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Paracausal deformations of Lorentzian metrics and Møller isomorphisms in algebraic quantum field theory

Valter Moretti[1] · Simone Murro[2] · Daniele Volpe[1]

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Abstract

Given a pair of normally hyperbolic operators over (possibnly different) globally hyperbolic spacetimes on a given smooth manifold, the existence of a geometric isomorphism, called *Møller operator*, between the space of solutions is studied. This is achieved by exploiting a new equivalence relation in the space of globally hyperbolic metrics, called *paracausal relation*. In particular, it is shown that the Møller operator associated to a pair of paracausally related metrics and normally hyperbolic operators also intertwines the respective causal propagators of the normally hyperbolic operators and it preserves the natural symplectic forms on the space of (smooth) initial data. Finally, the Møller map is lifted to a $*$-isomorphism between (generally off-shell) CCR-algebras. It is shown that the Wave Front set of a Hadamard bidistribution (and of a Hadamard state in particular) is preserved by the pull-back action of this $*$-isomorphism.

Keywords Paracausal deformation · Convex interpolation · Cauchy problem · Møller operators · Normally hyperbolic operators · Algebraic quantum field theory · Hadamard states · Globally hyperbolic manifolds

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✉ Simone Murro
murro@dima.unige.it

Valter Moretti
valter.moretti@unitn.it

Daniele Volpe
daniele.volpe@unitn.it

1 Dipartimento di Matematica, Università di Trento and INFN-TIFPA, Via Sommarive 14, 38123 Povo, Italy

2 Dipartimento di Matematica, Università di Genova and INdAM and INFN, sezione di Genova, Via Dodecaneso 35, 16146 Genova, Italy

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1 Introduction

Recently a great deal of progress has been made in comparing the spaces of solutions of hyperbolic partial differential equations on (possibly different) Lorentzian manifolds as well as in the comparison of the associated quantum field theories. More precisely, given a pair N and N' of Green hyperbolic differential operators on (possibly different) globally hyperbolic spacetimes (M, g) and (M, g'), a natural issue concerns the existence of a linear isomorphism $\mathsf{S} : \mathsf{Sol}_{\mathsf{N}} \to \mathsf{Sol}_{\mathsf{N}'}$ between the linear spaces of the solutions of the equations $\mathsf{N}\psi = 0$ and $\mathsf{N}'\psi' = 0$. Such an isomorphism, if it exists, is called a *Møller map*. Since the said space of solutions is the first step in the construction of corresponding (algebraic) free quantum field theories, a natural related issue concerns the possibility to promote the Møller map R to a $*$-isomorphism between the associated abstract operator algebras $\mathcal{A}$ and $\mathcal{A}'$ constructed out of N and N' respectively on (M, g) and (M, g'), in terms of corresponding generators given by *abstract field operators* $\Phi(\mathfrak{h})$ and $\Phi'(\mathfrak{h}')$ and the associated *causal propagators* G_{N}, $\mathsf{G}_{\mathsf{N}'}$. Actually, *off-shell* linear QFT can be used to build up a perturbative approach to interacting QFT, a final problem would concern the possibility to extend the Møller

isomorphism of algebras to an isomorphism of more physically interesting algebras, for instance including Wick powers or time-ordered powers.
These problems have been tackled in the past for special cases of metrics g, g' and several types of Green hyperbolic field operators which rule the dynamics of bosonic fields [20, 27] or fermionic fields [29, 65]. In the *loc. cit.*, the pairs of Lorentzian metrics g, g' had to satisfy one of the following assumptions: (I) they shared a common foliation of smooth spacelike Cauchy surfaces; (II) they coincided outside a compact set.

In this paper, we do not assume either of the above restrictions and instead, we consider a very wide family of pairs g, g' of globally hyperbolic metrics on the same manifold M. As a matter of fact, we consider a new type of relationship between globally hyperbolic metrics, which we call *paracausal relationship*. To the authors' knowledge this notion represents a complete novelty on the subject. Though the effective definition of paracausal equivalence relation in the set of globally hyperbolic metrics on M (Definition 2.19) is different and more effective for the issues regarding Møller maps raised above, a complete characterization of it can be stated as follows in terms of elementary Lorentzian geometry:

Theorem 1 (Theorem 2.24) *The globally hyperbolic metric g on M is paracausally related to the globally hyperbolic metric g' on M if and only if there is a finite sequence $g_0 := g, g_1, \dots, g_N := g'$ of globally hyperbolic metrics on M such that, at each step g_k, g_{k+1}, the future open light cones of these metrics have non-empty intersection $V_x^{g_k+} \cap V_x^{g_{k+1}+} \neq \varnothing$ at every point $x \in \mathsf{M}$.*

The class of paracausally related metrics on a given manifold M is very large, though some elementary counterexamples of topological nature can be constructed. A specific study on the properties of this equivalence relation is necessary and it will be done elsewhere.

Equipped with this notion, the paper specializes the analytic setup as follows: $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ are (2nd order) *normally hyperbolic* operators on a real or complex vector bundle E on M, respectively associated to a pair of globally hyperbolic metrics g, g' on M.

The first important achievement of this work is the proof of the existence of (infinitely many) *Møller operators*, i.e., isomorphisms $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$, which restrict to Møller maps S between the space of solutions when g and g' are paracausally related. The overall idea is inspired by the scattering theory in the special case of a pair of globally hyperbolic metrics g_0, g_1 over M such that the light cones of g_0 are included in the light cones of g_1 (this is the most elementary case of paracausal relation). We start with two "free theories", described by the space of solutions of normally hyperbolic operators N_0 and N_1 in corresponding spacetimes (M, g_0) and (M, g_1), respectively, and we intend to connect them through an "interaction spacetime" (M, g_χ) with a "temporally localized" interaction defined by interpolating the two metrics by means of a smoothing function χ. Here we need two Møller maps: Ω_+ connecting (M, g_0) and (M, g_χ) – which reduces to the identity in the past when χ is switched off – and a second Møller map connecting (M, g_χ) to (M, g_1) – which reduces to the identity in the future when χ constantly takes the value 1. The "*S*-

matrix" given by the composition $\mathsf{S} := \Omega_- \Omega_+$ will be the Møller map connecting N_0 and N_1.

The above construction generalizes to the case of a pair of globally hyperbolic metrics g, g' on M which are paracausally related and this fact is denoted by $g \simeq g'$. A summary of the main results obtained is the following where also a special notion of adjoint operator $\mathsf{R}^{\dagger_{gg'}}$ is used. It will be discussed in details in Sect. 4.

Theorem 2 (Theorems 4.5, 4.6, and 4.12) *Let* E *be* $\mathbb{K}$*-vector bundle over the smooth manifold* M *with a non-degenerate, real or Hermitian depending on* $\mathbb{K}$*, fiber metric* $\langle \cdot | \cdot \rangle$*. Consider* $g, g' \in \mathcal{GH}_M$ *with respectively associated normally hyperbolic formally-selfadjoint operators* N, N'.
If the metrics are paracausally related $g \simeq g'$*, then it is possible to define a (non-unique)* $\mathbb{K}$*-vector space isomorphism* $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$*, called* **Møller operator** *of* g, g' *(with this order), such that the following facts are true.*

(1) *The restrictions to the relevant subspaces of* $\Gamma(\mathsf{E})$ *respectively define symplectic Møller maps* S^0 *(see Definition 4.7) which preserve the symplectic forms* σ_g^{N}, $\sigma_{g'}^{\mathsf{N}'}$ *defined as in Eq. (3.7), namely*

$$\sigma_{g'}^{\mathsf{N}'}(\mathsf{S}^0\Psi, \mathsf{S}^0\Phi) = \sigma_g^{\mathsf{N}}(\Psi, \Phi) \quad \textit{for every } \Psi, \Phi \in \mathsf{Ker}_{sc}^g(\mathsf{N}).$$

(2) *The causal propagators* $\mathsf{G}_{\mathsf{N}'}$ *and* G_{N}*, respectively of* N' *and* N*, satisfy* $\mathsf{R}\mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{N}'}$*, where* $\mathsf{R}^{\dagger_{gg'}}$ *is the adjoint of the Møller operator (see Definition 3.9).*
(3) *By denoting* c' *the smooth function such that* $\mathrm{vol}_{g'} = c' \,\mathrm{vol}_g$*, we have* $c'\mathsf{N}'\mathsf{R} = \mathsf{N}$.
(4) *It holds* $\mathsf{R}^{\dagger_{gg'}}\mathsf{N}'|_{\Gamma_c(\mathsf{E})} = \mathsf{N}|_{\Gamma_c(\mathsf{E})}$.
(5) *The maps* $\mathsf{R}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ *and* $(\mathsf{R}^{\dagger_{gg'}})^{-1} = (\mathsf{R}^{-1})^{\dagger_{g'g}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ *are continuous with respect to the natural topologies of* $\Gamma_c(\mathsf{E})$ *in the domain and in the co-domain.*

Theorem 2 permits us to promote R to a $*$-isomorphism of the algebras of field operators $\mathcal{A}$, $\mathcal{A}'$ respectively associated to the paracausally related metrics g and g' (and the associated N, N') and generated by respective field operators $\Phi(\mathfrak{f})$ and $\Phi'(\mathfrak{f}')$ with $\mathfrak{f}, \mathfrak{f}'$ compactly supported smooth sections of E. These field operators satisfy respective CCRs

$$[\Phi(\mathfrak{f}), \Phi(\mathfrak{h})] = i\mathsf{G}_{\mathsf{N}}(\mathfrak{f}, \mathfrak{h})\mathbb{I}\,, \quad [\Phi'(\mathfrak{f}'), \Phi'(\mathfrak{h}')] = i\mathsf{G}_{\mathsf{N}'}(\mathfrak{f}', \mathfrak{h}')\mathbb{I}'$$

and the said unital $*$-algebra isomorphism $\mathcal{R} : \mathcal{A}' \to \mathcal{A}$ is determined by the requirement (Proposition 5.6)

$$\mathcal{R}(\Phi'(\mathfrak{f}')) = \Phi(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f})\,.$$

The final important result regards the properties of $\mathcal{R}$ for the algebras of a pair of paracausally related metrics g, g' when it acts on the states $\omega : \mathcal{A} \to \mathbb{C}$, $\omega' : \mathcal{A}' \to \mathbb{C}$ of the algebras in terms of pull-back.

$$\omega' = \omega \circ \mathcal{R}\,.$$

As is known, the most relevant (quasifree) states in algebraic QFT are *Hadamard states* characterized by a certain wavefront set of their two-point function. To this regard, we prove that the pull-back through $\mathcal{R}$ of a Hadamard state $\omega : \mathcal{A} \to \mathbb{C}$ is a Hadamard state of the off-shell algebra $\mathcal{A}'$, provided the metrics g, g' be paracausally related. The result is extended to a generic bidistribution ν (corresponding to the two-function of ω, dropping the remaining requirements included in the definition of state). The proof of the theorem below is both of geometrical and microlocal analytic nature (see also Theorem 5.14).

Theorem 3 (Theorem 5.19) *Let* E *be an* $\mathbb{R}$*-vector bundle on a smooth manifold* M *equipped with a non-degenerate, symmetric, fiberwise metric* $\langle \cdot | \cdot \rangle$. *Let* $g, g' \in \mathcal{GH}_{\mathsf{M}}$*, consider the corresponding formally-selfadjoint normally hyperbolic operators* $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *and refer to the associated CCR algebras* $\mathcal{A}$ *and* $\mathcal{A}'$.
Let $\nu \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ *be of Hadamard type and satisfy*

$$\nu(x, y) - \nu(y, x) = i\mathsf{G}_{\mathsf{N}}(x, y) \quad mod \quad C^{\infty} \,,$$

$\mathsf{G}_{\mathsf{N}}(x, y)$ *being the distributional Kernel of* G_{N}.
Assuming $g \simeq g'$*, let us define*

$$\nu' := \nu \circ \mathsf{R}^{\dagger_{gg'}} \otimes \mathsf{R}^{\dagger_{gg'}} \,,$$

for a Møller operator $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *of* g, g'. *Then the following facts are true.*

(i) ν *and* ν' *are bisolutions mod* C^{∞} *of the field equations defined by* N *and* N' *respectively,*
(ii) $\nu' \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$,
(iii) $\nu'(x, y) - \nu'(y, x) = i\mathsf{G}_{\mathsf{N}'}(x, y)$ *mod* C^{∞},
(iv) ν' *is of Hadamard type.*

As this crucial result concerns off-shell algebras, in principle, it could be exploited in perturbative constructions of interacting theories. Indeed the preservation of the Hadamard singularity structure plays a crucial role in the development of the perturbative theory [27]. Another work will be devoted to this investigation.

The work is organized as follows. Section 2 contains a recap on the relevant notions of Lorentzian geometry we exploit throughout. In particular, in Sect. 2.2 we introduce some (apparently new) results about convex interpolations of globally hyperbolic metrics which are preparatory to Sect. 2.3 where we present the definition of paracausal relation and we give some basic results about this equivalence relation. Section 3 is completely devoted to recalling some notions and fundamental results about Green hyperbolic operators, normal hyperbolic results and their interplay with convex combinations of Lorentzian metrics. Section 4 is the core of the paper. In the Sects. 4.1 and 4.2 Møller maps are introduced under the additional assumption that the metrics $g_0, g_1 \in \mathcal{GH}_{\mathsf{M}}$ satisfy $g_0 \preceq g_1$. The latter is removed in Sect. 4.3, where the analysis is extended to encompass paracausally deformed metrics $g \simeq g'$. In Sect. 4.4, it is shown that the Møller map preserves the natural symplectic form on the space of initial data and finally, in Sect. 4.5 the Møller operator for paracausally deformed metrics is introduced and analyzed in detail. Section 5 is devoted to the study of free quantum field theories on globally hyperbolic spacetimes. In particular, in Sect. 5.2 we lift the Møller operator to a $*$-isomorphism of algebras of observables and in Sect. 5.3 we show that the pull-back of any quasifree state along the Møller $*$-isomorphism preserves the Hadamard condition. Finally, we conclude our paper with Sect. 6, where open issues and future prospects are presented.

General notation and conventions

- $A \subset B$ permits the case $A = B$, otherwise we write $A \subsetneq B$.
- The symbol $\mathbb{K}$ denotes any element of $\{\mathbb{R}, \mathbb{C}\}$.
- Tensor fields and sections of $\mathbb{K}$-vector bundles on M are always supposed to be smooth.
- (M, g) denotes a $(n+1)$-dimensional spacetime (*cf.* Definition 2.4) and we adopt the convention that g has the signature $(-, + \ldots, +)$.
- $\sharp : \Gamma(\mathsf{T}^*\mathsf{M}) \to \Gamma(\mathsf{TM})$ and its inverse $\flat : \Gamma(\mathsf{TM}) \to \Gamma(\mathsf{T}^*\mathsf{M})$ denote the standard (fiberwise) **musical isomorphisms** (*cf.* Sect. 2.2.1) referred to a given metric g on M.
- $\mathcal{M}_{\mathsf{M}}$, $\mathcal{T}_{\mathsf{M}} \subset \mathcal{M}_{\mathsf{M}}$ and $\mathcal{GH}_{\mathsf{M}} \subset \mathcal{T}_{\mathsf{M}}$ denote respectively the sets of smooth Lorentzian metrics, **time-oriented** Lorentzian metric and **globally hyperbolic** metrics on M;
- $g \preceq g'$ denotes that $g, g' \in \mathcal{M}_{\mathsf{M}}$ and the open light cone V_p^g of g is a subset of the open lightcone $V_p^{g'}$ of g' at every point $p \in \mathsf{M}$;
- $g \simeq g'$ denotes that g and g' are **paracausally related** (*cf.* Definition 2.19).

2 Convex interpolation and paracausal deformations of Lorentzian metrics

The aim of this section is twofold. On the one hand, we shall investigate the properties of *convex interpolation of Lorentzian metrics*, on the other hand we introduce the notion of *paracausal deformations of globally hyperbolic metrics*. As we shall see, these mathematical tools rely on a certain preordering relation in the set of Lorentzian metrics on a given manifold, they are quite interesting on their own right and they will be exploited in the second part of this work to construct Møller operators and Møller $*$-isomorphisms of algebras of quantum fields.

2.1 Preliminaries on Lorentzian geometry

The aim of this section is to recall some basic results of Lorentzian geometry which we will need later on. For a more detailed introduction to Lorentzian geometry we refer to [3, 9, 66].

2.1.1 Lorentzian manifolds and cones

Let M be a smooth connected paracompact Hausdorff manifold and assume that M is noncompact or its Euler characteristic vanishes. Under these assumptions, M admits a Lorentzian metric and we denote the space of Lorentzian metrics on M by $\mathcal{M}_{\mathsf{M}}$ (see e.g. [9]). Once that a Lorentzian metric g is assigned to a smooth manifold $\mathsf{M} \ni p$, we can classify the vectors $v_p \in \mathsf{T}_p\mathsf{M}$ into three different types:

- **spacelike** *i.e.* $g(v_p, v_p) > 0$ or $v_p = 0$,
- **timelike** *i.e.* $g(v_p, v_p) < 0$,
- **lightlike** (also called **null**) *i.e.* $g(v_p, v_p) = 0$ *and* $v_p \neq 0$.

Remark 2.1 Notice that, with our definition, the tangent vector 0 is spacelike.

As usual, we denote as **causal vectors** any timelike or lightlike vector. Piecewise smooth curves are classified analogously according to the nature of their tangent vectors.

Keeping in mind this classification, the open **lightcone** of (M, g) at $p \in \mathsf{M}$ is the set

$$V_p^g := \{v_p \in \mathsf{T}_p\mathsf{M} \mid g(v_p, v_p) < 0\} .$$

It is not difficult to see that it is an open convex cone made of two disjoint open convex halves defining the two connected components of V_p^g.

The notion of *time orientation* is defined as in [3]: A smooth Lorentzian manifold (M, g) is said to be **time-orientable** if there is a continuous timelike vector field X on M.

If (M, g) is time orientable and a preferred continuous timelike vector field X has been chosen as above, the **future lightcone** $V_p^{g+} \subset V_p^g$ at $p \in \mathsf{M}$ is the connected component of V_p^g containing X_p. The other connected component V_p^{g-} is the **past lightcone** at p. V_p^{g+} and V_p^{g-} respectively includes the **future-directed** and **past-directed** timelike vectors at p. The terminology extends to the causal (lightlike) vectors which belong to the closures of the said halves. A classification of (piecewise smooth) causal curves into past-directed and future-directed curves (see [3]) arises according to their tangent vectors.

If (M, g) is time orientable, the continuous choice of one of the two halves of V_p^g for all $p \in \mathsf{M}$ through a continuous timelike vector field as above defines a **time orientation** of (M, g). (M, g) with this choice of preferred halves of cones is said to be **time oriented**. If (M, g) is connected and time orientable, then it admits exactly two time orientations.

Notation 2.2 In the following, we denote with $\mathcal{M}_{\mathsf{M}}$, the set of smooth Lorentzian metrics on the smooth manifold M and with $\mathcal{T}_{\mathsf{M}}$ the class of time-oriented Lorentzian metrics on M.

We have an elementary fact whose proof is immediate if working in a g-orthonormal basis.

Proposition 2.3 *Assume that $g \in \mathcal{T}_{\mathsf{M}}$, $p \in \mathsf{M}$, and $Y_p, Z_p \in V_p^g$. Then*

(i) $Y_p \in V_p^{g\mp}$ *and* $Z_p \in V_p^{g\pm}$ *if and only if* $g(Y_p, Z_p) > 0$,
(ii) $Y_p, Z_p \in V_p^{g\pm}$ *if and only if* $g(Y_p, Z_p) < 0$.

If $g \in \mathcal{M}_{\mathsf{M}}$, the associated standard (fiberwise) **musical isomorphism** $\sharp : \Gamma(\mathsf{T}^*\mathsf{M}) \to \Gamma(\mathsf{TM})$ is pointwise defined by

$$g(\sharp\omega_p, v_p) = \omega_p(v_p) \quad \text{for every } v \in \Gamma(\mathsf{TM}) \text{ and } \omega \in \Gamma(\mathsf{T}^*\mathsf{M}) \text{ and } p \in \mathsf{M},$$

and we denote the (fiberwise) **inverse musical isomorphism** by $\flat : \Gamma(\mathsf{TM}) \to \Gamma(\mathsf{T}^*\mathsf{M})$. The notation $g^\sharp \in \Gamma(\mathsf{TM} \otimes \mathsf{TM})$ indicates the Lorentzian metric induced on 1-forms from $\sharp$ as

$$g^\sharp(\omega_{1p}, \omega_{2p}) = g(\sharp\omega_{1p}, \sharp\omega_{2p}) \quad \text{for every } \omega_1, \omega_2 \in \Gamma(\mathsf{T}^*\mathsf{M}) \text{ and } p \in \mathsf{M}.$$

Once that a Lorentzian metric is introduced on 1-forms, we can distinguish three different type of co-vectors: $\omega_p \in \mathsf{T}_p^*\mathsf{M}$ is **spacelike**, **timelike**, **null** and **causal** if, respectively, $\sharp\omega_p \in \mathsf{T}_p\mathsf{M}$ is spacelike, timelike or null. With the definition, we can define the open **lightcone of 1-forms** at $p \in \mathsf{M}$ analogously to the case of vectors

$$V_p^{g^\sharp} := \{\omega_p \in \mathsf{T}_p^*\mathsf{M} \mid g^\sharp(\omega_p, \omega_p) < 0\}\,.$$

Analogously, if $g \in \mathcal{T}_{\mathsf{M}}$, the **future** and **past lightcones of 1-forms** at $p \in \mathsf{M}$ are defined as

$$V_p^{g^\sharp\pm} := \{\omega_p \in \mathsf{T}_p^*\mathsf{M} \mid \sharp\omega_p \in V_p^{g\pm}\}\,.$$

Let us finally recall that, embedded codimension-1 submanifold $\Sigma \subset \mathsf{M}$ of a Lorentzian manifold (M, g), also called **hypersurfaces**, are classified according to their normal covector n: They are **spacelike**, **timelike**, **null** if respectively n is timelike, spacelike, null everywhere in Σ. Notice that an embedded $n-1$ submanifold $\Sigma \subset \mathsf{M}$ is spacelike if and only if its tangent vectors are spacelike in (M, g). The restriction of g to the tangent vectors to a spacelike hypersurface Σ defines a Riemannian metric on it.

2.1.2 Spacetimes and causality

Definition 2.4 A **spacetime** is a $(n+1)$-dimensional $(n \geqslant 1)$, connected, time-oriented, smooth Lorentzian manifold (M, g)

Remark 2.5 Sometimes it is also assumed that M is orientable and oriented, but we do not adopt this hypothesis here. However, when we write that (M, g) *is a spacetime* we also mean that a *time-orientation* of (M, g) as Lorentzian manifold has been chosen. In this case, with a little misuse of language, we speak of the *time-orientation of the metric* g.

Let now $A \subset \mathsf{M}$ for a spacetime (M, g). The **causal sets** $J_{\pm}(A)$ and the **chronological sets** $I_{\pm}(A)$ are defined according to [3]: $J_{\pm}(A)$ is made of the points of A itself and all $p \in M$ such that there is a smooth future-directed/past-directed causal curve $\gamma : [a, b] \to \mathsf{M}$ with $\gamma(a) \in A$ and $\gamma(b) = p$. *Notice that* $J_{\pm}(A) \supset A$ *by definition*, while $I_{\pm}(A)$ is made of the points $p \in M$ such that there is a smooth future-directed/past-directed timelike curve $\gamma : [a, b] \to \mathsf{M}$ with $\gamma(a) \in A$ and $\gamma(b) = p$. As usual we define $J(A) := J_+(A) \cup J_-(A)$.

Let us recall that, on a spacetime (M, g), a smooth future-directed causal curve $\gamma : I \to \mathsf{M}$ with $I \subset \mathbb{R}$ open interval is said to be **future inextendible** [66] if there is no *continuous* curve $\gamma' : J \to \mathsf{M}$, defined on an open interval $J \subset \mathbb{R}$, such that $\sup J > \sup I$ and $\gamma'|_I = \gamma$. A **past inextendible** causal curve is defined analogously. A causal curve is said to be **inextendible** if it is both past and future inextendible.

We eventually define the **future Cauchy development** $D_+(A)$ of A to be the set of points $p \in \mathsf{M}$ such that every past inextendible future-directed smooth causal curve passing through p meets A in the past. Similarly, the **past Cauchy development** $D_-(A)$ is the set of points $p \in \mathsf{M}$ such that every future inextendible future-directed smooth causal curve passing through p meets A in the future.

On a generic Lorentzian manifold, the Cauchy problem for a differential operator is in general ill-posed: This can be a consequence of the presence of closed timelike curves or the presence of naked singularities. Therefore, it is convenient to restrict ourselves to the class of *globally hyperbolic spacetimes*.

Definition 2.6 A **globally hyperbolic spacetime** is a spacetime (M, g) such that

(i) there are no closed causal curves;
(ii) for all points $p, q \in \mathsf{M}$, $J_+(p) \cap J_-(q)$ is compact.

Notation 2.7 If M is a smooth connected $(n + 1)$-manifold, $\mathcal{GH}_\mathsf{M} \subset \mathcal{T}_\mathsf{M}$ denotes the class of Lorentzian metrics g such that (M, g) is globally hyperbolic for a time-orientation. Any $g \in \mathcal{GH}_\mathsf{M}$ is called **globally hyperbolic** metric on M.

In his seminal paper [51], Geroch established the equivalence for a Lorentzian manifold being globally hyperbolic and the existence of a *Cauchy hypersurface*.

Definition 2.8 A subset $\Sigma \subset \mathsf{M}$ of a spacetime (M, g) is called **Cauchy hypersurface** if it intersects exactly once any inextendible future-directed smooth timelike curve.

In particular, a Cauchy hypersurface is **achronal**: it intersects at most once every future-directed smooth timelike curve.

Theorem 2.9 ([51, Theorem 11]) *A spacetime* (M, g) *is globally hyperbolic if and only if it contains a Cauchy hypersurface.*

It turns out that Cauchy hypersurfaces of (M, g) are closed co-dimension 1 topological submanifolds of M homeomorphic one to each other. As a byproduct of Geroch's theorem, it follows that the globally hyperbolic manifold (M, g) admits a continuous foliation in Cauchy hypersurfaces Σ, namely M is homeomorphic to $\mathbb{R} \times \Sigma$. The proof of these facts was carried out by finding a *Cauchy time function*, i.e., a continuous function $t : \mathsf{M} \to \mathbb{R}$ which is strictly increasing on any future-directed timelike curve and such that its level sets $t^{-1}(t_0)$, $t_0 \in \mathbb{R}$, are Cauchy hypersurfaces homeomorphic to Σ. Geroch's splitting appears at a topological level, and the possibility to smooth them remained an open folk questions for many years. Only recently, in [13] Bernal and Sánchez "smoothened" the result of Geroch by introducing the notion of *Cauchy temporal function*.

Theorem 2.10 ([13, Theorems 1.1 and 1.2], [14, Theorem 1.2],) *For every globally hyperbolic spacetime* (M, g) *there is an isometry* $\psi : \mathsf{M} \to \mathbb{R} \times \Sigma$, *where the latter spacetime is equipped with the smooth Lorentzian metric*

$$-\beta^2 d\tau \otimes d\tau \oplus h_\tau\ , \tag{2.1}$$

and the time-orientation induced from (M, g) *through* ψ. *Above* τ *is the canonical projection*

$$\mathbb{R} \times \Sigma \ni (t, p) \mapsto t \in \mathbb{R}$$

and the following facts are valid:

(i) $\nabla\tau := \sharp d\tau$ *is past-directed timelike,*
(ii) $\beta : \mathbb{R} \times \Sigma \to (0, +\infty)$ *(called* **lapse function***) is a smooth function,*
(iii) h_t*(called* **spatial metric***) is a smooth Riemannian metric on each leaf* $\{t\} \times \Sigma$, $t \in \mathbb{R}$,
(iv) *every embedded co-dimension-*1 *submanifold* $\{t_0\} \times \Sigma = \tau^{-1}(t_0)$ *is a spacelike (smooth) Cauchy hypersurface.*

Finally, if $S \subset \mathsf{M}$ *is a spacelike Cauchy hypersurface of* (M, g), *then we can define an isometry* $\psi : \mathsf{M} \to \mathbb{R} \times S$, *and* τ, β, h *as above in order that* $S = \psi^{-1}(\{0\} \times S)$.

The characterization given by Bernal and Sánchez permits us to give some relevant definitions.

Definition 2.11 Given a spacetime (M, g), a smooth surjective function $t : \mathsf{M} \to \mathbb{R}$ with dt past-directed timelike is

(a) a **Cauchy temporal function** if

  (i) (M, g) is isometric, through some isometry $\psi : \mathsf{M} \to \mathbb{R} \times \Sigma$, to a spacetime $(\mathbb{R} \times \Sigma, h)$ with the time-orientation induced from (M, g),
  (ii) $t = \tau \circ \psi$ (where $\tau : \mathbb{R} \times \Sigma \ni (t, p) \mapsto t \in \mathbb{R}$),
  (iii) h has the form (2.1) as in Theorem 2.10 satisfying (i)-(iv);

(b) a **smooth Cauchy time function** if

  (i) (M, g) is isometric, through some isometry $\psi : \mathsf{M} \to \mathbb{R} \times \Sigma$, to a spacetime $(\mathbb{R} \times \Sigma, h)$ with the time-orientation induced from (M, g),
  (ii) $t = \tau \circ \psi$ (where $\tau : \mathbb{R} \times \Sigma \ni (t, p) \mapsto t \in \mathbb{R}$),
  (iii) every $\Sigma_{t_0} := t^{-1}(t_0) = \psi^{-1}(\{t_0\} \times \Sigma)$ is a spacelike Cauchy hypersurface of (M, g) for $t_0 \in \mathbb{R}$.

Remarks 2.12 (1) An intrinsic way to write (2.1) for a Cauchy temporal function t without making use to the splitting diffeomorphism ψ is, for $p \in \Sigma_s = t^{-1}(p)$

$$g_p(X, Y) = \frac{dt \otimes dt(X, Y)}{g^\sharp(dt, dt)} + h_s(\pi_{t,g}X, \pi_{t,g}Y)\,, \quad X, Y \in \mathsf{T}_p\mathsf{M} = L(\sharp_g dt) \oplus \Sigma_s$$

where

$$\mathsf{T}_p\mathsf{M} \ni X \mapsto \pi_{t,g}X := X - \frac{\langle dt, X\rangle\, \sharp_g dt}{g^\sharp(dt, dt)} \in \mathsf{T}_p\Sigma_s$$

defines the orthogonal projector onto $\mathsf{T}_p\Sigma_s$ associated to t and g, using $\sharp_g dt$ as normal (contravariant) vector to Σ_s.

(2) If an either smooth time or temporal Cauchy function t exists for (M, g), the level sets $\Sigma_{t_0} := t^{-1}(t_0)$ are smooth spacelike Cauchy surfaces diffeomorphic to each other and (M, g) is globally hyperbolic. Theorem 2.10 proves that temporal Cauchy functions – thus also smooth time Cauchy functions – exist for every globally hyperbolic spacetime. Furthermore, every smooth spacelike Cauchy hypersurface can be embedded in the foliation induced by a suitable temporal Cauchy function.

(3) A Cauchy temporal function is always a Cauchy time function, but even a smooth time function may not be a temporal one.

(4) A Cauchy hypersurface may meet a causal curve in more than a point (say, a segment), but this is not the case for the spacelike Cauchy hypersurfaces since they are **acausal**: they intersect *at most once* every future-directed smooth causal curve, as easily arises from Theorem 2.10.

We shall now give some notable examples of globally hyperbolic spacetimes to acquaint the reader with some concrete cases.

Examples 2.13 We shall list a few globally hyperbolic spacetimes which appear commonly in general relativity and quantum field theory over curved backgrounds. As one can infer per direct inspection, they all fulfill Theorem 2.10:

- The prototype example is Minkowski spacetime which isometric to $\mathbb{R}^{n+1}$ with Cartesian coordinates $(t, x^1, \ldots, x^n)$ and equipped with the Minkowski metric

$$-dt \otimes dt + \sum_{i=1}^{n} dx^i \otimes dx^i\ ;$$

- De Sitter spacetime, that is the maximally symmetric solution of Einstein's equations with a positive cosmological constant Λ. As a manifold it is topologically $\mathbb{R} \times \mathbb{S}^3$ and the metric reads:

$$g = -dt^2 + \frac{3}{\Lambda}\cosh^2\left(\sqrt{\frac{\Lambda}{3}}t\right)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)]$$

where $t \in \mathbb{R}$ while (χ, θ, φ) are the standard coordinates on $\mathbb{S}^3$;

- the Friedmann-Robertson-Walker spacetime, i.e., an isotropic and homogeneous manifold which is topologically $\mathbb{R} \times \Sigma$ and

$$g = -dt^2 + a(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

where k can be either 0 or ± 1 and function $a(t)$ is smooth and positive valued;
- The external Schwarzschild spacetime, i.e., a stationary spherically symmetric solution of vacuum Einstein's equations which is topologically $\mathbb{R}^2 \times \mathbb{S}^2$ with metric

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Here $M > 0$ is interpreted as the mass of the spherically symmetric source (a blackhole, a star,...) and the domain of definition of the coordinates is $t \in \mathbb{R}$, $r \in (2M, +\infty)$ and $(\theta, \varphi) \in \mathbb{S}^2$;
- Finally, given any n-dimensional complete Riemannian manifold (Σ, h), an open interval $I \subseteq \mathbb{R}$ and a smooth function $f : I \to (0, +\infty)$, the Lorentzian *warped product* defined topologically by $I \times \Sigma$ with metric $g = -dt^2 + f(t)h$ is a globally hyperbolic spacetime.

2.2 Convex interpolation of Lorentzian metrics

Due to motivations that will be clear later in the paper and related to the construction of Møller operators, we are now interested in the structure of the set $\mathcal{M}_{\mathsf{M}}$ of Lorentzian metrics on a given manifold M. In particular, we are interested in the following problem:

> *Are there some natural operations that s can be used to produce (globally hyperbolic) Lorentzian metrics starting from (globally hyperbolic) Lorentzian metrics?*

Given two globally hyperbolic metrics g, g', a linear combination of them is in general not a Lorentzian metric and, when it is, it fails to be globally hyperbolic in general. However, as shown in [16, Appendix B], if g and g' coincide outside a compact set, then there exists a sequence of 5 globally hyperbolic metrics, such that for each neighbouring pair all pointwise convex combinations are globally hyperbolic metrics. Therefore, this section aims to provide sufficient conditions for some kind of linear combination of globally hyperbolic metrics to be a globally hyperbolic Lorentzian metric. We shall see that convex combinations are an interesting case of study under suitable conditions. We point out the recent work [71] where, in addition to several related issues, the convex structure of the space of globally hyperbolic metrics on a given manifold is addressed with a number of results.

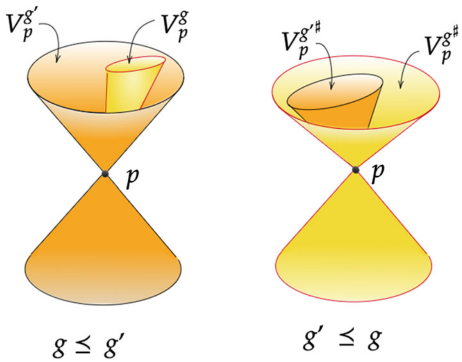


Fig. 1 Lorentzian metrics $\preceq$-comparable

2.2.1 A preorder relation of Lorentzian metrics

Definition 2.14 Let $g, g' \in \mathcal{M}_{\mathsf{M}}$ and denote

$$g \preceq g' \quad \text{iff} \quad V_p^g \subset V_p^{g'} \text{ for all } p \in \mathsf{M}.$$

We say that $g, g' \in \mathcal{M}_{\mathsf{M}}$ are **$\preceq$-comparable** if either $g \preceq g'$ or $g' \preceq g$ (see e.g. Fig. 1).

Remarks 2.15 (1) Let us remark that the definition above can be generalized by considering the so-called *causal diffeomorphisms*, namely a time-orientation preserving diffeomorphism $\varphi : \mathsf{M} \to \mathsf{N}$ such that the open light cone V_p^g of g is included in the open light cone $V_p^{\varphi g'}$ of $\varphi^* g'$ for every $p \in \mathsf{M}$. For further details and properties we refer to [37, 39, 40].

(2) The preorder relation introduce in Definition 2.14 has a corresponding for the associated metrics in the cotangent space: If $g, g' \in \mathcal{M}_{\mathsf{M}}$,

$$g^\sharp \preceq g'^\sharp \quad \text{iff} \quad V_p^{g^\sharp} \subset V_p^{g'^\sharp} \text{ for all } p \in \mathsf{M}.$$

We observe that if $g \preceq g'$ for $g, g' \in \mathcal{T}_{\mathsf{M}}$ and the two metrics share the same time-orientation – i.e., there is a continuous vector field on M which is timelike for both metrics and defines the same time-orientation for both of them – then $V_p^{g+} \subset V_p^{g'+}$ and $V_p^{g-} \subset V_p^{g'-}$ for every $p \in \mathsf{M}$. Similar inclusions hold when considering the closures of the considered half cones. As a consequence, we have both inclusions with obvious notations

$$I_\pm^g(A) \subset I_\pm^{g'}(A)\,, \quad J_\pm^g(A) \subset J_\pm^{g'}(A) \quad \text{for every } A \subset \mathsf{M}.$$

The relation $\preceq$ in $\mathcal{M}_{\mathsf{M}}$ has several consequences whose most elementary ones are established in the following proposition. We remind the reader that a closed set $A \subset \mathsf{M}$, with (M, g) time-oriented, is **past compact** if $J_-(p) \cap A$ is compact or empty for every $p \in \mathsf{M}$. The definition of **future compact** is analogous, just replacing J_- for J_+.

Then next lemma is just a routine computation, so we leave the proof to the reader.

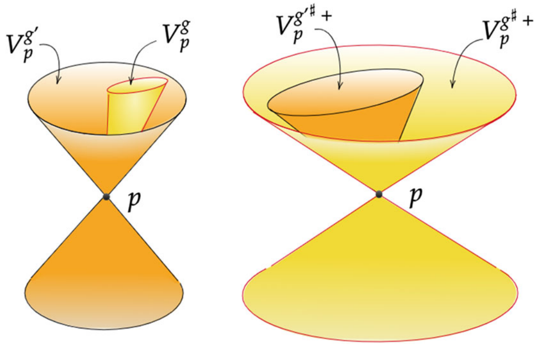


Fig. 2 Inclusion-of-cones relations

Lemma 2.16 *Let* M *be a smooth* $(n+1)$*-dimensional manifold and* $g, g' \in \mathcal{M}_{\mathsf{M}}$*. The following facts are valid for the preordering relation* $\preceq$ *in* $\mathcal{M}_{\mathsf{M}}$.

(1) *For* $p \in \mathsf{M}$ *and* $v \in \mathsf{T}_p\mathsf{M}$*, if* $g \preceq g'$ *then*

  (i) $g(v, v) = 0$ *implies* $g'(v, v) \leqslant 0$.
  (ii) $g'(v, v) > 0$ *implies* $g(v, v) > 0$.
  (iii) $g'(v, v) = 0$ *implies* $g(v, v) \geqslant 0$.

(2) *If* $g \preceq g'$ *with* $g \in \mathcal{T}_{\mathsf{M}}$ *and* $g' \in \mathcal{GH}_{\mathsf{M}}$*, then* (M, g) *is globally hyperbolic as well when, e.g., equipped with the same orientation and time-orientation of* (M, g') *and*

  (i) *a spacelike Cauchy hypersurface for* (M, g') *is also a spacelike Cauchy hypersurface for* (M, g)*;*
  (ii) *a smooth Cauchy time function for* (M, g') *is also a smooth Cauchy time function for* (M, g)*;*
  (iii) *a closed set* $A \subset \mathsf{M}$ *is past/future compact in* (M, g) *if it is respectively past/future compact in* (M, g').

(3) $g \preceq g'$ *if and only if* $g'^\sharp \preceq g^\sharp$.

(4) *If* $g, g' \in \mathcal{T}_{\mathsf{M}}$*,* $g \preceq g'$ *and* $p \in \mathsf{M}$*, then* $V_p^{g+} \subset V_p^{g'+}$ *if and only if* $V_p^{g'^\sharp +} \subset V_p^{g^\sharp +}$ *(see e.g. Fig. 2).*

Using the lemma above, we can immediately conclude the following.

Proposition 2.17 *If* $g \in \mathcal{M}_{\mathsf{M}}$ *and* $\mu : \mathsf{M} \to (0, +\infty)$ *is smooth, then*

(a) μg *and* $\mu^{-1} g$ *are Lorentzian,*
(b) $\mu g \preceq g \preceq \mu g$*,*
(c) $\mu^{-1} g \preceq g \preceq \mu^{-1} g$.
(d) μg *and* $\mu^{-1} g$ *are globally hyperbolic if* g *is and the spacelike Cauchy hypersurfaces of* g *are also spacelike Cauchy hypersurfaces for* μg *and* $\mu^{-1} g$.

2.2.2 Properties of convex combinations of Lorentzian metrics

A more interesting set of properties arises when focusing on smooth *convex combinations* of Lorentzian metrics. This is the first main result of this section.

Theorem 2.18 *Let* M *be a smooth* $(n+1)$*-dimensional manifold,* $g, g' \in \mathcal{M}_{\mathsf{M}}$*, and consider a smooth function* $\chi : \mathsf{M} \to [0,1]$*. If* $g \preceq g'$*, the following facts are valid*

(1) $(1-\chi)g + \chi g'$ *is a metric of Lorentzian type;*
(2) $g \preceq (1-\chi)g + \chi g' \preceq g'$*;*
(3) *if* $g^\sharp_\chi := (1-\chi)g^\sharp + \chi g'^\sharp$*, then* $g^\sharp_\chi := (g_\chi)^\sharp$ *for a (unique) metric* g_χ *of Lorentzian type;*
(4) $g \preceq g_\chi \preceq g'$*;*
(5) *If* g' *is globally hyperbolic and* g *time-orientable, then* $(1-\chi)g + \chi g'$ *and* g_χ *are globally hyperbolic.*

Proof (1) It is sufficient to prove the thesis point by point. Let q, q' be quadratic forms in a real $n+1$ dimensional linear space V of signature $(-,+,\dots,+)$ such that $q'(x) \leqslant 0$ implies $q(x) \leqslant 0$. We prove that the strict convex combination $q'' = cq + (1-c)q'$ for $c \in (0,1)$ has signature $(-,+,\dots,+)$. Indeed, there is a 1-dimensional subspace L on which $q'(x) < 0$ if $x \neq 0$. So $q(x) \leqslant 0$ on L and hence $q''(x) < 0$ on L for $x \neq 0$. There is also a n-dimensional subspace H on which $q(x) > 0$ if $x \neq 0$. Then $q'(x) > 0$ on H for $x \neq 0$ and hence $q''(x) > 0$ on H if $x \neq 0$. By construction, $L \cap H = \{0\}$ necessarily, so that $V = L \oplus H$. The bilinear form $Q'' : V \times V \to \mathbb{R}$ associated to q'', in a basis of V made of $0 \neq e_0 \in L$ and $\{e_k\}_{k=1,\dots,n} \in H$ with $Q''(e_k, e_h) = \delta_{kh}$, is represented by the $(n+1)\times(n+1)$ matrix $\begin{bmatrix} q''(e_0) & c^t \\ c & I \end{bmatrix}$. Since the determinant is $q''(e_0) - c^t c < 0$ and n eigenvalues are $+1$, its signature is $(-,+,\dots,+)$.
(2) Suppose that $g(v,v) < 0$, then $g'(v,v) < 0$ because $g \preceq g'$ and thus $(1-\chi)g(v,v) + \chi g'(v,v) < 0$ because $\chi, 1-\chi \geqslant 0$. We have obtained that $g \preceq (1-\chi)g + \chi g'$. Let us pass to the remaining inequality. If $(1-\chi)g(v,v) + \chi g'(v,v) < 0$ then $g'(v,v) < 0$ or $g(v,v) < 0$, in this second case also $g'(v,v) < 0$ because $g \preceq g'$. In both cases $g'(v,v) < 0$. Summing up, $(1-\chi)g + \chi g' \preceq g'$, concluding the proof of (2).
(3) $g^\sharp$ and $g'^\sharp$ are Lorentzian metrics on $\mathsf{T}^*\mathsf{M}$ and $g'^\sharp \preceq g^\sharp$ due to Lemma 2.16, we can recast the same argument used to establish (1) with trivial re-arrangements, obtaining that $g^\sharp_\chi$ is Lorentzian and $g'^\sharp \preceq g^\sharp_\chi = (1-\chi')g'^\sharp + \chi' g^\sharp \preceq g^\sharp$ with $\chi' := 1-\chi$. Notice that $g_\chi(v,v) := g^\sharp_\chi(\flat v, \flat v)$ defines a Lorentzian metric as well, since it has the same signature of h by construction, and $g^\sharp = h$ trivially (and it is the unique metric with this property since $\flat$ is an isomorphism).
(4) It immediately arises from Lemma 2.16 by using $g'^\sharp \preceq g^\sharp_\chi = (1-\chi')g'^\sharp + \chi' g^\sharp \preceq g^\sharp$ with $\chi' := 1-\chi$.
(5) A smooth timelike vector field of (M, g) is also timelike for $(1-\chi)g + \chi g'$ and g_χ for (2) and (4) respectively. Hence these metrics are time-orientable and the thesis follows from Lemma 2.16 point (2).

□

2.3 Paracausal deformation of Lorentzian metrics

The aim of this section is to provide a new definition that shall encode the idea to deform a Lorentzian metric equipped with a time-orientation into another Lorentzian metric with a corresponding time-orientation, taking advantage of a procedure consisting of a finite number of steps. At each step, the light cones of the final metric g_k is related to the initial one g_{k-1} through an inclusion relation, either $g_{k-1} \preceq g_k$ or $g_k \preceq g_{k-1}$ preserving the time-orientation at each step, i.e., the future cone of g_k, respectively, includes or is included in the future cone of g_{k-1}.

2.3.1 Paracausal relation

Definition 2.19 Consider a pair of globally hyperbolic spacetimes on the same manifold M with corresponding metrics $g, g' \in \mathcal{GH}_{\mathsf{M}}$ and corresponding time-orientations. We say that g is **paracausally related** to g – and we denote it by $g \simeq g'$ – or equivalently g' is a **paracausal deformation** of g, if there is a finite sequence $g_0 = g, g_1, \ldots, g_N = g' \in \mathcal{GH}_{\mathsf{M}}$ with corresponding time-orientations, such that either

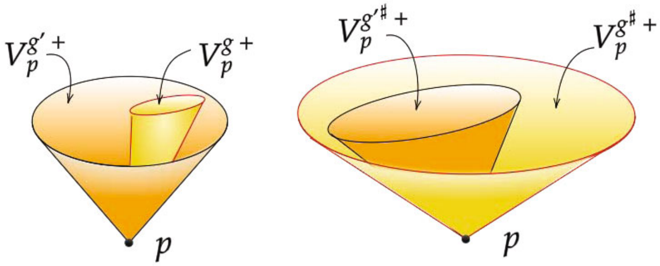
$$V_p^{g_k+} \subset V_p^{g_{k+1}+} \text{for all } p \in \mathsf{M}$$

or

$$V_p^{g_{k+1}+} \subset V_p^{g_k+} \quad \text{for all } p \in \mathsf{M},$$

where the choice may depend on $k = 0, \ldots, N-1$.

Remarks 2.20 (1) Let us remark that our notion of paracausally deformation implies in particular that g_k and g_{k+1} are always $\preceq$-comparable.
(2) Evidently, to be paracausally related is an *equivalence relation* in $\mathcal{GH}_{\mathsf{M}}$.
(3) We stress that paracausal deformations explicitly consider the time-orientations of the used sequences of globally hyperbolic spacetimes. So, even if we say that "metrics are paracausally related", the relation actually involves *the metrics equipped with corresponding time-orientations*.



(3) We shall show below a characterization of the paracausal relationship which seems more natural from a geometric and physical viewpoint. However, the definition above *as it stands* is much more directly suitable for the applications to Møller operators we shall introduce in the second part of this work.

Examples 2.21 (1) There two elementary cases of paracausally related (globally hyperbolic) metrics g_0, g_1 on M which are not directly $\preceq$-comparable:

1. There is a globally hyperbolic metric g on M such that, simultaneously $g \preceq g_0$ and $g \preceq g_1$ and the future lightcones are correspondingly included.
2. There is a globally hyperbolic metric g on M such that, simultaneously $g_0 \preceq g$ and $g_1 \preceq g$ and the future lightcones are correspondingly included.

In both cases, the existence of sequence g_0, g, g_1 proves that $g_0 \simeq g_1$.

(2) Let us give an elementary concrete example of paracausally related metrics. Consider the following smooth manifold $\mathbb{R}^n$ endowed with the Minkowski metrics

$$\eta_0 = -dt \otimes dt + \sum_{i=1}^{n} dx^i \otimes dx^i \qquad \eta_1 = -d\tau \otimes d\tau + \sum_{i=1}^{n} dy^i \otimes dy^i$$

where $(t, x_1, \ldots, x_n)$ and $(\tau, y_1, \ldots, y_n)$ are two different systems of Cartesian coordinates on $\mathbb{R}^{n+1}$. Here t and τ are Cauchy temporal functions associated to the respective Lorentzian metric and defining the time-orientation of the two metrics: dt and $d\tau$ are assumed to be past directed for the respective metric. More precisely, we assume that the two coordinate systems are related by means of a physically non-trivial permutation which interchanges space and time, as in Fig. 3, $\tau = x_1$, $y_1 = t$, and $y_k = x_k$ for $k > 1$. It is not difficult to see that even if $\eta_0 \neq \eta_1$ evidently, we have $\eta_0 \simeq \eta_1$: they are paracausally related by the sequence of metrics $\eta_0, \overline{g}_1, \overline{g}_2, \eta_1$ whose future cones are given as in Fig. 4. It is evident that by further implementing the procedure, it is possible to reverse the time-orientation of (M, η_0) through a sequence of paracausal deformations leaving the final metric identical to the initial one.

(3) We pass to present a case where a pair of globally hyperbolic metrics are *not* paracausally related. Consider the $2D$ Minkowski cylinder M obtained by identifying x and $x + L$ in $\mathbb{R}^2$ with coordinates x, y. The first globally hyperbolic spacetime is (M, η_1) where $\eta_1 = -dy \otimes dy + dx \otimes dx$, taking the identification into account, and with time-orientation defined by assuming that ∂_y is future-directed. The second globally hyperbolic spacetime is (M, η_2) where again $\eta_2 = -dy \otimes dy + dx \otimes dx$, taking the identification into account, but with the opposite time-orientation, i.e., defined by $-\partial_y$. See also Fig. 5.

These two metrics are not paracausally related. Any attempt to use the procedure as in the previous example to rotate the former into the latter faces the insurmountable obstruction that one of the auxiliary metrics would have Cauchy hypersurfaces given by the x-constant lines. This Lorentzian manifold is not globally hyperbolic because it admits closed temporal curves as in Fig. 6.

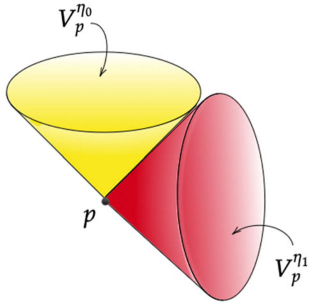


Fig. 3 Future light cones of different Minkowski metrics on $\mathbb{R}^{n+1}$

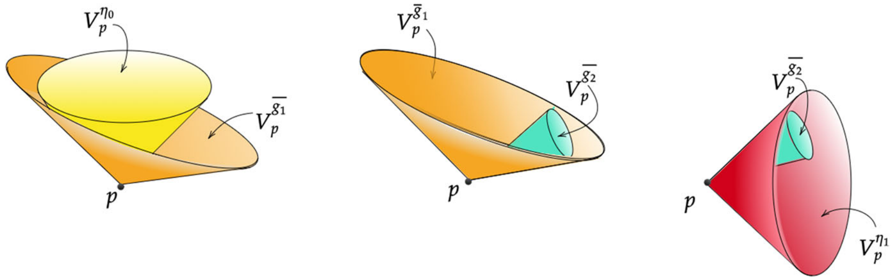


Fig. 4 Auxiliary future light cones to prove $\eta_0 \simeq \eta_1$

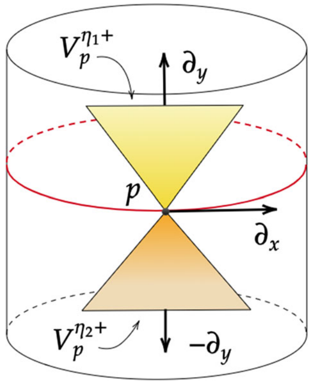


Fig. 5 2-D Minkowski cylinder

Notice that this obstruction does not take place without the identification $x \equiv x+L$.

2.3.2 Characterization of paracausal deformation in terms of future cones

There is a natural situation where two globally hyperbolic metrics g and g' on M are paracausally related. The generalization of the following result leads to a natural characterization of the paracausal relationship.

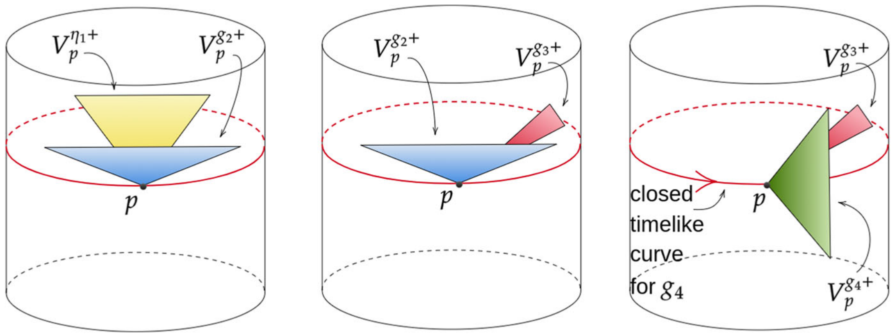


Fig. 6 Sequence of metrics where g_4 is not globally hyperbolic

Proposition 2.22 *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes on the same manifold* M. *If* $V_x^{g+} \cap V_x^{g'+} \neq \varnothing$ *for every* $x \in \mathsf{M}$, *then the metrics* g *and* g' *are paracausally related.*

Proof To prove the assertion it is sufficient to prove the existence of a Lorentzian metric $h \in \mathcal{T}_{\mathsf{M}}$ such that $h \preceq g$ and $h \preceq g'$. In this case, h would be globally hyperbolic according to (2) in Lemma 2.16 and the same argument as in (1) Examples 2.21 would prove the thesis.

Let us start by proving that a smooth vector field X on M exists such that $X_p \in V_p^{g+} \cap V_p^{g'+}$ for all $p \in \mathsf{M}$. Let us define the smooth functions

$$G : \mathsf{TM} \ni (p, v) \mapsto g_p(v, v) \in \mathbb{R}\,, \quad G_Y : \mathsf{TM} \ni (p, v) \mapsto g_p(v, Y) \in \mathbb{R}\,,$$

where Y is a smooth timelike future oriented vector field for g. By construction (with obvious notation) $\cup_{p\in\mathsf{M}} V_p^{g+} = G^{-1}(-\infty, 0) \cap G_Y^{-1}(-\infty, 0) \subset \mathsf{TM}$ is an open set. With the same argument, we have that also $\cup_{p\in\mathsf{M}} V_p^{g'+} \subset \mathsf{TM}$ is open. Finally, $\cup_{p\in\mathsf{M}} V_p^{g+} \cap \cup_{p\in\mathsf{M}} V_p^{g'+} = \cup_{p\in\mathsf{M}} V_p^{g+} \cap V_p^{g'+}$ is therefore open, non-empty by hypothesis, and projects onto the whole M by construction. As a consequence, given a local trivialization patch $\mathsf{T}U$ around $p \in U$, where (U, ψ) is a local chart on M (with $\dim(\mathsf{M}) = n + 1$), the set $\left(\cup_{p\in\mathsf{M}} V_p^{g+} \cap V_p^{g'+}\right) \cap \mathsf{T}U$ is diffeomorphic to an open subset $A \subset V \times \mathbb{R}^{n+1}$ with $V := \psi(U) \subset \mathbb{R}^{n+1}$ and $\pi_1(A) = V$ ($\pi_1 : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ being the standard projection onto the former factor). Working in coordinates, it is then trivially possible to pick out a smooth local section $X^{(U)}$ of $\mathsf{T}U$ such that $X_q^{(U)} \in V_q^{g+} \cap V_q^{g'+}$ if $q \in U$. To conclude, consider a partition of the unity $\{\chi_i\}_{i\in I}$ of M subordinated to a locally finite covering $\{U_i\}_{i\in I}$ of domains of local charts of M and let $X_p^{(U_i)} \in V_p^{g+} \cap V_p^{g'+}$ be constructed as above when $p \in U_i$ for every $i \in I$. The smooth vector field constructed as a locally finite convex linear combination $X := \sum_{i\in I} \chi_i X^{(U_i)}$ satisfies $X_p \in V_p^{g+} \cap V_p^{g'+}$ for every $p \in \mathsf{M}$ because the cones V_q^{g+}, $V_q^{g'+}$ are convex sets in a vector space and thus their intersection is also convex. X is the vector field we were searching for.

As the second step we construct a Lorentzian metric h, whose future cones V_p^{h+} satisfy $X_p \in V_p^{h+} \subset V_p^{g+} \cap V_p^{g'+}$ for every $p \in \mathsf{M}$. Notice that it means $h \in \mathcal{T}_{\mathsf{M}}$ since X is future directed for h (and also for the two metrics g and g') and thus it defines a time-orientation for (M, h). Since $h \preceq g, g'$, this would conclude our proof.

Let us construct h taking advantage of the vector field X. Consider $p \in M$ and define a g-pseudo orthonormal basis $e_0, \dots, e_n$ where $e_0 = \frac{X_p}{\sqrt{-g(X_p, X_p)}}$ and the remaining vectors are spacelike. If $v, v' \in \mathsf{T}_p\mathsf{M}$,

$$g(v, v') = -g(e_0, v)g(e_0, v') + \sum_{k=1}^{n} g(e_k, v)g(e_k, v') .$$

If $a \in (0, 1)$, the new Lorentzian scalar product in $\mathsf{T}_p\mathsf{M} \ni v, v'$

$$\begin{aligned} g^a(v, v') &:= -ag(e_0, v)g(e_0, v') + \sum_{k=1}^{n} g(e_k, v)g(e_k, v') \\ &= g(v, v') + (a-1)\frac{g(X_p, v)g(X_p, v')}{g(X_p, X_p)} \end{aligned} \tag{2.2}$$

trivially satisfies (the closure being taken in $\mathsf{T}_p\mathsf{M}\setminus\{0\}$)

$$X_p \in V_p^{g^a+} \subsetneq \overline{V_p^{g^a+}} \subsetneq V_p^{g+} \quad \text{for } a \in (0, 1).$$

The strong inclusions are due to the fact that the lightlike boundary of $V_p^{g^a+}$ is made of timelike vectors of g as it arises from the definition of g^a. Now note that $\partial V_p^{g^a+}$ becomes more and more concentrated around the set $\{\lambda X_p \mid \lambda > 0\}$ as a approaches 0 from above. (In particular, the limit and degenerate case $g_p^{a=0}(v, v) = 0$ implies v is parallel to X_p.) Since $X_p \in V_p^{g'+}$ which is also an open convex cone as $V_p^{g^a+}$, there must exist $a_p \in (0, 1)$ such that $V_p^{g^{a_p}+} \subset V_p^{g'+}$. This property is locally uniform in a as established in the following technical lemma:

Lemma.[1] *Within the hypotheses of the proposition, if* $x \in \mathsf{M}$, *there is a coordinate patch with domain* $V \ni x$, *an open set* $U \ni x$ *with compact closure* $\overline{U} \subset V$, *and a constant* $a_U \in (0, 1)$ *such that* $V_p^{g^{a_U}+} \subset V_p^{g'+}$ *for every* $p \in U$.

Proof If $x \in \mathsf{M}$, there is a coordinate patch with domain $V \ni x$ and coordinates $V \ni p \mapsto \varphi(p) = (x^0(p), \dots, x^n(p)) \in \mathbb{R}^{n+1}$ such that $U \ni x$ for some open subset $U \subset V$ such that $\overline{U}$ is compact. We will henceforth deal with U and the coordinates $(x^0, \dots, x^n)$ restricted to thereon. We will also take advantage of the compact set $K := \varphi(\overline{U}) \subset \mathbb{R}^{n+1}$ and identify $\mathsf{T}\overline{U}$ with $K \times \mathbb{R}^{n+1}$ using the coordinates. Finally,

[1] As noticed by the referee, a different strategy for proving this lemma would be showing that the function $M \ni p \mapsto a(p) = \sup\{a \in (0, 1) : V_p^{g^a+} \subset V_p^{g'+}\}$ is continuous. In that case, one canalternatively define $a_U := \min_{p \in \overline{U} a(p)}$. However the proof of continuity is not technically easy.

we will equip both K and $\mathbb{R}^{n+1}$ (representing $\mathsf{T}_p\mathsf{M}$ at each $p \in \overline{U} \equiv K$) with the standard Euclidean metric of $\mathbb{R}^{n+1}$ whose norm will be denoted by $||\cdot||$.

Let us start the proof by proving that the family of cones $V_p^{g'+}$ of g' has a minimal width $m > 0$ when p ranges in K. We henceforth view the above future-directed timelike vector field X and g' as geometric objects on K using the coordinate system. In particular, if $p \in K$, let us indicate by $v_p \in \mathbb{R}^{n+1}$ the unique future-directed timelike vector parallel to X_p (now viewed as a vector in $\mathbb{R}^{n+1}$) such that $||v_p|| = 1$. Consider the set made of future-directed elements of $T\mathsf{M}$

$$C := \{(p, u) \in K \times \mathbb{S}^n \mid g'_p(u, v_p) \leqslant 0\,,\, g'_p(u, u) = 0\}$$

(above $\mathbb{S}^n := \{z \in \mathbb{R}^{n+1} \mid ||z|| = 1\}$) and the continuous function

$$W : C \ni (p, u) \mapsto ||u - v_p|| \geqslant 0\,,$$

which computes the width of $\partial V_p^{g'+}$ (that is of $V_p^{g'+}$ itself) around X_p along the direction u by using the Euclidean distance induced by $||\cdot||$. Observe that C is compact since it is the intersection of preimages of a pair of closed sets along two corresponding continuous maps and it is included in a compact set. Since this map is continuous and C is compact, there exists

$$m := \min_C W > 0\,.$$

In particular, $m > 0$, otherwise $u = v_q$ for some $(q, u) \in C$ and this is not possible since it would imply $g'(v_q, v_q) = g'(u, u) = 0$, but v_q is timelike ($g'_q(v_q, v_q) < 0$) since it does not vanish ($||v_q|| = 1$) and it is proportional to the timelike vector X_q.

An analogous width-cone function can be defined for the cones of g^a (including the degenerated case $a = 0$) on a set C' which also embodies the dependence on a:

$$C' := \{(a, p, u) \in [0, 1/2] \times K \times \mathbb{S}^n \mid g^a_p(u, v_p) \leqslant 0\,,\, g^a_p(u, u) = 0\}\,.$$

We also define the continuous function

$$W' : C' \ni (a, p, u) \mapsto ||u - v_p|| \geqslant 0\,.$$

Observe that C' is again compact since it is the intersection of preimages of two closed sets along a pair of corresponding continuous maps of (a, p, u) and C' is included in a compact set.

We want to prove that there exists $a^m \in [0, 1/2]$ such that $W'(a^m, p, u) < m$ for all $(p, u) \in C$. If this were not the case, then for every $a_n := 1/n$ there would be a pair $(p_n, u_n) \in C$ such that $W'(a_n, p_n, u_n) \geqslant m$. Since C' is a compact metric space, we could extract a subsequence of triples $(a_{n_k}, p_{n_k}, u_{n_k}) \to (0, p_\infty, u_\infty) \in [0, 1/2] \times C$ for $k \to +\infty$ and some $(p_\infty, u_\infty) \in C$. By continuity $0 = g^{a_n}_{p_n}(u_n, u_n) \to g^0_{p_\infty}(u_\infty, u_\infty)$ where $||u_\infty|| = 1$. From (2.2), $g^0_{p_\infty}(u_\infty, u_\infty) = 0$ would entail that u_∞ is parallel to v_{p_∞} and thus $W'(0, p_\infty, u_{p_\infty}) = ||u_\infty - v_{p_\infty}|| = 0$. That is in contradiction with the requirement $W'(a_n, p_n, u_n) \geqslant m > 0$ for every $n = 1, 2, \ldots$ in view of the continuity of W'.

We have therefore established that there exists $a^m \in [0, 1/2]$ such that $W'(a^m, p, u) < m$ for all $(p, u) \in C$. From the definition of W and W', we have also obtained that $V_p^{g^{a^m}+} \subset V_p^{g'+}$ for all $p \in K$. It is enough to conclude that $V_p^{g^{a_U}+} \subset V_p^{g'+}$ for all $p \in U$ as wanted simply by taking $a_U := a^m$. This concludes our claim. □

Let us go on with the main proof. For every U as in the previous lemma, define the constant function $a(p) = a_U$ for $p \in U$. Since this can be done in a neighborhood of every point $p \in \mathsf{M}$, using a partition of the unity $\{\chi_i\}_{i\in I}$ subordinated to a locally finite covering of charts $\{U_i\}_{i\in I}$, we can construct the metric h, where now every $a_i := a_{U_i} : U_i \to (0, 1)$ is a constant in U and thus it is a smooth function therein.

$$\begin{aligned} h_p(v, v') &= \sum_i \chi_i(p) g_p^{a_i(p)}(v, v')r \\ &= \sum_i \chi_i(p) \left(g_p(v, v') + (a_i(p) - 1) \frac{g_p(X_p, v) g_p(X_p, v')}{g_p(X_p, X_p)} \right) \\ &= g_p(v, v') + \left(\sum_i \chi_i(p) a_i(p) - 1 \right) \frac{g_p(X_p, v) g_p(X_p, v')}{g_p(X_p, X_p)} \end{aligned}$$

Since $\sum_i \chi_i(p) a_i(p) \in (0, 1)$, this metric is still Lorentzian and of the form (2.2) point by point, where now $a(p) = \sum_i \chi_i(p) a_i(p)$. By construction $X_p \in V_p^{h+} \subset V_p^{g+}$ for every $p \in \mathsf{M}$, just because it happens point by point with the above choice of $a(p)$. In particular, we can endow h with the time-orientation induced by X as it happens for g, g' and all local metrics g^{a_i}. Finally, $V_p^{h+} \subset V_p^{g'+}$ because, if $h_p(v, v) < 0$, at least one of the values $g^{a_{i_0}(p)}(v, v)$ appearing in $\sum_i \chi_i(p) g_p^{a_i(p)}(v, v)$ must be negative and thus, if v is future-directed, $v \in V_+^{g^{a_i(p)}+} \subset V_p^{g'+}$. The proof is over because h satisfies all requirements $X_p \in V_p^{h+} \subset V_p^{g+} \cap V_p^{g'+}$ for every $p \in \mathsf{M}$. □

As an immediate byproduct, it is easy to see that for any globally hyperbolic metric g, there exists a paracausal deformation g' of g which is ultrastatic.

Corollary 2.23 *Let (M, g) be a globally hyperbolic spacetime. Then there exists a paracausal deformation g' of g such that (M, g') is an ultrastatic spacetime.*

Proof Let t be a Cauchy temporal function for the globally hyperbolic spacetime (M, g) so that M is isometric to $\mathbb{R} \times \Sigma$ with metric $-dt^2 + h_t$. We indicate by ∂_t the tangent vector to the submanifold $\mathbb{R}$. Let h be a complete Riemannian metric on Σ. Then the ultrastatic metric $g' := -dt^2 + h$ is globally hyperbolic [70] and the vector ∂_t is contained in the intersection of V_p^{g+} and $V_p^{g'+}$ for any $p \in \mathsf{M}$. Proposition 2.22 ends the proof. □

The result established in Proposition 2.22 leads to a crucial characterization of paracausally related metrics, which represent the second main result of this section.

Theorem 2.24 *Let* M *be a smooth manifold. Two metrics* $g, g' \in \mathcal{GH}_{\mathsf{M}}$ *are paracausally related if and only if there exists a finite sequence of globally hyperbolic metrics* $g_1 = g, g_2 \ldots, g_n = g'$ *on* M *such that all pairs of consecutive metrics* g_k, g_{k+1} *satisfy* $V_x^{g_k +} \cap V_x^{g_{k+1}+} \neq \varnothing$ *for every* $x \in \mathsf{M}$.

Proof If g, g' are paracausally related, then a sequence of metrics as in Definition 2.19 trivially satisfies the condition in the thesis. If that condition is *vice versa* satisfied, then the metrics of each pair g_k, g_{k+1} of the sequence are paracausally related in view of Proposition 2.22. Since paracausal relation is transitive, g and g' are paracausally related. □

2.3.3 Paracausal deformation and Cauchy temporal functions

We now study the interplay of the notion of Cauchy temporal function and the one of paracausal deformation.

A first result regards metrics that share a common foliation of Cauchy surfaces. We need a preliminary technical result.

Lemma 2.25 *Let* (M, g) *be a globally hyperbolic spacetime,* $t : \mathsf{M} \to \mathbb{R}$ *a Cauchy temporal function according to Definition 2.11, and* $\psi : \mathsf{M} \to \mathbb{R}\times\Sigma$ *a diffeomorphism mapping isometrically* (M, g) *to* $(\mathbb{R} \times \Sigma, -\beta^2 d\tau \otimes d\tau \oplus h_\tau)$. *Finally let* (M, g_α) *be a time oriented spacetime with time orientation such that* dt *is past directed.*
If ψ *maps* (M, g_α) *isometrically to* $(\mathbb{R} \times \Sigma, -d\tau \otimes d\tau \oplus \alpha^2(\tau)\beta^{-2}h_\tau)$ *with* $\alpha \in C^\infty(\mathbb{R}, (0, \infty))$*, then* (M, g_α) *is globally hyperbolic.*

Proof We will henceforth omit to write the isometry ψ and consider, without loss of generality, $\mathsf{M} = \mathbb{R}\times\Sigma, t = \tau, g = -\beta^2 dt\otimes dt\oplus h_t$ and $g_\alpha = -dt\otimes dt\oplus\alpha^2(t)\beta^{-2}h_t$.

We want to prove that Σ, viewed as the $t = 0$ slice of the temporal function t, is a spacelike Cauchy hypersurface for g_α. Evidently Σ is a spacelike hypersurface for g_α so that it suffices to prove that it meets exactly once every inextendible future directed g_α-timelike curve $\gamma : I \ni s \mapsto \gamma(s) \in \mathsf{M}$.

Since $\frac{dt}{ds} = g_\alpha(\partial_t, \dot\gamma) < 0$ by hypothesis, that γ can be re-parametrized by t itself as $\gamma' : J \ni t \mapsto \gamma'(t) \in M$ for some open interval $J \subset \mathbb{R}$. There must exist a finite $a > 0$ such that $(-a, a) \cap J \neq \varnothing$. Since $\gamma'|_{(-a,a)\cap J}$ is inextendible in the spacetime $(-a, a) \times \Sigma$ (otherwise it would not be inextendible in the whole spacetime), to conclude it is sufficient to prove that $(-a, a) \times \Sigma$ equipped with the metric g_α and the time-orientation induced by dt admits Σ as a Cauchy surface. Indeed, in that case, γ' must meet Σ exactly once in $(-a, a) \times \Sigma$ and thus Σ would be a Cauchy hypersurface for $(\mathbb{R} \times \Sigma, g_\alpha)$. Moreover, notice that it cannot meet $\Sigma = t^{-1}(0)$ again outside $(-a, a) \times \Sigma$ because γ' is parametrized by t. Global hyperbolicity of $((-a, a) \times \Sigma, g_\alpha)$ can be proved as follows. If $a > 0$, there exists a positive constant α_0 such that $\alpha(t) \geqslant \alpha_0 > 0$ for all $t \in [-a, a]$. We therefore have $g_\alpha \preceq g_{\alpha_0}$ on $(a, b) \times \Sigma$. In particular, with the time-orientation declared in the hypothesis, every future-directed causal tangent vector for g_α is a future-directed causal vector for g_{α_0}. Therefore, according to (2) in Lemma 2.16, it suffices to show that g_{α_0} is globally hyperbolic on $(-a, a) \times \Sigma$ and that Σ is a Cauchy hypersurface for g_{α_0}. To this end, consider an inextendible future-directed timelike curve $\gamma = (\gamma^0, \hat\gamma)$ in

$((-a,a)\times\Sigma, g_{\alpha_0})$. The curve $\widetilde{\gamma} := (\alpha_0^{-1}\gamma_0, \hat{\gamma})$ is future directed timelike w.r.t. g and still inextendible, therefore it meets $\Sigma = t^{-1}(0)$ exactly once, but $\widetilde{\gamma}$ and γ intersect in $t = 0$. Thus γ intersects Σ once. This shows g_{α_0} and therefore g_α to be globally hyperbolic on $(-a,a)\times\Sigma$. □

We can now state and prove a first result concerning Cauchy surfaces and the paracausal relation.[2]

Proposition 2.26 *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes on* M *which share a Cauchy temporal function* $t : \mathsf{M} \to \mathbb{R}$ *according to Definition 2.11. Then* $g \simeq g'$.

Proof As before, we will henceforth omit to write the isometries identifying the various spacetimes. However we may have two different isometries from M to $\mathbb{R}\times\Sigma$ for g and g'. Proposition 2.17 yields $g \preceq \hat{g} \preceq g$, $g' \preceq \hat{g}' \preceq g'$ if

$$\hat{g} := \beta_0^{-2} g = -dt\otimes dt + \beta_0^{-2} h_t \qquad \text{and} \qquad \hat{g}' := \beta_1^{-2} g' = -dt\otimes dt + \beta_1^{-2} h'_t,$$

where β_0^2, β_1^2 are the lapse function we choose in accordance with Theorem 2.10. The metrics $\hat{g}$ and $\hat{g}'$ are globally hyperbolic for Lemma 2.25 (with $\alpha = 1$). The proof ends if proving that $\hat{g}$ and $\hat{g}'$ are paracausally related. Referring to the splitting of M as $\mathbb{R}\times\Sigma$ induced by the Cauchy temporal function t, define the globally hyperbolic metric $-dt\otimes dt + h$, where h is a complete Riemannian metric on Σ (see, e.g., [70]). For every $\lambda \in (0,1)$, direct inspection proves that,

$$\begin{aligned} g_\lambda &:= \lambda(-dt\otimes dt + h) + (1-\lambda)\hat{g} \\ &= -dt\otimes dt + \lambda h + (1-\lambda)\beta_0^{-2} h_t \preceq -dt\otimes dt + \lambda h \end{aligned}$$

and

$$\begin{aligned} g_\lambda = \lambda(-dt\otimes dt + h) + (1-\lambda)\hat{g} = -dt\otimes dt + \lambda h \\ +(1-\lambda)\beta_0^{-2} h_t \preceq -dt\otimes dt + (1-\lambda)\beta_0^{-2} h_t\,. \end{aligned}$$

Since λh is complete, from the former line we conclude that the metric g_λ is globally hyperbolic due to (2) Lemma 2.16 and that it is paracausally related to $dt\otimes dt + \lambda h$. From the latter, since $-dt\otimes dt + (1-\lambda)\beta_0^{-2}h_t$ is globally hyperbolic in view of Lemma 2.25, we have that this metric and g_λ are paracausally related. Since $(1-\lambda) \in (0,1)$, the cones of $-dt\otimes dt + (1-\lambda)\beta_0^{-2}h_t$ include the cones of $-dt\otimes dt + \beta_0^{-2}h_t = \hat{g}$ so that these metrics are paracausally related as well. Transitivity implies that $-dt\otimes dt + \lambda h$ and $\hat{g}$ are paracausally related. The same argument proves that $-dt\otimes dt + \lambda h$ and $\hat{g}'$ are paracausally related so that $\hat{g} \simeq \hat{g}'$ and the thesis holds. □

Now we prove another non trivial result about paracausally related metrics for Cauchy compact spacetimes.

[2] The following proof is actually extracted by a result due to M. Sánchez who, with Theorem 3.4 of [71], improved a similar statement in a previous version of this work where we also assumed that the Cauchy surfaces were compact. We are grateful to M. Sánchez for providing this improved version of our result.

Proposition 2.27 *Let* (M, g) *and* (M, g') *be spacetimes such that* $g, g' \in \mathcal{GH}_{\mathsf{M}}$. *Suppose that* g *admits a Cauchy temporal function* $t : \mathsf{M} \to \mathbb{R}$ *whose spacelike Cauchy hypersurfaces are compact and are also* g'*-spacelike, then* $g \simeq g'$ *up to a change of the temporal orientation of* g'.

Proof First of all, by defining the g-normal $n_g = \frac{\sharp_g dt}{\sqrt{-g^\sharp(dt,dt)}}$, any vector field X can be written as $X = X_n n_g + \pi_g(X)$, where $X_n = g(n_g, X)$ and $\pi_g(X) = Id - g(n_g, X)n_g$ projects on the Cauchy surface.

The metric tensor $g = -\frac{dt^2}{g^\sharp(dt,dt)} + h(\pi(\cdot), \pi(\cdot))$ under the action of the diffeomorphism ψ_g gets recast in the orthogonal form $g_{ort} = -\beta dt^2 + h_t$. This metric is obviously, by 2.17, paracausally related to the conformal metric $g_c = -dt^2 + \frac{1}{\beta_t} h_t$, which is, in turn, paracausally related to the globally hyperbolic metric $\widetilde{g} = -dt^2 + h$ with h a complete Riemannian metric on the slice and if we choose coherently the time orientation. The last statement is a consequence of 2.22 which is proved exactly as 2.23.

Then we want look at the metric g' after the action of the isometric diffeomorphism ψ_g and define the $\widetilde{g}' = \psi_g^* g'$. The proof ends if we are able to find a globally hyperbolic metric g'' such that $\widetilde{g} \simeq g'' \simeq \widetilde{g}'$.

If we choose a function $\alpha \in C^\infty(\mathbb{R}, (0, \infty))$, then, by lemma 2.25 the metric tensor $g_\alpha = -dt^2 + \alpha(t)h$ is globally hyperbolic and, by 2.22, paracausally related to $\widetilde{g}$. We want to tune the fuction α in order to have that the cones of g_α intersect the cones of $\widetilde{g}'$.

First we define pointwise n' the smooth vector field g'-normal to the Cauchy hypersurfaces of g and decompose it with respect to the splitting of the tangent space induced by the metric g_α through its normal n_α. We get $n' = Zn_\alpha + W$ where $Z = g_\alpha(n_\alpha, n')$ and $W = \pi_{g_\alpha}(n')$.

Since the Cauchy hypersurfaces of g and g_α are spacelike also for g', we have that $Z \neq 0$. The cones of the two metrics intersct if α is such that n' is g_α-timelike i.e. iff

$$||n'||_{g_\alpha} = -|Z|^2 + \alpha(t)||W||_h^2 < 0 \iff \frac{1}{\alpha(t)} > \frac{||W||_h^2}{|Z|^2}.$$

The manifold $\mathbb{R} \times \Sigma$ can be covered by the time-strips $\mathcal{TS}_n = \{[-n, n] \times \Sigma\}_{n \in \mathbb{N}}$ which are obviously compact since Σ is compact by hypotesis.

This means that for all $n \in \mathbb{N}$ the smooth function $f : \mathsf{M} \to \mathbb{R}^+$ defined by $f := \frac{||W||_h^2}{|Z|^2}$ attains a maximum M_n and a minimum m_n when restricted to the strip $\mathcal{TS}_n$. So we construct the required function $\frac{1}{\alpha(t)} : \mathbb{R} \to \mathbb{R}^+$ such that

$$\frac{1}{\alpha(t)} = M_n + 1 \quad t \in [-n-1, -n) \cup (n, n+1].$$

This function isn't even continuous, but it piecewise constant. The maximum has been increased by one to avoid the possibility that this function gets null: it could happen if the normal n' and n_α get aligned in the first time-strip and then depart.

The last thing to do is to smoothen the function $\alpha(t)$, something which can of course be done by standard gluing arguments.

Now that we know that the cones of g_α and $\widetilde{g}'$ intersect, if the temporal orientation of g' is such that $V^+_{g_\alpha} \cap V^+_{\widetilde{g}'} \neq \varnothing$ we define $g'' := g_\alpha \simeq \widetilde{g}'$ and the proof is concluded. If $V^+_{g_\alpha} \cap V^+_{\widetilde{g}'} = \varnothing$ the metric g'', and therefore the metric $\widetilde{g}$, is paracausally related to $\widetilde{g}'$ with opposite time orientation. □

3 Normally hyperbolic operators and their properties

One of the main goals of this paper is to realize a geometric map to compare the space of solutions of *normally hyperbolic* operators defined on possibly different globally hyperbolic manifolds. Before starting to introduce our theory, we remind some general definitions and we fix the notation that will be used from now on. Let E be a vector bundle (always on $\mathbb{K}$ and of finite rank in this paper) over a spacetime (M, g), whose generic fiber (a $\mathbb{K}$ vector space isomorphic to a canonical fiber) is denoted by E_p for $p \in \mathsf{M}$. $\Gamma(\mathsf{E})$ is the $\mathbb{K}$-space of smooth sections E. $\Gamma(\mathsf{E})$ has a number of useful subspaces we list below.

(i) $\Gamma_c(\mathsf{E}) \subset \Gamma(\mathsf{E})$ is the subspace of compactly supported smooth sections.
(ii) $\Gamma_{pc}(\mathsf{E})$ and $\Gamma_{fc}(\mathsf{E})$ denote the subspaces of $\Gamma(\mathsf{E})$ whose elements have respectively past compact support and future compact support.
(iii) If (M, g) is globally hyperbolic, $\Gamma_{sc}(\mathsf{E}) \subset \Gamma(\mathsf{E})$ is the subspace of **spatially compact** sections: the smooth sections whose support intersects every spacelike Cauchy hypersurface in a compact set.

These spaces are equipped with natural topologies as discussed in [3]. In case there are several metrics on a common spacetime M basis of E, the used metric g will be indicated as well, for instance $\Gamma^g_{pc}(\mathsf{E})$, if the nature of the space of sections depends on the chosen metric (this is not the case for $\Gamma_c(\mathsf{E})$).

As usual, $\mathsf{E} \boxtimes \mathsf{E}'$ denotes the **external tensor product** of the two $\mathbb{K}$-vector bundles over M. This $\mathbb{K}$-vector bundle has basis $\mathsf{M} \times \mathsf{M} \ni (p, q)$ and fibers given by the pointwise tensor product $\mathsf{E}_p \otimes \mathsf{E}'_q$ of the fibers of the two bundles. Referring to $\Gamma(\mathsf{E}\boxtimes\mathsf{E}')$, if $\mathfrak{f} \in \mathsf{E}$ and $\mathfrak{f}' \in \mathsf{E}'$, then $\mathfrak{f}\otimes\mathfrak{f}' \in \Gamma(\mathsf{E}\boxtimes\mathsf{E}')$ denotes the section defined by $(\mathfrak{f}\otimes\mathfrak{f}')(p, q) := \mathfrak{f}(p) \otimes \mathfrak{f}'(q)$ where the tensor product on the right-hand side is the one of the fibers and $(p, q) \in \mathsf{M} \times \mathsf{M}$.

3.1 Normally hyperbolic operators

As for Sect. 2.1, if $g \in \mathcal{M}_\mathsf{M}$ we defined $g^\sharp$ as the induced metric on the cotangent bundle. If (M, g) is globally hyperbolic, by fixing a Cauchy temporal function $t : \mathsf{M} \to \mathbb{R}$ such that $g = -\beta^2 dt \otimes dt + h_t$, we have

$$g^\sharp = -\beta^{-2}\partial_t \otimes \partial_t + h^\sharp_t.$$

Definition 3.1 A linear second order differential operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is **normally hyperbolic** if its principal symbol σ_{N} satisfies

$$\sigma_{\mathsf{N}}(\xi) = -g^{\sharp}(\xi, \xi)\,\mathrm{Id}_{\mathsf{E}}$$

for all $\xi \in \mathsf{T}^*\mathsf{M}$, where Id_{E} is the identity automorphism of E.

Referring to a foliation of (M, g) as in Definition 2.11, in local coordinates (t, x) on M adapted to the foliation so that $x = (x_1, \dots, x_n)$ are local coordinates on Σ_t, and using a local trivialization of E, any normally hyperbolic operator N in a point $p \in \mathsf{M}$ reads as

$$\mathsf{N} = \frac{1}{\beta^2}\partial_t^2 - \sum_{i,j=1}^{n} h^{\sharp}_{t\,ij}\partial_{x_i}\partial_{x_j} + A_0(t,x)\partial_t + \sum_{j=1} A_j(t,x)\partial_{x_j} + B(t,x)$$

where A_0, A_j and B are linear maps $\mathsf{E}_{(t,x)} \to \mathsf{E}_{(t,x)}$ depending smoothly on (t, x).

Examples 3.2 In the class of normally hyperbolic operators we can find many operators of interest in quantum field theory:

- Fix E be the trivial real bundle, i.e. $\mathsf{E} = \mathsf{M} \times \mathbb{R}$, so that the space of smooth sections of E can be identified with the ring of smooth functions on M. The Klein-Gordon operator $\mathsf{N} = \Box + m^2$ is normally hyperbolic, where $\Box$ is the d'Alembert operator and m is a mass-term.
- Let now $\mathsf{E} = \Lambda^k \mathsf{T}^*\mathsf{M}$ be the bundle of k-forms and d (resp δ) the exterior derivative (resp. the codifferential). The operator $\mathsf{N} := d\delta + \delta d + m^2$ is normally hyperbolic and it is used to describe the dynamics of Proca fields, for further details we refer to [4, Example 2.17].
- Let SM be a spinor bundle over a globally hyperbolic spin manifold M_g and let ∇ be a spin connection. By denoting with $\gamma : \mathsf{TM} \to \mathrm{End}(\mathsf{SM})$ the Clifford multiplication, the classical Dirac operator reads as $\mathsf{D} = \gamma \circ \nabla : \Gamma(\mathsf{SM}) \to \Gamma(\mathsf{SM})$, see [29, 30, 53] for further details. By Lichnerowicz-Weitzenböck formula we get the spinorial wave operator $\mathsf{N} = \mathsf{D}^2 = \nabla^{\dagger}\nabla + \frac{1}{4}\mathrm{Scal}_g$, where Scal_g is the scalar curvature.

It is well-known that, once the Cauchy data are suitably assigned, the Cauchy problem for N turns out to be well-posed, see e.g. [3, 52].

Theorem 3.3 *Let E be a vector bundle (of finite rank) over a globally hyperbolic manifold (M, g), let N be a normally hyperbolic operator with a N-compatible connection ∇ (see (3.3) below) and Σ_0 a (smooth) spacelike Cauchy hypersurface of (M, g). Then the Cauchy problem for N is well-posed, i.e. for any $\mathfrak{f} \in \Gamma_c(\mathsf{E})$, $\mathfrak{h}_1, \mathfrak{h}_2 \in \Gamma_c(\mathsf{E}|_{\Sigma_0})$ there exists a unique solution $\Psi \in \Gamma_{sc}(\mathsf{E})$ to the initial value problem*

$$\begin{cases} \mathsf{N}\Psi = \mathfrak{f} \\ \Psi|_{\Sigma_0} = \mathfrak{h}_1 \\ (\nabla_{\mathsf{n}}\Psi)|_{\Sigma_0} = \mathfrak{h}_2 \end{cases}$$

being $\mathfrak{n}$ *the future directed timelike unit normal field along* Σ_0*, and it depends continuously on the data* $(\mathfrak{f}, \mathfrak{h}_1, \mathfrak{h}_2)$ *w.r.to the standard topologies of smooth sections and satisfies*

$$supp(\Psi) \subset J(supp(\mathfrak{f})) \cup J(supp(\mathfrak{h}_1)) \cup J(supp(\mathfrak{h}_2)) \,. \tag{3.1}$$

As a consequence of the well-posedness of the Cauchy problem of normally hyperbolic operators with "finite propagation of the solutions" stated in (3.1), one may establish the existence of Green operators. In order to recall this result, we need first a preparatory definition.

Definition 3.4 A linear differential operator $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is called **Green hyperbolic** if

(1) there exist linear maps, dubbed **advanced Green operator** $\mathsf{G}^+ : \Gamma_{pc}(\mathsf{E}) \to \Gamma(\mathsf{E})$ and **retarded Green operator** $\mathsf{G}^- : \Gamma_{fc}(\mathsf{E}) \to \Gamma(\mathsf{E})$, satisfying

  (i.a) $\mathsf{G}^+ \circ \mathsf{P}\,\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^+\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$,
  (ii.a) $\text{supp}\,(\mathsf{G}^+\mathfrak{f}) \subset J^+(\text{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$;
  (i.b) $\mathsf{G}^- \circ \mathsf{P}\,\mathfrak{f} = \mathsf{P} \circ \mathsf{G}^-\mathfrak{f} = \mathfrak{f}$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$,
  (ii.b) $\text{supp}\,(\mathsf{G}^-\mathfrak{f}) \subset J^-(\text{supp}\,\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$;

(2) the **formally dual operator** P^* admits advanced and retarded Green operators as well.

For sake of completeness, let us recall that the formally dual operator $\mathsf{P}^* : \Gamma(\mathsf{E}^*) \to \Gamma(\mathsf{E}^*)$ is the unique linear differential operator acting on the smooth sections of the dual bundle E^* satisfying

$$\int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{P}\mathfrak{f} \rangle \, \text{vol}_g = \int_{\mathsf{M}} \langle \mathsf{P}^*\mathfrak{f}', \mathfrak{f} \rangle \, \text{vol}_g$$

for every $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ and $\mathfrak{f}' \in \Gamma_c(\mathsf{E}^*)$ (which is equivalent to saying $\mathfrak{f} \in \Gamma(\mathsf{E})$ and $\mathfrak{f}' \in \Gamma(\mathsf{E}^*)$ such that $\text{supp}(\mathfrak{f}) \cap \text{supp}(\mathfrak{f})'$ is compact), vol_g being the volume form induced by g on M.

Remarks 3.5 (1) The Green operators we define below are the extensions to $\Gamma_{pc/fc}(\mathsf{E})$ of the analogs defined in [2] and indicated by $\overline{G}_\pm$ therein.
(2) It is possible to prove that the Green operators are unique for a Green hyperbolic operator (cf. [2, Corollary 3.12]). Furthermore as a consequence of [2, Lemma 3.21], it arises that if $\mathfrak{f}' \in \Gamma_c(\mathsf{E}^*)$ and $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$ or $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$ respectively,

$$\int_{\mathsf{M}} \langle \mathsf{G}^-_{P^*}\mathfrak{f}', \mathfrak{f} \rangle \, \text{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^+_P\mathfrak{f} \rangle \, \text{vol}_g \,, \quad \int_{\mathsf{M}} \langle \mathsf{G}^+_{P^*}\mathfrak{f}', \mathfrak{f} \rangle \, \text{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}', \mathsf{G}^-_P\mathfrak{f} \rangle \, \text{vol}_g \,, \tag{3.2}$$

where $\mathsf{G}^\pm_P$ indicate the Green operators of P and $\mathsf{G}^\pm_{P^*}$ indicate the Green operators of P^*.

Proposition 3.6 *If* P *is a Green hyperbolic operator on a vector bundle* E *over the globally hyperbolic spacetime* (M, g) *and* $\rho : \mathsf{M} \to (0, +\infty)$ *is smooth, then* $\rho\mathsf{P}$ *is Green hyperbolic as well and* $\mathsf{G}^{\pm}_{\rho\mathsf{P}} = \mathsf{G}^{\pm}_{\mathsf{P}}\rho^{-1}$.

Proof The thesis immediately follows form the fact that $\mathsf{G}^{\pm}_{\mathsf{P}}\rho^{-1}$ and $\rho^{-1}\mathsf{G}^{\pm}_{\mathsf{P}*}$ satisfy the properties of the Green operators for $\rho\mathsf{P}$ and $(\rho\mathsf{P})^* = \mathsf{P}^*\rho$ respectively. □

Proposition 3.7 ([3, Corollary 3.4.3]) *A normally hyperbolic operator* N *on a vector bundle* E *(of finite rank) on a globally hyperbolic manifold* (M, g) *is Green hyperbolic.*

Given a Green hyperbolic operator with Green operators $\mathsf{G}^{\pm}$, a relevant operator constructed out of $\mathsf{G}^{\pm}$ is the so-called **causal propagator**,

$$\mathsf{G} := \mathsf{G}^{+}|_{\Gamma_c(\mathsf{E})} - \mathsf{G}^{-}|_{\Gamma_c(\mathsf{E})} : \Gamma_c(\mathsf{E}) \to \Gamma(\mathsf{E})\,.$$

It satisfies remarkable properties we are going to discuss (see e.g. [3, Theorem 3.6.21]).

Theorem 3.8 *Let* G *be the causal propagator of a Green hyperbolic differential operator* $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *on the vector bundle* E *over a globally hyperbolic spacetime* (M, g)*. The following sequence is exact*

$$\{0\} \to \Gamma_c(\mathsf{E}) \stackrel{\mathsf{P}}{\to} \Gamma_c(\mathsf{E}) \stackrel{\mathsf{G}}{\to} \Gamma_{sc}(\mathsf{E}) \stackrel{\mathsf{P}}{\to} \Gamma_{sc}(\mathsf{E}) \to \{0\}\,.$$

Proof Injectivity of $\Gamma_c(\mathsf{E}) \stackrel{\mathsf{P}}{\to} \Gamma_c(\mathsf{E})$ easily arises from the well-posedness of the Cauchy problem stated in Theorem 3.3. Let us pass to the other parts of the sequence. First of all notice that $\mathsf{G}^{\pm}(\Gamma_c(\mathsf{E})) \subset \Gamma_{sc}(\mathsf{E})$ since $\mathrm{supp}(\mathsf{G}^{\pm}(\mathfrak{f})) \subset J_{\pm}(\mathrm{supp}(\mathfrak{f}))$ and the first assertion then follows form known facts of globally hyperbolic spacetimes. Let us prove that $\Gamma_c(\mathsf{E}) \stackrel{\mathsf{G}}{\to} \Gamma_{sc}(\mathsf{E})$ is surjective when the image is restricted to the kernel of $\Gamma_{sc}(\mathsf{E}) \stackrel{\mathsf{P}}{\to} \Gamma_{sc}(\mathsf{E})$. Suppose that $\mathsf{P}\Psi = 0$ for $\Psi \in \Gamma_{sc}(\mathsf{E})$. If t is a smooth Cauchy time function of (M, g) and $\chi : \mathsf{M} \to [0, 1]$ is smooth, vanishes for $t < t_0$ and is constantly 1 for $t > t_1$, then

$$\mathfrak{f}_{\Psi} := \mathsf{P}(\chi\Psi) \in \Gamma_c(\mathsf{E})$$

is such that $\Psi = \mathsf{G}\mathfrak{f}_{\Psi}$. Notice that $\mathrm{supp}(\mathfrak{f}_{\Psi})$ is included between the Cauchy hypersufaces $t^{-1}(t_0)$ and $t^{-1}(t_1)$. Indeed,

$$\begin{aligned}\mathsf{G}\mathfrak{f}_{\Psi} &= \mathsf{G}^{+}\mathsf{P}(\chi\Psi) - \mathsf{G}^{-}\mathsf{P}(\chi\Psi) = \mathsf{G}^{+}\mathsf{P}(\chi\Psi) + \mathsf{G}^{-}\mathsf{P}((1-\chi)\Psi)\\ &= \chi\Psi + (1-\chi)\Psi = \Psi\,.\end{aligned}$$

It is obvious that that $\mathfrak{f}_{\Psi}$ can be changed by adding a section of the form $\mathsf{P}\mathfrak{h}$ with $\mathfrak{h} \in \Gamma_c(\mathsf{E})$ preserving the property $\mathsf{G}\mathfrak{f}_{\Psi} = \Psi$. This exhaust the kernel of $\Gamma_c(\mathsf{E}) \stackrel{\mathsf{G}}{\to} \Gamma_{sc}(\mathsf{E})$ as asserted in the thesis. Indeed, if $\mathsf{G}\mathfrak{f} = 0$, then $\mathsf{G}^{+}\mathfrak{f} = \mathsf{G}^{-}\mathfrak{f}$. From the properties of the supports of $\mathsf{G}^{\pm}\mathfrak{f}$, we conclude that $\mathsf{G}^{\pm}\mathfrak{f} = \mathfrak{h}_{\pm} \in \Gamma_c(\mathsf{E}) \subset \Gamma_{pc}(\mathsf{E}) \cap \Gamma_{fc}(\mathsf{E})$.

Hence $\mathfrak{f} = \mathsf{P}\mathsf{G}^{\pm}\mathfrak{f} = \mathsf{P}\mathfrak{h}_{\pm}$. To conclude, we prove that $\Gamma_{sc}(\mathsf{E}) \xrightarrow{\mathsf{P}} \Gamma_{sc}(\mathsf{E})$ is surjective. If $\mathfrak{f} \in \Gamma_{sc}(\mathsf{E})$, with χ as above,

$$\mathfrak{f} = \chi\mathfrak{f} + (1-\chi)\mathfrak{f} = \mathsf{P}\mathsf{G}^{+}(\chi\mathfrak{f}) + \mathsf{P}\mathsf{G}^{-}((1-\chi)\mathfrak{f}) = \mathsf{P}[\mathsf{G}^{+}(\chi\mathfrak{f}) + \mathsf{G}^{-}((1-\chi)\mathfrak{f})]$$

and $\mathsf{G}^{+}(\chi\mathfrak{f}) + \mathsf{G}^{-}((1-\chi)\mathfrak{f}) \in \Gamma_{sc}(\mathsf{E})$. □

3.2 Formally selfadjoint normally hyperbolic operators and their symplectic form

Let E be a $\mathbb{K}$-vector bundle on a globally hyperbolic spacetime (M, g). As shown in [6, Lemma 1.5.5], for any normally hyperbolic operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ there exists a unique covariant derivative ∇ on E such that

$$\mathsf{N} = -\mathrm{tr}_g(\nabla\nabla) + c \tag{3.3}$$

for some some zero-order differential operator $c : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$. In the formula above the left ∇ is actually the connection induced on $\mathsf{T}^*\mathsf{M} \otimes \mathsf{E}$ by the *Levi-Civita connection* associated to g and the original connection ∇ (the one appearing as the right ∇) given on E Adopting the terminology of [3], we shall refer to ∇ as the **N-compatible connection** on E.

We stress that, if we suppose that E is equipped with a smooth assignment of a Hermitian fiber metric

$$\langle \cdot \mid \cdot \rangle_p : \mathsf{E}_p \times \mathsf{E}_p \to \mathbb{K}.$$

then the above ∇ is g-metric but not necessarily metric with respect to $\langle \cdot \mid \cdot \rangle$.

The physical relevance of the fiber metric is that it permits to equip $\mathsf{Ker}_{sc}(\mathsf{N})$ with a symplectic form with important physical properties in the formulation of QFT in curved spacetime. This symplectic form can be derived using the Green identity for a normally hyperbolic operator N and its formal adjoint. For sake of completeness let us remind the definition of formal adjoint.

Definition 3.9 The **formal adjoint** of a differential operator $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is the unique differential operator $\mathsf{P}^\dagger : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ satisfying

$$\int_{\mathsf{M}} \langle \mathfrak{f}' \mid \mathsf{P}\mathfrak{f} \rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathsf{P}^\dagger \mathfrak{f}' \mid \mathfrak{f} \rangle \,\mathrm{vol}_g$$

for every $\mathfrak{f}, \mathfrak{f}' \in \Gamma_c(\mathsf{E})$ (which is equivalent to saying $\mathfrak{f}, \mathfrak{f}' \in \Gamma(\mathsf{E})$ such that $\mathrm{supp}(\mathfrak{f}) \cap \mathrm{supp}(\mathfrak{f})'$ is compact). If $\mathsf{P} = \mathsf{P}^\dagger$ then N is said to be (formally) **selfadjoint**.

Remark 3.10 If $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is normally hyperbolic on the bundle E over (M, g), equipped with a non-degenerate, Hermitian fiber metric $\langle \cdot \mid \cdot \rangle$, P is Green hyperbolic as said above. In this case $P^\dagger$ has the same principal symbol as P and thus it is Green hyperbolic as well. Taking advantage of the natural (antilinear if $\mathbb{K} = \mathbb{C}$) isomorphism

$\Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}^*)$ induced by $\langle \cdot \mid \cdot \rangle$ and (3.2), it is not difficult to prove that, if $\mathfrak{f}' \in \Gamma_c(\mathsf{E})$ and $\mathfrak{f} \in \Gamma_{pc}(\mathsf{E})$ or $\mathfrak{f} \in \Gamma_{fc}(\mathsf{E})$ respectively,

$$\int_{\mathsf{M}} \langle \mathsf{G}^-_{P^\dagger}\mathfrak{f}' \mid \mathfrak{f} \rangle \, \mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}' \mid \mathsf{G}^+_P \mathfrak{f} \rangle \, \mathrm{vol}_g \, , \quad \int_{\mathsf{M}} \langle \mathsf{G}^+_{P^\dagger}\mathfrak{f}' \mid \mathfrak{f} \rangle \, \mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f}' \mid \mathsf{G}^-_P \mathfrak{f} \rangle \, \mathrm{vol}_g \, . \tag{3.4}$$

where $\mathsf{G}^\pm_P$ indicate the Green operators of P and $\mathsf{G}^\pm_{P^\dagger}$ indicate the Green operators of $\mathsf{P}^\dagger$.

Let us pass to introduce a Green-like identity where we explicitly exploit the N-compatible connection ∇.

Lemma 3.11 (Green identity) *Let* E *be an non-degenerate, Hermitian* $\mathbb{K}$ *vector bundle over a* $(n+1)$*-dimensional spacetime* (M, g) *and denote the fiber metric* $\langle \cdot \mid \cdot \rangle$*. Moreover, let* $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *be a normally hyperbolic operator with* N*-compatible connection* ∇*. Let* $\mathsf{M}_0 \subset \mathsf{M}$ *be a submanifold with continuous piecewise smooth boundary. Then for every* $\Phi, \Psi \in \Gamma_c(\mathsf{E})$)

$$\int_{\mathsf{M}_0} \left(\langle \Psi \mid \mathsf{N}\Phi \rangle - \langle \mathsf{N}\Psi \mid \Phi \rangle \right) \mathrm{vol}_g = \int_{\partial \mathsf{M}_0} \Xi^{\mathsf{N}}_{\partial \mathsf{M}_0}(\Psi, \Phi), \tag{3.5}$$

where $\Xi^{\mathsf{N}}_{\partial \mathsf{M}_0}$ *is the* n*-form in* $\partial \mathsf{M}_0$

$$\Xi^{\mathsf{N}}_{\partial \mathsf{M}_0}(\Psi, \Phi) := \iota^*_{\partial \mathsf{M}_0} \left[\sharp \Big(\langle \Psi \mid \nabla \Phi \rangle - \langle \nabla \Psi \mid \Phi \rangle \Big) \lrcorner \, \mathrm{vol}_g \right]$$

$\iota_{\partial \mathsf{M}_0} : \partial \mathsf{M}_0 \to \mathsf{M}$ *being the inclusion embedding. If the normal vectors to* $\partial \mathsf{M}_0$ *are either spacelike or timelike (up to zero-measure sets), then*

$$\Xi^{\mathsf{N}}_{\partial \mathsf{M}_0}(\Psi, \Phi) = \Big(\langle \Psi \mid \nabla_{\mathrm{n}} \Phi \rangle - \langle \nabla_{\mathrm{n}} \Psi \mid \Phi \rangle \Big) \mathrm{vol}_{\partial \mathsf{M}_0} \tag{3.6}$$

where n *is the outward unit normal vector to* $\partial \mathsf{M}_0$ *and* $\mathrm{vol}_{\partial \mathsf{M}_0} = \mathrm{n} \lrcorner \, \mathrm{vol}_g$ *is the volume form of* $\partial \mathsf{M}_0$ *induced by* g*.*

Proof Consider the n-form in M

$$Z := \sharp \Big(\langle \Psi \mid \nabla \Phi \rangle - \langle \nabla \Psi \mid \Phi \rangle \Big) \lrcorner \, \mathrm{vol}_g \, .$$

If the normal vectors to $\partial \mathsf{M}_0$ are either spacelike or timelike, some computations with the exterior differential of forms yields (3.6). In all cases it is easy to prove that

$$\begin{aligned} dZ &= \Big(\langle \Psi \mid g^{ij} \nabla_i \nabla_j \Phi \rangle - \langle g^{ij} \nabla_i \nabla_j \Psi \mid \Phi \rangle \Big) \mathrm{vol}_g \\ &= \Big(\langle \Psi \mid \mathsf{N}\Phi \rangle - \langle \mathsf{N}\Psi \mid \Phi \rangle \Big) \mathrm{vol}_g \, . \end{aligned}$$

At this juncture, Stokes' theorem for $(n+1)$-forms,

$$\int_{\partial \mathsf{M}_0} Z = \int_{\mathsf{M}_0} dZ \, ,$$

produces (3.5). □

We have the following crucial result when applying the previous lemma to the theory on globally hyperbolic spacetimes.

Proposition 3.12 *Let $\Sigma \subset \mathsf{M}$ be a smooth spacelike Cauchy hypersurface with its future-oriented unit normal vector field $\mathtt{n}$ in the globally hyperbolic spacetime (M, g) and its induced volume element vol_{Σ}. Furthermore, let N be a formally self-adjoint normally hyperbolic operator. Then*

$$\sigma^{\mathsf{N}}_{(\mathsf{M},g)} : \mathsf{Ker}_{sc}(\mathsf{N}) \times \mathsf{Ker}_{sc}(\mathsf{N}) \to \mathbb{C} \quad \textit{such that} \quad \sigma^{\mathsf{N}}_{(\mathsf{M},g)}(\Psi, \Phi) = \int_{\Sigma} \Xi^{\mathsf{N}}_{\Sigma}(\Psi, \Phi) \tag{3.7}$$

where $\Xi^{\mathsf{N}}_{\Sigma}$ is defined in Eq. (3.6), yields a non-degenerate symplectic form (Hermitian if $\mathbb{K} = \mathbb{C}$) which does not depend on the choice of Σ.

Proof First note that, referring to a spacelike Cauchy hypersurface Σ, $\mathrm{supp}(\Psi) \cap \Sigma$ is compact since $\mathrm{supp}(\Psi)$ is spacelike compact, so that the integral is well-defined. The fact that σ_{N} is not degenerate can be proved as follows. If $\sigma^{\mathsf{N}}_{(\mathsf{M},g)}(\Psi, \Phi) = 0$ for all $\Phi \in \Gamma_{sc}(\mathsf{E})$, from the definition of σ_{N} and non-degenerateness of $\langle \cdot \,|\, \cdot \rangle_p$ (passing to local trivializations referred to local coordinates on Σ re-writing $\langle \cdot \,|\, \cdot \rangle_p$ in terms of the pairing with E^*_p), we have that the Cauchy data of Ψ vanishes on every local chart on Σ and thus they vanish on Σ. According to Theorem 3.3, $\Psi = 0$. The other entry can be worked out similarly.

Let $\Psi, \Phi \in \mathsf{Ker}_{sc}(\mathsf{N})$ and Σ'_t and $\Sigma_{t''}$ be a pair of smooth spacelike Cauchy hypersurfaces associated to a smooth time Cauchy function t with $t'' > t'$. Let us focus on the submanifold with boundary $\mathsf{M}_0 = t^{-1}((t', t''))$. Its boundary is $\partial \mathsf{M}_0 = \Sigma_{t'} \cup \Sigma_{t''}$. The supports of Ψ and Φ between the two Cauchy surfaces are included in the sets of type J_+ of the compact supports of the Cauchy data on $\Sigma_{t'}$ of Ψ and Φ respectively, and these portions of causal sets are compact as (M, g) is globally hyperbolic (see e.g. [3, Proposition 1.2.56]) we end up with a pair of functions in $\Gamma_c(\mathsf{E})$ and we can apply the Green identity (see Lemma 3.11) to M_0. Using a smoothly vanishing function as a factor, we can make smoothly vanishing Ψ and Φ before $\Sigma_{t'}$ and after $\Sigma_{t''}$ without touching them between the two Cauchy surfaces. As a matter of fact the resulting sections constructed out Ψ and Φ in this way are smooth, compactly supported and coincide with Ψ and Φ between the two Cauchy surfaces. We can therefore apply Lemma 3.11, obtaining

$$\int_{\mathsf{M}_0} (\langle \Psi \,|\, \mathsf{N}\Phi \rangle - \langle \mathsf{N}\Psi \,|\, \Phi \rangle) \mathrm{vol}_g = \int_{\Sigma'} \Xi^{\mathsf{N}}_{\Sigma'} - \int_{\Sigma} \Xi^{\mathsf{N}}_{\Sigma} \, .$$

Since N is assumed to be self-adjoint, $\langle \Psi \mid \mathsf{N}\Phi\rangle - \langle \mathsf{N}\Psi \mid \Phi\rangle = \langle \Psi \mid \mathsf{N}\Phi\rangle - \langle \Psi \mid \mathsf{N}^\dagger \Phi\rangle = 0$. Therefore we can conclude that $\int_{\Sigma'} \Xi^{\mathsf{N}}_{\Sigma'} = \int_\Sigma \Xi^{\mathsf{N}}_\Sigma$. Finally consider the case of two spacelike Cauchy functions Σ and Σ' belonging to different foliations induced by different smooth Cauchy time functions (notice that a spacelike Cauchy hypersurface always belong to a foliation generated by a suitable smooth Cauchy time (actually temporal) function for Theorem 2.10). We sketch a proof of the identity

$$\int_{\Sigma'} \Xi^{\mathsf{N}}_{\Sigma'} = \int_\Sigma \Xi^{\mathsf{N}}_\Sigma .$$

Let $K \subset \Sigma$ a compact set including the Cauchy data of Ψ and Φ. If t is the smooth Cauchy time function such that $\Sigma_{t_1} = \Sigma'$, let $T = \max_K t$. If $t_1 < T$ we can always take $t_2 > T$ and to consider the symplectic form evaluated on Σ_{t_2}. In view of the previous part of our proof the symplectic form on Σ_{t_1} and Σ_{t_2} coincide, so that our thesis can be re-written

$$\int_{\Sigma_2} \Xi^{\mathsf{N}}_{\Sigma'} = \int_\Sigma \Xi^{\mathsf{N}}_\Sigma .$$

As $t_2 > \max_K t$, we conclude that Σ_{t_2} does not intersect Σ in the set K. Therefore we can define the solid set L_K made of the portion of $J_+(K)$ between Σ and Σ_{t_2}. L is compact (see e.g. [3, Proposition 1.2.56]) and is a "truncated cone" whose "lateral surface" is part of the boundary of $J_+(K)$ and whose "non-parallel bases" are parts of Σ_2 and Σ. We can include L in the interior of a larger manifold with boundary M_0 whose part of the boundary are portions of Σ and Σ_{t_2} including the support of the Cauchy data of Ψ and Φ. Notice that M_0 includes the supports of Ψ and Φ between the two Cauchy surfaces according to Theorem 3.3 and these supports do not touch the "lateral surface" of M_0. We can now apply the Green identity 3.11 to M_0 proving the thesis. □

There is a nice interplay of the causal propagator G of $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ as above and the symplectic form $\sigma^{\mathsf{N}}_{(\mathsf{M},g)}$.

Proposition 3.13 *With the same hypotheses as of Proposition 3.12, if* $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$ *and* $\Psi_{\mathfrak{f}} := \mathsf{G}\mathfrak{f}$, $\Psi_{\mathfrak{h}} := \mathsf{G}\mathfrak{h}$, *it holds*

$$\sigma^{\mathsf{N}}_{(\mathsf{M},g)}(\Psi_{\mathfrak{f}}, \Psi_{\mathfrak{h}}) = \int_{\mathsf{M}} \langle \mathfrak{f} \mid \mathsf{G}\mathfrak{h}\rangle \, \mathrm{vol}_g .$$

Proof If $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$, consider a smooth Cauchy time function t and fix $t_0 < t_1$ such that the supports of $\mathfrak{f}$ and $\mathfrak{h}$ are included in the interior of the submanifold with boundary M_0 contained between the spacelike Cauchy hypersurfaces $\Sigma_{t_0} := t^{-1}(t_0)$ and $\Sigma_{t_1} := t^{-1}(t_1)$. It holds

$$\int_{\mathsf{M}} \langle \Psi_{\mathfrak{f}} \mid \mathfrak{h}\rangle \, \mathrm{vol}_g = \int_{\mathsf{M}_0} \langle \Psi_{\mathfrak{f}} \mid \mathfrak{h}\rangle \, \mathrm{vol}_g = \int_{\mathsf{M}_0} \langle \Psi_{\mathfrak{f}} \mid \mathsf{N}\mathsf{G}^+\mathfrak{h}\rangle \, \mathrm{vol}_g$$

Since $\mathsf{N}\Psi_{\mathfrak{f}} = 0$, we have found that

$$\int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \Psi_{\mathfrak{h}} \rangle \, \mathrm{vol}_g = \int_{\mathsf{M}_0} \big(\langle \Psi_{\mathfrak{f}} \,|\, \mathsf{N}\mathsf{G}^{+}\mathfrak{h} \rangle - \langle \mathsf{N}\Psi_{\mathfrak{f}} \,|\, \mathsf{G}^{+}\Psi_{\mathfrak{h}} \rangle \big) \, \mathrm{vol}_g \, .$$

Applying Lemma 3.11, we find

$$\int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \Psi_{\mathfrak{h}} \rangle \, \mathrm{vol}_g = \int_{\partial \mathsf{M}_0} \Xi^{\mathsf{N}}_{\partial \mathsf{M}_0} (\Psi_{\mathfrak{f}}, \mathsf{G}^{+}\mathfrak{h}) = \int_{\Sigma_{t_1}} \Xi^{\mathsf{N}}_{\partial \mathsf{M}_0} (\Psi_{\mathfrak{f}}, \mathsf{G}^{+}\mathfrak{h}) \, ,$$

where we noticed that $\mathsf{G}^{+}\mathfrak{h}$ vanishes on the remaining part of the boundary Σ_{t_0}. On the other hand, we can replace $\mathsf{G}^{+}\mathfrak{h}$ for $\mathsf{G}^{+}\mathfrak{h} - \mathsf{G}^{-}\mathfrak{h} = \mathsf{G}\mathfrak{h}$ in the last integral, since $\mathsf{G}^{-}\mathfrak{h}$ gives no contribution to the integral on Σ_{t_1}. In summary,

$$\int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \mathsf{G}\mathfrak{h} \rangle \, \mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \Psi_{\mathfrak{h}} \rangle \, \mathrm{vol}_g = \int_{\Sigma_{t_1}} \Xi^{\mathsf{N}}_{\Sigma_{t_1}} (\Psi_{\mathfrak{f}}, \mathsf{G}\mathfrak{h}) = \sigma^{\mathsf{N}}_{(\mathsf{M},g)} (\Psi_{\mathfrak{f}}, \Psi_{\mathfrak{h}}) \, .$$

and this is the thesis. □

3.3 Convex combinations of normally hyperbolic operators

Let now $\mathsf{N}_0, \mathsf{N}_1$ be normally hyperbolic operators with respect to different Lorentzian metric g_0 and g_1 (the former time-orientable and the latter globally hyperbolic) on the same manifold M and assume that they are acting on the smooth sections of the same vector bundle E. It turns out, that a positive (and convex) combination $(1-\chi)\mathsf{N}_0 + \chi \mathsf{N}_1$ is also (a) normally hyperbolic with respect to the naturally associated metric g_χ – the unique Lorentzian metric in TM whose associated metric in $\mathsf{T}^*\mathsf{M}$ is $(1-\chi)g_0^\sharp + \chi g_1^\sharp$ according to Theorem 2.18 – and (b) Green hyperbolic with respect to g_1, everything provided that $g_0 \preceq g_1$. This is the main result of this section.

Theorem 3.14 *Let* E *be a* $\mathbb{K}$*-vector bundle over a smooth manifold* M*, let be* $g_0, g_1 \in \mathcal{GM}_{\mathsf{M}}$ *with* $g_0 \preceq g_1$*, and let* $\mathsf{N}_0, \mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *be normally hyperbolic operator with respect to* g_0 *and* g_1 *respectively. If* $\chi \in C^\infty(\mathsf{M}, [0,1])$*, define* g_χ *as the unique Lorentzian metric whose associated metric in* $\mathsf{T}^*\mathsf{M}$ *is* $(1-\chi)g_0^\sharp + \chi g_1^\sharp$ *according to Theorem 2.18. Then the second order differential operator defined by*

$$\mathsf{N}_\chi := (1-\chi)\mathsf{N}_0 + \chi \mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{3.8}$$

satisfies the following properties:

(1) *It is normally and Green hyperbolic over* (M, g_χ)*;*
(2) *It is Green hyperbolic over* (M, g_1) *and, with obvious notation,*

$$\Gamma^{g_1}_{pc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{pc}(\mathsf{E}) \, , \quad \Gamma^{g_1}_{fc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{fc}(\mathsf{E}) \, ,$$
$$\mathsf{G}^{g_1 +}_{\mathsf{N}_\chi} = \mathsf{G}^{g_\chi +}_{\mathsf{N}_\chi}|_{\Gamma^{g_1}_{pc}(\mathsf{E})} \, , \quad \mathsf{G}^{g_1 -}_{\mathsf{N}_\chi} = \mathsf{G}^{g_\chi -}_{\mathsf{N}_\chi}|_{\Gamma^{g_1}_{fc}(\mathsf{E})} \, .$$

In particular, (2) is true for N_0 *by choosing* $\chi = 0$.

Proof (1) Since N_0 is a normally hyperbolic operator for (M, g_0) and N_1 is a normally hyperbolic operator for (M, g_1), by linearity

$$\sigma_2(\mathsf{N}_\chi, \xi) = (1-\chi)\sigma_2(\mathsf{N}_0, \xi) + \chi\sigma_2(\mathsf{N}_1, \xi).$$

In particular, we have that N_χ is normally hyperbolic with respect to g_χ:

$$\sigma_2(\mathsf{N}_\chi, \xi) = -(1-\chi)g_0^\sharp(\xi,\xi)\mathrm{Id}_\mathsf{E} - \chi g_1^\sharp(\xi,\xi)\mathrm{Id}_\mathsf{E} = -g_\chi^\sharp(\xi,\xi)\mathrm{Id}_\mathsf{E}.$$

By Theorem 2.18, the metric g_χ is globally hyperbolic and, on account of Proposition 3.7 N_χ is Green-hyperbolic over (M, g_χ).

Regarding (2), and referring to the existence of Green operators of N_χ in (M, g_1) we can proceed as follows. Observe that, since $g_\chi \preceq g_1$, we have $J_\pm^{g_\chi}(A) \subset J_\pm^{g_1}(A)$ and, with obvious notation, $\Gamma_{pc}^{g_1}(\mathsf{E}) \subset \Gamma_{pc}^{g_\chi}(\mathsf{E})$ together with $\Gamma_{fc}^{g_1}(\mathsf{E}) \subset \Gamma_{fc}^{g_\chi}(\mathsf{E})$, in view of (iii) (2) Lemma 2.16. As a consequence, the Green operators of N_χ with respect to (M, g_χ) are also Green operators with respect to (M, g_1). Finally we pass to the existence of the Green operators of N_χ^* – *where* * *is here referred to the volume form of* g_1 *and not* g_χ – in (M, g_1). Since N_χ^* has the same principal symbol $g_\chi^\sharp(\xi,\xi)\mathrm{Id}_\mathsf{E}$ as N_χ it is normally hyperbolic in (M, g_χ) and hence Green hyperbolic thereon. With the same argument used above, we see that the Green operators of N_χ^* (with * always referred to g_1) in (M, g_χ) are also Green operators in (M, g_1). □

Remark 3.15 We stress that, when $g_0 \preceq g_1$ are globally hyperbolic, N_χ and N_0 are therefore Green-hyperbolic second-order differential operators on (M, g_1) though they are *not* normally hyperbolic thereon. These are examples of *second-order* linear differential operators which are Green hyperbolic but *not* normally hyperbolic in a given globally hyperbolic spacetime.

4 Møller maps and operators for normally hyperbolic operators

We are in the position to introduce the notion of so-called *Møller map*, which we shall later specialize to the case of a *Møller operator*, namely a (geometric) map which compares the space of solutions of different normally hyperbolic operators. The novelty of this approach consists in defining the notion of Møller map in a more general fashion. More in detail, in [20, 29, 31, 65] the Møller operator was constructed once that a foliation of M in Cauchy hypersurfaces was assigned and referring to the family of the metrics which are decomposed as in (2.1) with respect to *that* foliation. Here we shall see, that the construction of a Møller map still requires the choice of a foliation (associated to some smooth Cauchy time function), but the involved metrics do not have any particular relationship with the choice of the foliation. Instead they should enjoy some interplay concerning their light-cone structures which generalizes $g \preceq g'$ in the sense of paracausal deformations.

4.1 General approach to construct Møller maps when $g_0 \preceq g_1$

Let us consider a globally hyperbolic spacetime (M, g) equipped with a vector bundle $\mathsf{E} \to \mathsf{M}$ as before. If $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is a linear differential operator, a family of physically relevant solutions of the inhomogeneous equation $\mathsf{P}\mathfrak{f} = \mathfrak{h}$ is the linear vector space of spacelike compact smooth solutions with compactly supported source:

$$\mathsf{Sol}^g_{sc,c}(\mathsf{P}) := \{\mathfrak{f} \in \Gamma^g_{sc}(\mathsf{E}) \mid \mathsf{P}\mathfrak{f} \in \Gamma_c(\mathsf{E})\} .$$

Its subspace corresponding to the solutions of the homogeneous equation $\mathsf{P}\mathfrak{f} = 0$ is denoted by

$$\mathsf{Ker}^g_{sc}(\mathsf{P}) := \{\mathfrak{f} \in \Gamma^g_{sc}(\mathsf{E}) \mid \mathsf{P}\mathfrak{f} = 0\}$$

and it will play a pivotal role in the formulation of linear QFT.

We now specialize the operators P to 2nd-order normally-hyperbolic linear operators $\mathsf{N}_1, \mathsf{N}_0, \mathsf{N}_\chi$ (3.8) over $\Gamma(\mathsf{E})$ associated to globally hyperbolic metrics $g_0 \preceq g_1$ and g_χ on the common spacetime manifold M. Our goal is to construct several families of *Møller maps*, namely linear operators such that

(a) they are linear space isomorphisms between $\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)$, $\mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)$, $\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)$;
(b) they restrict to isomorphisms to the subspaces $\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)$, $\mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)$, $\mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)$.

For later convenience, we shall additionally require that the Møller maps preserve also the symplectic forms, which are of interest in applications to linear QFT.

The overall idea is inspired by the scattering theory. We start with two "free theories", described by the space of solutions of normally hyperbolic operators N_0 and N_1 in corresponding spacetimes (M, g_0) and (M, g_1), respectively, and we intend to connect them through an "interaction spacetime" (M, g_χ) with a "temporally localized" interaction defined by interpolating the two metrics by means of a smoothing function χ. Here we need two Møller maps: Ω_+ connecting (M, g_0) and (M, g_χ) – which reduces to the identity in the past when χ is switched off – and a second Møller map connecting (M, g_χ) to (M, g_1) – which reduces to the identity in the future when χ constantly takes the value 1. The "S-matrix" given by the composition $\mathsf{S} := \Omega_-\Omega_+$ will be the Møller map connecting N_0 and N_1.

4.2 Møller maps for metrics satisfying $g_0 \preceq g_1$

The first step consists of comparing N_0 and N_1 with N_χ separately to construct the Møller map. As usual, we denote with E the $\mathbb{K}$-vector bundle over a spacetime (M, g).

We first start with operators denoted by $\mathsf{R}_\pm$ defined on the whole space of smooth sections $\Gamma(\mathsf{E})$ which is in common for the three metrics on M and next we will restrict these operators to the special spaces of solutions with spatially compact support and compactly supported sources, proving that these restrictions $\Omega_\pm$ are still linear space isomorphisms.

Proposition 4.1 *Let $g_0, g_1 \in \mathcal{GM}_{\mathsf{M}}$ be such that $g_0 \preceq g_1$ and $V_x^{g_0+} \subset V_x^{g_1+}$ for all $x \in M$. Let E be a vector bundle over M and $\mathsf{N}_0, \mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be normally hyperbolic operators associated to g_0 and g_1 respectively. Choose*

(a) *a smooth Cauchy time g_1-function $t : \mathsf{M} \to \mathbb{R}$ and $\chi \in C^\infty(\mathsf{M}; [0,1])$ such that $\chi(p) = 0$ if $t(p) < t_0$ and $\chi(p) = 1$ if $t(p) > t_1$ for given $t_0 < t_1$;*
(b) *a pair of smooth functions $\rho, \rho' : \mathsf{M} \to (0, +\infty)$ such that $\rho(p) = 1$ for $t(p) < t_0$ and $\rho'(p) = \rho(p) = 1$ if $t(p) > t_1$. (Notice that $\rho = \rho' = 1$ constantly is allowed.)*

The following facts are true.

(1) *The operators*

$$\mathsf{R}_+ = \mathrm{Id} - \mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{4.1}$$

$$\mathsf{R}_- = \mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{4.2}$$

are linear space isomorphisms, whose inverses are given by

$$\mathsf{R}_+^{-1} = \mathrm{Id} + \mathsf{G}^+_{\mathsf{N}_0}(\rho\mathsf{N}_\chi - \mathsf{N}_0) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{4.3}$$

$$\mathsf{R}_-^{-1} = \mathrm{Id} + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})\,. \tag{4.4}$$

(2) *It holds*

$$\rho\mathsf{N}_\chi\mathsf{R}_+ = \mathsf{N}_0 \qquad \text{and} \qquad \rho'\mathsf{N}_1\mathsf{R}_- = \rho\mathsf{N}_\chi\,. \tag{4.5}$$

(3) If $\mathfrak{f} \in \Gamma(\mathsf{E})$, then

$$(\mathsf{R}_+\mathfrak{f})(p) = \mathfrak{f}(p) \quad \text{for} \quad t(p) < t_0, \tag{4.6}$$

$$(\mathsf{R}_-\mathfrak{f})(p) = \mathfrak{f}(p) \quad \text{for} \quad t(p) > t_1\,. \tag{4.7}$$

Proof Observe that $\rho\mathsf{N}_\chi$ and $\rho'\mathsf{N}_1$ are Green hyperbolic with respect to g_χ (as in Theorem 3.14) and g_1 respectively according to Theorem 3.14 and 3.6, and thus they are with respect to g_1. Moreover $\mathsf{G}^\pm_{\rho\mathsf{N}_\chi} = \mathsf{G}^\pm_{\mathsf{N}_\chi}\rho^{-1}$ and $\mathsf{G}^\pm_{\rho'\mathsf{N}_1} = \mathsf{G}^\pm_{\mathsf{N}_1}\rho'^{-1}$.
(1) If $\mathfrak{f} \in \Gamma(\mathsf{E})$, in view of the hypotheses $((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})(p) = 0$ and $((\mathsf{N}_1 - \mathsf{N}_\chi)\mathfrak{f})(p) = 0$ is respectively $t(p) < t_0$ and $t(p) > t_1$ where $t^{-1}(t_0)$ and $t^{-1}(t_1)$ are spacelike Cauchy hypersurfaces in common for the metrics g_0, g_χ, g_1. Therefore the operators R_- and R_+ are linear and well defined on the domain $\Gamma(\mathsf{E})$ because $(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f} \in \Gamma^{g_1}_{pc}(\mathsf{E}) \subset \Gamma^{g_\chi}_{pc}(\mathsf{E}) \subset \mathrm{Dom}(\mathsf{G}^+_{\rho\mathsf{N}_\chi})$ and $(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)\mathfrak{f} \in \Gamma^{g_1}_{fc}(\mathsf{E}) \subset \mathrm{Dom}(\mathsf{G}^-_{\rho'\mathsf{N}_1})$. A similar argument holds for $\mathsf{R}^{-1}_\pm$. To prove bijectivity of $\mathsf{R}_\pm$ it suffices to establish that R_-^{-1} in (4.4) is a two-sided inverse of R_- and that R_+^{-1} in (4.3) is a two-sided inverse of R_+ on $\Gamma(\mathsf{E})$:

$$\mathsf{R}_- \circ \mathsf{R}_-^{-1} = \mathsf{R}_-^{-1} \circ \mathsf{R}_- = \mathrm{Id} \quad \text{and} \quad \mathsf{R}_+ \circ \mathsf{R}_+^{-1} = \mathsf{R}_+^{-1} \circ \mathsf{R}_+ = \mathrm{Id}.$$

The proof of the well definiteness of R_-^{-1} and R_+^{-1} on $\Gamma(\mathsf{E})$ is analogous to the previous one for R_-. We prove that R_- defined as in (4.4) inverts R_- from the right by direct computation:

$$\begin{aligned}\mathsf{R}_- \circ \mathsf{R}_-^{-1} &= (\mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)) \circ (\mathrm{Id} + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)) = \\ &= \mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) \\ &\quad - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)\mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi).\end{aligned}$$

Now, by exploiting the identity

$$\mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi)\mathsf{G}^-_{\rho\mathsf{N}_\chi} = \mathsf{G}^-_{\rho\mathsf{N}_\chi} - \mathsf{G}^-_{\rho'\mathsf{N}_1} : \Gamma^{g_\chi}_{fc}(\mathsf{E}) \cap \Gamma^{g_1}_{fc}(\mathsf{E}) \to \Gamma(\mathsf{E}),$$

we can prove our claim

$$\begin{aligned}\mathsf{R}_- \circ \mathsf{R}_-^{-1} &= \mathrm{Id} - \mathsf{G}^-_{\rho'\mathsf{N}_1}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) + \mathsf{G}^-_{\rho\mathsf{N}_\chi}(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) \\ &\quad - (\mathsf{G}^-_{\rho\mathsf{N}_\chi} - \mathsf{G}^-_{\rho'\mathsf{N}_1})(\rho'\mathsf{N}_1 - \rho\mathsf{N}_\chi) = \mathrm{Id}\,.\end{aligned}$$

The proof that R_-^{-1} is also a left inverse is the same with obvious changes and analogous calculations show that R_+^{-1} is a left and right inverse of R^+.
(2) Taking advantage of (ia)-(iib) in Definition 3.4 and the definition of N_χ and the one of $\mathsf{R}_\pm$, a direct computation establishes (4.5).
(3) Let us prove (4.6). Consider a compactly supported smooth section $\mathfrak{h}$ whose support is included in the set $t^{-1}((-\infty, t_0))$. Taking advantage of the former in (3.2), we obtain

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = \int_{\mathsf{M}} \langle \mathsf{G}^-_{(\rho\mathsf{N}_\chi)^*}\mathfrak{h}, (\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = 0$$

since $\mathrm{supp}(\mathsf{G}^-_{(\rho\mathsf{N}_\chi)^*}\mathfrak{h}) \subset J_-^{g_\chi}(\mathrm{supp}(\mathfrak{h}))$ from Definition 3.4 and thus that support does not meet $\mathrm{supp}((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})$ because $((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})(p)$ vanishes if $t(p) < t_0$. As $\mathfrak{h}$ is an arbitrary smooth section compactly supported in $t^{-1}((-\infty, t_0))$,

$$\int_{\mathsf{M}} \langle \mathfrak{h}, \mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}\rangle \,\mathrm{vol}_{g_\chi} = 0$$

entails that $\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f} = 0$ on $t^{-1}((-\infty, t_0))$. Eventually, the very definition (4.1) of $\mathsf{G}^+_{\rho\mathsf{N}_\chi}$ implies (4.6). The proof of (4.7) is strictly analogous, so we leave it to the reader. □

We can now pass to the second step, namely we perform restrictions of $\mathsf{R}_\pm$ to the relevant subspaces of solutions.

Proposition 4.2 *With the same hypotheses as in Proposition 4.1 (in particular $\chi(p) = 0$ if $t(p) < t_0$ and $\chi(p) = 1$ if $t(p) > t_1$ for given $t_0 < t_1$), we have*

$$\mathsf{R}_+(\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)) = \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi) \quad \textit{and} \quad \mathsf{R}_-(\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)) = \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1) \tag{4.8}$$

and

$$\mathsf{R}_+(\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)) = \mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi) \quad \textit{and} \quad \mathsf{R}_-(\mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)) = \mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)\,. \tag{4.9}$$

As a consequence, the restrictions

$$\begin{aligned}
\Omega_+ := \mathsf{R}_+|_{\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)} &: \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0) \to \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)\,,\\
\Omega^0_+ : \mathsf{R}_+|_{\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)} &: \mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0) \to \mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)\,,\\
\Omega_- := \mathsf{R}_-|_{\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)} &: \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi) \to \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)\,,\\
\Omega^0_- : \mathsf{R}_-|_{\mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi)} &: \mathsf{Ker}^{g_\chi}_{sc}(\mathsf{N}_\chi) \to \mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)\,,
\end{aligned}$$

define linear space isomorphisms such that

$$\rho\mathsf{N}_\chi\Omega_+ = \mathsf{N}_0\,, \quad \rho'\mathsf{N}_1\Omega_- = \rho\mathsf{N}_\chi \tag{4.10}$$

and, for $\mathfrak{f}$ *in the respective domains,*

$$(\Omega_+\mathfrak{f})(p) = \mathfrak{f}(p)\,, \quad (\Omega^0_+\mathfrak{f})(p) = \mathfrak{f}(p) \quad \textit{for} \quad t(p) < t_0, \tag{4.11}$$

$$(\Omega_-\mathfrak{f})(p) = \mathfrak{f}(p)\,, \quad (\Omega^0_-\mathfrak{f})(p) = \mathfrak{f}(p) \quad \textit{for} \quad t(p) > t_1\,. \tag{4.12}$$

Before we prove our claim, we need a preparatory lemma.

Lemma 4.3 *Let* $\mathsf{P} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *be a 2nd order normally hyperbolic differential operator on the vector bundle* $\mathsf{E} \to \mathsf{M}$ *on the globally hyperbolic spacetime* (M, g). *Let* $\Psi \in \Gamma(\mathsf{E})$ *be such that* $\mathsf{P}\Psi \in \Gamma_c(\mathsf{E})$. *Then the following facts are equivalent.*

(a) $\Psi \in \Gamma^g_{sc}(\mathsf{E})$*;*
(b) *there is a spacelike Cauchy hypersurface of* (M, g) *such that* Ψ *have compactly supported Cauchy data thereon.*

Proof If $\Psi \in \Gamma^g_{sc}(\mathsf{E})$ then, by definition, (b) is true. Suppose that (b) is true for Σ_0. According to Theorem 3.3, Ψ is the unique solution of the Cauchy problem whose equation is $\mathsf{P}\Psi = \mathfrak{f}$, where $\mathfrak{f} \in \Gamma_c(\mathsf{E})$. As a consequence the support of Ψ completely lies in $J(\mathrm{supp}(\mathfrak{f})) \cup J(\mathrm{supp}(\mathfrak{h}_0)) \cup J(\mathrm{supp}(\mathfrak{h}_1)) \subset J(K)$ where $\mathfrak{h}_0$ and $\mathfrak{h}_1$ are the Cauchy data of Ψ on Σ_0 and $K := \mathrm{supp}(\mathfrak{f}) \cup \mathrm{supp}(\mathfrak{h}_0) \cup J(\mathrm{supp}(\mathfrak{h}_1)$. In particular K is compact. In view of well known properties of globally hyperbolic spacetimes (see e.g. [3, Proposition 1.2.56]), since K is compact $J(K) \cap \Sigma$ is compact for every Cauchy hypersurface Σ of (M, g) so that $\Psi \in \Gamma^g_{sc}(\mathsf{E})$. □

Proof of Proposition 4.2 $\mathsf{R}_\pm$ and $\mathsf{R}^{-1}_\pm$ are bijective on $\Gamma(\mathsf{E})$. As a consequence (4.8) and thesis for $\Omega_\pm$, including (4.10) which is a specialization of (4.5), immediately arise when proving that

$$\mathsf{R}_+(\mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)) \subset \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)\,, \quad \mathsf{R}^{-1}_+(\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)) \subset \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0) \tag{4.13}$$

and

$$\mathsf{R}_{-}(\mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)) \subset \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)\,, \quad \mathsf{R}_{-}^{-1}(\mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)) \subset \mathsf{Sol}^{g_\chi}_{sc,c}(\mathsf{N}_\chi)$$

The identities in (4.9) and the thesis for $\Omega^0_\pm$ immediately arise from the bijectivity of the linear maps $\Omega_\pm$ and (4.10) where we know that $\rho, \rho' > 0$. To conclude, let us establish the first inclusion in (4.13), the remaining three inclusions have a strictly analogous proof. Suppose that $\mathfrak{f} \in \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)$. Hence $\rho \mathsf{N}_\chi \mathsf{R}_+ \mathfrak{f} = \mathsf{N}_0 \mathfrak{f} \in \Gamma^{g_0}_c(\mathsf{E}) = \Gamma^{g_\chi}_c(\mathsf{E})$ and $\mathsf{N}_\chi \mathsf{R}_+ \mathfrak{f} = \rho^{-1}\mathsf{N}_0 \mathfrak{f} \in \Gamma^{g_\chi}_c(\mathsf{E})$. Next pass to consider the Cauchy hypersurfaces of t which are in common with the three considered metrics g_0, g_1, g_χ and choose $t' < t_0$. (3) in Proposition 4.1 yields $(\mathsf{R}_+\mathfrak{f})(t', x) = \mathfrak{f}(t', x)$ where $x \in \Sigma_{t'}$. The Cauchy data of $\mathfrak{f}$ on $\Sigma_{t'}$ have compact support because $\mathfrak{f} \in \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0)$. On the ground of Lemma 4.3, noticing that N_χ is normally hyperbolic in (M, g_χ), referring to the Cauchy problem on $\Sigma_{t'}$ for the equation $\mathsf{N}_\chi \mathsf{R}_+ \mathfrak{f} = \rho^{-1}\mathsf{N}_0\mathfrak{f} \in \Gamma^{g_\chi}_c(\mathsf{E})$ in the spacetime (M, g_χ), we conclude that $\mathsf{R}_+\mathfrak{f} \in \Gamma^{g_\chi}_{sc,c}(\mathsf{E})$ because its Cauchy data on $\Sigma_{t'}$ (now interpreted as a Cauchy hypersurface for g_χ) have compact support as they coincide with the ones of $\mathfrak{f}$ itself. □

4.3 General Møller maps for paracausally related metrics

We are now in a position to state a result regarding the existence of Møller maps between two normally hyperbolic operators N_0 and N_1 on respective globally hyperbolic spacetimes over the same manifold (and vector bundle) whose metrics are $\preceq$ comparable. The final goal is to extend the results to pairs of paracausally related metrics.

Proposition 4.4 *Let $g_0, g_1 \in \mathcal{GM}_\mathsf{M}$ be such that either $g_0 \preceq g_1$ or $g_1 \preceq g_0$ with, respectively, either $V^{g_0+}_x \subset V^{g_1+}_x$ for all $x \in M$ or $V^{g_1+}_x \subset V^{g_0+}_x$ for all $x \in M$. Let E be a vector bundle over M and $\mathsf{N}_0, \mathsf{N}_1 : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be normally hyperbolic operators associated to g_0 and g_1 respectively. There exist (infinitely many) vector space isomorphisms,*

$$\mathsf{S} : \mathsf{Sol}^{g_0}_{sc,c}(\mathsf{N}_0) \to \mathsf{Sol}^{g_1}_{sc,c}(\mathsf{N}_1)$$

such that, for some smooth function $\mu : \mathsf{M} \to (0, +\infty)$ depending on S (which can be chosen $\mu = 1$),

(1) *referring to the said domains,*

$$\mu \mathsf{N}_1 \mathsf{S} = \mathsf{N}_0 \quad \textit{and} \quad \mu^{-1}\mathsf{N}_0 \mathsf{S}^{-1} = \mathsf{N}_1$$

(2) *the restriction $\mathsf{S}^0 := \mathsf{S}|_{\mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0)}$ defines a vector space isomorphism*

$$\mathsf{S}^0 : \mathsf{Ker}^{g_0}_{sc}(\mathsf{N}_0) \to \mathsf{Ker}^{g_1}_{sc}(\mathsf{N}_1)\,.$$

Proof First consider the case $g_0 \preceq g_1$. Referring to a smooth Cauchy time function t of (M, g_1) and a smoothing function χ, $\mathsf{S} := \Omega_-\Omega_+$ constructed as in Proposition 4.2 satisfies all the requirements trivially for $\mu := \rho'$. The previous result is also valid for $g_1 \preceq g_0$. It is sufficient to construct $\Omega_\pm$ as in Proposition 4.2, *but using* g_1 *as the initial metric and* g_0 *as the final one*, and eventually defining $\mu := \rho^{-1}$, $\mathsf{S} := (\Omega_-\Omega_+)^{-1} = \Omega_+^{-1}\Omega_-^{-1}$, and $\mathsf{S}^0 := (\Omega_-^0\Omega_+^0)^{-1} = (\Omega_+^0)^{-1}(\Omega_-^0)^{-1}$. □

We can pass to the generic case $g \simeq g'$, obtaining the first main result of this work.

Theorem 4.5 *Let* (M, g) *and* (M, g') *be globally hyperbolic spacetimes,* E *a vector bundle over* M *and* $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *normally hyperbolic operators associated to* g *and* g' *respectively.*
If $g \simeq g'$*, then there exist (infinitely many) vector space isomorphisms, called* **Møller maps** *of* g, g' *(with this order),*

$$\mathsf{S} : \mathsf{Sol}^g_{sc,c}(\mathsf{N}) \to \mathsf{Sol}^{g'}_{sc,c}(\mathsf{N}')$$

such that

(1) *referring to the said domains,*

$$\mu\mathsf{N}'\mathsf{S} = \mathsf{N}$$

for some smooth $\mu : \mathsf{M} \to (0, +\infty)$ *(which can be always taken* $\mu = 1$ *constantly in particular),*

(2) *the restriction* $\mathsf{S}^0 := \mathsf{S}|_{\mathsf{Ker}^g_{sc}(\mathsf{N})}$ *(also called* **Møller map** *of* g', g'*) defines a vector space isomorphism*

$$\mathsf{S}^0 : \mathsf{Ker}^g_{sc}(\mathsf{N}) \to \mathsf{Ker}^{g'}_{sc}(\mathsf{N}')\ .$$

Proof First of all we notice that there always exists a normally hyperbolic operator N on E associated to every $g \in \mathcal{GM}_M$: For instance the *connection-d'Alembert operator* in [3, Example 2.1.5] referred to a generic connection ∇ on E, which always exists, and the Levi-Civita connection on (M, g). Let us consider a sequence $g_0 = g, g_1, \ldots, g_N = g'$ of globally hyperbolic metrics on M satisfying Definition 2.19 and a corresponding sequence of formally selfadjoint normally hyperbolic operators N_k with $\mathsf{N}_0 := \mathsf{N}$ and $\mathsf{N}_N := \mathsf{N}'$. We can apply Proposition 4.4 for each pair g_k, g_{k+1} for $k = 0, 1, \ldots, N-1$. It turns immediately out that, with an obvious notation,

$$\mathsf{S} := \mathsf{S}_0\mathsf{S}_1 \cdots \mathsf{S}_{N-1}\ , \quad \mu := \mu_0 \cdots \mu_{N-1}\ ,$$
$$\text{where} \quad \mu_k\mathsf{N}_k\mathsf{S}_k = \mathsf{S}_{k-1} \quad k = 0, \ldots N-1\ .$$

satisfy the thesis of the theorem, where either $S_k := \Omega_{k-}\Omega_{k+}$, $\mu_k := \rho_k$ or $S_k := (\Omega_{k+})^{-1}(\Omega_{k-})^{-1}$, $\mu_k := \rho_k^{-1}$ according to $g_k \preceq g_{k+1}$ or $g_{k+1} \preceq g_k$ respectively. With the same convention it results that $\mathsf{S}^0 = \mathsf{S}_0^0\mathsf{S}_1' \cdots \mathsf{S}_{N-1}^0$ where either $S_k^0 = \Omega_{k-}^0\Omega_{k+}^0$ or $S_k = (\Omega_{k+}^0)^{-1}(\Omega_{k-}^0)^{-1}$ according to the discussed cases. □

4.4 Preservation of symplectic forms

The Møller maps S^0 as in Theorem 4.5 preserve the symplectic forms of the normal operators they relate when these operators are formally selfadjoint.

Theorem 4.6 *Consider $g, g' \in \mathcal{GH}_M$ with respectively associated normally hyperbolic operators N, N' on the $\mathbb{K}$-vector bundle E over M. If $g' \simeq g$ and N and N' are formally selfadjoint with respect to a non-degenerate, Hermitian fiber metric $\langle \cdot | \cdot \rangle$, then there are Møller maps S^0 satisfying the thesis of Theorem 4.5 such that*

$$\sigma_{g'}^{\mathsf{N}'}(\mathsf{S}^0\Psi, \mathsf{S}^0\Phi) = \sigma_g^{\mathsf{N}}(\Psi, \Phi) \quad \textit{for every } \Psi, \Phi \in \mathsf{Ker}_{sc}^{g}(\mathsf{N}),$$

where we used the notation σ_g^{N} in place of $\sigma_{(\mathsf{M},g)}^{\mathsf{N}}$.

Proof It is sufficient to prove the thesis for the maps $\Omega_\pm^0$ referred to two metrics $g_0 \preceq g_1$, which immediately implies the thesis also for the inverse maps $(\Omega_\pm^0)^{-1}$ they being isomorphisms. Indeed, according the proof of Theorem 4.5, the isomorphisms S^0 are compositions of various copies of $\Omega_\pm^0$ and their inverses. Let us consider $\Omega_+^0 : \mathsf{Ker}_{sc}(\mathsf{N}_0) \to \mathsf{Ker}_{sc}(\mathsf{N}_\chi)$ and we prove the thesis for it, the other case being very similar. Consider a smooth Cauchy time function t for g_1 and the associated foliation made of spacelike Cauchy hypersurfaces Σ_t in common for g_0, g_1, and g_χ. If the smoothing function χ used to build up g_χ and N_χ vanishes before t_0 and we use Σ_t with $t < t_0$ to compute the relevant symplectic forms, due to (4.11),

$$\sigma_{g_\chi}^{\mathsf{N}_\chi}(\Omega_+^0\Psi, \Omega_+^0\Phi) = \sigma_{g_0}^{\mathsf{N}_0}(\Psi, \Phi) \quad \text{for every } \Psi, \Phi \in \mathsf{Ker}_{sc}^{g_0}(\mathsf{N}_0).$$

Above, we have used the definition of the symplectic form, we have noticed that $g_\chi = g_0$ around Σ_t and that the N_0 and N_χ compatible connections must coincide there as they are locally defined and uniquely determined by $\mathsf{N}_0\Psi = \mathsf{N}_\chi\Psi = (-\mathrm{tr}_g(\nabla\nabla)+c)\Psi$ for every smooth Ψ compactly supported around a point p with $t(p) < t_0$. Thinking of $\sigma_{g_\chi}^{\mathsf{N}_\chi}(\Omega_+^0\Psi, \Omega_+^0\Phi)$ as defined in (M, g_χ) and of $\sigma_{g_0}^{\mathsf{N}_0}(\Psi, \Phi)$ as defined in (M, g_0), though both computed on Σ_t with $t < t_0$, Proposition 3.12 concludes the proof. □

Definition 4.7 We call **symplectic Møller map** any linear isomorphism defined in accordance with Theorem 4.6.

4.5 Causal propagators and paracausally related metrics

In this section, we prove how is possible to choose the functions ρ and ρ' affecting the definitions (4.1), (4.2) of $\mathsf{R}_\pm$ in order to satisfy a further requirement with some crucial implications in QFT: the preservation of the causal propagator of two operators N and N' when the associated metrics are paracausally related. Essentially speaking, a Møller map satisfying this further requirement will be named *Møller operator*.

4.5.1 Adjoint operators

To study the relation between Møller maps and the causal propagator of normally hyperbolic operators defined on a vector bundle equipped with a non-degenerate (Hermitian) fiber metric, we need a suitable notion of *adjoint operator* which generalizes the notion of formal adjoint of differential operators.

Let E be a $\mathbb{K}$-vector bundle on the manifold M equipped with a non-degenerate, symmetric if $\mathbb{K} = \mathbb{R}$ or Hermitian if $\mathbb{K} = \mathbb{C}$, fiber metric $\langle \cdot \mid \cdot \rangle$. Suppose that g and g' (possibly $g \neq g'$) are Lorentzian metrics on M. Consider a $\mathbb{K}$-linear operator

$$\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E}) ,$$

where $\mathsf{Dom}(\mathsf{T}) \subset \Gamma(\mathsf{E})$ is a $\mathbb{K}$-linear subspace and $\mathsf{Dom}(\mathsf{T}) \supset \Gamma_c(\mathsf{E})$.

Definition 4.8 An operator

$$\mathsf{T}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$$

is said to be the **adjoint of T with respect to** g, g' (with the said order) if it satisfies

$$\begin{aligned} &\int_{\mathsf{M}} \langle \mathfrak{h}(x) \mid (\mathsf{T}\mathfrak{f})(x) \rangle \mathrm{vol}_{g'}(x) \\ &= \int_{\mathsf{M}} \langle \left(\mathsf{T}^{\dagger_{gg'}} \mathfrak{h} \right)(x) \mid \mathfrak{f}(x) \rangle \mathrm{vol}_g(x) \quad \forall \mathfrak{f} \in \mathsf{Dom}(\mathsf{T}) \,,\ \forall \mathfrak{h} \in \Gamma_c(\mathsf{E}). \end{aligned}$$

Notation 4.9 If $g = g'$ then we shall denote the adjoint of T with respect to g simply as $\mathsf{T}^{\dagger_g}$.

We prove below that $\mathsf{T}^{\dagger_{gg'}}$ is unique if exists so that calling it "the" adjoint operator of T is appropriate.

Remark 4.10 If $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ is defined as in Definition 4.8 and $\mathsf{T}^{\dagger_{gg'}}$ exists, then

$$\int_{\mathsf{M}} \langle \mathfrak{h} \mid \mathsf{T}\mathfrak{f}_n \rangle \mathrm{vol}_{g'} \to 0 \quad \forall \mathfrak{h} \in \Gamma_c(\mathsf{E}) \text{ as } \Gamma_c(\mathsf{E}) \ni \mathfrak{f}_n \to 0 \text{ for } n \to +\infty \text{ in the } \textit{topology of test sections } [3] \,.$$

Vice versa, this only condition is not sufficient to guarantee the existence of $\mathsf{T}^{\dagger_{gg'}}$ as a $\Gamma_c(\mathsf{E})$-valued operator. Using a straightforward extension of the Schwartz kernel theorem, the condition above just implies the existence of a weaker version of $\mathsf{T}^{\dagger_{gg'}}$ which is distribution-valued.

In the rest of the paper if $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ and $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{E})$, we define the **standard domains** of their compositions as follows, where $a \in \mathbb{K}$.

(a) $\mathsf{Dom}(a\mathsf{T}) := \mathsf{Dom}(\mathsf{T})$ – or $\mathsf{Dom}(a\mathsf{T}) := \Gamma(\mathsf{E})$ if $a = 0$ – is the domain of $a\mathsf{T}$ defined pointwise;

(b) $\mathsf{Dom}(\mathsf{T}+\mathsf{T}') := \mathsf{Dom}(\mathsf{T}) \cap \mathsf{Dom}(\mathsf{T}')$ is the domain of $a\mathsf{T}+b\mathsf{T}'$ defined pointwise;
(c) $\mathsf{Dom}(\mathsf{T}' \circ \mathsf{T}) := \{\mathfrak{f} \in \mathsf{Dom}(\mathsf{T}) \mid \mathsf{T}(\mathfrak{f}) \in \mathsf{Dom}(\mathsf{T}')\}$ is the domain of $\mathsf{T}' \circ \mathsf{T}$.

Proposition 4.11 *Referring to the notion of adjoint in Definition 4.8, the following facts are valid.*

(1) *If the adjoint $\mathsf{T}^{\dagger_{gg'}}$ of T exists, then it is unique.*
(2) *If $\mathsf{T} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is a differential operator and $g = g'$, then $\mathsf{T}^{\dagger_{gg}}$ exists and is the restriction of the formal adjoint to $\Gamma_c(\mathsf{E})$. (In turn, the formal adjoint of $\mathsf{T}^{\dagger}$ is the unique extension to $\Gamma(\mathsf{E})$ of the differential operator $\mathsf{T}^{\dagger}$ as a differential operator)*
(3) *Consider a pair of $\mathbb{K}$-linear operators $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$, $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{E})$ and $a, b \in \mathbb{K}$. Then*

$$(a\mathsf{T} + b\mathsf{T}')^{\dagger_{gg'}} = \overline{a}\mathsf{T}^{\dagger_{gg'}} + \overline{b}\mathsf{T}'^{\dagger_{gg'}}$$

provided $\mathsf{T}^{\dagger_{gg'}}$ and $\mathsf{T}'^{\dagger_{gg'}}$ exist.
(4) *Consider a pair of $\mathbb{K}$-linear operators $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ and $\mathsf{T}' : \mathsf{Dom}(\mathsf{T}') \to \Gamma(\mathsf{E})$ such that*

   (i) $\mathsf{Dom}(\mathsf{T}' \circ \mathsf{T}) \supset \Gamma_c(\mathsf{E})$,
   (ii) $\mathsf{T}^{\dagger_{gg'}}$ *and* $\mathsf{T}'^{\dagger_{g'g''}}$ *exist,*

   then $(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}}$ *exists and*

$$(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}} = \mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}} .$$

(5) *If $\mathsf{T}^{\dagger_{gg'}}$ exists, then $(\mathsf{T}^{\dagger_{gg'}})^{\dagger_{g'g}} = \mathsf{T}|_{\Gamma_c(\mathsf{E})}$.*
(6) *If $\mathsf{T} : \mathsf{Dom}(T) = \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ is bijective, admits $\mathsf{T}^{\dagger_{gg'}}$, and T^{-1} admits $(\mathsf{T}^{-1})^{\dagger_{g'g}}$, then $\mathsf{T}^{\dagger_{gg'}}$ is bijective and $(\mathsf{T}^{-1})^{\dagger_{g'g}} = (\mathsf{T}^{\dagger_{gg'}})^{-1}$.*

Proof We write below $\dagger$ in place of $\dagger_{gg'}$ if it is not strictly necessary to specify the metrics. To prove (1) let's assume that, fixed an operator $\mathsf{T} : \mathsf{Dom}(\mathsf{T}) \to \Gamma(\mathsf{E})$ there exist two different adjoints $\mathsf{T}_1^{\dagger}, \mathsf{T}_2^{\dagger} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ both satisfying definition 4.8, i.e.

$$\int_{\mathsf{M}} \langle \mathsf{T}_1^{\dagger}\mathfrak{h} \mid \mathfrak{f} \rangle \,\mathrm{vol}_g = \int_{\mathsf{M}} \langle \mathsf{T}_2^{\dagger}\mathfrak{h} \mid \mathfrak{f} \rangle \,\mathrm{vol}_g$$

for all $\mathfrak{f} \in \mathsf{Dom}(\mathsf{T})$ and all $\mathfrak{h} \in \Gamma_c(\mathsf{E})$. Then by linearity of the integration and (anti) linearity of the product, the former identity is equivalent to

$$\int_{\mathsf{M}} \langle \mathsf{T}_1^{\dagger}\mathfrak{h} - \mathsf{T}_2^{\dagger}\mathfrak{h} \mid \mathfrak{f} \rangle \,\mathrm{vol}_g = 0.$$

Since $\Gamma_c(\mathsf{E}) \subset \mathsf{Dom}(\mathsf{T})$, the thesis follows by reducing to every fixed local trivialization over every arbitrarily fixed coordinate patch U on M. Restricting to U, the equation above can be recast to

$$\int_U \sum_{a=1}^{N} (\mathsf{T}_1^{\dagger}\mathfrak{h} - \mathsf{T}_2^{\dagger}\mathfrak{h})^a(p)\mathfrak{f}_a(p) \,\mathrm{vol}_g(p) = 0.$$

where $\mathfrak{f}_a(p)$ is a fiber component of $\langle \cdot | \mathfrak{f} \rangle_p \in \mathsf{E}^*_p$ with $p \in U$. Since $U \ni p \mapsto (\mathsf{T}_1^\dagger \mathfrak{h} - \mathsf{T}_2^\dagger \mathfrak{h})^a(p)$ is continuous and $U \ni p \mapsto \mathfrak{f}_a(p)$ is smooth, compactly supported (with support in U) and arbitrary (because $\langle \cdot | \cdot \rangle$ is non-degenerate), the fundamental lemma of calculus of variations implies that $U \ni p \mapsto (\mathsf{T}_1^\dagger \mathfrak{h} - \mathsf{T}_2^\dagger \mathfrak{h})^a(p)$ is the zero function for $a = 1, \ldots, N$. Since U can be fixed as a neighborhood of every point of M, (1) follows.

The proof of (2) and (3) is obvious: (2) follows by comparing definitions 4.8 and 3.9, while (3) follows by direct computation checking that $\overline{a}\mathsf{T}^\dagger + \overline{b}\mathsf{T}'^\dagger$ satisfies the definition of $(\overline{a}\mathsf{T} + \overline{b}\mathsf{T}')^\dagger$ (notice that $\Gamma_c(\mathsf{E}) \subset \mathsf{Dom}(\overline{a}\mathsf{T}^\dagger + \overline{b}\mathsf{T}'^\dagger)$ if $\mathsf{T}^\dagger$ and $\mathsf{T}'^\dagger$ exist).

To prove (4), since the composition is well defined on a suitable domain, we can just use twice the definition 4.8

$$\int_\mathsf{M} \langle \mathfrak{h} | \mathsf{T}' \circ \mathsf{T}\mathfrak{f} \rangle \,\mathrm{vol}_{g''} = \int_\mathsf{M} \langle \mathsf{T}'^{\dagger_{g'g''}} \mathfrak{h} | \mathsf{T}\mathfrak{f} \rangle \,\mathrm{vol}_{g'} = \int_\mathsf{M} \langle \mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}} \mathfrak{h} | \mathfrak{f} \rangle \,\mathrm{vol}_{g}$$

for all $\mathfrak{f} \in \mathsf{Dom}(\mathsf{T}' \circ \mathsf{T})$ and all $\mathfrak{h} \in \Gamma_c(\mathsf{E})$: notice that using the definition of the adjoint in the second equality is possible because $\mathsf{T}'^{\dagger_{g'g''}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$. The found identity proves that $\mathsf{T}^{\dagger_{gg'}} \circ \mathsf{T}'^{\dagger_{g'g''}}$ satisfies the definition of $(\mathsf{T}' \circ \mathsf{T})^{\dagger_{gg''}}$ ending the proof of (4).

(5) is true because, if $\mathsf{T}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ exists, then $\mathsf{T}|_{\Gamma_c(\mathsf{E})}$ satisfies the definition of $(\mathsf{T}^{\dagger_{gg'}})^{\dagger_{g'g}}$.

Finally, (6) arises by taking the $^{\dagger_{gg}}$ adjoint of both sides of the identity $T \circ T^{-1} = I$ and the $^{\dagger_{g'g'}}$ adjoint of both sides of the identity $T^{-1} \circ T = I$ and taking (4) into account. ☐

4.5.2 Møller operators and causal propagators

We are in a position to state one of the most important results of this work by specializing the isomorphisms introduced in Theorem 4.5 by means of a suitable choice of the function μ. As a matter of fact (1) and (3) have been already established in Theorem 4.5.

Theorem 4.12 *Let* E *be* $\mathbb{K}$*-vector bundle over the smooth manifold* M *with a non-degenerate, real or Hermitian depending on* $\mathbb{K}$*, fiber metric* $\langle \cdot | \cdot \rangle$*. Consider* $g, g' \in \mathcal{GH}_M$ *with respectively associated normally hyperbolic formally-selfadjoint operators* N, N'.
If $g \simeq g'$*, then it is possible to define (in infinite ways) a* $\mathbb{K}$*-vector space isomorphism* $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$*, called* **Møller operator** *of* g, g' *(with this order), such that the following facts are true.*

(1) *The restrictions to the relevant subspaces of* $\Gamma(\mathsf{E})$ *respectively define Møller maps (hence linear isomorphisms) as in Theorem 4.5.*

$$\mathsf{R}|_{\mathsf{Sol}^g_{sc,c}(\mathsf{N})} = \mathsf{S} : \mathsf{Sol}^g_{sc,c}(\mathsf{N}) \to \mathsf{Sol}^{g'}_{sc,c}(\mathsf{N}')$$
$$\textit{and} \quad \mathsf{R}|_{\mathsf{Ker}^g_{sc}(\mathsf{N})} = \mathsf{S}^0 : \mathsf{Ker}^g_{sc}(\mathsf{N}) \to \mathsf{Ker}^{g'}_{sc}(\mathsf{N}').$$

(2) *The causal propagators* $\mathsf{G}_{\mathsf{N}'}$ *and* G_{N}*, respectively of* N' *and* N*, satisfy*

$$\mathsf{R}\mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}} = \mathsf{G}_{\mathsf{N}'} \,. \tag{4.14}$$

(3) *By denoting* c' *the smooth function such that* $\mathrm{vol}_{g'} = c' \, \mathrm{vol}_g$*, we have*

$$c'\mathsf{N}'\mathsf{R} = \mathsf{N} \,. \tag{4.15}$$

(4) *It holds*

$$\mathsf{R}^{\dagger_{gg'}}\mathsf{N}'|_{\Gamma_c(\mathsf{E})} = \mathsf{N}|_{\Gamma_c(\mathsf{E})} \,.$$

(5) *The maps* $\mathsf{R}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ *and* $(\mathsf{R}^{\dagger_{gg'}})^{-1} = (\mathsf{R}^{-1})^{\dagger_{g'g}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ *are continuous with respect to the natural topologies of* $\Gamma_c(\mathsf{E})$ *in the domain and in the co-domain.*

Remarks 4.13 Before proving our claim, we want to underline the following:

(1) Any Møller operator defines a symplectic Møller map (*cf.* Definition 4.7). Indeed, the preservation of the causal propagator (*cf.* (2) in Theorem 4.12) implies that the symplectic forms are preserved in view of Proposition 3.13. However, the converse is false since the preservation of the causal propagator relies upon a suitable choice of the function ρ, whereas this choice is immaterial for the preservation of the symplectic forms.
(2) Møller operators can be explicitly constructed as follows. If $g' \simeq g$, and referring to a finite sequence of metrics $g_0 := g, g_1, \ldots, g_N := g' \in \mathcal{GH}_{\mathsf{M}}$ as in Definition 2.19, then there exists a corresponding sequence of formally selfadjoint g_k-normally hyperbolic operators $\mathsf{N}_0 := \mathsf{N}, \mathsf{N}_1, \ldots, \mathsf{N}_N := \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ such that

$$\mathsf{R} = \mathsf{R}_0 \cdots \mathsf{R}_{N-1} \,, \tag{4.16}$$

is a Møller operator of g, g' where

$$\begin{aligned} \mathsf{R}_k &:= \mathsf{R}_-^{(k)}\mathsf{R}_+^{(k)} \quad \text{if } g_k \preceq g_{k+1} \qquad \text{or} \\ \mathsf{R}_k &:= (\mathsf{R}_+^{(k)})^{-1}(\mathsf{R}_-^{(k)})^{-1} \quad \text{if } g_{k+1} \preceq g_k. \end{aligned} \tag{4.17}$$

Above, for every given k, $\mathsf{R}_\pm^{(k)}$ are defined as $\mathsf{R}_\pm$ as in Eq. (4.1) and (4.2) where

(i) N_0 is replaced by N_k and N_1 is replaced by N_{k+1} if $g_k \preceq g_{k+1}$,
(ii) N_0 is replaced by N_{k+1} and N_1 is replaced by N_k if $g_{k+1} \preceq g_k$,
(iii) $\rho := c_0^\chi$, and $\rho' := c_0^1$ (assuming $\mathrm{vol}_{g_\chi} = c_0^\chi \mathrm{vol}_{g_0}$ and $\mathrm{vol}_{g_1} = c_\chi^1 \mathrm{vol}_{g_\chi}$).

The smooth Cauchy time function χ in (4.1) and (4.2) can be chosen arbitrarily and depending on k in general. The final Møller operator R of g, g' also depends on all the made choices.

Proof of Theorem 4.12 We divide the proof into several steps.

(1)-(3) Let us first prove the thesis for the special case of $g = g_0 \preceq g_1 = g'$, with $V_x^{g_0+} \subset V_x^{g_1+}$ for all $x \in \mathsf{M}$, and specialize the definition of the isomorphisms (4.1) and (4.2) to

$$\mathsf{R}_+ = \mathrm{Id} - \mathsf{G}^+_{c_0^\chi \mathsf{N}_\chi}(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{4.18}$$

$$\mathsf{R}_- = \mathrm{Id} - \mathsf{G}^-_{c_0^1 \mathsf{N}_1}(c_0^1 \mathsf{N}_1 - c_0^\chi \mathsf{N}_\chi) : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}) \tag{4.19}$$

where

$$\mathrm{vol}_{g_\chi} = c_0^\chi \mathrm{vol}_{g_0} \quad \text{and} \quad \mathrm{vol}_{g_1} = c_0^1 \mathrm{vol}_{g_0}$$

It is easy to see that

$$(c_0^\chi \mathsf{N}_\chi)^{\dagger_{g_0}} = c_0^\chi \mathsf{N}_\chi \quad \text{and} \quad (c_0^1 \mathsf{N}_1)^{\dagger_{g_0}} = c_0^1 \mathsf{N}_1 \,. \tag{4.20}$$

Our goal is to prove that the isomorphism $\mathsf{R} := \mathsf{R}_-\mathsf{R}_+ : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ satisfies the thesis.

Per direct inspection, applying the definition of adjoint operator and taking advantage of (4.20), Proposition 3.6, and (3.4), we almost immediately have that

$$\begin{aligned} \mathsf{R}_+^{\dagger_{g_0}} &= \mathrm{Id} - (c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})} \quad \text{and} \\ \mathsf{R}_-^{\dagger_{g_0}} &= \mathrm{Id} - (c_0^1 \mathsf{N}_1 - c_0^\chi \mathsf{N}_\chi)\mathsf{G}^+_{c_0^1 \mathsf{N}_1}|_{\Gamma_c(\mathsf{E})} \,. \end{aligned} \tag{4.21}$$

Again per direct inspection we see that

$$c_0^\chi \mathsf{N}_\chi \mathsf{R}_+ = \mathsf{N}_0 \quad \text{and} \quad c_0^1 \mathsf{N}_1 \mathsf{R}_- = c_0^\chi \mathsf{N}_\chi$$

and thus

$$c_0^1 \mathsf{N}_1 \mathsf{R} = c_0^1 \mathsf{N}_1 \mathsf{R}_- \mathsf{R}_+ = \mathsf{N}_0$$

as wanted.
As we prove below, the following identities are valid

$$\mathsf{R}_+ \mathsf{G}_{\mathsf{N}_0} \mathsf{R}_+^{\dagger_{g_0}} = \mathsf{G}_{c_0^\chi \mathsf{N}_\chi} \quad \text{and} \quad \mathsf{R}_- \mathsf{G}_{c_0^\chi \mathsf{N}_\chi} \mathsf{R}_-^{\dagger_{g_0}} = \mathsf{G}_{c_0^1 \mathsf{N}_1} = \mathsf{G}_{\mathsf{N}_1}(c_0^1)^{-1} \tag{4.22}$$

so that

$$\mathsf{R}_- \mathsf{R}_+ \mathsf{G}_{\mathsf{N}_0} R_+^{\dagger_{g_0}} R_-^{\dagger_{g_0}} = \mathsf{G}_{\mathsf{N}_1}(c_0^1)^{-1}$$

which is equivalent to

$$\mathsf{R}_- \mathsf{R}_+ \mathsf{G}_{\mathsf{N}_0} (\mathsf{R}_- \mathsf{R}_+)^{\dagger_{g_0}} c_0^1 = \mathsf{G}_{\mathsf{N}_1} \,.$$

On the other hand, we have

$$A^{\dagger_{g_0}} c_0^1 = A^{\dagger_{g_0 g_1}}$$

so that

$$\mathsf{R}\mathsf{G}_{\mathsf{N}_0}\mathsf{R}^{\dagger_{g_0 g_1}} = \mathsf{R}_-\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}(\mathsf{R}_-\mathsf{R}_+)^{\dagger_{g_0 g_1}} = \mathsf{G}_{\mathsf{N}_1} .$$

To conclude the proof of (1)-(3) for the case $g = g_0 \preceq g_1 = g'$ we prove (4.22). Since G_{N_0} is defined as the difference of the advanced and retarded Green operators restricted to compact sections, we perform the computation separately for the two operators.

We start from $\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}|_{\Gamma_c(\mathsf{E})}\mathsf{R}_+^{\dagger_{g_0}}$: the adjoint of the Møller operator is defined over $\Gamma_c(\mathsf{E})$ and gives back compactly supported sections, then the advanced Green operator maps a compactly supported section $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ to a solution such that $\mathrm{supp}\,(\mathsf{G}^+_{N_0}\mathfrak{f}) \subset J_0^+(\mathrm{supp}\,(\mathfrak{f})) \subset J_\chi^+(\mathrm{supp}\,(\mathfrak{f}))$, where the last inclusion is due to the crucial hypothesis $g_0 \preceq g_\chi \preceq g_1$. Now since $\mathrm{supp}\,(\mathfrak{f})$ is compact the smooth Cauchy time function t attains a minimum $t_0 \in \mathbb{R}$ therein, so we choose a common smooth Cauchy hypersurface Σ_{t_1} of the foliation induced by t such that $t_1 < t_0$ and deduce that $\mathrm{supp}\,(\mathsf{G}^+_{N_0}\mathfrak{f}) \subset J_\chi^+(\mathrm{supp}\,(\mathfrak{f})) \subset J_\chi^+(\Sigma_{t_1})$ which implies by [3, Lemma 1.2.61] that $\mathsf{G}^+_{N_0}\mathfrak{f} \in \Gamma^\chi_{pc}(\mathsf{E})$.

Omitting the restriction of the domain of the causal propagators from the notation for sake of clarity, but having in mind that it is crucial for the validity of the argument, we obtain:

$$\mathsf{R}_+\mathsf{G}^+_{\mathsf{N}_0} = \mathsf{G}^+_{\mathsf{N}_0} - G^+_{c_0^\chi \mathsf{N}_\chi} c_0^\chi \mathsf{N}_\chi \mathsf{G}^+_{\mathsf{N}_0} + G^+_{c_0^\chi \mathsf{N}_\chi} \mathsf{N}_0 \mathsf{G}^+_{\mathsf{N}_0} = \mathsf{G}^+_{c_0^\chi \mathsf{N}_\chi} .$$

A similar reasoning proves that

$$G^-_{\mathsf{N}_0}\mathsf{R}_+^{\dagger_{g_0}} = \mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi} .$$

where now the restriction of the domains of the causal propagators to compactly supported sections is assumed from the definition of the adjoint. Collecting together the two identities found, we have

$$\mathsf{R}_+\mathsf{G}_{\mathsf{N}_0}\mathsf{R}_+^{\dagger_{g_0}} = (\mathsf{R}_+\mathsf{G}^+_{\mathsf{N}_0} - G^-_{\mathsf{N}_0}\mathsf{R}_+^{\dagger_{g_0}}) + M = \mathsf{G}_{c_0^\chi \mathsf{N}_\chi} + M ,$$

with, where both sides have to be computed on compactly supported sections,

$$M := (\mathrm{Id} - \mathsf{R}_+)\mathsf{G}^-_{\mathsf{N}_0}\mathsf{R}_+^{\dagger_{g_0}} - \mathsf{R}_+\mathsf{G}^+_{\mathsf{N}_0}(\mathrm{Id} - \mathsf{R}_+^{\dagger_{g_0}}) .$$

A lengthy direct evaluation of M using (4.18) and the former in (4.21) shows that $M = 0$. All that establishes the first identity in (4.22), while the latter follows by almost identical facts.

Let us pass to prove (1)-(3) for the case $g_1 \preceq g_0$, with $V_x^{g_1+} \subset V_x^{g_0+}$ for all $x \in \mathsf{M}$. First of all we observe that from the previously treated case ($g_0 \preceq g_1$) we have $c_1^0 \mathsf{N}_0 \mathsf{R}^{-1} = \mathsf{N}_1$ where $c_1^0 = (c_1^0)^{-1}$ and $\mathrm{vol}_{g_0} = c_1^0 \mathrm{vol}_{g_1}$. Interchanging the names of g_0 and g_1, this result implies that (4.15) is true for $g_1 \preceq g_0$ when using R^{-1} in place of R. An analogous procedure proves (4.14) for the case $g_1 \preceq g_0$ from the same equation, already established, valid when $g_0 \preceq g_1$. Also in this case the relevant Møller operator is R^{-1}. To this end, we have only to prove that $(\mathsf{R}^{-1})^{\dagger_{g_1 g_0}}$ exists and coincides to $(\mathsf{R}^{\dagger_{g_0 g_1}})^{-1}$. Indeed, under these assumptions (4.14) implies

$$\mathsf{N}_1 = \mathsf{R}^{-1}\mathsf{N}_0(\mathsf{R}^{\dagger_{g_0 g_1}})^{-1} = \mathsf{R}^{-1}\mathsf{N}_0(\mathsf{R}^{-1})^{\dagger_{g_1 g_0}}$$

which is our thesis when interchanging g_0 and g_1. This fact that $(\mathsf{R}^{-1})^{\dagger_{g_1 g_0}} = (\mathsf{R}^{\dagger_{g_0 g_1}})^{-1}$ actually can be established exploiting (6) in 4.11: R is bijective over $\Gamma(\mathsf{E})$, and admits the adjoint $\mathsf{R}^{\dagger_{g_0 g_1}}$, so if the inverse R^{-1} admits the adjoint $(\mathsf{R}^{-1})^{\dagger_{g_1 g_0}}$, then $\mathsf{R}^{\dagger_{g_0 g_1}}$ is bijective and its inverse is such that $(\mathsf{R}^{\dagger_{g_0 g_1}})^{-1} = (\mathsf{R}^{-1})^{\dagger_{g_1 g_0}}$. Let us prove that R^{-1} admits adjoint (with respect to any metric among g_0, g_χ, g_1 since the existence of the adjoint with respect to one of them trivially implies the existence of the adjoint with respect to the other metrics) to end the proof for the case $g_1 \preceq g_0$. By recalling that $\mathsf{R}^{-1} = \mathsf{R}_+^{-1} \circ \mathsf{R}_-^{-1}$ it suffices to show that R_+^{-1} and R_-^{-1} both admit adjoints. We explicitly give the g_0-adjoint of R_+^{-1} the other case being analogous,

$$(\mathsf{R}_+^{-1})^{\dagger_{g_0}} = \mathrm{Id} + (c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}_{\mathsf{N}_0}^-|_{\Gamma_c(\mathsf{M})} .$$

Let us pass to the proof of (1)-(3) for the general case $g \simeq g'$ also establishing the last part of the thesis. In this case there is a sequence $g_0 = g, g_1, \ldots, g_N = g'$ of globally hyperbolic metrics on M satisfying Definition 2.19 and a corresponding sequence of selfadjoint normally hyperbolic operators N_k with $\mathsf{N}_0 := \mathsf{N}$ and $\mathsf{N}_N := \mathsf{N}'$. (This sequence always exists because, for every globally hyperbolic metric g, there is a normally hyperbolic operator N as proved in the proof of Theorem 4.5. The operator $\widetilde{\mathsf{N}} := \frac{1}{2}(\mathsf{N} + \mathsf{N}^{\dagger_g})$ is simultaneously formally selfadjoint with respect to $\langle \cdot | \cdot \rangle$ and normally hyperbolic.) Taking advantage of the validity of the thesis in the cases $g \preceq g'$ and $g' \preceq g$, using in particular (4) and (6) in Proposition 4.11, one immediately shows that we can build a Møller map for a paracausal deformation of metrics just by defining R as the composition of the various similar operators defined for each copy g_k, g_{k+1} as in (4.16) and (4.17).

(4) If $\mathfrak{f} \in \Gamma_c(\mathsf{E})$,

$$\mathsf{R}^{\dagger_{gg'}}\mathsf{N}'\mathfrak{f} = \mathsf{R}^{\dagger_{gg'}}\mathsf{N}'^{\dagger_{g'}}\mathfrak{f} = (\mathsf{N}'\mathsf{R})^{\dagger_{gg'}}\mathfrak{f} = (c'\mathsf{N})^{\dagger_{gg'}}\mathfrak{f} = \mathsf{N}^{\dagger_g}\mathfrak{f} = \mathsf{N}\mathfrak{f} .$$

(5) It is sufficient to prove the thesis for the case $g = g_0 \preceq g_1 = g'$ and for $\mathsf{R}_+^{\dagger_{g_0}}$. The case of $\mathsf{R}_-^{\dagger_{g_0}}$ is analogous. In the case $g_1 \preceq g_0$ one uses the inverses of the operators above, and all remaining cases are proved just by observing that the considered Møller operators are compositions of the elementary operators $\mathsf{R}_\pm^{\dagger_{g_0}}$ and/or their inverses and

smooth functions used as multiplicative operators. We know that

$$\mathsf{R}_+^{\dagger g_0} = \mathrm{Id} - (c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})} \,.$$

The identity operator has already the requested continuity property so we have only to focus on the second addend using the fact that a linear combination of continuous maps is continuous as well. The map $\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})} : \Gamma_c(\mathsf{E}) \to \Gamma(\mathsf{E})$ is continuous with respect to the natural topologies of the domain and co-domain (see e.g. [3, Corollary 3.6.19]). Since $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)$ is a smooth differential operator $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}|_{\Gamma_c(\mathsf{E})} : \Gamma_c(\mathsf{E}) \to \Gamma(\mathsf{E})$ is still continuous. To conclude the proof it is sufficient to prove that if $\Gamma_c(\mathsf{E})\mathfrak{f}_n \to 0$ in the topology of $\Gamma_c(\mathsf{E})$ and $K \supset supp(\mathfrak{f}_n)$ for all $n \in \mathbb{N}$ is a compact set, then there is a compact set K' such that $K' \supset supp((c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}\mathfrak{f}_n)$ for all $n \in \mathbb{N}$. If $t : \mathsf{M} \to \mathbb{R}$ is the Cauchy temporal function of g_1 used to construct R_+ and R_-, whose level sets $\Sigma_\tau := t^{-1}(\tau)$ are Cauchy hypersurfaces for g_0, g_χ, g_1 and $g_\chi = g_0$ in the past of Σ_{t_0}, then the set $J_-^{(\mathsf{M},g_\chi)}(K) \cap D_+^{(\mathsf{M},g_\chi)}(\Sigma_{t_0})$, which is compact for known properties of globally hyperbolic spacetimes, includes all supports of $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)\mathsf{G}^-_{c_0^\chi \mathsf{N}_\chi}\mathfrak{f}_n$ from the very definition of retarded Green operator also using the fact that $(c_0^\chi \mathsf{N}_\chi - \mathsf{N}_0)$ vanishes in the past of Σ_{t_0}. □

As a byproduct of Theorem 4.12 we get a technical, but important, corollary.

Corollary 4.14 *Consider $g, g', g'' \in \mathcal{GH}_\mathsf{M}$, corresponding formally selfadjoint and normally hyperbolic operators $\mathsf{N}, \mathsf{N}', \mathsf{N}''$ on the $\mathbb{K}$-vector bundle E on M equipped with a non-degenerate, Hermitian, fiberwise metric. Assume that $g \simeq g'$ and $g' \simeq g''$ and suppose that $\mathsf{R}_{gg'}$ is a Møller operator of g, g' and $\mathsf{R}_{g'g}$ is a Møller operator of g', g'' according to (4.16). The following facts are true.*

(1) $\mathsf{R}_{gg'}^{-1}$ *is a Møller operator of g', g.*
(2) $\mathsf{R}_{gg'}\mathsf{R}_{g'g''}$ *is a Møller operator of g', g''.*

Proof It is immediate form the construction of R described at the end of Theorem 4.12 relying on (4.16). □

Remark 4.15 Observe that the construction of the Møller operator R of g_0, g_1, for $g_0 \preceq g_1$, as $\mathsf{R} = \mathsf{R}_-\mathsf{R}_+$ we used several times in this work is nothing but an elementary case of (2). Indeed, in that case, $g_0 \preceq g_\chi \preceq g_1$ and R_+, R_- are, respectively, a Møller operator of g_0, g_χ and g_χ, g_1.

5 Møller ∗-isomorphisms in algebraic quantum field theory

The aim of this section is to investigate the role of the Møller operators at the quantum level. In order to achieve our goal, we will follow the so-called algebraic approach to quantum field theory, see e.g. [4, 5, 11, 36, 58]. In *loc. cit.* the quantization of a free field theory on a (curved) spacetime is interpreted as a two-step procedure:

1. The first consists of the assignment to a physical system of a $*$-algebra of observables which encodes structural properties such as causality, dynamics and canonical commutation relations.
2. The second step calls for the identification of an algebraic state, which is a positive, linear and normalized functional on the algebra of observables.

Using this framework, in this section we shall lift the action of the Møller operators on the algebras of the free quantum fields and then we will pull-back the action of the Møller operators on quantum states, showing that the maps preserve the Hadamard condition with quite weak hypotheses which, in principle, permit an extension of the theory to a perturbative approach.
For a more detailed introduction to the algebraic approach to quantum field theory we refer to [15, 41] for textbook and to [10–12, 17–27, 43–50] for some recent applications.

We begin first by recalling the construction of the free quantum field theories in curved spacetime for the general class of Green hyperbolic operators, which, as we have seen contains the class of the normally hyperbolic operators which are the ones under exam.

Notation 5.1 Through this section, E will always denote an $\mathbb{R}$-vector bundle over a globally hyperbolic spacetime (M, g). In particular, we denote the non-degenerate, symmetric, fiberwise metric by $\langle \cdot \mid \cdot \rangle$.

5.1 Algebras of free quantum fields and the Møller $*$-isomorphism

Given a formally-selfadjoint normally hyperbolic operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ and its causal propagator G, we first define the unital complex $*$-algebra $\mathcal{A}_f$ as the *free complex unital $*$-algebra* with abstract (distinct) generators $\phi(\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{E})$, identity 1, and involution * as discussed in [58]. (As a matter of fact $\mathcal{A}_f$ is made of finite linear complex combinations of 1 and finite products of generic elements $\phi(\mathfrak{f})$ and $\phi(\mathfrak{h})^*$.) Then we define a refined complex unital $*$-algebras by imposing the following relations by the quotient $\mathcal{A} = \mathcal{A}_f/\mathcal{I}$ where $\mathcal{I}$ is the two sided $*$-ideal generated by the following elements of $\mathcal{A}$:

- $\phi(a\mathfrak{f} + b\mathfrak{h}) - a\phi(\mathfrak{f}) - b\phi(\mathfrak{h})\,, \quad \forall a, b \in \mathbb{R} \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$
- $\phi(\mathfrak{f})^* - \phi(\mathfrak{f})\,, \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$
- $\phi(\mathfrak{f})\phi(\mathfrak{h}) - \phi(\mathfrak{h})\phi(\mathfrak{f}) - i\mathsf{G}_N(\mathfrak{f}, \mathfrak{h})1\,, \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E}),$

where we have used the notation

$$\mathsf{G}_\mathsf{N}(\mathfrak{f}, \mathfrak{h}) := \int_\mathsf{M} \langle \mathfrak{f}(x) \mid (\mathsf{G}_\mathsf{N}\mathfrak{h})(x)\rangle \mathrm{vol}_g(x)\,.$$

We have the further possibility to enrich the ideal with the generators:

- $\phi(\mathsf{N}\mathfrak{f})\,, \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$.

Notation 5.2 The equivalence classes $[\phi(\mathfrak{f})]$ will be denoted by $\Phi(\mathfrak{f})$ and they will be called **field operators** (**on-shell** if the ideal is enlarged by including the generators $\phi(\mathsf{N}\mathfrak{f})$), and we use the notation $\mathbb{I}$ for the identity $[1]$ of $\mathcal{A}_f/\mathcal{I}$.

Definition 5.3 Given a formally-selfadjoint normally hyperbolic operator $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ and its causal propagator G, we call **CCR algebra** of the quantum fields Φ, the unital $*$-algebra defined by $\mathcal{A} := \mathcal{A}_f/\mathcal{I}$. The algebra is said to be **on-shell** in case the ideal is enlarged by including the generators $\phi(\mathsf{N}\mathfrak{f})$. Furthermore, we call **observables** of $\mathcal{A}$ any Hermitian element of it.

With the above notation, the following properties are valid

- $\mathbb{R}$-**Linearity**. $\Phi(a\mathfrak{f} + b\mathfrak{h}) = a\Phi(\mathfrak{f}) + b\Phi(\mathfrak{h})\ , \quad \forall a, b \in \mathbb{R} \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$
- **Hermiticity**. $\Phi(\mathfrak{f})^* = \Phi(\mathfrak{f})\ , \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$
- **CCR**. $\Phi(\mathfrak{f})\Phi(\mathfrak{h}) - \Phi(\mathfrak{h})\Phi(\mathfrak{f}) = i\mathsf{G}_N(\mathfrak{f}, \mathfrak{h})\mathbb{I}\ , \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$.

The on-shell field operators also satisfy

- **Equation of motion**. $\Phi(\mathsf{N}\mathfrak{f}) = 0\ , \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$.

Remark 5.4 The idea behind the notation $\Phi(\mathfrak{f})$ is a formal smearing procedure which uses the scalar product

$$\Phi(\mathfrak{f}) = \int_{\mathsf{M}} \langle \Phi(x) \,|\, \mathfrak{f}(x) \rangle \mathrm{vol}_g(x)\ .$$

From this perspective, since N is formally selfadjoint, the identity $\Phi(\mathsf{N}\mathfrak{f}) = 0$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{E})$ has the distributional meaning $\mathsf{N}\Phi = 0$. Alternatively, as explained in [69], one may use a different representation where Φ is viewed as a "generalized section" of the dual bundle E^*. In that case the formal identity $\mathsf{N}\Phi = 0$ corresponding to the equation of motion has to be replaced by $\mathsf{N}^*\Phi = 0$.

Given different normally hyperbolic operators N, N' all the information about causality and dynamics is encoded in the ideal $\mathcal{I}, \mathcal{I}'$. In that case we have two corresponding initial unital $*$-algebras $\mathcal{A}_f$ and $\mathcal{A}'_f$ with respective generators $\phi(\mathfrak{f})$ and $\phi'(\mathfrak{f})$. Though the freely generated algebras are canonically isomorphic, under the unique unital $*$-isomorphism such that $\phi(\mathfrak{f}) \to \phi'(\mathfrak{f})$ for all $\mathfrak{f} \in \Gamma_c(\mathsf{E})$, the quotient algebras are intrinsically different because the CCR are different depending on the choice of the causal propagator G_N or $\mathsf{G}_{\mathsf{N}'}$. However there is an isomorphism between them as soon as a Møller operator exists. Indeed, the existence of the Møller operator discussed in the previous sections can be exploited to define first an isomorphism of the free algebras $\mathcal{A}_f$ and $\mathcal{A}'_f$ since the operator $\mathsf{R}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ is an isomorphism.

Definition 5.5 Let $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ be two formally-selfadjoint (with respect to a fiber metric $\langle \cdot | \cdot \rangle$) normally hyperbolic operators globally hyperbolic spacetimes (M, g) and (M, g'). If $g \simeq g'$, we define an isomorphism $\mathcal{R}_f : \mathcal{A}'_f \to \mathcal{A}_f$ as the unique unital $*$-algebra isomorphism between the said free unital $*$-algebras such that $\mathcal{R}_f(\phi'(\mathfrak{f})) = \phi(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}) \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$. where R is a Møller operator of g, g' (in this order) satisfying Theorem 4.12 and Eq. (4.16).

As we shall see in the next proposition, the isomorphism between freely generated algebras induces an isomorphism of the quotient algebras.

Proposition 5.6 *Let* N *and* N' *be two formally-selfadjoint normally hyperbolic operators acting on the sections of the* $\mathbb{R}$*-vector bundle* E *over* M*, and referred to respective* $g, g' \in \mathcal{GM}_{\mathsf{M}}$.
If $g \simeq g'$ *and* R *is a Møller operator of* g, g' *in the sense of Theorem 4.12 and Eq. (4.16), then the CCR algebras* $\mathcal{A}$ *and* $\mathcal{A}'$ *(possibly both on-shell) respectively associated to* N *and* N' *are isomorphic under the quotient isomorphism* $\mathcal{R} : \mathcal{A}'_f/\mathcal{I}' \to \mathcal{A}_f/\mathcal{I}$ *constructed out of* $\mathcal{R}_f$*, the unique unital* $*$*-algebra isomorphism satisfying* $\mathcal{R}(\Phi'(\mathfrak{f})) = \Phi(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}) \quad \forall \mathfrak{f} \in \Gamma_c(\mathsf{E})$.

Proof To prove the statement it suffices to show that the operator $\mathcal{R}_f$ maps the ideal $\mathcal{I}'$ to the ideal $\mathcal{I}$. Each ideal $\mathcal{I}$ and $\mathcal{I}'$ is the intersection of three (four) ideals corresponding to the requirements of linearity, Hermiticity, CCR (and equation of motion). The fact that $\mathcal{R}_f$ preserves the ideals due to linearity and hermiticity is an immediate consequence of the fact that $\mathcal{R}_f$ is a $*$-algebra homomorphism of the involved freely generates algebras. The ideal arising from the equation of motion condition is preserved due to the first statements of Theorem 4.12 and item (4) therein.
The situation is more delicate regarding the ideal generated by the CCR. Preservation of that ideal is actually an immediate consequence of $\mathcal{R}_f(\mathbb{I}') = \mathbb{I}$ ($\mathcal{R}_f$ is unital by hypothesis) and the structure of CCR together with (4.14):

$$\begin{aligned}
\mathsf{G}_{\mathsf{N}}(\mathfrak{f}', \mathfrak{h}') &= \mathsf{G}_{\mathsf{N}_0}(\mathsf{R}^{\dagger_{gg'}}\mathfrak{f}, \mathsf{R}^{\dagger_{gg'}}\mathfrak{h}) \\
&= \int_{\mathsf{M}} \langle \mathsf{R}^{\dagger_{gg'}}\mathfrak{f} \,|\, \mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}\rangle \mathrm{vol}_{g} \\
&= \int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \mathsf{R}\mathsf{G}_{\mathsf{N}}\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}\rangle \mathrm{vol}_{g'} \\
&= \int_{\mathsf{M}} \langle \mathfrak{f} \,|\, \mathsf{G}_{\mathsf{N}'}\mathfrak{h}\rangle \mathrm{vol}_{g'} \\
&= \mathsf{G}_{\mathsf{N}'}(\mathfrak{f}, \mathfrak{h})\,.
\end{aligned}$$

This concludes our proof. □

Definition 5.7 A unital $*$-isomorphism $\mathcal{R} : \mathcal{A}' \to \mathcal{A}$ defined in Proposition 5.6 out of the Møller operator R of g, g' as in Theorem 4.12 and (4.16) is called **Møller $*$-isomorphism** of the CCR algebras $\mathcal{A}$,$\mathcal{A}'$ (in this order)

5.2 Pull-back of algebraic states through the Møller $*$-isomorphism

As explained in the beginning of this section, the subsequent step in the quantization of a field theory consists in identifying a distinguished state on the $*$-algebra of the quantum fields. The *GNS construction* then guarantees the existence of a representation of the quantum field algebra through, in general unbounded, operators defined over a common dense subspace of some Hilbert space. We will not care about the explicit representation and recall some definitions (see [32] for a general discussion also pointing out several not completely solved standing issues).

Definition 5.8 We call an **(algebraic) state** over a unital $*$-algebra $\mathcal{B}$ a $\mathbb{C}$-linear functional $\omega : \mathcal{B} \to \mathbb{C}$ which is

(i) **Positive** $\omega(a^* a) \geqslant 0 \quad \forall a \in \mathcal{B}$,
(ii) **Normalized** $\omega(\mathbb{I}) = 1$

A generic element of the CCR algebras $\mathcal{A}$ of a quantum field Φ associated to the normally hyperbolic operators discussed before can be written as a finite polynomial of the generators $\Phi(f)$, where the zero grade term is proportional to $\mathbb{I}$, to specify the action of a state it's sufficient to know its action on the monomials, i.e its **n-point functions**

$$\omega_n(\mathfrak{f}_1, .., \mathfrak{f}_n) := \omega(\Phi(\mathfrak{f}_1)...\Phi(\mathfrak{f}_n)) \tag{5.1}$$

The map $\Gamma_c(\mathsf{E}) \times \cdots \times \Gamma_c(\mathsf{E}) \ni (\mathfrak{f}_1, \ldots, \mathfrak{f}_n) \mapsto \omega_n(\mathfrak{f}_1, .., \mathfrak{f}_n)$ can be extended by linearity to the space of finite linear combinations of sections $\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_n \in \Gamma_c(\mathsf{E}^{n\boxtimes})$, where $\mathsf{E}^{n\boxtimes}$ is n-times exterior tensor product of the vector bundle E with itself. If we impose continuity with respect to the usual topology on the space of compactly supported test sections, since the said linear combinations are dense, we can uniquely extend the n-point functions to distributions in $\Gamma'_c(\mathsf{E}^{n\boxtimes})$ we shall hereafter indicate by the same symbol ω_n. It has a formal integral kernel,

$$\omega_n(\mathfrak{f}_1, .., \mathfrak{f}_n) = \int_{\mathsf{M}^n} \widetilde{\omega}_n(x_1, ..., x_n)\mathfrak{f}_1(x_1)...\mathfrak{f}_n(x_n)\mathrm{vol}_{\mathsf{M}^n}(x_1, \ldots, x_n),$$

where

$$\mathrm{vol}_{\mathsf{M}^n}(x_1, \ldots, x_n) := \mathrm{vol}_g(x_1) \otimes \cdots (n \text{ times}) \cdots \otimes \mathrm{vol}_g(x_n)$$

henceforth. Notice that if more strongly $\omega_n \in \Gamma'_c(\mathsf{E}^{n\boxtimes})$, then

$$\omega_n(\mathfrak{h}) = \int_{\mathsf{M}^n} \widetilde{\omega}_n(x_1, ..., x_n)\mathfrak{h}(x_1, \ldots, x_n)\mathrm{vol}_{\mathsf{M}^n}(x_1, \ldots, x_n)$$

is also defined for $\mathfrak{h} \in \Gamma_c(\mathsf{E}^{n\boxtimes})$. The case $n = 2$ is the easiest one. The Schwartz kernel theorem implies $\Gamma_c(\mathsf{E}) \ni \mathfrak{f} \mapsto \omega_2(\mathfrak{h}, \mathfrak{f})$ is (sequentially) continuous at $\mathfrak{f} = 0$ for every fixed $\mathfrak{h} \in \Gamma_c(\mathsf{E})$ if and only if ω_2 continuously extends to a unique distribution we hereafter indicate with the same symbol $\omega_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$.

An important fact (see the comment after [69, Proposition 5.6]) is that, if the CCR algebra $\mathcal{A}$ admits states, then the fiberwise metric $\langle\cdot|\cdot\rangle$ must be positive. In other words, $\langle\cdot|\cdot\rangle$ is a real symmetric positive scalar product. We shall assume it henceforth.

Differently from a free quantum field theory on Minkowski spacetime, where the Poincaré invariant state – known as Minkowski vacuum – might be a natural choice, on a general curved spacetime there might be no choice of a natural state. However there is a class of states, known as quasifree (or Gaussian) states, whose GNS representation mimics the Fock representation of Minkowski vacuum (see e.g. [58]).

Definition 5.9 Let $\mathcal{A}$ be the CCR algebra. A state $\omega : \mathcal{A} \to \mathbb{C}$ is called **quasifree**, or equivalently **Gaussian**, if the following properties for its n-point functions hold

(i) $\omega_n(\mathfrak{f}_1, ..., \mathfrak{f}_n) = 0$, if $n \in \mathbb{N}$ is odd,
(ii) $\omega_{2n}(\mathfrak{f}_1, ..., \mathfrak{f}_{2n}) = \sum_{partitions} \omega(f_{i_1}, f_{i_2}) \cdots \omega(f_{i_{n-1}}, f_{i_n})$, if $n \in \mathbb{N}$ is even,

where "partitions" for even n refers to the class of all possible decompositions of the set $\{1, 2, \ldots, n\}$ into $n/2$ pairwise disjoint subsets of 2 elements $\{i_1, i_2\}$, $\{i_3, i_4\}$, ..., $\{i_n - 1, i_n\}$ with $i_{2k-1} < i_{2k}$ for $k = 1, 2, \ldots, n/2$.

For these states all the information is encoded in the two-point distribution, as one can expect in a free theory. It is not difficult to prove that, for a quasifree state in view of the definition above, $\omega_2 \in \Gamma'_c(\mathsf{E})$ entails that ω_n continuously extends to $\omega_n \in \Gamma'_c(\mathsf{E}^{n\boxtimes})$ obtained, for $n = 2k$, as a linear combination of tensor products of distributions ω_2 and trivial if $n = 2k + 1$.

Remark 5.10 If $\mathcal{A}$ is on-shell, then the n-point function satisfies trivially

$$\omega_n(\mathfrak{f}_1, \ldots, \mathsf{N}\mathfrak{f}_k, \ldots, \mathfrak{f}_n) = 0 \quad \text{for every } k = 1, \ldots, n \text{ and } f_k \in \Gamma_c(\mathsf{M}).$$

as a consequence of (5.1) and $\Phi(\mathsf{N}\mathfrak{f}) = 0$. However it may happen that these identities are valid (for some n) even if the algebra is not on-shell.

In the next proposition, we shall see that the action of the Møller isomorphism $\mathcal{R}$ between CCR-algebras can be pull-backed on the quantum states. Furthermore, the pull-back of a quasifree state is again a quasifree state.

Proposition 5.11 *Let be $g, g' \in \mathcal{GH}_{\mathsf{M}}$, consider the algebras $\mathcal{A}$, $\mathcal{A}'$ respectively associated to formally-selfadjoint normally hyperbolic operators $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ constructed out of g and g' and let $\omega : \mathcal{A} \to \mathbb{C}$ be a state. Assuming that $g \simeq g'$, we define a functional $\omega' : \mathcal{A}' \to \mathbb{C}$ by pull-back through a Møller $*$-isomorphism $\mathcal{R} : \mathcal{A}' \to \mathcal{A}$ of $\mathcal{A}$,$\mathcal{A}'$ as in Definition 5.7, i.e.*

$$\omega' = \omega \circ \mathcal{R}.$$

Then the following statements hold true:

(1) *ω' is a state on $\mathcal{A}'$;*
(2) *$\omega'_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ if and only if $\omega_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$;*
(3) *ω' is quasifree if and only if ω is.*

Proof (1) Linearity is obvious since we are composing linear maps. Normalization follows from 1 in 5.6 and from the fact that ω is normalized. Positivity follows from positivity of ω and the fact that $\mathcal{R}$ preserves the involutions, the products, and is surjective. (2) Since $\omega_2 \in \Gamma'_c(\mathsf{E} \times \mathsf{E})$, then it is $\Gamma_c(\mathsf{E})$-continuous in the right entry (taking values in $\Gamma'_c(\mathsf{E})$ and with respect to the corresponding topology). As a consequence, by composition of continuous functions, if $\mathfrak{h} \in \Gamma_c(\mathsf{E})$ is given,

$$\Gamma_c(\mathsf{E}) \ni \mathfrak{f} \mapsto \omega'_2(\mathfrak{h}, \mathfrak{f}) = \omega_2(\mathsf{R}^{\dagger_{gg'}}\mathfrak{h}, \mathsf{R}^{\dagger_{gg'}}\mathfrak{f})$$

is $\Gamma_c(\mathsf{E})$-continuous as well because $\mathsf{R}^{\dagger_{gg'}} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ is continuous in the $\Gamma_c(\mathsf{E})$ topology in domain and co-domain for (5) of Theorem 4.12. In other words $\Gamma_c(\mathsf{E}) \ni \mathfrak{f} \mapsto \omega'_2(\cdot, \mathfrak{f}) \in \Gamma'_c(\mathsf{E})$ is continuous. We conclude that $\omega'_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ due to the Schwartz kernel theorem. The result can be reversed swapping the role of the states and the metrics, noticing that $\omega = \omega' \circ \mathcal{R}^{-1}$ where $\mathcal{R}^{-1}$ is also a Møller $*$-isomorphism, the one constructed out of R^{-1} which is, in turn, a Møller operator associated to the pair g', g in this order in view of Corollary 4.14.
(3) The proof is immediate and follows by construction. □

5.3 Møller preservation of the microlocal spectrum condition for off-shell algebras

It is widely accepted that, among all possible (quasifree) states, the physical ones are required to satisfy the so-called Hadamard condition. The reasons for this choice are manifold: For example, it implies the finiteness of the quantum fluctuations of the expectation value of every observable and it allows us to construct Wick polynomials [55, 57] and other observables, as the stress-m energy tensor, relevant in semi-classical quantum gravity following a covariant scheme [54, 61], encompassing a locally covariant ultraviolet renormalization [56] (see also [58] for a recent pedagogical review). These states have been also employed, e.g. (the following list is far from being exhaustive) in the study of the Blackhole radiation [24, 42, 59, 64], in cosmological models [21, 23] and other applications to spacetime models [34, 35, 62], and to study energy quantum inequalities [33]. For later convenience, we decided to present the Hadamard condition as a microlocal condition on the wave-front set of the two-point distribution [67, 68] instead of the equivalent geometric version based on the Hadamard parametrix [1, 60, 63]. Let's briefly sketch what they are and why they are useful.

From now on we adopt the definitions of wave-front set $WF(\psi)$ of distribution ψ on $\mathbb{R}$-vector bundles equipped with a non-degenerate, symmetric, fiberwise metric[3] as in [69].

We shall use some very known definitions and results of microlocal analysis applied to distributions of $\Gamma'_c(\mathsf{F})$ where F is a $\mathbb{K}$-vector bundle, $\mathsf{F} = \mathsf{E} \boxtimes \mathsf{E}$ for instance (see [69] for details). In particular,

- $\psi \in \Gamma'_c(\mathsf{F})$ is a smooth section of the dual bundle F^*, indicated with the same symbol $\psi \in \Gamma(\mathsf{F}^*)$, if and only if $WF(\psi) = \varnothing$.
- We say that $\psi, \psi' \in \Gamma'_c(\mathsf{F})$ are **equal mod** C^∞, if $\psi - \psi' \in \Gamma(\mathsf{F}^*)$.
- Let us assume that $\mathsf{F} = \mathsf{E} \boxtimes \mathsf{E}$ where E is equipped with a non-degenerate, symmetric (Hermitian if $\mathbb{K} = \mathbb{C}$), fiberwise metric and let $\mathsf{P} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ be a formally selfadjoint smooth differential operator. We say that $\nu \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ is a **bi-solution** $\mathsf{P}\mathfrak{f} = 0$ **mod** C^∞, if there exist $\varphi, \varphi' \in \Gamma(\mathsf{F}^*)$ such that

$$\nu(\mathsf{P}\mathfrak{f} \otimes \mathfrak{h}) = \int_{\mathsf{M}} \langle \psi, \mathfrak{f} \otimes \mathfrak{h} \rangle \operatorname{vol}_g \otimes \operatorname{vol}_g ,$$

[3] The authors of [69] more generally study the case of a complex Hermitian vector bundle endowed with an antilinear involution (here the identity bundle map) there indicated by Γ.

$$\nu(\mathfrak{f} \otimes \mathsf{P}\mathfrak{h}) = \int_{\mathsf{M}} \langle \psi', \mathfrak{f} \otimes \mathfrak{h} \rangle \operatorname{vol}_g \otimes \operatorname{vol}_g \quad \forall \mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E}) .$$

We are in a position to state the definition of *microe local spectrum condition* and *Hadamard state*. Below, $\sim_{\parallel}$ is the equivalence relation in $T^*\mathsf{M}^2\setminus\{0\}$ such that $(x, k_x) \sim_{\parallel} (y, k_y)$ if there is a *null geodesic* passing through $x, y \in \mathsf{M}$ and the geodesic parallely transports the co-tangent vector to that geodesic $k_x \in T^*_x\mathsf{M}$ into the co-tangent vector to that geodesic $k_y \in T^*_y\mathsf{M}$. Finally, $k_x \triangleright 0$ means that the covector k_x is *future directed*.

Definition 5.12 With $\mathcal{A}$ as in Definition 5.9, a state $\omega : \mathcal{A} \to \mathbb{C}$ is called a **Hadamard state** if $\omega_2 \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ and the following **microlocal spectrum condition** is valid

$$WF(\omega_2) = \{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2\setminus\{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \triangleright 0\} . \tag{5.2}$$

More generally, a distribution $\nu \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ is said to be of **Hadamard type** if its wave-front set $WF(\nu)$ is the right-hand side of (5.2).

Remark 5.13 (1) Notice that $(x, k_x; x, -k_x) \in WF(\nu)$ for every future directed light-like covector $k_x \in T^*_x\mathsf{M}$ if $\nu \in \Gamma'_c(\mathsf{E} \boxtimes \mathsf{E})$ is of Hadamard type.
(2) It is possible to prove that a fiberwise scalar product $\langle\cdot|\cdot\rangle$ must be necessarily positive if $\mathcal{A}$ admits quasifree Hadamard states as proved in the comment after [69, Proposition 5.6]. We henceforth assume that $\langle\cdot|\cdot\rangle$ is positive.

The theorem below, which is the second main result of this paper, shows that the Hadamard condition is preserved under the pull-back along the Møller isomorphism.

Theorem 5.14 *Let* E *be an* $\mathbb{R}$*-vector bundle over smooth manifold* M *and denote with* $\langle\cdot\,|\,\cdot\rangle$ *positive, symmetric, fiberwise metric. Let be* $g, g' \in \mathcal{GH}_{\mathsf{M}}$*, consider the corresponding formally-selfadjoint normally hyperbolic operators* $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *and refer to the associated CCR algebras* $\mathcal{A}$ *and* $\mathcal{A}'$ *(off-shell in general). Finally, suppose that* $g \simeq g'$*.*
$\omega : \mathcal{A} \to \mathbb{C}$ *is a quasifree Hadamard state, if and only if*

$$\omega' := \omega \circ \mathcal{R} : \mathcal{A}' \to \mathbb{C} ,$$

constructed out of a Møller $*$*-isomorphism* $\mathcal{R}$ *of* $\mathcal{A}$, $\mathcal{A}'$*, is a quasifree Hadamard state of* $\mathcal{A}'$*.*

Remark 5.15 We stress that it is not required that the algebras are on-shell nor that the relevant two-point functions satisfy the equation of motion with respect to the corresponding normally hyperbolic operators.

The rest of this section is devoted to prove Theorem 5.14, a refinement of it stated in the last Theorem 5.19, and a proof of the existence of Hadamard states based on our formalism.

Our first observation is the following.

Lemma 5.16 *Let* $\mathsf{S} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *be any of the four operators* R_+, R_-, R_+^{-1}, R_-^{-1}, *defined as in (4.1), (4.2), (4.3), (4.4), and* $U \subset \mathbb{R}^m$ *an open set.*
If $\{\mathfrak{f}_z\}_{z\in U} \subset \Gamma(\mathsf{E})$ *is such that* $\mathsf{M} \times U \ni (x, z) \mapsto \mathfrak{f}_z(x)$ *is jointly smooth, then*

$$\mathsf{M} \times U \ni (x, z) \mapsto (\mathsf{S}\mathfrak{f}_z)(x)$$

is jointly smooth as well.

Proof We consider the case of R_+, the remaining three instances having a similar proof. What we have to prove is that $\mathsf{M} \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is smooth under the said hypotheses. Let us first consider the case where there is compact $K \subset \mathsf{M}$ such that $\mathrm{supp}(\mathfrak{f}_z) \subset K$ for all $z \in U$. In this case, defining $F(x, z) := (\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z(x)$, the projection $\pi : \mathrm{supp}(F) \ni (x, z) \mapsto z \in U$ is proper[4] and this fact will be used shortly. Interpreting $\mathsf{G}^+_{\rho\mathsf{N}_\chi} : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E})$ and thus as a Schwartz kernel, we can compute the wavefront set of the map $\mathsf{M} \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ viewed as the distributional kernel of the composition of the kernel $\mathsf{G}^+_{\rho\mathsf{N}_\chi}(x, y)$ and the smooth kernel $F(y, z)$. We know that (see, e.g. [58] for the scalar case, the vector case being analogous)

$$WF(\mathsf{G}^+_{\rho\mathsf{N}_\chi}) = \Big\{(x, k_x; y, -k_y) \in T^*\mathsf{M}^2\backslash\{0\} \mid (x, k_x) \sim_\parallel (y, k_y),$$
$$x \in J_+(y) \text{ or } k_x = k_y\ , x = y\Big\}$$

whereas, since F is jointly smooth,

$$WF(F) = \varnothing\,.$$

The known composition rules of wavefront sets of Schwartz kernels, which use in particular the fact that the projection π above is proper (in [58, Theorem 5.3.14] which is valid also in the vector field case), immediately yields

$$WF(\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F) \subset \varnothing\,.$$

It being $WF(\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F) = \varnothing$, we conclude that $\mathsf{M} \times U \ni (x, z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is a smooth function as desired.

Let us pass to consider the generic jointly smooth family $\{\mathfrak{f}_z\}_{z\in U} \subset \Gamma(\mathsf{E})$ without restrictions on the supports. First of all, we observe that $\mathfrak{f}'_z(x) := ((\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f})(x)$ is past compact by construction for every $z \in U$, because its support is contained in the future of Σ_{t_0} referring to the construction of N_χ. According to the proof of [3, Theorem 3.6.7], if $\mathfrak{h}$ is past compact, $x_0 \in \mathsf{M}$, and $A \supset \mathrm{supp}(\mathfrak{h}) \cap J_-(x_0)$ is an open relatively

[4] If $C \subset U$ is compact and thus closed, then $\pi^{-1}(C)$ is a closed set, π being continuous, contained in the compact $K \times C$, so that $\pi^{-1}(C)$ is compact as well.

compact set, for every compactly supported smooth function $s_A \in C_c^\infty(\mathsf{M};[0,1])$ such that $s_A(x) = 1$ if $x \in A$, it holds

$$(\mathsf{G}^+_{\rho\mathsf{N}_\chi}\mathfrak{h})(x_0) = (\mathsf{G}^+_{\rho\mathsf{N}_\chi} s_A\mathfrak{h})(x_0)\,.$$

We want to apply this identity for $\mathfrak{h} = \mathfrak{f}_z$. Take $t' < t_0$. Given $x_0 \in \mathsf{M}$ we can always define $A := I_-(\widetilde{x}_0) \cap I^+(\Sigma_{t'})$ where $\widetilde{x}_0 \in I_+(x_0)$ [5]. With this choice, A does not depend on $z \in U$ and the same A can be used for x varying in an open neighborhood A' of x_0, since $I_-(\widetilde{x}_0)$ is open. We conclude that, if $(x,z) \in A' \times U$, then

$$\begin{aligned}\left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x) = (\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F)(x,z) \quad \text{where}\\ F(x,z) = s_A(x)(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z(x)\,.\end{aligned} \tag{5.3}$$

In this case $K := \mathrm{supp}(s_A)$ includes all the supports of the maps $\mathsf{M} \ni x \mapsto F(x,z)$ for every $z \in U$. The first part of the proof is therefore valid for the map $\mathsf{M} \times U \ni (x,z) \mapsto (\mathsf{G}^+_{\rho\mathsf{N}_\chi} \circ F)$ which must be jointly smooth as a consequence. In particular, its restriction $A' \times U \ni (x,z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is jointly smooth as well. Since A' can be taken as a neighborhood of every point in M and $z \in U$ is arbitrary, from (5.3) the whole function $\mathsf{M} \times U \ni (x,z) \mapsto \left(\mathsf{G}^+_{\rho\mathsf{N}_\chi}(\rho\mathsf{N}_\chi - \mathsf{N}_0)\mathfrak{f}_z\right)(x)$ is jointly smooth. □

Relying on Lemma 5.16, we can notice the following.

Lemma 5.17 *Consider a pair of globally hyperbolic metrics g_0 and g_χ on M as in Proposition 4.1 and corresponding normally hyperbolic operators $\mathsf{N}_0, \mathsf{N}_\chi : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ for the $\mathbb{R}$-vector bundle on M equipped with the positive symmetric fiberwise metric $\langle\cdot|\cdot\rangle$.*
Then, $\nu_0 \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$ is a bisolution of $\mathsf{N}_0\mathfrak{f} = 0$ mod C^∞ if and only if $\nu_\chi := \nu \circ \mathsf{R}_+^{\dagger_{g_0 g_\chi}} \otimes \mathsf{R}_+^{\dagger_{g_0 g_\chi}}$ is a bisolution of $\mathsf{N}_\chi\mathfrak{f} = 0$ mod C^∞, where R_+ is defined in (4.18).

Proof We start by stressing that, as already noticed, in view of the known continuity properties of $\mathsf{R}_+^{\dagger_{g_0 g_\chi}}$ and its inverse and using Schwartz' kernel theorem, $\nu_0 \in \Gamma_c'(\mathsf{E}\boxtimes\mathsf{E})$ if and only if $\nu_\chi \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$.
We pass to prove that if ν_0 is a bisolution mod C^∞, then ν_χ is a bisolution mod C^∞, referring to the corresponding operators. Let us hence suppose that $\nu_0(\mathsf{N}_0\mathfrak{f},\mathfrak{h}) = \psi(\mathfrak{f} \otimes \mathfrak{h})$ and $\nu_0(\mathfrak{f}, \mathsf{N}_0\mathfrak{h}) = \psi'(\mathfrak{f} \otimes \mathfrak{h})$ for some smooth sections $\psi, \psi' \in \Gamma((\mathsf{E} \boxtimes \mathsf{E})^*)$ and all $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$. The identity

$$\mathsf{R}_+^{\dagger_{g_0 g_\chi}} \mathsf{N}_\chi|_{\Gamma_c(\mathsf{E})} = \mathsf{N}_0|_{\Gamma_c(\mathsf{E})}\,,$$

[5] Notice that since the spacetime is globally hyperbolic, $\overline{I_\pm(x)} = J_\pm(x)$ and $\overline{I_-(\widetilde{x}_0) \cap I^+(\Sigma_{t'})} = J_-(\widetilde{x}_0) \cap J^+(\Sigma_{t'})$ which is compact because $\Sigma_{t'}$ is a smooth spacelike Cauchy surface.

immediately implies that, if $\varphi(x, y) := c_0^\chi(x)c_0^\chi(y)\psi(x, y)$, $\varphi'(x, y) := c_0^\chi(x) c_0^\chi(y)\psi'(x, y)$,

$$\nu_\chi(\mathsf{N}_\chi \mathfrak{f}, \mathfrak{h}) = \int_{\mathsf{M}\times\mathsf{M}} \langle \varphi(x, y), (\mathrm{Id} \otimes \mathsf{R}_+^{\dagger_{g_0 g_\chi}}(\mathfrak{f} \otimes \mathfrak{h}))(x, y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

and

$$\nu_\chi(\mathfrak{f}, \mathsf{N}_\chi \mathfrak{h}) = \int_{\mathsf{M}\times\mathsf{M}} \langle \varphi'(x, y), (\mathsf{R}_+^{\dagger_{g_0 g_\chi}} \otimes \mathrm{Id}(\mathfrak{f} \otimes \mathfrak{h}))(x, y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)$$

The proof ends if proving that there are smooth sections $\varphi_1, \varphi_1' \in \Gamma((\mathsf{E} \boxtimes \mathsf{E})^*)$, such that

$$\begin{aligned}&\int_{\mathsf{M}\times\mathsf{M}} \langle \varphi, \mathrm{Id} \otimes \mathsf{R}_+^{\dagger_{g_0 g_\chi}}(\mathfrak{f} \otimes \mathfrak{h})\rangle \mathrm{vol}_{g_\chi} \otimes \mathrm{vol}_{g_\chi}\\ &= \int_{\mathsf{M}\times\mathsf{M}} \langle \varphi_1(x, y), \mathfrak{f}(x)\mathfrak{h}(y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathsf{M}\times\mathsf{M}} \langle \varphi, \mathsf{R}_+^{\dagger_{g_0 g_\chi}} \otimes \mathrm{Id}(\mathfrak{f} \otimes \mathfrak{h})\rangle \mathrm{vol}_{g_\chi} \otimes \mathrm{vol}_{g_\chi}\\ &= \int_{\mathsf{M}\times\mathsf{M}} \langle \varphi_1'(x, y), \mathfrak{f}(x)\mathfrak{h}(y)\rangle \mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)\end{aligned}$$

for every pair $\mathfrak{f}, \mathfrak{h} \in \Gamma_c(\mathsf{E})$. We prove the former identity only, the second one having an identical proof. To this end we pass to the index notation (also assuming Einstein's summing convention), the indices being referred to the fiber in the local trivialization,

$$\begin{aligned}&\int_{\mathsf{M}\times\mathsf{M}} \langle \varphi, \mathrm{Id} \otimes \mathsf{R}_+^{\dagger_{g_0 g_\chi}}(\mathfrak{f} \otimes \mathfrak{h})\rangle \mathrm{vol}_{g_\chi} \otimes \mathrm{vol}_{g_\chi}\\ &= \sum_{j,k} \int_{\mathsf{M}\times\mathsf{M}} \chi_j(x)\chi_k'(y)\varphi_{ab}(x, y)(\mathsf{R}_+^{\dagger_{g_0 g_\chi}}\mathfrak{f})^a(x)\mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(x) \otimes \mathrm{vol}_{g_\chi}(y)\end{aligned}$$

Above $\{\chi_j\}_{j\in J}$ and $\{\chi_k'\}_{k\in K}$ are partitions of the unity of M subordinated to corresponding locally finite coverings of M supporting local trivializations, whose fiber coordinates are labeled by ^{a} and ^{b}. Moreover, only a finite number of indices $(j, k) \in J \times K$ give a contribution to the sum, uniformly in x, y, in view of the compactness of the supports of $\mathfrak{f}$ and $\mathfrak{h}$ and the local finiteness of the used coverings. The right-hand side can be rearranged to

$$
\begin{aligned}
&= \sum_{k\in K}\int_{\mathsf{M}} \chi'_k(y)\left(\sum_{j\in J}\int_{\mathsf{M}} \chi_j(x)\varphi_{ab}(x,y)(\mathsf{R}_+^{\dagger_{g_0g_\chi}}\mathfrak{f})^a(x)\right)\mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(y)\\
&= \sum_{k\in K}\int_{\mathsf{M}} \chi'_k(y)\left(\int_{\mathsf{M}} \langle\varphi'_{yb}(x)\mid(\mathsf{R}_+^{\dagger_{g_0g_\chi}}\mathfrak{f})(x)\rangle\mathrm{vol}_{g_\chi}(x)\right)\mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(y)\\
&= \int_{\mathsf{M}}\sum_{k\in K} \chi'_k(y)\left(\int_{\mathsf{M}} \langle(\mathsf{R}_+\varphi'_{yb})(x)\mid\mathfrak{f}(x)\rangle\mathrm{vol}_{g_0}(x)\right)\mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(y)\\
&= \sum_{j,k}\int_{\mathsf{M}\times\mathsf{M}} \chi_j(x)\chi'_k(y)c_0^\chi(x)(\mathsf{R}_+\varphi'_{yb})_a(x)\mathfrak{f}^a(x)\mathfrak{h}^b(y)\mathrm{vol}_{g_\chi}(x)\otimes\mathrm{vol}_{g_\chi}(y)\\
&= \int_{\mathsf{M}\times\mathsf{M}} \langle\varphi_1(x,y),\mathfrak{f}\otimes\mathfrak{h}(x,y)\rangle\mathrm{vol}_{g_\chi}(x)\otimes\mathrm{vol}_{g_\chi}(y)\,,
\end{aligned}
$$

where we have locally defined ${\varphi'}^a_{yb}(x) := \xi^{ac}(x)\varphi_{cb}(x,y)$, with $\xi^{ab}(x)$ being the inverse fiber metric at $x \in M$ in any considered local trivialization. Above, $\varphi_{1\,ab}(x,y) := c_0^\chi(x)(\mathsf{R}_+\varphi'_{yb})_a(x)$ is the candidate section of $(\mathsf{E}\boxtimes\mathsf{E})^*$ we were looking for, represented in local coordinates of the atlas of the said trivialization. That section is smooth, i.e., $\varphi_1 \in \Gamma((\mathsf{E}\boxtimes\mathsf{E})^*)$ as desired. Indeed, the maps $\mathsf{M}\times U_k \ni (x,y) \mapsto \varphi'_{yb}(x)$ define a family of sections of $\Gamma(\mathsf{E})$ parametrized by $y \in U_k$ for every given $b \in \{1,\ldots,N\}$, where $U_k \subset \mathsf{M}$ is the projection onto M of the domain of the considered local trivialization. This family is jointly smooth in x, y as established in Lemma 5.16.

The converse statement, that ν_0 is a bisolution mod C^∞ if ν_χ is, can be proved with the same procedure simply replacing R_+ with $(\mathsf{R}_+)^{-1}$ and using Lemma 5.16 again. □

Before giving the proof of Theorem 5.14, we need a final lemma, which shows that any Hadamard distribution whose antisymmetric part is given by the causal propagator of a normally hyperbolic system N is actually a bisolution of N itself modulo smooth errors.

Lemma 5.18 *Let* $\mathsf{N} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *be a formally selfadjoint normally hyperbolic operator and suppose that* $\nu \in \Gamma'_c(\mathsf{E}\boxtimes\mathsf{E})$ *is of Hadamard type and satisfies*

$$\nu(x,y) - \nu(y,x) = i\mathsf{G}_{\mathsf{N}}(x,y) \quad mod \quad C^\infty$$

where $G_{\mathsf{N}}(x,y)$ *is the distributional kernel of the causal propagator* G_{N}. *In this case* ν *is a bisolution of* $\mathsf{N}\mathfrak{f} = 0$ *mod* C^∞.

Proof The proof is a straightforward re-adaptation of the proof appearing in the *Note added in proof* of [67]. □

We are finally in a position to prove Theorem 5.14.

Proof of Theorem 5.14 We have only to prove that ω' is Hadamard if and only if ω is, since the other preservation property has been already proved in (4) of Proposition 5.11. If $g_0 \simeq g_1$, there is a sequence of globally hyperbolic metrics $g'_0 = g_0, g'_1, \ldots, g'_N =$

g_1 such that either $g'_k \preceq g'_{k+1}$ or $g'_{k+1} \preceq g'_k$ and the future cones satisfy a corresponding inclusion. The Møller operator $\mathcal{R}$ of $\mathcal{A}$, $\mathcal{A}'$ is obtained as the composition of the Møller operators $\mathcal{R}_k$ of the formally-selfadjoint normally hyperbolic operators N'_k, N'_{k+1} associated to the pairs g'_k, g'_{k+1}:

$$\mathcal{R} := \mathcal{R}'_0\mathcal{R}'_1 \cdots \mathcal{R}'_{N-1}$$

as in the proof of Theorems 4.5, 4.12 and (4.16). The thesis is demonstrated if we prove that, with obvious notation, ω^{k+1} is Hadamard if and only if ω^k is. So in principle we have to prove the thesis only for a pair of metrics g_0, g_1 with the two cases $g_0 \preceq g_1$ and $g_1 \preceq g_0$. Actually the latter is a consequence of the former, using the fact that a Møller $*$-isomorphisms are bijective and that a Møller operator of the second case is the inverse operator of a Møller operator of the first case in accordance to Corollary 4.14. In summary, the proof is over if establishing the thesis for the case $g = g_0 \preceq g_1 = g'$ and we shall concentrate on that case only in the rest of the proof.

Recalling by (4.12) and (4.11) that $\mathsf{R}^{\dagger_{g_0 g_1}} = \mathsf{R}_+^{\dagger_{g_0 g_\chi}} \mathsf{R}_-^{\dagger_{g_\chi g_1}}$, we write

$$\omega_2^1(\mathfrak{f}_1, \mathfrak{f}_2) = \omega_2^0(\mathsf{R}^{\dagger_{g_0 g_1}}\mathfrak{f}_1, \mathsf{R}^{\dagger_{g_0 g_1}}\mathfrak{f}_2) = \omega_2^0(\mathsf{R}_+^{\dagger_{g_0 g_\chi}} \mathsf{R}_-^{\dagger_{g_\chi g_1}}\mathfrak{f}_1, \mathsf{R}_+^{\dagger_{g_0 g_\chi}} \mathsf{R}_-^{\dagger_{g_\chi g_1}}\mathfrak{f}_2).$$

To analyze the wave-front set of this bidistribution, we split again the operation in two steps. First we define a pull-back state on the algebra $\mathcal{A}_\chi$ of quantum fields defined for the formally-selfadjoint normally hyperbolic operator N_χ, i.e a normally hyperbolic operator on (M, g_χ). This intermediate pull-back states reads

$$\omega_2^\chi(\mathfrak{f}_1, \mathfrak{f}_2) = \omega_2^0(\mathsf{R}_+^{\dagger_{g_0 g_\chi}}\mathfrak{f}_1, \mathsf{R}_+^{\dagger_{g_0 g_\chi}}\mathfrak{f}_2). \tag{5.4}$$

We intend to prove that $\omega_2^\chi \in \Gamma'_c(\mathsf{E})$ is of Hadamard type if and only ω_2^0 is. Notice that both two-point functions have antisymmetric parts that coincide with $i\mathsf{G}_{\mathsf{N}_\chi}$ and $i\mathsf{G}_{\mathsf{N}_0}$, respectively, in view of the CCRs of the respective algebras. If $\omega_2^0 \in \Gamma'_c(\mathsf{E})$ is of Hadamard type, then it is a bisolution of $\mathsf{N}_0\mathfrak{f} = 0 \bmod C^\infty$ in view of Lemma 5.18. The same argument proves that, if $\omega_2^\chi \in \Gamma'_c(\mathsf{E})$ is of Hadamard type, then it is a bisolution of $\mathsf{N}_\chi\mathfrak{f} = 0 \bmod C^\infty$ due to 5.18. Applying Lemma 5.17 to both cases we have that,

(a) $\omega_2^0 \in \Gamma'_c(\mathsf{E})$ of Hadamard type implies that ω_2^0 is a bisolution $\mathsf{N}_0\mathfrak{f} = 0 \bmod C^\infty$ and ω_2^χ is a a bisolution of $\mathsf{N}_\chi\mathfrak{f} = 0 \bmod C^\infty$;
(b) $\omega_2^\chi \in \Gamma'_c(\mathsf{E})$ of Hadamard type implies that ω_2^χ is a bisolution $\mathsf{N}_\chi\mathfrak{f} = 0 \bmod C^\infty$ and ω_2^0 is a a bisolution of $\mathsf{N}_0\mathfrak{f} = 0 \bmod C^\infty$.

We are now in a position to apply the Hadamard singularity propagation theorem. Consider the smooth Cauchy time function t in common with g_0 and g_χ, such that $\chi(x) = 0$ if $t(x) < t_0$. As a preparatory remark we notice that $\mathsf{R}_+^{\dagger_{g_0 g_\chi}}\mathfrak{f} = \mathfrak{f}$ from (4.21) when the support of $\mathfrak{f}$ stays in the past of the Cauchy surface $\Sigma_{t_0} = t^{-1}(t_0)$. In that region $g_0 = g_\chi$ by definition of g_χ. Finally due to (5.4),

$$\omega_2^\chi(\mathfrak{f}, \mathfrak{h}) = \omega_2^0(\mathfrak{f}, \mathfrak{h}) \quad \text{if } t(\mathrm{supp}(\mathfrak{f})) < t_0, t(\mathrm{supp}(\mathfrak{h})) < t_0$$

Hence, in particular, ω_2^χ is of Hadamard type when the supports of the test functions are taken in that region if and only if ω_2^0 is of Hadamard type when the supports of the test functions are taken there. More precisely, it happens when the supports of the arguments $\mathfrak{f}, \mathfrak{h}$ are taken in a (globally hyperbolic) neighborhood of a Cauchy surface (for both metrics!) $\Sigma_\tau := t^{-1}(\tau)$ with $\tau < t_0$ between two similar slices. Since both distributions are bisolutions of the respective equation of motion mod C^∞ and the operators are normally hyperbolic, the theorem of propagation of Hadamard singularity (see, e.g., Theorem 5.3.17 in [58][6].) implies that ω_2^χ and ω_2^0 are of Hadamard type everywhere in (M, g_χ) and (M, g_0), respectively.
A similar reasoning shows that $\omega_2^1 \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$, with

$$\omega_2^1(\mathfrak{f}_1, \mathfrak{f}_2) = \omega_2^\chi(\mathsf{R}_-^{\dagger_{g_\chi g_1}} \mathfrak{f}_1, \mathsf{R}_-^{\dagger_{g_\chi g_1}} \mathfrak{f}_2) ,$$

is Hadamard on (M, g_1) if and only if ω_2^χ is on (M, g_χ). Combining the two results we have that $\omega' = \omega^1$ is Hadamard on $(\mathsf{M}, g' = g_1)$ if and only if $\omega = \omega^0$ is Hadamard on $(\mathsf{M}, g = g_0)$ concluding the proof. □

We are now in the position to prove our last result.

Theorem 5.19 *Let* E *be an* $\mathbb{R}$*-vector bundle on a smooth manifold* M *equipped with a positive, symmetric, fiberwise metric* $\langle \cdot | \cdot \rangle$. *Let* $g, g' \in \mathcal{GH}_\mathsf{M}$, *consider the corresponding formally-selfadjoint normally hyperbolic operators* $\mathsf{N}, \mathsf{N}' : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *and refer to the associated CCR algebras* $\mathcal{A}$ *and* $\mathcal{A}'$.
Let $\nu \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$ *be of Hadamard type and satisfy*

$$\nu(x, y) - \nu(y, x) = i\mathsf{G}_\mathsf{N}(x, y) \quad mod \quad C^\infty ,$$

$\mathsf{G}_\mathsf{N}(x, y)$ *being the distributional Kernel of* G_N.
Assuming $g \simeq g'$, *let us define*

$$\nu' := \nu \circ \mathsf{R}^{\dagger_{gg'}} \otimes \mathsf{R}^{\dagger_{gg'}} ,$$

for a Møller operator $\mathsf{R} : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ *of* g, g'. *Then the following facts are true.*

(i) ν *and* ν' *are bisolutions mod* C^∞ *of the field equations defined by* N *and* N' *respectively,*
(ii) $\nu' \in \Gamma_c'(\mathsf{E} \boxtimes \mathsf{E})$,
(iii) $\nu'(x, y) - \nu'(y, x) = i\mathsf{G}_{\mathsf{N}'}(x, y)$ *mod* C^∞,
(iv) ν' *is of Hadamard type.*

Proof Since we never exploited the fact that ω is positive, nor the fact that the antisymmetric part of its two points function is *exactly* the causal propagator, nor the fact that the relevant algebras of fields are *on-shell* (i.e., the equation of motion are satisfied

[6] The proof which appears there is valid for the on-shell algebra of the scalar real Klein-Gordon field, but the passage to normally hyperbolic operators also weakening the bisolution requirement to bisolution mod C^∞ is immediate, since it is based on standard Hörmander theorems about singularity propagation which works mod C^∞. See the comments in Remark 5.3.18 in [58].

by the two-point functions), we can use the same arguments as in the proof of the previous theorem to conclude. □

We conclude this section with the following straightforward result of existence of Hadamard quasifree states which apparently does not use the Hadamard singularity propagation argument (actually this argument was used in the proof of Theorem 5.14).

Corollary 5.20 *Let* (M, g) *be a globally hyperbolic spacetime,* N *be a formally-selfadjoint normally hyperbolic operator acting on the sections of the* $\mathbb{R}$*-vector bundle* E *over* M *and refer to the associated CCR algebras* $\mathcal{A}$*. Then there exists a Hadamard state on* $\mathcal{A}$*.*

Proof It is well-known [38] that, in a globally hyperbolic ultrastatic spacetime, the (unique) CCR quasifree ground state which is invariant under the preferred Killing time is Hadamard. Hence, combining Corollary 2.23 with Theorem 5.14 we can conclude. □

6 Conclusion and future outlook

We conclude this paper by discussing some open issues which are raised in this paper and we leave for future works.

Paracausally related metrics. One of the key ingredients in the realization of the Møller operator is the introduction of the new geometric notion that we have called *paracausal deformation*. In particular, we have seen in Sect. 4, that for any given two paracausally deformed metrics, there exists a Møller operator which realizes an algebraic equivalence between corresponding free quantum field theories defined on different curved spacetimes (on a given manifold). Therefore it seems very natural and important to classify the class of metrics which are not paracausally related in relation to the existence of inequivalent quantum field theories. We have already seen in Sect. 2.3, that if (M, g) and (M, g') admit a common foliation of Cauchy hypersurfaces, then g and g' are paracausally related. As suggested by part (3) of the Example 2.21, the claim that if (M, g) and (M, g') are Cauchy-compact globally hyperbolic spacetimes then g and g' are paracausally related, does not sound reasonable. The idea behind is that g and g' in part (3) of the Example 2.21 have somehow 'different time-orientation'. Since the time-orientation depends on the metric on M, we have to provide a criteria to translate the requirement that g and g' are both 'future-directed' in some sense. Keeping in mind what said above, a conjecture which urges to be proved or disproved is the following one (maybe adding some further hypothesis).

Conjecture 6.1 Let t and t' be Cauchy temporal functions for globally hyperbolic spacetimes (M, g) and (M, g'). Finally denote with $\langle \cdot, \cdot \rangle$ the natural paring between $\mathsf{T}^*\mathsf{M}$ and TM. Then

$$g \simeq g' \qquad \text{if and only if} \qquad \langle \partial_t, dt' \rangle > 0 \ \text{ and } \ \langle \partial_{t'}, dt \rangle > 0,$$

where ∂_t (resp. $\partial_{t'}$) is the dual of dt (resp. dt') with respect to g (resp. g').

Remark 6.2 The requirement $\langle \partial_t, dt' \rangle > 0$ implies that the integral curve $\gamma = \gamma(t)$ of ∂_t on (M, g') satisfies $t'(\gamma(t_2)) > t'(\gamma(t_1))$ if $t_2 > t_1$. This requirement is weaker than assuming the ∂_t is timelike and future-directed for g'. The reason why we also impose $\langle \partial_{t'}, dt \rangle > 0$ is that being paracausally related is an equivalence relation in $\mathcal{GH}_{\mathsf{M}}$.

A similar conjecture has to be established or disproved in the class of asymptotically flat spacetimes.

Conjecture 6.3 If (M, g) and (M, g') are asymptotically flat globally hyperbolic spacetimes then g and g' are paracausally related if $\langle \partial_t, dt' \rangle > 0$ and $\langle \partial_{t'}, dt \rangle > 0$.

Remark 6.4 Differently from Conjecture 6.1, the conjecture above would not provide a characterization of asymptotically flat spacetimes, since part (3) of the Example 2.21 provide a counterexample to the 'necessary' part of the statement. Indeed, the Minkowski metrics presented in Example 2.21 do not satisfy $\langle dt', \partial_t \rangle > 0$.

If the conjectures are proved to hold, the results would suggest that free quantum field theories on globally hyperbolic spacetimes are more sensitive to the topology of the manifold M with respect to the metric g endowing M. In particular, the physics encoded in the quasifree states ω for a quantum field propagating on curved spacetime $(\mathbb{R}^4, g)$ can be found in the pull-backed state $\omega' = \omega \circ \mathcal{R}$ on Minkowski spacetime. Loosely speaking, the quantum effects due to the interaction between the quantum field and the classical gravitational potential can be thought as special observable in spacetimes without gravitational interaction.

Homotopical properties of Møller operators Another issue which deserves to be investigated is the dependence of the Møller operator R of g, g' and the associated $*$-isomorphism from the finite sequence of globally hyperbolic metrics joining g, g' in the sense of paracausal relationship. It is clear from the construction of R that the natural "composition" of sequences corresponds to the composition of operators.

Question 6.5 Is there some homotopical notion in the space of globally hyperbolic metrics which is reflected in the space of Møller operators?

Pull-back of ground and KMS states through the Møller $*$-isomorphism. Another issue we have faced is the lack of control on the action of the group of $*$-automorphism induced by the isometry group of the spacetime M on ω. Let us remark, that the study of invariant states is a well-established research topic (*cf.* [7, 8]). Indeed, the type of factor can be inferred by analyzing which and how many states are invariant. From a more physical perspective instead, invariant states can represent equilibrium states in statistical mechanics e.g. KMS-states or ground states. The previous remark leads us to the following open question:

Question 6.6 Under which conditions it is possible to perform an adiabatic limit, namely when is $\lim_{\chi \to 1} \omega_1$ well-defined?

A priori we expect that there is no positive answer in all possible scenarios, since it is known that certain free-field theories, e.g., the massless and minimally coupled (scalar or Dirac) field on four-dimensional de Sitter spacetime, do not possess a ground state, even though their massive counterpart does. (Notice that this is not a no-go Theorem, but at least an indication that, in these situations, the map $\omega \to \omega \circ \mathcal{R}$ cannot be expected to preserve the ground state property.)

A partial investigation in this direction has been carried on in [20, 28] for the case of a scalar field theory on globally hyperbolic spacetimes with empty boundary. In this situation it has been shown that, under suitable hypotheses the adiabatic limit can be performed preserving the invariance property under time translation but spoiling in general the ground state or KMS property.

Møller $*$-isomorphism in perturbative AQFT. We conclude with the following observation. The results of Sect. 5 are valid also for off-shell algebras as well as for distribution of Hadamard type. Therefore, it could be possible to extend the action of the Møller operator also on the algebra of extended observables in a perspective of *deformation product quantization* (see for instance Section 2 of [27]), which include, e.g., the Wick polynomials of the underlying fields. Wick polynomial and time-ordered products of Wick polynomials are the building blocks for perturbative renormalization of quantum fields, both in Minkowski spacetime and in curved spacetime, where the metric is considered as a given external classical field. Although of utmost physical relevance, these formal operators as the *stress energy operator* do not belong to the algebra of observables generated by the smoothly smeared field operators (operator-valued distributions). This is because they correspond to products of distributions at a given point and this notion is not well-defined in general. The popular and perhaps most effective procedure to eliminate the short-distance divergences consists of simply subtracting a suitable Hadamard distribution. This procedure is systematically embodied in a product deformation quantization procedure which relies on a suitable set of functionals with a specific wavefront set. The following observation leads to the following conjecture:

Conjecture 6.7 Let $\mathcal{A}_0, \mathcal{A}'_0$ be the algebra of observables of the globally hyperbolic spacetimes (M, g) and (M, g') and $\mathcal{R}_0$ a Møller $*$-isomorphism of them. If $\mathcal{A}, \mathcal{A}'$ are corresponding extended algebras of observable (which include the Wick polynomials etc.) and $g \simeq g'$, then $\mathcal{R}_0$ extends to a (Møller) $*$-isomorphism $\mathcal{R} : \mathcal{A} \to \mathcal{A}'$.

Møller operators and gauge theory. Last but not least, all our results concern vector-valued fields of a vector bundle E over M. The fiber metric on the bundle does not depend on the globally hyperbolic metrics g chosen on M. A natural extension of the formalism would be the inclusion of that g-dependence in the fiber metric. This extension would allow to encompass the case of a *Proca field* and, possibly the case of the electromagnetic field, though issues with gauge invariance and gauge fixing are expected to pop out.

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