IDENTIFIABILITY FOR THE k-SECANT VARIETY OF THE SEGRE-VERONESE VARIETIES

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Abstract. Identifiability holds for the k-secant variety $\sigma_k(X)$ of an embedded variety $X \subset \mathbb{P}^r$ if a general $q \in \sigma_k(X)$ is in the linear span of a unique subset of X with cardinality k. We consider here the case in which X is a Segre-Veronese embedding of a multiprojective space, i.e. q corresponds to a partially symmetric tensor and X-rank is the partially symmetric tensor rank. To improve by 1 the known results we exclude the case of codimension one contact loci and handle cases in which the irreducible components of the tangential k-contact locus are linear spaces.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ with characteristic 0. Set $n := \dim X$. For each integer $k > 1$ the k-secant variety $\sigma_k(X)$ of X is the closure in $\mathbb{P}^r$ of $\cup_{S \subset X, |S|=k} \langle S \rangle$, where $\langle \quad \rangle$ denotes the linear span. Now assume $\dim \sigma_k(X) = k(n+1) - 1 < r$, i.e. assume that $\sigma_k(X)$ has the expected dimension and that $\sigma_k(X) \neq \mathbb{P}^r$. Fix a general $(p_1, \dots, p_k) \in X^k$ (in particular $p_i \in X_{\mathrm{reg}}$ for all i). By Terracini's lemma ([6, Corollary 1.11]) the linear space $\mathbb{L} := \mathbb{L}(p_1, \dots, p_k) := \langle \cup_{i=1}^k T_{p_i}(X) \rangle$ has dimension $k(n+1) - 1$. Consider the set $B := \{x \in X_{\mathrm{reg}} \mid \mathbb{L} \text{ is tangent to } X \text{ at } x\}$ and let B' be the closure of B in X. Following [10, 16, 17] the *tangential k-contact locus* $\Gamma_k(p_1, \dots, p_k)$ of X with respect to $\{p_1, \dots, p_k)$ is the union of the irreducible components of B' containing at least one of the points $p_1, \dots, p_k$. A key observation is that if $\dim \Gamma_k(p_1, \dots, p_k) = 0$, then *identifiability* holds for $\sigma_k(X)$, i.e. for a general $q \in \sigma_k(X)$ there is a unique $S \subset X$ such that $|S| = k$ and $q \in \langle S \rangle$ ([10, Lemma 4.2 and Corollary 4.3], [16, Theorem 2.4], [17, Proposition 3.9]). This observation was used in several papers and our first aim is to improve by 1 the results on the identifiability given in [10] at least for Segre-Veronese embeddings of multiprojective spaces (i.e. for partially symmetric tensors). We think that this apply to other varieties, too, and we put the proof of our results in a way that it may be applied to other varieties (Remark 2.1 and Propositions 2.2 and 2.3). In particular we get the following results.

Theorem 1.1. *Let $X \subset \mathbb{P}^r$ be a Segre-Veronese embedding of a multiprojective space. Set $n :=$ $\dim X$. Fix a positiver integer k such that $\dim \sigma_k(X) = k(n+1) - 1 < r$. Exclude the case $s = 2$, $n = 2$, $k = 8$ embedded by the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4)|$. Then the tangential k-locus of X is not a hypersurface.*

Theorem 1.2. *Fix positive integers $s \geq 2$, n_i, d_i, $1 \leq i \leq s$, and set $n := n_1 + \cdots + n_s$ and $r := -1 + \prod_{i=1}^s \binom{n_i + d_i}{n_i}$. Let $\mathcal{A}$ be the set of all $i \in \{1, \dots, s\}$ such that $d_i = 1$. If $\mathcal{A} \neq \emptyset$ assume that for each $i \in \mathcal{A}$ there is $j \in \mathcal{A}$ such that $j \neq i$ and $n_j = n_i$. Let $X \subset \mathbb{P}^r$ be the Segre-Veronese embedding of multidegree $(d_1, \dots, d_s)$. Fix an integer $k > 0$ such that $(k+n-2)(n+1) \leq r$ and assume $\dim \sigma_{k+n-2}(X) = (k+n-2)(n+1) - 1$. Exclude the case $s = 2$, $n_1 = n_2 = 1$, $d_1 = d_2 = 4$ and $k = 8$. Then X is not tangentially k-defective and identifiability holds for $\sigma_k(X)$.*

The following corollaries follow using respectively [31, Theorem 3.1] and [7].

Corollary 1.3. *Fix integers $s \geq 2$ and $d_i > 0$, $1 \leq i \leq s$, and set $r := -1 + \prod_{i=1}^s (d_i + 1)$. Take $n_i = 1$ for all i. Fix a positive integer k such that $k \leq \lfloor r/(s+1) \rfloor - s + 2$. Exclude the case $s = 2$,*

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$n_1 = n_2 = 1$, $d_1 = d_2 = 4$ and $k = 8$. Then X is not tangentially k-defective and identifiability holds for $\sigma_k(X)$.

Corollary 1.4. *Fix integers $m \ge 2$ and $s \ge 2$. Set $r := -1 + (m+1)^s$, $a := \lfloor (m+1)^s/(ms+1)\rfloor$, and let ε be the only integer such that $0 \le \varepsilon \le m$ and $a \equiv \varepsilon \pmod{m+1}$. If either $\varepsilon > 0$ or $(k+ms)(sm+1) \le (m+1)^s - 1$, set $a' := a - \varepsilon$. If $\varepsilon = 0$ and $(ms+1)a = (m+1)^s$ set $a' := a-1$. Let $X \subset \mathbb{P}^r$ be the Segre embedding of $(\mathbb{P}^m)^s$. Then for all positive integers $k \le a' - ms + 2$ X is not tangentially k-defective and for a general $q \in \sigma_k(X)$ there is a unique $S \subset X$ such that $|S| = k$ and $q \in \langle S\rangle$.*

See [10, §5] for other varieties to which the tools used to prove Theorem 1.1 may (perhaps) be applied.

Our second aim is to continue the study of the tangential k-contact loci in the sense that we give some cases in which the irreducible components of the tangential contact loci are not linear spaces. We prove the following result.

Proposition 1.5. *Fix positive integers k, x, $s \ge 2$, n_i, d_i, $1 \le i \le s$, and set $n := n_1 + \cdots + n_s$ and $r := -1 + \prod_{i=1}^{s} \binom{n_i+d_i}{n_i}$. Let $X \subset \mathbb{P}^r$ be the Segre-Veronese embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. Let $\mathcal{A}_x$ be the set of all $i \in \{1, \dots, n\}$ such that $d_i = 1$ and $n_i \ge x$. Assume $\mathcal{A}_x \neq \emptyset$ and let i_0 be the maximum of all $i \in \mathcal{A}_x$. Assume $\sigma_k(X) = k(n+1) - 1$ and $k \ge \frac{(r+1)(n_{i_0}-x+1)}{(n_{i_0}+1)(n-x+1)}$. Then the tangential k-locus of X is not the union of k x-dimensional linear subspaces.*

If $\mathcal{A}_x = \emptyset$ it is easy to check that a general k-contact locus contains no x-dimensional linear subspace (see the proof of Proposition 1.5).

From Proposition 1.5 we get the following corollary.

Corollary 1.6. *Let $X \subset \mathbb{P}^r$ be a Segre-Veronese embedding of a multiprojective space. Set $n :=$ $\dim X$ and fix a positive integer k such that $\dim \sigma_{k+n-2}(X) = (k+n-2)(n+1) - 1 < r$. Then X is not tangentially k-defective and identifiability holds for $\sigma_k(X)$.*

To show why in some sense our results improve by 1 some of the results of [10] we state here [10, theorem 5.6], which the reader may compare to Theorem 1.2 and Proposition 1.5.

Theorem 1.7. *([10, Theorem 5.6]) Fix integer $s \ge 2$, $n_i > 0$, $d_i > 0$, $1 \le i \le s$. Set $n :=$ $n_1 + \cdots + n_s$ and $r = -1 + \prod \binom{d_i+n_i}{d_i}$. Let $X \subset \mathbb{P}^r$ be the (n-dimensional) Segre-Veronese embedding of the multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, via the linear system $L_1^{d_1} \otimes \cdots \otimes L_s^{d_s}$, where each L_i is the pull-back of $\mathcal{O}_{\mathbb{P}^{n_i}}(1)$ in the projection onto the i-th factor. Assume that $\dim \sigma_{k+n-1}(X) =$ $(k+n-1)(n+1) - 1 < r$. Assume also that either:*

- *$d_i > 1$ for all i; or*
- *for all i with $d_i = 1$ there exists $j \neq i$ such that $d_j = 1$ and $n_i = n_j$.*

Then X is not weakly j-defective (hence it is j-identifiable) for every $j \leqq k$.

Thus in our paper we require $\sigma_{k+n-2}(X) \neq \mathbb{P}^r$ and that $\sigma_{k+n-2}(X)$ has the expected dimension, while Theorem 1.7 requires $\sigma_{k+n-1}(X) \neq \mathbb{P}^r$ and that $\sigma_{k+n-1}(X)$ has the expected dimension. Obviously Theorem 1.7 is not the key result of [10], the key ones being [10, Lemma 3.5 and Theorem 5.1]. Similarly, all the results of this paper follow from a key observation (Remark 2.1, Propositions 2.2 and 2.3) plus a little work for each of the results stated in the introduction.

There are a huge number of papers on the identifiability problem ([12, 13, 20, 21, 22, 23, 24, 25, 27, 28, 33]). Since a necessary condition for the identifiability for $\sigma_k(X)$ is that $\dim \sigma_k(X) =$ $k(\dim X + 1) - 1$ (and we explicitly assumed it in the statements of Theorem 1.2 and Corollary 1.6, while we use that it was known for Corollaries 1.3 and 1.4), we also list some of the papers proving that $\dim \sigma_k(X) = k(\dim X + 1) - 1$ in the case of Segre-Veronese varieties ([1, 2, 3, 4, 5, 7, 8, 11, 14, 30, 31, 32]. The key tool (the tangential contact loci) of our paper comes from [15, 16, 17]. See [29] for several topics (pure and applied) on tensors and symmetric tensors.

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2. The proofs

We recall ([15, 16, 17]) that a tangential k-contact locus $\Gamma_k(p_1, \dots, p_k)$ is said to be of type I (resp. type II) if it is irreducible (resp. it has k irreducible components, each of them containing exactly one point p_i and all of them of the same dimension). If $\Gamma_k(p_1,\dots,p_k)$ is irreducible, then $p_i\in\Gamma_k(p_1,\dots,p_k)$ for all i. For any $p\in X\subset\mathbb{P}^r$ let $2p$ denote the closed subscheme of X with $(\mathcal{I}_p)^2$ as its ideal sheaf; here $\mathcal{I}_p$ is the ideal sheaf of p on X, not on $\mathbb{P}^r$. The scheme $2p$ is a zero-dimensional scheme with $2p_{\mathrm{red}}=\{p\}$. If $p\in X_{\mathrm{reg}}$ and $\dim X=n$, then $\deg(2p)=n+1$ and the linear span $\langle 2p\rangle$ of $2p$ in $\mathbb{P}^r$ is the tangent space $T_p(X)\subset\mathbb{P}^r$ of X at p. Let $Z\subset\mathbb{P}^r$ be a zero-dimensional scheme. The linear span $\langle Z\rangle$ of Z in $\mathbb{P}^r$ is the intersection of all hyperplanes of $\mathbb{P}^r$ containing Z, with the convention $\langle Z\rangle=\mathbb{P}^r$ if there is no such hyperplane. We always have $\dim\langle Z\rangle\le\deg(Z)-1$. We say that Z is linearly independent if $\dim\langle Z\rangle=\deg(Z)-1$. Any subscheme of a linearly independent zero-dimensional scheme is linearly independent. For any closed subscheme $A\subset X$ and any $p\in A_{\mathrm{reg}}$ let $(2p,A)$ denote the closed subscheme of A with $(\mathcal{I}_{p,A})^2$ as its ideal sheaf. We have $(2p,A)\subseteq 2p$ (hence $(2p,A)$ is linearly independent) and $\langle(2p,A)\rangle=T_p(A)$; we also have $(2p,A)=2p\cap T_p(X)$ (scheme-theoretic intersection) and $(2p,A)$ is the minimal subscheme of $2p$ whose linear span contains $T_p(A)$.

Remark 2.1. Let V_1,V_2,V_3 be finite dimensional vector spaces over $\mathbb{K}$ with $V_1\ne 0$, $V_2\ne 0$, and let $u:V_1\otimes V_2\to V_3$ be the linear map associated to a bilinear map $f:V_1\times V_2\to V_3$. Assume that f is both left and right non-degenerate, i.e. assume that for all $a\in V_1\setminus\{0\}$ and $b\in V_2\setminus\{0\}$ we have $f(a,b)\ne 0$ (equivalently, the induced linear maps $f_a:V_2\to V_3$ and $f_b:V_1\to V_3$ are injective, where f_a and f_b are defined by the formulas $f_a(y)=f(a,y)$ and $f_b(x)=f(x,b)$). The classical bilinear lemma says that $\dim u(V_1\otimes V_2)\ge\dim V_1+\dim V_2-1$ (see [26, page 544] for some historical informations). D. Eisenbud extended this result in two ways and we only need one of them, but we feel that the other one may be used in a similar context (using k-genericity of the bilinear map f, the previous condition being called (by definition) 1-genericity). Let X be an integral quasi-projective variety, $\mathcal{L},\mathcal{R}\in\mathrm{Pic}(X)$ and $V\subseteq H^0(\mathcal{L})$, $W\subseteq H^0(\mathcal{R})$ finite-dimensional vector spaces. Consider the multiplication map $u:V\otimes W\to H^0(\mathcal{L}\otimes\mathcal{R})$ induced by the bilinear map $f:H^0(\mathcal{L})\times H^0(\mathcal{R})\to H^0(\mathcal{L}\otimes\mathcal{R})$. Since X is integral, f is right and left injective. Thus the bilinear lemma says that $\dim u(V\otimes W)\ge\dim V+\dim W-1$. D. Eisenbud classified in [26] the cases in which equality holds. Obviously equality holds if either $\dim V=1$ or $\dim W=1$. Thus we assume $v:=\dim V\ge 2$ and $w:=\dim W\ge 2$. The linear subspaces V and W induce rational maps $u_1:X\dashrightarrow\mathbb{P}^{v-1}$ and $u_2:X\dashrightarrow\mathbb{P}^{w-1}$ whose images generate the projective spaces $\mathbb{P}^{v-1}$ and $\mathbb{P}^{w-1}$. Let $U\subseteq X$ be a non-empty Zariski open subset of X on which both u_1 and u_2 are defined. The rational maps u_1 and u_2 define a morphism $\varphi:U\to\mathbb{P}^{v-1}\times\mathbb{P}^{w-1}$. If we identify $H^0(\mathcal{O}_{\mathbb{P}^{v-1}}(1))$ with $V^\vee$ and $H^0(\mathcal{O}_{\mathbb{P}^{w-1}}(1))$ with $W^\vee$, then $H^0(\mathcal{O}_{\mathbb{P}^{v-1}\otimes\mathbb{P}^{w-1}}(1,1)))\cong V^\vee\otimes W^\vee$ (by the Künneth's formula). By [26, last line of page 543] we have $\dim u(V\otimes W)\ge v+w+\dim\varphi(U)-2$. In particular if $\dim u(V\otimes W)=v+w-1$, then $\varphi(U)$ is a curve. Now assume that X is projective and that V and W span $\mathcal{L}$ and $\mathcal{R}$ respectively. In this case we get a surjective morphism $X\to C$ with C a projective curve ([26, last line of page 543]).

Proposition 2.2. *Let $X\subset\mathbb{P}^r$ be a smooth, non-degenerate and linearly normal variety such that $h^1(\mathcal{O}_X)=0$. Set $n:=\dim X$. Fix a positive integer k such that $\dim\sigma_k(X)=k(n+1)-1<r$ and assume that X is k-tangentially defective of type I with generic tangential k-contact locus a hypersurface A. Then:*

(1) $h^0(\mathcal{O}_X(A))=k+1$, $h^0(\mathcal{O}_X(1)(-2A))=r+1-(n+1)k$ and $h^0(\mathcal{O}_X(1)(-A))=r+1-nk$.
(2) If $\mathcal{O}_X(A)$ is base point free, then $|\mathcal{O}_X(A)|$ maps X onto $\mathbb{P}^1$. If $\mathcal{O}_X(A)$ and $\mathcal{O}_X(1)(-2A)$ are base point free, then $|\mathcal{O}_X(A)|$ and $|\mathcal{O}_X(1)(-2A)|$ map X onto $\mathbb{P}^1$ with the same fibers.

Proof. Since X is smooth, any non-constant morphism $X\to C$ with $\dim C=1$ factor through the normalization of C. Since $h^1(\mathcal{O}_X)=0$ we have $D\cong\mathbb{P}^1$ for any smooth curve D such that there is a non-constant morphism $X\to D$.

Since $h^1(\mathcal{O}_X)=0$, algebraic equivalence and linear equivalence coincide for divisors of X. Since any divisor algebraically equivalent to A is linearly equivalent to A and A contains k general points of X, we have $h^0(\mathcal{O}_X(A))\ge k+1$. Let $\mathbb{L}\subseteq H^0(\mathcal{O}_X(1))$ denote the linear subspace such that $\mathbb{PL}=\langle\cup_{i=1}^kT_{p_i}(X)\rangle$. Since $\dim\sigma_k(X)=k(n+1)-1$ and $(p_1,\dots,p_k)$ is general in X^k, Terracini's lemma gives $\dim\mathbb{L}=r-k(n+1)>0$ ([6, Corollary 1.11]). Note that $\mathbb{L}=H^0(\mathcal{O}_X(1)(-\sum_{i=1}^k2p_i))$. By the definition of tangential k-locus $\mathbb{L}$ is tangent to X at each point of A_{reg}. Thus $\mathbb{L}\supseteq H^0(\mathcal{O}_X(1)(-2A))$. Since $p_i\in A$ for all i, we have $H^0(\mathcal{O}_X(1)(-2A))\subseteq H^0(\mathcal{O}_X(1)(-\sum_{i=1}^k2p_i))$. Thus $\mathbb{L}=H^0(\mathcal{O}_X(1)(-2A))$. Since $\dim\sigma_k(X)=k(n+1)-1$, the zero-dimensional scheme $\cup_{i=1}^k 2p_i$ is linearly independent in $\mathbb{P}^r$. Thus $\cup_{i=1}^k(2p_i, A)$ is linearly independent, i.e. $\dim\langle\cup_{i=1}^k(2p_i, A)\rangle = nk - 1$, i.e. $h^0(\mathcal{O}_X(1)(-(\cup_{i=1}^k(2p_i, A)))) = r + 1 - nk$. Since $p_i \in A$ for all i, we have the inclusion $H^0(\mathcal{O}_X(1)(-(\cup_{i=1}^k(2p_i, A)))) \subseteq H^0(\mathcal{O}_X(1)(-A))$. Consider the multiplication map $u : H^0(\mathcal{O}_X(1)(-2A)) \otimes H^0(\mathcal{O}_X(A)) \to H^0(\mathcal{O}_X(1)(-A))$. Since $H^0(\mathcal{O}_X(1)(-2A)) \neq 0$, $h^0(\mathcal{O}_X(A)) \geq k+1$ and $h^0(\mathcal{O}_X(1)(-A)) - \dim \mathbb{L} \leq k$, the bilinear lemma (Remark 2.1) gives $h^0(\mathcal{O}_X(A)) = k+1$ and $h^0(\mathcal{O}_X(1)(-A)) = r+1-kn$. If $\mathcal{O}_X(A)$ is base point free and $r+1 \geq k(n+1)+2$, then the last sentence of Remark 2.1 gives that the linear systems $|\mathcal{O}_X(A)|$ and $|\mathcal{O}_X(1)(-2A)|$ map X onto a smooth curve C with the same fibers. The smooth curve C is isomorphic to $\mathbb{P}^1$, because $h^1(\mathcal{O}_X) = 0$. □

Proposition 2.3. *Let $X \subset \mathbb{P}^r$ be a smooth, non-degenerate and linearly normal variety such that $h^1(\mathcal{O}_X) = 0$. Set $n := \dim X$. Fix a positive integer k such that $\dim \sigma_k(X) = k(n+1) - 1 < r$ and assume that X is k-tangentially defective of type II with generic tangential contact locus $A_1 \cup \cdots \cup A_k$. All A_i are linearly equivalent, $h^0(\mathcal{O}_X(1)(-2kA_1)) \neq 0$, $h^0(\mathcal{O}_X(1)(-kA_1)) = r + 1 - kn$, $h^0(\mathcal{O}_X(1)(-2kA_1)) = r+1-(n+1)k$, $h^0(\mathcal{O}_X(A_1)) = 2$ and $h^0(\mathcal{O}_X(kA_1)) = k+1$. If $h^0(\mathcal{O}_X(1)(-2kA_1)) \geq 2$ and $\mathcal{O}_X(kA_1)$ has no base points, then $\mathcal{O}_X(A_1)$ has no base points and both $|\mathcal{O}_X(A_1)|$ and $|\mathcal{O}_X(1)(-2kA_1)|$ induce surjections $X \to \mathbb{P}^1$.*

Proof. Since $h^1(\mathcal{O}_X) = 0$ and each A_i is algebraically equivalent to A_1, each A_i is linearly equivalent to C. Thus $E := A_1 \cup \cdots \cup A_k$ is linearly equivalent to kA_1. Since $h^0(\mathcal{O}_X(A_1)) \geq 2$ and $h^0(\mathcal{O}_X(kA_1)) = k+1$, we have $H^0(\mathcal{O}_X(kA_1)) = S^k H^0(\mathcal{O}_X(A_1))$. Hence $\mathcal{O}_X(A_1)$ and $\mathcal{O}_X(kA_1)$ have the same base points. Then we may continue as in the proof of Proposition 2.2. □

Remark 2.4. Take a smooth, non-degenerate and linearly normal $X \subset \mathbb{P}^r$ such that $h^1(\mathcal{O}_X) = 0$. Fix an integer $k > 0$ such that $\dim \sigma_k(X) = (n+1)k - 1 < r$, $n := \dim X$.

(a) By Proposition 2.2 the tangential k-contact locus of X is not an irreducible hypersurface if either $H^0(\mathcal{O}_X(1)(-2A)) = 0$ for every hypersurface A of X or there is no non-constant morphism $X \to C$ with C is a curve. The first assumption is satisfied for the Segre embedding of a multiprojective space. The latter condition is satisfied if $n > 1$ and $\mathrm{Pic}(X) \cong \mathbb{Z}$ (e.g. when $n \geq 3$ for all complete intersections varieties).

(b) By Proposition 2.3 the tangential k-contact locus of X is not of type II and a hypersurface if either $h^0(\mathcal{O}_X(1)(-2kA)) = 0$ for every hypersurface A of X or $h^0(\mathcal{O}_X(-2kA)) \geq 2$ and there is no non-constant morphism $X \to C$ with C is a curve. Assume that $\mathrm{Pic}(X) \cong \mathbb{Z}$, and call $\mathcal{A}$ the positive generator of $\mathrm{Pic}(X)$. Assume $\mathcal{O}_X(1) \cong \mathcal{A}^{\otimes e}$ and $A \in |\mathcal{A}^{\otimes f}|$. We have $h^0(\mathcal{O}_X(1)(-2kA)) \neq 0$ if and only if $e \geq 2kf$. We have $h^0(\mathcal{O}_X(1)(-2kA)) = 1$ if and only if $e = 2kf$.

Remark 2.5. Take an integral projective variety D and assume the existence of a curve C and a non-constant morphism $f : T_1 \times D \to C$. For each $p \in D$ the map $f_{|T_1 \times \{p\}}$ is constant. Thus f is the composition of the projection $T_1 \times D \to D$ with a surjective morphism $D \to C$.

Tale $X \subset \mathbb{P}^r$. For any $q \in \mathbb{P}^r$ let $\mathcal{S}(X, q)$ denote the set of all $S \subset X$ such that $|S| = r_X(q)$ and $q \in \langle S\rangle$.

The following well-known example shows the existence of the exceptional case in the statements of Theorems 1.1 and 1.2 and of Corollary 1.3.

Example 2.6. Take $s = 2$, $n_1 = n_2 = 1$, $d_1 = d_2 = 4$ and $k = 8$. We prove that $|\mathcal{S}(X, q)| = 2$ for a general $q \in \sigma_8(X)$ and that a general tangential 8-contact locus is a smooth elliptic curve. Take a general $S \subset X$ with $|S| = 8$ and any $q \in \mathbb{P}^r$ with $S \in \mathcal{S}(X, q)$. Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)) = 9$, there is a unique $C \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)|$ such that $S \subset C$. Since $h^0(\mathcal{I}_{2S}(4, 4)) = 1$ and $2C \supset 2S$, we have $|\mathcal{I}_{2S}(4, 4)| = \{2C\}$. Hence $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $|\mathcal{O}_X(4, 4)|$ is tangentially 8-degenerate. Now we check that identifiability does not hold for $\sigma_8(X)$. Since X has a type I tangential contact locus, we have $|\mathcal{S}(X, q)| = \mathcal{S}(C, q)|$ ([10, Corollary 4.5]). We have $\dim\langle C\rangle = -1 + h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4)) - h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)) = 15$. Thus C is linearly normal with arithmetic genus 1 in $\langle S\rangle$ and $\sigma_8(C) = \langle C\rangle$. Since C is not a rational normal curve, $|\mathcal{S}(C, q)| > 1$ ([15, Theorem 3.1]). For a general S we get a general $C \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)|$, i.e. a smooth elliptic curve. In this case we have $|\mathcal{S}(C, q)| = 2$ ([9, Proposition 3.1 (d)]).

Proof of Theorem 1.1: Write $X = \prod_{i=1}^s \mathbb{P}^{n_i}$ with $s \geq 1$ and $n_i > 0$ for all i. Let $(d_1, \ldots, d_s)$ be the multidegree giving the Segre-Veronese embedding. We have $r + 1 = \prod_{i=1}^s \binom{n_i+d_i}{n_i}$.

(a) Assume for the moment the existence of a non-constant map $f : X \to C$, with C a curve. Since X is smooth, taking the normalization of C instead of C we may assume that C is a smooth curve. Since X is rational, this curve must be rational. Since any map from a projective space of dimension > 1 into a curve is constant, Remark 2.5 shows that $n_i = 1$ for at least one index i and f factors through the projection $\pi : X \to X_1$ where $X_1 = (\mathbb{P}^1)^m$ is the product of the one-dimensional factors of X, say $f = f_1 \circ \pi$ with $f_1 : X_1 \to \mathbb{P}^1$. The structure of base point free pencils on $(\mathbb{P}^1)^m$, $m > 1$, shows that f is a subsystem of a linear system $|\mathcal{O}_X(c_1, \dots, c_s)|$ such that there is $i_0 \in \{1, \dots, s\}$ with $c_i = 0$ for all $i \neq i_0$ and $n_{i_0} = 1$. Moreover a general fiber of f has c_{i_0} connected components.

(b) Assume that X has a type I tangential contact locus, i.e. that for a general $(p_1, \dots, p_k) \in X^k$, $A := \Gamma_k(p_1, \dots, p_k)$ is irreducible. Take $(a_1, \dots, a_s) \in \mathbb{N}^s$ such that $A \in |\mathcal{O}_X(a_1, \dots, a_s)|$. Note that $|\mathcal{O}_X(A)|$ is base point free. Assume for the moment $k(n+1) < r$. The last sentence of step (a) gives $k = 1$, a contradiction.

Now assume $r = k(n+1)$, i.e. $d_i = 2a_i$ for all i. Thus $a_i > 0$ for all i. By Proposition 2.2 we have $\prod_{i=1}^{s} \binom{n_i+a_i}{n_i} = k+1$ and $\prod_{i=1}^{s} \binom{n_i+2a_i}{n_i} = (n+1)k$. In particular we have

$$\prod_{i=1}^{s} \binom{n_i + 2a_i}{n_i} < (n+1) \prod_{i=1}^{s} \binom{n_i + a_i}{n_i} \tag{1}$$

Claim 1: For all integers $m \geq 1$ and $x \geq 1$ we have $\binom{m+2x}{m} / \binom{m+x}{m} \geq (m+2)/2$.

Proof of Claim 1: The left hand side of the inequality in Claim 1 is the ratio $\psi(x, m)$ between $\prod_{i=1}^{x} (m + 2x + 1 - i)$ and $\prod_{i=1}^{x} (2x + 1 - i)$. Since $\psi(x+1, m)/\psi(x, m) = (m + 2x + 2)(m + 2x + 1)/(2x + 2)(2x + 1) > 1$, it is sufficient to observe that $\psi(1, m) = (m + 2)/2$.

Claim 2: We have $\prod_{i=1}^{s} (n_i + 2) \geq 2^s (n_1 + \cdots + n_s + 1)$, unless $s = 2$ and $n_1 = n_2 = 1$.

Proof of Claim 2: First assume $n_i = 1$ for all i. We have $3^s \geq 2^s(s + 1)$ for all $s \geq 3$. Now assume $n_1 = 2$ and $n_i = 1$ for all $i > 1$. We have $4 \cdot 3^{s-1} \geq 2^s(s + 2)$ for all $s \geq 2$. Then use that the difference between the left hand side and the right hand side of the inequality in Claim 2 has positive partial derivatives with respect to the variable $n_1, \dots, n_s$ when $n_i \geq 1$ for all i.

By Claims 1 and 2 the inequality (1) fails, except at most when $s = 2$, $n_1 = n_2 = 1$, $d_1 = 2a_1$, $d_2 = 2a_2$ and $3k = r = 4a_1a_2 + a2a_2 + 2a_2$. Since $k + 1 = (a_1 + 1)(a_2 + 1)$ we get $a_1a_2 = a_1 + a_2$ which has only $(a_1, a_2) = (2, 2)$ as a solution with positive integers. This case was discussed in Example 2.6.

(c) Assume that X has a type II tangential contact locus, i.e. assume that $A_i \neq A_j$ for all $i \neq j$. Set $E := A_1 \cup \cdots \cup A_k$. Since $h^1(\mathcal{O}_X) = 0$ and each A_i is algebraically equivalent to A_1, each A_i is algebraically equivalent to A_1. Thus $\mathcal{O}_X(E) \cong \mathcal{O}_X(kA_1)$. Take $(b_1, \dots, b_s) \in \mathbb{N}^s$ such that $A_1 \in |\mathcal{O}_X(b_1, \dots, b_s)|$. By Proposition 2.3 we have $h^0(\mathcal{O}_X(b_1, \dots, b_s)) = 2$. Thus there is $i_0 \in \{1, \dots, s\}$ such that $b_i = 0$ for all $i \neq i_0$ and $n_{i_0} = b_{i_0} = 1$. If $r = (n + 1)k$, we also get $d_i = 2kb_i$ for all i, contradicting the assumption $s \geq 2$ and that the Segre-Veronese embedding is an embedding. If $r > k(n + 1)$ the last sentence of Proposition 2.3 gives that $d_i = 2kb_i$ for all $i \neq i_0$, contradicting the assumption $s \geq 2$. □

Proof of Theorem 1.2: Write $n := \dim X$. Since $\sigma_{k+n-2}(X)$ has the expected dimension and $(n + 1)(k + n - 2) \leq r$, we have $\dim \sigma_x(X) = x(n+1) - 1$ for all $x \leq k + n - 2$. Assume that $\sigma_k(X)$ has tangential locus of dimension $t > 0$ and call B a generic tangential k-locus. By [10, Lemma 3.5] X has tangential loci of dimension $t_x \geq t$ for each $k < x \leq k + n - 2$.

First assume that B is either of type I or of type II, but that no connected component of B is a linear subspace of $\mathbb{P}^r$. By [10, Lemma 3.5] we have $t_x \geq t + x - k$ for all $x = k + 1, \dots, k + n - 2$. Since $t_{k+n-2} \leq n - 2$ by Theorem 1.1, we get a contradiction. Now assume that B has type II, say $B = B_1 \cup \cdots \cup B_k$, with each B_i a linear space. We mimic the proofs of [10, Theorems 5.3 and 5.6]. For any $i \in \{1, \dots, s\}$ let ε_i denote the element $(c_1, \dots, c_s) \in \mathbb{N}^s$ such that $c_j = 0$ for all $j \neq i$ and $c_i = 1$. Since B_1 is a linear space, we have $\mathcal{A} \neq \emptyset$ and there is $i \in \mathcal{A}$ such that $\mathcal{O}_{B_1}(\varepsilon_i)$ is the positive generator of $\mathrm{Pic}(B_1)$, while $\mathcal{O}_{B_1}(\varepsilon_j) \cong \mathcal{O}_{B_1}$ for all $j \neq i$. Set $\mathcal{B}_: = \{h \in \mathcal{A} \mid n_h = n_i\}$. By assumption $|\mathcal{B}| \geq 2$. Since $d_j = 1$ for all $j \in \mathcal{B}$, there is a linear automorphism of $\mathbb{P}^r$ sending X into itself and inducing an automorphism $f : X \to X$ which maps $(o_1, \dots, o_s) \in X$, $o_i \in \mathbb{P}^{n_i}$ for all i, to some $(o'_1, \dots, o'_s)$ with $o'_h = o_h$ if $h \notin \{i, j\}$, $o'_j = o_i$ and $o'_i = o_j$. This automorphism does not fix the numerical invariants of a generic tangential k-locus, a contradiction. □

Proofs of Corollaries 1.3 and 1.4: The improvement by 1 of [10, Corollaries 5.4 and 5.5] is an easy consequence of Theorem 1.1, the last part of the proof of Theorem 1.2, the non-defectivity of almost all Segre-Veronese varieties of $(\mathbb{P}^1)^s$ ([31]) and of [7, Theorem 3.1]. $\square$

Proof of Proposition 1.5: For $i = 1,\dots,s$ let $\pi_i : X \to \mathbb{P}^{n_i}$ denote the projection onto the i-th factor of X. For any x-dimensional linear subspace $L \subset X$ there is a unique $i \in \{1,\dots,s\}$ such that $\pi_j(L)$ is a point for all $j \ne i$, while π_i induces an embedding of L into $\mathbb{P}^{n_i}$. Thus $n_i \ge x$. Since L is a linear subspace of X contained in the i-th factor $\mathbb{P}^{n_i}$ of X, we have $d_i = 1$. Thus $i \in \mathcal{A}_x$ and in particular $n_i \le n_{i_0}$.

Claim 1: We have $k \ge \frac{(r+1)(n_i-x+1)}{(n_i+1)(n-x+1)}$.

Proof of Claim 1: By assumption we have $k \ge \frac{(r+1)(n_{i_0}-x+1)}{(n_{i_0}+1)(n-x+1)}$. Since $n_i \le n_{i_0}$, we get Claim 1.

Now take $L = L_1$ with $L_1 \cup \cdots \cup L_k = \Gamma_k(p_1,\dots,p_k)$ for some general $(p_1,\dots,p_k) \in X^k$. It is easy to check that by the generality of $(p_1,\dots,p_k)$ we get $\pi_j(L_h) = 0$ for all $j \ne 1$ and all $1 \le h \le k$ (as in [10, Remark 5.2 and proofs of Theorems 5.3 and 5.6]), i.e. all $L_1,\dots,L_k$ are contained in the same ruling of X. Up to a permutation of the factors of X we may assume $i = 1$. Fix a general linear space $H \subset \mathbb{P}^{n_1}$ with $\dim H = n_1 - x$ and set $X_H := H \times \prod_{i=2}^s \mathbb{P}^{n_i}$ seen as a codimension x subvariety of X. For a general H we have $|X_H \cap L_h| = 1$ for all h. Set $\{q_h\} := X_H \cap L_h$.

Claim 2: $\dim\langle \cup_{i=1}^k T_{q_i} X_H\rangle = k(n_1 - x + 1) - 1$.

Proof of Claim 2: By assumption $\mathbb{L} := \langle \cup_{i=1}^k T_{p_i} X\rangle$ has dimension $k(n+1) - 1$, i.e. $T_{p_i}X \cap T_{p_j}X = \emptyset$ for all $i \ne j$ and the linear spaces $T_{p_1}X,\dots,T_{p_k}X$ are linearly independent. Fix $i \in \{1,\dots,k\}$ and any $o_i \in L_i$. Let $M[o_i]$ be any codimension x linear subspace of $\mathbb{P}^{n_1}$ such that $M[o_i] \cap L_i = \{o_i\}$. Set $X[o_i] = M[o_i] \times \prod_{i=2}^s \mathbb{P}^{n_i}$. Since $d_1 = 1$ and L_i and $M[o_i]$ are linear subspaces of $\mathbb{P}^{n_1}$ with $M[o_i] \cap L_i = \{o_i\}$ and $\langle M[o_i] \cup L_i\rangle = \mathbb{P}^{n_1}$, we have $L_i \cap T_{o_i}X[o_i] = \{o_i\}$ and $T_{p_i}X = \langle L_i \cup T_{o_i}X[o_i]\rangle$. In particular $T_{q_i}X_H \subset T_{p_i}X$ for all i. Thus $T_{q_i}X_H \cap T_{q_j}X_H = \emptyset$ for all $i \ne j$ and $T_{q_1}X_H,\dots,T_{q_k}X_H$ are linearly independent, concluding the proof of Claim 2.

Since $\dim X_H = n - x$, Claim 2 and Terracini's lemma ([6, Corollary 1.11]) give $\sigma_k(X_H) = k(n-x+1)-1$. Since X_H is a Segre-Veronese variety and $d_1 = 1$, we have $\dim\langle X_H\rangle = -1 + (n_1 - x+1)\prod_{i=2}^s \binom{n_i+d_i}{n_i} = -1+(r+1)(n_1-x+1)/(n_1+1)$. Thus $k(n-x+1) \le (r+1)(n_1-x+1)/(n_1+1)$. Thus $k \le \frac{(r+1)(n_1-x+1)}{(n_1+1)(n-x+1)}$. Claim 1 gives $k = \frac{(r+1)(n_1-x+1)}{(n_1+1)(n-x+1)}$. We get $\mathbb{L} \supseteq \langle \sigma_k(X_H)\rangle$. Since X_H is a covering family of subvarieties of X, we get $\mathbb{L} = \mathbb{P}^r$, contradicting the assumption $k(n+1) \le r$. $\square$

Proof of Corollary 1.6: Assume that X is tangentially k-degenerate with tangential k contact locus of dimension $e > 0$. Set $k_0 := \lfloor r/(n+1)\rfloor$. Since X is not defective and $k_0(n+1) \le r$, Theorem 1.1 says that the tangential k_0-locus of X is not a hypersurface, i.e it has dimension $f \le n-2$. By Proposition 1.5 the tangential k_0-locus of X is not of type II with linear spaces as its irreducible components. Since $k \le k_0$, we have $f \ge e > 0$. For each $x \in \{k,\dots,k_0\}$ let e_x be the dimension of the tangential x-contact locus Γ_x of X. Thus $e_k = e$ and $e_{k_0} = f$. If $k \le x < k_0$ we have $e_x \le e_{x+1}$ and equality holds only if Γ_x and Γ_{x+1} have type II with linear components ([10, Lemma 3.5]). Since $k \le k_0 - n + 2$ and $f - e \le f - 1 \le n - 3$, there is $t \in \{k,\dots,k_0-1\}$ such that $e_t = e_{t+1}$. By [10, Lemma 3.5] each irreducible component of Γ_t and Γ_{t+1} is an e_t-dimensional linear space. Fix any integer z such that $t \le z \le k_0$ and a general z-contact locus $\Gamma_z(p_1,\dots,p_z)$, $(p_1,\dots,p_z)$ general in X^z. Since $\Gamma_t(p_1,\dots,p_t) = L_1 \cup \cdots \cup L_t$ with each L_i a linear space and $p_i \in L_i$ for all i, permuting the points $p_1,\dots,p_z$ we see that $\Gamma_z(p_1,\dots,p_z) \supseteq L_1 \cup \cdots \cup L_z$ with each L_i an e_t-dimensional linear space containing p_i. Taking $z = k_0$ we get that the k_0-contact locus contains enough linear spaces, each of them containing a different point $p_1,\dots,p_{k_0}$, to get a contradiction using the proof of Proposition 1.5 (in that set-up we have $x = e_t$). $\square$

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