Holography and Conformal Symmetry near black hole horizons

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Abstract. We show here how it is possible to build a QFT on the horizon of a Schwarzschild-like spacetime. That theory, found by restricting bulk quantum fields on the horizon, is equivalent to QFT on the bulk. That fact is called Holography. Moreover the hidden conformal symmetry ($SL(2,\mathbb{R})$) found for the bulk theory becomes manifest on the horizon in terms of some of its diffeomorphisms. Then the extension of group of the generator of that symmetry to the Virasoro algebra is discussed.

1. *Introduction.* In the last fifty years much work was done in order to understand the statistical origin of black-hole entropy. The Holographic principle, proposed for the first time by 't Hooft and Susskind [1, 2, 3], is one of the most promising idea to deal with that problem. In few words the quantum theory responsible for the statistical black hole entropy should be suited on the event horizon, moreover it has to describe the events that take place in the spacetime. In some sense as a photograph describe a landscape. Starting from these ideas and using the machinery of string theory, Maldacena [4] conjectured that the quantum field theory in a, asymptotically AdS, $d+1$ dimensional spacetime (the "bulk") is in correspondence with a conformal theory in a d dimensional manifold (the (conformal) "boundary" at spacelike infinity). Notice that the d dimensional conformal group on the boundary acts as the asymptotic isometry group on the bulk. Afterwards, Witten [5] showed that that correspondence can be reset in terms of observables of the two theories. More recently Rehren [6, 7] proved rigorously some holographic theorems concerning boundary and bulk observables in AdS background, without using string theory. In the last year we have shown that there is also a bulk-boundary correspondence for QFT on Schwarzschild-like black holes [8, 9]. Similar ideas, concerning algebraic QFT on spacetimes with bifurcate Killing horizons and conformal symmetry, were presented by Guido, Longo, Roberts and Verch [10]. Also Schroer and Wiesbrock [11] have studied the relationship between horizons and ambient QFT. Moreover, they use the term "hidden symmetry" in a sense similar as we do here and we done in [12]. Schroer [13] and Schroer and Fassarella [14] by means of the Lightfront formalism they presented holography for Minkowski spacetime.
In this letter, discarding the technical details and stressing some of their physical implications, we want to summarize some of our results. The near horizon structure of every Schwarzschild-like spacetime is similar to the Cartesian product of a two dimensional Rindler spacetime and a sphere. The holographic properties of this spacetime are already exhibited discarding the sphere. That's because in the first part of that letter we deal with two dimensional spacetimes. We

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have shown recently [12] that the quantum field theory in a two-dimensional Rindler spacetime presents a "hidden" $SL(2,\mathbb{R})$ symmetry. $SL(2,\mathbb{R})$ symmetry is the one dimensional conformal group. This, suggests that, as in the AdS case, Rindler quantum fields are in holographic relation with one dimensional conformal fields. But here the situation is a little bit different, in fact, even if the quantum theory is invariant under $SL(2,\mathbb{R})$[3], the symmetry does not descend from the isometries of the spacetime, that is because we say that the symmetry is "hidden". We argue that these symmetry acquires a geometrical meaning in the holographic-dual conformal theory. We search for the dual conformal theory on the horizon. In fact on the horizon the metric is degenerate and $SL(2,\mathbb{R})$ can be seen as a subgroup of the horizon diffeomorphism. Moreover, compactifiing the horizon, it is possible to extend the symmetry generated by the Lie algebra $sl(2,\mathbb{R})$ to the whole Virasoro algebra, with a central charge equal to one. In the last section we consider the four dimensional case, we show that the holographic relation holds also in this case, even if, at least in general, the extension of the symmetry to the Virasoro algebra does not take place.

2. $SL(2,\mathbb{R})$ *symmetry from energy spectrum: a Hidden and a Manifest case.* We start our discussion with an abstract problem: Consider a quantum Hamiltonian H whose spectrum goes from zero to infinity, (for simplicity without degeneration). The corresponding Hilbert space is clearly $\mathcal{H} := L^2(\mathbb{R}^+, dE)$. In [12] we have shown that $\mathcal{H}$ is irreducible under unitary representation of $SL(2,\mathbb{R})$. The corresponding generators, enjoying the $sl(2,\mathbb{R})$ commutation relation, are the selfadjoint extensions of iH, iD and iC, defined below.

$$H := E\,, \qquad D := -i\left(\frac{1}{2} + E\frac{d}{dE}\right)\,, \qquad C := -\frac{d}{dE}E\frac{d}{dE} + \frac{(k-\frac{1}{2})^2}{E}\,. \tag{1}$$

k can be fixed arbitrarily in $\{1/2, 1, 3/2, \dots\}$. In the Heisenberg representation of (1), the expectation value of H, D, C are constant of motion. See [8] for details. Notice that, up to now, only H has a physical meaning as the Hamiltonian, (the generator of time translation) of the system, whereas the physical (geometrical) meaning of D and C has to be discussed in every particular case. We say that the found $SL(2,\mathbb{R})$ symmetry is *manifest* when D and C have geometrical meaning on the contrary we say that the symmetry is *hidden*. Notice that, since the action of $SL(2,\mathbb{R})$ is closed in the one particle Hilbert space $\mathcal{H}$, the $SL(2,\mathbb{R})$ symmetry is inherited by the Fock space $\mathfrak{F}(\mathcal{H})$. In the following we shall analyze some particular case.

a)*Free particles in Rindler space time.* We remind here that every Schwarzschild-like metric $ds^2_{\mathbf{S}} = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2d\Omega^2$, where Ω are the angular coordinates, reduces to the metric of a two-dimensional Rindler $\mathbf{R}$ space time near the bifurcate horizon at $r = r_h$. $ds^2_{\mathbf{R}} = -\kappa^2y^2dt^2 + dy^2$ with with $A'(r_h) = 2\kappa$, and $\kappa y^2 = 2(r - r_h)$, the angular coordinates are dropped in first approximation. Consider the free Klein-Gordon equation, $-\partial_t^2\phi + \kappa^2\left(y\partial_y y\partial_y - y^2m^2\right)\phi = 0$ for a free scalar field ϕ The one particle Hilbert space of the free quantum particle arises by decomposing any real solution ψ of the Klein-Gordon equation in t-stationary modes as follows.

$$\psi(t,y) = \int_0^{+\infty} \sum_\alpha \Phi_E^{(\alpha)}(t,y)\tilde{\psi}_+^{(\alpha)}(E)\, dE + c.c. \tag{2}$$

[3]The invariance is a dynamical invariance in the sense presented in [15, 8]

$E \in [0, +\infty) = \mathbb{R}^+$ is an element of the spectrum of the Rindler Hamiltonian H associated with ∂_t evolution. If $m = 0$ there are two values of α, corresponding to *in*going and *out*going modes, $\Phi_E^{(in)/(out)}$ whose expression are $e^{-iE(t \pm \ln(\kappa y)/\kappa)}/\sqrt{4\pi E}$. On the other hand if $m > 0$ there is a unique mode $\Phi_E^{(\alpha)} = \Phi_E$ whose expression is $\sqrt{2E \sinh(\pi E/\kappa)}/\sqrt{2\pi^2 \kappa E}\, e^{-iEt}\, K_{iE/\kappa}(my)$. Notice that in the massive case there is no energy degeneration and the one-particle Hilbert space $\mathcal{H}$ is isomorphic to $L^2(\mathbb{R}^+, dE)$. In the other case ($m = 0$), twofold degeneracy implies that $\mathcal{H} \cong L^2(\mathbb{R}^+, dE) \oplus L^2(\mathbb{R}^+, dE)$. Quantum field operators, acting in the symmetrized Fock space $\mathfrak{F}(\mathcal{H})$ and referred to the Rindler vacuum $|0\rangle$ – that is $|0\rangle_{in} \otimes |0\rangle_{out}$ if $m = 0$ – read

$$\hat{\phi}(t, y) = \int_0^\infty \sum_\alpha \Phi_E^{(\alpha)}(t, y) a_{E\alpha} + \overline{\Phi_E^{(\alpha)}(t, y)} a_{E\alpha}^\dagger dE. \tag{3}$$

As usual, the causal propagator Δ satisfies $[\hat{\phi}(x), \hat{\phi}(x')] = -i\Delta(x, x')$.
In this cases, in the sense discussed above, there is a hidden $SL(2, \mathbb{R})$ symmetry. Indeed, at least for the massive case, D and C have no local action on the Rindler wedge $\mathbf{R}$.
b) *Free particles in AdS_2 space time.* We analyze here a free massive particle moving in the portion of AdS_2, delimited by the (non-bifurcate) Killing horizon. This particular chart of AdS_2 describe a near horizon approximation of an extremal Reisner-N ordstrum black hole. We write the AdS_2 metric in the form $ds^2_{\mathbf{A}} = -x^2/\ell\, dt^2 + \ell/x^2\, dx^2$, where ℓ is related with the cosmological constant. As in the Rindler case, consider the free Klein-Gordon field ϕ satisfying the motion equation $-\ell^2 \partial_t^2 \phi + \left(x^2/\ell^2\, \partial_x\, x^2 \partial_x - x^2 m^2\right) \phi = 0$. The decomposition in stationary modes of the real solution of the Klein-Gordon field ψ reads

$$\psi(t, y) = \int_0^{+\infty} \sum_\alpha \Phi_E(t, x) \tilde{\psi}_+(E)\, dE + c.c. \tag{4}$$

where $\Phi_E(t, x) := J_\nu(-\ell^2 E/x)\sqrt{\ell^2/(2x)}$, and $\nu := \sqrt{1/4 + m^2 \ell^2}$. As in the Rindler case, there is a single mode for every value of E in $\mathbb{R}^+$. Moreover the modes Φ_E are complete, then the one-particle Hilbert space $\mathcal{H}_{\mathbf{A}}$ is isomorphic to $L^2(\mathbb{R}^+, dE)$ too. In the symmetrized Fock space $\mathfrak{F}(\mathcal{H}_{\mathbf{A}})$, equipped with the vacuum $|0\rangle_{\mathbf{A}}$, the quantum field operators, read

$$\hat{\phi}(t, x) = \int_0^\infty \Phi_E(t, x) b_E + \overline{\Phi_E(t, x)} b_E^\dagger dE. \tag{5}$$

Notice that $\mathcal{H}_{\mathbf{A}}$ is a unitary representation of $SL(2, \mathbb{R})$ as above, but in this case, studying the isometries of AdS_2 is possible to give geometrical meaning at the generators D and C too. The isometries of AdS_2 are generated by the following vector fields basis

$$h := \frac{\partial}{\partial t}\,, \qquad d := t\,\frac{\partial}{\partial t} - x\,\frac{\partial}{\partial x}\,, \qquad c := \left(t^2 + \frac{\ell^2}{x^2}\right)\frac{\partial}{\partial t} - 2tx\,\frac{\partial}{\partial x}\,. \tag{6}$$

They satisfy the commutation relation of the $sl(2, \mathbb{R})$ Lie algebra. There is an isomorphism between the generators of isometries and the generators of a particular unitary representation of

$SL(2,\mathbb{R})$ (1). This isomorphism singles out a particular $SL(2,\mathbb{R})$ representation and a particular value of $k>0$ in C: $(k-1/2)^2 = 1/4 + m^2\ell^2$. Notice that in the massless case $k=1$. The $SL(2,\mathbb{R})$ is manifest.

3. *Quantum fields on the horizon.* **a)** *The future horizon.* The hidden $SL(2,\mathbb{R})$ symmetry on Rindler particle is unsatisfactory, we have to deeper analyze the nature of that symmetry. To do that we want to show what happens exactly on the horizon, arguing that the hidden symmetry takes a geometrical meaning on the horizon. Consider the Rindler spacetime $\mathbf{R}$ naturally embedded in a Minkowski spacetime. The (Rindler) *light coordinates* $u = t - \log(\kappa y)/\kappa$, $v = t + \log(\kappa y)/\kappa$ (where $u, v \in \mathbb{R}$) that cover the Rindler space $\mathbf{R}$ are respectively well defined on the past $\mathbf{P}$ and the future horizon $\mathbf{F}$, (see figure). First of all, we consider the theory restricted on the future horizon $\mathbf{F}$ and we shall show that it is a well defined quantum theory. Take the wavefunction in (2) and consider the limit on the future horizon $u \to +\infty$. That is equivalent to restrict the wavefunction on the horizon when it is considered as a wavefunction in Minkowski spacetime, obtaining

$$\psi(v) = \int \frac{e^{-iEv}}{\sqrt{4\pi E}} e^{i\rho_{m,\kappa}(E)} \tilde{\psi}_+(E)\, dE + \text{c.c.} \tag{7}$$

$e^{i\rho_{m,\kappa}(E)}$ is a pure phase (see [8] for details). In coordinate $u \in \mathbb{R}$, the restriction of ψ to $\mathbf{P}$ is similar with the v replaced for u and $\rho_{m,\kappa}(E)$ replaced by $-\rho_{m,\kappa}(E)$. If $m=0$ the restrictions to $\mathbf{F}$ and $\mathbf{P}$ read respectively

$$\psi(v) = \int \frac{e^{-iEv}}{\sqrt{4\pi E}} \tilde{\psi}_+^{(in)}(E)\, dE + \text{c.c.}\ , \qquad \psi(u) = \int \frac{e^{-iEu}}{\sqrt{4\pi E}} \tilde{\psi}_+^{(out)}(E)\, dE + \text{c.c.} \tag{8}$$

Discarding the phase it is possible to consider the following real "field on the future Horizon":

$$\varphi(v) = \int_{\mathbb{R}^+} \frac{e^{-iEv}}{\sqrt{4\pi E}} \tilde{\varphi}_+(E)\, dE + \int_{\mathbb{R}^+} \frac{e^{+iEv}}{\sqrt{4\pi E}} \overline{\tilde{\psi}_+(E)}\, dE \tag{9}$$

as the basic object in defining a quantum field theory on the future event horizon. The same can be done for the past event horizon. The one-particle Hilbert space $\mathcal{H}_{\mathbf{F}}$ is defined as the space generated by positive frequency parts $\tilde{\psi}_+(E)$ and turns out to be isomorphic to $L^2(\mathbb{R}^+, dE)$ once again. The field operator reads, on the symmetrized Fock space $\mathfrak{F}(\mathcal{H}_{\mathbf{F}})$ with vacuum $|0\rangle_{\mathbf{F}}$,

$$\hat{\phi}_{\mathbf{F}}(v) = \int_0^\infty \frac{e^{-iEv}}{\sqrt{4\pi E}} a_E + \frac{e^{iEv}}{\sqrt{4\pi E}} a_E^\dagger dE\ . \tag{10}$$

The causal propagator $\Delta_{\mathbf{F}}$ is defined by imposing $[\hat{\phi}(v), \hat{\phi}(v')] = -i\Delta_{\mathbf{F}}(v, v')$ and it takes the form $(1/4) sign(v - v')$. In spite of the absence of any motion equation the essential features of free quantum field theory are preserved by that definition as proven in [8]. Degeneracy of the metric on the horizon prevents from smearing field operators by functions due to the ill-definiteness of the induced volume measure. However, employing the symplectic approach [16], a well-defined smearing-procedure is that of field operators and exact 1-forms $\eta = df$ where

$f = f(v)$ vanishes fast as $v \mapsto \pm\infty$. The integration of forms does not need any measure. In other words for a real exact 1-form η as said above

$$\hat{\phi}_{\mathbf{F}}(\eta) = \int_0^\infty \frac{dE}{\sqrt{4\pi E}} \left(\int_{\mathbb{R}} e^{-iEv} \eta(v) \right) a_E + \left(\int_{\mathbb{R}} e^{iEv} \eta(v) \right) a_E^\dagger \tag{11}$$

is well defined and diffeomorphism invariant. In a suitable domain the map $\eta(v) \mapsto \Delta_{\mathbf{F}}(\eta) = \frac{1}{4}\int_{\mathbb{R}} sign(v - v')\eta(v') = \psi_\eta(v)$ defines a one-to-one correspondence between exact one-forms and horizon wavefunctions of the form (2) and $\eta = 2d\psi_\eta$. Finally, similarly to usual quantum field theory [16], it holds

$$[\hat{\phi}_{\mathbf{F}}(\eta), \hat{\phi}_{\mathbf{F}}(\eta')] = -i\Delta_{\mathbf{F}}(\eta, \eta') = \int_{\mathbf{F}} \psi_{\eta'} d\psi_\eta - \psi_\eta d\psi_{\eta'}.$$

The last term define a diffeomorphism-invariant symplectic form on horizon wavefunctions. Concerning locality, notice that $\hat{\phi}(\eta)$ commute with $\hat{\phi}(\eta')$ if supports of η and η' are contained in disjoint segments.

b) *The compactified horizon.* As pointed out above on **F** there is no preferred measure, then we can consider the compactified case $\mathbb{S}^1 \cong \mathbf{F} \cup \{\infty\}$. Everything described above, can be translated in this case. Parametrize $\mathbb{S}^1$ by $\theta \in [-\pi..\pi]$. The circle wavefunction are

$$\rho(\theta) = \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{\sqrt{4\pi n}} \tilde{\rho}(n) + \text{c.c.} \,. \tag{12}$$

The complex combination of the positive frequency part of $\rho(\theta)$ form a Hilbert space $\mathcal{H}_{\mathbb{S}^1}$ isomorphic to $\ell^2(\mathbb{C})$. Moreover in [9] we have shown that $\mathcal{H}_{\mathbb{S}^1}$ turns to be isomorphic to $\mathcal{H}_{\mathbf{F}}$. The isomorphism acts in this way on the wavefunctions: $\rho(\theta) = \psi(v(\theta))$, where $v(\theta) = \beta \tan\theta/2$, with β a positive constant. Moreover exists a basis $\{Z_n(E)\}$[4] on $L^2(\mathbb{R}^+, dE)$ that realizes the isomorphism: $\tilde{\rho}_+(n) = \langle Z_n(E), \tilde{\psi}_+(E)\rangle$ and $\tilde{\psi}_+(E) = \sum_{n=1}^\infty Z_n(E)\tilde{\rho}_+(n)$. Notice that positive frequencies on the horizon **F** correspond to positive frequencies on the circle $\mathbb{S}^1$. In the Fock space of the circle $\mathfrak{F}(\mathcal{H}_{\mathbb{S}^1})$, with respect to the vacuum $|0\rangle_{\mathbb{S}^1}$, the quantum field operator reads

$$\hat{\phi}_{\mathbb{S}^1}(\theta) := \sum_{n=1}^{\infty} \frac{e^{-in\theta}}{\sqrt{4\pi n}} \alpha_n + \frac{e^{in\theta}}{\sqrt{4\pi n}} \alpha_n^\dagger. \tag{13}$$

The causal propagator is computed trough the relation $[\hat{\phi}_{\mathbb{S}^1}(\theta), \hat{\phi}_{\mathbb{S}^1}(\theta')] = -i\Delta_{\mathbb{S}^1}(\theta, \theta')$ and it take the simple form: $(1/4)\, sign(\theta - \theta') - (\theta - \theta')/\pi$. As before, to get local quantities, and due to ill-definiteness of the metric, the quantum fields need to be smeared by 1-forms.

$$\hat{\phi}_{\mathbb{S}^1}(\eta) := \sum_{n=1}^{\infty} \left(\int_{\mathbb{S}^1} e^{-in\theta}\eta(\theta) \right) \frac{\alpha_n}{\sqrt{4\pi n}} + \sum_{n=1}^{\infty} \left(\int_{\mathbb{S}^1} e^{in\theta}\eta(\theta) \right) \frac{\alpha_n^\dagger}{\sqrt{4\pi n}},$$

where $\eta = df$.

[4]The explicit form of $Z_n(E)$ involves the Laguerre polynomials, see [12] for details.

4. *The $SL(2,\mathbb{R})$ symmetry becomes manifest on the compactified horizon.* Consider a quantum field theory on (or equivalently on bP) mapped as explained above on $\mathbb{S}^1$. On Hilbert space $\mathcal{H}_{\mathbb{S}^1} \cong L^2(\mathbb{R}^+, dE)$ acts the tree operators H, D, C as defined on the right side in (1). As discussed above, the operators iH, iC, iD generates a unitary representation $\{U_g\}_{g\in SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$. We want to discuss here the *"geometrical nature"* of that representation. Consider the new basis of $SL(2,\mathbb{R}$:

$$K := \frac{1}{2}\left(\beta H + \frac{C}{\beta}\right), \qquad S := \frac{1}{2}\left(\beta H - \frac{C}{\beta}\right), \qquad D. \tag{14}$$

As shown in [9], in the case when $k = 1$ in (1) the representation U_g has the following geometrical meaning: consider a wavefunction $\rho(\theta)$, the state $\tilde{\rho}_+^g(n) := U_g\tilde{\rho}_+(n)$, with $g \in SL(2,\mathbb{R})$, corresponds to the wavefunction $\rho^g(\theta)$. Moreover $\rho^g(\theta)$, satisfy a geometric transformation:

$$\rho^g(\theta) = \rho(d_g^{-1}(\theta)). \tag{15}$$

Where d_g is a diffeomorphism of the circle $\mathbb{S}^1$. In particular, the following relation, between the Lie algebra of the unitary representation of $SL(2,\mathbb{R})$ and the diffeomorphisms group, holds.

$$iK \leftrightarrow \partial_\theta, \qquad iS \leftrightarrow \sin(\theta)\,\partial_\theta \qquad iD \leftrightarrow \cos(\theta)\,\partial_\theta. \tag{16}$$

The theory on the horizon $\mathbf{F}$. For completeness we analyze the geometrical nature of the $SL(2,\mathbb{R})$ symmetry on $\mathbf{F}$. On the horizon Hilbert space $\mathcal{H}_{\mathbf{F}} \cong L^2(\mathbb{R}^+, dE)$ acts unitarily a representation $\{U_g\}$ of $g \in SL(2,\mathbb{R})$ generated by (1). In the case of $k = 1$ the wavefunction $\psi_g(v)$ associated with $U_g\tilde{\psi}_+$ reads:

$$\psi_g(v) = \psi\left(\frac{av+b}{cv+d}\right) - \psi\left(\frac{b}{d}\right), \qquad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{17}$$

Notice that in the action of $SL(2,\mathbb{R})$ is not exactly geometric due to the term $-\psi(b/d)$. The added term disappears considering local $1-$form $d\psi$ instead of wavefunction ψ. In this case:

$$iH \leftrightarrow \partial_v, \qquad iD \leftrightarrow v\,\partial_v. \tag{18}$$

The action of iC is not exactly geometric due to the term $-\psi(b/d)$. Apart this term it corresponds to corresponds to $v^2\,\partial_v$.

5. *Virasoro algebra.* Above we have seen that the unitary $SL(2,\mathbb{R})$ symmetry is manifest on the circle $\mathbb{S}^1 \cong F \cup \{\infty\}$ without a measure. But due to the degeneracy of the metric on $\mathbf{F}$ and then on $\mathbb{S}^1$, we expect more symmetry. In fact $SL(e,\mathbb{R})$ is only a little part of a greater group of symmetry: the group of diffeomorphism preserving orientation $Diff^+(\mathbb{S}^1)$. Consider its Lie algebra: $Vect(\mathbb{S}^1)$ of its vector field. A algebraic basis the vector fields is made of the real part of the following smooth fields:

$$\mathcal{L}_n := ie^{in\theta}\partial_\theta. \tag{19}$$

The element of this basis, equipped with the usual bracket $\{\cdot,\cdot\}$, satisfy the *Virasoro commutation rules* with vanishing central charge:

$$\{\mathfrak{L}_n, \mathfrak{L}_m\} = (n-m)\mathfrak{L}_{n+m}. \tag{20}$$

Moreover a particular Hermiticity condition is fulfilled, by means of the involution $\omega : X \mapsto -\overline{X}$:

$$\omega(\mathfrak{L}_m) = \mathfrak{L}_{-m}.$$

The tree generator $\{\mathfrak{L}_{-1}, \mathfrak{L}_0, \mathfrak{L}_1\}$ satisfy the $sl(2,\mathbb{R})$ commutation relation, moreover they are a linear combination of the generators of the $SL(2,\mathbb{R})$ on the circle defined above (16)

$$-i\mathfrak{L}_0 = \partial_\theta \,, \qquad -\frac{\mathfrak{L}_1 - \mathfrak{L}_{-1}}{2} = \sin\theta\,\partial_\theta \,, \qquad \frac{\mathfrak{L}_1 + \mathfrak{L}_{-1}}{2i} = \cos\theta\,\partial_\theta \,, \tag{21}$$

then $\{\mathfrak{L}_{-1}, \mathfrak{L}_0, \mathfrak{L}_1\}$ are in relation with the tree particular operators $\{K, S, D\}$ generating a unitary representation of $SL(2,\mathbb{R})$ on the Hilbert space $\mathcal{H}_{\mathbb{S}^1}$. Now a question arises. Is it possible to extend the $SL(2,\mathbb{R})$ representation to the whole $Diff^+(\mathbb{S}^1)$ group? Instead of answering that question, we want to show here that, at least in some particular situations, the quantum $sl(2,\mathbb{R})$ algebra can be extended to the whole Virasoro algebra. Moreover, a central charge arises through this extension. In the case when $k=1$ in (1), on the Fock space $\mathfrak{F}(\mathcal{H}_{\mathbb{S}^1})$, consider $a_n = i\sqrt{n}\,\alpha_n$, $a_{-n} = -i\sqrt{n}\,\alpha_n^\dagger$ if $n>0$ and $a_0 = 0$ written in terms of the creation and annihilation operators. $\{a_n\}$ satisfy the commutation relation: $[a_n, a_m] = n\delta_{n,-m} I$ and $a_n^\dagger = a_{-n}$. $\{a_n\}$ with that commutation relation form the so called *oscillator algebra* [17]. The quantum field (13) take the simplest form

$$\hat{\phi}_{\mathbb{S}^1}(\theta) = \frac{1}{i\sqrt{4\pi}} \sum_{n\in\mathbb{Z}} \frac{e^{-in\theta}}{n}\, a_n. \tag{22}$$

Define the operator as in [17]

$$L_m := \frac{\epsilon_m}{2} a_{m/2}^2 + \sum_{n>m/2} a_{-n} a_{n+m}, \qquad m \in \mathbb{Z}, \tag{23}$$

where ϵ_m is equal to one if m is even and 0 otherwise. Notice that on the Fock space $\mathfrak{F}(\mathcal{H}_{\mathbb{S}^1})$, since the number of particles is finite, the sum in (23) are finite. Moreover it is possible to show that they are well defined quantum operators on $\mathfrak{F}(\mathcal{H}_{\mathbb{S}^1})$. In the following we analyze some properties of $\{L_n\}$. First of all we notice that they satisfy the Virasoro commutation relation with central charge c equal to one.

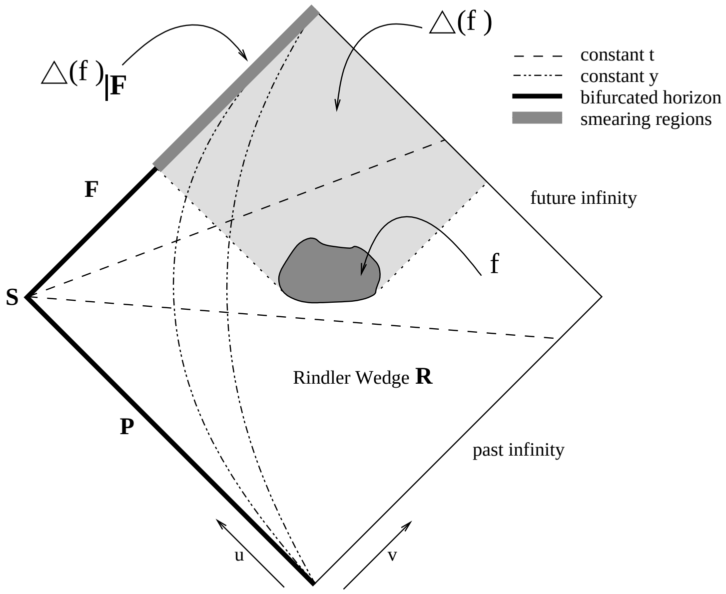
$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m} \frac{n^3 - n}{12} I.$$

Then the Hermiticity condition $L_n^\dagger = L_{-n}$ holds. The spectrum of L_0 is discrete and positive definite

$$\sigma(L_0) = \left\{ a_0^2/2 + N, \qquad N \in \mathbb{N} \right\}.$$

Notice that, if $a_0 = 0$, only the action of $\{L_{-1}, L_0, L_1\}$ is closed in the one particle Hilbert space $\mathcal{H}_{\mathbb{S}^1}$. By means of a difficult proof, it is possible to show that the action of $\{L_n\}$ on the fields $\hat{\phi}_{\mathbb{S}^1}(\theta)$ correspond to the action of $\{\mathfrak{L}_n\}$ on the point θ of the circle $\mathbb{S}^1$.

6. *Holography.* Above we have shown that there are some particular unitary relations between some different quantum theories. In particular these unitary relations were used to show the existence of $SL(2,\mathbb{R})$ "symmetry" also in some cases were it was not expected as for example on the Rindler free fields. We want to show here that there is a deeper relation between the Rindler $\mathbf{R}$ free fields and the fields defined on the line thought as the future Rindler horizon $\mathbf{F}$.



For simplicity we shall discuss here only the case of Rindler free massive fields. As mentioned above there is a unitary map $U_{\mathbf{F}} : \mathfrak{F}(\mathcal{H}_{\mathbf{R}}) \to \mathfrak{F}(\mathcal{H}_{\mathbf{F}})$ between the Rindler Fock $\mathfrak{F}(\mathcal{H}_{\mathbf{R}})$ space and the horizon Fock space $\mathfrak{F}(\mathcal{H}_{\mathbf{F}})$. $U_{\mathbf{F}}$ maps the Rindler vacuum horizon vacuum $U_{\mathbf{F}}|0\rangle = |0\rangle_{\mathbf{F}}$ and $U_{\mathbf{F}}^{-1}\hat{\phi}_{\mathbf{F}}(\eta)U_{\mathbf{F}} = \hat{\phi}(f)$ for any smooth compactly supported function f used to smear the bulk field, $\eta = 2d(\Delta(f)\restriction_{\mathbf{F}})$. (See figure.)

We want to show here the main idea of the above proposition. Details on the construction of are presented in [8]. The unitary action $U_{\mathbf{F}}$ has the following geometrical meaning: Consider a local function f used to smear the Rindler field $\hat{\phi}_{\mathbf{R}}$, then $\psi_f = \Delta(f)$ is the associated wavefunction. Taking its restriction on the horizon we obtain an horizon wavefunction as in (7) whose positive frequency part reads $e^{i\rho_{m,\kappa}}(E)\tilde{\psi}_{f+}(E)$. Then define a horizon wavefunction φ_f as in (9) with $\tilde{\varphi}_+$ replaced by $\tilde{\psi}_{f+}$. The map $\psi_f \mapsto \varphi_f$ corresponds to the unitary operator $U_{\mathbf{F}}\restriction_{\mathcal{H}}$ from $\mathcal{H}$ to $\mathcal{H}_{\mathbf{F}}$. Imposing $U_{\mathbf{F}}|0\rangle = |0\rangle_{\mathbf{F}}$, by taking tensor products of $U_{\mathbf{F}}\restriction_{\mathcal{H}}$, this map extends to a unitary map $U_{\mathbf{F}} : \mathfrak{F}(\mathcal{H}) \to \mathfrak{F}(\mathcal{H}_{\mathbf{F}})$. Finally, by direct inspection one finds that, if $\eta = 2d\varphi_f$, one also has $U_{\mathbf{F}}^{-1}\hat{\phi}_{\mathbf{F}}(\eta)U_{\mathbf{F}} = \hat{\phi}(f)$.

As a consequence, one has the holographic relation: the invariance of vacuum expectation values,

$$_{\mathbf{F}}\langle 0|\hat{\phi}_{\mathbf{F}}(\eta_1)\cdots\hat{\phi}_{\mathbf{F}}(\eta_n)|0\rangle_{\mathbf{F}} = \langle 0|\hat{\phi}(f_1)\cdots\hat{\phi}(f_n)|0\rangle \; . \tag{24}$$

There is an analogous relation between the Rindler free fields and quantum operators defined on the past Horizon **P**. The massless case is a little more difficult because one has to decompose the bulk fields in the ingoing and outgoing modes. Then the ingoing modes can be mapped, with a similar procedure, on the future horizon **F** whereas the outgoing are in holographic relation with a quantum field theory defined on the past horizon **P**. Before ending this section we want to remind here that the unitary holographic relation presented above is a particular case of the holographic relation existing between the abstract bulk observable algebra $\mathcal{A}_{\mathbf{R}}$ and the horizon observable algebra $\mathcal{A}_{\mathbf{F}}$. Such that $\phi_{\mathbf{R}}(f)$ is mapped onto $\phi_{\mathbf{F}}(\eta)$, where $\eta = 2d(\Delta(f) \restriction_{\mathbf{F}})$. The key point is that the algebraic holography preserve the causal propagator:

$$-i\Delta(f,g) = -i\Delta_{\mathbf{F}}(\eta_f, \eta_g).$$

Details on that can be found in [8, 9]. This analysis suggests that similar holographic relations seems to hold also for theories on more complex spacetime as for example Schwarzschild spacetime.

7. *Four dimensional case.* We want to extend the result suited above, concerning Rindler holography, to the four dimensional case. For this purpose we do not discard the angular coordinates Θ, ϕ in the near horizon approximation of a Schwarzschild-like spacetime as discussed in section 2. The metric reads:

$$ds^2 = -\kappa^2 y^2 dt^2 + dy^2 + r_h^2 d\Omega^2.$$

Every field takes an angular part described by the usual spherical harmonics $Y^l_m(\Theta, \phi)$. QFT in the bulk involves the one-particle Hilbert space $\oplus_{l=0}^{\infty}(\mathcal{H}_l \otimes \mathbb{C}^{2l+1})$ with $\mathcal{H}_l \cong L^2(\mathbb{R}^+, dE)$ if $l > 0$, $\mathbb{C}^{2l+1}$ being the space at fixed total angular momentum l and $\mathcal{H}_0 \cong L^2(\mathbb{R}^+, dE)$ in the massive case but $\mathcal{H}_0 \cong L^2(\mathbb{R}^+, dE) \oplus L^2(\mathbb{R}^+, dE)$ in the massless case. For wavefunctions with components in a fixed space $\mathbb{C}^{2l+1} \otimes L^2(\mathbb{R}^+, dE)$ Klein-Gordon equation reduces to the two-dimensional one with a positive contribution to the mass depending on l. Quantum field theory can also be constructed on the compactified future horizon $\mathsf{F} \cong (\mathbb{R} \cup \{\infty\}) \times \mathbb{S}^2$. The quantum fields on F take the following form:

$$\hat{\phi}_{\mathsf{F}}(\theta) := \sum_{n\geq 1, l, l \leq m \leq l} Y^l_m(\Theta, \phi) \frac{e^{-in\theta}}{\sqrt{4\pi n}} \alpha_{nlm} + \overline{Y}^l_m(\Theta, \phi) \frac{e^{in\theta}}{\sqrt{4\pi n}} \alpha^{\dagger}_{nlm}. \tag{25}$$

The appropriate causal propagator reads

$$\Delta_{\mathbf{F}}(x, x') = \left(\frac{1}{4}\, sign(\theta - \theta') - \frac{(\theta - \theta')}{\pi}\right) \delta(\Theta - \Theta')\delta(\phi - \phi')\sqrt{g_{\mathbb{S}^2}(\Theta, \phi)}\,.$$

Also in this case the holographic relation between the bulk and the horizon fields holds. We notice eventually that there is a difference concerning the extension of the $SL(2, \mathbb{R})$ symmetry to the Virasoro algebra. The Virasoro generators can be defines as follows:

$$L^{lm}_n := \frac{\epsilon_n}{2} a^2_{n/2lm} + \sum_{k>n/2} a_{(-k)lm} a_{(k+n)lm}, \qquad n \in \mathbb{Z}, \tag{26}$$

where $a_{nlm} := i\sqrt{n}\alpha_{nlm}$ if $n > 0$ and $a_{nlm} := -i\sqrt{-n}\alpha^\dagger_{(-n)lm}$ if $n < 0$ and $a_{0lm} := 0$. Notice that fixing (l, m) one gets a Virasoro algebra acting on the subspace $\mathcal{H}_l$. The Virasoro algebra acting on $\mathcal{H}_l \otimes \mathbb{C}^{2l+1}$ is

$$L_n^{(l)} := \sum_{-l\leq m\leq l} L_n^{lm}, \tag{27}$$

notice that $\{L_n^l\}$ forms a reducible Virasoro algebra on $\mathfrak{F}(\mathcal{H}_l \otimes \mathbb{C}^{2l+1})$ whose central charge is $c_l := 2l + 1$. To define the Virasoro generators on the whole Fock space $\mathfrak{F}(\mathcal{H}_{\mathsf{F}})$ we have to perform an infinite sum

$$L_n := \sum_l L_n^{(l)} \tag{28}$$

the corresponding central charge $c = \sum_l c_l$ becomes infinite and then the Virasoro algebra is not well defined.

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