$ introduced in Subsection 5.2.1 can be treated as a mapping between Banach spaces of functions defined in Q_∞.
We deal with the following problem.

Problem 5.2.2. *We search for a function $u \in \mathcal{M}((0,1);\mathcal{C}^0((0,\infty))) \cap L^2((0,\infty);V)$ such that $\overline{\mathcal{W}}(u) \in \mathcal{M}((0,1);\mathcal{C}^0((0,\infty))) \cap L^2(Q_\infty)$ and for all $\varphi \in L^2((0,\infty);V) \cap H^1((0,\infty);L^2(0,1))$ such that $\lim_{t\to\infty}\varphi(\cdot,t) = 0$ a.e. in $(0,1)$*

$$ \int_0^\infty\int_0^1 -(u+\overline{\mathcal{W}}(u))\,\frac{\partial\varphi}{\partial t}\,dx\,dt + \int_0^\infty\int_0^1 \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\,dx\,dt = \int_0^1 [u^0(x)+w^0(x)]\,\varphi(x,0)\,dx. \quad (5.2.2) $$

We notice that (5.2.2) can be interpreted in the following way

$$ \frac{\partial}{\partial t}(u+\overline{\mathcal{W}}(u)) - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in } \mathcal{D}'(0,1), \text{ a.e. in } (0,\infty) \quad (5.2.3) $$

together with the following (Neumann) boundary conditions

$$ \partial_x u(0,t) = \partial_x u(1,t) = 0 \qquad \text{for } t \in (0,\infty). $$

Actually (5.2.2) turns to be valid in $L^2((0,\infty);V')$, from what we deduce, by comparison, $u + \overline{\mathcal{W}}(u) \in H^1((0,\infty);V')$ and so $(u+\overline{\mathcal{W}}(u))_{|t=0} = u^0 + w^0$ in V' (in the sense of traces).
The first result we are able to prove is the following.

Theorem 5.2.3. *Suppose that Assumptions 5.2.1 hold. If $u^0 \in H^1(0,1)$ and $w^0 \in L^2(0,1)$ are given initial conditions, then there exist two constants $c > 0$ and $u_\infty \in \mathbb{R}$ such that the unique solution of Problem 5.2.2 has the following properties*

$$ \int_0^1 |\partial_x u(x,t)|^2\,dx \le e^{-ct}\int_0^1 |\partial_x u^0(x)|^2\,dx, \quad (5.2.4) $$

$$ \lim_{t\to\infty} u(x,t) = u_\infty \qquad \forall\, x \in [0,1]. $$

Before stating the second result we set

$$ \begin{aligned} B_\rho(u_\infty) &:= \{(r,v) \in \mathcal{P} : |v-u_\infty| + r \le \rho\} \\ A_\rho(u_\infty) &:= \operatorname{ess\,inf}\{\psi(r,v) : (r,v) \in B_\rho(u_\infty)\} \\ C_\rho(u_\infty) &:= \operatorname{ess\,sup}\left\{\frac{\partial\psi}{\partial t}(r,v) : |v-u_\infty| + r \le \rho\right\}, \end{aligned} \quad (5.2.5) $$

where u_∞ is given by Theorem 5.2.3 and $\mathcal{P}$ is the Preisach plane.
If the data are more regular we also have the following.

Theorem 5.2.4. *In the assumptions of Theorem* 5.2.3, *if we suppose in addition that* $u^0 \in W^{2,2}(0,1)$, $\partial_x u^0(0) = \partial_x u^0(1) = 0$ *and* ψ *is positive and continuous in* $\mathbb{R} \times (0,\infty)$, *then also* $\partial_t u$ *and* $\partial_x^2 u$ *decay to 0 exponentially in* $L^2(0,1)$ *as* $t \to \infty$.

5.2.3. Proof of Theorems 5.2.3 and 5.2.4

Proof of Theorem 5.2.3.

➜ UNIQUENESS. First of all we prove that, if Problem 5.2.2 admits at least a solution, then this solution is forced to be unique.
We fix an arbitrary $T > 0$ and consider any $t \in (0,T)$. Suppose by contradiction that u_1 and u_2 are both solutions of Problem 5.2.2. Then, using the regularity of u_1 and u_2 and by comparison, we surely have

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(u_1 + \overline{\mathcal{W}}(u_1)) - \frac{\partial^2 u_1}{\partial x^2} = 0 \\ &\frac{\partial}{\partial t}(u_2 + \overline{\mathcal{W}}(u_2)) - \frac{\partial^2 u_2}{\partial x^2} = 0 \end{aligned}\right. \qquad \text{for any } t \in (0,T), \text{ a.e. in } (0,1).$$

Taking the difference of the two previous equations we deduce

$$\frac{\partial}{\partial t}(u_1 - u_2 + \overline{\mathcal{W}}(u_1) - \overline{\mathcal{W}}(u_2)) - \frac{\partial^2}{\partial x^2}(u_1 - u_2) = 0. \tag{5.2.6}$$

At this point we would like to multiply equation (5.2.6) by the function $s_\varepsilon(u_1 - u_2) \in L^2((0,T) \times (0,1))$ in the scalar product of $L^2(0,1)$, where s_ε is a suitable approximation of the sign function, more precisely

$$s_\varepsilon(z) = \min\left\{1, \max\left\{-1, \frac{z}{\varepsilon}\right\}\right\}.$$

We remark that the function s_ε is nondecreasing and this fact, joint to the choice of Neumann boundary conditions, yields

$$\begin{aligned} 0 &= \int_0^1 \frac{\partial}{\partial t}(u_1 - u_2 + \overline{\mathcal{W}}(u_1) - \overline{\mathcal{W}}(u_2))\, s_\varepsilon(u_1 - u_2)\, dx + \int_0^1 s'_\varepsilon(u_1 - u_2) \left|\frac{\partial}{\partial x}(u_1 - u_2)\right|^2 dx \\ &\geq \int_0^1 \frac{\partial}{\partial t}(u_1 - u_2 + \overline{\mathcal{W}}(u_1) - \overline{\mathcal{W}}(u_2))\, s_\varepsilon(u_1 - u_2)\, dx. \end{aligned}$$

At this point we let $\varepsilon \to 0^+$ and we use (5.1.6), obtaining

$$\begin{aligned} 0 &\geq \int_0^1 \frac{\partial}{\partial t}(u_1 - u_2 + \overline{\mathcal{W}}(u_1) - \overline{\mathcal{W}}(u_2))\, \mathrm{sign}\,(u_1 - u_2)\, dx \\ &\geq \frac{\partial}{\partial t} \int_0^1 \left(|u_1 - u_2| + \int_0^\infty |g(r, \wp_r(u_1)(x,t)) - g(r, \wp_r(u_2)(x,t))|\, dr\right) dx. \end{aligned}$$

Now, if we integrate from 0 to t, we immediately get that Problem 5.2.2 admits a unique solution in $(0,T)$. As T was taken arbitrarily, we can conclude that the solution of our model problem is unique also in Q_∞.

➜ existence. As to concern existence, our assumptions on the hysteresis operator allow us to apply Theorem 1.1 contained in [39], Section IX.1 and to get the existence of a solution $u \in H^1(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1(0,1))$ for any fixed $T > 0$. In particular, this entails that, for any $n \in \mathbb{N}$ there exists a solution u_n, with the regularity we just made explicit, defined on $(0,1)\times(0,n)$. For $(x,t) \in Q_\infty$ we then define $n(t) = [t]+1$ where $[\cdot]$ is the integer part, and set $u(x,t) = u_{n(t)}(x,t)$. It follows from the uniqueness property that u is well defined on Q_∞ and solves Problem 5.2.2.

➜ asymptotic decay of $\partial_x u$: a first weak result. At this point we can go through the main result. If we set

$$\overline{\mathcal{U}}(u)(x,t) := \mathcal{U}(\lambda(x), u(x,\cdot))$$

(where λ is fixed, see Assumptions 5.2.1), it is then clear from condition *(ii)* of Theorem 1.5.13 that

$$\left[\frac{\partial}{\partial t}\overline{\mathcal{W}}(u)(x,t)\right] u(x,t) - \frac{\partial}{\partial t}\overline{\mathcal{U}}(u)(x,t) \geq 0 \qquad \text{a.e. in } Q_\infty. \tag{5.2.7}$$

Now we test equation (5.2.3) first by u and then by u_t, i.e. we multiply equation (5.2.3) first by u in the scalar product of $L^2(0,1)$ and then do the same for u_t. We deduce the following two inequalities which hold for almost any $t > 0$

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^1 \left(\frac{1}{2}|u(x,t)|^2 + \overline{\mathcal{U}}(u)(x,t)\right) dx + \int_0^1 \left|\frac{\partial}{\partial x}u(x,t)\right|^2 dx \\
\overset{(5.2.7)}{\leq} &\int_0^1 \left[\frac{\partial}{\partial t}u(x,t)\right] u(x,t)\,dx + \int_0^1 \left[\frac{\partial}{\partial t}\overline{\mathcal{W}}(u)(x,t)\right] u(x,t)\,dx + \int_0^1 \left|\frac{\partial}{\partial x}u(x,t)\right|^2 dx \;\leq\; 0
\end{aligned} \tag{5.2.8}$$

and

$$\int_0^1 \frac{\partial}{\partial t}u(x,t)\,\left[\frac{\partial}{\partial t}\left(u + \overline{\mathcal{W}}(u)\right)(x,t)\right] dx + \frac{1}{2}\frac{\partial}{\partial t}\int_0^1 \left|\frac{\partial}{\partial x}u(x,t)\right|^2 dx \;=\; 0. \tag{5.2.9}$$

First of all we observe that the function

$$h(t) = \int_0^1 \left|\frac{\partial}{\partial x}u(x,t)\right|^2 dx \tag{5.2.10}$$

is nonincreasing; this can be seen using the piecewise monotonicity for the operator $\overline{\mathcal{W}}$, i.e. property (3.1.4), and the previous equation (5.2.9). Moreover the function h belongs to

$L^1(0,\infty)$; this can be seen integrating in time, from 0 to an arbitrary $T > 0$, equation (5.2.8): in fact the potential energy of the Preisach operator is non negative, its initial value and the initial value of u are bounded by a constant independent of T and so also

$$\int_0^T h(t)\,dt$$

is bounded from above by a constant which is independent of T. These two facts entail this first asymptotic result, due to Lemma 5.1.1

$$\lim_{t\to\infty} t\,h(t) = 0. \tag{5.2.11}$$

➜ ASYMPTOTIC BEHAVIOUR OF $\partial_x u$: A SECOND STRONG RESULT. On the other hand, it turns out that the function u is uniformly bounded, i.e. $u \in L^\infty(0,\infty;L^\infty(0,1))$. This can be seen in the following way: if we integrate (5.2.8) in time we obtain a bound for u in $L^\infty(0,\infty;L^2(0,1))$; at this point, for any t, we exploit the fact that

$$||u(t)||_{L^\infty(0,1)} \leq\; c_1 + c_2 \left|\left|\frac{\partial}{\partial x}u(t)\right|\right|_{L^2(0,1)}$$

for c_1, c_2 costants which are independent of t and therefore u is uniformly bounded.
Let us fix some constants $R \geq K$, (where K is introduced in (5.2.1)) and $\Psi > 0$ such that, for all admissible arguments,

$$|u(x,t)| \leq R, \qquad \psi(r,z) \leq \Psi.$$

We recall that using (3.1.4), we have

$$\frac{\partial}{\partial t}u(x,t)\,\left[\frac{\partial}{\partial t}(u(x,t)+\overline{\mathcal{W}}(u)(x,t))\right] \geq \left|\frac{\partial}{\partial t}u(x,t)\right|^2 \qquad \text{a.e. in } Q_\infty;$$

as a complement of this, we have the following further inequality

$$\frac{\partial}{\partial t}u(x,t)\,\left[\frac{\partial}{\partial t}(u(x,t)+\overline{\mathcal{W}}(u)(x,t))\right] \geq \frac{1}{1+\Psi R}\left(\frac{\partial}{\partial t}[u(x,t)+\overline{\mathcal{W}}(u)(x,t)]\right)^2 \tag{5.2.12}$$

which comes from a direct differentiation of

$$\mathcal{W}[\lambda,u](t) = \int_0^\infty g(r,\wp_r[\lambda,u](t))\,dr$$

taking into account that in order to estimate the integral

$$\int_0^\infty \psi(r,\wp_r[\lambda,u](t))\,dr$$

over the unbounded domain $(0,\infty)$ from above, it is necessary to use the fact that for $r > R$ we have $\wp_r[\lambda,u](t) = 0$ for all t, (as $R \geq K$), hence we are actually integrating over $(0,R)$; that's here where the value R comes from.

At this point, by virtue of the Neumann boundary conditions, we deduce

$$\begin{aligned}\int_0^1 \left|\frac{\partial}{\partial x}u(x,t)\right|^2 dx &\leq \int_0^1 \left|\frac{\partial^2}{\partial x^2}u(x,t)\right|^2 dx \\ &\overset{(5.2.3)\ (5.2.12)}{\leq} (1+\Psi R)\int_0^1 \frac{\partial}{\partial t}u(x,t)\ \left[\frac{\partial}{\partial t}(u(x,t)+\overline{\mathcal{W}}(u)(x,t))\right] dx \qquad (5.2.13) \\ &\overset{(5.2.9)}{=} -\left(\frac{1+\Psi R}{2}\right)\frac{\partial}{\partial t}\int_0^1 \left|\frac{\partial}{\partial x}u(x,t)\right|^2 dx.\end{aligned}$$

This is enough to conclude that (5.2.4) holds with $c = \dfrac{2}{1+\Psi R}$. In fact, for this choice of the constant c, we deduce from (5.2.13)

$$h(t) + \frac{1}{c}h'(t) \leq\ 0,$$

where $h(t)$ was introduced in (5.2.10). Multiplying both sides of the inequality by e^{ct} we have

$$e^{ct}h'(t) + c\,e^{ct}h(t) \leq\ 0$$

which is equivalent to

$$\frac{d}{dt}(e^{ct}h(t)) \leq\ 0$$

and this yields

$$e^{ct}h(t) - h(0) \leq\ 0$$

and therefore

$$h(t) \leq\ h(0)\,e^{-ct}$$

which is exactly what we were looking for.

Remark 5.2.5. We want to remark that both (5.2.11) and (5.2.13) can be also obtained working in several dimensions. The first result can be seen more or less immediately; the second one is possible with an extra embedding constant depending on the domain Ω, more precisely on the smallest positive eigenvalue of the Laplacian operator on Ω (one can see this by expanding u into a Fourier series with respect to the eigenfunctions of the Laplacian).

➜ CONVERGENCE OF $u(x,t)$ TO u_∞. The most delicate part consists in proving the convergence of $u(x,t)$. This is an interesting result by itself but it is also essential to prove Theorem 5.2.4, in order to be sure that for small amplitude oscillations around the limit u_∞, the function u does not leave the convexity domain of $\mathcal{W}$. This allows to differentiate the equation and get therefore some further exponentially decay estimates.
We notice finally that these computations only hold in one dimensional space setting and they have, for the moment, no counterpart in several dimensions.

Suppose thus by contradiction that

$$\exists\, x \in [0,1] \text{ such that } \lim_{t\to\infty} u(x,t) \text{ does not exists.}$$

It is not restrictive to consider $x = 0$. Then we are assuming that

$$\liminf_{t\to\infty} u(0,t) < \limsup_{t\to\infty} u(0,t). \tag{5.2.14}$$

We know from the previous step that u is uniformly bounded. We fix constants $\varepsilon > 0$ and $a < b \in \mathbb{R}$ such that

$$\varepsilon < \frac{b-a}{4\Psi R} \tag{5.2.15}$$

and

$$a - \varepsilon < \liminf_{t\to\infty} u(0,t) < a < b < \limsup_{t\to\infty} u(0,t) < b + \varepsilon. \tag{5.2.16}$$

These notations will be clear later. Now, using (5.2.4), we have, for every $x \in [0,1]$,

$$|u(x,t) - u(0,t)| \leq \int_0^1 \left| \frac{\partial}{\partial y} u(y,t) \right| dy \to 0 \qquad \text{as } t \to \infty \tag{5.2.17}$$

and this implies that, for every $\bar{\varepsilon} > 0$ exists $\bar{t} > 0$ such that $\forall\, t \geq \bar{t}$

$$u(0,t) - \bar{\varepsilon} \leq\, u(x,t) \leq\, u(0,t) + \bar{\varepsilon} \qquad \forall\, x \in [0,1].$$

At this point, using (5.2.16) and taking into account that in (5.2.16) we have a strict inequality, we obtain

$$a - \varepsilon \leq u(x,t) \leq b + \varepsilon \qquad \forall\, x \in [0,1],\ \forall\, t \geq \bar{t}(\varepsilon),$$

where ε is exactly the one fixed in (5.2.15). On the other hand, the two facts

$$\liminf_{t\to\infty} u(0,t) < a \qquad \limsup_{t\to\infty} u(0,t) > b$$

allow us to find a sequence, which we may suppose to be increasing, $\bar{t} \leq t_0 < t_1 < t_2 < \ldots$, such that $\lim_{n\to\infty} t_n = \infty$ and for all $x \in [0,1]$ and for any $k = 0, 1, 2, \ldots$ we have

$$u(x, t_{2k}) \leq\, a \qquad u(x, t_{2k+1}) \geq\, b. \tag{5.2.18}$$

Consider now $x \in [0,1]$ be arbitrarily fixed. For any $r \geq 0$ and any $t \geq 0$, we set $\xi_r(x,t) = \wp_r[\lambda(x), u(x,t)]$. Our idea is now to estimate the difference between the values of $(u + \overline{\mathcal{W}}(u))$ at two consecutive instants t_{2k}, t_{2k+1} that is we try to estimate the quantity

$$u(x, t_{2k+1}) + \overline{\mathcal{W}}(u)(x, t_{2k+1}) - u(x, t_{2k}) - \overline{\mathcal{W}}(u)(x, t_{2k}).$$

Claim: we hope to find a constant c (independent of x and k) such that

$$u(x, t_{2k+1}) + \overline{\mathcal{W}}(u)(x, t_{2k+1}) - u(x, t_{2k}) - \overline{\mathcal{W}}(u)(x, t_{2k}) \geq\, c.$$

This will lead to a contradiction. In fact, the above inequality holds independently of x. If we integrate it over x, then we deduce

$$\int_0^1 (u(x,t_{2k+1}) + \overline{\mathcal{W}}(u)(x,t_{2k+1}) - u(x,t_{2k}) - \overline{\mathcal{W}}(u)(x,t_{2k}))\,dx \geq c.$$

On the other hand, as we have Neumann boundary conditions, integrating in the space variable equation (5.2.3) we also obtain

$$\frac{d}{dt}\int_0^1 (u(x,t) + \overline{\mathcal{W}}(u)(x,t))\,dx = 0 \qquad \text{a.e. in } Q_\infty \tag{5.2.19}$$

which is in contradiction with the previous inequality. This would finish our proof.
So let us prove the claim. We certainly have

$$\begin{aligned} &u(x,t_{2k+1}) + \overline{\mathcal{W}}(u)(x,t_{2k+1}) - u(x,t_{2k}) - \overline{\mathcal{W}}(u)(x,t_{2k}) \\ &\overset{(5.2.18)}{\geq} b - a + \int_0^R (g(r,\xi_r(x,t_{2k+1})) - g(r,\xi_r(x,t_{2k})))\,dr. \end{aligned}$$

As the function ψ is non-negative, the function g is nondecreasing in the second argument; for this reason we look for a lower bound for the quantity $\xi_r(x,t_{2k+1})$ and an upper bound for the term $\xi_r(x,t_{2k})$.
First of all we have the following two inequalities which directly come from the definition of the play operator

$$\xi_r(x,t_{2k+1}) \geq b - r \qquad \xi_r(x,t_{2k}) \leq a + r; \tag{5.2.20}$$

so one could start with this first rough estimate

$$u(x,t_{2k+1}) + \overline{\mathcal{W}}(u)(x,t_{2k+1}) - u(x,t_{2k}) - \overline{\mathcal{W}}(u)(x,t_{2k}) \geq b - a + \int_0^R (g(r,b-r) - g(r,a+r))\,dr.$$

At this point, the easier way to control this quantity is to exploit once again the fact that the function g is nondecreasing in the second argument, and therefore if $b - r \geq a + r$ then $\int_0^R (g(r,b-r) - g(r,a+r))\,dr \geq 0$. But this is possible only if $r \leq \frac{b-a}{2}$, otherwise a more accurate estimate for the lower bound of $\xi_r(x,t_{2k+1})$ and the upper bound for $\xi_r(x,t_{2k})$ is needed.
First of all we set for any $r \geq 0$ and any $t \geq 0$, $\lambda_0(x,r) = \xi_r(x,t_0)$ and $\lambda_1(x,r) = \xi_r(x,t_1)$. Consequently we apply Lemma 1.2.5 for $t \geq t_1$, getting

$$\min\{\lambda_1(x,r), a + r - \varepsilon\} \leq \xi_r(x,t) \leq \max\{\lambda_1(x,r), b - r + \varepsilon\}$$

which is equivalent to

$$\min\{0, a + r - \varepsilon - \lambda_1(x,r)\} \leq \xi_r(x,t) - \lambda_1(x,r) \leq \max\{0, b - r + \varepsilon - \lambda_1(x,r)\}. \tag{5.2.21}$$

As we already remarked, (see (5.2.20)), we have

$$\lambda_1(x,r) \geq b - r \qquad \text{for all } r > 0,$$

hence

$$0 \leq \max\{0, b - r + \varepsilon - \lambda_1(x,r)\} \leq \varepsilon$$

and this entails the first more accurate estimate

$$\xi_r(x,t) - \lambda_1(x,r) \leq \varepsilon. \tag{5.2.22}$$

On the other hand, we apply again Lemma 1.2.5, this time for $t \geq t_0$. We have

$$\xi_r(x,t) \leq \max\{\lambda_0(x,r), b - r + \varepsilon\} \overset{(5.2.20)}{\leq} \max\{a + r, b - r + \varepsilon\} \leq a + r + \varepsilon.$$

where in the third passage we used the fact that this time we are working with $r \geq \dfrac{b-a}{2}$. This tells us that $\xi_r(x,t_1) = \lambda_1(x,r) \leq a + r + \varepsilon$ and so

$$-2\varepsilon \leq \min\{0, a + r - \varepsilon - \lambda_1(x,r)\} \leq 0$$

which in turn gives, (using (5.2.21)), the second more accurate estimate we were looking for

$$\xi_r(x,t) \geq \lambda_1(x,r) - 2\varepsilon. \tag{5.2.23}$$

At this point we are able to conclude, since (for $k \neq 0$)

$$\begin{aligned}
&u(x,t_{2k+1}) + \overline{\mathcal{W}}(u)(x,t_{2k+1}) - u(x,t_{2k}) - \overline{\mathcal{W}}(u)(x,t_{2k}) \\
&\geq b - a + \int_0^R (g(r, \xi_r(x,t_{2k+1})) - g(r, \xi_r(x,t_{2k}))) \, dr \\
&\geq b - a + \int_{\frac{b-a}{2}}^R (g(r, \lambda_1(x,r) - 2\varepsilon) - g(r, \lambda_1(x,r) + \varepsilon) \, dr + \int_0^{\frac{b-a}{2}} (g(r, b - r) - g(r, a + r)) \, dr \\
&\geq b - a - 3\Psi R\varepsilon \overset{(5.2.15)}{\geq} \frac{b-a}{4}.
\end{aligned}$$

Therefore the previous claim is proved; this claim is in contradiction with the assumption from what our discussion started, i.e. (5.2.14) does not hold. Thus, using (5.2.17), we are able to prove the convergence $u(x,t) \to u_\infty$, where we set $u_\infty = \lim\limits_{t\to\infty} u(0,t)$. $\square$

Remark 5.2.6. We remark that the value u_∞, which is defined as the limit of $u(0,t)$ as t tends to infinity, is a constant and it appears clear that the substance of the whole development is to prove that the convergence takes place independently of x. On the other hand, one cannot expect the limit to depend on x as it is known from the general theory of dynamical systems that

if the solution asymptotically converges to something, then the limit is an equilibrium of the system. In our case, all equilibria are solutions of the Laplace equation with the homogeneous Neumann boundary conditions, hence constants.
This case is similar as in the case of the linear heat equation without hysteresis and with the homogeneous Neumann boundary conditions. Then the solution also in that case converges to a fixed constant, namely to the mean value of the initial condition. Also here the total energy is conserved (identity (5.2.19) can be interpreted as the energy conservation law), but only some part of the initial energy is stored in u itself, while the other part is stored in the hysteresis memory configuration, and there is a strong energy exchange between the two during the process. This is the reason why the convergence proof is not as trivial as in the Dirichlet case or as in the case without hysteresis.

Proof of Theorem 5.2.4.

The basic idea which undergoes the proof of this result is that for small amplitude oscillations of the solution around u_∞, $u(x,t)$ does not leave the convexity domain of the Preisach operator. This assertion can be made rigorous proving an analogue of Theorem 1.5.15. What is different now is that this time we have to take $A_\rho(u_\infty)$ instead of A_ρ and $C_\rho(u_\infty)$ instead of C_ρ (where we recall that the quantities $A_\rho(u_\infty)$ and $C_\rho(u_\infty)$ were introduced in (5.2.5)). From the assumptions we took on the function ψ, it turns out that $A_\rho(u_\infty) > 0$ for some $\rho > 0$ and so we can choose $R \in (0,\rho)$ sufficiently small such that

$$K_R(u_\infty) := \frac{1}{2}\,A_R(u_\infty) - R\,C_R(u_\infty) > 0.$$

The rest of the proof follows as in the case of Theorem 1.5.15 with the obvious modifications, so the radius R we found just now is such that the whole process takes place in $B_R(u_\infty)$ for large times. This is the convexity region of our Preisach operator. In this region we are allowed to differentiate our model equation and use the second order energy inequality (1.3.6). Actually, in order to differentiate in time our model equation, some higher regularity in time should be required. So we proceed in this manner: we first show all the computations, acting first only in a formal way, as everything would be regular enough; this in order to focus our attention on the estimates achieved. In a second step we will justify completely (by means of a suitable approximation procedure) what has been done so far.

➜ formal computations. We denote by $w(x,t) := u(x,t) + \overline{\mathcal{W}}(u)(x,t)$ a.e. in Q_∞ and by

$$P(t) := \int_0^1 \left(\frac{\partial u}{\partial t}\,\frac{\partial w}{\partial t}\right)(x,t)\,dx = \int_0^1 \left(\frac{\partial u}{\partial t}\,\frac{\partial}{\partial t}(u + \overline{\mathcal{W}}(u))\right)(x,t)\,dx.$$

By Theorem 1.3.6, we have

$$\frac{1}{2}\,(P(t) - P(s)) \;\le\; \int_s^t \int_0^1 (\partial_t u\,\partial_{tt} w)(x,\tau)\,dx\,d\tau$$

for a.a. $0 < s < t < \infty$. Hence, the function $P(t) - \int_0^t \int_0^1 (\partial_t u \, \partial_{tt} w)(x,\tau)\, dx\, d\tau$ is nonincreasing, so that P is equal almost everywhere to a a function of locally bounded variation. Differentiating then in a formal way equation (5.2.3) with respect to t, testing by $\partial_t u$, and using the above inequality, we obtain

$$\frac{1}{2}(P(t) - P(s)) + \int_s^t \int_0^1 |\partial_{xt} u|^2(x,\tau)\, dx\, d\tau \leq 0. \tag{5.2.24}$$

We now use formula (5.2.13) which states that, for a.a. $t \in (0,\infty)$

$$\int_0^1 |\partial_x u|^2(x,t)\, dx \ \leq \ (1 + \Psi R) P(t). \tag{5.2.25}$$

We put $\kappa = 1 + \Psi R$; then we consider equation (5.2.3), test (formally) it by $\partial_t u$ and integrate in time from s to t. We get

$$\int_s^t P(\tau)\, d\tau \leq \int_s^t \int_0^1 |\partial_x u \, \partial_{xt} u|(x,\tau)\, dx\, d\tau \leq \frac{1}{2\kappa} \int_s^t \int_0^1 |\partial_x u|^2(x,\tau)\, dx\, d\tau$$

$$+ \frac{\kappa}{2} \int_s^t \int_0^1 |\partial_{xt} u|^2(x,\tau)\, dx\, d\tau \overset{(5.2.25)}{\leq} \frac{1}{2} \int_s^t P(\tau)\, d\tau + \frac{\kappa}{2} \int_s^t \int_0^1 |\partial_{xt} u|^2(x,\tau)\, dx\, dt$$

from what we deduce

$$\int_s^t P(\tau)\, d\tau \leq \kappa \int_s^t \int_0^1 |\partial_{xt} u|^2(x,\tau)\, dx\, d\tau. \tag{5.2.26}$$

Combining the above inequalities we finally get

$$P(t) - P(s) + \mu \int_s^t P(\tau)\, d\tau \overset{(5.2.26)}{\leq} P(t) - P(s) + 2 \int_s^t \int_0^1 |\partial_{xt} u|^2(x,\tau)\, dx\, d\tau \overset{(5.2.24)}{\leq} 0$$

for a.a. $0 < s < t < \infty$, where we denote $\mu := 2/\kappa$ for the sake of simplicity. Set

$$f(t) \ = \ P(t) + \mu \int_0^t P(\tau)\, d\tau \quad \text{for } t \geq 0.$$

Then f is nonincreasing and by Proposition 5.1.3, we have, for every non-negative absolutely continuous test function η and every $t > 0$, that

$$\int_0^t \dot\eta(s)\, f(s)\, ds \ \geq \ f(t-)\, \eta(t) - f(0+)\, \eta(0).$$

We use this inequality with a special choice $\eta(t) = e^{\mu t}$ and obtain

$$e^{\mu t} \left(P(t-) + \mu \int_0^t P(\tau)\, d\tau \right) \leq P(0+) + \int_0^t \mu\, \mathrm{e}^{\mu s} \left(P(s) + \mu \int_0^s P(\tau)\, d\tau \right) ds$$

$$= P(0+) + \mu\, e^{\mu t} \int_0^t P(\tau)\, d\tau,$$

where in the last line we used a simple integration by parts in time. Hence

$$P(t-) \ \le \ \mathrm{e}^{-\mu t}\, P(0+) \qquad \forall\, t > 0. \tag{5.2.27}$$

The regularity of the data assures us that (5.2.27) is well defined; this allows us to conclude this part.

➜ APPROXIMATION AND CONCLUSION. As we said in the introduction, the computations made in the previous step are only formal as we don't have regularity enough to differentiate equation (5.2.3) in time. In this last part we briefly show that the previous formal computations actually become true under a suitable process of space-discretization of our model equation. First of all we notice that (5.2.3) can be rewritten as a system in the following way

$$\begin{cases} u_t + \varepsilon_t - v_x = 0 \\ u_x = v \\ \varepsilon = \overline{\mathcal{W}}(u) \end{cases} \qquad \text{a.e. in } Q_\infty. \tag{5.2.28}$$

We recall that

$$\overline{\mathcal{W}}(u)(x,t) := \mathcal{W}(\lambda(x), u(x,\cdot))(t)$$

so it is not restrictive to assume that the operator $\overline{\mathcal{W}}$ has the following form

$$\overline{\mathcal{W}}(u)(x,t) := W(x, u(x,\cdot))(t).$$

Now, fix $n \in \mathbb{N}$ and let $j = 0, \dots, n$. We denote by

$$u_j(t) := u\left(\frac{j}{n}, t\right), \qquad W_j(t) := W\left(\frac{j}{n}, t\right)$$

and do the same for the other quantities involved. Consider now the following system of O.D.E.s for any $t \in (0,\infty)$ which turns out to be the discrete counterpart of system (5.2.28)

$$\begin{cases} \dot u_j(t) + \dot\varepsilon_j(t) - \triangle_j\, \dot v(t) = 0 \\ v_j(t) = \triangle_j\, u(t) \\ \varepsilon_j(t) = W_j(u_j)(t), \end{cases} \tag{5.2.29}$$

where $\dot u_j = \partial_t u_j$ (and the same holds for $\dot\varepsilon_j$ and $\dot v$), while $\triangle_j\, u := n(u_{j+1} - u_j)$, together with the corresponding initial conditions

$$u_j(0) = u^0\left(\frac{j}{n}\right), \qquad w_j(0) = w^0\left(\frac{j}{n}\right).$$

At this point we are allowed to differentiate system (5.2.29) as now we have the right regularity in time; so (5.2.24) can be achieved for the corresponding discrete quantities (one can for example work as in [25], Section III.1, Lemma 1.8). Now the computations can be carried on for the discrete variables involved as we did in the previous step; as the estimate (5.2.27) is independent of the space discretization parameter, then it holds also for the solution of the original problem. This finishes the proof. □

5.3. The case $\mathcal{F}$ Prandtl-Ishlinskiĭ operator

Let us consider again a bounded interval Ω of $\mathbb{R}$, say $\Omega = (0,1)$ and let $Q_\infty := (0,1) \times (0,\infty)$. Let $\mathcal{F}_\varphi : \Lambda_0 \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ be a Prandtl-Ishlinskiĭ operator of play type (according to Definition 1.3.1)

$$\mathcal{F}_\varphi(\lambda, u) = h(0)u + \int_0^\infty \wp_r(\lambda, u)\, dh(r) \tag{5.3.1}$$

generated by the convex function

$$\varphi(r) = \int_0^r h(s)\, ds, \qquad r > 0,$$

where $\wp_r(\lambda, u)$ is the play operator introduced in (1.2.5) and recalled in the previous section and h is a given nondecreasing function. We fix any initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_K) \qquad \text{for some } K > 0, \tag{5.3.2}$$

(Λ_0 and Λ_K were introduced in (1.2.4), they have been already recalled in the previous section). We set

$$\overline{\mathcal{F}}_\varphi(u)(x,t) := \mathcal{F}_\varphi(\lambda(x), u(x,\cdot))(t) := h(0)u(x,t) + \int_0^\infty \overline{\wp}_r(\lambda, u)(x,t)\, dh(r); \tag{5.3.3}$$

where

$$\overline{\wp}_r(\lambda, u)(x,t) := \wp_r(\lambda(x), u(x,\cdot))(t);$$

working as in Section 1.6 it is easy to see that the operator $\overline{\mathcal{F}}_\varphi$ fulfills (3.1.1), (3.1.2), (3.1.3) and (3.1.4); moreover it satisfies also the content of Theorem 1.3.6.
Assume that the functional setting outlined in Subsection 5.2.2 still holds; with these notations the problem we would like to solve is Problem 5.2.2$_\varphi$ which is obtained by Problem 5.2.2 by replacing $\overline{\mathcal{W}}$ with $\overline{\mathcal{F}}_\varphi$.
The main result we have is then the following.

Theorem 5.3.1. *Suppose to have any fixed memory configuration λ as in* (5.3.2) *and consider the operator $\overline{\mathcal{F}}_\varphi$ introduced in* (5.3.3). *If $u^0 \in H^1(0,1)$ and $w^0 \in L^2(0,1)$ are given initial* *conditions, then there exists a constant* $c > 0$ *such that the unique solution of Problem* $5.2.2_\varphi$ *fulfills*

$$\int_0^1 |\partial_x u(x,t)|^2\,dx \le e^{-ct}\int_0^1 |\partial_x u^0(x)|^2\,dx \qquad \forall\, t > 0.$$

If we suppose in addition that $u^0 \in W^{2,2}(0,1)$, $\partial_x u^0(0) = \partial_x u^0(1) = 0$, *then also* $\partial_t u$ *and* $\partial_x^2 u$ *decay to 0 exponentially in* $L^2(0,1)$ *as* $t \to \infty$.

Proof. The proof of this result comes working as in the proof of Theorems 5.2.3 and 5.2.4; we have only to take into account two facts: first (5.2.12) still holds in this case as Prandtl-Ishlinskiĭ operators of type (5.3.1) with $a = 0$ and $h \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ belongs to the class of Preisach operators of type (5.1.5) with $\psi(r,v) = h'(r)$; moreover in this case (5.2.24) can be achieved directly using the convexity of the hysteresis loops, i.e. Theorem 1.3.6 works (in the case of our Prandtl-Ishlinskiĭ operator) for all times and not, as in the previous case, only for large times. □

APPENDIX A

Some complementary results

This appendix contains several results and concepts which are recalled and used in this thesis. Almost all are presented without proof, we quote in each case some references where the interested reader may find further details. We make an exception for the *generalized Poincaré inequality*, whose proof is very simple, and for the Riesz-Fréchet representation theorem, which is also presented complete with its proof because the idea of the proof of several results contained in this manuscript is the same employed for proving the Riesz-Fréchet representation theorem; that's why we choose to present here the computations instead of each time when needed in the previous chapters.

Throughout this appendix Ω will be an open subset of $\mathbb{R}^N$ ($N \geq 1$) and B an either real or complex Banach space, whose topological dual is denoted by B'. We say that Ω is of class $\mathcal{C}^{k,\nu}$ with $k \in \mathbb{N}$, $0 \leq \nu \leq 1$ if Ω is bounded, any point of its boundary Γ admits a neighbourhood U such that $U \cap \Omega$ stays only on one side of $U \cap \Gamma$ and finally $U \cap \Gamma$ is the graph of a function of class $\mathcal{C}^{k,\nu}$ (this last fact possibly after rotation of the axes). Here, if not otherwise specified, we assume that Γ is of Lipschitz class.
For any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in \mathbb{N}_0^N$, we set $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_N$ and $D^\alpha := \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}$.
Finally for any subset K, with the notation $K \subset\subset \Omega$ we mean that K is an open bounded subset of $\mathbb{R}^N$ such that $\overline{K} \subset \Omega$.

A.1. Spaces of functions with values in Banach spaces

For the results contained in this section we refer to [1], [27], [31], but also to [4] Chapter 5, [9] Chapter 2, [14] Chapter IV. In particular we assume to be known the definitions of the spaces of scalar functions of one real variable, for example the spaces $\mathcal{C}^0([0,T])$, $L^p(0,T)$ and $W^{k,p}(0,T)$, for all $k \in \mathbb{N}$, for all $p \in [1,+\infty]$, but also the Banach space $BV(0,T)$ of functions $[0,T] \to \mathbb{R}$ having finite total variation.

A.1.1. Continuous functions

We define the Banach space of vector-valued continuous functions in the following way

$$\mathcal{C}^0(\overline{\Omega}; B) := \{v : \overline{\Omega} \to B \text{ strongly continuous}\}.$$

This is a Banach space equipped with the norm $||v|| := \max_{x \in \Omega} ||v(x)||_B$.
For any $k \in \mathbb{N}$, we set

$$\mathcal{C}^k(\overline{\Omega}; B) := \{v \in \mathcal{C}^0(\overline{\Omega}; B) : \; D^\alpha v \in \mathcal{C}^0(\overline{\Omega}; B), \;\; \forall \, \alpha, \;\; |\alpha| \le \; k\}.$$

Also these are Banach spaces equipped with the respective graph norms. Here we mean that derivatives of functions $\overline{\Omega} \to B$ are strong limits in B of the corresponding incremental ratio.

A.1.2. Lebesgue functions

Let us denote by $\mathcal{S}(\Omega; B)$ the family of *simple functions* $\Omega \to B$, namely, functions with finite range, such that the inverse image of any element of B is measurable. We can then introduce the space of *strongly measurable functions* $\Omega \to B$:

$$\mathcal{M}(\Omega; B) := \{v : \Omega \to B : \;\; \exists \{v_n \in \mathcal{S}(\Omega; B)\}_{n \in \mathbb{N}} \text{ such that } v_n \to v \text{ strongly in } B, \text{ a.e. in } \Omega\}.$$

This is a Fréchét space, endowed with the quasi-norm

$$||v||_{\mathcal{M}(\Omega;B)} := \int_\Omega \frac{||v(x)||_B}{1 + ||v(x)||_B} \, dx.$$

Moreover,

$$v_n \to v \qquad \text{strongly in } \mathcal{M}(\Omega; B)$$

if and only if v_n converges to v in measure, that is, if we denote by λ_N the ordinary $N-$dimensional Lebesgue measure,

$$\lim_{n \to \infty} \lambda_N(\{x \in \Omega : \;\; ||v_n(x) - v(x)||_B \ge \; \varepsilon\}) = 0 \qquad \forall \, \varepsilon > 0.$$

It is natural to assume that B is a separable space because the range of any strongly measurable function is confined to a separable subspace.

The Lebesgue spaces of vector-valued functions

$$
\begin{aligned}
L^p(\Omega; B) &:= \left\{ v \in \mathcal{M}(\Omega; B) : \int_\Omega ||v||_B^p \, dx < +\infty \right\} \qquad \forall\, p \in [1, +\infty] \\
L^\infty(\Omega; B) &:= \{ v \in \mathcal{M}(\Omega; B) : \text{ ess sup}_\Omega ||v||_B < +\infty \},
\end{aligned}
$$

are (either real or complex) Banach spaces equipped with the norms

$$
||v||_{L^p(\Omega;B)} := \left(\int_\Omega ||v||_B^p \, dx \right)^{1/p}, \qquad ||v||_{L^\infty(\Omega;B)} := \text{ ess sup}_\Omega ||v||_B
$$

respectively. For any $p \in [1, +\infty]$, $L^p(\Omega; B)$ consists of classes of functions induced by the equivalence relation $u \sim v$ if and only if $u = v$ a.e. in Ω. Nevertheless, we write $X \subset \mathcal{C}^0(\overline{\Omega}; B)$ whenever $X \subset L^p(\Omega; B)$ and representatives of X can be selected in $\mathcal{C}^0(\overline{\Omega}; B)$. With abuse of notation, we moreover write for any set $A \subset B$, for example $L^p(\Omega; A)$ in place of $\{v \in L^p(\Omega; B) : v \in A, \text{ a.e. in } \Omega\}$.
It is important to remark that, if B is a Hilbert space, then also $L^2(0, T; B)$ is a Hilbert space endowed with the scalar product

$$
(u, v) = \int_0^T (u(t), v(t))_B \, dt.
$$

If $1 \leq p < \infty$ and B is a separable space, then the dual space of $L^p(0, T; B)$ can be identified to the space $L^{p'}(0, T; B')$ in the following way

$$
{}_{(L^p(0,T;B))'}\langle u, v \rangle_{L^p(0,T;B)} := \int_0^T {}_{B'}\langle u(t), v(t) \rangle_B \, dt
$$

for any $u \in L^{p'}(0, T; B')$ and $v \in L^p(0, T; B)$. If in addition $p > 1$ and B is reflexive, then also $L^p(0, T; B)$ is reflexive.

A.1.3. Distributions

For any $K \subset\subset \Omega$ let us denote by $\mathcal{D}_K(\Omega)$ the space of infinitely differentiable functions $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) whose support is included in K. This is a Fréchet space equipped with the family of seminorms

$$
|f|_{K,m} := \sum_{|\alpha| \leq m} \sup_K |D^\alpha f| \qquad (m \in \mathbb{N}_0).
$$

We then introduce the space of test functions

$$
\mathcal{D}(\Omega) := \bigcup_{K \subset\subset \Omega} \mathcal{D}_K(\Omega).
$$

This is a locally convex topological space equipped with the inductive limit topology, which is the finest topology on $\mathcal{D}(\Omega)$ that makes the injections $\mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$ continuous for any $K \subset\subset \Omega$.
The elements of $\mathcal{D}'(\Omega)$, i.e. the linear functionals $\mathcal{D}(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) that are continuous with respect to the inductive limit topology of $\mathcal{D}(\Omega)$, are called *distributions*. A sequence $\{T_n\}$ is said to converge to T in $\mathcal{D}'(\Omega)$ if and only if

$$\langle T_n, v\rangle := T_n(v) \to \langle T, v\rangle \qquad \forall\, v \in \mathcal{D}(\Omega).$$

Derivatives are defined in $\mathcal{D}'(\Omega)$ through the integration by parts formula

$$\langle D^\alpha T, v\rangle := (-1)^{|\alpha|}\,\langle T, D^\alpha v\rangle \qquad \forall\, T \in \mathcal{D}'(\Omega),\ \ \forall\, v \in \mathcal{D}(\Omega)$$

and they are linear and continuous operators in $\mathcal{D}'(\Omega)$.
Distributions taking values in Banach spaces can be defined by a similar construction (see for example [28], Chapter 1, Section 1.2, see also [19] Chapter 4 or [35] for more details on the topic).
The space of distributions $\mathcal{D}'(0, T; X)$ on $(0, T)$ taking values in the Banach space X in fact can be defined as follows

$$\mathcal{D}'(0, T; X) := \mathcal{L}(\mathcal{D}(0, T); X)$$

where $\mathcal{L}(X; Y)$ is the space of linear and continuous applications from X to Y. If $f \in \mathcal{D}'(0, T; X)$ one can define its distributional derivative in the following way

$$\frac{\partial f}{\partial t}(\varphi) = -f\,\frac{\partial \varphi}{\partial t} \qquad \forall\, \varphi \in \mathcal{D}(0, T). \tag{A.1.1}$$

For a function $f \in L^p(0, T; X)$ one can introduce a distribution (still denoted by f) on $(0, T)$ taking values in X in the following way

$$f(\varphi) = \int_0^T f(t)\,\varphi(t)\,dt \qquad \varphi \in \mathcal{D}(0, T);$$

this integral is an element of X. Also in this case we can define $\dfrac{\partial f}{\partial t}$ as an element of $\mathcal{D}'(0, T; X)$ using (A.1.1).

A.1.4. Sobolev spaces

We set

$$W^{k,p}(\Omega; B) := \{v \in L^p(\Omega; B) :\ D^\alpha v \in L^p(\Omega; B),\ \forall\, \alpha,\ |\alpha| \le\ k\} \qquad \forall\, k \in \mathbb{N},\ \forall\, p \in [1, +\infty],$$

where D^α is the derivative in the sense of distributions $\Omega \to B$.

For any $p \in [1, +\infty)$ the space $W^{k,p}(\Omega; B)$ coincides with the completion of $\mathcal{C}^\infty(\overline{\Omega}; B)$ with respect to the norm

$$||v||_{W^{k,p}(\Omega;B)} := \left(\sum_{|\alpha| \leq k} ||D^\alpha v||^p_{L^p(\Omega;B)} \right)^{1/p}.$$

We set, for all $k \in \mathbb{N}_0$, for all $\nu \in (0,1)$, for all $p \in [1, +\infty)$,

$$W^{k+\nu,p}(\Omega; B) := \left\{ v \in W^{k,p}(\Omega; B) : \sum_{|\alpha|=k} \iint_{\Omega^2} \frac{||D^\alpha v(x_1) - D^\alpha v(x_2)||^p_B}{|x_1 - x_2|^{(N+\nu p)}} \, dx_1 \, dx_2 < +\infty \right\}$$

and

$$W^{k+\nu,\infty}(\Omega; B) := \mathcal{C}^{k,\nu}(\overline{\Omega}; B);$$

moreover, denoting by $\mathcal{C}^\infty_0(\Omega; B)$ the space of infinitely differentiable functions $\Omega \to B$ with compact support,

$$W^{s,p}_0(\Omega; B) := \text{ closure of } \mathcal{C}^\infty_0(\Omega; B) \text{ in } W^{s,p}(\Omega; B), \qquad \forall\, s > 0, \ \forall p \in [1, +\infty).$$

These are (either real or complex) Banach spaces equipped with the respective graph norms. If either B is reflexive or B' is separable, we have $L^{p'}(\Omega; B') = L^p(\Omega; B)'$ for any $p \in [1, +\infty)$, where $p' := p/(p-1)$ for any $p \in (1, +\infty)$ and $1' := \infty$.
We then set

$$W^{-s,p'}(\Omega; B') := (W^{s,p}_0(\Omega; B))' \qquad \forall\, s > 0, \ \forall\, p \in [1, +\infty).$$

We also set

$$H^s(\Omega; B) := W^{s,2}(\Omega; B)$$

for any $s \in \mathbb{R}$; this is a Hilbert space if the same happens for B.

A.1.5. Some spaces of operators.

We assume that B_1 and B_2 are (real) Banach spaces, and introduce a space of operators:

$$\mathcal{C}^0(B_1; B_2) := \{F : B_1 \to B_2 \text{ strongly continuous}\}.$$

The following result can be easily proved

Proposition A.1.1. *Assume that $F \in \mathcal{C}^0(B_1; B_2)$ and set*

$$[\overline{F}(u)](x) := F(u(x)) \qquad \forall\, x \in \Omega, \ \forall\, u \in \mathcal{M}(\bar{\Omega}; B_1).$$

Then

$$\overline{F} : \mathcal{C}^0(\bar{\Omega}; B_1) \to \mathcal{C}^0(\bar{\Omega}; B_2)$$

and moreover

$$\overline{F} : \mathcal{M}(\Omega; B_1) \to \mathcal{M}(\Omega; B_2)$$

is continuous.
If B_1 and B_2 are (real) Banach spaces of time dependent functions and the operator F is either causal, or rate independent, or order preserving, or piecewise monotone, or fulfills a semigroup property, then the same holds for $\overline{F}$ respectively.

Proposition A.1.2. *Assume that $F : B_1 \to B_2$ is Lipschitz continuous. Then*

$$\overline{F} \in \mathcal{C}^0(\mathcal{C}^0(\bar{\Omega}; B_1); \mathcal{C}^0(\bar{\Omega}; B_2)) \cap \mathcal{C}^0(L^\infty(\Omega; B_1); L^\infty(\Omega; B_2))$$

and $\overline{F}$ maps relatively compact subsets of $\mathcal{C}^0(\bar{\Omega}; B_1)$ into relatively compact subsets of $\mathcal{C}^0(\bar{\Omega}; B_2)$. Moreover, for any $p \in [1, +\infty]$,

$$\overline{F} : L^p(\Omega; B_1) \to L^p(\Omega; B_2)$$

and is Lipschitz continuous,

$$\overline{F} : W^{\lambda,p}(\Omega; B_1) \to W^{\lambda,p}(\Omega; B_2)$$

and is linearly bounded, for any $\lambda \in [0, 1]$.

A.2. A characterization of the spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$

We recall the following characterization of the spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ which can be found in [7], Sections IX.1, IX.4.

Proposition A.2.1. *Suppose to have $u \in L^p(\Omega)$ with $1 < p \leq \infty$. The following properties are equivalent:*
(i) $u \in W^{1,p}(\Omega)$;
(ii) there exists a constant c such that

$$\left| \int_\Omega u \, \frac{\partial \varphi}{\partial x_i} \right| \leq c \, ||\varphi||_{L^{p'}(\Omega)} \qquad \forall \, \varphi \in \mathcal{C}_c^\infty(\Omega), \;\; \forall \, i = 1, \ldots, N;$$

(iii) there exists a constant c such that for any open set $\omega \subset\subset \Omega$ and any $h \in \mathbb{R}^N$ with $|h| < \operatorname{dist}(w, \Omega^c)$ we have

$$||\tau_h \, u - u||_{L^p(\omega)} \leq c \, |h|$$

where $\tau_h u(x) := u(x + h)$. We set $\delta_h u(x) := \tau_h u(x) - u(x) := u(x + h) - u(x)$. Moreover it is possible to choose $c := ||\nabla u||_{L^p(\Omega)}$ in the previous (ii) and (iii).

Suppose now that Ω is a bounded subset of $\mathbb{R}^N$ of class $\mathcal{C}^1$. Let $u \in L^p(\Omega)$ with $1 < p < \infty$. Then also the following properties are equivalent:
(iv) $u \in W_0^{1,p}(\Omega)$;
(v) there exists a constant c such that

$$\left| \int_\Omega u \, \frac{\partial \varphi}{\partial x_i} \right| \leq c \, ||\varphi||_{L^{p'}(\Omega)} \qquad \forall \, \varphi \in \mathcal{C}_c^1(\Omega), \;\; \forall \, i = 1, \ldots, N;$$

(vi) the function

$$\bar{u}(x) = \begin{cases} u(x) & x \in \Omega; \\ 0 & x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

belongs to the space $W^{1,p}(\mathbb{R}^N)$ *and in this case* $\dfrac{\partial \bar{u}}{\partial x_i} = \overline{\dfrac{\partial u}{\partial x_i}}$.

A.3. Some remarks on monotone operators

The results contained in this section can be found in [5] or [8], see also [28], Chapter 2, Sections 1 and 2.
Suppose to deal with a Hilbert triplet

$$V \subset H \equiv H' \subset V',$$

where V is a Banach space (endowed with the norm $|| \cdot ||_V$) contained in H with continuous and dense injection; let ${}_{V'}\langle \cdot, \cdot \rangle_V$ be the duality pairing between V and V'.
An operator $A : V \to V'$ is said to be *hemicontinuous* if the following holds

$$\forall \, u, v, w \in V, \;\text{ the function }\; \lambda \mapsto_{V'} \langle A(u + \lambda v), w \rangle_V \;\text{ is continuous from } \mathbb{R} \text{ to } \mathbb{R}.$$

An operator $A : V \to V'$ is *monotone* if verifies the following property

$$\forall \, u, v \in V \quad {}_{V'}\langle A(u) - A(v), u - v \rangle_V \; \geq \; 0.$$

An operator $A : V \to V'$ is said to be *maximal monotone* if and only if it is not properly included in any other monotone operator $V \to V'$.
An operator $A : V \to V'$ is *coercive* if

$$\frac{{}_{V'}\langle A(u), u \rangle_V}{||u||_V} \to +\infty \qquad\qquad \forall \, u \in V, \;\; ||u||_V \to +\infty.$$

The main theorem of the section is the following

Theorem A.3.1. *Let V be a reflexive Banach space and let $A : V \to V'$ be a monotone, everywhere defined and hemicontinuous operator. Then A is maximal monotone. If in addition A is coercive, then A is surjective.*

A.4. Generalized Poincaré inequality

Theorem A.4.1. *(Generalized Poincaré inequality)*
Assume that Ω is an open subset of $\mathbb{R}^N$ $(N \geq 1)$ of Lipschitz class and connected. Let $\Gamma_1 \subset \partial\Omega$ have positive $(N-1)$-dimensional Hausdorff measure. Then there exists a constant $c > 0$ such that

$$\forall v \in H^1(\Omega) \qquad \int_\Omega |v|^2 \, dx \leq c \left(\int_\Omega |\nabla v|^2 \, dx + \int_{\Gamma_1} |\gamma_0 v|^2 \, d\sigma \right).$$

Proof. By contradiction let us assume that for any $n \in \mathbb{N}$ there exists a sequence $v_n \in H^1(\Omega)$ such that the following holds

$$\int_\Omega |v_n|^2 \, dx \, \geq \, n \left(\int_\Omega |\nabla v_n|^2 \, dx \, + \, \int_{\Gamma_1} |\gamma_0 v_n|^2 \, d\sigma \right).$$

It is not restrictive to assume $||v_n||_{L^2(\Omega)} = 1$ for any $n \in \mathbb{N}$. From the previous inequality we deduce that the sequence $\{v_n\}$ is uniformly bounded in $H^1(\Omega)$, so there exists v such that, possibly passing to a subsequence, $v_n \to v$ weakly in $H^1(\Omega)$. The previous inequality also yields

$$\nabla v_n \to 0 \qquad \text{strongly in } L^2(\Omega; \mathbb{R}^N), \qquad\qquad \gamma_0 \, v_n \to 0 \qquad \text{strongly in } L^2(\Gamma_1).$$

Therefore $\nabla v = 0$ a.e. in Ω and $\gamma_0 v = 0$ a.e. on Γ_1. Hence $v = 0$ in Ω as this set is connected, and this is in contradiction with the fact that the functions v_n are normalized in $L^2(\Omega)$. □

A.5. The Green formulae

In this section we state one of the Green formulae which has been used several times in the previous chapters in order to interpret our model problems. We refer for example to [4] Chapter 18 or [9], Chapter 2, for more details on the topic.
We recall that the *divergence* of a vector-valued function w is defined as

$$\operatorname{div} w = \sum_i D_i \, w_i$$

where w_i are the components of w. We consider the space

$$L^p_{\text{div}}(\Omega) = \{ w \in (L^p(\Omega))^n : \, \operatorname{div} w \in L^p(\Omega) \}$$

endowed with the graph-norm. We notice that $L^p_{\text{div}}(\Omega) = (W^{1,p}(\Omega))^n$ if and only if $n = 1$, while in general $(W^{1,p}(\Omega))^n \subset L^p_{\text{div}}(\Omega)$. For $w \in (W^{1,p}(\Omega))^n$, every component of w has its trace; in the more general case $w \in L^p_{\text{div}}(\Omega)$, al least its *normal component* $w \cdot \nu$ has a trace. We denote

it by the notation $w \cdot \nu_{|\Gamma}$; more in general the symbol $\cdot_{|\Gamma}$ means that the quantity specified is understood in the sense of traces.
Then the following statement holds:

$$\int_\Omega w \cdot \nabla v \, dx + \int_\Omega v \operatorname{div} w \, dx = {}_{W^{-1/p,p}(\Gamma)}\langle w \cdot \nu_{|\Gamma}, v_{|\Gamma} \rangle_{W^{1/p,p'}(\Gamma)}$$
$$\text{for } w \in L^p_{\mathrm{div}}(\Omega),\ v \in W^{1,p'}(\Omega),\ 1 < p < \infty,$$

where Γ is the boundary of Ω, while $W^{-1/p,p}(\Gamma)$ and $W^{1/p,p'}(\Gamma)$ are the Sobolev spaces defined on the boundary of Ω (see again [9], Chapter 2 for more informations on these spaces).
If we apply the previous formula with $v = 1$, we get the generalized Gauss Theorem, i.e.

$$\int_\Omega \operatorname{div} w \, dx = {}_{W^{-1/p,p}(\Gamma)}\langle w \cdot \nu_{|\Gamma}, 1 \rangle_{W^{1/p,p'}(\Gamma)}, \qquad \text{for } w \in L^p_{\mathrm{div}}(\Omega),\ 1 < p < \infty. \tag{A.5.1}$$

A.6. Transposition

For more details on the topic of this section we refer to [4], Chapter 4.
Let X and Y be Banach spaces and $T : X \to Y$ be a linear and continuous operator. The transposed (or adjoint) $T^* : Y' \to X'$ is defined as:

$${}_{X'}\langle T^*\psi, f \rangle_X :=_{Y'} \langle \psi, Tf \rangle_Y \qquad \forall\, f \in X,\ \forall\, \psi \in Y'.$$

Theorem A.6.1. *Under the previous assumptions, we get that*

- T^* *is injective if and only if* $T(X) \subset Y$ *with continuous and dense injection;*
- T *is injective if* $T^*(Y') \subset X'$ *with continuous and dense injection. The converse also holds if* X *is reflexive.*

A.7. Gronwall's Lemma

We refer to [42], Section I.1, see also [16], Section B.2.

Lemma A.7.1. (Gronwall's Lemma - integral form)
Let $0 < T < +\infty$ *and* φ, α, $\beta : [0, T) \to \mathbb{R}$ *be continuous functions, with* α *nondecreasing and* $\beta \geq 0$. *If*

$$\varphi(t) \leq \alpha(t) + \int_0^t \beta(\tau)\, \varphi(\tau)\, d\tau \qquad \forall\, t \in [0, T), \tag{A.7.1}$$

then

$$\varphi(t) \leq \alpha(t) \exp\left(\int_0^t \beta(\tau)\, d\tau\right) \qquad \forall\, t \in [0,T).$$

Assumption (A.7.1) *can be replaced by the weaker condition*

$$\varphi(t) \leq \alpha(t) + \int_0^t \beta(t) \max_{[0,\tau]} |\varphi|\, d\tau$$

for any $t \in [0,T)$.

Lemma A.7.2. (Gronwall's Lemma - differential form)
Let $\eta(\cdot)$ *be a nonnegative, absolutely continuous function on* $[0,T]$, *which satisfies for a.e.* t *the differential inequality*

$$\eta'(t) \leq \phi(t)\, \eta(t) + \psi(t),$$

where $\phi(t)$ *and* $\psi(t)$ *are nonnegative, summable functions on* $[0,T]$, *then*

$$\eta(t) \leq \exp\left(\int_0^t \phi(s)\, ds\right)\left[\eta(0) + \int_0^t \psi(s)\, ds\right].$$

A.8. A theorem for evolution equations

We present now a result for evolution equations of parabolic type (see [7], Chapter X, Theorem X.9, see also [29])

Theorem A.8.1. *Let* H *be a Hilbert space endowed with the scalar product* $(\cdot,\cdot)$ *and with the norm* $|\cdot|$. *Assume that* H *is identified to its dual. Let* V *be another Hilbert space with the norm* $||\cdot||$ *and suppose that* $V \subset H$ *with continuous and dense inclusion, such that*

$$V \subset H \equiv H' \subset V'.$$

Let us consider a bilinear form $a(t;u,v) : V \times V \to \mathbb{R}$ *defined for a.e.* $t \in [0,T]$, *such that*
(i) *the function* $t \mapsto a(t;u,v)$ *is measurable for all* $u, v \in V$;
(ii) $|a(t;u,v)| \leq M\,||u||\,||v||$ *for a.e.* $t \in [0,T]$ *and all* $u, v \in V$;
(iii) $a(t;v,v) \geq \alpha\,||v||^2 - C\,|v|^2$ *for a.e.* $t \in [0,T]$ *and all* $v \in V$,
where $\alpha > 0$ *and* M, C *are given constants.*
Then for any $G \in L^2(0,T;V')$ *and for all* $u_0 \in H$ *there exists a unique function* u *such that*

$$u \in L^2(0,T;V) \cap \mathcal{C}^0(0,T;H) \qquad \frac{du}{dt} \in L^2(0,T;V')$$

and

$$\begin{cases} \langle \dfrac{du}{dt}(t), v\rangle + a(t;u(t),v) = \langle G(t), v\rangle & \text{for a.e. } t \in [0,T], \text{ for all } v \in V \\ u(0) = u_0. \end{cases}$$

A.9. A fixed point Theorem

The following result can be found for example in [28], Chapter 1, Section 4.3.

Theorem A.9.1. *Let us consider a map $\xi \to P(\xi)$ which is continuous from $\mathbb{R}^m$ into $\mathbb{R}^m$ itself, such that, for a suitable $\rho > 0$ we have*

$$(P(\xi), \xi) \geq 0 \qquad \forall\, \xi : |\xi| = \rho,$$

where $(\cdot, \cdot)$ is the scalar product in $\mathbb{R}^m$. Then there exists ξ such that $|\xi| \leq \rho$ and $P(\xi) = 0$.

A.10. The Riesz-Fréchet representation theorem

This theorem can be found for example in [7], Section V.2.

Theorem A.10.1. *(Riesz-Fréchet representation theorem)*
Consider an Hilbert space H, endowed with the scalar product $(\cdot, \cdot)_H$ and the norm $|\cdot|_H$; let H' be its dual and let ${}_{H'}\langle \cdot, \cdot \rangle_H$ be the duality pairing between H and H'. Then, for all $\varphi \in H'$, there exists a unique $f \in H$ such that

$${}_{H'}\langle \varphi, v \rangle_H = (f, v)_H \qquad \forall\, v \in H.$$

Moreover $|f|_H = ||\varphi||_{H'}$.

Proof. We present here two ways of proving this theorem, the first one using the theory of the reflexive spaces.
① Consider the map $T : H \to H'$ as follows: given $f \in H$, the map $v \mapsto (f, v)$ is a linear and continuous form on H; therefore it defines an element of H' which will be denoted by Tf, i.e. ${}_{H'}\langle Tf, v \rangle_H = (f, v)_H$ for all $v \in H$. Using the Cauchy-Schwartz inequality, we have $||Tf||_{H'} = |f|_H$. Therefore T is a linear, isometric, operator of H on $T(H)$, closed subspace of H'. In order to conclude, we have only to prove that $T(H)$ is dense in H'. Let us take $h \in H'' \equiv H$ (as H is reflexive) such that $\langle Tf, h \rangle = 0$ for all $f \in H$. We verify that $h = 0$. One has $(f, h)_H = 0$ for all $f \in H$ and so $h = 0$.
② The second proof does not use the theory of the reflexive spaces. Consider $M = \varphi^{-1}(0)$; M is a closed subspace of H. If $M = H$, then $\varphi = 0$ and we conclude taking $f = 0$. Suppose that $M \neq H$. We prove that there exists $g \in H$ such that $g \notin M$, $|g| = 1$ and $(g, w) = 0$ for all $w \in M$. Indeed, let $g_0 \in H$, $g_0 \notin M$ and consider $g_1 = P_M\, g_0$ (where P_M is the projection of g_0 on the convex set M). We set moreover $g = \dfrac{(g_0 - g_1)}{|g_0 - g_1|}$. For all $v \in H$ we can say that

$v = \lambda_1 g + w$ with $\lambda_1 \in \mathbb{R}$ and $w \in M$; in fact it is enough to set $\lambda_1 = \dfrac{\langle \varphi, v\rangle}{\langle \varphi, g\rangle}$ and $w = v - \lambda_1 g$. It follows that $0 = (g, w) = (g, v - \lambda_1 g)$, i.e. $(g, v) = \lambda_1 = \dfrac{\langle \varphi, v\rangle}{\langle \varphi, g\rangle}$. One concludes with $\langle \varphi, v\rangle = (f, v)$ for all $v \in H$, where $f = \langle \varphi, g\rangle\, g$.

A.11. Some basic concepts in fluid dynamics

The concepts presented in this section can be found in any introductory text of fluid dynamics, we refer for example to [32], Chapter 1.

In the study of the motion of a continuum, it is possible to adopt two points of view. In one case, the observer considers the locations of the particles of the continuum as functions of time. On the other hand, one can study the velocity field at a fixed point in space. The first approach is known as the Lagrangian point of view, the second as the Eulerian one.

In order to formulate these concepts in mathematical terms, let us introduce two different coordinates systems. The first one is based upon an orthonormal, right-handed Cartesian frame of reference, with base vectors e_i $(i = 1, 2, 3)$ such that $e_i \cdot e_j = \delta_{ij}$. In this frame of reference, the vector

$$\mathbf{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

determines the location of a point P and x_1, x_2, x_3 are the Cartesian coordinates of P.

Alternatively we may consider a right-handed system of *material coordinates* that is a system of curvilinear coordinates that moves with the continuum. We will denote them by ξ^α $(\alpha = 1, 2, 3)$. This means that a given set of values of the variables ξ^α corresponds to the same physical particle at all times. Suppose that the motion of the continuum can be represented by the function

$$\mathbf{x} = \mathbf{x}(\xi^\alpha, t) \tag{A.11.1}$$

which gives the location $\mathbf{x}$ of a material particle identified by the material coordinates ξ^α, as a function of time. Under suitable assumptions of regularity, equation (A.11.1) may be inverted to give

$$\xi^\alpha = \xi^\alpha(\mathbf{x}, t). \tag{A.11.2}$$

In the Lagrangian point of view, where one observes the motion of the material particles, the coordinates ξ^α are kept fixed as time varies; in this approach, for any quantity f, we have

$$f = \bar{f}(\xi^\alpha, t). \tag{A.11.3}$$

In the Eulerian point of view, instead, in which the motion is analyzed by studying its properties at a given point in space during the course of time, for any function f, combining (A.11.2) and

(A.11.3) we obtain

$$f = \bar{f}(\xi^\alpha(\mathbf{x}, t), t) = \hat{f}(\mathbf{x}, t) \tag{A.11.4}$$

and in a similar way, (A.11.3) can be obtained combining (A.11.2) and (A.11.4).
For our purposes, it is important to emphasize the difference between the time derivatives with the spatial coordinates kept constant, (i.e. in a fixed point in space, *Eulerian derivative*) and that with material coordinates kept constant. The latter is the time derivative performed by an observer travelling with a material particle, and is referred to as the *material derivative or substantial derivative* of f. Thus, for the Eulerian derivative, we will write

$$\frac{\partial f}{\partial t} = \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} = \frac{\partial \hat{f}}{\partial t}$$

where $\ldots|_{\mathbf{x}}$ indicates that the time derivative is taken with $\mathbf{x}$ constant. On the other hand, for the substantial derivative we will use the notation

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\xi^\alpha} = \frac{\partial \bar{f}}{\partial t}$$

where $\ldots|_{\xi^\alpha}$ indicates that the time derivative is taken with ξ^α constant.
The relationship between substantial derivative and the Eulerian time derivative is obtained by using their definitions and the Leibniz chain rule to get

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\xi^\alpha} = \left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} + \frac{\partial f}{\partial x_i} \cdot \left.\frac{\partial x_i}{\partial t}\right|_{\xi^\alpha} = \frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i}.$$

or

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla. \tag{A.11.5}$$

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