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On linear combinations of binomial OWA functions

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A B S T R A C T

We consider the binomial decomposition of ordered weighted averaging (OWA) functions proposed by Calvo and De Baets (1998) in the framework of Choquet integration. Our aim in the paper is to further investigate the equivalence between the two representations of OWA functions involved in the binomial decomposition: the usual canonical representation in terms of the order statistics, and the binomial representation in terms of the binomial OWA functions. We describe and discuss in detail the linear transformations that relate the coefficients of these two equivalent representations: the original expression of the weights in terms of the coefficients of the binomial representation, and its inverse, the expression of those coefficients in terms of the weights. In both cases simple and direct proofs are presented. Moreover, we consider the linear transformations between the two representations in the general linear algebra framework of unconstrained linear combinations of binomial OWA functions. In this perspective we obtain compact matrix expressions for the linear transformations, which also offer new insight on the geometry of the coefficient constraints in the binomial representation of OWA functions.

1. Introduction

In the general framework of averaging functions, and particularly in that of Choquet integration [16,17], the ordered weighted averaging functions [24,8] have an outstanding theoretical and practical relevance in a variety of modeling contexts [25,26].

The class of Choquet integrals includes both the standard and the ordered weighted averaging functions. The former are generated by additive capacities, whereas the latter are generated by symmetric capacities [9], which only depend on the cardinality of the coalitions considered. The plain mean is generated by the single capacity which is both additive and symmetric; all the other symmetric capacities are nonadditive and this one of the interesting features of ordered weighted averaging functions.

The weights of an ordered weighted averaging (OWA) function, expressed in terms of the values of the associated symmetric capacity, correspond to differences between the capacities of consecutive cardinalities. Alternatively, the weights can also be expressed in terms of the values of the symmetric Möbius transform [5,13,18] associated with that symmetric capacity. In this case, the weight dependence on the Möbius transform values is made explicit, thereby clarifying the way in which the interaction pattern contributes to the weighting structure. For this reason, the symmetric Möbius transform often plays a central role in the construction of nonadditive averaging models.

In the binomial decomposition of OWA functions, introduced in the seminal paper by Calvo and De Baets [4], the authors elaborate on the expression of the weights in terms of the symmetric Möbius transform values and introduce the binomial OWA functions, whose weights are conveniently defined by means of the binomial coefficients. In dimension $n \geq 2$, the OWA functions are

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naturally expressed as constrained linear combinations of the n binomial OWA functions C_j, where the coefficients are essentially the values of the associated symmetric Möbius transforms for coalitions of cardinality $j = 1, ..., n$. The constraints regarding the symmetric Möbius transform values reflect the usual weight constraints of OWA functions.

The binomial decomposition provides the ideal representation for the study of nonadditivity in OWA functions, particularly in relation with the various levels of k-additivity [12], with $k \le n$. In these terms, the binomial decomposition can be used to explore the k-additivity of OWA functions in the context of welfare and inequality [1]. The classical Gini welfare function, for instance, is a particular 2-additive OWA function, and the binomial decomposition is the natural framework to explore k-additive generalizations of welfare functions and their associated inequality measures.

In this paper we examine the binomial decomposition and further investigate the equivalence between two representations of OWA functions: the usual canonical representation in terms of the order statistics, and the binomial representation in terms of the binomial OWA functions. We describe and discuss in detail the linear transformations that relate the coefficients of these two representations: the original expression of the weights in terms of the coefficients of the binomial representation as in [4], and the inverse expression of those coefficients in terms of the weights.

A version of this inverse expression has recently been proposed in [20], with a long and elaborate proof based on strong induction. Here we present a common descriptive framework for both the expressions, with straightforward, direct proofs.

Moreover, in the paper we extend the description of the linear transformations between the two representations to the more general framework of unconstrained linear combinations of binomial OWA functions. The linear algebra approach provides simple and compact expressions for the linear transformations and offers further insight on the usual constrained framework of OWA functions. In particular, we discuss the geometry of the vertices and the orness distribution within the simplexes associated with the usual weight constraints as written in the binomial representation of OWA functions.

The paper is organized as follows. In Section 2, we recall the definitions and main properties of the binomial OWA functions introduced in [4]. In Section 3, we describe the construction of the binomial decomposition of OWA functions [4], focusing on the equivalence between the canonical and the binomial representations, and discussing in detail the linear relations which transform one representation into the other, with a straightforward, direct proof of the result in [20]. In Section 4, we extend the discussion to the case of general linear combinations of binomial OWA functions, with a classical linear algebra approach, and, in Section 5, we exploit this material to clarify some aspects of the binomial representation of OWA functions, including a number of illustrative graphical and numerical examples. Section 6 contains some concluding remarks.

2. The binomial OWA functions

Consider the standard framework of averaging functions on $\mathbb{R}^n$, with $n \ge 2$.

Notation. Points in $\mathbb{R}^n$ are denoted $\boldsymbol{x} = (x_1, \dots, x_n)$, with $\mathbf{1} = (1, \dots, 1)$ and $\mathbf{0} = (0, \dots, 0)$. Accordingly, for every $x \in \mathbb{R}$, we have $x \cdot \mathbf{1} = (x, \dots, x)$. Given $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $\tilde{\boldsymbol{x}} = (x_n, \dots, x_1) \in \mathbb{R}^n$. The coordinates of the point $\boldsymbol{x} \in \mathbb{R}^n$ arranged in ascending or descending order are denoted as $x_{(1)} \le \dots \le x_{(n)}$ or $x_{[1]} \ge \dots \ge x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \dots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \dots, x_n\} = x_{[1]}$. The arithmetic mean is denoted $\bar{x} = (x_1 + \dots + x_n)/n$.

We begin by presenting the definition and some of the fundamental properties of Ordered Weighted Averaging (OWA) functions on $\mathbb{R}^n$, with $n \ge 2$ throughout the text.

Definition 1. Given a weighting vector $\boldsymbol{w} = (w_1, \dots, w_n) \in [0, 1]^n$, satisfying the normalization condition $\sum_{i=1}^n w_i = 1$, the *Weighted Averaging (WA) function* associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as $A(\boldsymbol{x}) = \sum_{i=1}^n w_i\, x_i$. In turn, the *Ordered Weighted Averaging (OWA) function* associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_{(i)}. \tag{1}$$

The traditional form of OWA functions as introduced in Yager [24] is $A(\boldsymbol{x}) = \sum_{i=1}^n \tilde{w}_i\, x_{[i]}$ where $\tilde{w}_i = w_{n-i+1}$ for $i = 1, \dots, n$. Comprehensive reviews of the theory and applications of OWA functions can be found in [25,26].

The following is a classical result regarding a form of dominance relation between OWA functions, characterized in terms of the majorization relation between the associated weighted structures. The result is mentioned in Calvo and De Baets [4] on the basis of the extensive material in Skala [22], a direct proof can be found in Bortot and Marques Pereira [1].

Proposition 1. *Consider two OWA functions $A, B : \mathbb{R}^n \to \mathbb{R}$ associated with weighting vectors $\boldsymbol{w}^A = (w_1^A, \dots, w_n^A) \in [0, 1]^n$ and $\boldsymbol{w}^B = (w_1^B, \dots, w_n^B) \in [0, 1]^n$, respectively. It holds that $A(\boldsymbol{x}) \le B(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ if and only if*

$$\sum_{k=1}^{i} w_k^A \ge \sum_{k=1}^{i} w_k^B \qquad i = 1, \dots, n \tag{2}$$

where the case $i = n$ is an equality due to weight normalization.

Example 1. The following are simple examples for $n = 3$. Consider two OWA functions A and B associated with the following weighting vectors with decreasing weights $\boldsymbol{w}^A = (6/10, 3/10, 1/10) \in [0,1]^n$ and $\boldsymbol{w}^B = (5/10, 3/10, 2/10) \in [0,1]^n$, respectively. The weighting vectors satisfy conditions (2). We obtain

$$\begin{aligned} B(\boldsymbol{x}) - A(\boldsymbol{x}) &= 5x_{(1)} + 3x_{(2)} + 2x_{(3)} - (6x_{(1)} + 3x_{(2)} + x_{(3)}) \\ &= -x_{(1)} + x_{(3)} \geq 0 \end{aligned}$$

Consider now two OWA functions A and B associated with the following weighting vectors with increasing weights $\boldsymbol{w}^A = (2/10, 3/10, 5/10) \in [0,1]^n$ and $\boldsymbol{w}^B = (1/10, 3/10, 6/10) \in [0,1]^n$, respectively. The weighting vectors satisfy conditions (2). We obtain

$$B(\boldsymbol{x}) - A(\boldsymbol{x}) = -x_{(1)} + x_{(3)} \geq 0\,.$$

Interesting examples of OWA functions are the binomial OWA functions C_j, for $j = 1, \ldots, n$, introduced by Calvo and De Baets [4]. In what follows we recall the definition and the main properties of the binomial OWA functions.

In the paper we use the binomial coefficients $\binom{r}{s}$ with integer $r, s \geq 0$, under the usual convention that the binomial coefficients are null when $r < s$. Some identities and formulas of the binomial coefficients which are used in the paper are recalled in the Appendix.

Definition 2. The *binomial OWA functions* $C_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, for $j = 1, \ldots, n$, are defined as

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} = \sum_{i=1}^{n} \binom{n-i}{j-1} \binom{n}{j}^{-1} x_{(i)} \qquad j = 1, \ldots, n \tag{3}$$

where the binomial weights $w_{ji} \geq 0$, for $i, j = 1, \ldots, n$, are given by

$$w_{ji} = \binom{n-i}{j-1} \binom{n}{j}^{-1} \qquad i, j = 1, \ldots, n \tag{4}$$

and satisfy the normalization conditions $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$.

Consider a binomial OWA function C_j, for some $j = 1, \ldots, n$. An important feature of the binomial weights w_{ji}, for $i = 1, \ldots, n$, is that they are null when $i > n - j + 1$, for instance see the simple examples described in Example 2, in the cases $n = 3$ and $n = 4$. Moreover, the normalization condition

$$\begin{aligned} \sum_{i=1}^{n} w_{ji} &= \sum_{i=1}^{n} \binom{n-i}{j-1} \binom{n}{j}^{-1} \qquad j = 1, \ldots, n \\ &= \sum_{i=1}^{n-j+1} \binom{n-i}{j-1} \binom{n}{j}^{-1} = \binom{n}{j} \binom{n}{j}^{-1} = 1 \end{aligned} \tag{5}$$

is due to the column-sum formula (A.5) of binomial coefficients.

The first binomial OWA function C_1 has a uniform weighting vector and coincides with the plain mean, $C_1(\boldsymbol{x}) = \bar{x}$. The subsequent binomial OWA functions C_j, for $j = 2, \ldots, n$ have an increasing number $j - 1$ of null weights, in correspondence with $x_{(n-j+2)}, \ldots, x_{(n)}$. The fact that the last $j - 1$ weights of C_j are null, for $j = 1, \ldots, n$, means that each binomial OWA function C_j depends only on the $n - j + 1$ lowest coordinates of the point $\boldsymbol{x}$. The last binomial OWA function reduces to $C_n(\boldsymbol{x}) = x_{(1)}$.

Example 2. In the case $n = 3$ the binomial OWA functions are given by

$$\begin{aligned} C_1(\boldsymbol{x}) &= \frac{1}{3}\, x_{(1)} + \frac{1}{3}\, x_{(2)} + \frac{1}{3}\, x_{(3)} = \bar{x} \\ C_2(\boldsymbol{x}) &= \frac{2}{3}\, x_{(1)} + \frac{1}{3}\, x_{(2)} \\ C_3(\boldsymbol{x}) &= x_{(1)} \end{aligned}$$

and in the case $n = 4$ the binomial OWA functions are given by

$$C_1(\boldsymbol{x}) = \frac{1}{4}\,x_{(1)} + \frac{1}{4}\,x_{(2)} + \frac{1}{4}\,x_{(3)} + \frac{1}{4}\,x_{(4)} = \bar{x}$$
$$C_2(\boldsymbol{x}) = \frac{3}{6}\,x_{(1)} + \frac{2}{6}\,x_{(2)} + \frac{1}{6}\,x_{(3)}$$
$$C_3(\boldsymbol{x}) = \frac{3}{4}\,x_{(1)} + \frac{1}{4}\,x_{(2)}$$
$$C_4(\boldsymbol{x}) = x_{(1)}\,.$$

The examples above illustrate the fact that the binomial OWA functions C_j, for $j = 1, \dots, n$, have non-increasing weights. After the uniform weights of C_1, in the case of $j = 2, \dots, n-1$ the weight distribution of C_j progressively focuses on the first $n-j+1$ positive weights $w_{j1} > w_{j2} > \dots > w_{j,n-j+1}$, and finally with $j = n$ the single positive weight of C_n is $w_{n1} = 1$.

The detailed monotonicity structure of the binomial weights, which has been discussed in Calvo and De Baets [4] and Bortot and Marques Pereira [1], is as follows.

Proposition 2. *The weighting structures of the binomial OWA functions C_j, for $j = 1, \dots, n$, have the following properties,*

$$1/n = w_{j1} = w_{j2} = \dots = w_{jn} \qquad j = 1$$
$$j/n = w_{j1} > w_{j2} > \dots > w_{j,n-j+1} > w_{j,n-j+2} = \dots = w_{jn} = 0 \qquad j = 2, \dots, n-1$$
$$1 = w_{j1} > w_{j2} = \dots = w_{jn} = 0 \qquad j = n\,.$$

In general, the non-increasingness of the binomial weights $w_{j1} \ge w_{j2} \ge \dots \ge w_{jn}$, for $j = 1, \dots, n$, means that the binomial OWA functions are Schur-concave and can thus be regarded as welfare functions, see for instance Proposition 2 in Bortot and Marques Pereira [1] for a detailed discussion in the context of welfare and inequality.

The binomial OWA functions have a natural dominance hierarchy $C_1(\boldsymbol{x}) \ge C_2(\boldsymbol{x}) \ge \dots \ge C_n(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$. This result derives from the combination of a cumulative property of the binomial OWA weights, due to Calvo and De Baets [4], with the classical dominance theorem by Skala [22]. The latter is presented here as Proposition 1, formulated in conformity with our convention $x_{(1)} \le \dots \le x_{(n)}$ in the definition of OWA functions.

Given that the dominance hierarchy of the binomial functions is sensitive[1] to the underlying convention, a complete proof is presented here, concerning the cumulative property of the binomial weights plus the dominance hierarchy itself.

Proposition 3. *The binomial OWA functions C_j, $j = 1, \dots, n$, satisfy the inequalities $\bar{x} = C_1(\boldsymbol{x}) \ge C_2(\boldsymbol{x}) \ge \dots \ge C_n(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.*

Proof. The above inequalities follow from the fact that the cumulative weights of the binomial OWA functions C_j are greater or equal than the corresponding cumulative weights of the binomial OWA functions C_{j-1} for $j = 2, \dots, n$,

$$\sum_{k=1}^{i} w_{jk} \ge \sum_{k=1}^{i} w_{j-1,k} \qquad i = 1, \dots, n \quad j = 2, \dots, n\,. \tag{6}$$

We begin by proving the inequality (6) in the simple cases $i = 1$, $i = n$, and $j = n$. In the first case the inequality holds since $(j-1)\,w_{j1} = j\,w_{j-1,1}$ for $j = 2, \dots, n$, in the second case the inequality holds because both cumulative weights are equal to 1, and in the third case the inequality holds because all the cumulative weights $\sum_{k=1}^{i} w_{nk}$, for $i = 1, \dots, n$, are equal to 1.
In the remaining cases $i, j = 2, \dots, n-1$, the left-hand side of inequality (6) can be written as

$$\sum_{k=1}^{i} \binom{n-k}{j-1}\binom{n}{j}^{-1} = \sum_{k=n-i}^{n-1} \binom{k}{j-1}\binom{n}{j}^{-1} \tag{7}$$
$$= \left(\sum_{k=0}^{n-1} \binom{k}{j-1} - \sum_{k=0}^{n-i-1} \binom{k}{j-1}\right)\binom{n}{j}^{-1} \tag{8}$$
$$= \left(\binom{n}{j} - \binom{n-i}{j}\right)\binom{n}{j}^{-1} \tag{9}$$

where from (8) to (9) we have used the column-sum formula (A.5). Analogously, for $i, j = 2, \dots, n-1$, the right-hand side of inequality (6) can be written as

[1] The interested reader will notice that the dominance hierarchy in Calvo and De Baets [4] appears in the inverse order. This is due to the fact that Skala [22] refers to the original convention regarding the definition of OWA functions in Yager [24], in which the coordinates are arranged in descending order. In particular, the classical dominance theorem by Skala [22], presented in [4] as Theorem 3, is formulated in that original convention, which is different from the convention generally used in [4], for instance in the definition of OWA functions.

$$\sum_{k=1}^{i} \binom{n-k}{j-2} \binom{n}{j-1}^{-1} = \left(\binom{n}{j-1} - \binom{n-i}{j-1} \right) \binom{n}{j-1}^{-1}. \tag{10}$$

Therefore, after simplifying the common terms in (9) and (10), the inequality (6) reduces to

$$\binom{n}{j-1} \binom{n-i}{j} \leq \binom{n}{j} \binom{n-i}{j-1} \qquad i, j = 2, \dots, n-1 \tag{11}$$

which is a particular case of the balance property (A.20), with $r = n$, $s = n - i$, and $t = j$.
Finally, considering the cumulative property (6) in the framework of Proposition 1, with $w_{jk} = w_k^A$ and $w_{j-1,k} = w_k^B$, we conclude that $C_j(\boldsymbol{x}) \leq C_{j-1}(\boldsymbol{x})$ for $j = 2, \dots, n$. Accordingly, the binomial OWA functions satisfy the inequalities $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \dots \geq C_n(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$. □

Example 3. As a simple example consider the case $n = 3$, in which

$$C_1(\boldsymbol{x}) - C_2(\boldsymbol{x}) = \frac{1}{3}(x_{(3)} - x_{(1)}) \geq 0$$

$$C_2(\boldsymbol{x}) - C_3(\boldsymbol{x}) = \frac{1}{3}(x_{(2)} - x_{(1)}) \geq 0$$

using the explicit form of C_1, C_2, and C_3 in Example 2.

In the seminal paper by Yager [24], the author introduces the concept of orness as a characteristic index of the weighting structure of an OWA function.

Definition 3. The *orness* of an OWA function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined as

$$\text{Orness}(A) = \sum_{i=1}^{n} \frac{i-1}{n-1} w_i. \tag{12}$$

In particular, the orness of the binomial OWA functions is given by

$$\text{Orness}(C_j) = \frac{n-j}{(n-1)(j+1)} \qquad j = 1, \dots, n. \tag{13}$$

Example 4. In the case $n = 3$ we have

$$\text{Orness}(C_1) = \frac{1}{2}, \quad \text{Orness}(C_2) = \frac{1}{6}, \quad \text{Orness}(C_3) = 0$$

whereas in the case $n = 4$ we have

$$\text{Orness}(C_1) = \frac{1}{2}, \quad \text{Orness}(C_2) = \frac{2}{9}, \quad \text{Orness}(C_3) = \frac{1}{12}, \quad \text{Orness}(C_4) = 0.$$

In general, the orness of the binomial OWA functions is strictly decreasing with respect to $j = 1, \dots, n$, from $\text{Orness}(C_1) = 1/2$ to $\text{Orness}(C_n) = 0$, see Bortot, Fedrizzi, Marques Pereira and Nguyen [2].

3. The binomial representation of OWA functions

We begin with a brief review of some of the fundamental definitions of the Choquet integration framework, focusing on the relation between the capacity representation and the Möbius transform representation. Excellent reviews of Choquet integration can be found in Grabisch and Labreuche [14,16,17], and Grabisch, Kojadinovich, and Meyer [15].

We then consider the particular case of symmetric Choquet integration and recall the binomial decomposition of OWA functions due to Calvo and De Baets [4], as well as a complementary result due to Nguyen [20].

Our aim in this section is to present a comprehensive description of the binomial representation of OWA functions, explicitating both the linear transformations that relate the weights of the canonical representation to the coefficients of the binomial representation and vice versa, with a new straightforward proof of the complementary result by Nguyen [20].

Consider a finite set of interacting elements $N = \{1, 2, \dots, n\}$. The subsets $S, T \subseteq N$ with cardinalities $0 \leq s, t \leq n$ are usually called coalitions. The concepts of Choquet capacity and integral are due to Choquet [6], Sugeno [23], Denneberg [7], and Grabisch [10,11], the concept and properties of the Möbius transform in Choquet integration are due to Rota [21], Chateauneuf and Jaffray [5], Grabisch [13], Marichal [18], and Miranda and Grabisch [19].

A *capacity* on the set N is a set function $\mu : 2^N \longrightarrow [0,1]$ satisfying the *boundary* conditions $\mu(\emptyset) = 0$, $\mu(N) = 1$ and the *monotonicity* conditions $\mu(T \cup \{i\}) - \mu(T) \geq 0$ for each $i \in N$ and all coalitions $T \subseteq N \setminus \{i\}$.

A capacity μ can be equivalently represented by its associated *Möbius transform* $m_\mu : 2^N \longrightarrow \mathbb{R}$ which is defined as

$$m_\mu(T) = \sum_{S\subseteq T} (-1)^{t-s} \mu(S) \qquad T \subseteq N \tag{14}$$

where s and t denote the cardinality of the coalitions S and T, respectively. Conversely, given the Möbius transform m_μ, the associated capacity μ is obtained as

$$\mu(T) = \sum_{S\subseteq T} m_\mu(S) \qquad T \subseteq N\,. \tag{15}$$

In the Möbius representation, the boundary conditions are written as $m_\mu(\emptyset) = 0$, $\sum_{T\subseteq N} m_\mu(T) = 1$ and the monotonicity conditions are $\sum_{S\subseteq T} m_\mu(S \cup \{i\}) \geq 0$ for each $i \in N$ and all coalitions $T \subseteq N \setminus \{i\}$.

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [12] introduced the concept of k-additive capacities. A capacity μ on the set N is said to be k-*additive* if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.

A capacity μ is said to be *symmetric* if its values $\mu(T)$ for all $T \subseteq N$ depend only on the cardinality of the coalition considered, in which case we use the simplified notation $\mu(T) = \mu(t)$ where $t = |T|$. The boundary and monotonicity conditions take the form

$$\mu(0) = 0 \qquad \mu(n) = 1 \tag{16}$$

$$\mu(t) - \mu(t-1) \geq 0 \qquad t = 1,\dots,n\,. \tag{17}$$

Analogously, for the symmetric Möbius transform m_μ associated with a symmetric capacity μ we use the notation $m_\mu(T) = m_\mu(t)$ where $t = |T|$. The expression (14) for the symmetric Möbius transform m_μ in terms of the symmetric capacity μ reduces to

$$m_\mu(t) = \sum_{s=1}^{t} (-1)^{t-s} \binom{t}{s} \mu(s) \qquad t = 1,\dots,n \tag{18}$$

and the expression (15) for the symmetric capacity μ in terms of the symmetric Möbius transform m_μ reduces to

$$\mu(t) = \sum_{s=1}^{t} \binom{t}{s} m_\mu(s) \qquad t = 1,\dots,n\,. \tag{19}$$

Moreover, the boundary and monotonicity conditions take the form

$$m_\mu(0) = 0 \qquad \sum_{s=1}^{n} \binom{n}{s} m_\mu(s) = 1 \tag{20}$$

$$\sum_{s=1}^{t} \binom{t-1}{s-1} m_\mu(s) \geq 0 \qquad t = 1,\dots,n\,. \tag{21}$$

The *Choquet integral* with respect to a *symmetric* capacity μ reduces to an OWA function, see Fodor, Marichal, and Roubens [9], and Yager [24],

$$C_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} (\mu(n-i+1) - \mu(n-i))\, x_{(i)} = \sum_{i=1}^{n} w_i\, x_{(i)} = A(\boldsymbol{x}) \tag{22}$$

where the weights

$$w_i = \mu(n-i+1) - \mu(n-i) \qquad i = 1,\dots,n \tag{23}$$

satisfy $w_i \geq 0$ for $i = 1,\dots,n$ due to the monotonicity of the symmetric capacity μ, and $\sum_{i=1}^{n} w_i = 1$ due to the boundary conditions. Conversely, it holds that

$$\mu(i) = \sum_{j=1}^{i} w_{n-j+1} \qquad i = 1,\dots,n\,. \tag{24}$$

We will now discuss the direct relation between the weights and the symmetric Möbius transform values associated with the symmetric capacity μ.

We obtain the weights w_i for $i = 1,\dots,n$ in terms of the symmetric Möbius transform values $m_\mu(j)$, for $j = 1,\dots,n$, using (23) and (19), starting from the expression of the weights in terms of the symmetric capacity and then the expression of the symmetric capacity in terms of its symmetric Möbius transform,

$$w_i = \mu(n-i+1) - \mu(n-i) \tag{25}$$

$$= \sum_{j=1}^{n-i+1} \binom{n-i+1}{j} m_\mu(j) - \sum_{j=1}^{n-i} \binom{n-i}{j} m_\mu(j) \tag{26}$$

$$= \sum_{j=1}^{n-i+1} \left(\binom{n-i+1}{j} - \binom{n-i}{j} \right) m_\mu(j) \tag{27}$$

$$= \sum_{j=1}^{n-i+1} \binom{n-i}{j-1} m_\mu(j) \qquad i=1,\dots,n. \tag{28}$$

where from (27) to (28) we have used the recurrence identity (A.3).

Conversely, we obtain the symmetric Möbius transform values $m_\mu(j)$ for $j=1,\dots,n$ in terms of the weights w_i for $i=1,\dots,n$ using (18) and (24),

$$m_\mu(j) = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \mu(i) \tag{29}$$

$$= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \left(w_n + \dots + w_{n-i+1} \right) \tag{30}$$

$$= (-1)^{j-1} \binom{j}{1} (w_n) + (-1)^{j-2} \binom{j}{2} (w_n + w_{n-1})$$

$$+ \dots + (-1)^{j-i} \binom{j}{i} (w_n + \dots + w_{n-i+1}) + \dots \tag{31}$$

$$+ (-1)^{j-j} \binom{j}{j} (w_n + \dots + w_{n-i+1} + \dots + w_{n-j+1})$$

$$= \sum_{i=1}^{j} \left(\sum_{k=i}^{j} (-1)^{j-k} \binom{j}{k} \right) w_{n-i+1} \tag{32}$$

$$= \sum_{i=1}^{j} (-1)^{j-i} \binom{j-1}{j-1} w_{n-i+1} \tag{33}$$

$$= \sum_{i=1}^{j} (-1)^{j-i} \binom{j-1}{i-1} w_{n-i+1} \tag{34}$$

where from (31) to (32) we have collected the coefficients of w_{n-i+1} for $i=1,\dots,j$, and from (32) to (34) we have first used the formula (A.14) for the partial alternating row-sums of binomial coefficients, and then we have used the symmetry identity (A.1), see the Appendix.

In the framework of Choquet integration with respect to symmetric capacities, the above relations (25)-(28) and (29)-(34) between the weights of an OWA function A and the symmetric Möbius transform values of the associated symmetric capacity μ are the basis of the binomial decomposition of OWA functions due to Calvo and De Baets [4], see also Bortot and Marques Pereira [1] for a detailed discussion in the context of welfare and inequality.

The binomial decomposition establishes an equivalence between the canonical representation of an OWA function, that is, the usual constrained linear combination of the order statistics, and the binomial representation of the OWA function, which is a constrained linear combination of the binomial OWA functions. The original result in Calvo and De Baets [4] is based on the expression of the weights in terms of the values of the symmetric Möbius transform associated with the symmetric capacity which generates the OWA function.

More recently, the inverse expression of those coefficients in terms of the weights has been derived in Nguyen [20]. However, the proof in Nguyen [20] of this inverse relation is rather long and elaborate and this fact has motivated the search for a simpler proof. In the following proposition we present a comprehensive version of the binomial decomposition in which both the original and the inverse relations are derived with simple direct proofs.

Proposition 4 *(Binomial decomposition). An OWA function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ written in the canonical representation as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)} \qquad \boldsymbol{x} \in \mathbb{R}^n \tag{35}$$

where the weights w_i, for $i=1,\dots,n$, are subject to the constraints

$$w_i \geq 0 \quad i=1,\dots,n \qquad \text{and} \qquad \sum_{i=1}^{n} w_i = 1 \tag{36}$$

can be equivalently written in the binomial representation as

$$A(\boldsymbol{x}) = \sum_{j=1}^{n} \alpha_j \, C_j(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^n \tag{37}$$

where the coefficients α_j, for $j = 1, \ldots, n$, are in turn subject to the constraints

$$\sum_{j=1}^{n-i+1} \binom{n-i}{j-1} \binom{n}{j}^{-1} \alpha_j \geq 0 \quad i = 1, \ldots, n \qquad \text{and} \qquad \sum_{j=1}^{n} \alpha_j = 1 \,. \tag{38}$$

The equivalence between the canonical representation (35)-(36) and the binomial representation (37)-(38) is based on the linear transformations

$$w_i = \sum_{j=1}^{n-i+1} \binom{n-i}{j-1} \binom{n}{j}^{-1} \alpha_j \qquad i = 1, \ldots, n \tag{39}$$

$$\alpha_j = \binom{n}{j} \sum_{i=1}^{j} (-1)^{j-i} \binom{j-1}{i-1} w_{n-i+1} \qquad j = 1, \ldots, n \,. \tag{40}$$

which ensure the uniqueness of the correspondence between the two representations.

Proof. The proof is based on the expression of the weights in terms of the Möbius transform values (25)-(28), and conversely on the expression of the Möbius transform values in terms of the weights (29)-(34). For simplicity we extend all the summations involved in the proof to the full index range, under the usual convention that the binomial coefficients are null when the upper value is less than the lower value.

Using (25)-(28) we obtain

$$\sum_{i=1}^{n} w_i \, x_{(i)} = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \binom{n-i}{j-1} m_\mu(j) \Big) x_{(i)} \tag{41}$$

$$= \sum_{j=1}^{n} \binom{n}{j} m_\mu(j) \Big(\sum_{i=1}^{n} \binom{n-i}{j-1} \binom{n}{j}^{-1} x_{(i)} \Big) \tag{42}$$

$$= \sum_{j=1}^{n} \alpha_j \, C_j(\boldsymbol{x}) \tag{43}$$

where the coefficients α_j, for $j = 1, \ldots, n$, are conveniently defined as

$$\alpha_j = \binom{n}{j} m_\mu(j) \qquad j = 1, \ldots, n \,. \tag{44}$$

Conversely, using (29)-(34) we obtain

$$\sum_{j=1}^{n} \alpha_j \, C_j(\boldsymbol{x}) = \sum_{j=1}^{n} \binom{n}{j} m_\mu(j) \Big(\sum_{i=1}^{n} \binom{n-i}{j-1} \binom{n}{j}^{-1} x_{(i)} \Big) \tag{45}$$

$$= \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} (-1)^{j-k} \binom{j-1}{k-1} w_{n-k+1} \Big) \Big(\sum_{i=1}^{n} \binom{n-i}{j-1} x_{(i)} \Big) \tag{46}$$

$$= \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \Big(\sum_{j=1}^{n} (-1)^{j-k} \binom{j-1}{k-1} \binom{n-i}{j-1} \Big) w_{n-k+1} \Big) x_{(i)} \tag{47}$$

$$= \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \delta_{n-i,k-1} \, w_{n-k+1} \Big) x_{(i)} \tag{48}$$

$$= \sum_{i=1}^{n} w_i \, x_{(i)} \tag{49}$$

where from (47) to (48) we have used the binomial inversion property (A.8).

Moreover, the linear transformations (39) and (40) follow respectively from (25)-(28) and (29)-(34), plus the definition (44) of the coefficients α_j in terms of the Möbius transform values $m_\mu(j)$, for $j = 1, \ldots, n$.

Note that in (39) each weight w_i depends only on the coefficients $\alpha_1, \ldots, \alpha_{n-i+1}$ for $i = 1, \ldots, n$. In particular, w_n depends only on the coefficient α_1, w_{n-1} depends only on the coefficients α_1 and α_2, etc. Conversely, in (40) each coefficient α_j depends only on the weights $w_{n-j+1}, \ldots, w_n$ for $j = 1, \ldots, n$. In particular, α_1 depends only on the weight w_n, α_2 depends only on the weights w_{n-1} and w_n, etc. This form of nonsingular triangular dependency ensures the uniqueness of the linear transformations between the weights in the canonical representation (35)-(36) and the coefficients in the binomial representation (37)-(38).

Finally, the boundary and monotonicity constraints regarding the coefficients α_j in (38), for $j = 1, \dots, n$, are due to those concerning the weights w_i in (36), for $i = 1, \dots, n$. This follows immediately from the expression (39) of the weights in terms of the coefficients, plus the fact that the sum of the weights coincides with the sum of the coefficients,

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \binom{n-i}{j-1} \binom{n}{j}^{-1} \alpha_j \Big) \tag{50}$$

$$= \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \binom{n-i}{j-1} \Big) \binom{n}{j}^{-1} \alpha_j \tag{51}$$

$$= \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} \binom{k-1}{j-1} \Big) \binom{n}{j}^{-1} \alpha_j \tag{52}$$

$$= \sum_{j=1}^{n} \binom{n}{j} \binom{n}{j}^{-1} \alpha_j = \sum_{j=1}^{n} \alpha_j \tag{53}$$

where from (52) to (53) we have used the column-sum formula (A.5). □

We will now discuss the equivalence between the equation (40) and the original result in Nguyen [20],

$$\alpha_j = \binom{n}{j} \sum_{k=0}^{j-1} (-1)^{j-k-1} \binom{j-1}{j-k-1} w_{n-k} \qquad j = 1, \dots, n. \tag{54}$$

In terms of the new summation index $i = k + 1$, for $i = 1, \dots, j$, we obtain

$$\alpha_j = \binom{n}{j} \sum_{i=1}^{j} (-1)^{j-(i-1)-1} \binom{j-1}{j-(i-1)-1} w_{n-(i-1)} \tag{55}$$

$$= \binom{n}{j} \sum_{i=1}^{j} (-1)^{j-i} \binom{j-1}{j-i} w_{n-i+1} \tag{56}$$

$$= \binom{n}{j} \sum_{i=1}^{j} (-1)^{j-i} \binom{j-1}{i-1} w_{n-i+1} \qquad j = 1, \dots, n \tag{57}$$

as in (40), where from (56) to (57) we have used the binomial symmetry identity. We conclude that the original result in Nguyen [20], i.e. Proposition 3.17, with a long and elaborate proof based on the principle of strong induction, corresponds precisely to equation (40) in our Proposition 4.

The two linear transformations (39) and (40) translate between the canonical and the binomial representations. The linear transformation (40), in particular, provides the values of the symmetric Möbius transform associated with the weighting structure of an OWA function. This has an immediate relevance in the various contexts in which the weighting structure is known and one wishes to have some information on the underlying interactions as expressed by the symmetric Möbius transform. A simple example is that of the classical Gini welfare functions,

$$A^c(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^c x_{(i)} \qquad \text{where} \qquad w_i^c = \frac{2(n-i)+1}{n^2}. \tag{58}$$

Using the linear transformation (40), we obtain

$$\alpha_j = \binom{n}{j} \sum_{i=1}^{j} (-1)^{j-i} \binom{j-1}{i-1} \frac{2i-1}{n^2} \tag{59}$$

$$= \binom{n}{j} \sum_{i=1}^{j} (-1)^{(j-1)-(i-1)} \binom{j-1}{i-1} \frac{1}{n^2} \tag{60}$$

$$+ \binom{n}{j} \sum_{i=1}^{j} (-1)^{(j-1)-(i-1)} \binom{j-1}{i-1} \frac{2(i-1)}{n^2}$$

$$= \frac{1}{n^2} (-1)^{j-1} \binom{n}{j} \sum_{i=1}^{j} (-1)^{i-1} \binom{j-1}{i-1} \tag{61}$$

$$+ \frac{2}{n^2} (-1)^{j-1} \binom{n}{j} \sum_{i=1}^{j} (-1)^{i-1} \binom{j-1}{i-1} (i-1)$$

$$= \frac{1}{n^2}(-1)^{j-1}\binom{n}{j}(\delta_{j-1,0}) + \frac{2}{n^2}(-1)^{j-1}\binom{n}{j}(-\delta_{j-1,1}) \tag{62}$$

$$= \frac{1}{n}\,\delta_{j1} + \frac{n-1}{n}\,\delta_{j2} \tag{63}$$

where from (61) to (62) we have used (A.12) and (A.13). This means that $\alpha_1 = 1/n$ and $\alpha_2 = (n-1)/n$, with $\alpha_k = 0$ for $k = 3, \dots, n$. In other words, the classical Gini welfare function is 2-additive and can be written as

$$A^c(\boldsymbol{x}) = \frac{1}{n}\,C_1(\boldsymbol{x}) + \frac{n-1}{n}\,C_2(\boldsymbol{x}) \tag{64}$$

see Bortot and Marques Pereira [1] for further details.

The binomial decomposition of OWA functions, in the form (37), allows us to express the orness of an OWA function as in (12) in the following way,

$$\text{Orness}\,(A) = \sum_{j=1}^{n} \alpha_j\,\text{Orness}\,(C_j) = \sum_{j=1}^{n} \frac{n-j}{(n-1)\,(j+1)}\,\alpha_j \tag{65}$$

Example 5. Considering the orness of the binomial OWA functions in the cases $n = 3$ and $n = 4$ described in Example 4, we can thus write

$$\text{Orness}\,(A) = \frac{1}{2}\,\alpha_1 + \frac{1}{6}\,\alpha_2$$

$$\text{Orness}\,(A) = \frac{1}{2}\,\alpha_1 + \frac{2}{9}\,\alpha_2 + \frac{1}{12}\,\alpha_3$$

in the cases $n = 3, 4$, respectively. Notice that the missing contribution of α_n in each case is due to the fact that the orness of the corresponding OWA function C_n is null.

4. Linear combinations of binomial OWA functions

In this section we consider general linear combinations of the binomial OWA functions and we prove that they correspond precisely, on a one-to-one basis, to all the functions $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form $F(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\,x_{(i)}$ with $\boldsymbol{u} \in \mathbb{R}^n$. In other words, we prove that the binomial OWA functions constitute a basis for the linear space of such functions F.

We begin by introducing two lower triangular square matrices of order n and proving that both these matrices are nonsingular and reciprocally inverse.

Proposition 5. *The square matrices* $\mathbf{A} = (a_{ij} \in \mathbb{R})_{n\times n}$ *and* $\mathbf{B} = (b_{ij} \in \mathbb{R})_{n\times n}$ *defined as*

$$a_{ij} = \binom{i-1}{j-1}\binom{n}{j}^{-1} \qquad i, j = 1, \dots, n \tag{66}$$

$$b_{ij} = (-1)^{i-j}\binom{i-1}{j-1}\binom{n}{i} \qquad i, j = 1, \dots, n \tag{67}$$

are both nonsingular lower triangular and reciprocally inverse matrices, that is, $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$.

Proof. The diagonal elements of $\mathbf{A}$ and $\mathbf{B}$ are given by

$$a_{ii} = \binom{n}{i}^{-1} \qquad b_{ii} = \binom{n}{i} \qquad i = 1, \dots, n \tag{68}$$

and the (strictly) upper triangular elements of both matrices are null due to the binomial coefficients in their definition. Therefore, the determinant of both matrices, which is given by the product of their diagonal elements, is nonzero.

The proof that the product of the two matrices is the identity matrix,

$$\sum_{k=1}^{n} a_{ik}\,b_{kj} \tag{69}$$

$$= \sum_{k=1}^{n}(-1)^{k-j}\binom{i-1}{k-1}\binom{k-1}{j-1} \tag{70}$$

$$= \sum_{k=j}^{i}(-1)^{k-j}\binom{i-1}{k-1}\binom{k-1}{j-1} \tag{71}$$

$$= \delta_{i-1,j-1} = \delta_{ij} \qquad i, j = 1, \dots, n \tag{72}$$

relies on the binomial inversion property (A.8). □

We now consider functions $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the general form $F(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\, x_{(i)}$ with $\boldsymbol{u} \in \mathbb{R}^n$. In the central result of this paper we prove that the functions F of this general form are in one-to-one correspondence with the linear combinations $\sum_{j=1}^{n} v_j\, C_j$ of the binomial OWA functions, with $\boldsymbol{v} \in \mathbb{R}^n$. The square matrices $\mathbf{A}$, $\mathbf{B}$ in Proposition 5 translate from the vectors $\boldsymbol{u}$ to the vectors $\boldsymbol{v}$ and vice versa.

Proposition 6. *Given a vector $\boldsymbol{u} \in \mathbb{R}^n$ and the associated function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form*

$$F(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\, x_{(i)} \qquad \boldsymbol{x} \in \mathbb{R}^n \tag{73}$$

there exists a unique vector $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$F(\boldsymbol{x}) = \sum_{j=1}^{n} v_j\, C_j(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^n \tag{74}$$

and the two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ are related by the linear transformations

$$u_{n-i+1} = \sum_{j=1}^{n} a_{ij}\, v_j \quad \text{and} \quad v_j = \sum_{i=1}^{n} b_{ji}\, u_{n-i+1} \qquad i, j = 1, \ldots, n. \tag{75}$$

The linear transformations (75) can be written as $\tilde{\boldsymbol{u}} = \mathbf{A}\boldsymbol{v}$ and $\boldsymbol{v} = \mathbf{B}\tilde{\boldsymbol{u}}$, where $\mathbf{A}$ and $\mathbf{B}$ are the nonsingular and reciprocally inverse square matrices introduced in (66) and (67).

Proof. The fact that the function F can be written as a linear combination of the binomial OWA functions is expressed by

$$F(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\, x_{(i)} = \sum_{i,j=1}^{n} v_j\, w_{ji}\, x_{(i)} = \sum_{j=1}^{n} v_j\, C_j(\boldsymbol{x}) \tag{76}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. By means of appropriate choices of the points $\boldsymbol{x}$, for instance the n points $\boldsymbol{x}^{(1)} = (1, 0, 0, \ldots, 0)$, $\boldsymbol{x}^{(2)} = (1, 1, 0, \ldots, 0)$, ... , $\boldsymbol{x}^{(n)} = (1, 1, 1, \ldots, 1)$, one can easily show that the functional equation (76) is equivalent to the linear system

$$u_i = \sum_{j=1}^{n} v_j\, w_{ji} \qquad i = 1, \ldots, n \tag{77}$$

and therefore

$$u_{n-i+1} = \sum_{j=1}^{n} v_j\, w_{j,n-i+1} = \sum_{j=1}^{n} a_{ij}\, v_j \qquad i = 1, \ldots, n \tag{78}$$

where $a_{ij} = w_{j,n-i+1}$ according to definitions (4) and (66), for $i, j = 1, \ldots, n$.
The linear system (78) can be written in matrix form as

$$\tilde{\boldsymbol{u}} = \mathbf{A}\boldsymbol{v} \tag{79}$$

where $\tilde{u}_i = u_{n-i+1}$, for $i = 1, \ldots, n$. Given that $\mathbf{A}$ and $\mathbf{B}$ are two nonsingular and reciprocally inverse square matrices, as shown in Proposition 5, we can invert the linear system (79) as

$$\boldsymbol{v} = \mathbf{B}\tilde{\boldsymbol{u}} \tag{80}$$

which can be written as

$$v_j = \sum_{i=1}^{n} b_{ji}\, u_{n-i+1} \qquad j = 1, \ldots, n \tag{81}$$

thus concluding the proof. □

For illustrative purposes, we provide two simple numerical examples of this construction and we illustrate the form of the linear systems involved in (74) in dimensions $n = 3, 4$.

Example 6. Consider the case $n = 3$. The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ 1/3 & 2/3 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

and we can see that they are both lower triangular and reciprocally inverse matrices, with $\mathbf{B} = \mathbf{A}^{-1}$ and vice versa. The linear systems $\tilde{\boldsymbol{u}} = \mathbf{A}\boldsymbol{v}$ and $\boldsymbol{v} = \mathbf{B}\tilde{\boldsymbol{u}}$ are written respectively as

$$\begin{cases} u_3 = v_1/3 \\ u_2 = v_1/3 + v_2/3 \\ u_1 = v_1/3 + 2v_2/3 + v_3 \end{cases} \qquad \begin{cases} v_1 = 3u_3 \\ v_2 = 3u_2 - 3u_3 \\ v_3 = u_1 - 2u_2 + u_3 \end{cases}$$

and the explicit form of the linear systems shows clearly that the matrices **A** and **B** relate directly the vector $\tilde{\boldsymbol{u}} = (u_3, u_2, u_1)$ with the vector $\boldsymbol{v} = (v_1, v_2, v_3)$. As a result, the component u_i depends on the components $v_1, \ldots, v_{n-i+1}$, for $i = 1, 2, 3$. Conversely, the component v_i depends on the components $u_{n-i+1}, \ldots, u_n$, for $i = 1, 2, 3$.

Example 7. Consider the case $n = 4$. The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 1/4 & 1/6 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 \\ 1/4 & 1/2 & 3/4 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 \\ 4 & -8 & 4 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

and we can see that they are both lower triangular and reciprocally inverse matrices, with $\mathbf{B} = \mathbf{A}^{-1}$ and vice versa. The linear systems $\tilde{\boldsymbol{u}} = \mathbf{A}\boldsymbol{v}$ and $\boldsymbol{v} = \mathbf{B}\tilde{\boldsymbol{u}}$ are written respectively as

$$\begin{cases} u_4 = v_1/4 \\ u_3 = v_1/4 + v_2/6 \\ u_2 = v_1/4 + v_2/3 + v_3/4 \\ u_1 = v_1/4 + v_2/2 + 3v_3/4 + v_4 \end{cases} \qquad \begin{cases} v_1 = 4u_4 \\ v_2 = 6u_3 - 6u_4 \\ v_3 = 4u_2 - 8u_3 + 4u_4 \\ v_4 = u_1 - 3u_2 + 3u_3 - u_4 \end{cases}$$

and the explicit form of the linear systems shows clearly that the matrices **A** and **B** relate directly the vector $\tilde{\boldsymbol{u}} = (u_4, u_3, u_2, u_1)$ with the vector $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$. As a result, the component u_i depends on the components $v_1, \ldots, v_{n-i+1}$, for $i = 1, 2, 3, 4$. Conversely, the component v_i depends on the components $u_{n-i+1}, \ldots, u_n$, for $i = 1, 2, 3, 4$.

5. Back to the binomial representation

In the light of the general result presented in the previous section, that is, Proposition 6, the binomial decomposition described in Proposition 4 can be formulated as follows.

Given a weighting vector $\boldsymbol{w} \in [0, 1]^n$ and the associated OWA function $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form $A(\boldsymbol{x}) = \sum_{i=1}^n w_i\, x_{(i)}$, there exists a unique coefficient vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ such that

$$A(\boldsymbol{x}) = \sum_{j=1}^{n} \alpha_j\, C_j(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^n \tag{82}$$

where the two vectors $\boldsymbol{w} \in [0, 1]^n$ and $\boldsymbol{\alpha} \in \mathbb{R}^n$ are related by the linear transformations

$$w_{n-i+1} = \sum_{j=1}^{n} a_{ij}\, \alpha_j \quad \text{and} \quad \alpha_j = \sum_{i=1}^{n} b_{ji}\, w_{n-i+1} \qquad i, j = 1, \ldots, n \tag{83}$$

which can be written as

$$\tilde{\boldsymbol{w}} = \mathbf{A}\boldsymbol{\alpha} \qquad \boldsymbol{\alpha} = \mathbf{B}\tilde{\boldsymbol{w}} \tag{84}$$

where **A**, **B** are the nonsingular and reciprocally inverse square matrices introduced in (66) and (67). Moreover, the two vectors $\boldsymbol{w} \in [0, 1]^n$ and $\boldsymbol{\alpha} \in \mathbb{R}^n$ satisfy the constraints

$$w_{n-i+1} = \sum_{j=1}^{n} a_{ij}\, \alpha_j \geq 0 \quad i = 1, \ldots, n \qquad \sum_{i=1}^{n} w_i = 1 = \sum_{j=1}^{n} \alpha_j\, . \tag{85}$$

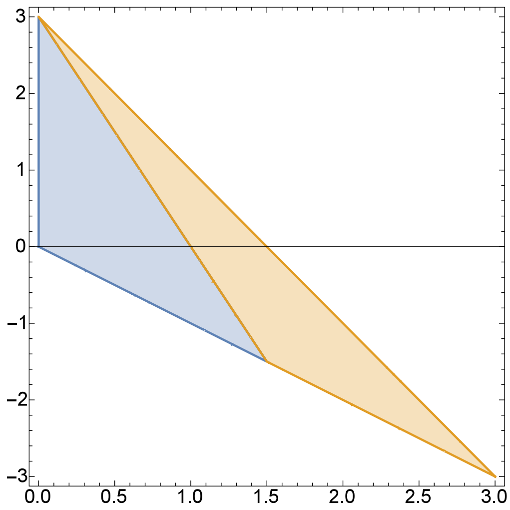

Fig. 1. Feasible region associated with conditions (86) in the case $n = 3$.

The weighting vectors $\boldsymbol{w}$ under constraints (85) constitute an $(n-1)$-dimensional simplex in n dimensions, whose n vertices are the unit vectors of the canonical basis $\boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}, \dots, \boldsymbol{w}^{(n)}$. The corresponding vectors $\boldsymbol{\alpha}$ containing the coefficients of the binomial decomposition satisfy constraints (85) and analogously constitute an $(n-1)$-dimensional simplex in n dimensions, whose n vertices are $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \dots, \boldsymbol{\alpha}^{(n)}$.

The two sets of n vertices are related by the linear transformations $\tilde{\boldsymbol{w}} = \mathbf{A}\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} = \mathbf{B}\tilde{\boldsymbol{w}}$ as indicated in (84). Denoting $\boldsymbol{\alpha}^{(k)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(k)} = \mathbf{B}\boldsymbol{w}^{(n-k+1)}$, for $k = 1, \dots, n$, note that the vertex $\boldsymbol{\alpha}^{(k)}$ coincides with the $(n-k+1)$ column of the matrix $\mathbf{B}$.

Given that the coefficients α_j, for $j = 1, \dots, n$ are constrained only by conditions (85), the binomial decomposition (82) does not express a simple convex combination of the binomial OWA functions, as the second part of (85) might suggest. In fact, conditions (85) allow for negative α values, as we see in the following examples.

Example 8. Consider the case $n = 3$. Using the condition $\alpha_1 + \alpha_2 + \alpha_3 = 1$, we can write the inequality constraints in (85) only in terms of α_1 and α_2,

$$\begin{cases} \alpha_1 \geq 0 \\ \alpha_1 + \alpha_2 \geq 0 \\ 2\,\alpha_1 + \alpha_2 \leq 3 \end{cases} \tag{86}$$

and the corresponding feasible region in the α_1, α_2 plane is illustrated in Fig. 1, where the two subregions of low (blue) and high (orange) orness are divided by the line associated with OWA functions with $\text{Orness}(A) = 1/2$.
The three vertices of the feasible region depicted in Fig. 1 are given by

$$\begin{array}{ll} \boldsymbol{w}^{(1)} = (1,0,0) & \boldsymbol{\alpha}^{(1)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(1)} = \mathbf{B}\boldsymbol{w}^{(3)} = (0,0,1) \\ \boldsymbol{w}^{(2)} = (0,1,0) & \boldsymbol{\alpha}^{(2)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(2)} = \mathbf{B}\boldsymbol{w}^{(2)} = (0,3,-2) \\ \boldsymbol{w}^{(3)} = (0,0,1) & \boldsymbol{\alpha}^{(3)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(3)} = \mathbf{B}\boldsymbol{w}^{(1)} = (3,-3,1) \end{array}$$

Note that each vertex $\boldsymbol{\alpha}^{(k)}$, for $k = 1, 2, 3$, corresponds to the $(n-k+1)$ column of the matrix $\mathbf{B}$ in dimension $n = 3$.

Two important particular cases are those of OWA functions with monotonic weights, which means non-decreasing weights $w_1 \leq w_2 \leq w_3$ or non-increasing weights $w_1 \geq w_2 \geq w_3$. Each case adds two inequalities to the basic three inequalities in (86), respectively

$$\begin{cases} 3\alpha_1 + 2\alpha_2 \geq 3 \\ \alpha_2 \leq 0 \end{cases} \qquad \begin{cases} 3\alpha_1 + 2\alpha_2 \leq 3 \\ \alpha_2 \geq 0 \end{cases} \tag{87}$$

where the inequalities on the left regard the case of non-decreasing weights and those on the right regard the case of non-increasing weights. The corresponding subregions of the basic feasible region in Fig. 1 are depicted in Fig. 2, where the subregion associated with non-decreasing weights is indicated in red and the subregion associated with non-increasing weights is indicated in green.

Example 9. Consider the case $n = 4$. Using the condition $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, we can write the inequality constraints in (85) only in terms of α_1, α_2, and α_3,

$$\begin{cases} \alpha_1 \geq 0 \\ 3\,\alpha_1 + 2\,\alpha_2 \geq 0 \\ 3\,\alpha_1 + 4\,\alpha_2 + 3\,\alpha_3 \geq 0 \\ 3\,\alpha_1 + 2\,\alpha_2 + \,\alpha_3 \leq 4 \end{cases} \tag{88}$$

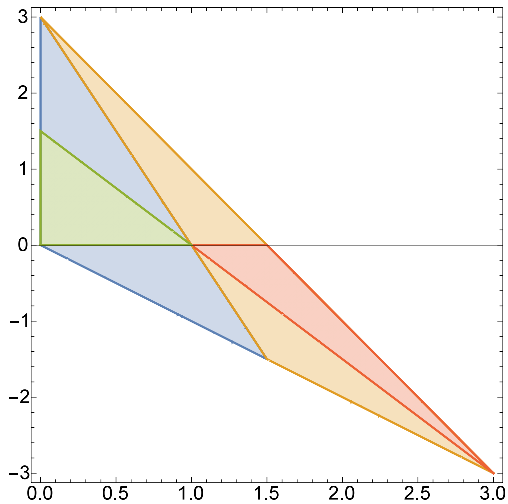

Fig. 2. Feasible region associated with conditions (86) and (87) and in the case $n = 3$.

The four vertices of the feasible region are given by

$$
\begin{array}{ll}
\boldsymbol{w}^{(1)} = (1,0,0,0) & \boldsymbol{\alpha}^{(1)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(1)} = \mathbf{B}\boldsymbol{w}^{(4)} = (0,0,0,1) \\
\boldsymbol{w}^{(2)} = (0,1,0,0) & \boldsymbol{\alpha}^{(2)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(2)} = \mathbf{B}\boldsymbol{w}^{(3)} = (0,0,4,-3) \\
\boldsymbol{w}^{(3)} = (0,0,1,0) & \boldsymbol{\alpha}^{(3)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(3)} = \mathbf{B}\boldsymbol{w}^{(2)} = (0,6,-8,3) \\
\boldsymbol{w}^{(4)} = (0,0,0,1) & \boldsymbol{\alpha}^{(4)} = \mathbf{B}\tilde{\boldsymbol{w}}^{(4)} = \mathbf{B}\boldsymbol{w}^{(1)} = (4,-6,4,-1)
\end{array}
$$

Note that each vertex $\boldsymbol{\alpha}^{(k)}$, for $k = 1, \dots, 4$, corresponds to the $(n - k + 1)$ column of the matrix $\mathbf{B}$ in dimension $n = 4$.

As before, two important particular cases are those of OWA functions with monotonic weights, which means non-decreasing weights $w_1 \leq w_2 \leq w_3 \leq w_4$ or non-increasing weights $w_1 \geq w_2 \geq w_3 \geq w_4$. Each case adds three inequalities to the basic four inequalities in (88), respectively

$$
\begin{cases} 6\alpha_1 + 5\alpha_2 + 3\alpha_3 \geq 6 \\ 2\alpha_2 + 3\alpha_3 \leq 0 \\ \alpha_2 \leq 0 \end{cases} \qquad \begin{cases} 6\alpha_1 + 5\alpha_2 + 3\alpha_3 \leq 6 \\ 2\alpha_2 + 3\alpha_3 \geq 0 \\ \alpha_2 \geq 0 \end{cases} \tag{89}
$$

where the inequalities on the left regard the case of non-decreasing weights and those on the right regard the case of non-increasing weights.

In the previous examples we have examined the cases $n = 3$ and $n = 4$ and in each case we have determined that the feasible regions associated with the weights w_i for $i = 1, \dots, n$ and the coefficients α_j for $j = 1, \dots, n$ are $(n - 1)$ dimensional simplexes with n vertices $\boldsymbol{w}^{(1)}, \dots, \boldsymbol{w}^{(n)}$ and $\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(n)}$, respectively. Moreover, as illustrated in Fig. 1, the simplexes associated with the α coefficients contain points with negative coefficients.

An interesting result is that those points $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ whose coefficients are all non-negative are associated to points $\boldsymbol{w} = (w_1, \dots, w_n)$ whose weights are non-increasing, although the converse is not true. The fact that the non-negativity of the coefficients implies the non-increaseness of the weights is easily proved using the linear systems on the left of (83) as follows,

$$
\begin{aligned}
w_{n-i} - w_{n-i+1} &= \sum_{j=1}^{n} a_{i+1,j}\, \alpha_j - \sum_{j=1}^{n} a_{ij}\, \alpha_j \\
&= \sum_{j=1}^{n} \left(\binom{i}{j-1} - \binom{i-1}{j-1} \right) \binom{n}{j}^{-1} \alpha_j \\
&= \sum_{j=2}^{n} \binom{i-1}{j-2} \binom{n}{j}^{-1} \alpha_j \geq 0 \qquad i = 1, \dots, n-1
\end{aligned} \tag{90}
$$

due to the recurrence identity (A.3). Conversely, using the linear systems on the right of (83), in the simple case $n = 3$ we see that the point $\boldsymbol{w} = (w_1 = 0.5, w_2 = 0.4, w_3 = 0.1)$ with decreasing weights is associated with the point $\boldsymbol{\alpha} = (\alpha_1 = 0.3, \alpha_2 = 0.9, \alpha_3 = -0.2)$ in which one of the coefficients is negative. This means that non-increasing weights can also be obtained when some of the α coefficients are negative. Clear numerical and graphical evidence of this fact, in the context of generalized Gini welfare functions, is discussed in Bortot, Fedrizzi, Marques Pereira, and Nguyen [2] and Bortot, Marques Pereira, and Nguyen [3].

These properties of the relations between the points $\boldsymbol{w}$ and the corresponding points $\boldsymbol{\alpha}$ in the equivalent canonical and binomial representations of OWA functions are relevant, for instance, in the context of welfare and inequality. The reason is that OWA functions which non-increasing weights are Schur-concave [1] and constitute the class of generalized Gini welfare functions.

6. Conclusions

The material presented in the paper has a twofold character. On the one hand, we have considered the binomial decomposition of OWA functions introduced in [4] and we have examined it from the point of view of the equivalence between the canonical and the binomial representations of OWA functions. In this perspective, we have discussed the derivation of the linear transformations from one representation to the other, including expressions, the original one in [4] and its the inverse expression recently proposed in [20], with a long and elaborate proof based on strong induction. The material of this discussion, in a uniform formulation which includes a straightforward, direct proof of the inverse expression, is presented in Proposition 4.

On the other hand, we have extended the original framework of the binomial decomposition, involving constrained linear combinations of the binomial OWA functions, to the general framework of unconstrained linear combinations of binomial OWA functions. In this different perspective, in which the linear transformation from the canonical to the binomial representation corresponds to a change of basis in the description of a functional vector space, we have used the natural matrix formulation of linear algebra in order to express the linear transformations from the canonical to the binomial representation, and vice versa, by means of simple matrix multiplication. This material is presented in Proposition 6.

The linear algebra formulation of the general unconstrained framework can be useful, back in the original constrained framework of the binomial decomposition of OWA functions, to provide new insight on the geometry of the vertices and the orness distribution within the simplexes described by (38) in the binomial representation of OWA functions.

The paper contains various graphical and numerical examples, and along the text we mention some natural applications [1,2] in the context of generalized Gini welfare functions.

CRediT authorship contribution statement

Silvia Bortot: Investigation. **Ricardo Alberto Marques Pereira:** Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A

In the paper we use the binomial coefficients $\binom{r}{s}$ for $r, s \geq 0$, with the understanding that they are null when $r < s$. For the convenience of the readers, we present here, in a uniform notation, the basic binomial identities and formulas used in the paper.

The relevant binomial identities referred in the text are

$$\textit{symmetry identity} \qquad \binom{r}{s} = \binom{r}{r-s} \qquad r \geq s \geq 0 \tag{A.1}$$

$$\textit{product identity} \qquad \binom{r}{t}\binom{t}{s} = \binom{r-s}{t-s}\binom{r}{s} \qquad r \geq t \geq s \geq 0 \tag{A.2}$$

$$\textit{recurrence identity} \qquad \binom{r+1}{s+1} = \binom{r}{s} + \binom{r}{s+1} \qquad r \geq s \geq 0 \tag{A.3}$$

and the relevant binomial formulas involving summations are

$$\textit{row-sum formula} \qquad \sum_{t=0}^{r} \binom{r}{t} = 2^r \qquad r \geq 0 \tag{A.4}$$

$$\textit{column-sum formula} \qquad \sum_{t=0}^{r} \binom{t}{s} = \sum_{t=s}^{r} \binom{t}{s} = \binom{r+1}{s+1} \qquad r \geq s \geq 0 \tag{A.5}$$

$$\textit{alternating row-sum formula} \qquad \sum_{t=0}^{r} (-1)^t \binom{r}{t} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{otherwise} \end{cases} \qquad r \geq 0 \tag{A.6}$$

$$\textit{partial alternating row-sum formula} \qquad \sum_{t=0}^{r-s} (-1)^t \binom{r}{t} = (-1)^{r-s} \binom{r-1}{r-s} \qquad r \geq s \geq 1\,. \tag{A.7}$$

In the paper we use the following binomial inversion formula,

$$\sum_{t=s}^{r}(-1)^{t-s}\binom{r}{t}\binom{t}{s}=\delta_{rs} \qquad r\geq s\geq 0 \tag{A.8}$$

where δ_{rs} is the Kronecker delta, with $\delta_{rs}=1$ if $r=s$, and 0 otherwise. The proof proceeds as follows,

$$\sum_{t=s}^{r}(-1)^{t-s}\binom{r}{t}\binom{t}{s}=\sum_{t=s}^{r}(-1)^{t-s}\binom{r-s}{t-s}\binom{r}{s} \tag{A.9}$$

$$=\binom{r}{s}\sum_{l=0}^{r-s}(-1)^{l}\binom{r-s}{l} \tag{A.10}$$

$$=\binom{r}{s}\delta_{rs}=\delta_{rs} \tag{A.11}$$

where in (A.9) we have used the product identity (A.2), and from (A.10) to (A.11) we have used the alternating row-sum formula (A.6), which can be written as

$$\sum_{t=0}^{r}(-1)^{t}\binom{r}{t}=\delta_{r0} \qquad r\geq 0\,. \tag{A.12}$$

Note that this formula corresponds to the particular case $s=0$ of the general formula (A.8). Another interesting particular case of (A.8) corresponds to $s=1$, in which case we have

$$\sum_{t=1}^{r}(-1)^{t}\binom{r}{t}\,t=-\delta_{r1} \qquad r\geq 1\,. \tag{A.13}$$

In the paper we also use the following formula for the *partial alternating row-sums of binomial coefficients*,

$$\sum_{t=s}^{r}(-1)^{r-t}\binom{r}{t}=(-1)^{r-s}\binom{r-1}{s-1} \qquad r\geq s\geq 1\,. \tag{A.14}$$

The proof proceeds as follows,

$$\sum_{t=s}^{r}(-1)^{r-t}\binom{r}{t}=\sum_{t=s}^{r}(-1)^{r-t}\binom{r}{r-t} \tag{A.15}$$

$$=\sum_{l=0}^{r-s}(-1)^{l}\binom{r}{l} \tag{A.16}$$

$$=(-1)^{r-s}\binom{r-1}{r-s}=(-1)^{r-s}\binom{r-1}{s-1} \tag{A.17}$$

where from (A.16) to (A.17) we have used the partial alternating row-sum formula (A.7) and the symmetry identity (A.1). Note that the full summation in (A.15) is null, that is,

$$\sum_{t=0}^{r}(-1)^{r-t}\binom{r}{t}=(-1)^{r}\binom{r}{0}+\sum_{t=1}^{r}(-1)^{r-t}\binom{r}{t} \tag{A.18}$$

$$=(-1)^{r}+(-1)^{r-1}=0 \tag{A.19}$$

where from (A.18) to (A.19) we have used (A.14) with $s=1$.
Finally, in the paper we use the following *balance property*,

$$\binom{r}{t-1}\binom{s}{t}\leq\binom{r}{t}\binom{s}{t-1} \qquad r\geq s\geq 1 \text{ and } t\geq 1\,. \tag{A.20}$$

The proof proceeds as follows. Let $r\geq s\geq 1$. If $t\leq s$, the various factorial terms can be almost entirely simplified and the inequality (A.20) reduces to $s\leq r$, which is assumed to be true. Otherwise, if $t>s$, the inequality holds trivially in any case.

References

[1] S. Bortot, R.A. Marques Pereira, The binomial Gini inequality indices and the binomial decomposition of welfare functions, Fuzzy Sets Syst. 255 (2014) 92–114.

[2] S. Bortot, M. Fedrizzi, R.A. Marques Pereira, T.H. Nguyen, The binomial decomposition of generalized Gini welfare functions, the S-Gini and Lorenzen cases, Inf. Sci. 460–461 (2018) 555–577.

[3] S. Bortot, R.A. Marques Pereira, T.H. Nguyen, The binomial decomposition of OWA functions, the 2-additive and 3-additive cases in n dimensions, Int. J. Intell. Syst. 33 (2018) 187–212.

[4] T. Calvo, B. De Baets, Aggregation operators defined by k-order additive/maxitive fuzzy measures, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 6 (6) (1998) 533–550.

[5] A. Chateauneuf, J.Y. Jaffray, Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion, Math. Soc. Sci. 17 (3) (1989) 263–283.
[6] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953) 131–295.
[7] D. Denneberg, Non-additive Measure and Integral, Kluwer Academic Publishers, Dordrecht, 1994.
[8] A. Emrouznejad, M. Marra, Ordered weighted averaging operators 1988–2014: a citation-based literature survey, Int. J. Intell. Syst. 29 (11) (2014) 994–1014.
[9] J. Fodor, J.L. Marichal, M. Roubens, Characterization of the ordered weighted averaging operators, IEEE Trans. Fuzzy Syst. 3 (2) (1995) 236–240.
[10] M. Grabisch, Fuzzy integral in multicriteria decision making, Fuzzy Sets Syst. 69 (3) (1995) 279–298.
[11] M. Grabisch, The application of fuzzy integrals in multicriteria decision making, Eur. J. Oper. Res. 89 (3) (1996) 445–456.
[12] M. Grabisch, k-Order additive discrete fuzzy measures and their representation, Fuzzy Sets Syst. 92 (2) (1997) 167–189.
[13] M. Grabisch, Alternative representations of discrete fuzzy measures for decision making, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 5 (5) (1997) 587–607.
[14] M. Grabisch, C. Labreuche, Fuzzy measures and integrals in MCDA, in: J. Figueira, S. Greco, M. Ehrgott (Eds.), Multiple Criteria Decision Analysis, Springer, Heidelberg, 2005, pp. 563–604.
[15] M. Grabisch, I. Kojadinovich, P. Meyer, A review of methods for capacity identification in Choquet integral based multi-attribute utility theory: applications of the Kappalab R package, Eur. J. Oper. Res. 186 (2) (2008) 766–785.
[16] M. Grabisch, C. Labreuche, A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, 4OR 6 (1) (2008) 1–44.
[17] M. Grabisch, C. Labreuche, A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, Ann. Oper. Res. 175 (1) (2010) 247–286.
[18] J.L. Marichal, Aggregation operators for multicriteria decision aid, Ph.D. Thesis, University of Liège, 1998.
[19] P. Miranda, M. Grabisch, Optimization issues for fuzzy measures, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 7 (6) (1999) 545–560.
[20] T.H. Nguyen, The algebraic representation of OWA functions in the binomial decomposition framework and its applications in large-scale problems, Ph.D. Thesis, University of Trento, Italy, 2019.
[21] G.C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, Z. Wahrscheinlichkeitstheor. Verw. Geb. 2 (4) (1964) 340–368.
[22] H.J. Skala, Concerning ordered weighted averaging aggregation operators, Stat. Pap. 32 (1) (1991) 35–44.
[23] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Thesis, Tokyo Institut of Technology, 1974.
[24] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1) (1988) 183–190.
[25] R.R. Yager, J. Kacprzyk (Eds.), The Ordered Weighted Averaging Operators, Theory and Applications, Kluwer Academic Publisher, Dordrecht, 1997.
[26] R.R. Yager, J. Kacprzyk, G. Beliakov (Eds.), Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice, Studies in Fuzziness and Soft Computing, vol. 265, Springer, Heidelberg, 2011.