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AN INNOVATIVE APPROACH BASED ON A TREE-SEARCHING ALGORITHM FOR THE OPTIMAL MATCHING OF INDEPENDENTLY OPTIMUM SUM AND DIFFERENCE EXCITATIONS

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An Innovative Approach Based on a Tree-Searching Algorithm for the Optimal Matching of Independently Optimum Sum and Difference Excitations

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Abstract

An innovative approach for the optimal matching of independently optimum sum and difference patterns through sub-arrayed monopulse linear arrays is presented. By exploiting the relationship between the independently optimal sum and difference excitations, the set of possible solutions is considerably reduced and the synthesis problem is recast as the search of the best solution in a non-complete binary tree. Towards this end, a fast resolution algorithm that exploits the presence of elements more suitable to change sub-array membership is presented. The results of a set of numerical experiments are reported in order to validate the proposed approach pointing out its effectiveness also in comparison with state-of-the-art optimal matching techniques.

Key words: Linear Arrays, Monopulse Antennas, Sum and Difference Pattern Synthesis, Tree-Searching Algorithm.

1 Introduction

A tracking radar system using the monopulse technique [1] can be realized through an antenna array able to generate two different patterns, namely the difference pattern and the sum pattern. These patterns are required to satisfy some constraints as narrow beamwidth, low side lobe level (SLL) and high directivity. In particular, as far as the sum pattern is concerned, there is the need of maximizing the gain. On the other hand, the more critical issues to be addressed dealing with difference patterns are concerned with both the first null beamwidth and the normalized difference slope on boresight direction, since they are strongly related to the sensitivity of the radar (i.e., to the angular error).
The optimal excitation coefficients for the sum and the difference patterns can be independently computed by using analytical methods as described in [2] and in [3], respectively. Nevertheless, the implementation of two independent feed networks is generally unacceptable because of the costs, the occupied physical space, the circuit complexity and the arising interferences. Thus, it is necessary to find a suitable compromise between the feed network complexity and the closeness of the synthesized sum and difference patterns to the optimal ones. Since the sum pattern is used in both signal transmission and reception, the most common way to solve the problem consists in generating an optimal sum pattern and a sub-optimal difference pattern [4], the latter synthesized by applying a sub-arraying technique. Accordingly, the synthesis is aimed at optimizing pre-specified sub-array layouts by sinthesizing sub-array and radiating element weights, but not the actual beamforming network.
In such a framework, several approaches for defining how the elements could be grouped and the sub-arrays weights computed have been proposed. As far as linear arrays are concerned, McNamara proposed in [4] the *Excitation Matching* method (EMM) aimed at determining a *best compromise* difference pattern close as much as possible to the optimum in the Dolph-Chebyshev sense [5] (i.e., narrowest first null beamwidth and largest normalized difference slope on the boresight for a specified sidelobe level). Towards this end, for each possible grouping, the corresponding sub-arrays coefficients are iteratively computed through pseudo-inversion of an overdetermined system of linear equations. It

is evident that since the best sub-array configuration is not *a-priori* known, the whole process is extremely time-expensive due to the exhaustive evaluations. Moreover, because of the ill-conditioning of the matrix system, large arrays cannot be easily managed.

In order to overcome the ill-conditioning and related issues, optimization approaches have been widely used [6][7][8][9][10]. Although such techniques allows a significant advancement in the framework of sum-difference pattern synthesis, they are still time-consuming when dealing with large arrays. As a matter of fact, even though the solution space is sampled with efficient searching criteria, the dimension of the solution space is very large.

In order to overcome such drawbacks allowing an effective choice of the array elements grouping as well as a fast and simple solution procedure, this paper proposes an innovative approach that, likewise [4] and unlike [6][7][8][9][10], is aimed at obtaining a compromise difference pattern optimum in the Dolph-Chebyshev sense [5] starting from the observation that the sub-arraying is not *blind*. As a matter of fact, it can be guided by considering similarity properties among the array elements, thus significantly reducing the dimension of the solution space. Starting from such an idea and by representing each solution by means of a path in a non-complete binary tree, the synthesis problem is then recast as the searching of the minimal-cost path from the root to the leafs of the solution tree. In graph theory, a tree is a graph defined as a non-empty finite set of vertices or nodes in which any two nodes are connected by exactly one path. The nodes are labeled such that there is only one node called the *root* of the tree, and the remaining nodes are partitioned in subtrees. In our case, since the tree is either empty or each node has not more than two subtrees, it is a *binary tree*. Accordingly, each node of a binary tree has either (I) no children, or (*ii*) one left/right child (i.e., non-complete binary tree), or (*iii*) a left child and a right child (i.e., complete binary tree), each child being the root of a binary tree called a subtree [11][12]. In order to solve the problem at hand, thus efficiently exploring the solution tree, suitable cost functions or *metrics* are defined and an innovative algorithm for the exploration of the solution space is defined by exploiting the *closeness* (to a sub-array) property of some elements, called *border elements*, of the array.

The paper is organized as follows. In Section 2, the problem is mathematically formulated defining a set of metrics aimed at quantifying the closeness of each solution to the optimal one (Sect. 2.1) as well as the tree structure (Sect. 2.2) and the algorithm for effectively exploring the solution space (Sect. 2.3). In Section 3, the results of selected numerical experiments are reported and compared with those from state-of-the-art optimal matching solutions. Conclusions and future possible trends are drawn in Section 4.

2 Mathematical Formulation

Let us consider a linear uniform array of $N = 2M$ elements $\{\xi_m;\ m = -M, ..., -1, 1, ..., M\}$. Following a sub-optimal strategy, the sum pattern is generated by means of the symmetric set of the real optimal [(1)] excitations $\underline{A}^{opt} = \{\alpha_m;\ m = 1, ..., M\}$ [2][13], while the difference pattern is defined through an anti-symmetric real excitation set $\underline{B} = \{b_m = -b_{-m};\ m = 1, ..., M\}$ [5]. Thanks to such symmetry properties, one half of the elements of the array $\underline{S} = \{\xi_m;\ m = 1, ..., M\}$ is descriptive of the whole array.

Grouping operation yields to a sub-array configuration mathematically described in terms of the grouping vector $\underline{C} = \{c_m;\ m = 1, \ldots, M\}$, $c_m \in [1, Q]$ being the sub-array index of the m-th element of the array [7]. Successively, a weight coefficient w_q is associated to each sub-array, $q = 1, ..., Q$, and, as a consequence, the sub-optimal difference excitation set is given by

$$\underline{B} = \{b_m = w_{mq}\alpha_m;\ m = 1, ..., M;\ q = 1, ..., Q\} \tag{1}$$

where $w_{mq} = \delta_{c_m q} w_q$ ($\delta_{c_m q} = 1$ if $c_m = q$, $\delta_{c_m q} = 0$ otherwise) is the weight associated to the m-th array element belonging to the q-th sub-array.

Accordingly, the original problem is recast as the definition of a sub-array configuration $\underline{C}$ and the corresponding set of weights $\underline{W} = \{w_q;\ q = 1, ..., Q\}$ such that the sub-optimal difference pattern $\underline{B}$ is as close as possible to the optimal one, $\underline{B}^{opt} = \{\beta_m;\ m = 1, ..., M\}$. Towards this end, let us formally proceed as follows. Firstly, two different metrics are defined in order to quantify the closeness of the sub-optimal solution to the optimal one. Then, exploiting some properties of the sub-array configurations, a non-complete binary

[(1)] In the Dolph-Chebyshev sense [5], unless mentioned elsewhere.

tree, where each path codes a possible elements grouping, is built. Finally, a simple algorithm for a fast search of the lowest cost path in the binary tree is presented for defining the best sub-optimal solution of the problem in hand.

2.1 Definition of the Solution-Metric

In order to find the optimal solution, let us define a suitable cost function or *metric* that quantifies the closeness of every candidate/trial solution $\underline{C}_t$ to the optimal one,

$$\Psi\left\{\underline{C}_t\right\} = \sum_{m=1}^{M}\left[v_m - d_m\left\{\underline{C}_t\right\}\right]^2, \tag{2}$$

where v_m and d_m are reference and estimated parameters, respectively. The estimated parameters $d_m\left\{\underline{C}_t\right\}$ are defined as the arithmetic mean of the reference parameters v_m related to the array elements belonging to the same sub-array. As far as the reference parameters $\underline{V} = \{v_m;\, m = 1, ..., M\}$ and the sub-arrays weights $\underline{W} = \{w_q;\, q = 1, ..., Q\}$ are concerned, they are defined according to two different strategies, namely the *Gain Sorting* (GS) algorithm and the *Residual Error* (RES) algorithm.

Concerning the GS technique, the reference parameters $v_m^{(GS)}$ are set to the *optimal gain*s

$$v_m^{(GS)} = \frac{\beta_m}{\alpha_m}, \qquad m = 1, \dots, M, \tag{3}$$

while the sub-array weights are assumed to be equal to the *computed gains* $d_m^{(GS)}$

$$w_q^{(GS)} = \delta_{c_m q} d_m^{(GS)}\left\{\underline{C}_t^{(ess)}\right\}, \qquad q = 1, ..., Q,\ m = 1, \dots, M. \tag{4}$$

Concerning the RES algorithm, the reference parameters are equal to the the so-called *optimal residual errors* $v_m^{(RES)}$ given by

$$v_m^{(RES)} = \frac{\alpha_m - \beta_m}{\beta_m}, \qquad m = 1, \dots, M. \tag{5}$$

Accordingly, since $\frac{\beta_m}{\alpha_m} = \frac{1}{1+v_m^{(RES)}}$, $m = 1, \dots, M$, the sub-array weights are expressed in terms of the *computed residual errors* $d_m^{(RES)}$ as follows

$$w_q^{(RES)} = \frac{1}{1 + \delta_{c_m q} d_m^{(RES)} \left\{ \underline{C}_t^{(ess)} \right\}}, \qquad q = 1, \ldots, Q, \; m = 1, \ldots, M. \tag{6}$$

2.2 Definition of the Solution-Tree

In general, the total number of sub-array configurations is equal to $T = Q^M$ since each of them might be expressed as a sequence of M digits in a Q-based notation system. Without any loss of information, such a number can be reduced by considering only the *admissible* (or *reliable*) solutions, i.e., grouping where there are no empty sub-arrays. Moreover, let us observe that if an *equivalence* relationship [(2)] among sub-array configurations holds true, it is convenient to consider just one sub-array configuration for each set (instead of the whole set), therefore obtaining a set of *non-redundant* solutions.

Now, let us sort the known reference parameters $\{v_m;\ m = 1, ..., M\}$ [computed according to either the GS (3) or the RES algorithm (5)] for obtaining a ordered list $\underline{L} = \{l_m;\ m = 1, ..., M\}$, where $l_i \leq l_{i+1}$, $i = 1, ..., M-1$, $l_1 = min_m \{v_m\}$, and $l_M = max_m \{v_m\}$. Since the cost function is minimized provided that elements belonging to each sub-array are consecutive elements of the ordered list $\underline{L}$ (see ***Appendix*** A for a detailed proof), the solution space can be further reduced to the so-called *essential* solution space $\Re^{(ess)}$ composed by *allowed* solutions. Consequently, the dimension T of the solution space turns out to be reduced from $T = Q^M$ up to $T^{(ess)} = \begin{pmatrix} M-1 \\ Q-1 \end{pmatrix}$ (see ***Appendix B*** for a detailed proof) and the essential solution space $\Re^{(ess)}$ can be formally represented by means of the non-complete binary tree depicted in Figure 1. In particular, each complete path in the tree codes an *allowed* sub-array configuration $\underline{C}_t^{(ess)} \in \Re^{(ess)}$ and the positive integer q inside each node at the l_m-th level indicates that the array element identified by l_m is a member of the q-th sub-array. Thanks to this formulation,

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(2) A sub-array configuration $\underline{C}_i$ is equivalent to the configuration $\underline{C}_j$ when it is possible to obtain the one from the other just using a different numbering for the same c_m coefficients. As an example, the sub-array configuration $\underline{C}_i = \{1, 2, 3, 3, 2, 3, 2, 1\}$ is equivalent to $\underline{C}_j = \{2, 3, 1, 1, 3, 1, 3, 2\}$.

the original minimization problem (i.e., $\underline{C}_{opt} = arg\left\{min_{t=1,...,T}\left[\Psi\left(\underline{C}_t\right)\right]\right\}$) is recast as that of finding the optimal path in the solution tree.

2.3 Tree-Searching Procedure

Although the set of candidate solutions has been considerably reduced by limiting the solution space to the *essential* space, its dimension $T^{(ess)}$ becomes very large when $M \gg Q$ and an exhaustive searching would be computationally expensive. In order to overcome such a drawback, let us observe that only some elements of the list L are candidate to change their sub-array membership without violating the sorting condition of the *allowed* sub-array configurations, $\left\{\underline{C}_t^{(ess)};\, t = 1, ..., T^{(ess)}\right\}$ [see Eq. (14) - ***Appendix*** B]. These elements, referred to as *border elements*, satisfy the following property: an array element related to l_m is a *border element* if one of the elements whose list value is l_{m-1} or/and l_{m+1} belongs to a different sub-array. Therefore, the aggregation $\underline{C}_{opt} \in \Re^{(ess)}$ minimizing the cost function Ψ is found starting from an initial path randomly chosen among the set of paths in the solution tree and iteratively updating the candidate solution just modifying the membership of the border elements. More in detail, the iterative procedure (k being the iteration index) consists of the following steps.

- *Step* 0 - ***Initialization***. Initialize the iteration counter ($k = 0$) and the sequence index ($m = 0$). Randomly generate a trial path in the solution tree corresponding to a candidate sub-arrays configuration $\underline{C}^{(0)} \in \Re^{(ess)}$. Set the optimal path to $\left.\underline{C}_{opt}^{(k)}\right\rfloor_{k=0} = \underline{C}^{(0)}$.

- *Step* 1 - ***Cost Function Evaluation***. Compute the cost function value of the current candidate path $\underline{C}^{(k)}$ by means of (2), $\Psi^{(k)} = \Psi\left\{\underline{C}^{(k)}\right\}$. Compare the cost of the aggregation $\underline{C}^{(k)}$ to the best cost function value attained at any iteration up to the current one, $\Psi_{opt}^{(k-1)} = min_{h=1,...,k-1}\left(\Psi\left\{\underline{C}^{(h)}\right\}\right)$ and update the optimal trial solution $\underline{C}_{opt}^{(k)} = \underline{C}^{(k)}$ if $\Psi\left\{\underline{C}^{(k)}\right\} < \Psi\left\{\underline{C}_{opt}^{(k-1)}\right\}$.

- *Step* 2 - ***Convergence Check***. If the *termination criterion*, based on a maximum number of iterations K or on a stationary condition for the fitness value (i.e.,

$\frac{\left|K_{window}\Psi_{opt}^{(k-1)}-\sum_{j=1}^{K_{window}}\Psi_{opt}^{(j)}\right|}{\Psi_{opt}^{(k)}} \leq \eta$, K_{window} and η being a fixed number of iterations and a fixed numerical threshold, respectively), is satisfied then set $\underline{C}_{opt} = \underline{C}_{opt}^{(k)}$ and stop the minimization process. Otherwise, go to Step 3.

- *Step* 3 - ***Iteration Updating***. Update the iteration index ($k \leftarrow k+1$) and reset the sequence index ($m = 0$).

- *Step* 4 - ***Sequence Updating***. Update the sequence index ($m \leftarrow m+1$). If $m > M$ then go to Step 3 else go to Step 5.

- *Step* 5 - ***Aggregation Updating***. If the array element related to $l_m^{(k)}$ is a border element belonging to the q-th sub-array then define a new grouping $\underline{C}^{(k,m)}$ by aggregating such an element to the $(q-1)$-th sub-array [if the array element corresponding to $l_{m-1}^{(k)}$ is a member of the $(q-1)$-th sub-array] or to the $(q+1)$-th sub-array [if the array element corresponding to $l_{m+1}^{(k)}$ is a member of the $(q+1)$-th sub-array]. If $\Psi^{(k,m)} = \Psi\left\{\underline{C}^{(k,m)}\right\} < \Psi\left\{\underline{C}^{(k)}\right\}$ then set $\underline{C}^{(k)} = \underline{C}^{(k,m)}$ and go to Step 1. Otherwise, go to Step 4.

3 Numerical Simulations and Results

In order to assess the effectiveness of the proposed method, an exhaustive set of numerical experiments has been performed and some representative results will be shown in the following.

For a quantitative evaluation, a set of beam pattern indexes has been defined and computed. More in detail, (a) the *pattern matching* Δ that quantifies the *distance* between the synthesized sub-optimal pattern and the optimal one

$$\Delta = \frac{\int_0^{\pi}\left|\left|AF\left(\psi\right)\right|_n^{opt} - \left|AF\left(\psi\right)\right|_n^{rec}\right| d\psi}{\int_0^{\pi}\left|AF\left(\psi\right)\right|_n^{opt} d\psi}, \tag{7}$$

where $\psi = (2\pi d/\lambda)\, sin\theta$, $\theta \in [0, \pi/2]$, (λ and d being the free-space wavelength and the inter-element spacing, respectively), $\left|AF\left(\psi\right)\right|_n^{opt}$ and $\left|AF\left(\psi\right)\right|_n^{rec}$ are the normalized optimal and generated array patterns, respectively; (b) the main lobes *beamwidth* B_W and (c) the *power slope* P_{slo} that give some indications on the slope on the boresight direction

$$P_{slo} = 2 \times \left[\max_{\psi} \left(\left| AF\left(\psi\right) \right|_n \right) \times \psi_{max} - \int_0^{\psi_{max}} \left| AF\left(\psi\right) \right|_n d\psi \right], \tag{8}$$

ψ_{max} being the angular position of the maximum in the array pattern; (d) the *sidelobes power* P_{sll}

$$P_{sll} = \int_{\psi_1}^{\pi} \left| AF\left(\psi\right) \right|_n d\psi, \tag{9}$$

where ψ_1 is the angular position of the *first null* in the difference beam pattern.
The remaining of this section is organized as follows. Firstly, some experiments aimed at showing the asymptotic behaviour of the proposed solution are presented (Sect. 3.1) and a comparative study is carried out (Sect. 3.2). Furthermore, some experiments devoted at showing the potentialities of the proposed solution in dealing with large arrays are discussed in Sect. 3.3. Finally, the computational issues are analyzed (Sect. 3.4).

3.1 Asymptotic Behavior Analysis

In order to assess that increasing the number of sub-arrays Q the synthesized difference patterns get closer and closer to the optimal one, let us consider a linear array of $N = 2 \times M = 20$ elements characterized by a $d = \frac{\lambda}{2}$ inter-element spacing. The optimal sum pattern excitations, $\{\alpha_m,\ m = 1, ..., M\}$, have been fixed to that of the linear Villeneuve pattern [13] with $\overline{n} = 4$ and $25\,dB$ sidelobe ratio (Fig. 2 - *Villeneuve*, 1984), while the optimal difference weights $\{\beta_m,\ m = 1, ..., M\}$, have been chosen equal to those of a Zolotarev difference pattern [5] with a sidelobe level $SLL = -30\,dB$ (Fig. 24 - *McNamara*, 1993). Then, Q has been varied between 2 and M and both GS and RES techniques have been applied. For sake of space, selected results concerned with $Q = 3$, $Q = 6$, and $Q = 9$ are reported in terms of difference excitations [Fig. 2(a) - GS approach; Fig. 2(b) - RES approach]. As expected, the coefficients obtained with both the GS and RES converge to the optimal ones and, starting from $Q = 6$, the differences between generated and reference difference patterns turn out to be smaller and smaller.

3.2 Comparative Assessment

For comparison purposes and in the framework of synthesis techniques aimed at determining the *best compromise* difference pattern as close as possible to the optimal one, let us consider the EMM by *McNamara* [4] as reference (3). As far as the test cases are concerned, the same benchmark investigated in [4] has been taken into account. The array geometry and the optimal sum excitations was as in Sect. 3.1, while the optimal difference excitation vector $\underline{B}^{opt}$ has been chosen for generating a modified Zolotarev difference pattern with $\overline{n} = 4$, $\varepsilon = 3$ and a sidelobe ratio of $25\,dB$ [3].

The first test case deals with a uniform sub-arraying over the antenna with $Q = 5$. The values of the sub-arrays weights optimized with the GS and the RES are $\underline{W}^{GS} = \{0.2951,$ $0.8847,\ 1.1885,\ 1.3994,\ 1.4878\}$ and $\underline{W}^{RES} = \{0.3411,\ 0.8915,\ 1.1193,\ 1.4016,\ 1.4881\}$, respectively. Moreover, the synthesized difference patterns are shown in Figure 3, while the computed beam-pattern indexes are reported in Table I. The advantages on the use of the tree-based approaches are evident, as confirmed by the values of both the SLL (almost $4\,dB$ below the level achieved by the EMM, $SLL^{EMM} = -17.00\,dB$ vs. $SLL^{GS} = -21.00$ and $SLL^{RES} = -20.50$) and the *pattern matching* index ($\frac{\Delta^{EMM}}{\Delta^{RES}} \simeq 1.4$ and $\frac{\Delta^{EMM}}{\Delta^{GS}} \simeq 1.5$ - Tab. I). Moreover, it is worth noting that, thanks to the structure of the solution tree, the dimension of the *essential* space reduces to $T^{(ess)} = 1$ (since l_1 and l_2 belong to the first sub-array, l_3 and l_4 to the second one, and so on), thus allowing a significant saving of computational resources. As a matter of fact, the EMM requires the solution of an overdetermined system of linear equations in correspondence with any possible uniform grouping [4], i.e., a number of $T = 945$ evaluations.

Second and third test cases consider non-uniform sub-arraying. The former configuration is an example of the limited number of sub-arrays ($Q = 3$) that might be used with a small monopulse antenna. The latter has the same number of sub-arrays as that of the first configuration ($Q = 5$). The tree-based algorithms have been applied and the following sub-array configurations have been determined. In particular the same grouping

(3) No comparison with optimization-based procedures (i.e., [6][7][8][9][10]) have been reported since they are aimed at minimizing a pattern parameter (e.g., the SLL) and not at better matching an optimal difference pattern.

$\underline{C}_{opt}^{GS,RES} = \{1, 2, 3, 3, 4, 5, 5, 5, 4, 3\}$ has been synthesized when $Q = 5$, while $\underline{C}_{opt}^{GS} = \{1, 1, 2, 2, 3, 3, 3, 3, 3, 2\}$ and $\underline{C}_{opt}^{RES} = \{1, 2, 3, 3, 3, 3, 3, 3, 3, 3\}$ have been obtained for $Q = 3$. The obtained beam patterns are shown in Fig. 4 and the corresponding values of the pattern indexes are reported in Tab. II. As it can be noticed, the GS and RES improve the performances of the EMM in matching the optimal difference pattern as pointed out by the behavior of the *global* matching index Δ ($\frac{\Delta^{EMM}}{\Delta^{GS}}\Big\rfloor_{Q=3} = 1.33$ and $\frac{\Delta^{EMM}}{\Delta^{RES}}\Big\rfloor_{Q=3} = 1.42$; $\frac{\Delta^{EMM}}{\Delta^{GS}}\Big\rfloor_{Q=5} = 1.63$ and $\frac{\Delta^{EMM}}{\Delta^{RES}}\Big\rfloor_{Q=5} = 1.68$). Concerning the smaller configuration, it is further confirmed (as already pointed out in Section 3.1) the flexibility and reliability of the GS algorithm in dealing also with complex cases where a limited number of sub-arrays is taken into account. As a matter of fact, for $Q = 3$ the GS gives the best performances getting the highest sidelobe ratio of$SLL = 18.63\,dB$ and synthesizing a main lobe very close to the optimal one, i.e., $B_w^{GS} = B_w^{opt} = 0.3735$ and $P_{slo}^{GS} = 0.1800$ vs. $P_{slo}^{opt} = 0.1802$.

3.3 Large Arrays Analysis

This section is aimed at analyzing the performances of the proposed tree-based techniques when dealing with large arrays. As far as the optimal setup is concerned, sum $\{\alpha_m,\, m = 1, ..., M\}$ and difference $\{\beta_m,\, m = 1, ..., M\}$ optimal excitations have been chosen to generate a Dolph-Chebyshev pattern [15] with $SLL = -25\,dB$ and a Zolotarev pattern [5] with $SLL = -30\,dB$, respectively.

As a first experiment, a linear array of $N = 200$ elements with $\lambda/2$ spacing has been used by considering various sub-arraying configurations. Figure 5 shows the optimal difference pattern (i.e., the synthesis target) and the patterns obtained when $Q = 4$ and $Q = 6$ by using both GS and RES. For completeness, the values of the synthesized difference excitations are displayed in Figure 6. It is worth noting that the GS algorithm outperforms the RES. As a matter of fact, although both approaches satisfactorily approximate the optimal main lobe characteristics in terms of both B_W and P_{slo}, the solutions computed with the gain-based logic present higher sidelobe ratios ($SLL^{GS}\Big\rfloor_{Q=4} = -21.90$ and $SLL^{GS}\Big\rfloor_{Q=6} = -25.13$) with an enhancement of more than $10\,dB$ and $5\,dB$ with respect to the RES approach ($SLL^{RES}\Big\rfloor_{Q=4} = -10.10$ and $SLL^{RES}\Big\rfloor_{Q=6} = -19.95$), respectively. Moreover, the overall matching performances turn out significantly increased as further confirmed by the values of Δ ($\frac{\Delta^{RES}}{\Delta^{GS}}\Big\rfloor_{Q=4} \simeq 3.77$ and $\frac{\Delta^{RES}}{\Delta^{GS}}\Big\rfloor_{Q=6} \simeq 2.47$).

The last test case (and second experiment dealing with large structures) is concerned with a linear array of $N = 2 \times M = 500$ elements ($d = \lambda/2$). As a representative example, the case of $Q = 4$ is reported and analyzed (Tab. III). The arising beam patterns allow one to drawn similar conclusions to those from the previous scenario, since once again the effectiveness of the GS technique in dealing with a limited number of sub-arrays is pointed out. As a matter of fact, the ratio between the matching indexes turns out quite large and equal to $\frac{\Delta^{RES}}{\Delta^{GS}}\Big\rfloor_{Q=4} \simeq 4.1$ (Tab. III). On the other hand, it is worth noting that unlike tree-based procedures the EMM is not reliable in dealing with large arrays since it requires the numerical processing of overdetermined linear systems, whose ill-conditioning get worse when the ratio $\frac{M}{Q}$ grows.

3.4 Computational Issues

Now, let us analyze the computational costs of the tree-based approaches, providing a comparison with the EMM, as well. Towards this end, let us firstly consider the dependence of the dimension of the solution space on the number of elements of the array M. As a representative case, let us analyze the behavior of T and $T^{(ess)}$ when $Q = 3$ ($K = 100$ and $\eta = 10^{-3}$) (Fig. 7). As it can be observed, the dimension of the solution space T of the EMM grows exponentially with M, while, as expected [see ***Appendix A***], $T^{(ess)}$ shows a polynomial behavior. Obviously, the same behavior holds true also for different values of Q (Fig. 7).

On the other hand, the computational effectiveness of the *Tree-Searching* procedure in sampling the solution space is further pointed out from the evaluation of the CPU-time, t, needed for reaching the convergence (Fig. 8). As a matter of fact, $max_Q\{t_Q\} = 70$ $[sec]$ ($k_{opt} = 90$) in correspondence with the largest array ($M = 250$), while $max_Q\{t_Q\} = 12.8$ $[sec]$ ($k_{opt} = 8$) and $max_Q\{t_Q\} = 2.3$ $[sec]$ ($k_{opt} = 4$) when $M = 100$ and $M = 50$, respectively.

4 Conclusions

In this paper, an innovative approach for the synthesis of sub-arrayed monopulse antennas by matching independently-optimum sum and difference excitations has been proposed. By exploiting some properties of the sub-array configurations, the problem of finding a "best compromise" difference pattern by grouping array elements has been recast as the search of the optimum, in terms of either the GS or the RES logic, path inside a non-complete binary tree. Towards this purpose, a fast resolution algorithm has been defined and assessed by means of several numerical experiments.

Concerning the methodological novelties of this work, the main contribution is concerned with the following issues: (a) an appropriate definition of the solution space; (b) an original and innovative formulation of the sum-difference problem in terms of a search in a non-complete binary tree; (c) a simple and fast solution procedure based on swapping operations among border elements and cost function evaluations.

Moreover, the main features of the proposed tree-based techniques are the following: (i) a reduction of the dimensionality $T^{(ess)}$ of the synthesis problem, by exploiting the information content of independently optimal sum and difference excitations; (ii) a significant reduction of the computational burden, by applying a fast solution algorithm for exploring the solution tree (i.e., sampling the solution space); (iii) the capability to deal with large-arrays synthesis in an effective and reliable way.

Because of the favorable trade-off between complexity/costs and effectiveness, the proposed tree-based strategy seems a promising tool to be further analyzed and extended to other geometries and synthesis problems. Towards this purpose, further methodological studies will be oriented in two different directions: (I) improving the solution procedure by developing a customized combinatorial approach, thus further reducing the computational costs as well as improving the convergence rate; (II) re-formulating the sum/difference optimization problem (dealt with in [6][7][8]) in terms of a binary-tree exploration.

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Appendix A

This appendix is aimed at proving that, given Q sub-arrays, the value of the cost function (2) is minimum provided that the elements belonging to each sub-array are consecutive elements of the ordered list $L = \{l_m;\ m = 1, \dots, M;\ l_m \leq l_{m+1}\}$. With reference to a set of elements $\underline{V} = \{v_m;\ m = 1, ..., M\}$ be to be divided in Q sub-sets, the thesis to be proved is that the partition minimizing the cost function (2) is a *contiguous partition* (i.e., if two elements v_i and v_n belong to the same class and $v_i < v_j < v_n$, then element v_j is assigned to the same subset of elements). Towards this end, the proof follows the guidelines reported in [16].

Let us consider a *non-contiguous* partition $\underline{P}_Q = \{V_q;\ q = 1, ..., Q\}$ of the set $\underline{V}$ and three elements v_i, v_j, v_n such that $v_i < v_j < v_n$. Let elements v_i and v_n belong to a subset with mean value d_r and let v_j belong to a different subset having mean value d_s. Whatever the values of d_r and d_s, at least one the following statements holds true

$$\begin{cases} |v_j - d_s| \geq |v_j - d_r| > 0, \\ |v_i - d_r| \geq |v_i - d_s| > 0, \\ |v_n - d_r| \geq |v_n - d_s| > 0. \end{cases} \tag{10}$$

Let us denote with v_t the element satisfying (10) and its own subset as $\underline{V}_k = \{v_k;\ k = 1, ..., N_k\}$. Moreover, let us refer to the other subset as $\underline{V}_h = \{v_h;\ h = 1, ..., N_h\}$. Accordingly, the cost function (2) associated to the partition $\underline{P}_Q$ may be written as:

$$\Psi = \sum_{m=1}^{M} v_m^2 - N_k \cdot d_k^2 - N_h \cdot d_h^2 - \sum_{q=1;\, q \neq h,k}^{Q} N_q \cdot d_q^2 \tag{11}$$

N_q and d_q being the number of elements and the mean value of the q-th sub-array, respectively.

Now, let us consider a new partition $\underline{P}_Q^{(1)}$ obtained by moving the element v_t from the subset $\underline{V}_k$ to the subset $\underline{V}_h$. We obtain two new subsets $\underline{V}_k^{(1)} = \underline{V}_k \setminus \{v_t\}$ and $\underline{V}_h^{(1)} = \underline{V}_k \cup \{v_t\}$ [4] with mean values equal to $d_k^{(1)} = \frac{N_k d_k - v_t}{N_k - 1}$ and $d_h^{(1)} = \frac{N_h d_h + v_t}{N_h + 1}$, respectively. Accordingly, the cost function associated to the partition $\underline{P}_Q^{(1)}$ can be written as:

$$\Psi^{(1)} = \sum_{m=1}^{M} v_m^2 - \frac{(N_k d_k - v_t)^2}{N_k - 1} - \frac{(N_h d_h - v_t)^2}{N_h - 1} - \sum_{q=1;\, q\neq h,k}^{Q} N_q d_q^2. \tag{12}$$

Now, by subtracting (12) from (11), after some manipulations, it turns out that

$$\Psi - \Psi^{(1)} = \frac{N_k}{N_k - 1}\left(v_t - d_k\right)^2 - \frac{N_h}{N_h + 1}\left(v_t - d_h\right)^2. \tag{13}$$

According to (10), $\Psi > \Psi^{(1)}$ and it can be concluded that for every *non-contiguous* partition we can find another one with the same number of subsets, but with a smaller cost. Hence, the partition minimizing the cost function (2) is a *contiguous* partition.

Appendix B

This section is devoted at quantifying the dimension $T^{(ess)}$ of the *essential* solution space $\Re^{(ess)} = \left\{\underline{C}_t^{(ess)};\ t = 1, ..., T^{(ess)}\right\}$, thus pointing out the computational saving allowed by the proposed approach compared to exhaustive or global sampling solution procedures. More in detail, the aim is that of determining the number $T^{(ess)}$ of candidate solutions or, in an equivalent fashion, the number of *allowed paths* in the solution tree.

Generally speaking, since a sub-array configuration $\underline{C}$ can be mathematically described by a sequence of M digits of a Q-symbols alphabet, the whole number of aggregations is equal to $T = Q^M$. Thanks to the *equivalence* relationship, the set of candidate solutions can be limited to the number of paths in a complete binary tree of depth M, thus the

[4] We explicitly note that the new partition $\underline{P}_Q^{(1)}$ has the same number of subsets as $\underline{P}_Q$. As a matter of fact, according to (10), the element v_t cannot be equal to the mean value d_k and thus, $\underline{V}_k$ has cardinality greater than one. It follows that the sub-set $\underline{V}_k^{(1)}$ has at least one element.

number of *non-redundant* solutions results $T = 2^{M-1}$. Moreover, by taking into account only *admissible* (i.e., grouping where there is at least one element in each sub-array) and *allowed* (i.e., sorted aggregations) complete sequences, the set of solution can be further reduced. With reference to the ordered list $L = \{l_m;\ m = 1, \ldots, M;\ l_m \leq l_{m+1}\}$, the *allowed* paths are mathematically described as

$$\underline{C}_t^{(ess)} = \left\{ c_{t,m}^{(ess)} \,\middle|\, c_{t,m}^{(ess)} \leq c_{t,m+1}^{(ess)},\ c_{t,1}^{(ess)} = 1,\ c_{t,M}^{(ess)} = Q \right\}, \qquad t = 1, ..., T^{(ess)}, \tag{14}$$

where $c_m^{(ess)}$ denotes the sub-array number to which the m-th element l_m of the ordered list L belongs.

In order to determine the essential dimension $T^{(ess)} = T^{(ess)}(Q, M)$ of the solution space, let us consider the "recursive" nature of the binary solution tree and, as a reference example, the case $Q = 2$. In such a situation, the grouping vector $\underline{C}_t^{(ess)}$ is a sequence of M symbols from the set $\{1, 2\}$ that satisfies the following constraints: (a) $c_{t,1}^{(ess)} = 1$, (b) $c_{t,M}^{(ess)} = 2$, and (c) if $c_{t,m}^{(ess)} = 2$ then $c_{t,m+1}^{(ess)} = c_{t,M}^{(ess)} = 2$. Thus, each possible solution $\underline{C}_t^{(ess)}$ is made up of a sub-sequence of consecutive symbols 1 followed by a sub-sequence of symbols 2. Accordingly, the trial solutions $\underline{C}_t^{(ess)}$, $t = 1, ..., T^{(ess)}$, are obtained by moving the starting point of the sub-sequence of symbols 2 from $m = 2$ (being $c_1 = 1$) up to $m = M$,

$$T^{(ess)}\left(Q, M\right)\Big|_{Q=2} = \begin{pmatrix} M-1 \\ 1 \end{pmatrix} = M - 1. \tag{15}$$

As far as the case $Q = 3$ is concerned, similar considerations hold true. In particular, each *allowed* trial solution $\underline{C}_t^{(ess)}$ ends with a sub-sequence of successive symbols 3. The number of elements of such a sub-sequence ranges from 1 to $M - 2$, leading to a complementary sub-sequence of symbols 1 and 2 of length $M - i$. Accordingly,

$$T^{(ess)}\left(Q, M\right)\Big|_{Q=3} = \sum_{i=1}^{M-2} T^{(ess)}\left(Q, M-i\right)\Big|_{Q=2} \tag{16}$$

Generalizing, since the smallest and largest number of occurrences of the symbol Q in a sequence is 1 and $M - (Q - 1)$, respectively, the essential dimension of the solution space when a M elements array is partitioned into Q sub-arrays is equal to

$$T^{(ess)}(Q, M) = \sum_{i=1}^{M-(Q-1)} T^{(ess)}(Q-1, M-i) = \left(\begin{array}{c} M-1 \\ Q-1 \end{array}\right). \quad (17)$$

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FIGURE CAPTIONS

- **Figure 1.** *Solution-Tree* structure representing the essential solution space $\Re^{(ess)}$.

- **Figure 2.** *Asymptotic Behavior* ($M = 10$, $d = \frac{\lambda}{2}$) - Sum $\{\alpha_m;\, m = 1, ..., M\}$ and difference $\{\beta_m;\, m = 1, ..., M\}$ optimal excitations. Compromise difference coefficients $\{b_m;\, m = 1, ..., M\}$ for different values of Q when (a) the GS algorithm and (b) the RES algorithm are applied.

- **Figure 3.** - *Uniform sub-arraying* ($M = 10$, $d = \frac{\lambda}{2}$, $Q = 5$) - Reference optimum and normalized difference patterns obtained by means of the EMM, the GS, and the RES approaches.

- **Figure 4.** *Non-uniform sub-arraying* ($M = 10$, $d = \frac{\lambda}{2}$) - Reference optimum and normalized difference patterns obtained by means of the EMM, the GS, and the RES approaches when (a) $Q = 3$ and (b) $Q = 5$.

- **Figure 5.** *Large Arrays* ($M = 100$, $d = \frac{\lambda}{2}$) - Reference optimum and normalized difference patterns obtained by means of the GS and RES techniques when $Q = 4$ and $Q = 6$.

- **Figure 6.** *Large Arrays* ($M = 100$, $d = \frac{\lambda}{2}$) - Difference excitations determined by the tree-based techniques when $Q = 4$ (a) and $Q = 6$ (b).

- **Figure 7.** *Computational Analysis* - *Computational Analysis* - Behavior of T versus M when the tree-based searching is applied $[T = T^{(ess)}]$.

- **Figure 8.** *Computational Analysis* - Behavior of t versus M for different values of Q (GS Approach).

TABLE CAPTIONS

- **Table I.** *Uniform sub-arraying* ($M = 10$, $d = \frac{\lambda}{2}$, $Q = 5$) - Beam pattern indexes.
- **Table II.** *Non-uniform sub-arraying* ($M = 10$, $d = \frac{\lambda}{2}$, $Q = 3, 5$) - Beam pattern indexes.
- **Table III.** *Large Arrays* ($M = 250$, $d = \frac{\lambda}{2}$, $Q = 4$) - Beam pattern indexes.

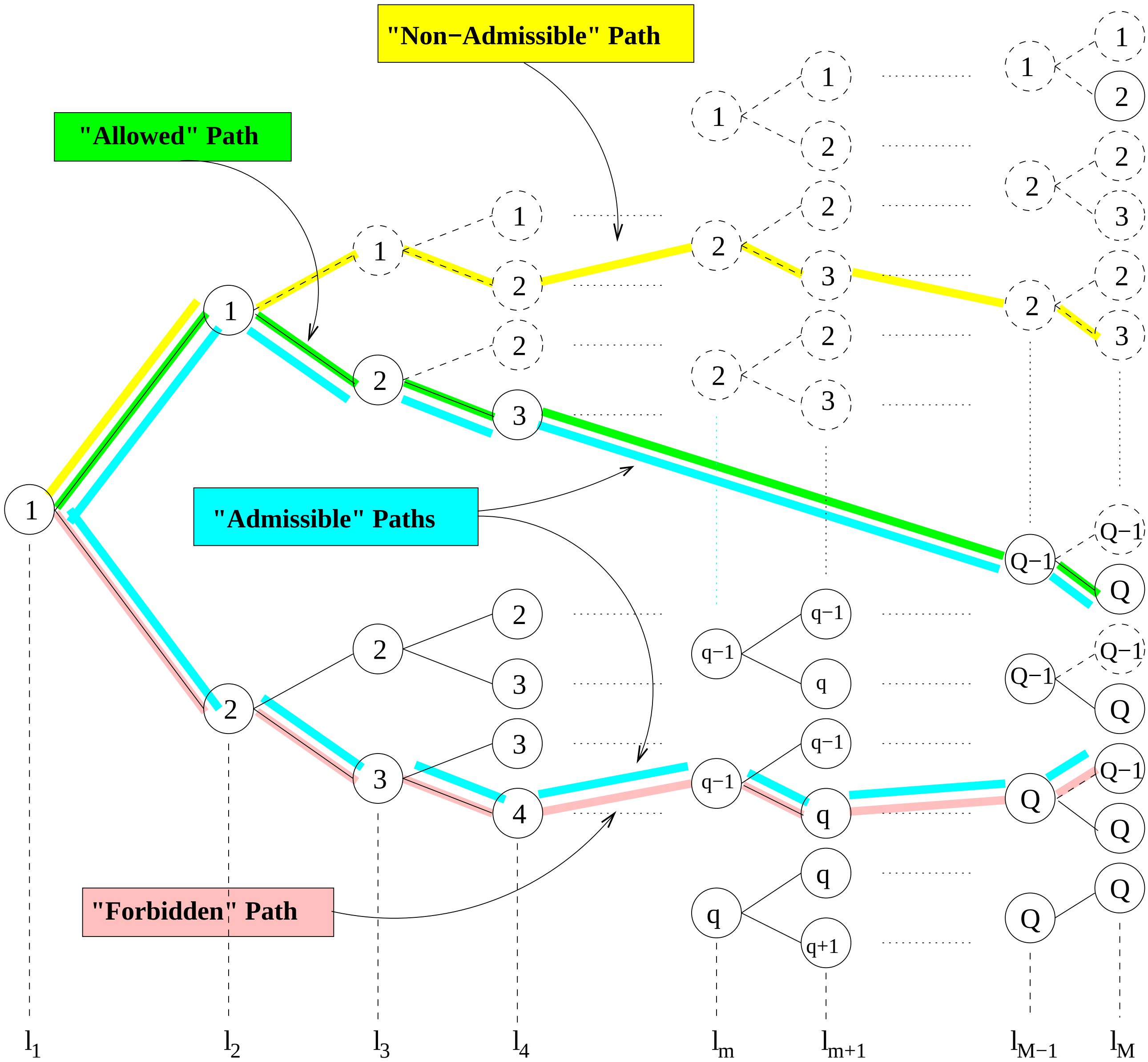
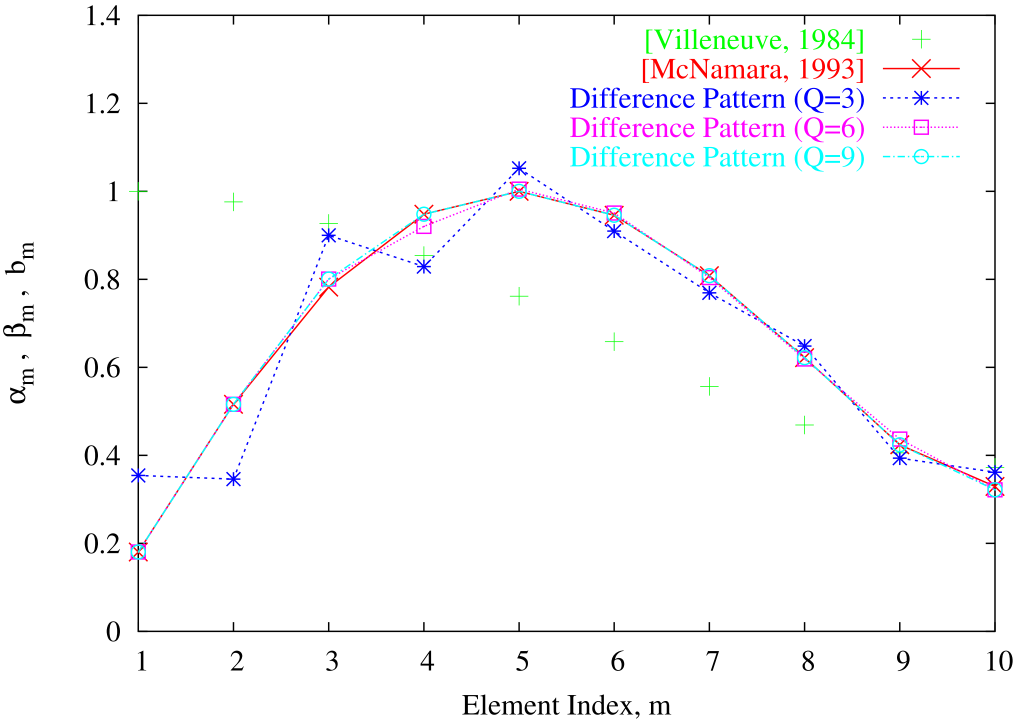
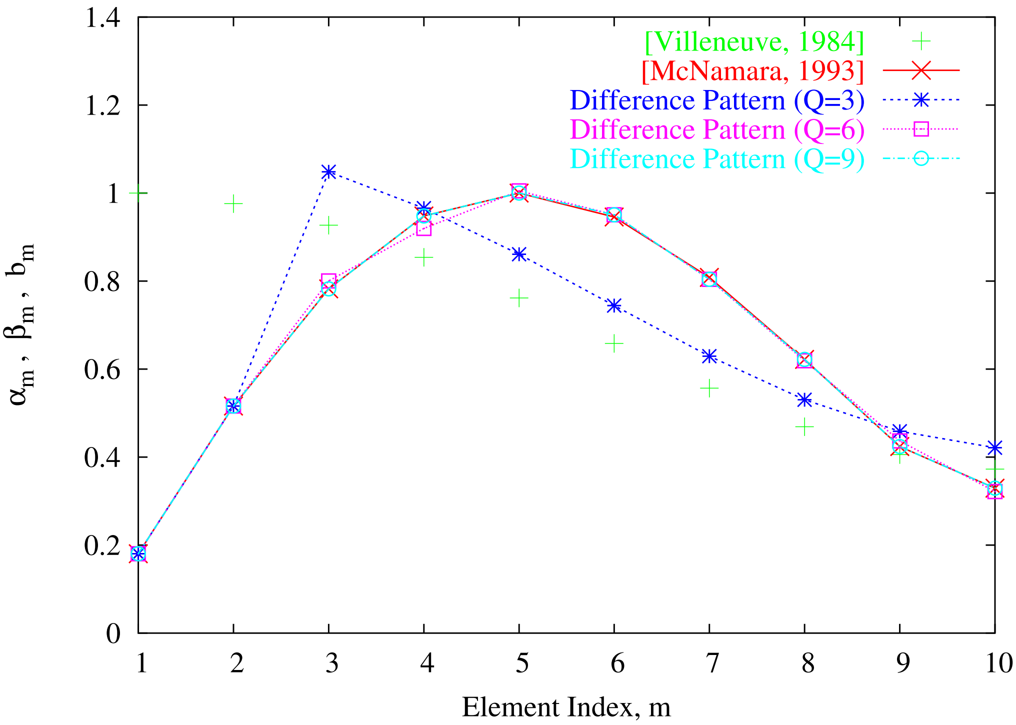


Fig. 1 - L. Manica *et al.*, "An Innovative Approach Based on ..."



(a)



(b)

Fig. 2 - L. Manica *et al.*, "An Innovative Approach Based on ..."

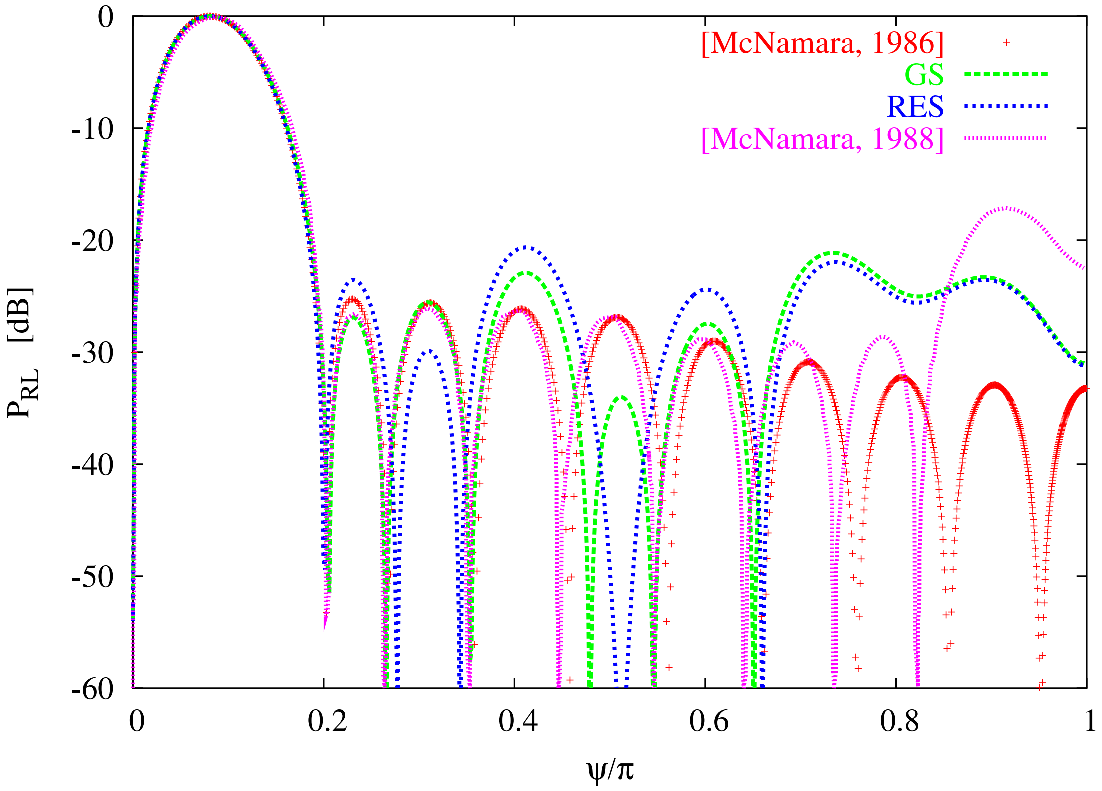
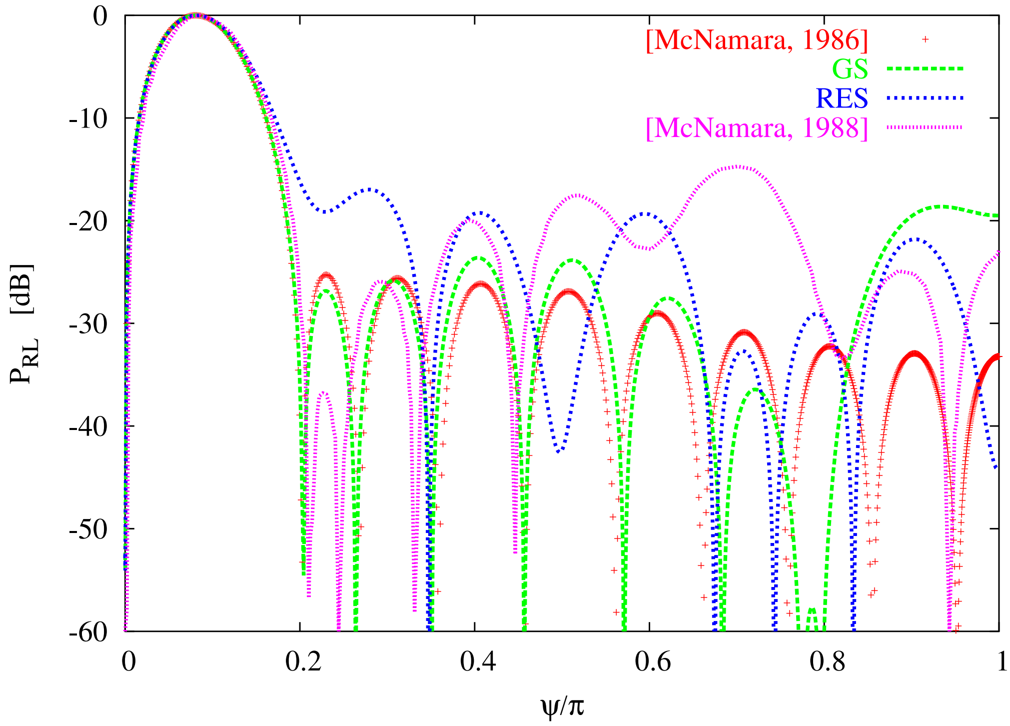
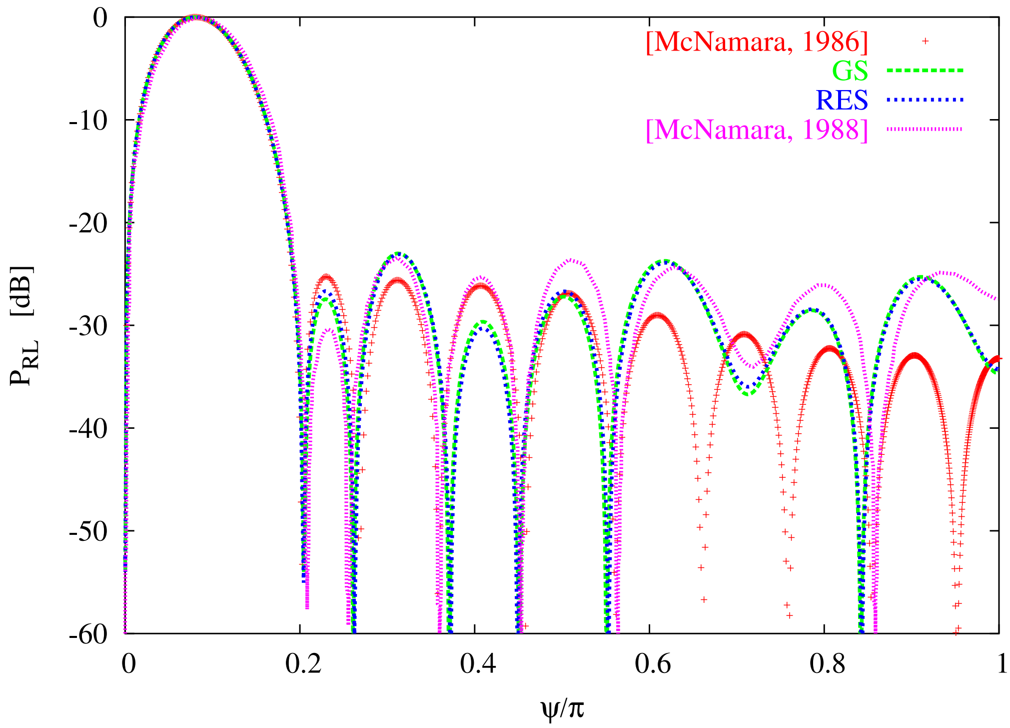


Fig. 3 - L. Manica *et al.*, "An Innovative Approach Based on ..."



(a)



(b)

Fig. 4 - L. Manica *et al.*, "An Innovative Approach Based on ..."

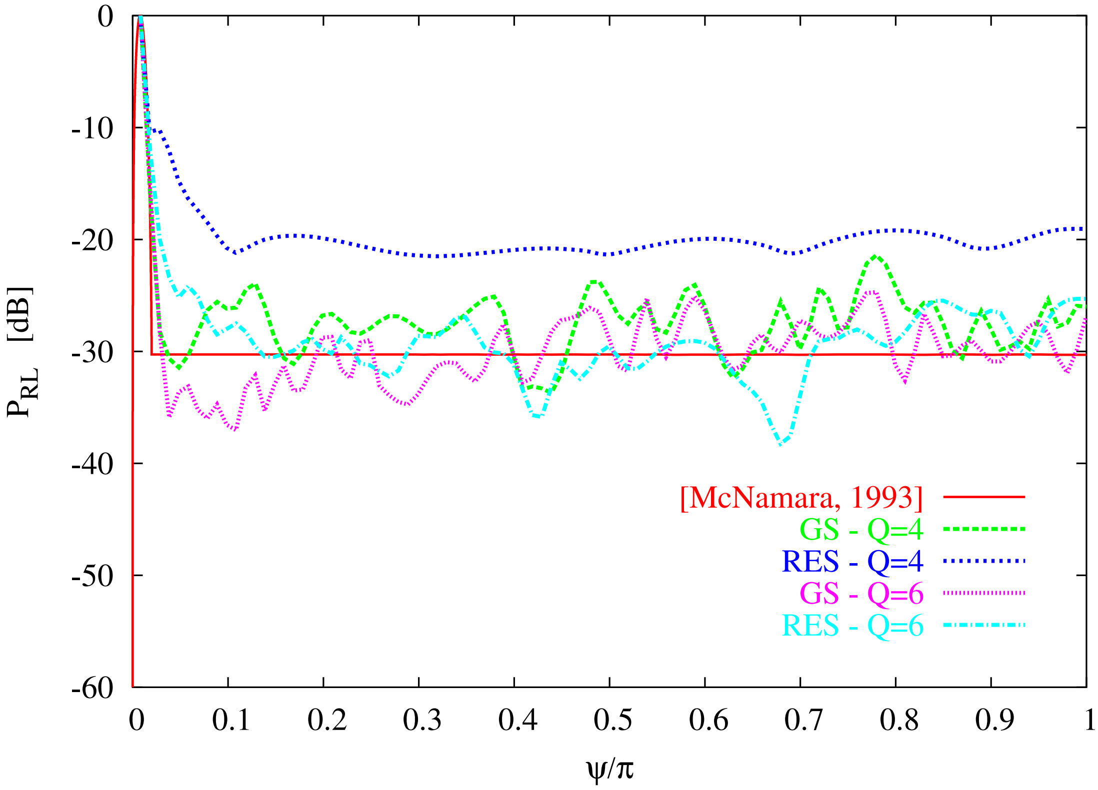
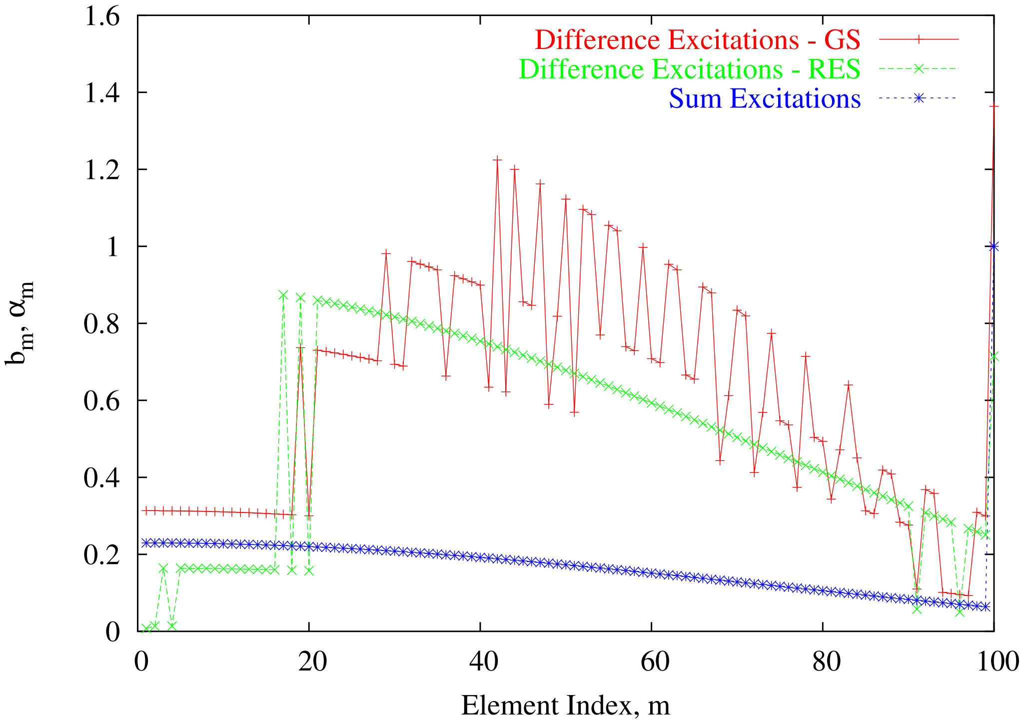
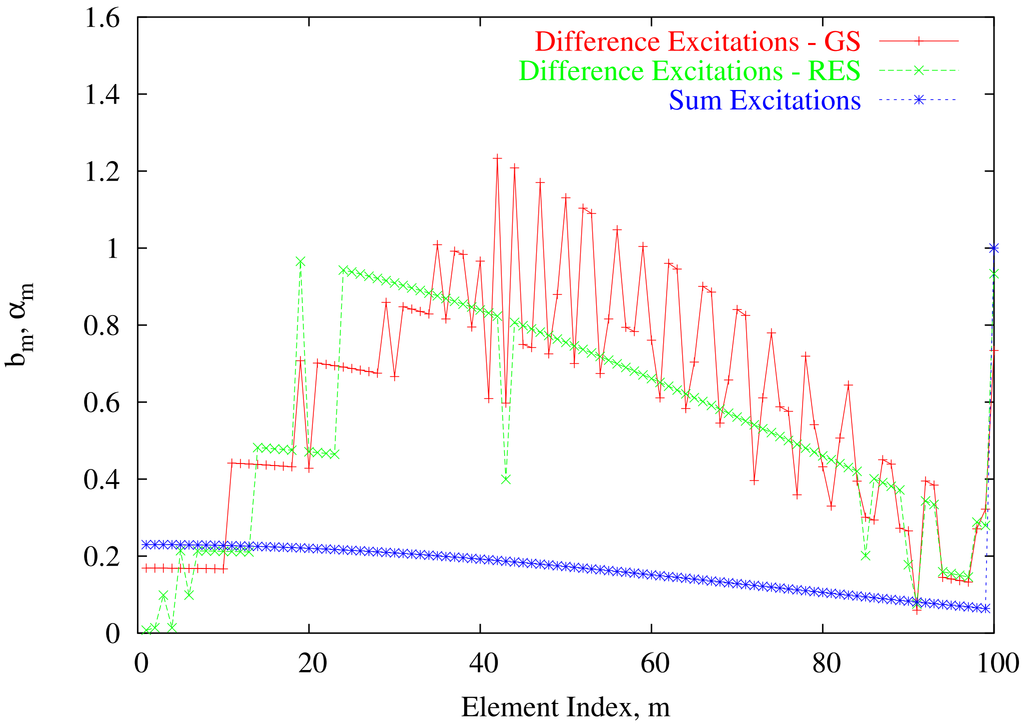


Fig. 5 - L. Manica *et al.*, "An Innovative Approach Based on ..."



(a)



(b)

Fig. 6 - L. Manica *et al.*, "An Innovative Approach Based on ..."

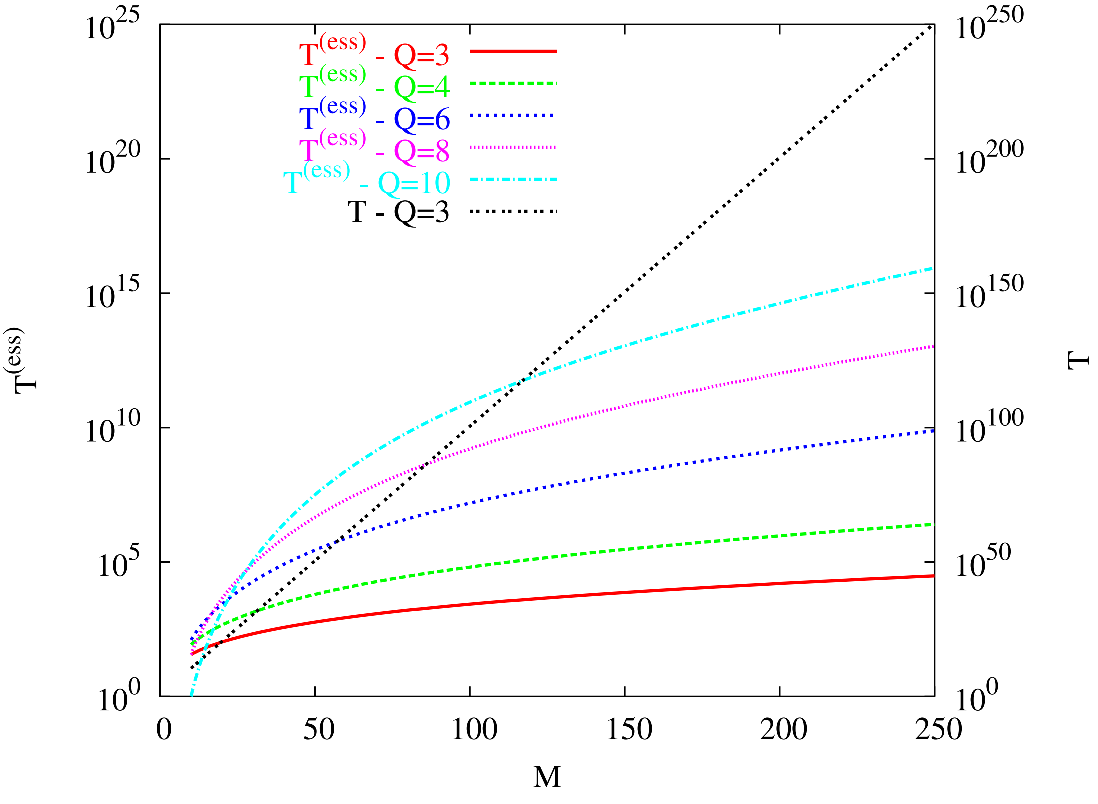


Fig. 7 - L. Manica *et al.*, "An Innovative Approach Based on ..."

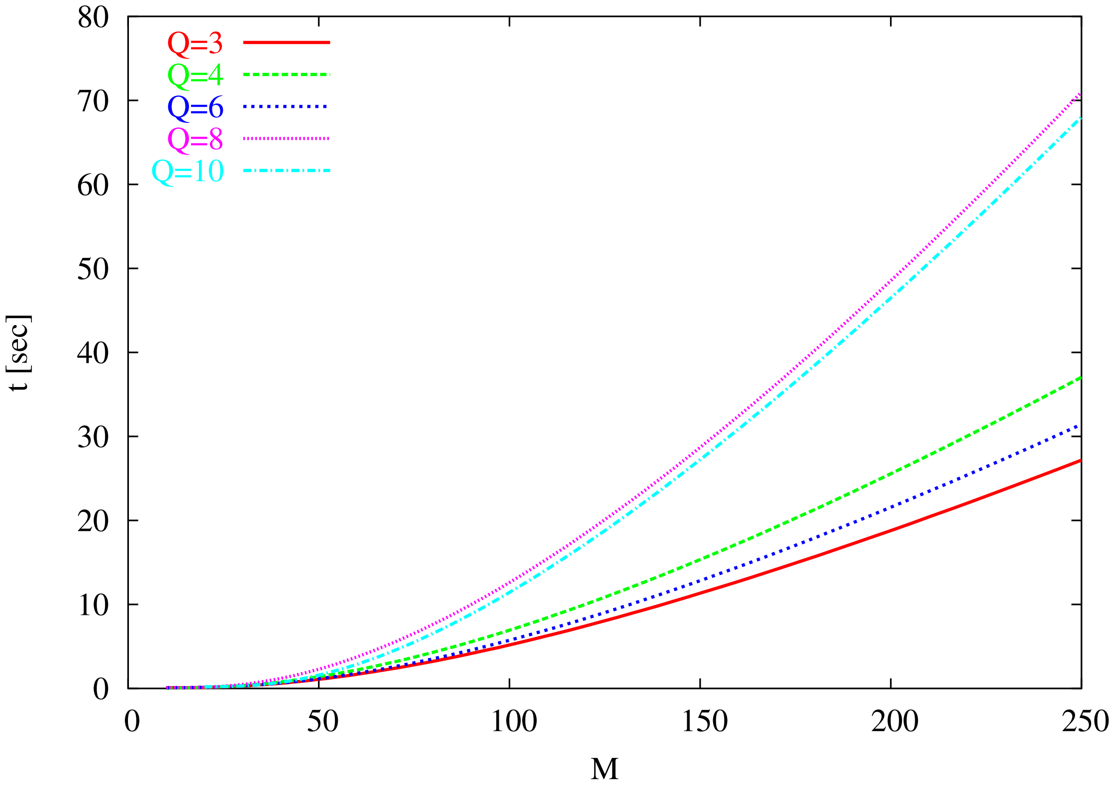


Fig. 8 - L. Manica *et al.*, "An Innovative Approach Based on ..."

$Approach$	P_{slo}	B_W	P_{sll}	$max\{SLL\}$	Δ
EMM [4]	0.1970	0.3610	0.1038	-17.00	0.4015
GS	0.1811	0.3784	0.1082	-21.10	0.2633
RES	0.1805	0.3735	0.1160	-20.50	0.2831
$Optimal$ [3]	0.1802	0.3735	0.0598	-25.00	$-$

Tab. I - L. Manica *et al.*, "An Innovative Approach Based on ..."

	$Q=3$			$Q=5$			
	EMM [4]	GS	RES	EMM [4]	GS	RES	$Optimal$ [5]
P_{slo}	0.2117	0.1800	0.1822	0.2000	0.1806	0.1805	0.1802
B_W	0.3745	0.3735	0.3930	0.3854	0.3735	0.3735	0.3735
P_{sll}	0.1798	0.1054	0.1365	0.0950	0.0823	0.0827	0.0598
$max\left\{SLL\right\}$	-14.70	-18.63	-17.00	-23.40	-23.00	-23.00	-25.00
Δ	0.5438	0.4073	0.3829	0.2562	0.1571	0.1517	$-$

Tab. II - L. Manica *et al.*, "An Innovative Approach Based on ..."

	GS	RES	$Optimal\ Difference$ [5]
P_{slo}	0.0066	0.0064	0.0066
B_W	0.0148	0.0158	0.0151
P_{sll}	0.0868	0.1797	0.0824
$max\{SLL\}$	-18.00	-10.05	-30.00
Δ	0.2921	1.1934	$-$

Tab. III - L. Manica *et al.*, "An Innovative Approach Based on ..."