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# Semilinear stochastic Volterra equations with dissipative nonlinearity

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## 1 Introduction

The purpose of this note is to study the existence, uniqueness and LDP for a class of integral Volterra equations perturbed by a Gaussian noise

$$u(t) = x - \int_0^t a(t-s)Au(s)\,\mathrm{d}s + \int_0^t a(t-s)F(u(s))\,\mathrm{d}s + B\,W(t), \quad (1.1)$$

in a real separable Hilbert space $H$. The case of a Lipschitz nonlinearity $F$ was considered in Bonaccorsi & Fantozzi (2002); in this paper, we assume that $F$ is a nonlinear operator defined on a subset of the Hilbert space $H$. Following Da Prato & Zabczyk (1992), we shall consider (1.1) in a smaller state space $X \subset H$, on which the operator $F$ is well defined and sufficiently regular. This method requires also that the initial condition takes values in the smaller space $X$.

Let us fix the relevant framework for our construction. On the kernel $a(t)$ we impose the following condition.

**Hypothesis 1.1.** *The kernel* $a : (0,\infty) \to \mathbb{R}$ *is completely monotone,* $a \in L^1_{loc}(0,\infty)$*, and there exists a Bernstein function* $k(t) = k_0 + \int_0^t k_1(s)\,\mathrm{d}s$ *associated to* $a(t)$*, the relation between* $a(t)$ *and* $k(t)$ *being given by*

$$k_0 a(t) + \int_0^t k_1(t-s)a(s)\,\mathrm{d}s = 1, \qquad t \in (0,\infty). \quad (1.2)$$

We assume that $X$ is a reflexive Banach space, densely and continuously embedded in $H$. Then we shall assume the following conditions on the operators $A$ and $F$.

**Hypothesis 1.2.**

**1.2a** *$A$ is a linear operator in $H$, with domain $D(A) \subset H$, of type $\omega$: $A$ belongs to $\widetilde{\Lambda}_{mc}(H)$,* i.e., *for some $\omega \geq 0$, $A + \omega I$ is $m$-accretive in $H$; the eigenvalues $\{\mu_k\}_{k\geq 1}$ form a nondecreasing sequence with* $\lim_{k\to\infty} \mu_k = \infty$ *and the corresponding eigenvectors $\{e_k\}_{k\geq 1}$ form a complete orthonormal system in $H$;*

**1.2b** *the part on $X$ of $A + \omega$ is $m$-accretive on $X$.*

The theory of accretive (and dissipative) operators is well known in the literature; as a general reference we mention the monograph of Da Prato (1976); for an introduction to the results used here, see also the paper Bonaccorsi & Fantozzi (2003).

**Hypothesis 1.3.** *The perturbation term $F$ maps $C(\mathbb{R}_+; X)$ into $C(\mathbb{R}_+; X)$, it is uniformly continuous on bounded sets of $\mathbb{R}_+ \times X$ and, for each $t \geq 0$, $F(t, \cdot)$ is $m$-dissipative on $X$.*

We are left with the conditions on the noise.

**Hypothesis 1.4.** *We are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and a $H$-valued Wiener process $W(t)$, $t \geq 0$, defined on it. With no loss of generality, we assume that there exists a sequence $\{\lambda_k\}$ of positive real numbers and a sequence $\{\beta_k(t)\}_{k\geq 1}$ of real standard Brownian motions such that*

$$BW(t) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t)\, e_k, \qquad t \geq 0.$$

Let us denote $S(t)$ the *resolvent operator* associated with the linear Volterra equation

$$u(t) = x - \int_0^t a(t-s) A u(s)\, \mathrm{d}s;$$

we define the *stochastic convolution process* $W_S(t)$ to be the mild solution to the equation

$$u(t) = -\int_0^t a(t-s) A u(s)\, \mathrm{d}s + B\, W(t).$$

In this paper, opposite to the case of Lipschitz nonlinearity treated in Bonaccorsi & Fantozzi (2002), we shall assume a space-time regularity for the noise.

**Hypothesis 1.5.** *The process $W_S(t)$, $t \geq 0$, has a $X$-continuous version.*

As we shall discuss in Section 2, this hypothesis may be given in terms of further assumptions on $A$ and $W$.

Now, we define $v(t) = u(t) - W_S(t)$ and note that for the $X$-valued processes $v(t)$ equation (1.1) may be written in the form

$$v(t) = x - \int_0^t a(t-s) Av(s)\,\mathrm{d}s + \int_0^t a(t-s) F(v(s) + z(s))\,\mathrm{d}s, \qquad (1.3)$$

where $z(t) = W_S(t) \in C(\mathbb{R}_+; X)$ is a trajectory of the stochastic convolution process.

Let $a(t)$ be a completely monotone kernel and $k(t)$ be the associated creep function; if we introduce the linear Volterra operator

$$Lu(t) = \frac{d}{dt}\left[k_0 u(t) + \int_0^t k_1(t-s) u(s)\,\mathrm{d}s\right] \qquad t \in \mathbb{R}_+,$$

then (1.3) is equivalent to the problem

$$\begin{cases} L[v-x](t) + Av(t) = F(v(t) + z(t)), \\ \\ k_0 v(0) + (k_1 * v)(0) = k_0 x, \end{cases} \qquad (1.4)$$

We arrive, therefore, to a pathwise deterministic equation settled in a real separable Banach space $X$, that we shall study via techniques of Volterra equations. With applications to (1.4) in mind, we have studied the problem of existence and uniqueness of a generalized solution in a previous paper Bonaccorsi & Fantozzi (2003), following the ideas of Gripenberg (1985).

**Theorem 1.6.** *Assume $X$ is a real Banach space and let the coefficients in* (1.1) *satisfy Hypotheses 1.1, 1.2, 1.3, 1.4 and 1.5. Then, for any $x \in \widehat{D}(A)$, there exists a unique generalized solution $v(t)$ to the abstract non-linear Volterra equation* (1.4).

*Then, we shall say that $u(t) = v(t) + W_S(t)$ is a generalized mild solution to (1.1) in $C(0,T;X)$.*

We shall discuss the transfer functional $\Psi$ that associates the trajectories of the stochastic convolution process to a solution of the nonlinear problem (1.4). We begin with this first remark.

**Remark 1.7.** The estimates involved in the proof of Theorem 1.6 do not depend on the whole path of the function $z(t)$ but only on its supremum norm, in other words the estimates do not change if we change the function $z(t)$ without modifying its supremum norm.

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We may be more precise on the regularity of $\Psi$; in Section 4 we prove the following.

**Theorem 1.8.** *Suppose the assumptions of Theorem 1.6 hold, then the functional* $\Psi : C([0,T];X) \to C([0,T];X)$ *that associates a trajectory of the stochastic convolution process to the solution of the* (1.4) *is continuous.*

In Section 5 we shall consider equation (1.1) with $B$ replaced by $\sqrt{\varepsilon}B$, bringing up a family of solutions $u_\varepsilon(t)$. We denote by $\nu_\varepsilon$ the law of $u_\varepsilon(\cdot)$ on the space $C([0,T];X)$. From Theorem 1.8, via the contraction principle, we shall prove

**Theorem 1.9.** *Suppose the assumptions of Theorem 1.6 hold, then the family of laws* $\nu_\varepsilon$ *satisfies the large deviation principle with respect to the following explicit functional* $J : C([0,T];X) \to [0;+\infty]$

$$J(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|B^{-1}\tfrac{d}{dt}[f(\vartheta)+(a*(Af(\cdot)-F(f(\cdot)))(\vartheta))]\right|^2 ds \\ \qquad\qquad\qquad \text{for } f \in \widetilde{R} \\ +\infty \qquad\qquad\quad \text{otherwise.} \end{cases} \tag{1.5}$$

*where* $\widetilde{R}$ *is the subset of* $C([0,T];X)$ *defined as*

$$\widetilde{R} = \Big\{ f \in C([0,T];X) \Big| \exists g \in L^2(0,T;H) : f(t) = S(t)x \\ + \tfrac{d}{dt}\left[\textstyle\int_0^t S(t-\vartheta)\,(a*F(f(\cdot)))(\vartheta)\;\mathrm{d}\vartheta\right] + \textstyle\int_0^t S(t-\vartheta)Bg(\vartheta)\;\mathrm{d}\vartheta \Big\}. \tag{1.6}$$

The large deviation principle for laws of solutions of stochastic differential equations has a wide literature and it was studied in different settings. Varadhan ((Varadhan 1966)) formulated the large deviation principle, and in the finite dimensional case it was established by Freĭdlin & Wentzell ((Freĭdlin and Wentzell 1984)) and Azencott ((Azencott 1980)), and studied, later, by Doss & Dozzi ((Doss and Dozzi 1986)) and Tudor ((Tudor 1998)).

The case of stochastic differential equations with additive gaussian perturbation in a Banach subspace $E$ of $H$ was studied by Smoleński *et al.* ((Smoleński, Sztencel, and Zabczyk 1986)), by applying Varadhan's contraction principle, see also Da Prato & Zabczyk ((Da Prato and Zabczyk 1992, Theorem 12.15)). There the problem is solved assuming that the semilinear part $F$ is locally Lipschitz in $E$.

In the paper Bonaccorsi & Fantozzi (2002) we started considering abstract Volterra equations with additive noise, in the framework of equations in Hilbert spaces. The main technique is the contraction principle; it requires the continuity of transfer functional $\Psi$, which is formally analogous to the one used for differential equations, see Fantozzi (2002) and Fantozzi (2003).

The rate functional (1.5) resembles the one in Bonaccorsi & Fantozzi (2002). This is related to the essence of the rate functional: it expresses the $L^2$-norm of a good sample trajectory of $\dot{W}$, which is the *energy* given to the system to stay out of the deterministic system.

## 2 Stochastic convolution

In this section we shall discuss the applicability of the abstract setting introduced in Section 1. We shall, following Clément & Da Prato (1996), discuss an example arising in applications in mathematical physics, such as viscoelasticity and heat flow in materials with memory.

Let us consider the linear Volterra integral equation

$$u(t) = x - \int_0^t a(t-\theta) A u(\theta) \,\mathrm{d}\theta, \qquad t \in [0,T]. \tag{2.1}$$

A family $\{S(t),\ t \in [0,T]\}$ of bounded linear operators in a Banach space $X$ is called a resolvent of (2.1) if the following conditions are satisfied:

1. $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
2. $S(t)$ commutes with $A$;
3. the resolvent equation holds: for all $x \in D(A)$, $t \in [0,T]$:

$$S(t)x = x - \int_0^t a(t-s) A S(s) x \,\mathrm{d}s. \tag{2.2}$$

Notice that Volterra equation (2.1) is well posed if and only if it admits a resolvent.

It is possible to show that the resolvent admits a decomposition in the basis $\{e_k\}$ of $H$. Let us denote $\{\mu_k\}$ the sequence of eigenvalues of $A$ with respect to the basis $\{e_k\}$:

$$A e_k = \mu_k e_k, \qquad \mu_k \geq -\omega. \tag{2.3}$$

We introduce now the solution $s(\alpha; \cdot)$ of the scalar integral equation

$$s_\alpha(t) + \alpha \int_0^t a(t-\vartheta) s_\alpha(\vartheta) \, d\vartheta = 1, \qquad t \geq 0. \tag{2.4}$$

Let $\mu_k$ be an eigenvalue of $A$ with eigenvector $e_k$. Then

$$S(t) e_k = s(\mu_k; t) e_k, \qquad t \geq 0. \tag{2.5}$$

### Scalar differential resolvent

Let us denote by $r_\mu(\cdot)$ the solution to the integral equation

$$r_\mu(t) + \mu \int_0^t r_\mu(t-s)a(s)\,ds = \mu a(t). \tag{2.6}$$

By Lemma 4.1 in Prüss (1993), since $a(t)$ is completely monotonic, we know that for any $\mu > 0$, $r_\mu(t)$ belongs to $L^1(\mathbb{R}_+) \cap C(0,\infty)$, it is completely monotonic, $0 \le r_\mu(t) \le \mu a(t)$ and

$$\int_0^\infty r_\mu(s)\,ds = \widehat{r}_\mu(0) = \frac{\mu \widehat{a}(0)}{1 + \mu \widehat{a}(0)} \le 1.$$

Moreover, if $\mu < 0$, then $r_\mu(t)$ belongs to $L^1_{loc}(\mathbb{R}_+) \cap C(0,\infty)$ and $r_\mu(t) \le \mu a(t) < 0$, compare also Friedman (1963).

The relation between $s_\mu(t)$ and $r_\mu(t)$ is clarified in the following statement.

**Proposition 2.1.** *It holds that*

$$s_\mu(t) = \left(1 - \int_0^t r_\mu(\tau)\,d\tau\right), \qquad t > 0. \tag{2.7}$$

We shall resume, in the next proposition, some results about the limit behaviour of $r_\mu(\cdot)$ and $s_\mu(\cdot)$ as $\mu \to \infty$.

**Proposition 2.2.** *The following relation holds between $s_\mu(t)$ and the function $k(t)$:*

$$\mu s_\mu(t) = (r_\mu * k_1)(t) + k_0 r_\mu(t). \tag{2.8}$$

*Moreover,*

$$\mu \int_0^t s_\mu(\tau)\,d\tau \to k(t)$$

*for a.e. $t > 0$.*

*Proof.* Let us briefly sketch the idea of the proof. Taking convolution in (2.6) with $k_1(t)$, recalling from Hypothesis 1.1 that $k_0 a(t) + (k_1 * a)(t) = 1$, we obtain

$$(r_\mu * k_1)(t) + \mu\big((r_\mu * k_1) * a\big)(t) = \mu(a * k_1)(t) = \mu(1 - k_0 a(t)).$$

On the other hand, again from (2.6) and (2.4) it follows that

$$(\mu s_\mu - k_0 r_\mu)(t) + \mu\big((\mu s_\mu - k_0 r_\mu) * a\big)(t) = \mu(1 - k_0 a(t)),$$

and comparing this expression with the previous one we prove (2.8).

The Laplace transform of $\int_0^t r_\mu(s)\,ds$ verifies

$$\left[\widehat{\int r_\mu(s)\,ds}\right](\lambda) = \frac{1}{\lambda}\frac{\mu\widehat{a}(\lambda)}{1+\mu\widehat{a}(\lambda)} \to \frac{1}{\lambda}$$

as $\mu \to \infty$, hence

$$\int_0^t r_\infty(s)\,ds = 1 \qquad \text{for a.e. } t > 0. \tag{2.9}$$

Integrating both sides of (2.8) in $(0,t)$, using the limit in (2.9), yields the thesis. □

### Stochastic convolution

Let us consider the stochastic convolution process $W_S(t)$. Recall from Hypothesis 1.4 that $BW(t)$ has the form

$$BW(t) = \sum_{k=1}^\infty \lambda_k \beta_k(t)\, e_k, \qquad t \ge 0;$$

then an explicit computation shows that a necessary assumption in order to have well-posedness of the stochastic convolution process $W_S(t)$ is

$$\int_0^T \|S(t)B\|^2_{\mathcal{L}_2(H)}\,\mathrm{d}\theta < +\infty. \tag{2.10}$$

**Proposition 2.3.** *Assume (2.10). Then for any $t \ge 0$, $W_S(t)$ is a $H$-valued Gaussian random variable, with mean 0 and covariance operator $Q = \int_0^t S(\theta)BB^*S(\theta)^*\,\mathrm{d}\theta$.*

We remark that it is possible to express condition (2.10) in terms of the operators $A$ and $B$; to obtain this, we first estimate the $L^2(0,T)$-norm of $s(\alpha,t)$ in terms of $\alpha$, as follows, for any $\alpha > 0$:

$$\int_0^T |s(\alpha,t)|^2\,\mathrm{d}t \le C\frac{1}{\alpha^{\delta_0}}.$$

**Remark 2.4.** Note that for a kernel $a(t)$ satisfying Hypothesis 1.1 it is proved in Clément & Da Prato (1996) that we can take $\delta_0 = 1$. However, in some cases it is possible to improve this result; for instance, if we take $a(t) = \frac{1}{\Gamma(\rho)}t^{\rho-1}$, for $0 < \rho < 1$, it holds that $\delta_0 = \min(\frac{1}{\rho}, 2)$. △

With this notation, we write condition (2.10) as

$$\sum_{k=k_0}^{\infty} \lambda_k \frac{1}{\mu_k^{\delta_0}} < +\infty$$

where $\mu_{k_0}$ is the first positive eigenvalue.

We shall now discuss regularity of the stochastic convolution in spaces of continuous functions. It turns out that we can express hypothesis 1.5 in terms of $s(\mu_k, \cdot)$ and $e_k$. To fix the framework, we shall assume henceforth that $H = L^2(\mathcal{O})$, for a bounded open subset $\mathcal{O} \subset \mathbb{R}^n$.

**Proposition 2.5.** *Assume that for some $\gamma_1 > 0$ there exists $\delta_1 < \delta_0$ with*

$$\int_0^u |s(\mu; t-\vartheta) - s(\mu; u-\vartheta)|^2 \, d\vartheta + \int_u^t |s(\mu; t-\vartheta)|^2 \, d\vartheta \leq C \frac{1}{\mu^{\delta_1}} |t-u|^{2\gamma_1}, \quad (2.11)$$

*and that:*

$$\sum_{k=1}^{\infty} \lambda_k^2 \frac{1}{\mu_k^{\delta_1}} < \infty. \quad (2.12)$$

*Assume further that there exists $M > 0$ such that*

$$\begin{cases} |e_k(x)| \leq M, & k \in \mathbb{N}, \quad x \in \mathcal{O}, \\ |\nabla e_k(x)| \leq M \mu_k^{1/2}, & k \in \mathbb{N}, \quad x \in \mathcal{O}. \end{cases} \quad (2.13)$$

*Then the trajectories of $W_S(t,x)$ are almost surely continuous in $(t,x)$.*

Let us consider the following example, which is taken from Clément & Da Prato (1996).

Let $H = L^2(0,1)$ and $X = C(\mathcal{O})$, and set $-Au = D^2 u$, for any $u \in D(A) = H^2(0,1) \cap H_0^1(0,1)$. Then a complete orthonormal system in $H$ defined by eigenvectors of $A$ is given by $e_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$, $x \in (0,1)$, $k \in \mathbb{N}$, with $\mu_k = \pi^2 k^2$, $k \in \mathbb{N}$. It is a simple computation to show that (2.13) holds.

Let $a(t)$ be a completely positive kernel: then we can choose $\delta_0 = 1$ and inequality (2.11) holds for any $\delta_1 > 0$. Now, from the definition of $\mu_k$, the series in (2.12) converges for any $\delta_1 > \frac{1}{2}$, hence we may apply Proposition 2.5. and in this case Hypothesis 1.5 is verified.

## 3 Semilinear Volterra equations

In this section we briefly recall the ideas in Bonaccorsi & Fantozzi (2003) that lead to the proof of Theorem 1.6.

The problem under consideration is the following Volterra integral equation

$$L[u-x](t) + Au(t) = F(t, u(t)). \tag{3.1}$$

The problem was introduced since the early 1970s in case where $F(t,u) = f(t)$; this case, that we shall call the "inhomogeneous problem", is an important step also in our construction.

The next step in the literature is to consider functional perturbations of such problem, see for instance Crandall & Nohel (1978) or Gripenberg (1985). In Bonaccorsi & Fantozzi (2003) we consider perturbation operators acting on $X$, but we can allow such operators to be non-autonomous. The study of (3.1) with the operator $F(t,u)$ is based on the results for the inhomogeneous problem $F = f(t)$ and a fixed point argument; this should justify the appellative of "perturbation term" given to $F(t,u)$.

### Volterra operators

We discuss first some properties of the linear Volterra operator

$$Lu(t) = \frac{d}{dt}\left[k_0 u(t) + \int_0^t k_1(t-s)u(s)\,ds\right], \qquad t > 0. \tag{3.2}$$

The operator $L$ is $m$-accretive in $L^p(\mathbb{R}_+; X)$, for any $p \geq 1$, see Clément & Nohel (1981). There is a natural representation of its inverse operator $L^{-1}$ in terms of the kernel $a(t)$:

$$L^{-1}v(t) = \int_0^t a(t-s)v(s)\,ds. \tag{3.3}$$

We now proceed to analyze the Yosida approximation $L_\mu = L(I + \frac{1}{\mu}L)^{-1}$. The following result is proved in Bonaccorsi & Fantozzi (2003).

**Lemma 3.1.** *The operator $L_\mu = L(I + \frac{1}{\mu}L)^{-1}$ is given by*

$$L_\mu v(t) = \mu \frac{d}{dt}(v(\cdot) * s_\mu(\cdot))(t). \tag{3.4}$$

### Inhomogeneous problem

We consider the inhomogeneous problem

$$L[u-x](t) + Au(t) = f(t). \tag{3.5}$$

In order to define a generalized solution to (3.5), we shall consider an approximate equation, where the operator $L$ is replaced by its Yosida approximation $L_\mu$, $\mu > 0$. Let $u_\mu$ be the solution of the following equation

$$L_\mu[u_\mu(\cdot) - x](t) + Au_\mu(t) = f(t). \tag{3.6}$$

In the next theorem, we establish the existence of a generalized solution of (3.5).

**Theorem 3.2.** *Assume that the coefficients in (3.5) satisfy Hypotheses 1.1 and 1.2, 1.3 and let $x \in \overline{D(A)}$ and $f \in C(\mathbb{R}_+; X)$. Then, for every $\mu > 0$ equation (3.6) has a unique solution $u_\mu(\cdot) \in C(\mathbb{R}_+; X)$.*

*As $\mu \to \infty$, there exists a function $u = U(x, f)$ with $u \in L^1_{loc}(\mathbb{R}_+; X)$ such that $u_\mu \to u$ in $L^1_{loc}(\mathbb{R}_+; X)$.*

*If $x \in \widehat{D}(A)$ then the convergence takes place also in $L^\infty_{loc}(\mathbb{R}_+; X)$ and the limit function $u$ belongs to $C(\mathbb{R}_+; X)$.*

The function $u = U(x, f)$, that exists according to Theorem 3.2, is said the generalized solution for problem (3.5).

### Non-autonomous perturbations

Now we return to equation (3.1). Before we discuss the case of dissipative non-linearities, that is the object of Theorem 1.6, we shall consider the case of a Lipschitz non-linearity. We shall say that $u(t)$ is a generalized solution of (3.1) if $u = U(x, F(\cdot, u))$.

**Theorem 3.3.** *Let the assumptions of Theorem 3.2 be fulfilled and assume that the non linear term $F(t, \cdot)$ satisfies*

$$F : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X) \tag{3.7}$$

*and there exists a function $\eta(t) \in L^\infty_{loc}(\mathbb{R}_+)$ such that, for any $t \in \mathbb{R}_+$*

$$\|F(t, u) - F(t, v)\| \leq \eta(t)\|u - v\|. \tag{3.8}$$

*Then there exists a unique generalized solution to equation (3.1)*

$$\begin{cases} L[u(\cdot) - x](t) + Au(t) = F(t, u(t)), \\ t \in (0, \infty), \qquad u(0+) = x. \end{cases}$$

We conclude this section with a few remarks about (3.1). Notice that we are concerned with a continuous and $m$-dissipative operator $F$; however, since this term is non-autonomous, we cannot consider the sum $A - F$ as a unique operator, even if we assume that $A - F$ is $m$-accretive.

Finally, we shall remark that the techniques in the proof of Theorem 1.6 are those usually applied in the theory of dissipative systems, cf. the proof of Theorem 7.13 in Da Prato & Zabczyk (1992).

# 4 Proof of Theorem 1.8

In this section we shall consider the functional $\Psi$ that associates to any $z \in C([0,T];X)$ the solution $v(t)$ of equation (1.4). We can now prove the Theorem 1.8.

*Proof.* We shall prove this theorem in two steps. In the first step we suppose that the non linear term $F$ is locally Lipschitz on $X$, then in the next step we prove the theorem in the general case. For the good definition of the functional $\Psi$ in both cases we refer to Theorem 3.3 and Theorem 1.6 respectively.

Let $z_1(t)$ a continuous function on $X$, we want to show that $\Psi$ is continuous in the point $z_1$ of $C([0,T];X)$, so we can restrict ourselves to a bounded subset $B$ around $z_1$. Then, since $F$ is locally Lipschitz, we can suppose, without loss of generality, that $F$ is totally Lipschitz on $B$, with Lipschitz constant equal to $\Lambda$.

Let $z_2$ belong to $B$, then from definition of generalized solution we have that there exist two sequences $v_{1,\mu}, v_{2,\mu}$ such that

$$v_{i,\mu} \to v_i \in L^1_{loc}(\mathbb{R}_+;X) \cap C(\mathbb{R}_+;X),$$

where $v_1 = \Psi(z_1)$ and $v_2 = \Psi(z_2)$ respectively, and

$$L_\mu(v_{i,\mu} - x)(t) + Av_{i,\mu}(t) = F(v_i(t) + z_i(t))$$

Then subtracting we have

$$\begin{aligned} L_\mu(v_{1,\mu} - v_{2,\mu})(t) + A(v_{1,\mu}(t) - v_{2,\mu}) \\ = F(v_1(t) + z_1(t)) - F(v_2(t) + z_2(t)). \quad (4.1) \end{aligned}$$

Let us consider $y^* \in \partial\|v_{1,\mu}(t) - v_{2,\mu}(t)\|$, then, applying $y^*$ to the previous equation, we have

$$\begin{aligned} \langle L_\mu(v_{1,\mu} - v_{2,\mu})(t), y^*\rangle + \langle A(v_{1,\mu}(t) - v_{2,\mu}), y^*\rangle \\ = \langle F(v_1(t) + z_1(t)) - F(v_2(t) + z_2(t)), y^*\rangle, \quad (4.2) \end{aligned}$$

expliciting the definition of $L_\mu$, we get

$$\begin{aligned} \mu\Big(\|v_{1,\mu}(t) - v_{2,\mu}\| - \big(\|v_{1,\mu}(\cdot) - v_{2,\mu}\| * r_\mu\big)(t)\Big) - \omega\|v_{1,\mu}(t) - v_{2,\mu}(t)\| \\ \le \Lambda(\|v_1(t) - v_2(t)\| + \|z_1(t) - z_2(t)\|). \quad (4.3) \end{aligned}$$

From Lemma 4.1 below, we get the estimate

$$\begin{aligned} \|v_{1,\mu}(t) - v_{2,\mu}(t)\| \le \Lambda\tfrac{\omega_\mu}{\omega}\frac{d}{dt}\Bigg(\Big(\tfrac{1}{\mu}\big[\|v_1(t) - v_2(t)\| + \|z_1(t) - z_2(t)\|\big] \\ + \big[a * (\|v_1(\cdot) - v_2(\cdot)\| + \|z_1(\cdot) - z_2(\cdot)\|)\big]\Big) * s_{-\omega_\mu}\Bigg)(t), \end{aligned}$$

and passing to the limit as $\mu \to \infty$ we obtain

$$\|v_1(t) - v_2(t)\| \le \Lambda \frac{d}{dt}\bigg(\Big(a * (\|v_1(\cdot) - v_2(\cdot)\| + \|z_1(\cdot) - z_2(\cdot)\|)\Big) * s_{-\omega}\bigg)(t)$$

that becomes

$$\begin{aligned}\|v_1(t) - v_2(t)\| &\le \Lambda\big(a * (\|v_1(\cdot) - v_2(\cdot)\| + \|z_1(\cdot) - z_2(\cdot)\|)\big)(t)\\ &\quad - \Lambda\big(a * (\|v_1(\cdot) - v_2(\cdot)\| + \|z_1(\cdot) - z_2(\cdot)\|)\big)(t)\\ &\quad + \Lambda\big( - \tfrac{1}{\omega} r_{-\omega} * (\|v_1(\cdot) - v_2(\cdot)\| + \|z_1(\cdot) - z_2(\cdot)\|)\big)\\ &= \Lambda\big( - \tfrac{1}{\omega} r_{-\omega} * (\|v_1(\cdot) - v_2(\cdot)\| + \|z_1(\cdot) - z_2(\cdot)\|)\big).\end{aligned}$$

Now since $-\frac{1}{\omega} r_{-\omega}(t)$ is a completely monotone kernel, we can apply again Lemma 4.1 and we have

$$\|v_1(t) - v_2(t)\| \le \big( - \widetilde{r}_{-\Lambda} * \|z_1(\cdot) - z_2(\cdot)\|\big)(t),$$

where $\widetilde{r}_{-\Lambda}(t)$ satisfies

$$\widetilde{r}_{-\Lambda} + \tfrac{\Lambda}{\omega} \widetilde{r}_{-\Lambda} * r_{-\omega} = \tfrac{\Lambda}{\omega} r_{-\omega}.$$

Since $-\widetilde{r}_{-\Lambda}(t) \le -r_{-(\omega+\Lambda)}(t)$ for all $t \ge 0$, we have

$$\|\Psi(z_1)(t) - \Psi(z_2)(t)\| \le \big( - r_{-(\omega+\Lambda)} * \|z_1(\cdot) - z_2(\cdot)\|\big)(t). \tag{4.4}$$

from what we have that $\Psi$ is continuous in the Lipschitz case.

Let us pass to step two: in the general case we can approximate $F$ with its Yosida approximations $F_\alpha$, so denoting with $\Psi_\alpha$ the functional corresponding to $\Psi$ in the (1.4) with $F$ substituted by $F_\alpha$ we have:

$$\begin{aligned}\|\Psi(z_1) - \Psi(z_2)\| &\le \|\Psi(z_1) - \Psi_\alpha(z_1)\|\\ &\qquad + \|\Psi_\alpha(z_1) - \Psi_\alpha(z_2)\| + \|\Psi_\alpha(z_2) - \Psi(z_2)\|.\end{aligned}$$

From Remark 1.7 and Theorem 1.6 we have that for all $\varepsilon$ there exists a $\alpha$ such that

$$\begin{aligned}\|\Psi(z_1) - \Psi_\alpha(z_1)\| &\le \varepsilon\\ \|\Psi_\alpha(z_2) - \Psi(z_2)\| &\le \varepsilon\end{aligned}$$

for all $z_1(t), z_2(t)$ in the same bounded set of $C([0,T];X)$.

Now since $F_\alpha$ is Lipschitz from (4.4) we have that for all $\varepsilon$ there exists $\delta$ such that $\|\Psi_\alpha(z_1) - \Psi_\alpha(z_2)\| \le \varepsilon$ if $\|z_1 - z_2\| \le \delta$.

This ends the proof of theorem. □

### A Gronwall-type lemma

In the last part of the section, we state a Gronwall-type lemma that allows to prove estimates for the solution of a Volterra equation. The proof of this lemma is given in Bonaccorsi & Fantozzi (2003).

**Lemma 4.1.** *Let $v(t)$ be a continuous, non negative function which satisfies the estimate*

$$v(t) \leq s_\lambda(t)x + \tfrac{1}{\lambda} f(t) + \tfrac{\omega}{\lambda} v(t) + r_\lambda * v(t), \tag{4.5}$$

*where $s_\lambda(t)$ and $r_\lambda(t)$ are defined in* (2.4) *and* (2.6) *respectively. Then*

$$v(t) \leq \frac{d}{dt}\left(\tfrac{\omega_\lambda}{\omega}(x + \tfrac{1}{\lambda} f + a * f) * s_{-\omega_\lambda}\right)(t), \tag{4.6}$$

*where $s_{-\omega_\lambda}(t)$ is defined as in (2.4) with $\omega_\lambda = \frac{\lambda\omega}{\lambda-\omega}$.*

**Remark 4.2.** In case $f \equiv 0$ we obtain from the above lemma the following estimate:

$$v(t) \leq \tfrac{\omega_\lambda}{\omega} x\, s_{-\omega_\lambda}(t). \tag{4.7}$$

If we consider, instead, the case $\omega = 0$, then estimate (4.6) becomes

$$v(t) \leq x + \tfrac{1}{\lambda} f(t) + (a * f)(t). \tag{4.8}$$

△

# 5 Large deviations

In this section we shall give the proof for the Theorem 1.9.

First of all we recall same preliminary results. For any $\varepsilon > 0$, we consider the laws of the processes $\sqrt{\varepsilon}W_S(\cdot)$ on the space $L^2(0,T;H)$.

**Theorem 5.1.** *Suppose that Hypotheses 1.1, 1.2 and 1.4 hold, and let $\mu$ be the law of the stochastic convolution process $W_S(\cdot)$. Then the family $\mu_\varepsilon$ of laws of $\sqrt{\varepsilon}W_S(\cdot)$ satisfies a large deviation priciple with respect to the rate functional $I$ given by*

$$I(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|B^{-1}\tfrac{d}{dt}[f(\vartheta) + (a * Af(\cdot))(\vartheta)]\right|^2 d\vartheta \\ \qquad\qquad\qquad \text{for } f \in R \\ +\infty \qquad\qquad \text{otherwise.} \end{cases}$$

*where $R$ is the subspace of $L^2(0,T;H)$ defined as*

$$R = \left\{ f \in L^2(0,T;H) \middle| \exists g \in L^2(0,T;H) :\ f(t) = -\int_0^t S(t-\vartheta)Bg(\vartheta)\,\mathrm{d}\vartheta \right\}.$$

For the proof of this result, based on the fact that $W_S(\cdot)$ is a centered Gaussian variable in $L^2(0,T;H)$, see Theorem 3.4 of Bonaccorsi & Fantozzi (2002).

Under Hypothesis 1.5, *i.e.*, that the process $W_S(\cdot)$ has continuous trajectories on $X$, we denote as before $\mu_\varepsilon$, for any $\varepsilon > 0$, the laws of the processes $\sqrt{\varepsilon} W_S(\cdot)$ on the space $C([0,T];X)$.

**Theorem 5.2.** *Assume that Hypoteses 1.1, 1.2, 1.4 and 1.5 hold, and let $\mu$ be the law of the stochastic convolution process $W_S(\cdot)$ on the space of continuous functions $C([0,T];X)$. Then the family $\mu_\varepsilon$ of laws of $\sqrt{\varepsilon} W_S(\cdot)$ satisfies a large deviation priciple with respect to the rate functional $I$ given by*

$$I(f) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^T \left| B^{-1} \tfrac{d}{dt} [f(\vartheta) + (a * Af(\cdot))(\vartheta)] \right|^2 d\vartheta \\ \qquad\qquad\qquad \textit{for } f \in R \\ +\infty \qquad\qquad \textit{otherwise,} \end{cases} \tag{5.1}$$

*where $R$ is the subspace of $C([0,T];X)$ defined as*

$$R = \left\{ f \in C([0,T];X) \middle| \exists g \in L^2(0,T;H) : f(t) = -\int_0^t S(t-\vartheta) B g(\vartheta)\, \mathrm{d}\vartheta \right\}. \tag{5.2}$$

*Proof.* Since $X$ is dense and continuosly embedded in $H$, the same holds for $C([0,T];X)$ in $L^2(0,T;H)$.

We know that the Gaussian process $W_S(\cdot)$ has a Gaussian law on the space $L^2(0,T;H)$ but, since it has support on the space $C([0,T];X)$, we have that $\mu$ is a Gaussian variable also on $C([0,T];X)$. So a large deviation principle holds for the family $\mu_\varepsilon$ on the space $C([0,T];X)$. By uniqueness of reproducing kernel, see Proposition 2.8 in Da Prato & Zabczyk (1992), the rate functional has the same explicit formulation as for trajectories in $L^2(0,T;H)$. □

**Remark 5.3.** The rate functional has another formulation. Assume that $g \in L^2(0,T;U)$ and consider the following integral control system

$$h^g(t) = -\int_0^t a(t-\vartheta) A h^g(\vartheta)\, d\vartheta + \int_0^t B g(\vartheta)\, d\vartheta, \qquad t \in [0,T],$$

that is solved by

$$h^g(t) = \int_0^t S(t-\vartheta) B g(\vartheta)\, d\vartheta.$$

It follows that the rate functional can be expressed in terms of $g$:

$$I(f) = \inf \left\{ \frac{1}{2} \int_0^T |g(\vartheta)|^2\, d\vartheta \ : h^g = f \right\} \tag{5.3}$$

This formulation has the following interpretation: $I$ can be viewed as the minimal "energy" given by a trajectory $g$ to force the sistem to remain in $f$.

$\triangle$

We can now proceed to the proof of Theorem 1.9.

*Proof.* We have that $\nu_\varepsilon = \Psi \circ \mu_\varepsilon$, where from Theorem 1.8 the functional $\Psi$ is continuous. Thus, from Theorem 5.2 and Proposition 12.3 in Da Prato & Zabczyk (1992), the family of laws $\nu_\varepsilon$ has the large deviation property with respect to the functional $J = \Psi^{-1}(I)$. Eventually the result follows since the definition of $\Psi$ implies that $J$ has the explicit formlulation (1.5). $\square$

**Remark 5.4.** As in the linear case the rate functional is related to the control system given by

$$h^g(t) = x - \int_0^t a(t-\vartheta)[Ah^g(\vartheta) - F(h^g(\vartheta))]\,\mathrm{d}\vartheta + \int_0^t Bg(\vartheta)\,\mathrm{d}\vartheta,$$

so it is possible to give the following definition for the rate in terms of $g$:

$$J(f) = \inf\left\{\frac{1}{2}\int_0^T |g(\vartheta)|^2\,d\vartheta \ : h^g = f\right\}$$

and this formulation brings us to the same interpretation for $I$ as the minimal "energy" given by a trajectory $g$ to force the sistem to remain in $f$.

$\triangle$

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