STRUCTURE RESULTS FOR THE INTEGRAL SET OF A SUBMANIFOLD WITH RESPECT TO A NON-INTEGRABLE EXTERIOR DIFFERENTIAL SYSTEM

SILVANO DELLADIO

ABSTRACT. Let $\mathcal{N}$ and $\mathcal{O}$ be, respectively, a C^2 manifold and an arbitrary family of C^1 differential forms on $\mathcal{N}$. Moreover assume that

(0.1) *For all $y \in \mathcal{N}$ and for all M-dimensional integral elements Σ of $\mathcal{O}$ at y, there is $\omega \in \mathcal{O}$ such that $(d\omega)_y|_\Sigma \neq 0$.*

If $\mathcal{M}$ is any M-dimensional C^1 imbedded submanifold of $\mathcal{N}$, then we expect that condition (0.1) prevents the existence of interior points in the integral subset of $\mathcal{M}$ with respect to $\mathcal{O}$

$$\mathcal{I}(\mathcal{M},\mathcal{O}) := \bigcap_{\omega\in\mathcal{O}} \{\omega|_\mathcal{M} = 0\}.$$

Actually, the structure of $\mathcal{I}(\mathcal{M},\mathcal{O})$ can be described much more precisely by invoking the notion of superdensity. Indeed, under the previous hypotheses, the following structure result holds: *There are no $(M+1)$-density points of $\mathcal{I}(\mathcal{M},\mathcal{O})$ relative to $\mathcal{M}$.*
If we now consider $\mathcal{M}$ in the smaller class of C^2 imbedded submanifolds of $\mathcal{N}$, then it becomes natural to expect a further "slimming" of $\mathcal{I}(\mathcal{M},\mathcal{O})$. Indeed we have the following second structure result: *If $\mathcal{O}$ is countable, then $\mathcal{I}(\mathcal{M},\mathcal{O})$ is an $(M-1)$-dimensional C^1 rectifiable subset of $\mathcal{M}$.* These results are immediate corollaries of two general structure theorems, which are the main goal of this paper. Applications in the context of non-involutive distributions and in the context of Pfaff problem are provided.

1. Introduction

1.1. **The main goal.** Let $\mathcal{N}$ and $\mathcal{O}$ be, respectively, a C^2 manifold and an arbitrary family of C^1 differential forms on $\mathcal{N}$. Moreover, let $V_M(\mathcal{O})_y$ denote the set of all M-dimensional integral elements of $\mathcal{O}$ at $y \in \mathcal{N}$ (see [BCG91, Y92] and Section 2.2 below) and assume that

(1.1)
For all $y \in \mathcal{N}$ and $\Sigma \in V_M(\mathcal{O})_y$ there is $\omega \in \mathcal{O}$ such that $(d\omega)_y|_\Sigma \neq 0$.

If $\mathcal{M}$ is any M-dimensional C^1 imbedded submanifold of $\mathcal{N}$ (see Section 2.2 below), then we expect that condition (1.1) prevents the existence of

2020 *Mathematics Subject Classification.* 58A15, 58A10, 28A75, 58A30;
Key words and phrases. Non-integrable exterior differential systems, Non-solvable systems of PDEs, Non-involutive distributions, Frobenius theorem, Pfaff problem.

interior points in the integral subset of $\mathcal{M}$ with respect to $\mathcal{O}$, i.e.,

$$\mathcal{I}(\mathcal{M},\mathcal{O}) := \bigcap_{\omega\in\mathcal{O}} \{\omega|_{\mathcal{M}} = 0\}.$$

Actually, the structure of $\mathcal{I}(\mathcal{M},\mathcal{O})$ can be described much more precisely by invoking the notion of superdensity (see Subsection 2.5). Indeed, under the previous hypotheses, the following structure result holds: *There are no $(M+1)$-density points of $\mathcal{I}(\mathcal{M},\mathcal{O})$ relative to $\mathcal{M}$* (see Corollary 3.4). As a side note to this main result, we observe that the existence of C^1 imbedded submanifolds $\mathcal{M}$ of $\mathcal{N}$ for which the measure of $\mathcal{I}(\mathcal{M},\mathcal{O})$ is large could be a somewhat common event: a very notable example occurs when $\mathcal{O}$ is a family of defining 1-forms for a non-involutive distribution, see [AMS22] and some references therein (in addition to Example 3.1).
If we now consider $\mathcal{M}$ in the smaller class of C^2 imbedded submanifolds of $\mathcal{N}$, then it becomes natural to expect a further "slimming" of $\mathcal{I}(\mathcal{M},\mathcal{O})$. Indeed the following property holds: *If $\mathcal{O}$ is countable, then $\mathcal{I}(\mathcal{M},\mathcal{O})$ is an $(M-1)$-dimensional C^1 rectifiable subset of $\mathcal{M}$* (see Corollary 4.1).

The results just mentioned are immediate corollaries of the following two general theorems, which are the main goal of this paper.

First structure theorem (see Theorem 3.2). *Let $\mathcal{N}$ and $\mathcal{O}$ be, respectively, a C^2 manifold and a family of C^1 differential forms on $\mathcal{N}$. Then, for every M-dimensional C^1 imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$, one has*

$$\mathcal{I}(\mathcal{M},\mathcal{O})^{(M+1)} \subset \bigcap_{\omega\in\mathcal{O}} \{(d\omega)|_{\mathcal{M}} = 0\}$$

where $\mathcal{I}(\mathcal{M},\mathcal{O})^{(M+1)}$ denotes the set of all $(M+1)$-density points of $\mathcal{I}(\mathcal{M},\mathcal{O})$ relative to $\mathcal{M}$.

Second structure theorem (see Theorem 4.2). *Let $\mathcal{N}$ and $\mathcal{O}$ be, respectively, a C^{k+1} manifold and a countable family of C^k differential forms on $\mathcal{N}$ (with $k \geq 1$). Moreover, let condition* (1.1) *be satisfied and define*

$$\rho_M := \min_{\substack{y\in\mathcal{N} \\ \Sigma\in V_M(\mathcal{O})_y}} \max\{m \geq 1 | ((d\omega)^m)_y|_\Sigma \neq 0 \text{ for some } \omega\in\mathcal{O}\}.$$

Then $\rho_M \geq 1$ and this property holds: For every M-dimensional C^{k+1} imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$, the set $\mathcal{I}(\mathcal{M},\mathcal{O})$ is covered by countably many C^k imbedded submanifolds of $\mathcal{M}$ of dimension less or equal than $M-\rho_M$ (in particular, $\mathcal{I}(\mathcal{M},\mathcal{O})$ is an $(M-\rho_M)$-dimensional C^k rectifiable subset of $\mathcal{M}$).

1.2. Application in the context of non-involutive distributions.

This theory can be applied very easily to describe the structure of the tangency $\mathcal{T}(\mathcal{M},\mathcal{D})$ of an M-dimensional imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$ with respect to a non-involutive distribution $\mathcal{D}$ of rank M on $\mathcal{N}$. In this case, in fact, if $\mathcal{O}$ is a family of linearly independent defining 1-forms for $\mathcal{D}$ in $\mathcal{N}$, then we have $\mathcal{I}(\mathcal{M},\mathcal{O}) = \mathcal{T}(\mathcal{M},\mathcal{D})$ and these facts follow immediately from the first and second structure theorems:

- *Let $\mathcal{N}$ be of class C^2, $\mathcal{D}$ be of class C^1 and $\mathcal{M}$ be of class C^1. Then $\mathcal{T}(\mathcal{M},\mathcal{D})$ has no $(M+1)$-density points relative to $\mathcal{M}$* (see Corollary 5.1);
- *Let $\mathcal{N}$ be of class C^{k+1}, $\mathcal{D}$ be of class C^k and $\mathcal{M}$ be of class C^{k+1} (with $k \geq 1$). Then $\mathcal{T}(\mathcal{M},\mathcal{D})$ is an $(M-\rho_M)$-dimensional C^k rectifiable subset of $\mathcal{M}$* (see Corollary 5.2).

The following further application in this context concerns the dimensional estimate [BPR11, Theorem 1.3]. Let $\mathcal{D}$ be a C^k distribution of rank M on an N-dimensional C^{k+1} manifold $\mathcal{N}$, with $N > M \geq 1$ (and $k \geq 1$). Moreover consider a family $\omega_1, \dots, \omega_{N-M}$ of defining 1-forms for $\mathcal{D}$ in $\mathcal{N}$ and, for $m = 1, \dots, N$, let $\mathcal{A}_m$ denote the set of all points $P \in \mathcal{M}$ such that there exists an m-dimensional vector subspace X of $T_P\mathcal{N}$ satisfying

$$(\omega_h)_P|_X = 0, \quad (d\omega_h)_P|_X = 0$$

for all $h = 1, \dots, N-M$. It is obvious that the family $\mathcal{A}_1, \dots, \mathcal{A}_N$ is decreasing, with $\mathcal{A}_1 = \mathcal{N}$ and $\mathcal{A}_{M+1} = \emptyset$ (see [BPR11, Section 1]). In Theorem 6.1 we get a new very easy proof of the following result proved in [D18]: *If $1 \leq m \leq N$ and $\mathcal{U}$ denotes the image of any injective C^{k+1} immersion $\varphi : U \subset \mathbb{R}^N \to \mathcal{N}$, then the set $\mathcal{R}_m := \mathcal{T}(\mathcal{U},\mathcal{D}) \setminus \mathcal{A}_{m+1}$ is covered by finitely many C^k imbedded submanifolds of $\mathcal{N}$ of dimension less or equal than m.* From this result we easily obtain the structure formula for the tangency set

$$\mathcal{T}(\mathcal{U},\mathcal{D}) \subset \bigcup_{m=1}^{M} [\mathcal{R}_m \cap (\mathcal{A}_m \setminus \mathcal{A}_{m+1})],$$

hence the dimensional estimate [BPR11, Theorem 1.3] trivially follows:

$$\dim \mathcal{T}(\mathcal{U},\mathcal{D}) \leq \max_{1 \leq m \leq M} \{\min\{\dim(\mathcal{A}_m \setminus \mathcal{A}_{m+1}), m\}\}$$

where dim denotes the Hausdorff dimension. In the special case when $\mathcal{D}$ is non-involutive (so that $\mathcal{A}_M(\mathcal{D}) = \emptyset$), we find

$$\mathcal{T}(\mathcal{U},\mathcal{D}) \subset \bigcup_{m=1}^{M-1} [\mathcal{R}_m \cap (\mathcal{A}_m \setminus \mathcal{A}_{m+1})]$$

and

$$\dim \mathcal{T}(\mathcal{U},\mathcal{D}) \leq \max_{1 \leq m \leq M-1} \{\min\{\dim(\mathcal{A}_m \setminus \mathcal{A}_{m+1}), m\}\}.$$

1.3. **Application in the context of Pfaff problem.** Given integers N and m, with $3 \le 2m+1 \le N$, let us consider an N-dimensional C^{k+1} manifold $\mathcal{N}$ and a C^k differential 1-form ω on an open set $\mathcal{U} \subset \mathcal{N}$ (with $k \ge 1$) satisfying the following property: *There exists a C^{k+1} diffeomorphism $\varphi : U \subset \mathbb{R}^N \to \mathcal{U}$ such that*

$$\varphi^*\omega = \pi_m := dx_1 + x_2 dx_3 + \cdots + x_{2m} dx_{2m+1}$$

where $x_1, \dots, x_N$ are the coordinates of $\mathbb{R}^N$. For this type of differential forms we have

$$(d\omega)^m \wedge \omega \ne 0, \quad (d\omega)^{m+1} = 0 \qquad \text{(everywhere in } \mathcal{U}) \tag{1.2}$$

hence, in particular, ω has constant rank m. Using Corollary 4.6, we easily obtain the following couple of facts about situations in which the condition (1.2), necessary for the above property to be true, is violated:

- Let $(d\omega)^{m+1} \ne 0$ everywhere. Then, for every C^{k+1} diffeomorphism $\varphi : U \subset \mathbb{R}^N \to \mathcal{U}$, the set
$$\{x \in U | (\varphi^*\omega)_x = (\pi_m)_x\} \tag{1.3}$$
is covered by finitely many C^k imbedded submanifolds of U of dimension less or equal than $N-1$ (see Theorem 7.2).
- Assume that there exist a positive integer m and a C^{k+1} diffeomorphism $\varphi : U \subset \mathbb{R}^N \to \mathcal{U}$ such that $2(2m+1) \le N$ and
$$(d\varphi^*\omega)^{m+1} \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{2m+1} \ne 0, \quad (d\varphi^*\omega)^{m+2} = 0$$
everywhere in U. Then the set (1.3) is covered by finitely many C^k imbedded submanifolds of U of dimension less or equal than $N - 2m - 1$ (see Theorem 7.3).

2. Basic notation and notions

2.1. **Basic notation.** The coordinates of $\mathbb{R}^M$ are denoted by $(x_1, \dots, x_M)$ so that $dx_1, \dots, dx_M$ is the standard basis of the dual space of $\mathbb{R}^M$. We set $D_i := \partial/\partial x_i$. If p is any positive integer not exceeding M, then $I(M,p)$ is the family of integer multi-indices $\alpha = (\alpha_1, \dots, \alpha_p)$ such that $1 \le \alpha_1 < \cdots < \alpha_p \le M$. Given a generic map $\Phi : A \to \mathbb{R}^n$ and $v \in \mathbb{R}^n$, we set for simplicity $\{\Phi = v\} := \{P \in A \,|\, \Phi(P) = v\}$. Let $\mathcal{L}^M$ and $\mathcal{H}^s$ denote, respectively, the Lebesgue measure and the s-dimensional Hausdorff measure on $\mathbb{R}^M$. The open ball of radius r centered at $x \in \mathbb{R}^M$ will be denoted by $B_r(x)$.

2.2. **Manifolds, differential forms, integral elements.** In relation to this topic, we will adopt the notations commonly used in the main bibliographic references (see, e.g., [L13, N85]). We report here, quickly, just a few of them.

A C^k manifold is a topological manifold (that is a locally Euclidean second-countable Hausdorff space) endowed with a C^k structure.

Let $\mathcal{M}$ be an M-dimensional C^k manifold. Then a C^k differential p-form (or C^k differential form of degree p) on $\mathcal{M}$ is a map $\omega : \mathcal{M} \to \Lambda^p T^*\mathcal{M}$ with the following property: If

$$\sum_{\alpha \in I(M,p)} f_\alpha dx_\alpha \qquad (dx_\alpha := dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_p})$$

is any local representation of ω, then f_α is of class C^k. For any given $P \in \mathcal{M}$, we will use the standard notation ω_P instead of $\omega(P)$. As we did for real-valued maps, let us set $\{\omega = 0\} := \{P \in \mathcal{M} \,|\, \omega_P = 0\}$ for simplicity.

Let $\mathcal{N}$ be a C^k manifold. Then a set $\mathcal{M} \subset \mathcal{N}$ is said to be a C^k imbedded submanifold of $\mathcal{N}$ if $\mathcal{M}$ is a manifold without boundary in the subspace topology, endowed with a C^k structure with respect to which the inclusion map $\mathcal{M} \hookrightarrow \mathcal{N}$ is a C^k imbedding.

Let $\mathcal{M}$ be a C^k imbedded submanifold of a C^k manifold $\mathcal{N}$ and let $\iota : \mathcal{M} \hookrightarrow \mathcal{N}$ be the inclusion map. If ω is a C^{k-1} differential p-form on $\mathcal{N}$, then the C^{k-1} differential p-form $\iota^*\omega$ (i.e., the restriction of ω to $\mathcal{M}$) will be denoted by $\omega|_{\mathcal{M}}$.

If ω is a C^1 differential p-form and m is a positive integer, then ω^m will denote the m-fold wedge product $\omega \wedge \cdots \wedge \omega$. The rank of ω at x is the integer r (depending on x) such that

$$(d\omega)^r \wedge \omega \neq 0, \quad (d\omega)^{r+1} \wedge \omega = 0$$

at x, see [BCG91, Ch. II, Sect. 3].

Let m be a positive integer satisfying $2m+1 \leq M$ and consider the following C^∞ differential 1- form on $\mathbb{R}^M$

$$\pi_m := dx_1 + x_2 dx_3 + \cdots + x_{2m} dx_{2m+1}.$$

Then a standard computation shows that

$$(d\pi_m)^m = m!\, dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{2m+1}, \quad (d\pi_m)^{m+1} = 0 \tag{2.1}$$

and

$$(d\pi_m)^m \wedge \pi_m = m!\, dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{2m+1}. \tag{2.2}$$

In particular, π_m has constant rank m.

Let us recall that a C^1 Riemannian manifold $(\mathcal{N}, g)$ with the associated Riemannian distance function is a metric space whose topology coincides to the original manifold topology, see [L13, Theorem 13.29]. Hence one can define the corresponding s-dimensional Hausdorff measure $\mathcal{H}^s_g$, see [F69, Section 2.10.2], [T06, Chapter 12]. The open metric ball of radius r centered at $P \in \mathcal{N}$ will be denoted by $\mathcal{B}_g(P, r)$.

Let $\mathcal{N}$ and $\mathcal{O}$ be, respectively, a C^1 manifold and a family of C^1 differential forms on $\mathcal{N}$. Moreover, let $y \in \mathcal{N}$.Then a linear subspace Σ of $T_y\mathcal{N}$ is said to be an integral element of $\mathcal{O}$ at y if $\omega_y|_\Sigma = 0$ for all $\omega \in \mathcal{O}$ (see [BCG91, Y92]). If M is a positive integer (not exceeding the dimension of $\mathcal{N}$), then the set of all M-dimensional integral elements of $\mathcal{O}$ at y is denoted by $V_M(\mathcal{O})_y$.

2.3. **Distributions.** Let $\mathcal{N}$ and $\mathcal{M}$ be, respectively, a C^1 manifold and a C^1 submanifold of $\mathcal{N}$. Set for simplicity

$$M := \dim \mathcal{M}, \quad N := \dim \mathcal{N}, \quad H := N - M$$

and consider a C^1 distribution $\mathcal{D}$ of rank M on $\mathcal{N}$. Then we define the tangency set of $\mathcal{M}$ with respect to $\mathcal{D}$ as

$$\mathcal{T}(\mathcal{M}, \mathcal{D}) := \{y \in \mathcal{M} \mid T_y\mathcal{M} = \mathcal{D}_y\}.$$

If $H \geq 1$ and $\omega_1, \ldots, \omega_H$ is a family of linearly independent C^1 differential 1-forms defining $\mathcal{D}$ on an open set $\mathcal{V} \subset \mathcal{N}$, then one has

$$\mathcal{V} \cap \mathcal{T}(\mathcal{M}, \mathcal{D}) = \mathcal{T}(\mathcal{V} \cap \mathcal{M}, \mathcal{D}) = \bigcap_{h=1}^{H} \{\omega_h|_\mathcal{M} = 0\}, \tag{2.3}$$

that is

$$\mathcal{V} \cap \mathcal{T}(\mathcal{M}, \mathcal{D}) = \{y \in \mathcal{V} \cap \mathcal{M} \mid T_y\mathcal{M} \in V_M(\mathcal{O})_y\}, \text{ with } \mathcal{O} := \{\omega_h\}_{h=1}^H.$$

Recall that the distribution $\mathcal{D}$ is said to be involutive at $y \in \mathcal{V}$ if

$$(d\omega_h)_y|_{\mathcal{D}_y} = 0, \text{ for all } h = 1, \ldots, H.$$

One can verify that such a definition of involutivity does not depend on the choice of the local defining 1-forms. The distribution $\mathcal{D}$ is called involutive (in $\mathcal{N}$) if it is involutive at every $y \in \mathcal{N}$. Also recall that, if $\mathcal{N}$ and $\mathcal{D}$ are of class C^k, then a non-empty C^k imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{T}(\mathcal{M}, \mathcal{D}) = \mathcal{M}$ is called a C^k integral manifold of $\mathcal{D}$. As a celebrated theorem by Frobenius establishes, the involutivity of $\mathcal{D}$ is a necessary and sufficient condition for the existence of an integral manifold of $\mathcal{D}$ through every point of $\mathcal{N}$. This topic is extensively covered in many books of differential geometry, for example in [CCL99, Sect. 3.2], [L13, Ch. 19], [N85, Sect. 2.11].

2.4. **Basic geometric measure theory on manifolds: rectifiable sets and Hausdorff dimension.** The notions of rectifiable set and Hausdorff measure on a Riemannian manifold are essentially obvious to anyone familiar with classical geometric measure theory (see [F69, Mo88, S84, KP08]). Perhaps for this reason we are not able to provide satisfactory basic bibliographic information on this subject. It is therefore for the convenience of the reader not familiar with GMT that we devote this subsection to a very quick presentation of these notions.

First of all, let us recall the following well-known properties of the Hausdorff measure $\mathcal{H}^s_g$ on a C^1 Riemannian manifold $(\mathcal{N}, g)$:

- If $s = \dim \mathcal{N}$, then $\mathcal{H}^s_g(B) = V_g(B)$ for all Borel sets $B \subset \mathcal{N}$, where V_g denotes the standard volume form of $(\mathcal{N}, g)$, see [F69, Section 3.2.46], [T06, Proposition 12.6].
- If $\mathcal{M}$ is a C^1 imbedded submanifold of $\mathcal{N}$ and $g_{\mathcal{M}}$ denotes the induced metric, then one has $\mathcal{H}^s_{g_{\mathcal{M}}}(B) = \mathcal{H}^s_g(B)$ for all Borel sets $B \subset \mathcal{M}$, see [T06, Proposition 12.7].
- If g denotes the standard Euclidean metric on $\mathbb{R}^N$, then one obviously has $\mathcal{H}^s_g = \mathcal{H}^s$. In particular, $\mathcal{H}^N_g$ is the N-dimensional Lebesgue measure.

By using again [F69, Section 3.2.46] it is not difficult to prove the following result.

Proposition 2.1. *Let k, n, N be positive integers, with $n \le N$. Moreover let $\mathcal{N}$ be an N-dimensional C^k manifold and $\mathcal{R}$ be a subset of $\mathcal{N}$. The following are equivalent:*

(1) For every $y \in \mathcal{R}$ there is a C^k chart $(\mathcal{W}, \Phi)$ of $\mathcal{N}$ such that $y \in \mathcal{W}$ and $R := \Phi(\mathcal{R} \cap \mathcal{W})$ is a $(\mathcal{H}^n, n)$-rectifiable set of class C^k, namely R is a Borel subset of $\mathbb{R}^N$ and there is a countable family $S_1, S_2, \dots$ of n-dimensional C^k imbedded submanifolds of $\mathbb{R}^N$ such that

$$\mathcal{H}^n\big(R \setminus \bigcup_i S_i\big) = 0.$$

(2) $\mathcal{R}$ is a Borel subset of $\mathcal{N}$ and there is a countable family $\mathcal{S}_1, \mathcal{S}_2, \dots$ of n-dimensional C^k imbedded submanifolds of $\mathcal{N}$ such that

$$\mathcal{H}^n_g\big(\mathcal{R} \setminus \bigcup_i \mathcal{S}_i\big) = 0$$

for every C^1 Riemannian metric g on $\mathcal{N}$.

(3) *$\mathcal{R}$ is a Borel subset of $\mathcal{N}$ and there is a countable family $\mathcal{S}_1, \mathcal{S}_2, \dots$ of n-dimensional C^k imbedded submanifolds of $\mathcal{N}$ such that*

$$\mathcal{H}^n_g\big(\mathcal{R} \setminus \bigcup_i \mathcal{S}_i\big) = 0$$

for a certain C^1 Riemannian metric g on $\mathcal{N}$.

Definition 2.2. If $\mathcal{R}$ satisfies any or, equivalently, all of the conditions of Proposition 2.1, then we say that $\mathcal{R}$ is an n-dimensional C^k rectifiable subset of $\mathcal{N}$, see [AS94, A94].

Remark 2.3. Let $\mathcal{M}$ be a C^k imbedded submanifold of a C^k manifold $\mathcal{N}$ and let $\mathcal{R}$ be an m-dimensional C^k rectifiable subset of $\mathcal{M}$ ($m \leq \dim \mathcal{M}$). Then $\mathcal{R}$ is also an m-dimensional C^k rectifiable subset of $\mathcal{N}$.

Another property which follows readily from [F69, Section 3.2.46] is this one.

Proposition 2.4. *Let $\mathcal{N}$ be a C^1 manifold, $\mathcal{E} \subset \mathcal{N}$ and $s \in [0, +\infty)$. The following are equivalent:*

(1) *For every C^1 chart $(\mathcal{W}, \Phi)$ of $\mathcal{N}$, one has $\mathcal{H}^s(\Phi(\mathcal{W} \cap \mathcal{E})) = 0$.*

(2) *For every C^1 Riemannian metric g on $\mathcal{N}$, one has $\mathcal{H}^s_g(\mathcal{E}) = 0$.*

(3) *There exists a C^1 Riemannian metric g on $\mathcal{N}$ such that $\mathcal{H}^s_g(\mathcal{E}) = 0$.*

Definition 2.5. Let $I(\mathcal{E})$ denote the set of $s \in [0, +\infty)$ satisfying any or, equivalently, all of the conditions of Proposition 2.4. Then the Hausdorff dimension in $\mathcal{N}$ of $\mathcal{E}$ is defined as the number $\dim_{\mathcal{N}} \mathcal{E} := \inf I(\mathcal{E})$ (note: $I(\mathcal{E})$ is a right half-line containing $(\dim \mathcal{N}, +\infty)$, so that $\dim_{\mathcal{N}} \mathcal{E} \leq \dim \mathcal{N}$).

Remark 2.6. The Hausdorff dimension in a C^1 Riemannian manifold $\mathcal{N}$ has the following properties, see [Ma95, Chapter 4]:

- It is monotone, that is: $\dim_{\mathcal{N}} \mathcal{E} \leq \dim_{\mathcal{N}} \mathcal{F}$, whenever $\mathcal{E} \subset \mathcal{F} \subset \mathcal{N}$;
- It is stable with respect to countable unions. This means that, for any given countable family $\mathcal{E}_1, \mathcal{E}_2, \dots$ of subsets of $\mathcal{N}$, one has

$$\dim_{\mathcal{N}} \bigcup_i \mathcal{E}_i = \sup_i (\dim_{\mathcal{N}} \mathcal{E}_i).$$

Remark 2.7. Let $\mathcal{M}$ be a C^1 imbedded submanifold of a C^1 manifold $\mathcal{N}$ and let $\mathcal{E}$ be a subset of $\mathcal{M}$. Then $\dim_{\mathcal{N}} \mathcal{E} = \dim_{\mathcal{M}} \mathcal{E}$. By virtue of this remark, we can avoid distinguishing the notations $\dim_{\mathcal{M}} \mathcal{E}$ and $\dim_{\mathcal{N}} \mathcal{E}$, identifying them (if desired) with the simpler notation $\dim \mathcal{E}$.

2.5. **Superdensity.** Also the following proposition is a consequence of [F69, Section 3.2.46], see [D19c, Proposition 3.3].

Proposition 2.8. *Let $\mathcal{N}$ be an N-dimensional C^1 manifold, $\mathcal{E} \subset \mathcal{N}$, $P \in \mathcal{N}$ and $m \in [N, +\infty)$. The following are equivalent:*

(1) There is a C^1 chart $(\mathcal{W}, \Phi)$ of $\mathcal{N}$ such that $P \in \mathcal{W}$ and

$$\mathcal{L}^N(B_r(\Phi(P)) \setminus \Phi(\mathcal{E} \cap \mathcal{W})) = o(r^m) \qquad (\text{as } r \to 0+).$$

(2) For every C^1 Riemannian metric g on $\mathcal{N}$, one has

$$\mathcal{H}^N_g(\mathcal{B}_g(P, r) \setminus \mathcal{E}) = o(r^m) \qquad (\text{as } r \to 0+).$$

(3) There exists a C^1 Riemannian metric g on $\mathcal{N}$ such that

$$\mathcal{H}^N_g(\mathcal{B}_g(P, r) \setminus \mathcal{E}) = o(r^m) \qquad (\text{as } r \to 0+).$$

Definition 2.9. If any or, equivalently, all of the conditions of Proposition 2.8 are satisfied, then we say that P is an m-density point of $\mathcal{E}$ (relative to $\mathcal{N}$). The set of all m-density points of $\mathcal{E}$ is denoted by $\mathcal{E}^{(m)}$, see [D19c].

Remark 2.10. Let $\mathcal{N}$ and $\mathcal{E}$ be as in Proposition 2.8. The following facts occur:

- Every interior point of $\mathcal{E}$ is an m-density point of $\mathcal{E}$, for all $m \in [N, +\infty)$. Thus, whenever $\mathcal{E}$ is open, one has $\mathcal{E} \subset \mathcal{E}^{(m)}$ for all $m \in [N, +\infty)$.
- If $N \le m_1 \le m_2 < +\infty$, then $\mathcal{E}^{(m_2)} \subset \mathcal{E}^{(m_1)}$. In particular, one has $\mathcal{E}^{(m)} \subset \mathcal{E}^{(N)}$ for all $m \in [N, +\infty)$.
- Let $\{\mathcal{E}_j\}_{j \in J}$ be any family of subsets of $\mathcal{N}$ and $m \in [N, +\infty)$.
  - One has
  $$\Big(\bigcap_{j \in J} \mathcal{E}_j\Big)^{(m)} \subset \bigcap_{j \in J} \mathcal{E}_j^{(m)};$$
  - If J is finite, then
  $$\Big(\bigcap_{j \in J} \mathcal{E}_j\Big)^{(m)} = \bigcap_{j \in J} \mathcal{E}_j^{(m)}; \tag{2.4}$$
  - If J is countable infinite, then (2.4) can fail to be true, e.g., $\mathcal{N} = \mathbb{R}^2$ and
  $$\mathcal{E}_j := B_{1/j}(O) \qquad (j = 1, 2, \ldots).$$

Remark 2.11. For convenience of the reader, we recall some known results in the special case when $\mathcal{N} = \mathbb{R}^N$ (which actually could be easily generalized):

- If $E \subset \mathbb{R}^N$ is $\mathcal{L}^N$-measurable then: $x \in E^{(N)}$ if and only if x is a Lebesgue density point of E, hence $\mathcal{L}^N(E\Delta E^{(N)}) = 0$. In particular, it follows that $(E^{(N)})^{(N)} = E^{(N)}$.
- If $E \subset \mathbb{R}^N$, then $E^{(m)}$ is $\mathcal{L}^N$-measurable, for all $m \in [N, +\infty)$ (see [D16, Proposition 3.1]).
- Every open set $U \subset \mathbb{R}^N$ can be approximated in measure by uniformly N-dense closed subsets of $\overline{U}$. More precisely: For all $C < \mathcal{L}^N(U)$ there exists a closed set $F \subset \overline{U}$ such that $\mathcal{L}^N(F) > C$ and $F^{(m)} = \emptyset$ for all $m > N$ (obviously one has $F^{(N)} \subset F$ and $\mathcal{L}^N(F \setminus F^{(N)}) = 0$), see [D19a, Proposition 5.4].
- Let $N \geq 2$ and $E \subset \mathbb{R}^N$ be a set of finite perimeter, so that $\mathcal{H}^{N-1}(\partial^* E) < +\infty$ (where $\partial^* E$ is the reduced boundary of E, see [M12, Theorem 15.9]). Then $\mathcal{L}^N(E \setminus E^{(m_0)}) = 0$, with
$$m_0 := N + 1 + \frac{1}{N-1},$$
see Theorem 1 in [EG92, Section 6.1.1] (compare also [D12, Lemma 4.1]). Moreover, the number m_0 is the maximum order of density common to all sets of finite perimeter. More precisely, the following property holds (see [D16, Proposition 4.1]): For all $m > m_0$ there exists a compact set F_m of finite perimeter in $\mathbb{R}^N$ such that $\mathcal{L}^N(F_m) > 0$ and $F_m^{(m)} = \emptyset$.

3. The integral set of a C^1 submanifold with respect to a family of C^1 differential forms: first structure theorem

3.1. **Introduction to the first structure theorem.** Throughout this section, $\mathcal{N}$ and $\mathcal{O}$ will denote, respectively, a C^2 manifold (ambient manifold) and a family of C^1 differential forms on $\mathcal{N}$. If $\mathcal{M}$ is a C^1 imbedded submanifold of $\mathcal{N}$, then we define
$$\mathcal{I}(\mathcal{M}, \mathcal{O}) := \bigcap_{\omega \in \mathcal{O}} \{\omega|_{\mathcal{M}} = 0\}$$
and call it the integral subset of $\mathcal{M}$ with respect to $\mathcal{O}$. We observe that:

- When $\mathcal{M}$ is a C^2 integral manifold of $\mathcal{O}$, one has
$$\bigcap_{\omega \in \mathcal{O}} \{(d\omega)|_{\mathcal{M}} = 0\} = \mathcal{I}(\mathcal{M}, \mathcal{O}) = \mathcal{M},$$
see [N85, Proposition 2.6.7] with $k = 2$.
- If $\mathcal{O}$ is a family of linearly independent C^1 differential 1-forms defining a distribution $\mathcal{D}$ on $\mathcal{N}$, then one has

$$\mathcal{I}(\mathcal{M}, \mathcal{O}) = \mathcal{T}(\mathcal{M}, \mathcal{D}). \tag{3.1}$$

Let us now recall an interesting example.

Example 3.1. Let $U := (0,1)^M$ and consider a C^1 differential 1-form on $\mathcal{N} := U \times \mathbb{R}$

$$\omega_{(x,t)} = F_1(x)dx_1 + \cdots + F_M(x)dx_M - dt, \quad x = (x_1, \ldots, x_M, t) \in \mathcal{N}$$

with $F := (F_1, \ldots, F_M) \in C^1(U, \mathbb{R}^M) \cap L^\infty(U, \mathbb{R}^M)$ and such that

$$DF(x)^T \neq DF(x), \text{ for all } x \in U. \tag{3.2}$$

Then ω defines a distribution $\mathcal{D}$ of rank M on $\mathcal{N}$ which is noninvolutive everywhere, see [N85, Section 2.11.14], but this does not prevent the existence of C^1 imbedded submanifolds $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{I}(\mathcal{M}, \mathcal{O})$ is "large". More precisely, for every $\varepsilon > 0$:

- There exists $u_\varepsilon \in C_0^1(U)$ such that
$$\mathcal{L}^M(A_\varepsilon) \leq \varepsilon, \quad A_\varepsilon := \{\nabla u_\varepsilon \neq F\},$$
and
$$\|\nabla u_\varepsilon\|_\infty \leq \frac{C}{\varepsilon}\|F\|_\infty$$
see [A91, Theorem 1].
- Hence, denoting the graph of u_ε by $\mathcal{M}_\varepsilon$, we find
$$\mathcal{I}(\mathcal{M}_\varepsilon, \{\omega\}) = \mathcal{T}(\mathcal{M}_\varepsilon, \mathcal{D}) = \{(x, u_\varepsilon(x)) \,|\, x \in U \setminus A_\varepsilon\}$$
and we have
$$\mathcal{H}^M(\mathcal{M}_\varepsilon \setminus \mathcal{I}(\mathcal{M}_\varepsilon, \{\omega\})) = \int_{A_\varepsilon} (1 + |\nabla u_\varepsilon|^2)^{1/2} d\mathcal{L}^M \leq \varepsilon + C\|F\|_\infty.$$

Observe that

$$(d\omega)_{(x,u_\varepsilon(x))}\left(\frac{\partial}{\partial x_i} + D_i u_\varepsilon(x)\frac{\partial}{\partial t}, \frac{\partial}{\partial x_j} + D_j u_\varepsilon(x)\frac{\partial}{\partial t}\right) = D_i F_j(x) - D_j F_i(x)$$

for all $x \in U$ and $1 \leq i, j \leq M$. Hence the set $\{(d\omega)|_{\mathcal{M}_\varepsilon} = 0\}$ is empty, by (3.2). Unfortunately, the lack of regularity does not allow us to use the standard elementary arguments (such as the invariant formula for the exterior derivative, see [N85, Proposition 2.6.6] with $k = 2$) to infer that $\mathcal{I}(\mathcal{M}_\varepsilon, \{\omega\}) = \mathcal{T}(\mathcal{M}_\varepsilon, \mathcal{D})$ has no interior points relative to $\mathcal{M}_\varepsilon$. Actually, we can be much more precise. Indeed, by [D19c, Theorem 1.3], one has $\mathcal{T}(\mathcal{M}, \mathcal{D})^{(M+1)} = \emptyset$ for all M-dimensional C^1 imbedded submanifolds $\mathcal{M}$ of $\mathcal{N}$. Hence in particular $\mathcal{I}(\mathcal{M}_\varepsilon, \{\omega\})^{(M+1)} = \mathcal{T}(\mathcal{M}_\varepsilon, \mathcal{D})^{(M+1)} = \emptyset$.

In general, we know little about the size of $\mathcal{I}(\mathcal{M}, \mathcal{O})$ when $\mathcal{M}$ is chosen from the family of C^1 imbedded submanifolds. Based on Example 3.1, however, we are inclined to think that the existence of $\mathcal{M}$ belonging to such a family for which $\mathcal{I}(\mathcal{M}, \mathcal{O})$ has large size is not an uncommon event (even without integrability assumptions on $\mathcal{O}$). In addition we have the following result whose proof is postponed to Subsection 3.2:

Theorem 3.2 (First structure theorem)**.** *Let $\mathcal{M}$ be any C^1 imbedded submanifold of $\mathcal{N}$ and put $M := \dim \mathcal{M}$. Then one has*

$$\mathcal{I}(\mathcal{M}, \mathcal{O})^{(M+1)} \subset \bigcap_{\omega \in \mathcal{O}} \{(d\omega)|_{\mathcal{M}} = 0\}$$

where $\mathcal{I}(\mathcal{M}, \mathcal{O})^{(M+1)}$ denotes the set of all $(M+1)$-density points of $\mathcal{I}(\mathcal{M}, \mathcal{O})$ relative to $\mathcal{M}$.

This corollary is an immediate consequence of Theorem 3.2. Although it establishes a very natural and most likely known property, we are unable to provide a reference for it.

Corollary 3.3. *Let $\mathcal{M}$ be a C^1 imbedded submanifold of $\mathcal{N}$ such that $\mathcal{I}(\mathcal{M}, \mathcal{O}) = \mathcal{M}$ (i.e., $\mathcal{M}$ is an integral manifold of $\mathcal{O}$). Then one has*

$$\bigcap_{\omega \in \mathcal{O}} \{(d\omega)|_{\mathcal{M}} = 0\} = \mathcal{M}.$$

Another trivial consequence of Theorem 3.2 is the following one.

Corollary 3.4. *Let $V_M(\mathcal{O})_y$ denote the set of all M-dimensional integral elements of $\mathcal{O}$ at $y \in \mathcal{N}$ (see* [BCG91, Y92]*) and assume that:*

(3.3)

For all $y \in \mathcal{N}$ and $\Sigma \in V_M(\mathcal{O})_y$ there is $\omega \in \mathcal{O}$ such that $(d\omega)_y|_\Sigma \neq 0$.

Then one has

$$\mathcal{I}(\mathcal{M}, \mathcal{O})^{(M+1)} = \emptyset$$

whenever $\mathcal{M}$ is an M-dimensional C^1 imbedded submanifold of $\mathcal{N}$.

3.2. **Proof of the first structure theorem.** Theorem 3.2 follows at once from Theorem 3.5 below (obtained by adapting the argument of [D12, Theorem 2.1]) and Remark 2.10.

Theorem 3.5. *Let ω be a C^1 differential p-form on $\mathcal{N}$ and $\varphi : U \subset \mathbb{R}^M \to \mathcal{N}$ be a C^1 map (so that $M \leq \dim \mathcal{N}$). Then*

$$U \cap \{d\lambda = \varphi^*\omega\}^{(M+1)} \subset \{\varphi^* d\omega = 0\}$$

for every C^1 differential $(p-1)$-form λ on U. In particular

$$U \cap \{\varphi^*\omega = 0\}^{(M+1)} \subset \{\varphi^* d\omega = 0\}.$$

Proof. Observe that the assertion is trivial for $p \geq M$, hence we can assume $p \leq M-1$. Let $\rho \in (0,1)$ and consider $g \in C_c^2(B_1(0))$, with $B_1(0) \subset \mathbb{R}^M$, such that

$$0 \leq g \leq 1, \qquad g|B_\rho(0) \equiv 1$$

and

$$|D_i g| \leq \frac{2}{1-\rho} \qquad (i = 1, \dots, M).$$

Now let $\overline{x} \in U \cap \{d\lambda = \varphi^*\omega\}^{(M+1)}$ and for $r > 0$ define $g_r \in C_c^2(B_r(\overline{x}))$ by

$$g_r(x) := g\left(\frac{x - \overline{x}}{r}\right), \quad x \in B_r(\overline{x}).$$

Then

$$D_i g_r(x) = \frac{1}{r} D_i g\left(\frac{x - \overline{x}}{r}\right), \quad x \in B_r(\overline{x})$$

hence

$$|D_i g_r| \leq \frac{2}{r(1-\rho)}. \tag{3.4}$$

We now consider r sufficiently small for the following facts to hold:

- $B_r(\overline{x}) \subset U$;
- There exists a sequence $\psi_1, \psi_2, \ldots \in C^2(B_r(\overline{x}), \mathcal{N})$ converging to $\varphi|_{B_r(\overline{x})}$ with respect to the C^1 topology.

Observe that, given an arbitrary C^2 differential $(M-1-p)$-form θ on $B_r(\overline{x})$, one has

$$d(g_r\, \psi_j^*\omega \wedge \theta) = dg_r \wedge \psi_j^*\omega \wedge \theta + g_r\, d\psi_j^*\omega \wedge \theta + (-1)^p g_r\, \psi_j^*\omega \wedge d\theta \qquad (\text{on } B_r(\overline{x}))$$

for all j, by the differentiation formula for the wedge product of forms (e.g. [L13, Proposition 14.23]). Since

$$\int_{B_r(\overline{x})} d(g_r\, \psi_j^*\omega \wedge \theta) = \int_{\partial B_r(\overline{x})} g_r\, \psi_j^*\omega \wedge \theta = 0$$

by the Stokes theorem, we obtain

$$\begin{aligned}
\int_{B_r(\overline{x})} g_r\, \psi_j^* d\omega \wedge \theta &= \int_{B_r(\overline{x})} g_r\, d\psi_j^*\omega \wedge \theta \\
&= -\int_{B_r(\overline{x})} dg_r \wedge \psi_j^*\omega \wedge \theta \\
&\quad + (-1)^{p+1} \int_{B_r(\overline{x})} g_r\, \psi_j^*\omega \wedge d\theta.
\end{aligned} \tag{3.5}$$

Now let $F_\theta : B_r(\overline{x}) \to \mathbb{R}$ be the continuous function such that

$$\varphi^* d\omega \wedge \theta = F_\theta\, dx \qquad (dx := dx_1 \wedge \cdots \wedge dx_M)$$

in $B_r(\overline{x})$. Then, setting $K := \{d\lambda = \varphi^*\omega\}$ and passing to the limit ($j \to +\infty$) in (3.5), we find

$$\begin{aligned}\int_{B_r(\overline{x})} g_r F_\theta \, dx &= \int_{B_r(\overline{x})} g_r \, \varphi^* d\omega \wedge \theta \\ &= -\int_{B_r(\overline{x})} dg_r \wedge \varphi^*\omega \wedge \theta + (-1)^{p+1} \int_{B_r(\overline{x})} g_r \, \varphi^*\omega \wedge d\theta \\ &= -\int_{B_r(\overline{x})\cap K} dg_r \wedge d\lambda \wedge \theta - \int_{B_r(\overline{x})\setminus K} dg_r \wedge \varphi^*\omega \wedge \theta + \\ &\quad + (-1)^{p+1} \int_{B_r(\overline{x})\cap K} g_r \, d\lambda \wedge d\theta + (-1)^{p+1} \int_{B_r(\overline{x})\setminus K} g_r \, \varphi^*\omega \wedge d\theta \\ &= \int_{B_r(\overline{x})} -dg_r \wedge d\lambda \wedge \theta + (-1)^{p+1} g_r \, d\lambda \wedge d\theta + \\ &\quad + \int_{B_r(\overline{x})\setminus K} dg_r \wedge (d\lambda - \varphi^*\omega) \wedge \theta + (-1)^{p+1} g_r \, (\varphi^*\omega - d\lambda) \wedge d\theta.\end{aligned}$$

But in the last member of this equality the integral over $B_r(\overline{x})$ is zero. Indeed, since g_r and θ are of class C^2, the differentiation formula for the wedge product of forms (e.g. [L13, Proposition 14.23]) yields

$$-dg_r \wedge d\lambda \wedge \theta + (-1)^{p+1} g_r \, d\lambda \wedge d\theta = d\big(dg_r \wedge \lambda \wedge \theta + (-1)^{p+1} g_r \, \lambda \wedge d\theta\big)$$

hence

$$\int_{B_r(\overline{x})} -dg_r \wedge d\lambda \wedge \theta + (-1)^{p+1} g_r \, d\lambda \wedge d\theta = 0$$

by the Stokes theorem.

Thus

$$\int_{B_r(\overline{x})} g_r F_\theta \, dx = \int_{B_r(\overline{x})\setminus K} dg_r \wedge (d\lambda - \varphi^*\omega) \wedge \theta + (-1)^{p+1} g_r \, (\varphi^*\omega - d\lambda) \wedge d\theta.$$

By recalling also (3.4), it follows that there exists a number C, not depending on r and ρ, such that

$$\left| \int_{B_r(\overline{x})} g_r F_\theta \, dx \right| \leq C \, \mathcal{L}^M(B_r(\overline{x})\setminus K) \left(\frac{1}{r(1-\rho)} + 1 \right).$$

On the other hand, the triangle inequality implies

$$\left| \int_{B_r(\overline{x})} g_r F_\theta \, dx \right| \geq \left| \int_{B_{\rho r}(\overline{x})} g_r F_\theta \, dx \right| - \left| \int_{B_r(\overline{x})\setminus B_{\rho r}(\overline{x})} g_r F_\theta \, dx \right|$$

hence there is a number C_1, which does not depend on r and ρ, such that

$$\begin{aligned}\rho^M \left| \fint_{B_{\rho r}(\overline{x})} F_\theta \, dx \right| &\leq \left| \fint_{B_r(\overline{x})} g_r \, F_\theta \, dx \right| + \frac{1}{\mathcal{L}^M(B_r(\overline{x}))} \left| \int_{B_r(\overline{x}) \setminus B_{\rho r}(\overline{x})} g_r \, F_\theta \, dx \right| \\ &\leq \frac{C_1 \mathcal{L}^M(B_r(\overline{x}) \setminus K)}{r^M} \left(\frac{1}{r(1-\rho)} + 1 \right) + \frac{C_1 (r^M - \rho^M r^M)}{r^M} \\ &= \frac{C_1 \mathcal{L}^M(B_r(\overline{x}) \setminus K)}{r^{M+1}} \left(\frac{1}{1-\rho} + r \right) + C_1(1-\rho^M).\end{aligned}$$

Passing to the limit for $r \downarrow 0$ and recalling that $\overline{x} \in K^{(M+1)}$, we obtain

$$\rho^M |F_\theta(\overline{x})| \leq C_1(1-\rho^M).$$

Then, letting $\rho \uparrow 1$, we find $F_\theta(\overline{x}) = 0$, that is

$$(\varphi^* d\omega)_{\overline{x}} \wedge \theta_{\overline{x}} = 0.$$

From the arbitrariness of θ it follows that $(\varphi^* d\omega)_{\overline{x}} = 0$. □

Remark 3.6. In general, under the assumptions of Theorem 3.5, one cannot expect that

$$U \cap \{d\lambda = \varphi^* \omega\}^{(M)} \subset \{\varphi^* d\omega = 0\}.$$

For example (see [D12, Remark 2.2]), if

$$U = \mathcal{N} = \mathbb{R}^2, \quad \varphi(x) = x, \quad \omega_x = -x_2 dx_1 + x_1 dx_2,$$

then the set $\{\varphi^* d\omega = 0\}$ is empty, while a simple application of [A91, Theorem 1] provides $\lambda \in C^1(\mathbb{R}^2)$ such that $\mathcal{L}^2(\{d\lambda = \varphi^* \omega\}^{(2)}) > 0$.

4. The integral set of a C^{k+1} submanifold with respect to a family of C^k differential forms: second structure theorem

4.1. **Introduction to the second structure theorem.** We turn our attention to Corollary 3.4. If under the assumptions of that result we are restricted to considering imbedded submanifolds $\mathcal{M}$ of class C^2, then it is natural to expect a consequent further "slimming" of $\mathcal{I}(\mathcal{M}, \mathcal{O})$. In fact the following result holds:

Corollary 4.1. *Let $\mathcal{N}$ be a C^2 manifold, $\mathcal{O}$ be a countable family of C^1 differential forms on $\mathcal{N}$ and assume condition* (3.3). *Then this property holds: For every M-dimensional C^2 imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$, the set $\mathcal{I}(\mathcal{M}, \mathcal{O})$ is covered by countably many C^1 imbedded submanifolds of $\mathcal{M}$ of dimension less or equal than $M-1$ (in particular, $\mathcal{I}(\mathcal{M}, \mathcal{O})$ is a $(M-1)$-dimensional C^1 rectifiable subset of $\mathcal{M}$).*

Corollary 4.1 is a special case of the more general structure result below, which will be proved in Subsection 4.2:

Theorem 4.2 (Second structure theorem)**.** *Let $\mathcal{N}$ and $\mathcal{O}$ be, respectively, a C^{k+1} manifold and a countable family of C^k differential forms on $\mathcal{N}$ (with $k \geq 1$). Assume that condition* (3.3) *is satisfied and define*

$$\rho_M := \min_{\substack{y\in\mathcal{N} \\ \Sigma\in V_M(\mathcal{O})_y}} \max\{m \geq 1 | ((d\omega)^m)_y|_\Sigma \neq 0 \text{ for some } \omega \in \mathcal{O}\}. \tag{4.1}$$

Then $\rho_M \geq 1$ and this property holds: For every M-dimensional C^{k+1} imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$, the set $\mathcal{I}(\mathcal{M},\mathcal{O})$ is covered by countably many C^k imbedded submanifolds of $\mathcal{M}$ of dimension less or equal than $M - \rho_M$ (in particular, $\mathcal{I}(\mathcal{M},\mathcal{O})$ is an $(M-\rho_M)$-dimensional C^k rectifiable subset of $\mathcal{M}$).

Example 4.3. We now return to the special case considered in Example 3.1, namely $\mathcal{N} := U \times \mathbb{R} = (0,1)^M \times \mathbb{R}$ and

$$\omega_{(x,t)} = F_1(x)dx_1 + \cdots + F_M(x)dx_M - dt, \quad (x,t) = (x_1,\ldots,x_M,t) \in \mathcal{N}.$$

This time, in addition to assuming (3.2), we require $F = (F_1,\ldots,F_M) \in C^k(U,\mathbb{R}^M)$ with $k \geq 1$. Since $V_M(\{\omega\})_{(x,t)} = \{\mathcal{D}_{(x,t)}\}$ for all $(x,t) \in \mathcal{N}$ and condition (3.3) is trivially satisfied, we can apply Theorem 4.2 (we also recall (3.1)). We conclude that

$$\rho_M = \min_{(x,t)\in\mathcal{N}} \max\{m \geq 1 \,|\, ((d\omega)^m)_{(x,t)}|_{\mathcal{D}_{(x,t)}} \neq 0\} \geq 1$$

and the following property holds: For every M-dimensional C^{k+1} imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$, the set $\mathcal{T}(\mathcal{M},\mathcal{D})$ is covered by countably many C^k imbedded submanifolds of $\mathcal{M}$ of dimension less or equal than $M - \rho_M$ (in particular, $\mathcal{T}(\mathcal{M},\mathcal{D})$ is a $(M-\rho_M)$-dimensional C^k rectifiable subset of $\mathcal{M}$).

4.2. **Proof of the second structure theorem.** Let us first introduce some notation useful for stating Theorem 4.5 and Corollary 4.6 below. From Corollary 4.6 we will then obtain the proof of Theorem 4.2.

Let

$$\theta_h = \sum_{\alpha\in I(M,p_h)} g_\alpha^{(h)}\, dx_\alpha \qquad (h = 1,\ldots,H)$$

be a family of homogeneous differential forms of class C^k on an open set $U \subset \mathbb{R}^M$, so that $g_\alpha^{(h)} \in C^k(U)$ for all α, h. To proceed with the discussion, we now need to order the sets $I(M,p_h)$ in any way (for example lexicographically), so that we can write

$$I(M,p_h) = \{\alpha^{(h;1)}, \alpha^{(h;2)}, \ldots, \alpha^{(h;K_h)}\}, \quad K_h := K(p_h) := \binom{M}{p_h}.$$

and

$$\theta_h = \sum_{i=1}^{K_h} g^{(h)}_{h;\alpha^{(h;i)}}\, dx_{\alpha^{(h;i)}}.$$

Moreover set $K := K_1 + \cdots + K_H$ and consider $G = (G_1, \ldots, G_K)^T \in C^k(U, \mathbb{R}^K)$ defined as follows:

$$G_i := g^{(1)}_{\alpha^{(1;i)}} \qquad (i = 1, \ldots, K_1)$$

and

$$G_{K_1+\cdots+K_{h-1}+i} := g^{(h)}_{\alpha^{(h;i)}} \qquad (h = 2, \ldots, H; i = 1, \ldots, K_h).$$

Remark 4.4. In the special case that $p_h = 1$ for all h (hence $K = HM$), it is natural to set $\alpha^{(h;i)} := i$, so that $G_{(h-1)M+i} := g^{(h)}_i$. In particolar, we get the following identity that will be useful below:

$$\begin{aligned} d\theta_h(u,v) &= \sum_{i=1}^{M} (dG_{(h-1)M+i} \wedge dx_i)(u,v) \\ &= \sum_{i,j=1}^{M} D_j G_{(h-1)M+i}(u_j v_i - u_i v_j) \\ &= \sum_{i=1}^{M} v_i(\nabla G_{(h-1)M+i}) \cdot u - \sum_{i=1}^{M} u_i(\nabla G_{(h-1)M+i}) \cdot v \end{aligned}$$

for all $u, v \in \mathbb{R}^M$ and $h = 1, \ldots, H$.

Now let $(y_1, \ldots, y_K)$ denote the coordinates of $\mathbb{R}^K$ and define the C^k differential 1-form on $U \times \mathbb{R}^K$

$$\Theta := \sum_{l=1}^{K} \widetilde{G}_l\, dy_l,$$

where $\widetilde{G}_l$ denotes the function

$$(x,y) \mapsto G_l(x), \quad (x,y) \in U \times \mathbb{R}^K.$$

Finally, we set $U_0 := \{x \in U \,|\, (d\Theta)_{(x,0)} = 0\}$,

$$U_r := \{x \in U \,|\, (d\Theta)^r_{(x,0)} \neq 0,\, (d\Theta)^{r+1}_{(x,0)} = 0\} \qquad (r = 1, \ldots, M)$$

and

$$D_\lambda G_\beta := \begin{pmatrix} D_{\lambda_1} G_{\beta_1} & \cdots & D_{\lambda_r} G_{\beta_1} \\ \vdots & \ddots & \vdots \\ D_{\lambda_1} G_{\beta_r} & \cdots & D_{\lambda_r} G_{\beta_r} \end{pmatrix} \qquad (\beta \in I(K,r), \lambda \in I(M,r)).$$

Theorem 4.5. *The sets $U_0, U_1, \ldots, U_M$ form a pairwise disjoint decomposition of U. Moreover, for $r = 1, \ldots, M$, the following facts hold:*

(1) *One has*

$$(d\Theta)^r_{(x,0)} = \pm\, r! \sum_{\beta\in I(K,r)} \sum_{\lambda\in I(M,r)} \det(D_\lambda G_\beta(x))\, dx_\lambda \wedge dy_\beta$$

for all $x \in U$, *hence*

$$U_r = \{\mathrm{rank}\, DG = r\}.$$

(2) *The set*

$$U_r \cap \left(\bigcap_{h=1}^{H} \{\theta_h = 0\} \right)$$

is covered by finitely many $(M-r)$*-dimensional* C^k *imbedded submanifolds of* U.

Proof. The first assertion about $U_0, U_1, \ldots, U_M$ is obvious. From

$$d\Theta = \sum_{l=1}^{K} d\widetilde{G}_l \wedge dy_l$$

we obtain

$$\begin{aligned}(d\Theta)^r &= \pm \sum_{l_1,\ldots,l_r=1}^{K} d\widetilde{G}_{l_1} \wedge \cdots \wedge d\widetilde{G}_{l_r} \wedge dy_{l_1} \wedge \cdots \wedge dy_{l_r} \\ &= \pm\, r! \sum_{\beta\in I(K,r)} d\widetilde{G}_{\beta_1} \wedge \cdots \wedge d\widetilde{G}_{\beta_r} \wedge dy_\beta .\end{aligned}$$

Moreover one has (for all $x \in U$)

$$\begin{aligned}(d\widetilde{G}_{\beta_1} \wedge \cdots \wedge d\widetilde{G}_{\beta_r})_{(x,0)} &= \sum_{j_1,\ldots,j_r=1}^{M} D_{j_1} G_{\beta_1}(x) \cdots D_{j_r} G_{\beta_r}(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_r} \\ &= \sum_{\lambda\in I(M,r)} \det(D_\lambda G_\beta(x))\, dx_\lambda .\end{aligned}$$

This proves (1). As for (2), we observe that (1) yields

$$\begin{aligned}U_r \cap \left(\bigcap_{h=1}^{H} \{\theta_h = 0\} \right) &= \{G = 0\} \cap \{\mathrm{rank}\, DG = r\} \\ &\subset \bigcup_{\beta\in I(K,r)} \{G_\beta = 0\} \cap \{\mathrm{rank}\, DG_\beta = r\}\end{aligned}$$

where $G_\beta := (G_{\beta_1}, \ldots, G_{\beta_r})^T$. The conclusion follows from standard literature on submanifolds, e.g., [N85, Corollary 2.5.5], [KP13, Theorem 4.3.1]. □

Corollary 4.6. *Let* θ *and* r *be, respectively, a* C^k *differential* p*-form on* U *and a positive integer. Then the set* $\{\theta = 0\} \cap \{(d\theta)^r \neq 0\}$ *is covered* *by finitely many C^k imbedded submanifolds of U of dimension less or equal than $M-r$.*

Proof. Since $(d\theta)^r$ has degree $r(p+1)$, we can assume $r(p+1)\le M$ (otherwise, the statement is trivially verified). We set $\theta_1 := \theta$ and adopt the notation above, so that

$$\theta=\theta_1=\sum_{i=1}^{K} g^{(1)}_{\alpha^{(i)}}\, dx_{\alpha^{(i)}}=\sum_{i=1}^{K} G_i\, dx_{\alpha^{(i)}}$$

with $K=\binom{M}{p}$. Then one has

$$\begin{aligned}(d\theta)^r=(d\theta_1)^r&=\left(\sum_{i=1}^{K} dG_i\wedge dx_{\alpha^{(i)}}\right)^r\\ &=\pm\sum_{i_1,\dots,i_r=1}^{K} dG_{i_1}\wedge\cdots\wedge dG_{i_r}\wedge dx_{\alpha^{(i_1)}}\wedge\cdots\wedge dx_{\alpha^{(i_r)}}\\ &=\pm\, r!\sum_{\beta\in I(K,r)} dG_{\beta_1}\wedge\cdots\wedge dG_{\beta_r}\wedge dx_{\alpha^{(\beta_1)}}\wedge\cdots\wedge dx_{\alpha^{(\beta_r)}}\end{aligned}$$

where

$$\begin{aligned}dG_{\beta_1}\wedge\cdots\wedge dG_{\beta_r}&=\sum_{j_1,\dots,j_r=1}^{M} D_{j_1}G_{\beta_1}\cdots D_{j_r}G_{\beta_r}\, dx_{j_1}\wedge\cdots\wedge dx_{j_r}\\ &=\sum_{\lambda\in I(M,r)}\det(D_\lambda G_\beta)\, dx_\lambda.\end{aligned}$$

Thus

$$(d\theta)^r=\pm\, r!\sum_{\beta\in I(K,r)}\sum_{\lambda\in I(M,r)}\det(D_\lambda G_\beta)\, dx_\lambda\wedge dx_{\alpha^{(\beta_1)}}\wedge\cdots\wedge dx_{\alpha^{(\beta_r)}}$$

which implies

$$\begin{aligned}\{\theta=0\}\cap\{(d\theta)^r\neq 0\}&\subset\{\theta_1=0\}\cap\{\operatorname{rank} DG\ge r\}\\ &=\bigcup_{j=r}^{M}(\{\theta_1=0\}\cap\{\operatorname{rank} DG=j\}).\end{aligned}$$

Hence the conclusion follows by Theorem 4.5. □

Remark 4.7. When p is even and $r\ge 2$, the set $\{(d\theta)^r\neq 0\}$ is empty and thus Corollary 4.6 is trivial and uninteresting.

We are finally ready to prove the second structure theorem:

Proof of Theorem 4.2. From assumption (3.3) we get immediately $\rho_M\ge 1$ and

$$\bigcup_{\omega\in\mathcal{O}}\{(d\omega)^{\rho_M}|_{\mathcal{M}}\neq 0\}=\mathcal{M}.$$

Hence

$$\begin{aligned}\mathcal{I}(\mathcal{M},\mathcal{O}) &= \bigcap_{\omega\in\mathcal{O}}\{\omega|_{\mathcal{M}}=0\}\\ &= \left(\bigcap_{\omega\in\mathcal{O}}\{\omega|_{\mathcal{M}}=0\}\right)\cap\left(\bigcup_{\omega\in\mathcal{O}}\{(d\omega)^{\rho_M}|_{\mathcal{M}}\neq 0\}\right)\\ &\subset \bigcup_{\omega\in\mathcal{O}}\left(\{\omega|_{\mathcal{M}}=0\}\cap\{(d\omega)^{\rho_M}|_{\mathcal{M}}\neq 0\}\right).\end{aligned}$$

The conclusion follows at once from Corollary 4.6 and a standard localization argument. □

5. Application in the context of non-involutive distributions I

5.1. **From the EDS point of view.** Let $\mathcal{N}$ be a N-dimensional C^{k+1} manifold, with $k \geq 1$. Consider a C^k distribution $\mathcal{D}$ of rank M on $\mathcal{N}$ which is non-involutive at each point of $\mathcal{N}$, so that $H := N - M \geq 1$. If $\mathcal{O} := \{\omega_1, \ldots, \omega_H\}$ is a family of linearly independent C^k defining 1-forms for $\mathcal{D}$ in $\mathcal{N}$, then the following property holds: For all $y \in \mathcal{N}$, one has $V_M(\mathcal{O})_y = \{\mathcal{D}_y\}$ and there exists h (depending on y) such that

$$(d\omega_h)_y|_{\mathcal{D}_y} \neq 0,$$

see Section 2.3. Hence condition (3.3) is satisfied and (recalling the definition of ρ_M given in Theorem 4.2) we have

$$\rho_M = \min_{y\in\mathcal{N}} \max\{m \geq 1 | ((d\omega_h)^m)_y|_{\mathcal{D}_y} \neq 0 \text{ for some } h\} \geq 1. \tag{5.1}$$

Observe that ρ_M depends on $\mathcal{D}$ but not on the choice of the family $\mathcal{O}$ defining $\mathcal{D}$.

The next two results, the first of which is known, describe the structure of the tangency of an imbedded submanifold with respect to $\mathcal{D}$. They follow trivially from Corollary 3.4 and Theorem 4.2, respectively (by recalling also (3.1)).

Corollary 5.1 (Theorem 1.3 in [D19c])**.** *Under the assumptions above with $k = 1$, one has*

$$\mathcal{T}(\mathcal{M},\mathcal{D})^{(M+1)} = \emptyset$$

whenever $\mathcal{M}$ is M-dimensional C^1 imbedded submanifold of $\mathcal{N}$.

Corollary 5.2. *Under the assumptions above, the following property holds: For every M-dimensional C^{k+1} imbedded submanifold $\mathcal{M}$ of $\mathcal{N}$, the set $\mathcal{T}(\mathcal{M},\mathcal{D})$ is covered by countably many C^k imbedded submanifolds of $\mathcal{M}$*

of dimension less or equal than $M-\rho_M$ (in particular, $\mathcal{T}(\mathcal{M},\mathcal{D})$ is an $(M-\rho_M)$-dimensional C^k rectifiable subset of $\mathcal{M}$).

5.2. **From the PDE point of view.** Let $U\subset\mathbb{R}^M$, $V\subset\mathbb{R}^H$ be two open sets and consider a family of functions

$$\Phi^{(1)},\dots,\Phi^{(M)}\in C^k(U\times V,\mathbb{R}^H),$$

with $k\geq 1$, satisfying the following condition:

For every $(x,t)\in U\times V$ there exist $i,j\in\{1,\dots,M\}$ such that

$$\left(\frac{\partial\Phi^{(i)}}{\partial x_j}-\frac{\partial\Phi^{(j)}}{\partial x_i}+\sum_{l=1}^{H}\frac{\partial\Phi^{(i)}}{\partial t_l}\Phi_l^{(j)}-\sum_{l=1}^{H}\frac{\partial\Phi^{(j)}}{\partial t_l}\Phi_l^{(i)}\right)_{(x,t)}\neq 0. \tag{5.2}$$

It is well known that condition (5.2) is equivalent to the local unsolvability of the partial differential system

$$\begin{cases}D_i f(x)=\Phi^{(i)}(x,f(x)) & (i=1,\dots,M)\\ f(x^0)=t^0\end{cases} \tag{5.3}$$

for all $(x^0,t^0)\in U\times V$, see Theorem 1 in [Sp79, Chapter 6]. In geometric terms, the system (5.3) expresses the property that the graph of f is an integral manifold through (x^0,t^0) of the C^k distribution $\mathcal{D}$ of rank M defined by the family of C^k differential 1-forms on $U\times V$

$$\mathcal{O}:=\{\omega_1,\dots,\omega_H\},\quad \omega_h:=\sum_{i=1}^{M}\Phi_h^{(i)}\,dx_i-dt_h$$

and indeed it is well known that the non-involutivity of $\mathcal{D}$ at each point of $U\times V$ is equivalent to condition (5.2), see [Sp79, Chapter 6], [N85, Section 2.11], [D18, Section 3]. Thus, correspondingly to Corollary 5.1 and Corollary 5.2, we obtain the following couple of results:

Corollary 5.3. *Under the assumptions above with $k=1$, let $f\in C^1(U,V)$. Then one has*

$$\left(\bigcap_{i=1}^{M}\{x\in U\mid D_i f(x)=\Phi^{(i)}(x,f(x))\}\right)^{(M+1)}=\emptyset.$$

Corollary 5.4. *Under the assumptions above, let ρ_M be defined as in* (4.1) *(so that* (5.1) *holds). Then, for all $f\in C^{k+1}(U,V)$, the set*

$$\bigcap_{i=1}^{M}\{x\in U\mid D_i f(x)=\Phi^{(i)}(x,f(x))\}$$

is covered by countably many C^k imbedded submanifolds of U of dimension less or equal than $M-\rho_M$ (in particular, it is a $(M-\rho_M)$-dimensional C^k rectifiable subset of U).

Remark 5.5. Consider the very special case when $H = 1$ and the functions $\Phi^{(i)}$ do not depend explicitly on t, that is

$$\Phi^{(i)}(x,t) = F_i(x), \text{ for all } (x,t) \in U \times \mathbb{R}$$

with $F_i \in C^k(U)$, for all $i = 1, \dots, M$. Then condition (5.2) turns to

$$DF(x)^T \neq DF(x), \text{ for all } x \in U \qquad (F := (F_1, \dots, F_M)^T) \tag{5.4}$$

and we can define the positive integer ρ_M as above. Thus Corollary 5.3 and Corollary 5.4 provide the following properties, which in a much more general version were proved in [D21] (see Corollary 3.8 and Corollary 3.4 of [D21]):

- Let $k = 1$ and assume condition (5.4). Then, for every function $f \in C^1(U)$, one has $\{\nabla f = F\}^{(M+1)} = \emptyset$;
- Let us assume condition (5.4). Then, for all $f \in C^{k+1}(U)$, the set $\{\nabla f = F\}$ is a $(M - \rho_M)$-dimensional C^k rectifiable subset of U.

A remarkable example is given by $F \in C^k(U, \mathbb{R}^M)$ of the form

$$F(x_1, \dots, x_M) = (-x_{r+1}, \dots, -x_{2r}, x_1, \dots, x_r, F_{2r+1}(x_{2r+1}), \dots, F_M(x_M))^T$$

so that

$$(\omega_1)_{(x,t)} = -\sum_{i=1}^{r} x_{r+i} dx_i + \sum_{i=r+1}^{2r} x_{i-r} dx_i + \sum_{i=2r+1}^{M} F_i(x_i) dx_i - dt.$$

Hence

$$(d\omega_1)^r = \pm 2^r \, r! \, dx_1 \wedge \cdots \wedge dx_{2r}$$

which implies $\rho_M = r$. In particular, when $M = 2r$ and $k = 1$, we obtain [D19b, Corollary 4.1] which improves the second statement in [B03, Theorem 3.1] and yields immediately the following property: The characteristic set of a codimension 1 submanifold of class C^2 in the Heisenberg group $\mathbb{H}^r$ has Hausdorff dimension less or equal than r, see [B03, Theorem 1.2]. Related to this subject, we point out the recent paper [Tu21] where the results of [D19b] are extended to the context of gradient-like vector fields associated to a general geometric structure on $\mathbb{R}^{2p}$.

6. Application in the context of non-involutive distributions II

The main purpose of this section is to provide (in Theorem 6.1 below) a very simple new proof of [D18, Theorem 5.1], which in turn almost trivially implies [D18, Corollary 5.2] about the structure of tangency sets with respect to a distribution (see Corollary 6.5 below).

Throughout this section, $\mathcal{D}$ denotes a C^k distribution of rank M on an $(M + H)$-dimensional C^{k+1} manifold $\mathcal{N}$ (with $M, H, k \geq 1$).

Theorem 6.1. *Let $\omega_1, \dots, \omega_H$ be a family of linearly independent C^k differential 1-forms defining $\mathcal{D}$ in an open subset $\mathcal{V}$ of $\mathcal{N}$. For $m = 1, \dots, M+H$, let $\mathcal{A}_m(\omega_1, \dots, \omega_H)$ (or simply $\mathcal{A}_m$) denote the set of all points $P \in \mathcal{V}$ satisfying the following property: There exists an m-dimensional vector subspace X of $T_P\mathcal{N}$ such that*

$$(\omega_h)_P|_X = 0, \quad (d\omega_h)_P|_X = 0 \qquad (\text{for all } h = 1, \dots, H). \tag{6.1}$$

The following facts hold:

(1) One has

$$\mathcal{A}_M \subset \mathcal{A}_{M-1} \subset \cdots \subset \mathcal{A}_1 = \mathcal{V}, \quad \mathcal{A}_{M+1} = \cdots = \mathcal{A}_{M+H} = \emptyset;$$

(2) Let U be an open subset of $\mathbb{R}^M$ and $\varphi : U \to \mathcal{N}$ be an injective C^{k+1} immersion such that $\mathcal{U} := \varphi(U) \subset \mathcal{V}$. Then, for all $m = 1, \dots, M$, the set $\mathcal{T}(\mathcal{U}, \mathcal{D}) \setminus \mathcal{A}_{m+1}$ is covered by finitely many C^k imbedded submanifolds of $\mathcal{N}$ of dimension less or equal than m.

Proof. Assertion (1) is actually obvious, see [BPR11]. To prove assertion (2), we will use Theorem 4.5. So let us define

$$\theta_h := \varphi^*\omega_h \qquad (h = 1, \dots, H)$$

and adopt the notation of Subsection 4.2. We shall prove the following inclusion

$$\varphi^{-1}(\mathcal{T}(\mathcal{U}, \mathcal{D}) \setminus \mathcal{A}_{m+1}) \subset \bigcup_{r=M-m}^{M} \left(U_r \cap \left(\bigcap_{h=1}^{H} \{\theta_h = 0\} \right) \right)$$

that implies the conclusion, by (2) of Theorem 4.5. For this purpose, we consider x in the left-hand side set, namely $x \in U$ such that

$$\varphi(x) \in \mathcal{T}(\mathcal{U}, \mathcal{D}), \quad \varphi(x) \notin \mathcal{A}_{m+1}. \tag{6.2}$$

From the first one and (2.3), we obtain

$$x \in \bigcap_{h=1}^{H} \{\theta_h = 0\}$$

hence it remains to prove that

$$x \in \bigcup_{r=M-m}^{M} U_r. \tag{6.3}$$

Let us assume, by contradiction, that (6.3) does not happen. Then, by Theorem 4.5, one has $\operatorname{rank} DG(x) \leq M - m - 1$ and thus

$$M = \dim \ker DG(x) + \operatorname{rank} DG(x) \leq \dim \ker DG(x) + M - m - 1$$

which yields

$$K := \dim \ker DG(x) \geq m + 1. \tag{6.4}$$

Now consider the K-dimensional vector space $X := d\varphi_x(\ker DG(x)) \subset T_{\varphi(x)}\mathcal{N}$. Recalling the first formula in (6.2), we obtain

$$X \subset d\varphi_x(\mathbb{R}^M) = T_{\varphi(x)}\mathcal{U} = \mathcal{D}_{\varphi(x)}$$

hence

$$(\omega_h)_{\varphi(x)}|_X = 0 \qquad (h = 1, \ldots, H).$$

Moreover if ξ and η are two arbitrary vectors in X, so that $\xi = d\varphi_x(u)$ and $\eta = d\varphi_x(v)$ with $u, v \in \ker DG(x)$, then

$$\begin{aligned}
(d\omega_h)_{\varphi(x)}(\xi, \eta) &= (d\omega_h)_{\varphi(x)}(d\varphi_x(u), d\varphi_x(v)) \\
&= (\varphi^* d\omega_h)_x(u, v) \\
&= (d\theta_h)_x(u, v) \\
&= \sum_{i=1}^{M} v_i \nabla G_{(h-1)M+i}(x) \cdot u - \sum_{i=1}^{M} u_i \nabla G_{(h-1)M+i}(x) \cdot v \\
&= 0
\end{aligned}$$

for all $h = 1, \ldots, H$, by Remark 4.4. Thus one has also

$$(d\omega_h)_{\varphi(x)}|_X = 0 \qquad (h = 1, \ldots, H),$$

so that $\varphi(x) \in \mathcal{A}_K$. Finally, the contradiction arises by recalling (6.2), (6.4) and statement (1). □

Remark 6.2. Let $\omega_1, \ldots, \omega_H$ be as in Theorem 6.1. The following properties holds, simply by definition:

- If $\mathcal{D}$ is involutive in $\mathcal{V}$, then $\mathcal{A}_M(\omega_1, \ldots, \omega_H) \cap \mathcal{V} = \mathcal{V}$;
- If $\mathcal{D}$ is non-involutive at each point of $\mathcal{V}$, then $\mathcal{A}_M(\omega_1, \ldots, \omega_H) \cap \mathcal{V} = \emptyset$.

Remark 6.3. One can easily verify that identities in (6.1) are independent from the family $\omega_1, \ldots, \omega_H$ of local defining 1-forms for $\mathcal{D}$, in the following sense: If $\omega_1, \ldots, \omega_H$ (resp. $\omega'_1, \ldots, \omega'_H$) is a family of defining 1-forms for $\mathcal{D}$ in an open subset $\mathcal{V}$ (resp. $\mathcal{V}'$) of $\mathcal{N}$ and if $1 \leq m \leq M + H$, then

$$\mathcal{A}_m(\omega_1, \ldots, \omega_H) \cap \mathcal{V}' = \mathcal{A}_m(\omega'_1, \ldots, \omega'_H) \cap \mathcal{V}.$$

It follows that, if define (for $m = 1, \ldots, M + H$)

$$\mathcal{A}_m(\mathcal{D}) := \bigcup \{\mathcal{A}_m(\omega_1, \ldots, \omega_H) \,|\, \omega_1, \ldots, \omega_H \text{ are local defining 1-forms for } \mathcal{D}\}$$

then for every family $\omega_1, \dots, \omega_H$ of defining 1-forms for $\mathcal{D}$ in an open set $\mathcal{V} \subset \mathcal{N}$ one has

$$\mathcal{A}_m(\mathcal{D}) \cap \mathcal{V} = \mathcal{A}_m(\omega_1, \dots, \omega_H). \tag{6.5}$$

Corollary 6.4. *The following facts hold:*

(1) One has

$$\mathcal{A}_M(\mathcal{D}) \subset \mathcal{A}_{M-1}(\mathcal{D}) \subset \cdots \subset \mathcal{A}_1(\mathcal{D}) = \mathcal{N}$$

and

$$\mathcal{A}_{M+1}(\mathcal{D}) = \cdots = \mathcal{A}_{M+H}(\mathcal{D}) = \emptyset;$$

(2) Let $\mathcal{M}$ be an M-dimensional C^{k+1} imbedded submanifold of $\mathcal{N}$ and $1 \leq m \leq M$. Then

$$\mathcal{R}_m := \mathcal{T}(\mathcal{M}, \mathcal{D}) \setminus \mathcal{A}_{m+1}(\mathcal{D})$$

is a C^k rectifiable set of dimension less or equal than m.

Proof. The first statement follows trivially from (1) of Theorem 6.1. To prove the second one, let $P \in \mathcal{R}_m$ and consider a family $\omega_1, \dots, \omega_H$ of defining 1-forms for $\mathcal{D}$ in a neighbourhood $\mathcal{V}$ of P. Moreover, let $(\widetilde{\mathcal{U}}, \Psi)$ be a C^{k+1} chart of $\mathcal{N}$ with the following properties:

- $P \in \widetilde{\mathcal{U}} \subset \mathcal{V}$;
- If we define $\mathcal{U} := \mathcal{M} \cap \widetilde{\mathcal{U}}$, then one has

$$\Psi(\mathcal{U}) = \{(x_1, \dots, x_{M+H}) \in \Psi(\widetilde{\mathcal{U}}) \mid x_{M+1} = \ldots = x_{M+H} = 0\} = U \times \{0\}^H,$$

  where U is an open subset of $\mathbb{R}^M$;
- The map

$$\psi := (\Psi_1|_{\mathcal{U}}, \dots, \Psi_M|_{\mathcal{U}}) : \mathcal{U} \to U$$

  is a C^{k+1} diffeomorphism.

Now observe that

$$\begin{aligned}\mathcal{R}_m \cap \mathcal{U} &= (\mathcal{T}(\mathcal{M}, \mathcal{D}) \cap \mathcal{U}) \setminus (\mathcal{A}_{m+1}(\mathcal{D})) \cap \mathcal{U}) \\ &= \mathcal{T}(\mathcal{U}, \mathcal{D}) \setminus (\mathcal{A}_{m+1}(\omega_1, \dots, \omega_H) \cap \mathcal{U}) \\ &= \mathcal{T}(\mathcal{U}, \mathcal{D}) \setminus \mathcal{A}_{m+1}(\omega_1, \dots, \omega_H)\end{aligned}$$

by (6.5). From Theorem 6.1 it follows that $\mathcal{R}_m \cap \mathcal{U}$ is covered by finitely many C^k imbedded submanifolds of $\mathcal{N}$ of dimension less or equal than m. Hence $\psi(\mathcal{R}_m \cap \mathcal{U})$ is covered by finitely many C^k imbedded submanifolds of U of dimension less or equal than m. We conclude by recalling Proposition 2.1. $\square$

As an almost immediate consequence of Corollary 6.4 and Remark 2.6, we obtain the following well-known result that provides, in particular, an upper bound for the function $\mathcal{M} \mapsto \dim \mathcal{T}(\mathcal{M}, \mathcal{D})$, see [BPR11, D18].

Corollary 6.5. *Let $\mathcal{M}$ be a M-dimensional C^{k+1} imbedded submanifold of $\mathcal{N}$. Then one has*

$$\begin{aligned} \mathcal{T}(\mathcal{M}, \mathcal{D}) &= \bigcup_{m=1}^{M} [\mathcal{R}_m \cap (\mathcal{A}_m(\mathcal{D}) \setminus \mathcal{A}_{m+1}(\mathcal{D}))] \\ &= [\mathcal{T}(\mathcal{M}, \mathcal{D}) \cap \mathcal{A}_M(\mathcal{D})] \cup \bigcup_{m=1}^{M-1} [\mathcal{R}_m \cap (\mathcal{A}_m(\mathcal{D}) \setminus \mathcal{A}_{m+1}(\mathcal{D}))] \end{aligned} \tag{6.6}$$

hence

$$\dim \mathcal{T}(\mathcal{M}, \mathcal{D}) \leq \max_{1 \leq m \leq M} \{\min\{\dim(\mathcal{A}_m(\mathcal{D}) \setminus \mathcal{A}_{m+1}(\mathcal{D})), m\}\}.$$

Remark 6.6. Let $\mathcal{D}$ be non-involutive at each point of $\mathcal{N}$ and $\mathcal{M}$ be a M-dimensional C^{k+1} imbedded submanifold of $\mathcal{N}$. Then, by Remark 6.2 and Remark 6.3, one has $\mathcal{A}_M(\mathcal{D}) = \emptyset$ and (6.6) reduces to

$$\mathcal{T}(\mathcal{M}, \mathcal{D}) = \bigcup_{m=1}^{M-1} \mathcal{R}_m^* \tag{6.7}$$

with

$$\mathcal{R}_m^* := \mathcal{R}_m \cap (\mathcal{A}_m(\mathcal{D}) \setminus \mathcal{A}_{m+1}(\mathcal{D})).$$

If define

$$\mu_M := M - \max\{m \in \{1, \ldots, M-1\} \,|\, \mathcal{H}^m(\mathcal{R}_m^*) > 0\},$$

then we obviously have $\mu_M \geq 1$ and (by (2) of Corollary 6.4)

$$\mathcal{H}^{M-\mu_M}(\mathcal{R}_m^*) = 0, \text{ for all } m \neq M - \mu_M. \tag{6.8}$$

From (6.7) and (6.8) (recalling again (2) of Corollary 6.4) it follows that $\mathcal{T}(\mathcal{M}, \mathcal{D})$ is a $(M - \mu_M)$-dimensional C^k rectifiable subset of $\mathcal{M}$. Now, it is natural to ask whether or not the equality $\mu_M = \rho_M$ is true (see Corollary 5.2). Unfortunately, we do not currently have an answer for this question.

7. An application in the context of Pfaff problem

Let us consider an N-dimensional C^2 manifold $\mathcal{N}$ (with $N \geq 3$), an open set $\mathcal{U} \subset \mathcal{N}$ and a C^2 differential 1-form ω on $\mathcal{U}$ with the following property: There exists a C^2 diffeomorphism $\varphi : U \subset \mathbb{R}^N \to \mathcal{U}$ such that

$$\varphi^* \omega = \pi_m = dx_1 + x_2 dx_3 + \cdots + x_{2m} dx_{2m+1} \qquad (2m+1 \leq N). \tag{7.1}$$

Then one has

$$(d\omega)^m \wedge \omega \neq 0, \quad (d\omega)^{m+1} = 0 \tag{7.2}$$

by (2.1) and (2.2). Recall that the following converse result by G. Darboux holds, see [BCG91, Ch. II, Theorem 3.4], [IL03, Theorem 1.9.17] and [K05, Theorem 8].

Theorem 7.1. *Let $\mathcal{N}$ be an N-dimensional C^2 manifold (with $N \geq 3$) and ω be a C^2 differential 1-form on a neighborhood $\mathcal{V}$ of $P \in \mathcal{N}$. If ω satisfies condition (7.2) everywhere in $\mathcal{V}$, then there is a C^2 diffeomorphism $\varphi : U \subset \mathbb{R}^N \to \mathcal{U} \subset \mathcal{V}$ such that $P \in \mathcal{U}$ and (7.1) holds.*

Thus, it becomes natural to ask what the sets of the type $\{\varphi^*\omega = \pi_m\}$ are reduced to when condition (7.2) is not satisfied. The following results provide some answers to this question in the case when $\mathcal{N}$ is an N-dimensional C^{k+1} manifold (with $k \geq 1$ and $N \geq 3$) and ω is a C^k differential 1-form on an open set $\mathcal{U} \subset \mathcal{N}$.

Theorem 7.2. *Assume that there exists a positive integer m such that $(d\omega)^{m+1} \neq 0$ everywhere in $\mathcal{U}$. Then, for every C^{k+1} diffeomorphism $\varphi : U \subset \mathbb{R}^N \to \mathcal{U}$, the set $\{\varphi^*\omega = \pi_m\}$ is covered by finitely many C^k imbedded submanifolds of U of dimension less or equal than $N-1$.*

Proof. Since $(d\omega)^{m+1} \neq 0$ everywhere in $\mathcal{U}$ (by assumption) and recalling the second identity of (2.1), we find

$$\{d\varphi^*\omega = d\pi_m\} \subset \{(d\varphi^*\omega)^{m+1} = (d\pi_m)^{m+1}\} = \emptyset.$$

Hence

$$\{\varphi^*\omega = \pi_m\} = \{\varphi^*\omega = \pi_m\} \cap \{d\varphi^*\omega \neq d\pi_m\}$$

and the conclusion follows by applying Corollary 4.6 with $r = 1$ to the C^k differential 1-form $\varphi^*\omega - \pi_m$. □

This further result of "low Pfaffianity" is a special case of Theorem 7.2 (stronger assumptions, smaller dimension of $\{\varphi^*\omega = \pi_m\}$).

Theorem 7.3. *Assume that there exist a positive integer m and a C^{k+1} diffeomorphism $\varphi : U \subset \mathbb{R}^N \to \mathcal{U}$ such that $2(2m+1) \leq N$ and*

$$(d\varphi^*\omega)^{m+1} \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{2m+1} \neq 0, \quad (d\varphi^*\omega)^{m+2} = 0 \tag{7.3}$$

at each point of U. Then the set $\{\varphi^*\omega = \pi_m\}$ is covered by finitely many C^k imbedded submanifolds of U of dimension less or equal than $N - 2m - 1$.

Proof. If $\theta := \varphi^*\omega - \pi_m$, then

$$\begin{aligned}(d\theta)^{2m+1} &= (d\varphi^*\omega - d\pi_m)^{2m+1} \\ &= \sum_{h=0}^{2m+1} (-1)^{2m+1-h} \binom{2m+1}{h} (d\varphi^*\omega)^h \wedge (d\pi_m)^{2m+1-h},\end{aligned}$$

where:

- For $h \geq m+2$ the addends are zero, by the second identity in (7.3);
- For $h \leq m$ the addends are zero, by the second identity in (2.1).

Hence, by also recalling the first identity in (2.1) and the first one in (7.3), we obtain

$$(d\theta)^{2m+1} = (-1)^m \binom{2m+1}{m+1} (d\varphi^*\omega)^{m+1} \wedge (d\pi_m)^m \neq 0$$

in U. The conclusion follows by applying Corollary 4.6 with $r = 2m+1$. $\square$

8. Declarations

8.1. **Ethical Approval.** This declaration is not applicable.

8.2. **Competing interests.** There is no interests of a financial or personal nature.

8.3. **Authors' contributions.** This declaration is not applicable (there is only one author).

8.4. **Funding.** No funding was received.

8.5. **Availability of data and materials.** This declaration is not applicable.

References

[A91] G. Alberti: *A Lusin type theorem for gradients.* J. Funct. Anal. 100 (1991), 110-118.

[A94] G. Alberti: *On the structure of singular sets of convex functions.* Calc. Var. 2 (1994), 17-27.

[AMS22] G. Alberti, A. Massaccesi, E. Stepanov: *On the geometric structure of currents tangent to smooth distributions.* J. Differential Geom. 122 (2022), no. 1, 1–33.

[AS94] G. Anzellotti, R. Serapioni: *$\mathcal{C}^k$-rectifiable sets.* J. Reine Angew. Math. 453 (1994), 1-20.

[B03] Z.M. Balogh: *Size of characteristic sets and functions with prescribed gradient.* J. reine angew. Math. 564 (2003), 63-83.

[BPR11] Z.M. Balogh, C. Pintea, H. Rohner: *Size of tangencies to non-involutive distributions.* Indiana Univ. Math. J. 60 (2011), no. 6, 2061-2092.

[BCG91] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, P.A. Griffiths: *Exterior Differential Systems.* Mathematical Sciences Research Institute Publications, 18. Springer-Verlag, New York, 1991. ISBN: 0-387-97411-3

[CCL99] S.S. Chern, W.H. Chen, K.S. Lam: *Lectures on Differential Geometry.* Series On University Mathematics Vol. 1, World Scientific 1999.

[D12] S. Delladio: *Functions of class C^1 subject to a Legendre condition in an enhanced density set.* Rev. Mat. Iberoam. 28 (2012), n. 1, 127-140.

[D16] S. Delladio: *A note on some topological properties of sets with finite perimeter.* Glasg. Math. J. 58 (2016), n. 3, 637-647.

[D18] S. Delladio: *Structure of tangencies to distributions via the implicit function theorem.* Rev. Mat. Iberoam. 34 (2018), n. 3, 1387-1400.

[D19a] S. Delladio: *Density-degree function for subsets of* $\mathbb{R}^n$*.* Houston J. Math. 45 (2019), n. 3, 743-762.

[D19b] S. Delladio: *Structure of prescribed gradient domains for non-integrable vector fields.* Ann. Mat. Pura ed Appl. (1923 -) 198 (2019), n.3, 685-691 DOI: 10.1007/s10231-018-0793-1.

[D19c] S. Delladio: *The tangency of a* C^1 *smooth submanifold with respect to a non-involutive* C^1 *distribution has no superdensity points.* Indiana Univ. Math. J. 68 (2019), n. 2, 393-412.

[D21] S. Delladio: *The identity* $G(D)f = F$ *for a linear partial differential operator* $G(D)$*. Lusin type and structure results in the non-integrable case.* Proc. Roy. Soc. Edinburgh Sect. A 151 (2021), no. 6, 1893–1919.

[EG92] L.C. Evans, R.F. Gariepy: *Lecture Notes on Measure Theory and Fine Properties of Functions.* (Studies in Advanced Math.) CRC Press 1992.

[F69] H. Federer: *Geometric Measure Theory.* Springer-Verlag 1969.

[IL03] T.A. Ivey, J.M. Landsberg: *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems.* Graduate Studies in Mathematics, 61. American Mathematical Society, Providence, RI, 2003. ISBN: 0-8218-3375-8

[K05] N. Kamran: *Exterior differential systems and Cartan-Kähler theory.* Acta Appl. Math. 87 (2005), no. 1-3, 147-164.

[KP08] S.G. Krantz, H.R. Parks: *Geometric Integration Theory.* Cornerstones, Birkhäuser 2008.

[KP13] S.G. Krantz, H.R. Parks: *The Implicit Function Theorem. History, Theory and Applications.* Modern Birkhäuser Classics, Birkhäuser 2013.

[L13] J.M. Lee: *Introduction to Smooth Manifolds (second edition).* Graduate Texts in Mathematics 218, Springer Verlag 2013.

[M12] F. Maggi: *Sets of Finite Perimeter and Geometric Variational Problems, an Introduction to Geometric Measure Theory.* Cambridge studies in advanced mathematics 135, Cambridge University Press 2012.

[Ma95] P. Mattila: *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge studies in advanced mathematics 44, Cambridge University Press 1995.

[Mo88] F. Morgan: *Geometric Measure Theory, a Beginner's Guide.* Academic Press Inc. 1988.

[N85] R. Narasimhan: *Analysis on Real and Complex Manifolds.* North-Holland Math. Library 35, North-Holland 1985.

[S84] L. Simon: *Lectures on Geometric Measure Theory.* Proceedings of the Centre for Mathematical Analysis, Canberra, Australia, vol. 3, 1984.

[Sp79] M. Spivak: *A Comprehensive Introduction to Differential Geometry.* Vol. I. Second edition. Publish or Perish, Inc., Wilmington, Del., 1979. xiv+668 pp. ISBN: 0-914098-83-7

[T06] M.E. Taylor: *Measure Theory and Integration.* Graduate Studies in Mathematics, 76. American Mathematical Society, Providence, RI, 2006.

[Tu21] R.M. Tudoran: *A note on the structure of prescribed gradient-like domains of non-integrable vector fields.* Math. Methods Appl. Sci. 44 (2021), no.14, 1128–11292.

[Y92] K. Yang: *Exterior Differential Systems and Equivalence Problems.* Mathematics and its Applications, 73. Kluwer Academic Publishers Group, Dordrecht, 1992. ISBN: 0-7923-1593-6

Department of Mathematics, University of Trento, via Sommarive 14, 38123 Trento, Italy

Email address: silvano.delladio@unitn.it