Second Order Homological Obstructions and Global Sullivan–type Conditions on Real Algebraic Varieties

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Abstract

It is well-known that the existence of non–algebraic $\mathbb{Z}/2$–homology classes of a real algebraic manifold Y is equivalent to the existence of non–algebraic elements of the unoriented bordism group of Y and generates (first order) obstructions which prevent the possibility of realizing algebraic properties of smooth objects defined on Y. The main aim of this paper is to investigate the existence of smooth maps $f : X \longrightarrow Y$ between a real algebraic manifold and Y not homotopic to any regular map when Y has totally algebraic homology, i.e, when the first order obstructions on Y do not occur. In this situation, we also discover that the homology of Y generates obstructions: the second order obstructions on Y. In particular, our results establish a clear distinction between the property of a smooth map f to be bordant to a regular map and the property of f to be homotopic to a regular map. As a byproduct, we obtain two global versions of Sullivan's condition on the local Euler characteristic of a real algebraic set and give obstructions to the existence of algebraic tubular neighborhoods of algebraic submanifolds of $\mathbb{R}^n$.

Key words: Real algebraic homotopy classes, Second order obstructions, Sullivan-type conditions, Real algebraic manifolds, Real algebraic sets.

Introduction

A $\mathbb{Z}/2$–homology class of an affine real algebraic manifold Y is called algebraic if it is represented by a Zariski closed subset of Y. The algebraic homology group $H_*^{alg}(Y, \mathbb{Z}/2)$ of Y is the subgroup of $H_*(Y, \mathbb{Z}/2)$ generated by all algebraic homology classes of Y. This concept plays a crucial role in the study of the classical problem of making smooth objects algebraic (see chapters 11–14 of [11]). Let us recall two of the main aspects of this fact. Let M be a compact smooth submanifold of Y and let $[M]$ be the $\mathbb{Z}/2$–homology class of Y represented by M. In order to approximate M in Y by algebraic submanifolds, the condition $[M] \in H_*^{alg}(Y, \mathbb{Z}/2)$ is necessary. There are cases in which this condition is also sufficient: for example, when Y is compact and M has codimension one or when Y is a compact 3–fold and M is a curve (see [5], [12] and section 12.4 of [11])). Examples of submanifolds M which do not verify the previous necessary condition can be obtained as follows. By a famous result of Thom [24], we know that, if $m \leq \frac{1}{2}\dim(Y)$ or $m = \dim(Y) - 1$, each homology class in $H_m(Y, \mathbb{Z}/2)$ is realizable by smooth submanifolds of Y. In this way, for each non–algebraic element of $H_m(Y, \mathbb{Z}/2)$, there is a m–dimensional compact smooth submanifold M of Y which is not approximable in Y by algebraic submanifolds. Consider now a compact affine real algebraic manifold X and a smooth map $f : X \longrightarrow Y$ (recall that X and Y have natural structures of smooth manifolds). In the real algebraic setting, it is natural to wonder whether f has at least one of the following algebraic properties:

1. *the unoriented bordism class of f is algebraic, i.e., there exist a compact smooth manifold W with boundary and a smooth map $F : W \longrightarrow Y$ such that $\partial W = X \sqcup X'$ where X' is an affine real algebraic manifold, $F|_X = f$ and $F|_{X'}$ is a regular map,*

2. *f is homotopic to a regular map*

where, evidently, *(2)* $\Longrightarrow$ *(1)*. Property *(1)* is closely related to the algebraic properties of the homology of Y. Evidently, the truthfulness of each of the previous properties is subordinated to the fact that f sends the fundamental $\mathbb{Z}/2$–homology class of X into $H_*^{alg}(Y, \mathbb{Z}/2)$. From the Steenrod Representability Theorem [24], it follows that, for each $\mathbb{Z}/2$–homology class α of Y, there is a smooth map $f : X \longrightarrow Y$ such that $f_*([X]) = \alpha$. If α is not algebraic, then f cannot satisfy property *(1)*. In particular, if the unoriented bordism class of each smooth map from a compact affine real algebraic manifold to Y is algebraic, then the homology of Y is totally algebraic, i.e., $H_*^{alg}(Y, \mathbb{Z}/2) = H_*(Y, \mathbb{Z}/2)$. The converse of the latter fact is also true. It follows from deep results of Differential Topology.

Theorem (Thom [24], Milnor [21], Conner–Floyd [13]). *Let Y be an affine real algebraic manifold with totally algebraic homology. Then, the unoriented bordism class of each smooth map from a compact affine real algebraic manifold to Y is algebraic.*

The previous arguments show how the existence of non–algebraic homology classes of Y generates obstructions to the possibility of realizing algebraic properties of smooth objects defined on Y. Because of their importance and their relationships with property *(1)*, we call the obstructions induced by the inequality $H_*^{alg}(Y, \mathbb{Z}/2) \neq H_*(Y, \mathbb{Z}/2)$ *first order obstructions on* Y. In literature, there are many examples of algebraic manifolds without totally algebraic homology (see section 3, chapter 11 of [11] and the references at the end of that chapter).

The main purpose of this paper is to investigate the existence of smooth maps $f : X \longrightarrow Y$ between affine algebraic manifolds not homotopic to any regular map when Y has totally algebraic homology, i.e, when the first order obstructions on Y do not occur. In this situation, we also discover that the homology of Y generates obstructions which we call *second order obstructions on* Y. In particular, our results establish a clear distinction between the property of a smooth map f to be bordant to a regular map and the property of f to be homotopic to a regular map. As a byproduct, we obtain two global versions of Sullivan's condition on the local Euler characteristic of real algebraic sets and give obstructions to the existence of semi–local and global algebraic tubular neighborhoods of algebraic submanifolds of $\mathbb{R}^n$.

The remainder of the paper is subdivided into three sections. In section 1, we present the main theorems. Section 2 is devoted to the introduction and the study of the crucial notion of obstructive system of an algebraic manifold. In section 3, we prove the main theorems making use of the results of section 2.

1 The main theorems

Let us start recalling some classical notions and fixing some notations. Let V be a real algebraic set equipped with the Zariski topology. A point p of V is nonsingular of dimension r is the ring of germs of regular functions on V at p is a regular local ring of dimension r. The dimension $\dim(V)$ of V is the largest dimension of nonsingular points of V and $\mathrm{Nonsing}(V)$ indicates the set of all nonsingular points of V of dimension $\dim(V)$. If $V = \mathrm{Nonsing}(V)$, then V is called nonsingular. By *algebraic manifold*, we mean a nonsingular real algebraic set. Unless otherwise indicated, all algebraic manifolds are considered equipped with the euclidean topology. As is usual, the notion of irreducibility of an algebraic manifold refers to the Zariski topology. Let Y be an algebraic manifold. By algebraic submanifold of Y, we mean a nonsingular Zariski closed subset of Y. Let us specify the meaning of algebraic homology of an algebraic manifold. Since we consider only homology with coefficients in $\mathbb{Z}/2$, we use the symbol $H_m(\cdot)$ in place of $H_m(\cdot, \mathbb{Z}/2)$. Let Y be a r–dimensional algebraic manifold and let $m \in \{0, 1, \ldots, r\}$. Let K be a m–dimensional finite subpolyhedron of Y, i.e., the topological subspace of Y associated with a m–dimensional finite subcomplex of a certain triangulation of Y. K determinates uniquely a m–cycle of Y and hence an unique element $[K]$ of $H_m(Y)$ called homology class of Y represented by K. If $K = Y$, then $[K]$ is called fundamental class of Y. The latter notion extends to all compact smooth manifolds. Recall that every compact semi–algebraic subset of Y is a semi–algebraic subpolyhedron of Y. A m^{th}–homology class α of Y is said to be algebraic if there is a m–dimensional Zariski closed subset Z of Y such that the euclidean closure Z^* of $\mathrm{Nonsing}(Z)$ in Y is compact and $[Z^*] = \alpha$ (remark that, when Z is compact, $[Z^*] = [Z]$). The null element of $H_m(Y)$ is considered algebraic. If each element of $H_m(Y)$ is algebraic, $H_m(Y)$ is said to be algebraic. When $H_m(Y)$ is algebraic for each integer m, the homology of Y is said to be totally algebraic. Let $g : X \longrightarrow Y$ be a continuous map from an algebraic manifold to Y and let m be an integer. We indicate by $H_m(g) : H_m(X) \longrightarrow H_m(Y)$ the homomorphism induced by g and, when no confusion is possible concerning the index m, we use g_* instead of $H_m(g)$. We denote by $\mathbb{N}$ the set of all nonnegative integers and define $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Let us present the main theorems of this paper.

Second order homological obstructions. We have the following results.

Theorem 1.1 *Let Y be a smooth manifold, let $m \in \mathbb{N}^*$ and let α be a nonzero m^{th}–homology class of Y. Then, for each $d \in \mathbb{N}^*$, there exist a $(m+d)$–dimensional compact irreducible algebraic manifold $X_{\alpha,d}$ (which can be choosen connected if $d \geq 2$) and a smooth map $f_{\alpha,d} : X_{\alpha,d} \longrightarrow Y$ such that, for every homotopy equivalence $h : Y \longrightarrow Y'$ between Y and a compact algebraic manifold Y' with totally algebraic homology, the composition map $h \circ f_{\alpha,d}$ is not homotopic to any regular map.*

Theorem 1.1′ *Let Y, m and α be as above. Then, for each $d \in \mathbb{N}^*$, there exist a $(m+d)$–dimensional compact irreducible algebraic manifold $X_{\alpha,d}$ (which can be choosen connected if $d \geq 2$) and a smooth map $f_{\alpha,d} : X_{\alpha,d} \longrightarrow Y$ with the following property: given any s–dimensional compact algebraic manifold Y' and any continuous map $h : Y \longrightarrow Y'$ such that $H_{s-m}(Y')$ is algebraic and the homomorphism $H_m(h) : H_m(Y) \longrightarrow H_m(Y')$ is injective, the composition map $h \circ f_{\alpha,d}$ is not homotopic to any regular map.*

Theorem 1.2 *Let Y be a smooth manifold and let M be a positive dimensional compact connected smooth submanifold of Y such that the cobordism class of M is null and M represents a nonzero homology class of Y. Then, for each positive dimensional compact smooth manifold D, there exist an algebraic manifold $X_{M,D}$ diffeomorphic to $D \times M$ and a smooth map $f_{M,D} : X_{M,D} \longrightarrow Y$ such that, for every homotopy equivalence $h : Y \longrightarrow Y'$ between Y and a compact algebraic manifold Y' with totally algebraic homology, the composition map $h \circ f_{M,D}$ is not homotopic to any regular map.*

Theorem 1.2′ *Let Y and M be as in the statement of Theorem 1.2. Let m be the dimension of* $\dim(M)$. *Then, for each positive dimensional compact smooth manifold D, there exist an algebraic manifold $X_{M,D}$ diffeomorphic to $D \times M$ and a smooth map $f_{M,D} : X_{M,D} \longrightarrow Y$ with the following property: given any s–dimensional compact algebraic manifold Y' and any continuous map $h : Y \longrightarrow Y'$ such that $H_{s-m}(Y')$ is algebraic and the homomorphism $H_m(h)$ sends the homology class of Y represented by M into a nonzero homology class of Y', the composition map $h \circ f_{M,D}$ is not homotopic to any regular map.*

Let Y be a noncompact algebraic manifold. Let us define the notion of link of infinity in Y by an explicit construction. Suppose that Y is a noncompact algebraic submanifold of $\mathbb{R}^k$. For each $\lambda \in \mathbb{R}^+ := \{x \in \mathbb{R} \,|\, x > 0\}$, we denote by $\overline{B}_k(\lambda)$ the closed ball of $\mathbb{R}^k$ centered in the origin 0 with radious λ and by $S^{k-1}(\lambda)$ the sphere $\partial\overline{B}_k(\lambda)$. Increasing k and using a translation if needed, we may suppose that 0 does not belong to Y. Let $\xi : \mathbb{R}^k \setminus \{0\} \longrightarrow \mathbb{R}^k \setminus \{0\}$ be the inversion defined by $\xi(x) := x/\|x\|^2$. The algebraic subset $\dot{Y} := \{0\} \sqcup \xi(Y)$ of $\mathbb{R}^k$ is the algebraic one point compactification of Y. By Sards's theorem and the Local Conic Structure Theorem for semi–algebraic sets (see Theorem 9.3.6 of [11]), there is $\mu \in \mathbb{R}^+$ (arbitrarily large) such that, for each $\lambda \geq \mu$, $S^{k-1}(\lambda)$ intersects transversally

Y in $\mathbb{R}^k$, $S^{k-1}(\lambda) \cap Y$ is diffeomorphic to $S^{k-1}(\mu) \cap Y$ and $\dot{Y} \cap \overline{B}_k(\frac{1}{\lambda})$ is semi–algebraically homeomorphic to the cone with basis $S^{k-1}(\lambda) \cap Y$. Fixed $\lambda \geq \mu$, we define the *link* $\mathrm{lk}_\infty(Y)$ *of infinity in* Y by $\mathrm{lk}_\infty(Y) := S^{k-1}(\lambda) \cap Y$. This definition is consistent up to diffeomorphism.

Theorem 1.3 *Let Y be a r–dimensional noncompact algebraic manifold and let $i : \mathrm{lk}_\infty(Y) \hookrightarrow Y$ be the inclusion map. Suppose that there is $m \in \mathbb{N}^*$ such that $H_m(i) : H_m(\mathrm{lk}_\infty(Y)) \longrightarrow H_m(Y)$ is not surjective and $H_{r-m-1}(i) : H_{r-m-1}(\mathrm{lk}_\infty(Y)) \longrightarrow H_{r-m-1}(Y)$ is injective. Then, for each $d \in \mathbb{N}^*$, there exist a $(m+d)$–dimensional compact irreducible algebraic manifold X_d (which can be choosen connected if $d \geq 2$) and a smooth map $f_d : X_d \longrightarrow Y$ such that, for every diffeomorphism $h : Y \longrightarrow Y'$ between Y and an algebraic manifold Y' with totally algebraic homology, the composition map $h \circ f_d$ is not homotopic to any regular map.*

Remark 1.4 *Without a doubt, among the first order obstructions on compact algebraic manifold, the one revealed by the Benedetti–Dedò–Teichner theorem [9], [23] is the deepest. This result asserts the following: "For each $r \geq 6$, there are a r–dimensional compact connected smooth manifold Y and a homology class α in $H_{d-2}(Y)$ such that, for every homeomorphism $h : Y \longrightarrow Y'$ between Y and an algebraic manifold Y', the class $h_*(\alpha)$ is not algebraic. In particular, there are a $(r-2)$–dimensional compact connected algebraic manifold X and a smooth map $f : X \longrightarrow Y$ such that, for each diffeomorphism $h : Y \longrightarrow Y'$ with Y' as above, the unoriented bordism class of $h \circ f$ is not algebraic". The five results presented above extend the Benedetti–Dedò–Teichner theorem to the setting of second order obstructions and generalize Theorem 1 of [16]. In particular, when Y is an algebraic manifold with totally algebraic homology, Theorem 1.1, Theorem 1.2 and Theorem 1.3 fully describe the notion of second order obstruction on Y.*

Let X and Y be algebraic manifolds and let $f : X \longrightarrow Y$ be a smooth map. As we have seen, the previous theorems establish a clear distinction between the property of f to represent an algebraic unoriented bordism class and the property of f to be homotopic to a regular map. In the following result, we make progress in this direction by showing how the condition "f is homotopic to a regular map" is strong when compared with other remarkable algebro–analytic properties of f.

Theorem 1.5 *Let Y be a r–dimensional compact algebraic manifold with totally algebraic homology and $r \geq 1$, let Z be a s–dimensional algebraic manifold with totally algebraic homology and let $d \in \mathbb{N}^*$ such that $s \geq r + 2d + 1$. Indicate by W the product variety $Y \times Z \times \mathbb{R}^2$. Then, there exist a $(r+d)$–dimensional compact irreducible algebraic manifold X_d (which can be choosen connected if Y is connected and $d \geq 2$) and a map $\psi_d : X_d \longrightarrow W$ such that:*

1. *ψ_d is a Nash embedding, i.e., it is a real analytic embedding and is a semi–algebraic map,*

2. *the unoriented bordism class of ψ_d is algebraic,*

3. *$\psi_d(X_d)$ is an algebraic submanifold of W,*

but ψ_d is not homotopic to any regular map.

One might think that the previous result depends on some pathological algebraic property of the algebraic manifold X_d. This is not true in the sense specified by the following result: *Let $d, q_1, \ldots, q_h \in \mathbb{N}^*$ such that $k := \sum_{i=1}^h q_i$ is even, q_i is odd for some i and $d \geq k+1$. Indicate by S^n the standard n–sphere. Then, there is a map ψ_d from the product variety $X_d := \prod_{i=1}^h S^{q_i}$ to the product variety $W := S^k \times S^d \times \mathbb{R}^2$ which satisfies properties (1), (2) and (3) of the previous theorem, but it is not homotopic to any regular map.* Such result follows easily from Theorem 13.5.1 of [11] and the proof of Theorem 1.5 given in section 3.

For a study of the homotopy classes of maps into standard spheres represented by regular maps, we refer the reader to sections 13.4 and 13.5 of [11].

Global Sullivan–type conditions. Let Y be an algebraic manifold, let Z be a Zariski closed subset of Y and let $\varphi : M \longrightarrow Y$ be a smooth map between a smooth manifold and Y. The map φ *is transverse to Z in Y* if $\varphi(M) \cap Z \subset \mathrm{Nonsing}(Z)$ and φ is transverse to $\mathrm{Nonsing}(Z)$ in Y in the usual way. Let V be a smooth submanifold of Y. If the inclusion $V \hookrightarrow Y$ is transverse to Z in Y, then V is said to be transverse to Z in Y also. Let N be a smooth manifold and let W be a subset of N. A stratification of W is a locally finite partition $\mathcal{W}$ of W into smooth submanifolds of N called strata of $\mathcal{W}$. Such a stratification is a Whitney stratification of W if any stratum of $\mathcal{W}$ is Whitney regular over any other stratum of $\mathcal{W}$ (see pages 10–11 of [18]). A Whitney stratified set is a pair $(W, \mathcal{W})$ formed by a subset W of a smooth manifold and a Whitney stratification $\mathcal{W}$ of W. We emphasize that every semi–algebraic subset of an algebraic manifold has a Whitney stratification (see page 20 of [18]). Let $f : N \longrightarrow N'$ be a smooth map between N and a smooth manifold, let W' be a subset of N' and let $\mathcal{W}'$ be a stratification of W'. If f is transverse to all strata of $\mathcal{W}'$ in N', we say that f *is transverse to $\mathcal{W}'$ in N'*. Let P' be a smooth submanifold of N'. If the inclusion $P' \hookrightarrow N'$ is transverse to $\mathcal{W}'$ in N', then P' is said to be transverse to $\mathcal{W}'$ in N' also.

The following theorem describes two general features of real algebraic sets.

Theorem 1.6 *Let Z be an algebraic subset of $\mathbb{R}^n$ with $\dim(Z) < n$ and let M be a compact smooth submanifold of $\mathbb{R}^n$. The following assertions are true.*

a) *If M is transverse to Z in $\mathbb{R}^n$, then the compact smooth manifold $M \cap Z$ is a boundary, i.e., its cobordism class is null.*

b) *If M is transverse to some Whitney stratification of Z in $\mathbb{R}^n$, then the Euler characteristic of the compact polyhedron $M \cap Z$ is even.*

As an immediate consequence, we rediscover well-known Sullivan's condition on the local Euler characteristic of a real algebraic set.

Corollary 1.7 (Sullivan [22]) *Let Z be a real algebraic set and let $z \in Z$. The Euler characteristic of the link of z in Z is even.*

We remind the reader of the key role played by Sullivan's condition and its generalizations [7], [15], [19] in the study of the topology of real algebraic sets (see [2], [8] and the excellent survey articles [14] and [20]). At the moment, it is not known whether any obstructions of global nature on the topology of real algebraic sets exist (see Question 2 in section 5 of [20]).

Question Is it possible to obtain these kinds of obstructions by means of results in the line of Theorem 1.6?

As a first step in the direction of the previous question, one can ask whether there exist a compact subpolyhedron P of $\mathbb{R}^n$, a Whitney stratification $\mathcal{P}$ of P and a compact smooth submanifold M of $\mathbb{R}^n$ such that M is transverse to $\mathcal{P}$ in $\mathbb{R}^n$, $\chi(M \cap P)$ is odd and, for each $p \in P$, the link of p in P is homeomorphic to an algebraic set.

Algebraic tubular neighborhoods. Let Y be a subset of $\mathbb{R}^n$ and let S be a subset of Y. We say that Y has an *algebraic tubular neighborhood locally at S in $\mathbb{R}^n$* if there are a neighborhood U of S in $\mathbb{R}^n$ and a regular retraction of U on Y, i.e., a regular map $\rho : U \longrightarrow Y$ such that $\rho(y) = y$ for each $y \in U \cap Y$. If there is a regular retraction from a neighborhood of Y in $\mathbb{R}^n$ on Y, then we say that Y has an *algebraic tubular neighborhood in $\mathbb{R}^n$*.

Theorem 1.8 *Let Y be a compact algebraic submanifold of $\mathbb{R}^n$ with totally algebraic homology and let K be a positive dimensional subpolyhedron of Y representing a nonzero homology class of Y. Then, Y does not have any algebraic tubular neighborhood locally at K in $\mathbb{R}^n$.*

The next corollary is a particular case of Theorem 2 of [16].

Corollary 1.9 *For each $n \in \mathbb{N}$, the unique compact algebraic submanifold of $\mathbb{R}^n$ which has an algebraic tubular neighborhood in $\mathbb{R}^n$ is the single point.*

2 Obstructive systems and their algebro–topological nature

In this section, we introduce and study the notion of obstructive system of an algebraic manifold. Thanks to this notion, we are able to generate second order obstructions on algebraic manifolds.

Definition 2.1 *Let Y be an algebraic manifold, let M be a compact smooth manifold and let $\varphi : M \longrightarrow Y$ be a smooth map. Consider the following two conditions on φ:*

(∂) *there exists a Zariski closed subset Z of Y such that $\varphi(M) \not\subset Z$, φ is transverse to Z in Y and the compact smooth manifold $\varphi^{-1}(Z)$ is not a boundary, i.e., its cobordism class is not null,*

(χ) *there are a Zariski closed subset Z of Y and a Whitney stratification $\mathscr{Z}$ of Z such that $\varphi(M) \not\subset Z$, φ is transverse to $\mathscr{Z}$ in Y and the Euler characteristic $\chi(\varphi^{-1}(Z))$ of $\varphi^{-1}(Z)$ is odd.*

We say that: φ is a regular obstructive system of Y if it satisfies condition (∂), φ is a singular obstructive system of Y if it satisfies condition (χ) and φ is an obstructive system of Y if it satisfies at least one of these two conditions. Suppose that $\varphi : M \longrightarrow Y$ is an obstructive system of Y. If φ verifies (∂) or (χ) with a set Z having dimension equal to $\dim(Y)$*, then we say that the obstructive system φ is trivial. Otherwise, φ is said to be non–trivial. We define the dimension of φ as the dimension of M. Let $m \in \mathbb{N}$. We indicate by $\mathfrak{Obstr}_m^{\mathrm{reg}}(Y)$ the family of all m–dimensional regular obstructive systems of Y, by $\mathfrak{Obstr}_m^{\mathrm{sing}}(Y)$ the family of all m–dimensional singular obstructive systems of Y and define $\mathfrak{Obstr}_m(Y) := \mathfrak{Obstr}_m^{\mathrm{reg}}(Y) \cup \mathfrak{Obstr}_m^{\mathrm{sing}}(Y)$. A subset S of Y is called m–obstructive if there is $\varphi \in \mathfrak{Obstr}_m(Y)$ whose image is S and is called obstructive if it is m–obstructive for some $m \in \mathbb{N}$.*

Remark 2.2 *The void smooth manifold has null cobordism class and null Euler characteristic.*

Remark 2.3 *Let Y be an algebraic manifold. If Y is compact and contains at least two points, then Y is an obstructive subset of itself. This is false if Y consists of a single point. Let S be a compact smooth submanifold of Y which intersects transversally a Zariski closed subset Z of Y in such a way that $S \cap Z$ is a proper smooth submanifold of S and its cobordism class is not null (for example, when the Euler characteristic of $S \cap Z$ is odd). Since the inclusion map $S \hookrightarrow Y$ is a regular obstructive system of Y, it follows that S is an obstructive subset of Y.*

Remark 2.4 (trivial obstructive systems) *The trivial obstructive systems of an algebraic manifold can be easily described as follows. Let Y be an algebraic manifold. A smooth map φ between a compact smooth manifold M and Y is a trivial obstructive system of Y if and only if the following is true: there are a union N of connected components of M and a union Z of irreducible components of Y such that N is not a boundary, $N \neq M$, $\varphi(N) \subset Z$ and $\varphi(M \setminus N) \subset Y \setminus Z$. Finally, let us remark that every* 0*–dimensional obstructive system of Y is trivial and, if Y is irreducible, every obstructive system of Y is non–trivial.*

The importance of the notion of obstructive system is described by the following two crucial lemmas which will be proved in the next section.

Lemma 2.5 *Let Y be an algebraic manifold, let $m \in \mathbb{N}$ and let $\varphi : M \longrightarrow Y$ be an element of $\mathfrak{Obstr}_m(Y)$. Then, for each $d \in \mathbb{N}^*$, there exist a $(m + d)$–dimensional* *compact irreducible algebraic manifold $X_{\varphi,d}$ (which can be choosen connected if M is connected and $d \geq 2$) and a smooth map $f_{\varphi,d} : X_{\varphi,d} \longrightarrow Y$ such that $f_{\varphi,d}(X_{\varphi,d}) = \varphi(M)$ and $f_{\varphi,d}$ is not homotopic to any regular map.*

Lemma 2.5$'$. *Let Y be an algebraic manifold, let M be a compact smooth manifold with null cobordism class and let $\varphi : M \longrightarrow Y$ be an obstructive system of Y. Then, for each positive dimensional compact smooth manifold D, there exist an algebraic manifold $X_{\varphi,D}$ diffeomorphic to $D \times M$ and a smooth map $f_{\varphi,D} : X_{\varphi,D} \longrightarrow Y$ such that $f_{\varphi,D}(X_{\varphi,D}) = \varphi(M)$ and $f_{\varphi,D}$ is not homotopic to any regular map.*

The remainder of this section is devoted to the study of the connection between the existence of obstructive systems of an algebraic manifold Y and the algebro-topological properties of Y itself. In particular, we will see that the nature of the regular obstructive systems is homological.

Proposition 2.6 *Let Y be an algebraic manifold and let $m \in \mathbb{N}^*$. If φ is a m–dimensional non–trivial regular obstructive system of Y, then φ is not unoriented bordant to any locally constant map. In particular, if $H_k(Y) = \{0\}$ for each $k \in \{1, \ldots, m\}$, then every element of $\mathfrak{Obstr}^{\mathrm{reg}}_m(Y)$ is trivial.*

Proof. Let $\varphi : M \longrightarrow Y$ be a m–dimensional non–trivial regular obstructive system of Y. By definition, there are a Zariski closed subset Z of Y such that $\varphi(M) \not\subset Z$, φ is transverse to Z in Y, $\varphi^{-1}(Z)$ is a not boundary and $\dim(Z) < \dim(Y)$. By the Hironaka Resolution Theorem, there is an algebraic multiblowup $\pi : \widetilde{Z} \longrightarrow Z$ with centers over $\mathrm{Sing}(Z) := Z \setminus \mathrm{Nonsing}(Z)$ such that $\widetilde{Z}$ is a compact algebraic manifold. Since π restricts to a biregular isomorphism from $\pi^{-1}(\mathrm{Nonsing}(Z))$ to $\mathrm{Nonsing}(Z)$, the maps φ and π are transverse. Let P be the fibered product of φ and π and let $\varrho : P \longrightarrow M$ be the natural projection. Since $\varphi(M) \cap Z \subset \mathrm{Nonsing}(Z)$, ϱ induces a diffeomorphism from P to $\varphi^{-1}(Z)$. In particular, P is not a boundary. Let Δ be the diagonal of $Y \times Y$, let $j : Z \hookrightarrow Y$ be the inclusion map, let $\pi' : \widetilde{Z} \longrightarrow Y$ be the composition $j \circ \pi$ and let $\varphi \times \pi' : M \times \widetilde{Z} \longrightarrow Y \times Y$ be the product map of φ and π'. Remark that $(\varphi \times \pi')(M \times \widetilde{Z}) \not\subset \Delta$, $\varphi \times \pi'$ is transverse to Δ in $Y \times Y$ and $(\varphi \times \pi')^{-1}(\Delta) = P$. Suppose that φ is unoriented bordant to a locally constant map. Then, there is a compact smooth manifold W with boundary and a smooth map $\Phi : W \longrightarrow Y$ such that ∂W is the disjoint union of M and a compact smooth manifold M', $\Phi|_M = \varphi$ and $\varphi' := \Phi|_{M'}$ is locally constant. Let $M'_1, \ldots, M'_e$ be the connected components of M' and, for each $i \in \{1, \ldots, e\}$, let q_i be the point of Y such that $\varphi'(M'_i) = \{q_i\}$. Since $\dim(Z) < \dim(Y)$, we may suppose that $\{q_1, \ldots, q_e\} \cap Z = \emptyset$. Consider a copy $\Phi' : W' \longrightarrow Y$ of $\Phi : W \longrightarrow Y$ and identify ∂W with $\partial W'$ in the natural way obtaining a compact smooth manifold W^* and a smooth map $\Phi^* : W^* \longrightarrow Y$ which doubles the bordism Φ. W is now a subset of W^*. Remark that the product map $\Phi^* \times \pi' : W^* \times \widetilde{Z} \longrightarrow Y \times Y$ restricted to $M \times \widetilde{Z}$ coincides with $\varphi \times \pi'$ and, for each $i \in \{1, \ldots, e\}$, $(\Phi^* \times \pi')(M'_i \times \widetilde{Z}) \subset \{q_i\} \times Z$ so $(\Phi^* \times \pi')(M'_i \times \widetilde{Z})$ is a compact subset of $Y \times Y$ disjoint from Δ. Applying the Thom Transversality Theorem to $\Phi^* \times \pi'$, we obtain a smooth map $\Psi : W^* \times \widetilde{Z} \longrightarrow Y \times Y$

arbitrarily close to $\Phi^* \times \pi'$ in $C^\infty(W^* \times \widetilde{Z}, Y \times Y)$ such that $\Psi|_{M \times \widetilde{Z}} = \varphi \times \pi'$, Ψ is transverse to Δ in $Y \times Y$ and $\Psi(M' \times \widetilde{Z}) \cap \Delta = \emptyset$. Since there is a neighborhood of $M \times \widetilde{Z}$ in $W^* \times \widetilde{Z}$ diffeomorphic to $M \times \widetilde{Z} \times (-1, 1)$, making use of Theorem 14.1.1 of [11], we obtain that $N := (W \times \widetilde{Z}) \cap \Psi^{-1}(\Delta)$ is a compact smooth submanifold of $W^* \times \widetilde{Z}$ with boundary such that $\partial N = (M \times \widetilde{Z}) \cap \Psi^{-1}(\Delta) = (\varphi \times \pi')^{-1}(\Delta) = P$. This is impossible because P is not a boundary. In this way, we have proved that φ cannot be unoriented bordant to any locally constant map. Let us conclude the proof. Let $\mathfrak{N}_*(Y)$ be the unoriented bordism group of Y. By [24] and [13], we know that $\mathfrak{N}_*(Y)$ is generated by maps $\rho_{ijk} : V_i \times W_{jk} \longrightarrow Y$ obtained by composition as follows: $\rho_{ijk} : V_i \times W_{jk} \xrightarrow{\pi_{ijk}} W_{jk} \xrightarrow{\varphi_{jk}} Y$ where $\{V_i\}_i$ is a family of compact smooth manifolds which generates $\mathfrak{N}_*(\text{point})$, $\{\pi_{ijk}\}_{i,j,k}$ are the natural projections and, for each $k \in \mathbb{N}$, $\{\varphi_{jk} : W_{jk} \longrightarrow Y\}_j$ is a family of smooth maps from k–dimensional compact smooth manifolds to Y such that $\{(\varphi_{jk})_*([W_{jk}])\}_j$ generates $H_k(Y)$ (see Lemma 2.7.1 of [7]). Let $m \in \mathbb{N}^*$ and let $\varphi \in \mathfrak{Obstr}^{\text{reg}}_m(Y)$. Bearing in mind the definition of the previous generators ρ_{ijk} of $\mathfrak{N}_*(Y)$, if $H_k(Y) = \{0\}$ for each $k \in \{1, \ldots, m\}$, it follows that $\mathfrak{N}_m(Y)$ is generated by the family $\{\rho_{ij0} = \varphi_{j0} \circ \pi_{ij0} \mid \dim(V_i) = m\}$ where each φ_{j0} (and hence each ρ_{ij0}) is constant. In this way, φ is unoriented bordant to a locally constant map. By the previous part of this proof, we know that φ must be trivial. The proof is complete. $\square$

Corollary 2.7 *Let Y be an irreducible algebraic manifold. If there is $m \in \mathbb{N}^*$ such that $H_k(Y) = \{0\}$ for each $k \in \{1, \ldots, m\}$, then $\mathfrak{Obstr}^{\text{reg}}_m(Y) = \emptyset$. In particular, if $H_k(Y) = \{0\}$ for each $k \in \mathbb{N}^*$, then Y does not have any regular obstructive system.*

Proposition 2.8 *Let Y be a r–dimensional compact algebraic manifold with $r \geq 1$, let $m \in \{1, \ldots, r\}$, let K be a m–dimensional subpolyhedron of Y with $[K] \neq 0$ and let U be a neighborhood of K in Y. Suppose $H_{r-m}(Y)$ algebraic. Then, there exists $\varphi \in \mathfrak{Obstr}^{\text{reg}}_m(Y) \cap \mathfrak{Obstr}^{\text{sing}}_m(Y)$ such that the image of φ is contained in U. In particular, every neighborhood of K in Y contains an obstructive subset of Y.*

Proof. By [24], $[K]$ has a Steenrod representation in Y, i.e., there are a compact smooth manifold N and a smooth map $\xi : N \longrightarrow Y$ such that $\xi(N) \subset U$ and $\xi_*([N]) = [K]$. Since $[K] \neq 0$, there exists a connected component M of N such that $\xi_*([M])$ is not null in $H_m(Y)$. Define the map $\varphi : M \longrightarrow Y$ as the restriction of ξ to M. Let $D : H_*(Y) \longrightarrow H^*(Y)$ be the Poincaré duality isomorphism. Since $H_{r-m}(Y)$ is algebraic, there is a $(r-m)$–dimensional Zariski closed subset Z of Y such that $D(\varphi_*([M])) \cup D([Z]) \neq 0$. Using the Thom Transversality Theorem, we may also suppose that φ is transverse to Z in Y so it holds: $0 \neq D^{-1}(D(\varphi_*([M])) \cup D([Z])) = \varphi_*([\varphi^{-1}(Z)])$. It follows that $\varphi^{-1}(Z)$ is a finite set formed by an odd number of points. In particular, $\varphi^{-1}(Z)$ has odd Euler characteristic. $\square$

The previous proof ensures that, under the hypothesis of Proposition 2.8, if $[K]$ has a Steenrod representation $\xi : N \longrightarrow Y$ in Y with N connected (for example, when K is a m–dimensional connected smooth submanifold of Y), then there exists a

smooth map $\varphi : N \longrightarrow Y$ arbitrarily close to ξ in $C^\infty(N,Y)$ which is an obstructive system of Y. In particular, φ can be choosen in such a way that $\varphi(N) \subset U$ and $\varphi_*([N]) = [K]$.
Let $f : N \longrightarrow N'$ be a smooth map between smooth manifolds and let Z' be a subset of N' equipped with a Whitney stratification $\mathcal{Z}'$. Suppose f transverse to $\mathcal{Z}'$ in N' and define the partition $f^{-1}(\mathcal{Z}')$ of $f^{-1}(Z')$ by $f^{-1}(\mathcal{Z}') := \{f^{-1}(V)\}_{V \in \mathcal{Z}'}$. It is easy to see that $f^{-1}(\mathcal{Z}')$ is a Whitney stratification of $f^{-1}(Z')$ (see (1.4), page 14 of [18]). Let P' be a smooth submanifold of N' transverse to $\mathcal{Z}'$ in N' and let $j : P' \hookrightarrow N'$ be the inclusion map. We indicate by $P' \cap \mathcal{Z}'$ the Whitney stratification $j^{-1}(\mathcal{Z}')$ of P'.

Proposition 2.9 *Let Y be an algebraic manifold with totally algebraic homology and let $m \in \mathbb{N}^*$. If φ is a m-dimensional non-trivial singular obstructive system of Y, then φ is not homotopic to any locally constant map.*

PROOF. Let $\varphi : M \longrightarrow Y$ be a m-dimensional non-trivial singular algebraic system of Y. By definition, there are a Zariski closed subset Z of Y and a Whitney stratification $\mathcal{Z}$ of Z such that $\varphi(M) \not\subset Z$, φ is transverse to $\mathcal{Z}$ in Y, $\chi(\varphi^{-1}(Z))$ is odd and $\dim(Z) < \dim(Y)$. Applying Proposition 2.8 of [1] (or Theorem 2.8.4 of [7]) to φ and using Théorème 2.D.2 of [25], we may suppose that M is a compact algebraic manifold and φ is a regular map. Let us show that φ is not homotopic to any locally constant map. Suppose on the contrary that there is a homotopy H from φ to a locally constant map. Let S^1 be the standard 1-sphere. Making H smooth and doubling H, we find a smooth map $G : M \times S^1 \longrightarrow Y$ and two distinct points a and b of S^1 such that, identifying M with $M \times \{a\}$ in $M \times S^1$, $G|_{M\times\{a\}} = \varphi$ and $G|_{M\times\{b\}}$ is locally constant. Since $\dim(Z) < \dim(Y)$, we may suppose that $G(M \times \{b\}) \cap Z = \emptyset$. Let $\pi_2 : M \times S^1 \longrightarrow S^1$ be the natural projection and let $G \times \pi_2 : M \times S^1 \longrightarrow Y \times S^1$ be the product map of G and π_2. Remark that S^1 has totally algebraic homology so, by Kunneth formula, $Y \times S^1$ has totally algebraic homology also. Embed $M \times S^1$ into some $\mathbb{R}^n$ with $n \geq 2(m+1)+1$. Applying Proposition 2.8 of [1] to $G \times \pi_2$, we obtain an algebraic submanifold T of $\mathbb{R}^n$, a diffeomorphism π from T to $M \times S^1$ and two regular maps $P : T \longrightarrow Y$ and $\xi : T \longrightarrow S^1$ such that $M \times \{a\} \subset T$, P is arbitrarily close to $G \circ \pi$ in $C^\infty(T,Y)$, $P|_{M\times\{a\}} = \varphi$, ξ is arbitrarily close to $\pi_2 \circ \pi$ in $C^\infty(T,S^1)$ and $\xi(M \times \{a\}) = \{a\}$. Let $W := P^{-1}(Z)$. Since $G|_{M\times\{a\}} = \varphi$ is transverse to $\mathcal{Z}$ in Y and $G(M \times \{b\}) \cap Z = \emptyset$, choosing P sufficiently close to $G \circ \pi$ and ξ sufficiently close to $\pi_2 \circ \pi$, it is easy to find an open neighborhood U of a in S^1 and an open neighborhood V of b in S^1 with the following two properties: (1) $\xi^{-1}(V) \cap W = \emptyset$, (2) setting $U' := \xi^{-1}(U)$, $U' \cap P^{-1}(\mathcal{Z})$ is a Whitney stratification of $U' \cap W$ and ξ restricted to each stratum of $U' \cap P^{-1}(\mathcal{Z})$ is a submersion (the reader bears in mind section D of chapter II of [25]). Let $\xi' : W \longrightarrow S^1$ be the restriction of ξ to W. By Thom's First Isotopy Lemma (see Theorem (5.2), page 58 of [18]), we have that ξ' is trivial over U. In particular, $(\xi')^{-1}(z)$ is homeomorphic to $\varphi^{-1}(Z)$ (hence $\chi((\xi')^{-1}(z))$ is odd) for each $z \in U$ and $(\xi')^{-1}(z) = \emptyset$ (hence $\chi((\xi')^{-1}(z))$ is zero) for each $z \in V$. This contradicts Lemma 5.2 of [4] (such lemma coincides with Theorem 3.9 of section 3). $\square$

Corollary 2.7 and Proposition 2.9 have the following corollary.

Corollary 2.10 *Let Y be an irreducible algebraic manifold with totally algebraic homology. Suppose that each connected component of Y is contractible. Then, Y does not have any obstructive system. In particular, this is true for each Euclidean space $\mathbb{R}^n$.*

Proposition 2.11 *Let Y be a r–dimensional noncompact algebraic manifold with $r \geq 1$, let $m \in \mathbb{N}^*$ and let $i : \mathrm{lk}_\infty(Y) \hookrightarrow Y$ be the inclusion map. Suppose that $H_{r-m}(Y)$ is algebraic, $H_m(i) : H_m(\mathrm{lk}_\infty(Y)) \longrightarrow H_m(Y)$ is not surjective and $H_{r-m-1}(i) : H_{r-m-1}(\mathrm{lk}_\infty(Y)) \longrightarrow H_{r-m-1}(Y)$ is injective. Then, the intersection $\mathfrak{Obstr}_m^{\mathrm{reg}}(Y) \cap \mathfrak{Obstr}_m^{\mathrm{sing}}(Y)$ is non–void.*

Proof. We may suppose that: Y is a nonsingular Zariski closed subset of $\mathbb{R}^k \setminus \{0\}$, $\dot{Y} := \{0\} \sqcup Y$ is the algebraic one point compactification of Y and, for some small $\varepsilon > 0$, $S^{k-1}(\varepsilon) \cap Y$ is the link of infinity in Y and $\overline{B}_k(3\varepsilon) \cap \dot{Y}$ is semi-algebraically homeomorphic to the cone with vertex 0 and basis $S^{k-1}(3\varepsilon) \cap Y$. Applying the Hironaka Resolution Theorem to $\dot{Y}$, we obtain an algebraic multiblowup $\pi : \widetilde{Y} \longrightarrow \dot{Y}$ with center over $\{0\}$ such that $\widetilde{Y}$ is a compact algebraic manifold. Define $U := \widetilde{Y} \setminus \pi^{-1}(0)$, $V := \pi^{-1}(\overline{B}_k(2\epsilon) \cap \dot{Y})$ and $A := U \cap V$. Since π induces a biregular isomorphism from U to Y, we can identify U with Y and $\pi^{-1}(\mathrm{lk}_\infty(Y))$ with $\mathrm{lk}_\infty(Y)$. Remark that $\mathrm{lk}_\infty(Y)$ is a deformation retract of A. Let $i : \mathrm{lk}_\infty(Y) \hookrightarrow Y$, $j : \mathrm{lk}_\infty(Y) \hookrightarrow V$, $a : Y \hookrightarrow \widetilde{Y}$ and $b : V \hookrightarrow \widetilde{Y}$ be the inclusion maps. Let $s := r - m$. Consider the following two portions of the Mayer–Vietoris sequence associated with the triad $(\widetilde{Y}, Y, V)$:

$$H_s(Y) \oplus H_s(V) \xrightarrow{\Phi_s} H_s(\widetilde{Y}) \xrightarrow{\Delta_s} H_{s-1}(\mathrm{lk}_\infty(Y)) \xrightarrow{\Psi_{s-1}} H_{s-1}(Y) \oplus H_{s-1}(V) \qquad (1)$$

and

$$\cdots \longrightarrow H_m(\mathrm{lk}_\infty(Y)) \xrightarrow{\Psi_m} H_m(Y) \oplus H_m(V) \xrightarrow{\Phi_m} H_m(\widetilde{Y}) \longrightarrow \cdots . \qquad (2)$$

Let us recall that, for each $k \in \mathbb{N}$, $\Psi_k = H_k(i) \oplus H_k(j)$ and $\Phi_k(\alpha, \beta) = H_k(a)(\alpha) + H_k(b)(\beta)$. By hypothesis, $H_{s-1}(i)$ is injective so Ψ_{s-1} is injective also. From the exactness of (1), it follows that Φ_s is surjective. Let us prove that the homomorphism $H_m(a)$ is not null. Suppose this is false. From the definition of Φ_m, we obtain that $\ker(\Phi_m) = H_m(Y) \oplus \ker(H_m(b))$. By the exactness of (2), we have that $\mathrm{Image}(\Psi_m) = H_m(Y) \oplus \ker(H_m(b))$. In particular, $H_m(i)$ is surjective. This contradicts our assumptions, hence $H_m(a)$ is not null. Let $V' := \pi^{-1}(\overline{B}_k(3\varepsilon) \cap \dot{Y})$. Since $H_m(a)$ is not null and $\widetilde{Y} \setminus V'$ is a deformation retract of Y, there is a m–dimensional finite subpolyhedron K of $\widetilde{Y} \setminus V'$ such that the homology class of $\widetilde{Y}$ represented by K is not null. Let B be a neighborhood of K in $\widetilde{Y}$ disjoint from V. Let us conclude the proof following the second part of the proof of Proposition 2.8. Using [24], we find a m–dimensional compact smooth manifold N and a smooth map $\xi : N \longrightarrow \widetilde{Y}$ such that $\xi(N) \subset B$ and $\xi_*([N]) = [K]$. Since $[K] \neq 0$, there is a connected component M of N such that $\xi_*([M]) \neq 0$. Let

$\psi : M \longrightarrow \widetilde{Y}$ be the restriction of ξ to M. Let $D : H_*(\widetilde{Y}) \longrightarrow H^*(\widetilde{Y})$ be the Poincaré duality isomorphism. Since Φ_s is surjective and $H_s(Y)$ is algebraic, there are a s–dimensional Zariski closed subset W of $\widetilde{Y}$ and a s–dimensional subpolyhedron H of $\widetilde{Y}$ contained in V such that $D(\psi_*([M])) \cup D([W]+[H]) \neq 0$. Since $\psi(M) \subset B$, $H \subset V$ and $B \cap V = \emptyset$, $D(\psi_*([M])) \cup D([H]) = 0$ and hence $D(\psi_*([M])) \cup D([W]) \neq 0$. Using the Thom Transversality Theorem, we may also suppose that ψ is transverse to W in Y and $\psi(M) \subset Y$. In particular, we have: $0 \neq D^{-1}(D(\psi_*([M])) \cup D([W])) = \psi_*([\psi^{-1}(W)])$. It follows that $\psi^{-1}(W)$ is formed by an odd number of points. Let $\varphi : M \longrightarrow Y$ be the composition $\pi \circ \psi$ and let $Z := \pi(W) \setminus \{0\}$. Since Z is a Zariski closed subset of Y, φ is transverse to Z in Y, $\varphi(M) \not\subset Z$ and $\varphi^{-1}(Z) = \psi^{-1}(W)$, it follows that $\varphi \in \mathfrak{Obstr}_m^{\mathrm{reg}}(Y) \cap \mathfrak{Obstr}_m^{\mathrm{sing}}(Y)$. $\square$

We have two immediate corollaries of the previous results.

Corollary 2.12 *Let Y be a r–dimensional compact irreducible algebraic manifold with totally algebraic homology and $r \geq 1$. The following is true.*

1. $\mathfrak{Obstr}_0^{\mathrm{reg}}(Y) = \emptyset$,
2. $\mathfrak{Obstr}_m^{\mathrm{reg}}(Y) \neq \emptyset$ *for some* $m \in \{1, \ldots, r-1\}$ *if and only if* $H_k(Y) \neq \{0\}$ *for some* $k \in \{1, \ldots, r-1\}$,
3. $\mathfrak{Obstr}_r^{\mathrm{reg}}(Y) \neq \emptyset$.

Let Y be a noncompact algebraic manifold and let $i : \mathrm{lk}_\infty(Y) \hookrightarrow Y$ be the inclusion map. We say that Y is *nonsingular at infinity* if $H_m(Y) = \{0\}$ for each $m \in \mathbb{N}^*$ or the following is true: $H_m(i)$ is not surjective for each $m \in \{1, \ldots, r-1\}$ and is injective for each $m \in \{0, 1, \ldots, r-2\}$.

Corollary 2.13 *Let Y be an irreducible algebraic manifold with totally algebraic homology. Suppose Y noncompact and nonsingular at infinity. The following is true:*

1. $\mathfrak{Obstr}_0^{\mathrm{reg}}(Y) = \emptyset$,
2. $\mathfrak{Obstr}_m^{\mathrm{reg}}(Y) \neq \emptyset$ *for some* $m \in \mathbb{N}^*$ *if and only if* $H_k(Y) \neq \{0\}$ *for some* $k \in \mathbb{N}^*$.

3 Proofs of the main theorems

We need to recall some well–known theorems concerning the classical problem of making smooth objects algebraic.

Theorem 3.1 (Tognoli [26]) *Every compact smooth manifold is diffeomorphic to an algebraic manifold.*

Theorem 3.2 (Tognoli [27]) *Every compact smooth manifold of positive dimension is diffeomorphic to an irreducible algebraic manifold.*

The following is an immediate consequence of Theorem 3.2 of [10].

Theorem 3.3 (Benedetti–Tognoli [10]) *Let V be a r–dimensional compact smooth submanifold of $\mathbb{R}^n$ where $n \geq 2r+1$, let W be an algebraic submanifold of $\mathbb{R}^n$ contained in V and let $\beta : W \longrightarrow \mathbb{G}_{n,n-r}(\mathbb{R})$ be the map which sends $p \in W$ into the orthogonal vector space of the tangent space of V at p in $\mathbb{R}^n$. If β is a regular map, then there is a smooth embedding ψ of V into $\mathbb{R}^n$ such that $\psi(p) = p$ for each $p \in W$ and $\psi(V)$ is an algebraic submanifold of $\mathbb{R}^n$.*

The next theorem is a version of Lemma 2.4 of [1] (for a proof adapted to this version, see Lemma 4 of [16]).

Theorem 3.4 (Akbulut–King [1]) *Let V be a compact algebraic manifold, let W be an algebraic manifold and let $f : V \longrightarrow W$ be a smooth map. Then, there are a compact algebraic manifold X, an open subset X_0 of X, a diffeomorphism $\pi : X_0 \longrightarrow V$ and a regular map $R : X \longrightarrow W$ such that $R|_{X_0}$ is arbitrarily close to $f \circ \pi$ in $C^\infty(X_0, W)$.*

Theorem 3.5 (Akbulut–King [1]) *Let V and W be as above and let $f : V \longrightarrow W$ be a smooth map whose unoriented bordism class is algebraic. Then, there are a compact algebraic manifold X, a diffeomorphism $\pi : X \longrightarrow V$ and a regular map $R : X \longrightarrow W$ such that R is arbitrarily close to $f \circ \pi$ in $C^\infty(X, W)$.*

Theorem 3.6 (Akbulut–King [3]) *Let V and W be as above and let $f : V \longrightarrow W$ be a smooth map homotopic to a regular map. Then, there exist a compact algebraic manifold T, a diffeomorphism $\xi : T \longrightarrow V$ and a regular map $P : T \longrightarrow W$ such that ξ is a regular map and P is arbitrarily close to $f \circ \xi$ in $C^\infty(T, W)$.*

Theorem 3.7 (Akbulut–King [6]) *Let V and W be as above and let $\psi : V \longrightarrow W$ be a smooth embedding whose unoriented bordism class is algebraic. Identify W with $W \times \{0\} \subset W \times \mathbb{R}^2$. Then, there exists a smooth embedding ψ' of V into $W \times \mathbb{R}^2$ arbitrarily close to ψ in $C^\infty(V, W \times \mathbb{R}^2)$ such that $\psi'(V)$ is an algebraic submanifold of $W \times \mathbb{R}^2$.*

We shall also need the following two remarkable theorems.

Theorem 3.8 (Bochnak–Kucharz [12]) *Let V be a compact algebraic manifold, let W be an irreducible algebraic manifold, let $f : V \longrightarrow W$ be a regular map and let w_1 and w_2 be regular values of f. Then, the compact smooth manifolds $f^{-1}(w_1)$ and $f^{-1}(w_2)$ are cobordant.*

Theorem 3.9 (Akbulut-King [4]) *Let V and W be real algebraic sets with W irreducible and let $f : V \longrightarrow W$ be a regular map. Then, there is a Zariski closed subset Z of W with $\dim(Z) < \dim(W)$ such that the Euler characteristic $\chi(f^{-1}(w))$ of $f^{-1}(w)$ is constant mod 2 for each $w \in W \setminus Z$.*

Let N be a smooth manifold and let $\mathcal{N}$ be a Whitney stratification of N. Let J be an interval of $\mathbb{R}$ containing 0 and let $h : N \times J \longrightarrow N$ be a continuous map. For each $t \in J$, define the map $h_t : N \longrightarrow N$ by $h_t(x) := h(x,t)$ and the family $h_t(\mathcal{N})$ of subsets of N by $h_t(\mathcal{N}) := \{h_t(V)\}_{V \in \mathcal{N}}$. We say that h is a stratified isotopy of $(N, \mathcal{N})$ if h_0 is the identity map on N and, for each $t \in J$, the following is true: h_t is a homeomorphism of N into itself, $h_t(\mathcal{N})$ is a Whitney stratification of N and h_t sends diffeomorphically each stratum V of $\mathcal{N}$ into the stratum $h_t(V)$ of $h_t(\mathcal{N})$. We indicate this isotopy by $\{h_t\}_{t \in J}$. Suppose that N is a compact smooth submanifold of $\mathbb{R}^n$ and indicate by $\|v\|$ the usual norm of a vector v of $\mathbb{R}^n$. Remark that, by compactness, $\mathcal{N}$ contains only finitely many strata. In this situation, by the expression "the isotopy $\{h_t\}_{t \in J}$ of $(N, \mathcal{N})$ is arbitrarily small", we mean the following: for each $k \in \mathbb{N}$ and for each $\varepsilon > 0$, $\{h_t\}_{t \in J}$ can be choosen in such a way that, for each $t \in J$ and for each stratum $V \in \mathcal{N}$, there are an open neighborhood U_V of V in $\mathbb{R}^n$ and a smooth extension $\overline{h}_t : U_V \longrightarrow \mathbb{R}^n$ of $h_t|_V$ such that $\sup_{|\alpha| \leq k,\, x \in U_V} \|D_\alpha \overline{h}_t(x) - D_\alpha i_{U_V}(x)\| < \varepsilon$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \sum_{i=1}^n \alpha_i$, D_α indicates the partial derivative $\partial^{|\alpha|}/\partial x_{\alpha_1} \cdots \partial x_{\alpha_n}$ and i_{U_V} is the inclusion map $U_V \hookrightarrow \mathbb{R}^n$. Let Z be a subset of N, let $\mathcal{Z}$ be a Whitney stratification of Z and let M be a smooth manifold. We indicate by $\mathcal{Z} \times M$ the Whitney stratification of the subset $Z \times M$ of $N \times M$ defined by $\mathcal{Z} \times M := \{V \times M\}_{V \in \mathcal{Z}}$.

We are now in position to prove Lemma 2.5. The proof will be constructive. The idea of the proof was originally contained in our preprint [17] (see [16] also). A similar idea has been used independently by Bochnak and Kucharz to prove an interesting obstructive result (see Proposition 1.2 of [12]).

Proof of Lemma 2.5. We subdivide the proof into three steps.

Step I (*preliminary construction*). Let W be an irreducible algebraic manifold having two connected components W_1 and W_2 both diffeomorphic to the standard d–sphere S^d, let ψ be a diffeomorphism between S^d and W_1 and let $j : W_1 \hookrightarrow W$ be the inclusion map. Let M be a compact algebraic manifold, let $\varphi : M \longrightarrow Y$ be a smooth map and let Z be a subset of Y such that $\varphi(M) \not\subset Z$. Let $\pi_1 : S^d \times M \longrightarrow S^d$ and $\pi_2 : S^d \times M \longrightarrow M$ be the natural projections. Applying Theorem 3.4 to $j \circ \psi \circ \pi_1$, we obtain a compact algebraic manifold X, an open subset X_0 of X, a diffeomorphism $\pi : X_0 \longrightarrow S^d \times M$ and a regular map $R : X \longrightarrow W$ such that $R|_{X_0}$ is arbitrarily close to $j \circ \psi \circ \pi_1 \circ \pi$ in $C^\infty(X_0, W)$. Remark that X_0 is the union of certain connected components of X. Fix a point $y \in \varphi(M) \setminus Z$ and define $f : X \longrightarrow Y$ as follows: $f := \varphi \circ \pi_2 \circ \pi$ on X_0 and $f(x) := y$ for each $x \in X \setminus X_0$.

Step II (*regular case*). Let $\varphi : M \longrightarrow Y$ be an element of $\mathfrak{Obstr}_m^{\mathrm{reg}}(Y)$. By definition, there is a Zariski closed subset Z of Y such that $\varphi(M) \not\subset Z$, φ is transverse to Z in Y and $\varphi^{-1}(Z)$ is not a boundary. By Theorem 3.1, we may suppose that

M is a compact algebraic manifold. Repeat word for word *Step I* with such φ and Z. Let us prove that f is not homotopic to any regular map. Suppose this is false. Applying Theorem 3.6 to f, we obtain a compact algebraic manifold T, a diffeomorphism $\xi : T \longrightarrow X$ which is also a regular map and a regular map $P : T \longrightarrow Y$ such that P is arbitrarily close to $f \circ \xi$ in $C^\infty(T, Y)$. Define $Z' := \xi^{-1}\pi^{-1}(S^d \times \varphi^{-1}(Z))$ and $Z'' := P^{-1}(Z)$. From the definition of f and the transversality between φ and Z, it follows that $Z' = (f \circ \xi)^{-1}(Z)$ and $f \circ \xi$ is transverse to Z. Choosing P sufficiently close to $f \circ \xi$, by transversality (see Theorem 14.1.1 of [11]), we have that Z'' is a compact algebraic submanifold of T and there is a smooth embedding η of Z'' in T arbitrarily close to the inclusion map $Z'' \hookrightarrow T$ in $C^\infty(Z'', T)$ such that $\eta(Z'') = Z'$. Let $\alpha : Z'' \longrightarrow W$ be the smooth submersion defined by $\alpha := j \circ \psi \circ \pi_1 \circ \pi \circ \xi \circ \eta$. Remark that $\alpha(Z'') = W_1$. Fix $w_1 \in W_1$ and $w_2 \in W_2$. Since $\alpha^{-1}(w_1) = \eta^{-1}\xi^{-1}\pi^{-1}(\psi^{-1}(w_1) \times \varphi^{-1}(Z))$, we have that $\alpha^{-1}(w_1)$ is diffeomorphic to $\varphi^{-1}(Z)$, hence it is not a boundary. Bearing in mind that $R|_{X_0}$ is arbitrarily close to $j \circ \psi \circ \pi_1 \circ \pi$ and η is arbitrarily close to the inclusion map $Z'' \hookrightarrow T$, we have that the regular map $R'' : Z'' \longrightarrow W$ defined by $R'' := R \circ \xi|_{Z''}$ is arbitrarily close to α in $C^\infty(Z'', W)$ also. It follows that: R'' is a submersion, $R''(Z'') = W_1$ and the fiber $(R'')^{-1}(w_1)$ is diffeomorphic to $\alpha^{-1}(w_1)$ (and hence to $\varphi^{-1}(Z)$). In particular, it holds: w_1 and w_2 are regular values of R'', the cobordism class of $(R'')^{-1}(w_1)$ is not null, while the cobordism class of $(R'')^{-1}(w_2)$ is null because $(R'')^{-1}(w_2) = \emptyset$. This contradicts Theorem 3.8. We have just proved that f is not homotopic to any regular map. Remark that there exists at least one irreducible component X' of X such that $f|_{X'}$ is not homotopic to any regular map also. Otherwise, whole f would be homotopic to some regular map which is impossible. Setting $X_{\varphi,d} := X'$ and $f_{\varphi,d} := f|_{X'}$, we obtain the desired map.

Step II (*singular case*). Let $\varphi : M \longrightarrow Y$ be an element of $\mathfrak{Obstr}_m^{\mathrm{sing}}(Y)$. By definition, there are a Zariski closed subset Z of Y and a Whitney stratification $\mathcal{Z}$ of Z such that $\varphi(M) \not\subset Z$, φ is transverse to $\mathcal{Z}$ in Y and $\chi(\varphi^{-1}(Z))$ is odd. Let us proceed as above. By Theorem 3.1, we may suppose that M is a compact algebraic manifold. Repeat word for word *Step I* with such φ and Z. We must prove that f is not homotopic to any regular map. Suppose this is not true. We apply Theorem 3.6 to f as in *Step II* (*regular case*) defining $\xi : T \longrightarrow X$ and $P : T \longrightarrow Y$ in such a way that P is arbitrarily close to $f \circ \xi$ in $C^\infty(T, Y)$. Choosing P sufficiently close to $f \circ \xi$, it is easy to find a smooth map $H : T \times \mathbb{R} \longrightarrow Y$ such that, defining $H_t : T \longrightarrow Y$ by $H_t(x) := H(x, t)$ for each $t \in \mathbb{R}$, $H_t = f \circ \xi$ for each $t \leq 0$, $H_t = P$ for each $t \geq 1$ and, for each $t \in \mathbb{R}$, H_t is arbitrarily close to $f \circ \xi$ in $C^\infty(T, Y)$ (hence H_t is transverse to $\mathcal{Z}$ in Y). Let $\mathcal{Y}$ be the Whitney stratification of Y formed by all strata of $\mathcal{Z}$ and the connected components of $Y \setminus Z$. Define $\mathcal{T} := (f \circ \xi)^{-1}(\mathcal{Y})$. Suppose T biregularly embedded into some $\mathbb{R}^n$. By Théorème 2.D.2 of [25], there is a stratified isotopy $\{h_t\}_{t \in \mathbb{R}}$ of $(T, \mathcal{T})$ arbitrarily small such that $h_t(\mathcal{T}) = H_t^{-1}(\mathcal{Y})$. Define $Z' := (f \circ \xi)^{-1}(Z) = \xi^{-1}\pi^{-1}(S^d \times \varphi^{-1}(Z))$, $\mathcal{Z}' := (f \circ \xi)^{-1}(\mathcal{Z})$, $Z'' := P^{-1}(Z)$ and $\mathcal{Z}'' := P^{-1}(\mathcal{Z})$. Remark that $h_t(\mathcal{Z}') = \mathcal{Z}'$ for each $t \leq 0$ and $h_t(\mathcal{Z}') = \mathcal{Z}''$ for each $t \geq 1$. Let $T_0 := \xi^{-1}(X_0)$ and $g : T_0 \longrightarrow W$ be the composition $j \circ \psi \circ \pi_1 \circ \pi \circ \xi|_{T_0}$. Since each stratum of $\mathcal{Z}'$ is of the form $\xi^{-1}\pi^{-1}(S^d \times \varphi^{-1}(V))$ with $V \in \mathcal{Z}$, $g(T_0) = W_1$

and, for each $w \in W_1$, $g^{-1}(w) = \xi^{-1}\pi^{-1}(\psi^{-1}(w) \times M)$, we have that g restricted to each stratum of $\mathscr{Z}'$ is a submersion onto W_1. Since $R \circ \xi|_{T_0}$ can be choosen arbitrarily close to g in $C^\infty(T_0, W)$, there is a smooth map $G : T_0 \times \mathbb{R} \longrightarrow W$ such that, defining $G_t : T_0 \longrightarrow W$ by $G_t(x) := G(x,t)$ for each $t \in \mathbb{R}$, $G_t = g$ for each $t \leq 0$, $G_t = R \circ \xi|_{T_0}$ for each $t \geq 1$ and G_t is arbitrarily close to g in $C^\infty(T_0, W)$ for each $t \in \mathbb{R}$. Let $\widetilde{G} : T_0 \times \mathbb{R} \longrightarrow W \times \mathbb{R}$ be the smooth map defined by $\widetilde{G}(x,t) := (G_t(x), t)$. Define $\widetilde{Z} := H^{-1}(Z)$ and $\widetilde{\mathscr{Z}} := H^{-1}(\mathscr{Z})$. The map H is transverse to $\mathscr{Z}$ in Y so $(\widetilde{Z}, \widetilde{\mathscr{Z}})$ is a Whitney stratified set. Remark that $\widetilde{\mathscr{Z}} \subset T_0 \times \mathbb{R}$ and $\widetilde{\mathscr{Z}} \cap (T_0 \times \{t\}) = h_t(\mathscr{Z}')$ for each $t \in \mathbb{R}$. Since $R \circ \xi|_{T_0}$ can be choosen arbitrarily close to g and $\{h_t\}_{t \in \mathbb{R}}$ can be choosen arbitrarily small, we may suppose that $R \circ \xi(T_0) = W_1$ and $\widetilde{G}$ restricted to each stratum of $\widetilde{\mathscr{Z}}$ is a submersion onto $W_1 \times \mathbb{R}$. Applying Thom's First Isotopy Lemma to $\widetilde{G}$ and $(\widetilde{Z}, \widetilde{\mathscr{Z}})$, we obtain that $g^{-1}(w) \cap Z' = \xi^{-1}\pi^{-1}(\psi^{-1}(w) \times \varphi^{-1}(Z))$ is homeomorphic to $(R \circ \xi|_{T_0})^{-1}(w) \cap Z''$ for each $w \in W_1$. In particular, $(R \circ \xi|_{T_0})^{-1}(w) \cap Z''$ is homeomorphic to $\varphi^{-1}(Z)$ for each $w \in W_1$. Let $R'' : Z'' \longrightarrow W$ be the regular map defined by $R'' := R \circ \xi|_{Z''}$. Remark that Z'' is a Zariski closed subset of T (and hence a real algebraic set) and it holds: $\chi((R'')^{-1}(w)) = \chi(\varphi^{-1}(Z))$ is odd for each $w \in W_1$ and $\chi((R'')^{-1}(w)) = 0$ for each $w \in W_2$. The latter fact contradicts Theorem 3.9. It follows that f is not homotopic to any regular map as desired.

Step III (*connectedness*). Suppose that M is connected and $d \geq 2$. We must prove that $X_{\varphi,d}$ can be choosen connected. For simplicity, we assume that $\varphi \in \mathfrak{Obstr}^{\mathrm{reg}}_m(Y)$. Repeating *Step I* (*preliminary construction*) and *Step II* (*regular case*) with $(d-1)$ instead of d, we obtain a $(m+d-1)$–dimensional compact irreducible algebraic manifold X, a connected component X_0 of X diffeomorphic to $S^{d-1} \times M$, a point y of $\varphi(M)$ and a smooth map $f : X \longrightarrow Y$ with the following properties: $f(X_0) = \varphi(M)$, $f(X \setminus X_0) = \{y\}$ and f is not homotopic to any regular map. First, suppose that $X_0 = X$. Let $X_{\varphi,d} := X \times S^1$, let $\rho : X_{\varphi,d} \longrightarrow X$ be the natural projection and let $f_{\varphi,d} : X_{\varphi,d} \longrightarrow Y$ be the composition map $f \circ \rho$. Fix $a \in S^1$ and identify X with $X \times \{a\} \subset X_{\varphi,d}$. Since f is not homotopic to any regular map from X to Y and $X \subset X_{\varphi,d}$, we have that $f_{\varphi,d}$ is not homotopic to any regular map from $X_{\varphi,d}$ to Y. Suppose now that $X_0 \neq X$. Indicate by $X_1, \ldots, X_e$ the connected components of $X \setminus X_0$. Define $A := X \times S^1$ and, for each $i \in \{0, 1, \ldots, e\}$, $A_i := X_i \times S^1$. Embed biregularly A into some $\mathbb{R}^N$ with $N \geq 2(m+d)+1$. Fix again a point a of S^1 and identify X with $X \times \{a\} \subset A$. Let $\rho : A \longrightarrow X$ be the natural projection and let $g : A \longrightarrow Y$ be the composition map $f \circ \rho$. Fix $b_0 \in A_0$, $c_e \in A_e$ and, for each $i \in \{1, \ldots, e-1\}$, two distinct points b_i and c_i of A_i in such a way that $g(b_0) = y$ and $X \cap \{b_0, b_1, c_1, \ldots, b_{e-1}, c_{e-1}, c_e\} = \emptyset$. For each $i \in \{0, 1, \ldots, e-1\}$, choose a small ball B_i of A_i centered at b_i and, for each $i \in \{1, \ldots, e\}$, choose a small ball C_i of A_i centered at c_i in such a way that $g|_{\partial B_0}$ is homotopic to the constant map which sends ∂B_0 into y and both the distance between $\bigsqcup_{i=0}^{e-1} B_i$ and $\bigsqcup_{i=1}^{e} C_i$ in A and the distance between $D := \bigsqcup_{i=0}^{e-1} B_i \cup \bigsqcup_{i=1}^{e} C_i$ and X in A are positive. Let $A' := A \setminus D$. For each $i \in \{0, 1, \ldots, e-1\}$, we attack a handle $H_i = [0,1] \times S^{m+d-1}$ to A' identifying $\{0\} \times S^{m+d-1}$ with ∂B_i and $\{1\} \times S^{m+d-1}$

with ∂C_{i+1}. We obtain a compact connected smooth submanifold $X_{\varphi,d}$ of $\mathbb{R}^N$ and a smooth map $f_{\varphi,d} : X_{\varphi,d} \longrightarrow Y$ such that $X_{\varphi,d}$ contains X and $f_{\varphi,d}$ is an extension of $g|_{A'}$. In particular, $f_{\varphi,d}|_X = f$. Applying Theorem 3.3 to $X_{\varphi,d}$, we may suppose that $X_{\varphi,d}$ is an algebraic submanifold of $\mathbb{R}^N$ also. Since f is not homotopic to any regular map from X to Y and $f_{\varphi,d}$ extends f, we have that $f_{\varphi,d}$ is not homotopic to any regular map from $X_{\varphi,d}$ to Y. $\square$

Proof of Lemma 2.5′. Let us modify *Step I* (*preliminary construction*) of the previous proof.

Step I′ (*preliminary construction*). Let D be a positive dimensional compact smooth manifold, let M be a compact algebraic manifold which is the boundary of a compact smooth manifold N with boundary, let $\varphi : M \longrightarrow Y$ be a smooth map and let Z be a subset of Y such that $\varphi(M) \not\subset Z$. By Theorem 3.2, there is an irreducible algebraic manifold W having two connected components W_1 and W_2 both diffeomorphic to D. Let ψ be a diffeomorphism between D and W_1 and let $j : W_1 \hookrightarrow W$ be the inclusion map. Let $\pi_1 : D \times M \longrightarrow D$, $\varrho_1 : D \times N \longrightarrow D$ and $\pi_2 : D \times M \longrightarrow M$ be the natural projections. Since $\partial(D \times N) = D \times M$ and $j \circ \psi \circ \varrho_1|_{D\times M} = j \circ \psi \circ \pi_1$, we have that the unoriented bordism class of $j \circ \psi \circ \pi_1$ is null (and hence algebraic). Applying Theorem 3.5 to $j \circ \psi \circ \pi_1$, we obtain a compact algebraic manifold X, a diffeomorphism $\pi : X \longrightarrow D \times M$ and a regular map $R : X \longrightarrow W$ such that R is arbitrarily close to $j \circ \psi \circ \pi_1 \circ \pi$ in $C^\infty(X, W)$. Define the smooth map $f : X \longrightarrow Y$ by the composition $\varphi \circ \pi_2 \circ \pi$.

In order to complete the proof, it suffices to repeat *Step II* (*regular case*), *Step II* (*singular case*) and *Step III* (*connectedness*) of the previous proof. $\square$

Second order homological obstructions. Theorem 1.1 and Theorem 1.2 are particular cases of Theorem 1.1′ and Theorem 1.2′ respectively. We must prove Theorem 1.1′, Theorem 1.2′, Theorem 1.3 and Theorem 1.5.

Proof of Theorem 1.1′. *First part*) By Steenrod Representability Theorem [24], there are a m–dimensional compact smooth manifold N and a smooth map $\eta : N \longrightarrow Y$ such that $\eta_*([N]) = \alpha$. Since α is a nonzero element of $H_m(Y)$, there exists a connected component M of N such that $\eta_*([M]) \neq 0$. Let $\varphi : M \longrightarrow Y$ be the restriction of η to M. Let Y' be a compact algebraic manifold and let $h : Y \longrightarrow Y'$ be a continuous map as in the statement of the theorem. Since each continuous map between Y and Y' is homotopic to a smooth map, we may assume that h is smooth. By hypothesis, it follows that $(h\circ\varphi)_*([M])$ is a nonzero element of $H_m(Y')$. Proceeding as in the proof of Proposition 2.8, we find a $(s-m)$–dimensional Zariski closed subset Z' of Y' and a smooth map $\varphi' : M \longrightarrow Y'$ arbitrarily close to $h \circ \varphi$ in $C^\infty(M, Y')$ such that g is transverse to Z' in Y' and $(\varphi')^{-1}(Z')$ is a finite set formed by an odd number of points.

Second part) Let $d \in \mathbb{N}^*$. Following *Step I* (*preliminary construction*) of the proof of Lemma 2.5, we repeat the definitions of W, j, π_1, π_2, X, X_0, π and R. Fix a point $y \in \varphi(M)$ and define a smooth map $f : X \longrightarrow Y$ as follows: $f := \varphi \circ \pi_2 \circ \pi$

on X_0 and $f(x) := y$ for each $x \in X \setminus X_0$. Choose $y' \in Y' \setminus Z'$ arbitrarily close to $h(y)$ in Y'. Let $g : X \longrightarrow Y'$ be the smooth map defines as follows: $g := \varphi' \circ \pi_2 \circ \pi$ on X_0 and $g(x) := y'$ for each $x \in X \setminus X_0$. Since g is arbitrarily close to $h \circ f$ in $C^\infty(X, Y')$, we have that g is also homotopic to $h \circ f$. Repeating *Step II* (*regular case*) and *Step III* (*connectedness*) of the just mentioned proof with f replaced with g and φ replaced by φ', it follows that: g is not homotopic to any regular map, X can be choosen irreducible and X can be choosen connected when $d \geq 2$. In particular, $h \circ f$ is not homotopic to any regular map and the proof of Theorem 1.1$'$ is complete. □

Proof of Theorem 1.2$'$. It suffices to repeat the previous proof using *Step I$'$* (*preliminary construction*) of the proof of Lemma 2.5$'$ instead of *Step I* (*preliminary construction*) of the proof of Lemma 2.5. □

Proof of Theorem 1.3. We may assume that Y has totally algebraic homology. Let $h : Y \longrightarrow Y'$ be a map as in the statement of the theorem. By the proof of Proposition 2.11, we know that there are a smooth map $\varphi : M \longrightarrow Y$ from a m-dimensional compact smooth manifold to Y and a $(r-m)$–dimensional Zariski closed subset Z of Y such that: φ is transverse to Z in Y, $\varphi^{-1}(Z)$ is a finite set formed by an odd number of points, the euclidean closure Z^* of Nonsing(Z) in Y is compact and $[Z^*]$ is a nonzero homology class of Y. By the Hironaka Resolution Theorem, there is an algebraic multiblowup $H : \widetilde{Z} \longrightarrow Z$ with centers over $\mathrm{Sing}(Z) = Z \setminus \mathrm{Nonsing}(Z)$ such that $\widetilde{Z}$ is an algebraic manifold. Since Z^* is compact, we have that $\widetilde{Z}$ is compact also. The map H restricts to a biregular isomorphism from $H^{-1}(\mathrm{Nonsing}(Z))$ to $\mathrm{Nonsing}(Z)$ so H and φ are transverse in Y. Choose a compact neighborhood U of $H^{-1}(\mathrm{Sing}(Z))$ in $\widetilde{Z}$ such that $H(U) \cap \varphi(M) = \emptyset$ (recall that $\varphi(M) \cap Z \subset \mathrm{Nonsing}(Z)$). Applying Theorem 3.5 to $h \circ H$, we obtain a compact algebraic manifold Z^*, a diffeomorphism $\varrho : Z^* \longrightarrow \widetilde{Z}$ and a regular map $h' : Z^* \longrightarrow Y'$ such that h' is arbitrarily close to $h \circ H \circ \varrho$ in $C^\infty(Z^*, Y')$. Let $U' := \varrho^{-1}(U)$. Choosing h' sufficiently close to $h \circ H \circ \varrho$, we have that: h' is transverse to $h \circ \varphi$ in Y', the fibered product of h' and $h \circ \varphi$ is diffeomorphic to $\varphi^{-1}(Z)$, $h'(U') \cap h(\varphi(M)) = \emptyset$ and $h'|_{Y' \setminus U'}$ is injective. Let Z' be the Zariski closure of $h'(Z^*)$ in Y'. Since $h'|_{Y' \setminus U'}$ is injective, the dimension of Z' is equal to $\dim(Z^*) = r - m$ so $\dim(\mathrm{Sing}(Z')) < r - m$. By the Thom Transversality Theorem, there is a smooth map $\varphi' : M \longrightarrow Y'$ arbitrarily close to $h \circ \varphi$ in $C^\infty(M, Y')$ such that $\varphi'(M) \cap \mathrm{Sing}(Z') = \emptyset$ and φ' is transverse to $\mathrm{Nonsing}(Z')$ in Y'. Choosing φ' sufficiently close to $h \circ \varphi$, it follows that: φ' is homotopic to $h \circ \varphi$, h' is transverse to φ' in Y', the fibered product P' of h' and φ' is diffeomorphic to $\varphi^{-1}(Z)$ and $h'(U') \cap \varphi'(M) = \emptyset$. Using again the injectivity of $h'|_{Y' \setminus U'}$, we have that $(\varphi')^{-1}(Z')$ and P' have the same cardinality so $(\varphi')^{-1}(Z')$ is a finite set formed by an odd number of points. Repeating the second part of the proof of Theorem 1.1$'$, we complete the proof of the present theorem. □

Proof of Theorem 1.5. Fix $p \in Y$ and $q \in Z$. Let $\varphi : Y \longrightarrow Y \times Z$ be the map which sends y into (y, q). Since φ is transverse to $\{p\} \times Z$ and $\varphi^{-1}(\{p\} \times Z) = \{p\}$, it follows that φ is a r–dimensional obstructive system of $Y \times Z$. By Lemma 2.5, there exist a $(r+d)$–dimensional irreducible compact algebraic manifold X_d (which can be choosen connected if Y is connected and $d \geq 2$) and a smooth map $f_d : X_d \longrightarrow Y \times Z$ which is not homotopic to any regular map. Remark that $r + s \geq 2(r + d) + 1$. By the Whitney Embedding Theorem, there is a smooth embedding f'_d from X_d to $Y \times Z$ arbitrarily close to f_d in $C^\infty(X_d, Y \times Z)$ (and hence homotopic to f_d). Identify $Y \times Z$ with $Y \times Z \times \{0\} \subset W$ and view f'_d as a map from X_d to W. Let $\pi : W \longrightarrow Y \times Z$ be the natural projection. Since π is a regular map, f'_d is not homotopic to any regular map from X_d to W. By the Kunneth formula, we know that W has totally algebraic homology so its unoriented bordism group is algebraic. Applying Theorem 3.7 to f'_d, we obtain a smooth embedding ψ_d from X_d to W arbitrarily close to f'_d in $C^\infty(X_d, W)$ such that $\psi_d(X_d)$ is an algebraic submanifold of W. Since ψ_d is homotopic to f'_d, we have that ψ_d is not homotopic to any regular map. Making use of the Stone–Weierstrass Approximation Theorem and the Nash Tubular Neighborhood Theorem, we may suppose that ψ_d is a Nash embedding also. $\square$

Global Sullivan–type conditions. Theorem 1.6 follows immediate from Definition 2.1 and Corollary 2.10 or from Lemma 2.5 and the Stone–Weierstrass Approximation Theorem.

Algebraic tubular neighborhoods. The following result is an easy consequence of Lemma 2.5 and the Stone–Weierstrass Approximation Theorem.

Proposition 3.10 *Let Y be an algebraic submanifold of $\mathbb{R}^n$ and let S be an obstructive subset of Y. Then, Y does not have any algebraic tubular neighborhood locally at S in $\mathbb{R}^n$.*

Theorem 1.8 follows immediately from Proposition 3.10 and the last part of Proposition 2.8. Let us give the proof of Corollary 1.9: Let Y be a compact algebraic submanifold of $\mathbb{R}^n$. If Y contains at least two points, then Y is an obstructive subset of itself so Proposition 3.10 prevents the existence of any algebraic tubular neighborhood of Y in $\mathbb{R}^n$.

Acknowledgements

I would like to thank Alberto Tognoli and Riccardo Benedetti for their invaluable uggestions. A special thank goes to Fulvio Lazzeri who helped me to better understand the nature of obstructive systems and to Masahiro Shiota for his useful remarks.

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