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OPTIMAL CONTROL FOR STOCHASTIC VOLTERRA EQUATIONS WITH COMPLETELY MONOTONE KERNELS

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Abstract

In this paper, we study a class of stochastic optimal control problems, where the drift term of the equation has a linear growth on the control variable, the cost functional has a quadratic growth, and the control process belongs to the class of square integrable, adapted processes with no bound assumed on it.

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1. Introduction

Integro partial differential equations are widely used in literature for modelling various technological applications related to the study of creep of metal, plastic materials, concrete, rock, and other bodies. In particular, we recall the applications in viscoelasticity theory (see for instance [21] for several examples on how the state of a mechanical system depends on the whole history of actions that were performed on it) and fractional diffusion-wave equations.

Let us recall the model for a linear fractional diffusion-wave equation

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) = b \frac{\partial^2}{\partial x^2} u(t,x),$$

where α is a parameter describing the order of the fractional derivative, that is taken in the sense of Caputo fractional derivative. Altough theoretically α can be any number, we consider here only the case $0 < \alpha < 1$; notice that for $\alpha = 1$ the above equation represents the standard diffusion equation. There has been a growing interest to investigate this equation, for various reasons; for instance [1] quote “modeling of anomalous diffusive and subdiffusive systems” and “description of fractional random walk”.

The introduction of a stochastic perturbation term is generally motivated in the literature as a model for random environment or rapidly varying perturbing term or as a model for chaotic behaviour of the system. In this paper we assume that the noise enters the system when we introduce a *control*. The object of this paper is really to study the optimal control problem for an

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integro partial differential equation with a stochastic perturbation of the control; we remark that several concrete examples, in the fields mentioned above, can be handled within our theory. Let us briefly introduce the general equation we will deal with.

Throughtout the paper we assume that H, Ξ and U are real separable Hilbert spaces. We are concerned with the following class of stochastic integral Volterra equation on H

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s &= Au(t) + g(t,u(t))\,\Big[r(t,u(t),\gamma(t)) + \dot{W}(t)\Big], \qquad t\in[0,T] \\ u(t) &= u_0(t), \qquad t \le 0. \end{aligned} \tag{1.1}$$

The control problem consists of minimizing a cost functional of the form

$$\mathbb{J}(u_0,\gamma) = \mathbb{E}\int_0^T l(t,u(t),\gamma(t))\,\mathrm{d}t + \mathbb{E}[\phi(u(T))]. \tag{1.2}$$

To our knowledge, this paper is the first attempt to study optimal control problems for stochastic Volterra equations in infinite dimensions. In order to handle the control problem, we first restate equation (1.1) in an evolution setting, by the state space setting first introduced in [19, 9] and recently revised, for the stochastic case, in [17, 3, 4]. Thus we obtain a stochastic evolution equation in a (different) Hilbert space. Then we associate to this equation a backward stochastic equation and we try to solve the control problem via this forward-backward system (FBSE). Nonlinear backward stochastic differential equations (BSDEs) were first introduced by Pardoux and Peng [20]. A major application of BSDEs is in stochastic control: see, e.g., [10, 22]. We also refer the reader to [16, 11] and [15].

In comparison with the existing literature, we must underline the following two facts. The first is typical of our approach: the forward system has an unbounded operator in the diffusion term, which makes it harder to solve. Second, we consider a degenerate control problem (since nothing is assumed on the image of g), essentially in the setting assumed in [13]: we suppose that r has a linear growth in γ and it is not bounded, l has quadratic growth in u and γ, and ϕ has quadratic growth in u; but, differently from them, we treat the infinite dimensional case.

We work under the following set of assumptions:

Hypothesis 1.1.

(i) The kernel $a : (0,\infty) \to \mathbb{R}$ is completely monotonic, locally integrable, with $a(0+) = +\infty$. The singularity in 0 shall satisfy some technical conditions that we make precise in Section 2. As an example, we can consider the fractional derivative kernel $a(t) = \frac{1}{\Gamma[\rho]}t^{\rho-1}$ for $0 < \rho < \frac{1}{2}$ that was discussed above.

(ii) $A : D(A) \subset H \to H$ is a sectorial operator in H. Thus A generates an analytic semigroup e^{tA}. Interpolation and extrapolation spaces H_η of H will always be constructed with respect to A.

(iii) The mapping $g : [0,T]\times H \to L_2(U,H)$ (the space of Hilbert-Schmidt operators from U into H) is measurable; moreover there exists a constant $C > 0$ such that for every $t \in [0,T]$, $u, u' \in H$

$$\|g(t,u) - g(t,u')\|_{L_2(U,H)} \le C|u-u'|,$$

$$\|g(t,u)\|_{L_2(U,H)} \le C;$$

(iv) *The function* $r : [0, T] \times H \times \Xi \to U$ *is measurable and there exists* $C > 0$ *such that, for* $t \in [0, T]$, $u, u' \in H$ *and* $\gamma \in \Xi$, *it holds*

$$|r(t, u, \gamma) - r(t, u', \gamma)| \le C(1 + |\gamma|)|u - u'|;$$

moreover $r(t, u, \cdot)$ *has sub-linear growth: there exists* $C > 0$ *such that, for* $t \in [0, T]$ *and* $u \in H$, *it holds*

$$|r(t, u, \gamma)| \le C(1 + |\gamma|). \tag{1.3}$$

Finally, for all $t \in [0, T]$, $u \in H$, $r(t, u, \cdot)$ *is a continuous function from* Ξ *to* U.

(v) *The initial condition* $u_0(t)$ *belongs to the space* $\tilde{X}_0$ *of admissible initial conditions:*

$$\tilde{X}_0 = \left\{ u : \mathbb{R}_- \to H, \text{ there exists } M > 0 \text{ and } \omega > 0 \text{ such that } |u(t)| \le M e^{-\omega t} \right\}$$

and it satisfies further the assumption:

(a) $u_0(0)$ *belongs to* H_ϵ *for some* $\epsilon \in (0, \frac{1}{2})$ *and there exist* $M_2 > 0$ *and* $\tau > 0$ *such that* $|u_0(t) - u_0(0)| \le M|t|$ *for all* $t \in [-\tau, 0]$).

(vi) *The process* $\{W_t,\ t \in [0, T]\}$ *is a cylindrical Wiener process defined on a complete probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ *with values in the Hilbert space* U. *This means that* $W(t)$ *is a linear mapping* $W(t) : U \to L^2(\Omega)$ *such that* (a) *for every* $u \in U$, $\{W(t) \cdot u,\ t \ge 0\}$ *is a real-valued Brownian motion and* (b) *for every* $u, v \in U$ *and* $t \ge 0$, $\mathbb{E}[(W(t) \cdot u)(W(t) \cdot v)] = \langle u, v \rangle_U$.

(vii) *We say that a* Ξ*-valued,* $\mathcal{F}_t$*-adapted process* γ *belongs to the class of admissible controls if*

$$\mathbb{E} \int_0^T |\gamma(s)|^2 \, \mathrm{d}s < +\infty.$$

(viii) *The functions* l *and* ϕ *which enters the definition of the cost functional are measurable mappings* $l : [0, T] \times H \times \Xi \to R$, $\phi : H \to \mathbb{R}$, *satisfying the bounds*

$$0 \le l(t, u, \gamma) \le C(1 + |u|^2 + |\gamma|^2) \tag{1.4}$$

and

$$0 \le \phi(u) \le C(1 + |u|^2) \tag{1.5}$$

for given constants c, C.
Moreover for all $t \in [0, T]$, $u \in H$, $l(t, u, \cdot)$ *is a continuous function from* Ξ *to* $\mathbb{R}$.

(ix) *There exist* $c > 0$ *and* $R > 0$ *such that for every* $t \in [0, T]$, $x \in H$ *and every control* γ *satisfying* $|\gamma| > R$ *then*

$$l(t, u, \gamma) \ge c(1 + |\gamma|^2) \tag{1.6}$$

Hence, a control process which is not square summable would have infinite cost.

We consider the following notion of solution for the Volterra equation (1.1).

Definition 2.1 *We say that a process* $u = \{u(t),\ t \in [0, T]\}$ *is a solution to (1.1) if* u *is an* H*-valued predictable process with*

$$\mathbb{E} \int_0^T |u(s)|^2 \, \mathrm{d}s < +\infty$$

and the identity

$$\int_{-\infty}^t a(t-s)\langle u(s),\zeta\rangle_H \,\mathrm{d}s = \langle \bar u,\zeta\rangle_H + \int_0^t \langle u(s), A^\star\zeta\rangle_H \,\mathrm{d}s \\ + \int_0^t \langle g(s,u(s)) r(s,u(s),\gamma(s)),\zeta\rangle_H \,\mathrm{d}s + \int_0^t \langle g(s,u(s))\,\mathrm{d}W(s),\zeta\rangle_H \quad (1.7)$$

holds $\mathbb{P}$-a.s. for arbitrary $t \in [0,T]$, where $A^\star$ is the adjoint of the operator A and

$$\bar u = \int_{-\infty}^0 a(-s)u_0(s)\,\mathrm{d}s.$$

Our first result is the existence and uniqueness of the solution for problem (1.1). This will be the object of Section 4.4; notice that, in order to get this result, we must first prove a general result concerning existence and uniqueness of the solution for a stochastic evolution equation with unbounded operator terms. This is an extension of the results in [8], compare also [4].

Theorem 4.2 *For every admissible control γ, there exists a unique solution u to problem (1.1).*

We proceed with the study of the optimal control problem associated to the stochastic Volterra equation (1.1)(Section 5). The control problem consists of minimizing a cost functional as introduced in (1.2). While proceeding in the proof of Theorem 4.2, we shall show in Section 3 that we can associate to (1.1) the controlled state equation

$$\begin{aligned} \mathrm{d}v(t) &= \big[Bv(t) + (I-B)Pg(t,J(v(t)))\,r(t,J(v(t)),\gamma(t))\big]\,\mathrm{d}t \\ &\qquad\qquad + (I-B)Pg(t,J(v(t)))\,\mathrm{d}W_t \\ v(0) &= v_0. \end{aligned} \quad (1.8)$$

Such approach, called state space setting, have the following interpretation. On a different Hilbert space X, the internal state of the system at time t is recorded into an random variable $v(t)$, which contains all the informations about the solution up to time t; the state space X is quite large and does not have a direct interpretation in terms of the original space H. $B : D(B) \subset X \to X$ is the operator which governs the evolution from paste into future; it is proven that B generates an analytic, strongly continuous semigroup on X; we again appeal to interpolation theory in order to define the spaces $X_\eta = (X, D(B))_{\eta,2}$. The operator $J : D(J) \subset X \to H$ recovers the original variable u in terms of v. $P : H \to X$ is a linear operator which acts as a sort of projection into the state space X. For the details, we refer to section 2 or the original papers [9, 3].

Also the control problem can be translated in the state setting: our purpose is to minimize over all admissible controls the cost functional

$$\mathbb{J}(v_0,\gamma) = \mathbb{E}\int_0^T l(t,J(v(t)),\gamma(t))\,\mathrm{d}t + \mathbb{E}[\phi(J(v(T))]. \quad (1.9)$$

Now, to solve the control problem we use the forward-backward system approach. We define in a classical way the Hamiltonian function relative to the above problem:

$$\psi(t,v,z) = \inf\{l(t,J(v),\gamma) + z\cdot r(t,J(v),\gamma) : \gamma \in \Xi\} \quad (1.10)$$

We take an arbitrary, complete probability space $(\Omega,\mathcal{F},\mathbb{P}^\circ)$ and a Wiener process W° in U with respect to $\mathbb{P}^\circ$. We denote by $(\mathcal{F}_t^\circ)$ the associated Brownian filtration, i.e., the filtration generated by W° and augmented by the $\mathbb{P}^\circ$-null sets of $\mathcal{F}$; $(\mathcal{F}_t^\circ)$ satisfies the usual conditions. We introduce the forward equation

$$\begin{aligned} \mathrm{d}v(t) &= Bv(t)\,\mathrm{d}t + (I-B)Pg(t, J(v(t)))\,\mathrm{d}W_t^\circ \\ v(0) &= v_0 \end{aligned} \tag{1.11}$$

whose solution is a continuous $(\mathcal{F}_t^\circ)$-adapted process, which exists and is unique by the results in Section 3. Next we consider the associated backward equation of parameters $(\psi, T, \phi(J(v_T)))$

$$\begin{aligned} dY_t &= \psi(t, v_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t^\circ, \qquad t \in [0,T], \\ Y_T &= \phi(J(v_T)). \end{aligned} \tag{1.12}$$

The solution of (1.12) exists in the sense specified by the following proposition.

Proposition 5.3 *There exist Borel measurable functions θ and ζ with values in $\mathbb{R}$ and U^*, respectively, with the following property: for an arbitrarily chosen complete probability space $(\Omega, \mathcal{F}, \mathbb{P}^\circ)$ and Wiener process W° in U, denoting by v the solution of (1.11), the processes Y, Z defined by*

$$Y_t = \theta(t, v_t), \quad Z_t = \zeta(t, v_t)$$

satisfy

$$\mathbb{E}^\circ \sup_{t\in[0,T]} |Y_t|^2 < \infty, \quad \mathbb{E}^\circ \int_0^T |Z_t|^2\,\mathrm{d}t < \infty;$$

moreover, Y is continuous and nonnegative, and $\mathbb{P}^\circ$-a.s.,

$$Y_t + \int_t^T Z_s\,\mathrm{d}W_s^\circ = \phi(v_T) + \int_t^T \psi(s, v_s, Z_s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{1.13}$$

Finally, this solution is the maximal solution among all the solutions (Y', Z') of (1.12) satisfying

$$\mathbb{E}^\circ \sup_{t\in[0,T]} |Y_t^{'}|^2 < \infty.$$

The difficulty here is that the Hamiltonian corresponding to the control problem has quadratic growth in the gradient and consequently the associated BSDE has quadratic growth in the Z variable. Well-posedness for this class of BSDEs has been proved in [18] in the case of bounded terminal value. Since we allow for unbounded terminal cost, to treat such equations we have to apply the techniques introduced in [5] and used in [13]. This point require a particular attention, because we can not use directly a monotone stability result (see Prop. 2.4 in [18]), well-known in finite dimensional framework. We notice that for such BSDEs no general uniqueness results are known: we replace uniqueness with the selection of a maximal solution.

Our main result is to prove that the optimal feedback control exists and the optimal cost is given by the value Y_0 of the maximal solution (Y, Z) of the BSDE (1.12) with quadratic growth and unbounded terminal value.

Corollary 5.5 *For every admissible control γ and any initial datum x, we have $\mathbb{J}(\gamma) \geq \theta(0,x) = Y_0$, and the equality holds if and only if the following feedback law holds $\mathbb{P}$-a.s. for almost every $t \in [0,T]$:*

$$\psi(t, v_t, \zeta(t, v_t)) = \zeta(t, v_t) \cdot r(t, J(v_t), \gamma_t) + l(t, J(v_t), \gamma_t),$$

where v is the trajectory starting at x and corresponding to control γ.

Finally we address the problem of finding a weak solution to the so-called closed loop equation. If we assume that the infimum in (1.10) is attained, we can prove that there exists a measurable function μ of t, v, z such that

$$\psi(t, v, z) = l(t, J(v), \mu(t, v, z)) + z \cdot r(t, J(v), \mu(t, v, z)). \tag{1.14}$$

We define $\bar{\gamma}(t, v) = \mu(t, v, \zeta(t, v))$, where ζ is defined in Proposition 5.3. The closed loop equation is

$$\begin{aligned} \mathrm{d}v(t) &= \big[Bv(t) + (I - B)Pg(t, J(v(t)))\Phi r(t, J(v(t)), \bar{\gamma}(t, v))\big]\,\mathrm{d}t \\ &\qquad\qquad + (I - B)Pg(t, J(v(t)))\,\Phi\,\mathrm{d}W_t \\ v(0) &= v_0 \end{aligned} \tag{1.15}$$

By a weak solution we mean a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions, a Wiener process W in U with respect to $\mathbb{P}$ and $(\mathcal{F}_t)$, and a continuous $(\mathcal{F}_t)$-adapted process v with values in X satisfying, $\mathbb{P}$-a.s.,

$$\mathbb{E}\int_0^T |\bar{\gamma}(t, v_t)|^2 dt < \infty, \tag{1.16}$$

and such that (1.15) holds. First of all, we prove the following.

Proposition 5.6 *There exists a weak solution of the closed loop equation, satisfying in addition*

$$\mathbb{E}\int_0^T |\bar{\gamma}(t, v_t)|^2 dt < \infty. \tag{1.17}$$

Moreover, we show that we can construct an optimal feedback in terms of the process Z.

Corollary 5.7 *If v is the solution to (1.15) and we set $\gamma^*_s = \bar{\gamma}(s, v_s)$, then $\mathbb{J}(\gamma^*) = \theta(0, x)$, and consequently v is an optimal state, γ^*_s is an optimal control, and $\bar{\gamma}$ is an optimal feedback.*

2. The state equation

The convolution kernel $a : (0, \infty) \to \mathbb{R}$ in equation (1.1) is completely monotone, with $a(0+) = \infty$ and $\int_0^1 a(s)\,\mathrm{d}s < \infty$. In particular, by Bernstein's Theorem there exists a measure ν on $[0, \infty)$ such that

$$a(t) = \int_{[0,\infty)} e^{-\kappa t}\,\nu(\mathrm{d}\kappa). \tag{2.1}$$

From the required singularity of a at $0+$ we obtain that $\nu([0, \infty)) = a(0+) = \infty$ while for $s > 0$ the Laplace transform $\hat{a}$ of a verifies

$$\hat{a}(s) = \int_{[0,\infty)} \frac{1}{s + \kappa}\,\nu(\mathrm{d}\kappa) < \infty.$$

As stated in the introduction, we also require an assumption on the singularity of a at $0+$.

Hypothesis 2.1. *For the completely monotone kernel a we define the following numbers:*

$$\begin{aligned} \alpha(a) &:= \sup\left\{\rho \in (0,1) \mid \int_c^\infty s^{\rho-2}\frac{1}{\hat{a}(s)}\,\mathrm{d}s < \infty\right\}, \\ \delta(a) &:= \inf\left\{\rho \in (0,1) \mid \int_c^\infty s^{-\rho}\hat{a}(s)\,\mathrm{d}s < \infty\right\}, \end{aligned} \tag{2.2}$$

for some $c > 0$. Then we require $\alpha(a) > \dfrac{1}{2}$.

The definitions of $\alpha(a)$ and $\delta(a)$ are independent of the choice of the number $c > 0$. It is always true that $\alpha(a) \le \delta(a)$, but there are completely monotone kernels a with $\alpha(a) < \delta(a)$.

These quantities are related to the power of the singularity of the kernel at $0+$ as the following example shows.

Remark 2.1. Let $a(t) = e^{-bt}\frac{t^{-\rho}}{\Gamma[1-\rho]}$, where $b \ge 0$. This kernel is completely monotone, with Laplace transform $\hat{a}(\lambda) = (b+\lambda)^{\rho-1}$; an easy computation shows that $\alpha(a) = \rho$, hence we satisfy Assumption 2.1 whenever we take $\rho \in (\frac{1}{2}, 1)$.

Under the assumption of complete monotonicity of the kernel, a semigroup approach to a type of abstract integro-differential equations encountered in linear viscoelasticity was introduced in [9]. We recall the extension given in [3] to the case of Hilbert space valued equations. We start for simplicity with the equation

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s &= Au(t) + f(t), \qquad && t \in [0,T] \\ u(t) &= u_0(t), && t \le 0. \end{aligned} \tag{2.3}$$

The starting point is the following identity, which follows by Bernstein's theorem

$$\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s = \int_{-\infty}^t \int_{[0,\infty)} e^{-\kappa(t-s)}\,\nu(\mathrm{d}\kappa)\,u(s)\mathrm{d}s = \int_{[0,\infty)} v(t,\kappa)\,\nu(\mathrm{d}\kappa)$$

where we introduce the state variable

$$v(t,\kappa) = \int_{-\infty}^t e^{-\kappa(t-s)}u(s)\,\mathrm{d}s. \tag{2.4}$$

Formal differentiation yields

$$\frac{\partial}{\partial t}v(t,\kappa) = -\kappa v(t,\kappa) + u(t), \tag{2.5}$$

while the integral equation (1.1) can be rewritten

$$\int_{[0,\infty)} (-\kappa v(t,\kappa) + u(t))\,\nu(\mathrm{d}\kappa) = Au(t) + f(t). \tag{2.6}$$

Now, the idea is to use equation (2.5) as the state equation, with $Bv = -\kappa v(\kappa) + u$, while (2.6) enters in the definition of the domain of B.

In our setting, the function $v(t,\cdot)$ will be considered the state of the system, contained in the state space X that consists of all Borel measurable functions $\tilde{x} : [0,\infty) \to H$ such that the seminorm

$$\|\tilde{x}\|_X^2 := \int_{[0,\infty)} (\kappa+1)|\tilde{x}(\kappa)|_H^2 \, \nu(\mathrm{d}\kappa)$$

is finite. We shall identify the classes $\tilde{x}$ with respect to equality almost everywhere in ν.

Let us consider the initial condition. On the space $\tilde{X}_0$ introduced in Hypothesis 1.1 we define a positive inner product $\langle u, v\rangle_{\tilde{X}} = \int\int [a(t+s) - a'(t+s)]\langle u(-s), v(-t)\rangle_H \, \mathrm{d}s \, \mathrm{d}t$; then, setting $\tilde{N}_0 = \{u \in \tilde{X}_0 \ : \ \langle u, u\rangle_{\tilde{X}} = 0\}$, $\langle \cdot,\cdot\rangle_{\tilde{X}}$ is a scalar product on $\tilde{X}_0/\tilde{N}_0$; we define $\tilde{X}$ the completition of this space with respect to $\langle \cdot,\cdot\rangle_{\tilde{X}}$. We let the operator $Q : \tilde{X} \to X$ be given by

$$v(0,\kappa) Q u_0(\kappa) = \int_{-\infty}^0 e^{\kappa s} u_0(s) \, \mathrm{d}s. \tag{2.7}$$

We quote from [3] the main result concerning the state space setting for stochastic Volterra equations in infinite dimensions.

Theorem 2.2. (State space setting.) *Let A, a, $\alpha(a)$, W be given above; choose numbers $\eta \in (0,1)$, $\theta \in (0,1)$ such that*

$$\eta > \frac{1}{2}(1-\alpha(a)), \quad \theta < \frac{1}{2}(1+\alpha(a)), \quad \theta - \eta > \frac{1}{2}.$$

Then there exist

1) *a separable Hilbert space X and an isometric isomorphism $Q : \tilde{X} \to X$,*
2) *a densely defined sectorial operator $B : D(B) \subset X \to X$ generating an analytic semigroup e^{tB},*
3) *its real interpolation spaces $X_\rho = (X, D(B))_{(\rho,2)}$ with their norms $\|\cdot\|_\rho$,*
4) *linear operators $P : H \to X_\theta$, $J : X_\eta \to H$*

such that the following holds:

a) *For each $v_0 \in X$, the problem (2.3) is equivalent to the evolution equation*

$$\begin{aligned} v'(t) &= Bv(t) + (I-B)Pf(t) \\ v(0) &= v_0 \end{aligned} \tag{2.8}$$

in the sense that if $u_0 \in \tilde{X}_0$ and $v(t;v_0)$ is the weak solution to Problem (2.8) *with $v_0 = Qu_0$, then $u(t;u_0) = Jv(t;v_0)$ is the unique weak solution to Problem* (2.3).

With the same notation as above, but taking into account that we are interested to Eq. (1.1), we obtain the following stochastic evolution equation in X

$$\begin{aligned} \mathrm{d}v(t) &= [Bv(t) + (I-B)Pg(t, J(v(t)))\, r(t, J(v(t)), \gamma(t))]\, \mathrm{d}t \\ &\qquad + (I-B)Pg(t, J(v(t)))\, \mathrm{d}W_t \\ v(0) &= v_0. \end{aligned} \tag{2.9}$$

Definition 2.1. We say that a process $u = \{u(t),\ t \in [0,T]\}$ is a solution of (2.9) if u is an H-valued predictable process with

$$\mathbb{E} \int_0^T |u(s)|^2 \,\mathrm{d}s < +\infty$$

and the identity

$$\int_{-\infty}^t a(t-s)\langle u(s), \zeta\rangle_H \,\mathrm{d}s = \langle \bar{u}, \zeta\rangle_H + \int_0^t \langle u(s), A^\star \zeta\rangle_H \,\mathrm{d}s \\ + \int_0^t \langle g(s,u(s)) r(s,u(s),\gamma(s)), \zeta\rangle_H \,\mathrm{d}s + \int_0^t \langle g(s,u(s)) \,\mathrm{d}W(s), \zeta\rangle_H \quad (2.10)$$

holds $\mathbb{P}$-a.s. for arbitrary $t \in [0,T]$, where $A^\star$ is the adjoint of the operator A and

$$\bar{u} = \int_{-\infty}^0 a(-s) u_0(s) \,\mathrm{d}s.$$

Then we state the main result of this section. Its proof will be given in Section 4.4.

Theorem 2.3. *There exists a unique solution* $\{v(t),\ t \in [0,T]\}$ *of Equation* (2.9)*. Further, the solution depends continuously on the initial condition* $v_0 \in X_\eta$.

3. Stochastic differential equations with unbounded diffusion operator

We first study the uncontrolled equation

$$\begin{aligned} \mathrm{d}v(t) &= Bv(t)\,\mathrm{d}t + (I-B)Pg(t, J(v(t)))\,\mathrm{d}W_t \\ v(s) &= x \end{aligned} \quad (3.1)$$

for $0 \le s \le t \le T$ and initial condition $x \in X_\eta$. The above expression is only formal in X_η since the coefficients do not belong to the state space; however, we can give a meaning to the mild form of the equation:

$$v(t; s, v_0) = e^{(t-s)B} x + \int_s^t e^{(t-\sigma)B}(I-B)Pg(\sigma, J(v(\sigma)))\,\mathrm{d}W(\sigma). \quad (3.2)$$

Let us state the main existence result for the solution of equation (3.1).

Theorem 3.1. *Under the assumptions of Hypothesis 1.1, for every* $p \ge 2$ *there exists a unique process* $v \in L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$ *solution of (3.1). Moreover, the solution* $v(t; s, v_0)$ *depends continuously on the initial conditions* $(s, v_0) \in [0,T] \times X_\eta$ *and the estimate*

$$\mathbb{E} \sup_{t \in [s,T]} \|v(t)\|_\eta^2 \le C(1 + \|v_0\|_\eta)^p$$

holds for some constant C *depending on* T *and the parameters of the problem.*

Proof. The result is basically known, see e.g. [4], but we include the proof for completeness and because it will be useful in the following. The argument is as follows: we define a mapping $\mathcal{K}$ from $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$ to itself by the formula

$$\mathcal{K}(v)(t) = e^{(t-s)B} v_0 + \int_s^t e^{(t-\tau)B}(I-B)Pg(\tau, J(v(\tau)))\,\mathrm{d}W(\tau), \qquad t \in [s,T]$$

and show that it is a contraction, under an equivalent norm. The unique fixed point is the required solution.

Let us introduce the norm

$$\|v\|_\eta^p = \mathbb{E} \sup_{t\in[0,T]} e^{-\beta pt}\|v(t)\|_\eta^p,$$

where $\beta > 0$ will be chosen later. In the space $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$ this norm is equivalent to the original one.

We define the mapping

$$\Lambda(v,s)(t) = \int_s^t e^{(t-\tau)B}(I-B)Pg(\tau, J(v(\tau)))\,\mathrm{d}W(\tau).$$

Our first step is to prove that Λ is a well defined mapping on $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$ and to give estimates on its norm. Let us stress that in the sequel $\Lambda(v)$ will be defined for $t \in [0,T]$ by setting $\Lambda(v)(t) = 0$ for $t < s$.

Lemma 3.2. *For every $\frac{1}{p} < \theta - \eta - \frac{1}{2}$ the operator Λ maps $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$ into itself.*

Proof. Let $v \in L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$; for simplicity we fix the initial time $s = 0$ and write $\Lambda(t) = \Lambda(v,0)(t)$. Further, we remark that the thesis is equivalent to the statement: $t \mapsto (I - B)^\eta\Lambda(t) \in L^p_{\mathcal{F}}(\Omega; C([0,T]; X))$. The proof is based on the classical factorization method by Da Prato and Zabczyk.

Step 1. For given $\gamma \in (0,1)$ and $\eta \in (0,1)$, the following identity holds:

$$(I_H - B_{-1})^\eta\Lambda(t) = c_\gamma \int_0^t e^{-\beta(t-\tau)}(t-\tau)^{\gamma-1}e^{(t-\tau)B}y_\eta(\tau)\,\mathrm{d}\tau$$

where y_η is the process

$$y_\eta(\tau) = \int_0^\tau e^{-\beta\sigma}(\tau-\sigma)^{-\gamma}e^{-\beta(\tau-\sigma)}e^{(\tau-\sigma)B}(I-B_{-1})^\eta Pg(\sigma, J(v(\sigma)))\,\mathrm{d}W(\sigma).$$

We shall estimate the $L^p(\Omega; X)$-norm of this process:

$$\mathbb{E}|y_\eta(\tau)|^p = \mathbb{E}\left|\int_0^\tau e^{-\beta\sigma}(\tau-\sigma)^{-\gamma}e^{-\beta(\tau-\sigma)}e^{(\tau-\sigma)B}(I-B_{-1})^\eta Pg(\sigma, J(v(\sigma)))\,\mathrm{d}W(\sigma)\right|^p.$$

Proceeding as in [7, Lemma 7.2] this leads to

$$\mathbb{E}|y_\eta(\tau)|^p \\ \le C\,\mathbb{E}\left[\int_0^\tau \|e^{-\beta\sigma}e^{(\tau-\sigma)B}(I-B_{-1})^\eta Pg(\sigma, J((\sigma)))\,(\tau-\sigma)^{-\gamma}e^{-\beta(\tau-\sigma)}\|^2_{L_2(U,X)}\,\mathrm{d}\sigma\right]^{p/2}.$$

Since the semigroup e^{tB} is analytic, P maps H into X_θ for arbitrary $\theta < \frac{1+\alpha(a)}{2}$ and g takes values in $L_2(U,H)$, the following estimate holds:

$$\|e^{(\tau-\sigma)B}(I-B_{-1})^\eta Pg(\sigma, J(v(\sigma)))\|_{L_2(U,X)} \le C(\tau-\sigma)^{\theta-1-\eta}\|g(\sigma, J(v(\sigma)))\|_{L_2(U,H)}$$

and the process y_η is estimated by

$$\mathbb{E}|y_\eta(\tau)|^p \le C\,\mathbb{E}\left(\int_0^\tau e^{-2\beta\sigma}\|g(\sigma, J(v(\sigma)))\|^2_{L_2(U,H)}\,e^{-2\beta(\tau-\sigma)}\,(\tau-\sigma)^{-2(\gamma+1+\eta-\theta)}\,\mathrm{d}\sigma\right)^{p/2}$$

We apply Young's inequality to get

$$\begin{aligned}
|y_\eta|^p_{L^p_{\mathcal{F}}(\Omega;L^p(0,T;X))} &= \left(\mathbb{E} \int_0^T |y_\eta(\tau)|^p \,\mathrm{d}\tau \right) \\
&\le C\mathbb{E} \left[\left(\int_0^T e^{-2\beta\sigma} \|g(\sigma, J(v(\sigma)))\|^2_{L_2(U,H)} \,\mathrm{d}\sigma \right)^{2/p} \right]^{p/2} \\
&\qquad \left(\int_0^\infty e^{-\tau} (2\beta)^{1+2\gamma-2(\theta-\eta)} \tau^{-2(\gamma+1+\eta-\theta)} \,\mathrm{d}\tau \right)^{p/2}
\end{aligned}$$

hence, for any $\gamma < (\theta-\eta) - 1/2$ (notice that we can always choose $\gamma > 0$ small enough such that this holds) we obtain

$$\|y_\eta\|_{L^p_{\mathcal{F}}(\Omega;L^p(0,T;H))} \le C_T\,(2\beta)^{1+2\gamma-2(\theta-\eta)} \left(\mathbb{E} \left[\int_0^T e^{-p\beta\sigma} \|g(\sigma, J(v(\sigma)))\|^p_{L_2(U,H)} \,\mathrm{d}\sigma \right] \right)^{1/p} \tag{3.3}$$

Now, taking into account the assumptions on g and the choice of $v \in L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$, we estimate the integral term above and we finally arrive at

$$\|y_\eta\|_{L^p_{\mathcal{F}}(\Omega;L^p(0,T;X))} \le C(2\beta)^{1+2\gamma-2(\theta-\eta)}(1 + \|v\|_\eta).$$

Step 2. In [6, Appendix A] it is proved that for any $\gamma \in (0,1)$, p large enough such that $\gamma - \frac{1}{p} > 0$, the linear operator

$$R_\gamma \phi(t) = \int_0^t (t-\sigma)^{\gamma-1} S(t-\sigma)\phi(\sigma)\,\mathrm{d}\sigma$$

is a bounded operator from $L^q(0,T;X)$ into $C([0,T];X)$. Using the results in Step 1. the thesis follows.

We have proven that Λ is a well defined mapping in the space $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$. In order to conclude the proof that $\mathcal{K}$ maps $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$ into itself it is sufficient to recall that the initial condition x belongs to X_η, hence $t \mapsto e^{(t-s)B}x$, extended to a constant for $t < s$:

$$S(t-s)x = x \qquad \text{for } t < s$$

belongs to $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$.

Next, we prove that $\mathcal{K}$ is a contraction in the space $L^p_{\mathcal{F}}(\Omega; C([0,T]; H_\eta))$. If v, v_1 are processes belonging to this space, similar passages as those in Lemma 3.2 show that

$$\|\Lambda(v) - \Lambda(v_1)\|_\eta \le C(2\beta)^{1+2\gamma-2(\theta-\eta)} \|v - v_1\|_\eta). \tag{3.4}$$

Moreover, we can find β large enough such that

$$C(2\beta)^{1+2\gamma-2(\theta-\eta)} \le \delta < 1$$

so that $\mathcal{K}$ becomes a contraction on the time interval $[0,T]$ and by a classical fixed point argument we get that there exists a unique solution of the mild equation (3.2) on $[0,T]$.

Since the solution to (3.2) verifies $v = \mathcal{K}(v)$ we also deduce from the above computations that

$$\|v\|_\eta = \|\mathcal{K}(v)\|_\eta \le \delta(1 + \|v\|_\eta) + C(T)\|x\|_\eta$$

hence

$$\|v\|_\eta \le C(1 + \|x\|_\eta).$$

Regular dependence on the initial conditions

In the last part of the proof we are concerned with the regular dependence of the solution on the initial conditions. The results in this section rely on Proposition 2.4 in [14] where a parameter depending contraction principle is provided.

As before, we introduce the map

$$\mathcal{K}(v,s,x)(t) = e^{(t-s)B}x + \Lambda(v,s)(t)$$

and we set $\mathcal{K}(v,s,x)(t) = x$ for $t < s$. For a suitable $T > 0$, we have shown that this mapping is a contraction in the space $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$. Now the thesis follows if we prove that, for every $v \in L^p_{\mathcal{F}}(\Omega; C([0,T]; H_\eta))$, the mapping $(s,x) \mapsto \mathcal{K}(v,s,x)$ is continuous as a map from $[0,T] \times X_\eta \to L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$.

We introduce two sequences $\{s_n^-\}$ and $\{s_n^+\}$ such that $s_n^- \nearrow s$ and $s_n^+ \searrow 0$. First, recalling that we extend $S(t-s)x = x$ for $t < s$, we have

$$\sup_{t\in[0,T]} \|e^{(t-s_n^+)B}x - e^{(t-s_n^-)B}x\|_\eta = \sup_{t\in[s_n^-,T]} \|e^{(t-s_n^+)B}[x - e^{(s_n^+-s_n^-)B}x\|_\eta \to 0$$

and also the map $x \to \{t \mapsto e^{(t-s)B}x\}$ is clearly continuous from X_η into $C([0,T]; X_\eta)$.

Next, we consider the mapping Λ. Recall that

$$\Lambda(v,s)(t) = \int_s^t e^{(t-\tau)B}(I-B)Pg(\tau, J(v(\tau)))\,\mathrm{d}W(\tau)$$

and we set $\Lambda(v,s)(t) = 0$ for $t < s$. Our aim is to prove that

$$\mathbb{E} \sup_{t\in[0,T]} \|\Lambda(v,s_n^+) - \Lambda(v,s_n^-)\|_\eta^p \longrightarrow 0.$$

If $t < s_n^-$ then both terms are zero; for $s_n^- < t < s_n^+$, the first term disappears and only the second one remains; finally, for $t > s_n^+$ the first integral compensates a part of the second one, and we get

$$\mathbb{E} \sup_{t\in[0,T]} \|\Lambda(v,s_n^+) - \Lambda(v,s_n^-)\|_\eta^p = \mathbb{E} \sup_{t\in[s_n^-,s_n^+]} \left\| \int_{s_n^-}^t e^{(t-\tau)B}(I-B)Pg(\tau, J(v(\tau)))\,\mathrm{d}W(\tau)\right\|_\eta^p$$

using Burkhölder-Davis-Gundy inequality

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \|\Lambda(v,s_n^+) - \Lambda(v,s_n^-)\|_\eta^p &\le c\,\mathbb{E}\left| \int_{s_n^-}^{s_n^+} \|e^{(t-\tau)B}(I-B)Pg(\tau, J(v(\tau)))\|^2_{L_2(U,X_\eta)}\,\mathrm{d}\tau\right|^{p/2} \\
&\le c\,\mathbb{E}\left| \int_{s_n^-}^{s_n^+} \|e^{(t-\tau)B}(I-B)^{1+\eta-\theta}\|^2_{L(X)} \|g(\tau, J(v(\tau)))\|^2_{L_2(U,H)}\,\mathrm{d}\tau\right|^{p/2} \\
&\le c\,\mathbb{E}\left[\sup_{\sigma\in[s_n^-,s_n^+]} \|g(\sigma, J(v(\sigma)))\|^p_{L_2(U,H)}\right] \left(\int_{s_n^-}^{s_n^+} (t-\sigma)^{2(\theta-\eta-1)}\,\mathrm{d}\sigma\right)^{p/2} \\
&\le c\,(s_n^+ - s_n^-)^{2(\theta-\eta)-1}(1 + \|v\|_\eta^p).
\end{aligned}$$

Collecting all the above estimates, the continuity of the mapping $(s,x) \mapsto \mathcal{K}(X,s,x)$ is proved.

The last result in this section provides the existence, for every admissible control, of the solution for the stochastic problem

$$\begin{aligned} \mathrm{d}v(t) &= \big[Bv(t) + (I-B)Pg(t,J(v(t)))\, r(t,J(v(t)),\gamma(t))\big]\,\mathrm{d}t + (I-B)Pg(t,J(v(t)))\,\mathrm{d}W_t \\ v(s) &= x \end{aligned} \tag{3.5}$$

Theorem 3.3. *Let γ be an admissible control. Then there exists a unique mild solution v of equation (3.5) with $v \in L^2_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$.*

Proof. We shall use an approximation procedure in order to handle the growth bound of the function $r(t,J(v),\gamma)$ in γ. We introduce the sequence of stopping times

$$\tau_n = \inf\left\{ t \in [0,T] \mid \int_0^t |\gamma(s)|^2\,\mathrm{d}s > n \right\}$$

with the convention that $\tau_n = T$ if the indicated set is empty. Since γ is an admissible control, i.e., $\mathbb{E}\int_0^T |\gamma(s)|^2\,\mathrm{d}s < +\infty$, for $\mathbb{P}$-almost every $\omega \in \Omega$ there exists an integer $N(\omega)$ such that

$$n \geq N(\omega) \text{ implies } \tau_n(\omega) = T. \tag{3.6}$$

Let us fix $\gamma_0 \in \Xi$ and let us define, for every $n \in \mathbb{N}$,

$$\gamma_n(t) = \gamma(t)\,1_{t \leq \tau_n} + \gamma_0\,1_{t \geq \tau_n} \tag{3.7}$$

and we consider the family of equations

$$\begin{aligned} \mathrm{d}v_n(t) &= \big[Bv_n(t) + (I-B)Pg(t,J(v_n(t)))\, r(t,J(v_n(t)),\gamma_n(t))\big]\,\mathrm{d}t \\ &\qquad\qquad + (I-B)Pg(t,J(v_n(t)))\,\mathrm{d}W_t \\ v(s) &= x. \end{aligned} \tag{3.8}$$

For simplicity of notation we fix the initial time $s = 0$; the solution v_n of (3.8) is the fixed point of the mapping

$$v \mapsto \mathcal{K}_n(v,x) = e^{tB}x + \Lambda(v)(t) + \Gamma_n(v)(t),$$

where $\Gamma_n(v)$ is the process defined by

$$\Gamma_n(v)(t) = \int_0^t e^{(t-\tau)B}(I-B)Pg(\tau,J(v(\tau)))\, r(\tau,J(v(\tau)),\gamma_n(\tau))\,\mathrm{d}\tau.$$

We study the properties of the mapping Γ_n in the following lemma.

Lemma 3.4. *For every p the operator Γ_n maps $L^p_{\mathcal{F}}(\Omega; C([0,T], e^{-\nu t}; H_\eta))$ into itself and it is a contraction for every β large enough.*

Proof. As in the proof of Lemma 3.2, we consider only the case $s = 0$ and we write $\Gamma_n(t)$ for

$\Gamma_n(v)(t)$ for $v \in L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$. The following computation leads to the thesis:

$$\begin{aligned}
\sup_{t\in[0,T]} e^{-\beta p t} \|\Gamma_n(t)\|^p_\eta & \\
&\le \left| \sup_{t\in[0,T]} \int_0^t e^{-\beta\sigma} \|e^{-\beta(t-\sigma)} e^{(t-\sigma)B} (I-B)^{1+\eta} P g(\sigma, J(v(\sigma))) r(\sigma, J(v(\sigma)), \gamma_n)\| \,\mathrm{d}\sigma \right|^p \\
&\le \left(\int_0^\infty e^{-2\beta\sigma} \sigma^{-2(1+\eta-\theta)} \,\mathrm{d}\sigma \right)^{p/2} \mathbb{E} \left(\int_0^T \|e^{-\beta\sigma} g(\sigma, J(v(\sigma))) r(\sigma, J(v(\sigma)), \gamma_n(\sigma))\|^2 \,\mathrm{d}\sigma \right)^{p/2} \\
&\le C \left(\Gamma[2(\theta-\eta)-1] (2\beta)^{1-2(\theta-\eta)} \right)^{p/2} \mathbb{E} \left(\int_0^T e^{-\beta\sigma} |1 + |\gamma_n(\sigma))|^2 \,\mathrm{d}\sigma \right)^{p/2}
\end{aligned}$$

by the construction of τ_n it holds that the last quantity is bounded by $C(1+n)(2\beta)^{\frac{p}{2}(1-2(\theta-\eta))}$ hence $\Gamma_n(v)$ is well defined.

With the same computation we obtain that

$$|||\Gamma_n(v) - \Gamma_n(v')|||^p_\eta \le C(1+n)(2\beta)^{\frac{p}{2}(1-2(\theta-\eta))} |||v - v'|||^p_\eta \tag{3.9}$$

which implies that Γ_n is a contraction with norm decreasing with β.

Putting together estimates (3.4) and (3.9) we obtain that the mapping $v \mapsto \mathcal{K}_n(v,x)$ is a contraction on the space $L^p_{\mathcal{F}}(\Omega; C([0,T]; X_\eta))$ and for every $n \in \mathbb{N}$ there exists a unique solution v_n of the approximate problem (3.8).

Notice that $v_n(t) = v_{n+1}(t)$ coincide on the time interval $[0,\tau_n]$ and $\tau_n \nearrow T$ almost surely as $n \to \infty$. Hence we can define a process $v(t)$ as

$$v(t) = v_n(t) \text{ on } [0,\tau_n]$$

and clearly v is the required solution of (3.5).

It remains to prove that $v \in L^2_{\mathcal{F}}(\Omega; C([0,T]; X_\eta)$. The claim follows if we prove that

$$\sup_{n\in\mathbb{N}} \mathbb{E} \sup_{t\le T} \|v_n(t)\|^2_\eta < \infty.$$

From the mild form of equation (3.8) we obtain

$$\begin{aligned}
v_n(t) = e^{tB}x + \int_0^t e^{(t-\sigma)B}(I-B) P g(\sigma, Jv_n(\sigma)) r(\sigma, Jv_n(\sigma), \gamma_n(\sigma)) \,\mathrm{d}\sigma & \\
+ \int_0^t e^{(t-\sigma)B}(I-B) P g(\sigma, Jv_n(\sigma)) \,\mathrm{d}W(\sigma).&
\end{aligned}$$

We first notice that the first integral term on the right hand side can only be estimated in $L^2(\Omega)$-norm:

$$\begin{aligned}
\mathbb{E} \sup_{t\le T} \left\| \int_0^t e^{(t-\sigma)B}(I-B) P g(\sigma, Jv_n(\sigma)) r(\sigma, Jv_n(\sigma), \gamma_n(\sigma)) \,\mathrm{d}\sigma \right\|^2_\eta & \\
\le \mathbb{E} \sup_{t\le T} \left| \int_0^t (t-\sigma)^{-(1+\eta-\theta)} (1 + |\gamma_n(\sigma)|) \,\mathrm{d}\sigma \right|^2 \le C T^{2(\theta-\eta)-1} \left(\int_0^T (1 + \mathbb{E}|\gamma(s)|^2) \,\mathrm{d}s \right) \le C &
\end{aligned}$$

thanks to the assumptions on the admissible control γ.

As far as the stochastic term is concerned, we can give the estimate also in the $L^{2p}(\Omega)$-norm, for $p \geq 1$; although we only need $p = 1$ here, the general case will be useful later. It follows, from an application of Burkholder-Davis-Gundy inequality, that

$$\mathbb{E} \sup_{t \leq T} \left\| \int_0^t e^{(t-\sigma)B}(I - B)Pg(\sigma, Jv_n(\sigma))\,\mathrm{d}W(\sigma) \right\|_\eta^{2p}$$
$$\leq C \left(\int_0^T (t-\sigma)^{-2(1+\eta-\theta)} \|g(\sigma, Jv_n(\sigma))\|^2_{L_2(U,H)}\,\mathrm{d}\sigma \right)^{p/2} \leq C \quad (3.10)$$

which is bounded, independently of n, using the bound of g in Hypothesis 1.1. We have thus proven the thesis.

Corollary 3.5. *The family of random variables* $\sup_{t \leq T} |v_n(t)|^2$ *is uniformly integrable.*

Proof. Proceeding as above, we have that

$$\sup_{t \leq T} \|v_n(t)\|_\eta^2 \leq C \left(1 + \int_0^T |\gamma(s)|^2\,\mathrm{d}s + \sup_{t \leq T} \|\Lambda(v_n, t)\|_\eta^2 \right)$$

and we notice that the first two terms are integrable, the last is a uniformly integrable family of random variables, since we have proven in (3.10) that it is uniformly bounded in $L^{2p}(\Omega)$-norm for some $p > 1$. Therefore, also $\sup_{t \leq T} \|v_n(t)\|_\eta^2$ is uniformly integrable and the claim is proven.

4. The solution of the controlled stochastic Volterra equation

In the next section we show that there exists a unique solution of the original equation (1.1). The proof follows the line of Theorem 4.4 in [4]; however, under the assumptions in Hypothesis 1.1, it requires to pass through the approximation sequence $\{v^n\}$.

Theorem 4.1. *For every* $n \in \mathbb{N}$*, the process*

$$u_n(t) = \begin{cases} u_0(t), & t \leq 0 \\ Jv^n(t), & t \in [0, T] \end{cases} \quad (4.1)$$

where v^n *is the solution of problem (3.8) given in Theorem 3.3, is the unique solution of the equation*

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s &= Au(t) + g(t, u(t)) \left[r(t, u(t), \gamma_n(t)) + \dot{W}(t) \right], \qquad t \in [0, T] \\ u(t) &= u_0(t), \qquad t \leq 0. \end{aligned} \quad (4.2)$$

where γ_n *is the control process defined in (3.7).*

Proof. We propose to fulfill the following steps.

Step I The linear equation

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s &= Au(t), \qquad t \in [0, T] \\ u(t) &= 0, \qquad t \leq 0. \end{aligned} \quad (4.3)$$

has a unique solution $u \equiv 0$.

Step II The affine equation

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{t} a(t-s) u(s) \,\mathrm{d}s = Au(t) + g(t, \tilde{u}(t)) \left[r(t, \gamma_n(t), \tilde{u}(t)) + \dot{W}(t) \right], \qquad t \in [0,T] \\ u(t) = u_0(t), \qquad t \le 0. \end{aligned} \tag{4.4}$$

defines a contraction mapping $\mathcal{Q} : \tilde{u} \mapsto u$ on the space $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$. Therefore, equation (4.2) admits a unique solution.

Step III The process u_n defined in (4.1) satisfies equation (4.2). Accordingly, by the uniqueness of the solution, the thesis of the theorem follows.

4.1. Step I. The linear equation

Let us take the Laplace transform in both sides of the linear equation (4.3); we obtain

$$\lambda \hat{a}(\lambda) \hat{u}(\lambda) = A \hat{u}(\lambda), \qquad \Re\lambda \ge 0, \ \lambda \ne 0;$$

therefore, $\hat{u}(\lambda) = R(\lambda \hat{a}(\lambda), A) 0$.

Let $\lambda = x + iy$; recall that, by Bernstein's theorem, ν is the unique measure associated with the kernel a. Using [17, Lemma 1.1.7] we have

$$\lambda \hat{a}(\lambda) = \int_{[0,\infty)} \frac{(x+iy)\kappa + (x^2+y^2)}{(\kappa+x)^2 + y^2} \, \nu(\mathrm{d}\kappa)$$

hence $\Re(\lambda \hat{a}(\lambda)) \ge 0$ for all $\Re\lambda \ge 0$, $\lambda \ne 0$, which means that $\lambda \hat{a}(\lambda) \in \rho(A)$ and

$$\hat{u}(\lambda) = 0, \qquad \Re\lambda \ge 0, \ \lambda \ne 0.$$

The *complex inversion formula* for the Laplace transform therefore leads to

$$u(t) = 0, \qquad \text{for a.a. } t \ge 0$$

as claimed.

4.2. Step II. Stochastic Volterra equation with non-homogeneous terms

In this section, we consider problem (1.1) with coefficients $g(t)$ and $f(t) = g(t) r(t, \gamma_n(t))$ independent of u. The case $f(t) \equiv 0$ is treated in [3, Theorem 3.7]; we recall here the proof and extend it to the general case.

Theorem 4.2. *In our assumptions, let $v_0 \in X_\eta$ for some $\frac{1-\alpha(a)}{2} < \eta < \frac{1}{2}\alpha(a)$. Given the process*

$$v(t) = e^{tB} v_0 + \int_0^t e^{(t-s)B} (I-B) P f(s) \,\mathrm{d}s + \int_0^t e^{(t-s)B} (I-B) P g(s) \,\mathrm{d}W(s) \tag{4.5}$$

we define the process

$$u(t) = \begin{cases} Jv(t), & t \ge 0, \\ u_0(t), & t \le 0. \end{cases} \tag{4.6}$$

Then $u(t)$ is a weak solution to problem (1.1).

Proof. Again, in our assumption we have by [3, Lemma 3.11] that the operator J_0 can be extended to a bounded operator $J : X_\eta \to H$. We define B_η to be the restriction of B as an operator $B_\eta : X_{\eta+1} \to X_\eta$. As usual, B^* is the adjoint of B.

For fixed $\zeta \in D(A^*)$, we define the vector $\xi \in X$ by $\xi(\kappa) = \frac{1}{1+\kappa}\zeta$. We claim that $\xi \in D(B_\eta^*)$. Actually, for $x \in X_{\eta+1}$ we have

$$\begin{aligned}\langle \xi, B_\eta x\rangle_X &= \int_{[0,\infty)} (\kappa+1) \left\langle \frac{1}{1+\kappa}\zeta, (-\kappa x(\kappa) + J_0(x)) \right\rangle_H \nu(\mathrm{d}\kappa) \\ &= \left\langle \zeta, \int_{[0,\infty)} (-\kappa x(\kappa) + J_0(x))\, \nu(\mathrm{d}\kappa) \right\rangle_H = \langle \zeta, A J_0(x)\rangle_H = \langle A^*\zeta, J_0(x)\rangle. \end{aligned} \tag{4.7}$$

Moreover, by [3, Lemma 3.10]

$$\Psi(t) = (I - B)Pg(t) \in L^2_{\mathcal{F}}(\Omega \times (0,T); L_2(U, X_{\theta-1})) \quad \text{for any } \theta < \frac{\alpha(a)+1}{2}. \tag{4.8}$$

We have that the process v defined in (4.5) is a weak solution of problem (2.9), hence

$$\langle v(t), \xi\rangle_X = \langle v_0, \xi\rangle_X + \int_0^t \langle B_\eta^*\xi, v(s)\rangle\,\mathrm{d}s + \int_0^t \langle \xi, (I-B)Pf(s)\rangle_X\,\mathrm{d}s + \int_0^t \langle \xi, \Psi(s)\,\mathrm{d}W(s)\rangle_X. \tag{4.9}$$

We use the above representation in order to prove that u is a weak solution of (1.1).

Let us consider separately the several terms. The initial condition yields

$$\langle v_0, \xi\rangle_X = \int_{[0,\infty)} \langle v(0,\kappa), \zeta\rangle_H\, \nu(\mathrm{d}\kappa) = \langle \int_{[0,\infty)} \int_{[0,\infty)} e^{-\kappa s} u_0(-s)\,\mathrm{d}s\, \nu(\mathrm{d}\kappa), \zeta\rangle_H = \langle \bar{u}, \zeta\rangle_H$$

where $\bar{u}$ is defined in Definition 2.1; next, the second integral can be evaluated using equality (4.7)

$$\int_0^t \langle B_\eta^*\xi, v(s)\rangle_X\,\mathrm{d}s = \int_0^t \langle A^*\zeta, Jv(s)\rangle_H\,\mathrm{d}s = \int_0^t \langle A^*\zeta, u(s)\rangle_H\,\mathrm{d}s.$$

The third integral leads to

$$\int_0^t \langle \xi, (I-B)Pf(s)\rangle_X\,\mathrm{d}s = \int_0^t \langle \xi, Pf(s)\rangle_X\,\mathrm{d}s - \int_0^t \langle B^*\xi, Pf(s)\rangle_X\,\mathrm{d}s \tag{4.10}$$

and we consider the two terms separately:

$$\langle \xi, Pf(s)\rangle_X = \int_{[0,\infty)} (\kappa+1) \left\langle \frac{1}{\kappa+1}\zeta, \frac{1}{\kappa+1} R(\hat{a}(1), A) f(s) \right\rangle_H \nu(\mathrm{d}\kappa) = \langle \zeta, \hat{a}(1) R(\hat{a}(1), A) f(s)\rangle_H$$

and

$$\langle B^*\xi, Pf(s)\rangle_X = \langle \zeta, A J_0 P f(s)\rangle_H = \langle \zeta, A R(\hat{a}(1), A) f(s)\rangle_H$$

hence

$$\begin{aligned}&\int_0^t \langle \xi, (I-B)Pf(s)\rangle_X\,\mathrm{d}s \\ &\quad = \int_0^t \langle \zeta, \hat{a}(1) R(\hat{a}(1), A) f(s)\rangle_H\,\mathrm{d}s - \int_0^t \langle \zeta, A R(\hat{a}(1), A) f(s)\rangle_H\,\mathrm{d}s = \int_0^t \langle \zeta, f(s)\rangle_H\,\mathrm{d}s.\end{aligned}$$

We finally turn to the stochastic integral. For an orthonormal basis $\{e_j\}$ in U we let $\beta_j(t) = \langle W(t), e_j\rangle_U$ and $\Psi_j(t) = (I-B)Pg(s)\cdot e_j$; then

$$\begin{aligned}\int_0^t \langle \xi, \Psi(s)\,\mathrm{d}W(s)\rangle_X &= \sum_{j=1}^\infty \int_0^t \langle \xi, \Psi_j(s)\rangle_X \,\mathrm{d}\beta_j(s)\\ &= \sum_{j=1}^\infty \int_0^t \langle \xi, (I-B)Pg(s)\cdot e_j\rangle_X \,\mathrm{d}\beta_j(s)\end{aligned}$$

This quantity can be treated as we have done for (4.10) and we obtain

$$\begin{aligned}\int_0^t \langle \xi, \Psi(s)\,\mathrm{d}W(s)\rangle_X &= \sum_{j=1}^\infty \int_0^t \langle \zeta, (\hat{a}(1)-A)R(\hat{a}(1),A)g(s)\cdot e_j\rangle_H \,\mathrm{d}\beta_j(s)\\ &= \sum_{j=1}^\infty \int_0^t \langle \zeta, g(s)\cdot e_j\rangle_H \,\mathrm{d}\beta_j(s) = \int_0^t \langle \zeta, g(s)\,\mathrm{d}W(s)\rangle_H.\end{aligned}$$

We have proved so far that

$$\langle v(t), \xi\rangle_X = \langle \bar{u}, \zeta\rangle_H + \int_0^t \langle A^*\zeta, u(s)\rangle_H \,\mathrm{d}s + \int_0^t \langle \zeta, f(s)\rangle_H \,\mathrm{d}s + \int_0^t \langle \zeta, g(s)\,\mathrm{d}W(s)\rangle_H. \tag{4.11}$$

It only remains to prove

$$\int_{[0,\infty)} \langle v(t,\kappa), \zeta\rangle_H \,\nu(\mathrm{d}\kappa) = \int_{-\infty}^t \langle a(t-s)u(s), \zeta\rangle_H \,\mathrm{d}s.$$

If we recall the definition of $u(t) = \begin{cases} Jv(t), & t>0\\ u_0(t), & t\le 0,\end{cases}$ we obtain

$$\int_{-\infty}^t \langle a(t-s)u(s), \zeta\rangle_H \,\mathrm{d}s = \int_{-\infty}^0 \langle a(t-s)u_0(s), \zeta\rangle_H \,\mathrm{d}s + \int_0^t \langle a(t-s)Jv(s), \zeta\rangle_H \,\mathrm{d}s.$$

We then exploit the definition of $a(t)$; the first term becomes

$$\begin{aligned}\int_{-\infty}^0 \langle a(t-s)u_0(s), \zeta\rangle_H \,\mathrm{d}s &= \langle \int_{-\infty}^0 \int_{[0,\infty)} e^{-\kappa(t-s)}\nu(\mathrm{d}\kappa)u_0(s)\,\mathrm{d}s, \zeta\rangle_H\\ &= \langle \int_{[0,\infty)} e^{-\kappa t}\int_{-\infty}^0 e^{-\kappa(-s)}u_0(s)\,\mathrm{d}s\nu(\mathrm{d}\kappa), \zeta\rangle_H = \langle \int_{[0,\infty)} e^{-\kappa t}v(0,\kappa)\nu(\mathrm{d}\kappa), \zeta\rangle_H,\end{aligned}$$

the second term becomes

$$\begin{aligned}\int_0^t \langle a(t-s)Jv(s), \zeta\rangle_H \,\mathrm{d}s &= \langle \int_0^t \int_{[0,\infty)} e^{-\kappa(t-s)}\,\nu(\mathrm{d}\kappa)\,Jv(s)\,\mathrm{d}s, \zeta\rangle_H\\ &= \langle \int_{[0,\infty)} \int_0^t e^{-\kappa(t-s)}Jv(s)\,\mathrm{d}s\,\nu(\mathrm{d}\kappa), \zeta\rangle_H,\end{aligned}$$

and the thesis follows from the identity of the processes

$$\begin{aligned} v(t,\kappa) &= e^{tB} v_0(\kappa) + \int_0^t [e^{(t-s)B}(I-B)Pf(s)](\kappa)\,\mathrm{d}s \\ &\qquad + \int_0^t [e^{(t-s)B}(I-B)Pg(s)](\kappa)\,\mathrm{d}W(s) \\ \tilde v(t,\kappa) &= e^{-\kappa t} v_0(\kappa) + \int_0^t e^{-\kappa(t-s)} Jv(s)\,\mathrm{d}s. \end{aligned} \tag{4.12}$$

proved in next lemma.

Lemma 4.3. *In our assumptions, let $v_0 \in X_\eta$ for some $\frac{1-\alpha(a)}{2} < \eta < \frac{1}{2}\alpha(a)$. Consider the processes v and $\tilde v$ defined in* (4.12)*. Then $\tilde v(t)$ is a modification of $v(t)$.*

Proof. The proof follows by a Laplace transform argument which adapts to our case the ideas in [3, Proposition 3.8].

We shall denote $\mathcal{L}[f](s) = \displaystyle\int_{[0,\infty)} e^{-st} f(t)\,\mathrm{d}t$ and similarly $\mathcal{L}[g\dot W](s) = \displaystyle\int_{[0,\infty)} e^{-st} g(t)\,\mathrm{d}W(t)$; if we apply the Laplace transform in first line of (4.12) we get

$$\mathcal{L}[v(\cdot,\kappa)](s) = R(s,B)v_0(\kappa) + [R(s,B)(I-B)P\mathcal{L}[f](s)](\kappa) + [R(s,B)(I-B)P\mathcal{L}[g\dot W](s)](\kappa)$$

Now we use the representation formulas stated in [3, (2.14) and (2.17)] to get

$$\begin{aligned} \mathcal{L}[v(\cdot,\kappa)](s) &= \frac{1}{\kappa+s} v_0(\kappa) + R(s\hat a(s), A) \int_{[0,\infty)} \frac{\bar\kappa}{\bar\kappa + s} v_0(\bar\kappa)\,\nu(\mathrm{d}\bar\kappa) \\ &\qquad + \frac{1}{\kappa+s} R(s\hat a(s), A)\mathcal{L}[f](s) + \frac{1}{\kappa+s} R(s\hat a(s), A)\mathcal{L}[g\dot W](s). \end{aligned} \tag{4.13}$$

Now we turn to the second process in (4.12); we obtain that the Laplace transform is

$$\mathcal{L}[\tilde v](s) = \frac{1}{\kappa+s} v_0(\kappa) + \frac{1}{\kappa+s}\mathcal{L}[Jv](s)$$

and a direct computation shows that the above quantity is equal to (4.13).

Now we proceed to define the mapping

$$\mathcal{Q} : L^p(\Omega; C([0,T];H)) \to L^p(\Omega; C([0,T];H))$$

where $\mathcal{Q}(\tilde u) = u$ is the solution of the problem

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s &= Au(t) + g(t,\tilde u(t))\left[r(t,\tilde u(t),\gamma_n(t)) + \dot W(t)\right], \qquad t \in [0,T] \\ u(t) &= u_0(t), \qquad t \le 0. \end{aligned} \tag{4.14}$$

Theorem 4.4. *Let $\beta > 0$ be a parameter to be chosen later. Let*

$$|||u|||_H^p = \mathbb{E} \sup_{t\in[0,T]} e^{-\beta p t}|u(t)|^p$$

be a norm on $L^p(\Omega; C([0,T]; H))$; notice that this norm is equivalent to the natural one. Then there exists $\delta < 1$ such that

$$\lvert\lvert\lvert u_1 - u_2 \rvert\rvert\rvert_H = \lvert\lvert\lvert \mathcal{Q}(\tilde u_1) - \mathcal{Q}(\tilde u_2) \rvert\rvert\rvert_H \le \delta \lvert\lvert\lvert \tilde u_1 - \tilde u_2 \rvert\rvert\rvert_H$$

for every $\tilde u_1, \tilde u_2 \in L^p(\Omega; C([0,T]; H))$.

Proof. It follows from the uniqueness of the solution, proved in Step I, that the solution $u_i(t)$ $(i = 1, 2)$ has the representation

$$u_i(t) = \begin{cases} Jv_i(t), & t \in [0,T] \\ u_0(t), & t \le 0 \end{cases}$$

where

$$v_i(t) = e^{tB} v_0 + \int_0^t e^{(t-s)B}(I - B)Pg(s, \tilde u_i(s)) r(s, \tilde u_i(s), \gamma_n(s))\, \mathrm{d}s \\ + \int_0^t e^{(t-s)B}(I - B)Pg(s, \tilde u_i(s))\, \mathrm{d}W(s).$$

In particular,

$$U(t) = u_1(t) - u_2(t) = \begin{cases} J(v_1(t) - v_2(t)), & t \in [0,T] \\ 0, & t \le 0; \end{cases}$$

then

$$\mathbb{E} \sup_{t \in [0,T]} e^{-\beta p t} |U(t)|^p \le \|J\|^p_{L(X_\eta, H)} \mathbb{E} \sup_{t \in [0,T]} e^{-\beta p t} \|v_1(t) - v_2(t)\|^p_\eta$$

the quantity on the right hand side can be treated as in Theorem 3.3 and the claim follows.

4.3. Step III. The solution of the stochastic Volterra equation

It follows from Theorem 4.4 that there exists a unique solution u of problem (4.2); in order to prove Theorem 4.1 it only remains to prove the representation formula (4.1).

Let $\tilde f(s) = g(s, Jv^n(s)) r(s, Jv^n(s), \gamma_n(s))$ and $\tilde g(s) = g(s, Jv^n(s))$; it is a consequence of Theorem 4.2 that u, defined in (4.1), is a weak solution of the problem

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^t a(t-s) u(s)\, \mathrm{d}s = Au(t) + \tilde f(t) + \tilde g(t) \dot W(t), \qquad t \in [0,T] \\ u(t) = u_0(t), \qquad t \le 0, \end{aligned} \tag{4.15}$$

and the definition of $\tilde f$ and $\tilde g$ implies that u is a weak solution of

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^t a(t-s) u(s)\, \mathrm{d}s = Au(t) + g(s, Jv^n(s)) \left[r(s, Jv^n(s), \gamma_n(s)) + \dot W(t) \right], \qquad t \in [0,T] \\ u(t) = u_0(t), \qquad t \le 0, \end{aligned} \tag{4.16}$$

that is problem (4.2).

4.4. The stochastic Volterra equation (1.1)

The proof of the main result concerning existence of the solution for the controlled Volterra equation (1.1) relies again on the approximation procedure introduced in Section 3, see Theorem 3.3.

Theorem 4.1 states the existence of a family of processes $\{u_n(t),\ t \in [0,T]\}_{n\in\mathbb{N}}$, such that $u_n(t) = u_{n+1}(t)$ on the time interval $[0,\tau_n]$ and $\tau_n \nearrow T$ almost surely as $n \to \infty$. Hence we can define a process $u(t)$ as

$$u(t) = u_n(t) \text{ on } [0,\tau_n]$$

and clearly u is the required solution of (1.1). Further, by the uniqueness of the solution, it follows that $u(t) = Jv(t)$ for $t \in [0,T]$, where v is the process constructed in Theorem 3.3.

It remains to verify that $u \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H)$, which follows from the representation $u(t) = Jv(t)$ for $t \in [0,T]$ and the claim $v \in L^2_{\mathcal{F}}(\Omega; C([0,T]; X_\eta)$ proved in Theorem 3.3.

5. The optimal control problem

We define in a classical way the Hamiltonian function relative to the above problem: for all $t \in [0,T]$, $v \in X_\eta$, $z \in U^*$,

$$\psi(t,v,z) = \inf\{l(t,J(v),\gamma) + zr(t,J(v),\gamma)\ :\ \gamma \in \Xi\} \tag{5.1}$$

and the set of minimizers in (5.1):

$$\Gamma(t,v,z) = \{\gamma \in \Xi\ :\ l(t,J(v),\gamma) + zr(t,J(v),\gamma) = \psi(t,v,z)\}\,. \tag{5.2}$$

The map ψ ia a Borel measurable function from $[0,T]\times X_\eta \times U^*$ to $\mathbb{R}$. In fact, by the continuity of r and l with respect to γ, we have

$$\psi(t,v,z) = \inf_{\gamma\in K}[l(t,J(v),\gamma) + zr(t,J(v),\gamma)], \qquad \text{for } t \in [0,T],\ v \in X_\eta,\ z \in U^*,$$

where K is any countable dense subset of Ξ.

Moreover, by a direct computation using the assumptions on l and r (see also [13, Lemma 3.1]) we can show that there exists a constant $C > 0$ such that

$$-C(1+|z|^2) \le \psi(t,v,z) \le l(t,J(v),\gamma) + C|z|(1+|\gamma|) \qquad \forall \gamma \in \Xi. \tag{5.3}$$

We require moreover

Hypothesis 5.1. *$\Gamma(t,v,z)$ is non empty for all $t \in [0,T]$, $v \in X_\eta$ and $z \in U^*$.*

We can prove that, if Hypothesis 5.1 holds, then

$$\psi(t,v,z) = \min_{\gamma\in\Xi, |\gamma|\le C(1+|v|+|z|)}[l(t,J(v),\gamma) + zr(t,J(v),\gamma)] \quad t \in [0,T],\ v \in X_\eta,\ z \in U^*, \tag{5.4}$$

that is the infimum in (5.1) is attained in a ball of radius $C(1+|v|+|z|)$, and

$$\psi(t,v,z) < l(t,J(v),\gamma) + zr(t,J(v),\gamma) \quad \text{if } |\gamma| > C(1+|v|+|z|). \tag{5.5}$$

Moreover from (5.4) it follows that for every $t \in [0,T]$ and $v \in X_\eta$, the map $z \to \psi(t,v,z)$ is continuous on U^* .

Before giving the proof of Proposition 5.3, we state the following proposition which gives the main argument of the existence of the solution of (1.12).

Proposition 5.2. *Let τ be a stopping time and let (ψ, τ, ξ) be a set of parameters and let $(\psi_n, \tau, \xi_n)_n$ be a sequence of parameters such that:*

(i) *There exists $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $T > 0$, $k \in L^1[0,T]$ and there exists $C > 0$ such that*

$$\forall n \in \mathbb{N},\ \forall (t, v, z) \in \mathbb{R}^+ \times X_\eta \times U^*, \quad |\psi_n(t, v, z)| \le k_t + C|z|^2. \tag{5.6}$$

(ii) *For each n, the BSDE with parameters (ψ_n, τ, ξ_n) has a solution (Y^n, Z^n) such that Y^n is almost surely bounded, for almost every t, and $\mathbb{E} \int_0^T |Z_t^n|^2 \, \mathrm{d}t < \infty$; the sequence $(Y^n)_n$ is monotonic, and there exists $M > 0$ such that for all $n \in \mathbb{N}$, $||Y^n||_\infty \le M$;*

(iii) *for almost all $\omega \in \Omega$ and $t \in [0, \tau]$*

$$\lim_{n\to\infty} \psi_n(t, Y_t^n, Z_t^n) = \psi(t, Y_t, Z_t); \tag{5.7}$$

$\xi_n \in L^\infty(\Omega)$ and ξ_n converges to ξ in $L^\infty(\Omega)$.

(iv) *The stopping time τ is such that $\tau < \infty$ $\mathbb{P}$-a.s.*

Then there exists a pair of processes (Y, Z) such that Y is almost surely bounded, for almost every t, while

$$\mathbb{E} \int_0^T |Z_t|^2 \, \mathrm{d}t < \infty$$

and for all $T \in \mathbb{R}^+$

$$\lim_{n\to\infty} Y^n = Y \ \textit{ uniformly on } [0,T]$$
$$(Z^n)_n \textit{ converges to } Z \textit{ in } L^2(\Omega \times [0,\tau]; U^*)$$

and (Y, Z) is a solution of the BSDE with parameters (ψ, τ, ξ). In particular, if for each n, Y^n has continuous paths, also the process Y has continuous paths.

Remark 5.1. We note that in previous proposition we have essentially the same assumptions of [18, Proposition 2.4 (Monotone stability)], although we require condition (5.7) instead of locally uniform convergence of $(\psi_n)_n$ to ψ. The proof is similar. For the sake of completeness, we shall sketch it in the appendix.

Proposition 5.3. *Assume that g, l, r, ϕ satisfy Hypothesis 1.1. Then there exist Borel measurable functions θ and ζ with values in $\mathbb{R}$ and U^*, respectively,*

$$\theta : [0,T] \times X_\eta \to \mathbb{R} \ \textit{and}\ \zeta : [0,T] \times X_\eta \to U^*$$

with the following property: for an arbitrarily chosen complete probability space $(\Omega, \mathcal{F}, \mathbb{P}^\circ)$ and Wiener process W° in U, denoting by v the solution of (1.11), the processes Y, Z defined by

$$Y_t = \theta(t, v(t)), \quad Z_t = \zeta(t, v(t))$$

satisfy

$$\mathbb{E}^\circ \sup_{t\in[0,T]} |Y_t|^2 < \infty, \quad \mathbb{E}^\circ \int_0^T |Z_t|^2 \, \mathrm{d}t < \infty;$$

moreover, Y is continuous and nonnegative, and $\mathbb{P}^\circ$-a.s.,

$$Y_t + \int_t^T Z_s \,\mathrm{d}W_s^\circ = \phi(v(t)) + \int_t^T \psi(s, v(s), Z_s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{5.8}$$

Finally, this solution is the maximal solution among all the solutions (Y', Z') of (1.12) satisfying

$$\mathbb{E}^\circ \sup_{t\in[0,T]} |Y_t^{'}|^2 < \infty.$$

Proof. We proceed as in [13, Proposition 3.2]. We adopt the same strategy as that in [5] to construct a maximal solution to

$$\begin{aligned} dY_t &= \psi(t, v(t), Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t^\circ, \quad t \in [0,T], \\ Y_T &= \phi(J(v(t))). \end{aligned} \tag{5.9}$$

v is the solution of the forward equation

$$\begin{aligned} \mathrm{d}v(t) &= Bv(t)\,\mathrm{d}t + (I - B)Pg(t, J(v(t)))\,\mathrm{d}W_t^\circ \\ v(0) &= v_0 \end{aligned} \tag{5.10}$$

whose solution is a continuous $(\mathcal{F}_t^\circ)$-adapted process, which exists and is unique by the results in Section 3. First we note that

$$\mathbb{E}^\circ \sup_{t\in[0,T]} ||v(t)||_\eta^2 < \infty \quad \forall p \geq 2.$$

Moreover, from the (5.3) there exists a constant $C > 0$ such that

$$-C(1 + |z|^2) \leq \psi(t, v, z) \leq l(t, J(v), \gamma_0) + C(1 + |\gamma_0|)|z|. \tag{5.11}$$

For each $n \geq C$, we define the globally Lipschitz continuous function,

$$\psi_n(t, v, z) = \sup\{\psi(t, v, q) - n|q - z| : q \in U^* \cap \mathcal{H}\},$$

where $\mathcal{H}$ is a numerable subset dense in U^*. ψ_n is decreasing and converges to ψ; then by (Y^n, Z^n) we denote the unique solution to the BSDE with Lipschitz coefficient ψ_n,

$$\begin{aligned} dY_t^n &= -\psi_n(t, v(t), Z_t^n)\,\mathrm{d}t + Z_t^n\,\mathrm{d}W_t^\circ, \quad t \in [0,T], \\ Y_T^n &= \phi(J(v(t))), \end{aligned}$$

and by (Y^S, Z^S) the unique solution to the BSDE,

$$\begin{aligned} dY_t^S &= -[l(t, J(v(t)), \gamma_0) + C(1 + |\gamma_0|)|Z_t^S|]\,\mathrm{d}t + Z_t^S\,\mathrm{d}W_t^\circ, \quad t \in [0,T], \\ Y_T^S &= \phi(J(v(t))), \end{aligned}$$

where C is the same as in (5.3). We notice that, since $\psi_n(t, v, 0) \geq \psi(t, v, 0) \geq 0$ and by an application of the comparison theorem (see [4]), it holds $0 \leq Y_t^n \leq Y_t^S$. Let us introduce the following stopping times: for $k \geq 1$,

$$\tau_k = \inf\{t \in [0,T] : \max(|v(t)|, Y_t^S) > k\},$$

with the convention that $\tau_k = T$ if the indicated set is empty.

Then $(Y^n_k, Z^n_k) := (Y^n_{t\wedge\tau_k}, Z^n_{t\wedge\tau_k})$ satisfies the following BSDE:

$$Y^n_k(t) = \xi^n_k + \int_t^T 1_{s\le\tau_k}\psi_n(s, v(s), Z^n_k(s))\,\mathrm{d}s - \int_t^T Z^n_k(s)\,\mathrm{d}W^\circ_s,$$

where of course $\xi^n_k = Y^n_k(T) = Y^n_{\tau_k}$.

Now we fix k and prove, using Proposition 5.2, that there exists a process (Y_k, Z_k) such that Y_k is a continuous process, $E\int_0^T |Z_k(s)|^2\,\mathrm{d}s < \infty$,

$$\lim_{n\to\infty} \sup_{[0,T]} |Y^n_k(t) - Y_k(t)| = 0 \qquad \lim_{n\to\infty} \mathbb{E}^\circ \int_0^T |Z^n_k(t) - Z_k(t)|^2\,\mathrm{d}t = 0$$

and (Y_k, Z_k) solves the BSDE

$$Y_k(t) = \xi_k + \int_t^T 1_{s\le\tau_k}\psi(s, v(s), Z_k(s))\,\mathrm{d}s - \int_t^T Z_k(s)\,\mathrm{d}W^\circ_s, \tag{5.12}$$

where $\xi_k = \inf_n Y^n_{\tau_k}$.

We note that, for fixed k, Y^n_k is decreasing in n and remains bounded by k. Moreover, as in the proof of Proposition 5.2 (see Appendix), we can show that Z^n_k converges in $L^2(\Omega\times[0,T])$ to a process which we denote Z_k. In order to apply Proposition 5.2, we have only to check that for almost all $\omega\in\Omega$ and $t\in[0,\tau_k]$,

$$\lim_{n\to\infty} \psi_n(t, v(t), Z^n_k(t)) = \psi(t, v(t), Z_k(t)).$$

For all n we set $\bar Z^n_k(t) = \operatorname{argsup}\{\psi(t, v(t), q) - n|q - Z^n_k(t)|\}$, so that

$$\psi_n(t, v(t), Z^n_k(t)) = \psi(t, v(t), \bar Z^n_k(t)) - n|\bar Z^n_k(t) - Z^n_k(t)|$$

and

$$n|\bar Z^n_k(t) - Z^n_k(t)| = \psi(t, v(t), \bar Z^n_k(t)) - \psi_n(t, v(t), Z^n_k(t)) \ge 0. \tag{5.13}$$

It follows, by (5.11) and by the fact that $\psi_n(t, v, z) \ge \psi(t, v, z)$, that

$$\begin{aligned} n|\bar Z^n_k(t) - Z^n_k(t)| &= \psi(t, v(t), \bar Z^n_k(t)) - \psi_n(t, v(t), Z^n_k(t)) \\ &\le l(t, J(v(t)), \gamma_0) + C(1+|\gamma_0|)|\bar Z^n_k(t)| + C(1+|Z^n_k(t)|^2) \\ &\le C(1+|v(t)|^2+|\gamma_0|^2) + C(1+|\gamma_0|)|\bar Z^n_k(t) - Z^n_k(t)| \\ &\quad + C(1+|\gamma_0|)|Z^n_k(t)| + C(1+|Z^n_k(t)|^2) \end{aligned} \tag{5.14}$$

hence for $n > C(1+|\gamma_0|)$:

$$\begin{aligned} (n - C(1+|\gamma_0|))|\bar Z^n_k(t) - Z^n_k(t)| \le C(1+|v(t)|^2+|\gamma_0|^2) + C(1+|\gamma_0|)|Z^n_k(t)| \\ + C(1+|Z^n_k(t)|^2). \end{aligned} \tag{5.15}$$

We state that

$$\forall\epsilon>0\,\exists\, n_0 \text{ s.t. } \forall n\ge n_0 \quad |\bar Z^n_k(t) - Z^n_k(t)| < \epsilon, \quad \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{-a.s.} \tag{5.16}$$

If it were not true there shall exist $\epsilon > 0$ such that $\forall n_0$ we can find $n \geq n_0$ with

$$|\bar{Z}_k^n(t) - Z_k^n(t)| \geq \epsilon \quad \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{-a.s.}$$

hence we can construct a sequence $\{n_j\}_{j\in\mathbb{N}}$ increasing to infinity and such that

$$|\bar{Z}_k^{n_j}(t) - Z_k^{n_j}(t)| \geq \epsilon \quad \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{-a.s.}$$

but then by (5.15) we get

$$\begin{aligned} 0 < \epsilon &\leq |\bar{Z}_k^{n_j}(t) - Z_k^{n_j}(t)| \\ &\leq \frac{1}{(n_j - C(1+|\gamma_0|))} C(1+|v(t)|^2 + |\gamma_0|^2) + C(1+|\gamma_0|)|Z_k^{n_j}(t)| + C(1+|Z_k^{n_j}(t)|^2). \end{aligned} \tag{5.17}$$

Now, if in the previous inequality we take the expectation and send j to infinity we have a contradiction. In fact we recall that there exists a constant $\tilde{K}$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{E}\int_0^T |Z_k^n(s)|^2\,\mathrm{d}s \leq \tilde{K}.$$

So we can conclude that $\bar{Z}_k^n$ converges to Z_k $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-a.s.: in fact by (5.16) and from the fact that Z_k^n converges to Z $dt \otimes d\mathbb{P}$-a.s. we have

$$|\bar{Z}_k^n(t) - Z_k(t)| \leq |\bar{Z}_k^n(t) - Z_k^n(t)| + |Z_k^n(t) - Z_k(t)| \leq \epsilon.$$

It follows by (5.13) and from the definition of ψ_n

$$\psi(t, v(t), \bar{Z}_k^n(t)) \geq \psi_n(t, v(t), Z_k^n(t)) \geq \psi(t, v(t), Z_k^n(t))$$

and by continuity of ψ with respect to z we get

$$\psi(t, v(t), Z_k(t)) \geq \lim_{n\to\infty} \psi_n(t, v(t), Z_k^n(t)) \geq \psi(t, v(t), Z_k(t))$$

and the claim is proved.

Now we fix our attention on the equation (5.12). From the definition of (Y_k^n, Z_k^n), noting that $\tau_k \leq \tau_{k+1}$, we have

$$Y_{k+1}^n(t \wedge \tau_k) = Y_k^n(t), \quad Z_{k+1}^n(t)1_{t\leq\tau_k} = Z_k^n(t).$$

Sending n to infinity, we get

$$Y_{k+1}(t \wedge \tau_k) = Y_k(t), \quad Z_{k+1}(t)1_{t\leq\tau_k} = Z_k(t).$$

Now we define Y and Z on $[0, T]$ by setting

$$Y_t = Y_k(t), \quad Z_t = Z_k(t) \text{ if } t \in [0, \tau_k].$$

For $\mathbb{P}^\circ$-a.s. ω, there exists an integer $K(\omega)$ such that for $k \geq K(\omega)$, $\tau_k(\omega) = T$.

Thus Y is a continuous process, $Y_T = \phi(v(t))$, and $\int_0^T |Z_t|^2\,\mathrm{d}s < \infty$ $\mathbb{P}^\circ$-a.s. From (5.12), (Y, Z) satisfies

$$Y_{t\wedge\tau_k} = Y_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} \psi(s, v(s), Z(s))\,\mathrm{d}s - \int_{t\wedge\tau_k}^{\tau_k} Z(s)\,\mathrm{d}W_s^\circ.$$

By sending k to infinity, we deduce that (Y, Z) is a solution of (5.8) and

$$\lim_n \sup_{t\in[0,T]} |Y_t^n - Y_t| = 0, \quad \lim_n \int_0^T |Z_t^n - Z_t|^2 \,\mathrm{d}t = 0, \quad \mathbb{P}^\circ\text{-a.s.}$$

Thus $|Z^n - Z|$ converges to zero in measure $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$, and passing, if needed, to a subsequence (that by abuse of language we still denote Z^n), we can assume that $|Z^n - Z| \to 0$, $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$ almost everywhere.

Now, as ψ_n is globally Lipschitz continuous, from [12] there exist Borel measurable functions

$$\theta^n : [0,T] \times X_\eta \to \mathbb{R}, \quad \zeta^n : [0,T] \times X_\eta \to U$$

such that

$$Y_t^n = \theta^n(t, v(t)), \quad Z_t^n = \zeta(t, v(t)).$$

It suffices to define

$$\theta(t,v) = \liminf_{n\to\infty} \theta^n(t,v) \quad \text{and} \quad \zeta(t,x) = \liminf_{n\to\infty} \zeta^n(t,x)$$

to get

$$Y_t = \theta(t, v(t)), \quad Z_t = \zeta(t, v(t)),$$

which implies that (θ, ζ) is the Borel function we look for. Finally, $0 \le Y_t \le Y_t^S$ implies that

$$\mathbb{E} \sup_{t\in[0,T]} |Y_t|^2 < \infty,$$

and from the equation

$$|Y_t|^2 + \int_t^{\tau_k} |Z_s|^2 \,\mathrm{d}s = 2\int_t^{\tau_k} Y_s \psi(s, v(s), Z_s) \,\mathrm{d}s - 2\int_t^{\tau_k} Y_s Z_s \,\mathrm{d}W_s^\circ,$$

taking into consideration that

$$\begin{aligned} Y_s \psi(s, v(s), Z_s) &\le Y_s(l(s, J(v(s)), \gamma_0) + C(1 + |\gamma_0|)|Z_s|) \\ &\le Y_s^S(l(s, J(v(s)), \gamma_0) + C(1 + |\gamma_0|)|Z_s|), \end{aligned}$$

we deduce, by standard arguments, that

$$\mathbb{E}^\circ \int_0^T |Z_t|^2 \,\mathrm{d}t < \infty.$$

Moreover, this solution is the maximal solution among all the solutions (Y', Z') satisfying

$$\mathbb{E}^\circ[\sup_{t\in[0,T]} |Y_t'|^2] < +\infty,$$

and it suffices to apply [5, Proposition 5] to deduce that $Y^n \ge Y'$ and then $Y \ge Y'$.

5.1. The fundamental relation.

In this section we still assume that Hypothesis 1.1 holds.

Proposition 5.4. *Let θ, ζ denote the functions in the statement of Proposition 5.3. Then for every admissible control γ and for the corresponding trajectory v starting at v_0, we have*

$$\mathbb{J}(\gamma) = \theta(0, v_0) + \mathbb{E}\int_0^T [-\psi(t, v(t), \zeta(t, v(t))) + \zeta(t, v(t)) \cdot r(t, J(v(t)), \gamma_t) + l(t, J(v(t)), \gamma_t)]\,\mathrm{d}t.$$

Proof. We introduce stopping times τ_n and control processes γ_n as in the proof of Theorem 3.3, and we denote by v_n the solution to (3.8). Let us define

$$W_t^n = W_t + \int_0^t r(s, J(v_n(s)), \gamma_n(s))\,\mathrm{d}s.$$

From the definition of τ_n and from (1.3), it follows that

$$\begin{aligned}\int_0^T |r(s, J(v_n(s)), \gamma_n(s))|^2\,\mathrm{d}s &\le C\int_0^T (1 + |\gamma_n(s)|)^2\,\mathrm{d}s \\ &\le C\int_0^{\tau_n} (1 + |\gamma_n(s)|)^2\,\mathrm{d}s + C \le C + Cn. \end{aligned} \tag{5.18}$$

Therefore, by defining

$$\rho_n = \exp\left(\int_0^T r(s, J(v_n(s)), \gamma_n(s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^T |r(s, J(v_n(s)), \gamma_n(s)|^2\,\mathrm{d}s\right),$$

the Novikov condition implies that $\mathbb{E}\rho_n = 1$. Setting $\mathrm{d}\mathbb{P}^n = \rho_n \mathrm{d}\mathbb{P}$, by Girsanov's theorem W^n is a Wiener process under $\mathbb{P}^n$. Let us denote by $(\mathcal{F}_t^n)$ its natural augmented filtration. Since for all n

$$\begin{aligned} \mathrm{d}v_n(t) &= Bv_n(t)\,\mathrm{d}t + (I - B)Pg(t, J(v_n(t)))\,\mathrm{d}W_t^n \\ v_n(0) &= v_0 \end{aligned}$$

has a solution by Theorem 3.1, the process v_n is also $\mathcal{F}_t^n$ adapted. Let us define

$$Y_t^n = \theta(t, v_n(t)), \qquad Z_t^n = \zeta(t, v_n(t));$$

then, by Proposition 5.3, we have

$$\begin{aligned} \mathrm{d}Y_t^n &= Z_t^n\,\mathrm{d}W_t^n - \psi(t, v_n(t), Z_t^n)\,\mathrm{d}t, \qquad t \in [0, T], \\ Y_T^n &= \phi(J(v_n(T))) \end{aligned} \tag{5.19}$$

and $\mathbb{E}^n \int_0^T |Z_t^n|^2\,\mathrm{d}t < \infty$, where $\mathbb{E}^n$ denotes expectation with respect to $\mathbb{P}^n$. It follows that

$$Y_{\tau_n}^n = \phi(J(v_n(T))) + \int_{\tau_n}^T \psi(t, v_n(t), Z_t^n)\,\mathrm{d}t - \int_{\tau_n}^T Z_t^n\,\mathrm{d}W_t - \int_{\tau_n}^T Z_t^n r(t, J(v_n(t)), \gamma_n(t))\,\mathrm{d}t. \tag{5.20}$$

We note that for every $p \in [1, \infty)$ we have

$$\rho_n^{-p} = \exp\left(p \int_0^T r(s, J(v_n(s)), \gamma_n(s)) \, \mathrm{d}W_s^n - \frac{p^2}{2} \int_0^T |r(s, J(v_n(s)), \gamma_n(s))|^2 \, \mathrm{d}s \right) \cdot \exp\left(\frac{p^2 - p}{2} \int_0^T |r(s, J(v_n(s)), \gamma_n(s))|^2 \, \mathrm{d}s \right). \quad (5.21)$$

By (5.18) the second exponential is bounded by a constant depending only on n and p, while the first one has $\mathbb{P}^n$-expectation equal to 1. So we conclude that $\mathbb{E}^n \rho_n^{-p} < \infty$. It follows that

$$\mathbb{E} \left(\int_0^T |Z_t^n)|^2 \, \mathrm{d}t \right)^{1/2} \le \mathbb{E}^n \rho_n^{-2} \left(\int_0^T |Z_t^n|^2 \, \mathrm{d}t \right)^{1/2} \le (\mathbb{E}^n \rho_n^{-2})^{1/2} \mathbb{E}^n \left(\int_0^T |Z_t^n|^2 \, \mathrm{d}t \right)^{1/2} < \infty,$$

and the stochastic integral in (5.20) has zero $\mathbb{P}$-expectation. Hence we obtain

$$\mathbb{E} Y_{\tau_n}^n = \mathbb{E} \phi(J(v_n(T))) + \mathbb{E} \int_{\tau_n}^T [\psi(t, v_n(t), Z_t^n) - Z_t^n r(t, J(v_n(t)), \gamma_n(t))] \, \mathrm{d}t.$$

Since by definition it is $\psi(t, v, z) - z r(t, J(v), \gamma) - l(t, J(v), \gamma) \le 0$, we have

$$\mathbb{E} Y_{\tau_n}^n \le \mathbb{E} \phi(J(v_n(T))) + \mathbb{E} \int_{\tau_n}^T l(t, J(v_n(t)), \gamma_n(t)) \, \mathrm{d}t. \quad (5.22)$$

Now we let $n \to \infty$: by the definition of γ_n, from (1.4) and using the fact that J is bounded from X_η to H (see [3, Lemma 3.11]) we get

$$\mathbb{E} \int_{\tau_n}^T l(t, J(v_n(t)), \gamma_n(t)) \, \mathrm{d}t = \mathbb{E} \int_0^T \mathbf{1}_{\{s > \tau_n\}} l(t, J(v_n(t)), \gamma_0) \, \mathrm{d}t$$
$$\le C \mathbb{E} \int_0^T \mathbf{1}_{\{s > \tau_n\}} (1 + |J(v_n(t))|^2 + |\gamma_0|^2) \, \mathrm{d}s \le C \, \mathbb{E}\big[(T - \tau_n)\big(1 \ + \sup |v_n(t)|^2\big)\big] \quad (5.23)$$

and the right-hand side tends to 0 by the uniform integrability of $\sup\limits_{t \in [0,T]} |v_n(t)|^2$ (see Corollary 3.5) and by (3.6).

Next we note that, again by (3.6), for $n \ge N(\omega)$ we have $\tau_n(\omega) = T$ and

$$\phi(J(v_n(T))) = \phi(J(v_n(\tau_n))) = \phi(J(v(\tau_n))) = \phi(v(t)).$$

We deduce, thanks to (1.5) and again to [3, Lemma 3.11], that

$$|\phi(J(v_n(T)))| \le C(1 + |v_n(T)|^2) \le C(1 + \sup_{t \in [0,T]} |v_n(t)|^2),$$

and by the uniform integrability of $\sup\limits_{t \in [0,T]} |v_n(t)|^2$, the right hand side is uniformly integrable. We deduce that $\mathbb{E} \phi(J(v_n(T))) \to \mathbb{E} \phi(J(v(T)))$, and from (5.22) we conclude that $\limsup\limits_{n \to \infty} \mathbb{E} Y_{\tau_n}^n \le \mathbb{E} \phi(J(v(T)))$. On the other hand, for $n \ge N(\omega)$ we have $\tau_n(\omega) = T$ and

$$Y_{\tau_n}^n = Y_T^n = \phi(J(v_n(T))) = \phi(J(v(T))).$$

Since Y^n is positive, by an application of Fatou's lemma it follows that $\mathbb{E}\phi(J(v(T))) \le \liminf_{n\to\infty} \mathbb{E}Y^n_{\tau_n}$. We have thus proved that

$$\lim_{n\to\infty} \mathbb{E}Y^n_{\tau_n} = \mathbb{E}\phi(J(v(T))). \tag{5.24}$$

Now we return to (5.19) and write

$$Y^n_{\tau_n} = Y^n_0 + \int_0^{\tau_n} -\psi(t, v_n(t), Z^n_t)\,\mathrm{d}t + \int_0^{\tau_n} Z^n_t\,\mathrm{d}W_t + \int_0^{\tau_n} Z^n_t r(t, J(v_n(t)), \gamma_n(t))\,\mathrm{d}t.$$

Arguing as before, we conclude that the stochastic integral has zero $\mathbb{P}$-expectation. Moreover, we have $Y^n_0 = \theta(0, v_0)$, and, for $t \le \tau_n$, we also have $\gamma_n(t) = \gamma(t) = \gamma_t$, $v_n(t) = v(t)$, and $Z^n_t = \zeta(t, v(t))$. Thus we obtain

$$\mathbb{E}Y^n_{\tau_n} = \theta(0, v_0) + \mathbb{E}\int_0^{\tau_n} [-\psi(t, v(t), \zeta(t, v(t))) + \zeta(t, v(t)) r(t, J(v_t), \gamma_t)]\,\mathrm{d}t,$$

and by adding to both sides the quantity $\mathbb{E}\int_0^{\tau_n} l(t, J(v(t)), \gamma_t)\,\mathrm{d}t$ we get

$$\begin{aligned}\mathbb{E}\int_0^{\tau_n} l(t, J(v(t)), \gamma_t)\,\mathrm{d}t + \mathbb{E}Y^n_{\tau_n} &= \theta(0, v_0)\\ &+ \mathbb{E}\int_0^{\tau_n} [-\psi(t, v(t), \zeta(t, v(t))) + \zeta(t, v(t)) r(t, J(v_t), \gamma_t) + l(t, J(v(t)), \gamma_t)]\,\mathrm{d}t\end{aligned}$$

Noting that $-\psi(t, v(t), \zeta(t, v(t))) + \zeta(t, v(t))\, r(t, J(v_t), \gamma_t) + l(t, J(v(t)), \gamma_t) \ge 0$ and recalling that $l(t, u, \gamma) \ge 0$, by (5.24) and the monotone convergence theorem we obtain, for $n \to \infty$,

$$\begin{aligned}&\mathbb{E}\int_0^{T} l(t, J(v(t)), \gamma_t)\,\mathrm{d}t + \mathbb{E}\phi(J(v(t)))\\ &\quad= \theta(0, v_0) + \mathbb{E}\int_0^{T} [-\psi(t, v(t), \zeta(t, v(t))) + \zeta(t, v(t)) r(t, J(v_t), \gamma_t) + l(t, J(v(t)), \gamma_t)]\,\mathrm{d}t \end{aligned}\tag{5.25}$$

which gives the required conclusion.

Corollary 5.5. *For every admissible control γ and any initial datum v_0, we have $\mathbb{J}(v_0, \gamma) \ge \theta(0, v_0)$, and the equality holds if and only if the following feedback law holds $\mathbb{P}$-a.s. for almost every $t \in [0, T]$:*

$$\psi(t, v(t), \zeta(t, v(t))) = \zeta(t, v(t))\, r(t, J(v(t)), \gamma_t) + l(t, J(v(t)), \gamma_t),$$

where v is the trajectory starting at v_0 and corresponding to control γ.

5.2. Existence of optimal controls: the closed loop equation.

The aim of this subsection is to find a weak solution to the so-called closed loop equation. We are assuming that Hypothesis 5.1 holds. Then, by the Filippov Theorem (see, e.g., [2, Thm. 8.2.10, p. 316]) there exists a measurable selection of Γ, a Borel measurable function $\mu : [0, T] \times X_\eta \times U \to \Xi$ such that

$$\psi(t, v, z) = l(t, J(v), \mu(t, v, z)) + z\, r(t, J(v), \mu(t, v, z)) \quad t \in [0, T],\ v \in X_\eta,\ z \in U^*. \tag{5.26}$$

By (5.5), we have

$$|\mu(t, v, z)| \le C(1 + |v| + |z|). \tag{5.27}$$

We define

$$\bar{\gamma}(t,v) = \mu(t,v,\zeta(t,v)) \quad t \in [0,T], v \in X_\eta,$$

where ζ is defined in Proposition 5.3. The closed loop equation is

$$\begin{aligned} dv(t) &= \big[Bv(t) + (I-B)Pg(t,J(v(t)))r(t,J(v(t)),\bar{\gamma}(t,v))\big]\,dt \\ &\qquad\qquad\qquad\qquad + (I-B)Pg(t,J(v(t)))\,dW_t \\ v(0) &= v_0 \end{aligned} \tag{5.28}$$

By a weak solution we mean a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, a Wiener process W in U with respect to $\mathbb{P}$ and $(\mathcal{F}_t)$, and a continuous $(\mathcal{F}_t)$-adapted process v with values in H satisfying, $\mathbb{P}$-a.s.,

$$\mathbb{E}\int_0^T |\bar{\gamma}(t,v(t))|^2\,dt < \infty, \tag{5.29}$$

and such that (5.28) holds. We note that by (1.3) it also follows that

$$\mathbb{E}\int_0^T |r(t,v(t),\bar{\gamma}(t,v(t)))|\,dt < \infty, \quad \mathbb{P}\text{-a.s.}, \tag{5.30}$$

so that (5.28) makes sense.

Proposition 5.6. *Assume that Hypothesis 1.1 holds. Then there exists a weak solution of the closed loop equation, satisfying in addition*

$$\mathbb{E}\int_0^T |\bar{\gamma}(t,v(t))|^2\,dt < \infty. \tag{5.31}$$

Proof. Let us take an arbitrary complete probability space $(\Omega,\mathcal{F},\mathbb{P}^\circ)$ and a Wiener process W° in U with respect to $\mathbb{P}^\circ$. Let $(\mathcal{F}_t^\circ)$ be the associated Brownian filtration. We define the process v as the solution of the equation

$$\begin{aligned} dv(t) &= Bv(t)\,dt + (I-B)Pg(t,J(v(t))\,dW_t^\circ \qquad \in t \in [0,T], \\ v(0) &= v_0 \end{aligned} \tag{5.32}$$

The solution is a continuous $(\mathcal{F}_t^\circ)$-adapted process, which exists and is unique by Theorem 3.1. Moreover, it satisfies $\mathbb{E}^\circ\left[\sup_{t\in[0,T]} ||v(t)||^p_{X_\eta}\right] < \infty$ for every $p \in [2,\infty)$. By Proposition 5.3, setting

$$Y_t = \theta(t,v(t)) \quad Z_t = \zeta(t,v(t)),$$

the following backward equation holds:

$$\begin{aligned} dY_t &= -\psi(t,v(t),Z_t)\,dt + Z_t\,dW_t^\circ, \qquad t \in [0,T], \\ Y_T &= \phi(J(v(t))), \end{aligned} \tag{5.33}$$

and we have

$$\mathbb{E}^\circ\int_0^T |Z_t|^2\,dt < \infty. \tag{5.34}$$

By (1.3) we have $|r(t, v(t), \bar{\gamma}(t, v(t)))| \leq C(1 + |\bar{\gamma}(t, v(t))|)$, and by (5.3),

$$|\bar{\gamma}(t, v(t))| = |\mu(t, v(t), \zeta(t, v(t)))| \leq C\,(1 + |v(t)| + |\zeta(t, v(t))|) = C\,(1 + |v(t)| + |Z_t|)\,. \tag{5.35}$$

Now let us define the family of stopping times

$$\tau_n = \inf\left\{ t \in [0, T] : \int_0^t |\bar{\gamma}(s, v(s)))|^2 \,\mathrm{d}s > n \right\}$$

with the convention that $\tau_n = T$ if the indicated set is empty. By (5.34) and (5.35), for $\mathbb{P}^\circ$-a.e. $\omega \in \Omega$, there exists an integer $N(\omega)$ depending on ω such that $\tau_(\omega) = T$ for $n \geq N(\omega)$. Let us fix $\gamma_0 \in \Xi$, and for every n let us define

$$\begin{aligned}
\gamma_n(t) &= \bar{\gamma}(t, v(t)))\,\mathbf{1}_{\{t \leq \tau_n\}} + \gamma_0\,\mathbf{1}_{\{t \geq \tau_n\}} \\
M_t^n &= \exp\left(\int_0^t r(s, J(v(s)), \gamma_n(s))\,\mathrm{d}W_s^\circ - \frac{1}{2}\int_0^t |r(s, J(v(s)), \gamma_n(s))|^2 \,\mathrm{d}s \right), \\
M_t &= \left(\int_0^t r(s, J(v(s)), \bar{\gamma}(t, v(t)))\,\mathrm{d}W_s^\circ - \frac{1}{2}\int_0^t |r(s, J(v(s)), \bar{\gamma}(t, v(t)))|^2 \,\mathrm{d}s \right), \\
W_t^n &= W_t^\circ - \int_0^t r(s, J(v(s)), \gamma_n(s))\,\mathrm{d}s, \\
W_t &= W_t^\circ - \int_0^t r(s, J(v(s)), \bar{\gamma}(s, v(s)))\,\mathrm{d}s.
\end{aligned}$$

By previous estimates, M^n, M, W^n, and W are well defined; moreover,

$$\int_0^T |r(s, J(v(s)), \gamma_n(s)) - r(s, J(v(s)), \bar{\gamma}(s, v(s)))|^2 \,\mathrm{d}s \to 0 \qquad \mathbb{P}^\circ\text{-a.s.}$$

and consequently $M_T^n \to M_T$ in probability and $\sup_{t \in [0,T]} |W_t^n - W_t| \to 0$, $\mathbb{P}^\circ$-a.s.

We will conclude the proof by showing that there exists a probability $\mathbb{P}$ such that W is a Wiener process with respect to P and $(\mathcal{F}_t^\circ)$. The definition of τ_n and the Novikov condition imply that $\mathbb{E}^\circ[M_T^n] = 1$. Setting $\mathrm{d}\mathbb{P}^n = M_T^n \mathrm{d}\mathbb{P}^\circ$, by Girsanov's theorem W^n is a Wiener process with respect to $\mathbb{P}^n$ and $(\mathcal{F}_t^\circ)$. Writing the backward equation with respect to W^n, we obtain

$$Y_{\tau_n} = Y_0 + \int_0^{\tau_n} -\psi(t, v(t), Z_t)\,\mathrm{d}t + \int_0^{\tau_n} Z_t\,\mathrm{d}W_t^n + \int_0^{\tau_n} Z_t^n r(t, J(v_t), \gamma_t^n)\,\mathrm{d}t.$$

Arguing as in the proof of Proposition 5.4, we conclude that the stochastic integral has zero expectation with respect to $\mathbb{P}^n$. Taking into account that $\gamma_n(t) = \bar{\gamma}(t, v(t))$ for $t \leq \tau_n$, we obtain

$$\begin{aligned}
&\mathbb{E}^n Y_{\tau_n} + \mathbb{E}^n \int_0^{\tau_n} l(t, v(t), \bar{\gamma}(t, v(t)))\,\mathrm{d}t \\
&\qquad = Y_0 + \mathbb{E}^n \int_0^{\tau_n} [-\psi(t, v(t), Z_t)\,\mathrm{d}t + Z_t \cdot r(t, J(v_t), \bar{\gamma}(t, v(t))) + l(t, v(t), \bar{\gamma}(t, v(t))]\,\mathrm{d}t = Y_0
\end{aligned}$$

with the last equality coming from the definition of $\bar{\gamma}$. Recalling that Y is nonnegative, it follows that

$$\mathbb{E}^n \int_0^{\tau_n} l(t, v(t), \bar{\gamma}(t, v(t)))\,\mathrm{d}t \leq C$$

for some constant C independent of n. By (1.4) we also deduce

$$\mathbb{E}^n \int_0^{\tau_n} |\bar\gamma(t, v(t))|^2 \,\mathrm{d}t \le C. \tag{5.36}$$

Next we prove that the family $(M_T^n)_{n\ge 1}$ is uniformly integrable by showing that $\mathbb{E}^\circ[M_T^n \mathbf{1}_{\{M_T^n > c\}}] \to 0$ as $c \to \infty$, uniformly with respect to n. We have

$$\mathbb{E}^\circ[M_T^n \mathbf{1}_{\{M_T^n > c\}}] = \mathbb{E}^\circ[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n = T\}}] + \mathbb{E}^\circ[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n < T\}}]. \tag{5.37}$$

The first term in the right-hand side tends to 0 uniformly with respect to n: it is

$$\mathbb{E}^\circ[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n = T\}}] = \mathbb{E}^\circ[M_T \mathbf{1}_{\{M_T > c, \tau_n = T\}}] \le \mathbb{E}^\circ[M_T \mathbf{1}_{\{M_T > c\}}] \to 0$$

since it holds $\mathbb{E}^\circ[M_T^n] = 1$ and then Fatou's lemma implies that $\mathbb{E}^\circ[M_T] \le 1$. The second term in the right-hand side of (5.37) can be estimated as follows:

$$\begin{aligned}\mathbb{E}^\circ[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n < T\}}] &\le \mathbb{E}^\circ[M_T^n \mathbf{1}_{\{\tau_n < T\}}] = \mathbb{P}^n(\tau_n < T) \\ &\le \mathbb{P}^n\left(\int_0^{\tau_n} |\bar\gamma(t, v(t))|^2 \,\mathrm{d}t > n\right) \le \frac{1}{n}\mathbb{E}^n \int_0^{\tau_n} |\bar\gamma(t, v(t))|^2 \,\mathrm{d}t \le \frac{C}{n}\end{aligned} \tag{5.38}$$

with the last inequality coming from (5.36). The required uniform integrability follows immediately. Recalling that $M_T^n \to M_T$ in probability, we conclude that $\mathbb{E}^\circ|M_T^n - M_T| \to 0$, and in particular $\mathbb{E}^\circ[M_T] = 1$, and M is a $\mathbb{P}^\circ$-martingale. Thus we can define a probability $\mathbb{P}$ by setting $\mathrm{d}P = M_T \,\mathrm{d}\mathbb{P}^\circ$ and by Girsanov's theorem we conclude that W is a Wiener process with respect to $\mathbb{P}$ and $(\mathcal{F}_t^\circ)$.

It remains to prove (5.31). We define the stopping times

$$\sigma_n = \inf\left\{t \in [0, T] : \int_0^t |Z_s|^2 \,\mathrm{d}s > n\right\}$$

with the convention that $\sigma_n = T$ if the indicated set is empty. Writing the backward equation with respect to W, we obtain

$$Y_{\sigma_n} = Y_0 - \int_0^{\sigma_n} \psi(t, v(t), Z_t) \,\mathrm{d}t + \int_0^{\sigma_n} Z_t \,\mathrm{d}W_t + \int_0^{\sigma_n} Z_t r(t, J(v_t), \bar\gamma(t, v(t))) \,\mathrm{d}t$$

from which we deduce that

$$\begin{aligned}&\mathbb{E}^n Y_{\sigma_n} + \mathbb{E}^n \int_0^{\sigma_n} l(t, v(t), \bar\gamma(t, v(t))) \,\mathrm{d}t \\ &\quad = Y_0 + \mathbb{E}^n \int_0^{\sigma_n} [-\psi(t, v(t), Z_t) + Z_t \cdot r(t, J(v_t), \bar\gamma(t, v(t))) + l(t, J(v(t)), \bar\gamma(t, v(t))] \,\mathrm{d}t = Y_0\end{aligned}$$

with the last equality coming from the definition of $\bar\gamma$. Recalling that Y is nonnegative, it follows that

$$\mathbb{E} \int_0^{\sigma_n} l(t, J(v(t)), \bar\gamma(t, v(t))) \,\mathrm{d}t \le C$$

for some constant C independent of n. By (1.4) and by sending n to infinity, we finally prove (5.31).

Corollary 5.7. *By Corollary 5.5 it immediately follows that if v is the solution to (5.28) and we set $\gamma_s^* = \bar{\gamma}(s, v_s)$, then $\mathbb{J}(v_0, \gamma^*) = \theta(0, v_0)$, and consequently v is an optimal state, γ_s^* is an optimal control, and $\bar{\gamma}$ is an optimal feedback.*

Now we prove uniqueness in law for the closed loop equation. We remark that condition (5.29) is part of our definition of a weak solution.

Proposition 5.8. *Assume that g, l, r, ϕ satisfy Hypothesis 1.1. Fix $\mu : [0,T] \times X_\eta \times U^* \to \Xi$ satisfying (5.26) (and consequently (5.27)) and let $\bar{\gamma}(t, v) = \mu(t, v, \zeta(t, v))$. Then the weak solution of the closed loop equation (5.28) is unique in law.*

Proof. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, (W_t)_{t\in[0,T]}, (v(t))_{t\in[0,T]})$ be a weak solution of (5.28). Let us define

$$\begin{aligned} M_T &= \exp\left(-\int_0^T r(s, v(s), \bar{\gamma}(s, v(s)))\, \mathrm{d}W_s - \frac{1}{2}\int_0^T |r(s, v(s), \bar{\gamma}(s, v(s)))|^2\, \mathrm{d}s\right), \\ W_t^\circ &= W_t + \int_0^T r(s, v(s), \bar{\gamma}(s, v(s)))\, \mathrm{d}s. \end{aligned}$$

By (1.3) and (5.29), M_T and W° are well defined. We claim that $\mathbb{E}M_T = 1$. Assuming the claim for a moment, and setting $\mathrm{d}P^\circ = M_T\, \mathrm{d}\mathbb{P}$, we know by Girsanov's theorem that W° is a Wiener process under P°, and further v solves

$$\begin{aligned} \mathrm{d}v(t) &= Bv(t)\, \mathrm{d}t + (I - B)Pg(t, J(v(t)))\, \mathrm{d}W_t^\circ \\ v(0) &= v_0 \end{aligned}$$

and

$$M_T = \exp\left(-\int_0^T r(s, v(s), \bar{\gamma}(s, v(s)))\, \mathrm{d}W_s^\circ + \frac{1}{2}\int_0^T |r(s, v(s), \bar{\gamma}(s, v(s)))|^2\, \mathrm{d}s\right).$$

By Theorem 3.1 the law of (v, W°) under P° is uniquely determined by g and v_0. Taking into account the last displayed formula, we conclude that the law of (v, W°, M_T) under P° is also uniquely determined, and consequently so it is the law of v under $\mathbb{P}$.

To conclude the proof it remains to show that $\mathbb{E}[M_T] = 1$. We define the stopping times

$$\tau_n = \inf\left\{t \in [0,T] : \int_0^t |\bar{\gamma}(s, v(s))|^2\, \mathrm{d}s > n\right\},$$

with the convention that $\tau_n = T$ if the indicated set is empty. By (5.29), for $\mathbb{P}$-almost every $\omega \in \Omega$ there exists an integer $N(\omega)$ depending on ω such that $\tau_n(\omega) = T$ for $n \geq N(\omega)$.

Let us fix $\gamma_0 \in \Xi$ and let us define, for every $n \in \mathbb{N}$,

$$\begin{aligned} \gamma_n(t) &= \bar{\gamma}(t, v(t))\, 1_{t\leq\tau_n} + \gamma_0\, 1_{t\geq\tau_n} \\ M_T^n &= \exp\left(-\int_0^T r(s, J(v_n(s)), \gamma_n(s)\, \mathrm{d}W_s - \frac{1}{2}\int_0^T |r(s, J(v_n(s)), \gamma_n(s)|^2\, \mathrm{d}s\right). \end{aligned}$$

By (1.3) and the definition of τ_n, the Novikov condition shows that $\mathbb{E}M_T^n = 1$. Moreover, we have

$$\int_0^T |r(s, v(s), \gamma_n(s)) - r(s, v(s), \bar{\gamma}(s, v(s)))|^2\, \mathrm{d}s \to 0, \qquad \mathbb{P}\text{-a.s.},$$

and consequently $M_T^n \to M_T$ in probability. In order to conclude the proof it is therefore enough to show that the family $(M_T^n)_{n\geq 1}$ is uniformly integrable. To prepare for this, let us set $\mathrm{d}\mathbb{P}^n = M_T^n\,\mathrm{d}\mathbb{P}$ and note that, by Girsanov's theorem, the process $W_t^n = W_t + \int_0^t r(s, v(s), \gamma_s^n)\,\mathrm{d}s$ is a Wiener process under $\mathbb{P}^n$. Since v solves

$$\begin{aligned} \mathrm{d}v(t) &= Bv(t)\,\mathrm{d}t + (I-B)Pg(t, J(v(t)))\,\mathrm{d}W_t^n \\ v(0) &= v_0, \end{aligned}$$

it follows that v is adapted to the Brownian filtration $(\mathcal{F}_t^n)$ associated to W^n, and its law under $\mathbb{P}^n$ is uniquely determined by g and v_0. In particular, the quantities

$$C' := \mathbb{E}^n \int_0^T |v(t)|^2\,\mathrm{d}t, \qquad C'' := \mathbb{E}^n \int_0^T |\zeta(t, v(t))|^2\,\mathrm{d}t$$

do not depend on n (here $\mathbb{E}^n$ denotes, of course, the expectation with respect to $\mathbb{P}^n$). C' is clearly finite. By Proposition 5.3, setting $Z_t = \zeta(t, v(t))$, we have $\mathbb{E}^n \int_0^T |\zeta(t, v(t))|^2\,\mathrm{d}t = \mathbb{E}^n \int_0^T |Z_t|^2\,\mathrm{d}t < \infty$, and it follows that C'' is also finite. Now let us prove the uniform integrability of the family $(M_T^n)_{n\geq 1}$ by showing that that $\mathbb{E}[M_T^n \mathbf{1}_{\{M_T^n > c\}}] \to 0$ as $c \to \infty$, uniformly with respect to n. We have

$$\mathbb{E}[M_T^n \mathbf{1}_{\{M_T^n > c\}}] = \mathbb{E}[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n = T\}}] + \mathbb{E}[M_T^n 1_{\{M_T^n > c, \tau_n < T\}}].$$

The first term in the right-hand side tends to 0 uniformly with respect to n, since

$$\mathbb{E}[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n = T\}}] = \mathbb{E}[M_T \mathbf{1}_{\{M_T > c, \tau_n = T\}}] \leq \mathbb{E}[M_T \mathbf{1}_{\{M_T > c\}}] \to 0,$$

due to the fact that the equality $\mathbb{E}[M_T^n] = 1$ and Fatou's lemma imply that $\mathbb{E}[M_T] \leq 1$. The second term in the right-hand side of (5.39) can be estimated as follows:

$$\begin{aligned} \mathbb{E}[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n < T\}}] &\leq \mathbb{E}[M_T^n \mathbf{1}_{\{\tau_n < T\}}] = \mathbb{P}^n(\tau_n < T) \\ &\leq \mathbb{P}^n\left(\int_0^{\tau_n} |\bar{\gamma}(t, v(t))|^2\,\mathrm{d}t > n\right) \leq \frac{1}{n}\mathbb{E}^n \int_0^{\tau_n} |\bar{\gamma}(t, v(t))|^2\,\mathrm{d}t. \end{aligned} \tag{5.39}$$

By (5.27) we have

$$|\bar{\gamma}(t, v(t))|^2 = |\mu(t, v(t), \zeta(t, v(t)))|^2 \leq C(1 + |v(t)|^2 + |\zeta(t, v(t))|^2)$$

for some constant C, and it follows that

$$\mathbb{E}[M_T^n \mathbf{1}_{\{M_T^n > c, \tau_n < T\}}] \leq \frac{C}{n}\mathbb{E}^n \int_T^0 (1 + |v(t)|^2 + |\zeta(t, v(t))|^2)\,\mathrm{d}t = \frac{C}{n}(T + C' + C''),$$

with C' and C'' defined as above. The required uniform integrability follows immediately and this completes the proof.

6. Appendix. Proof of Proposition 5.2

Proposition 5.2 gives a result of monotone stability that is very close to the result contained in [18, Proposition 2.4]; the main difference is that we require that for almost all $\omega \in \Omega$ and $t \in [0, \tau]$

$$\lim_{n\to\infty} \psi_n(t, Y_t^n, Z_t^n) = \psi(t, Y_t, Z_t)$$

instead of locally uniform convergence of $(\psi_n)_n$ to ψ.

We sketch the proof.

Since for all $t \in \mathbb{R}^+$ the sequence Y_t^n is monotonic and bounded, it has a limit which we denote Y_t. In view of an extension to the infinite dimensional case of [18, Proposition 2.1], there exists a constant $\tilde{K}$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{E} \int_0^T |Z^n(s)|^2 \, \mathrm{d}s \le \tilde{K}.$$

Therefore there exists a process $Z_k \in L^2(\Omega \times [0,T], U^*)$ and a subsequence Z^{n_j} of Z such that

$$Z^{n_j} \rightharpoonup Z \text{ weakly in } L^2(\Omega \times [0,T], U^*). \tag{6.1}$$

The point is now to show that in fact the whole sequence converges strongly to Z_k in $L^2(\Omega \times [0,T], U^*)$. Since

$$|\psi_n(t,v,z)| \le \tilde{C}(1+|z|^2), \tag{6.2}$$

setting $K = 5\tilde{C}$ we have

$$|\psi_n(t,v,z) - \psi_m(t,v',z')| \le 2\tilde{C} + K(|z-z'|^2 + |z'-z''|^2 + |z''|^2).$$

Step 1. The strong convergence of Z_k^n in $L^2(\Omega \times [0,T], U^*)$. We have, for all $n, m \in \mathbb{N}$,

$$|\psi_n(t,v_t,Z^n(t)) - \psi_m(t,v_t,Z^m(t))| \le 2\tilde{C} + K(|Z^n - Z^m|^2 + |Z^m - Z|^2 + |Z|^2).$$

Let us apply Ito's formula to the process $(Y^n(t) - Y^m(t))_{t \in [0,T]}$ for $n, m \in \mathbb{N}$, $n \le m$ and to an increasing function $F \in C^2[0,2k]$, such that $F'(0) = 0$ and $F(0) = 0$.

The function F is yet to be chosen: (we omit the dependence on k for the moment and write $Y_k^n(t) = Y_t^n$)

$$\begin{aligned} F(Y_0^n - Y_0^m) &= F(Y_T^n - Y_T^m) + \int_0^{T\wedge\tau} F'(Y_s^n - Y_s^m)(\psi_n(s,v_s,Z_s^n) - \psi_m(s,v_s,Z_s^m)) \, \mathrm{d}s \\ &\quad - \frac{1}{2} \int_0^{T\wedge\tau} F''(Y_s^n - Y_s^m)|Z_s^n - Z_s^m|^2 \, \mathrm{d}s - \int_0^{T\wedge\tau} F'(Y_s^n - Y_s^m)(Z_s^n - Z_s^m) \, \mathrm{d}W_s. \end{aligned} \tag{6.3}$$

As $F'(Y_s^n - Y_s^m) \ge 0$, it follows

$$\begin{aligned} F(Y_0^n - Y_0^m) &\le F(Y_T^n - Y_T^m) \\ &\quad + \int_0^{T\wedge\tau} F'(Y_s^n - Y_s^m)(2C + K(|Z_s^n - Z_s^m|^2 + |Z_s^m - Z_s|^2 + |Z_s|^2)) \, \mathrm{d}s \\ &\quad - \frac{1}{2} \int_0^{T\wedge\tau} F''(Y_s^n - Y_s^m)|Z_s^n - Z_s^m|^2 \, \mathrm{d}s - \int_0^{T\wedge\tau} F'(Y_s^n - Y_s^m)(Z_s^n - Z_s^m) \, \mathrm{d}W_s. \end{aligned} \tag{6.4}$$

Now since $Y^n - Y^m$ is bounded,

$$\mathbb{E} \int_0^{T\wedge\tau} F'(Y_s^n - Y_s^m)(Z_s^n - Z_s^m) \, \mathrm{d}W_s = 0$$

and we get

$$\mathbb{E}F(Y_0^n - Y_0^m) + \mathbb{E}\int_0^{T\wedge\tau} [\frac{1}{2}F'' - KF'](Y_s^n - Y_s^m)|Z_s^n - Z_s^m|^2 - KF'](Y_s^n - Y_s^m)|Z_s^m - Z_s|^2 \,\mathrm{d}s$$
$$\leq \mathbb{E}F(Y_T^n - Y_T^m) + \mathbb{E}\int_0^{T\wedge\tau} F'(Y_s^n - Y_s^m)(2C + K|Z_s|^2)ds. \quad (6.5)$$

We want to pass to the limit as $m \to \infty$ along the subsequence n_j defined in (6.1). The convergence of $Y^m \to Y$ being pointwise, and Y^m being bounded, one has, by Lebesgue's dominated convergence theorem,

$$\mathbb{E}F(Y_0^n - Y_0) + \liminf_{m\to\infty} \mathbb{E}\int_0^{T\wedge\tau} [\frac{1}{2}F'' - KF'](Y_s^n - Y_s)|Z_s^n - Z_s^m|^2 \,\mathrm{d}s$$
$$- \mathbb{E}\int_0^{T\wedge\tau_k} KF'(Y_s^n - Y_s)|Z_s^n - Z_s|^2 \,\mathrm{d}s$$
$$\leq \mathbb{E}F(Y_T^n - Y_T) + \mathbb{E}\int_0^{T\wedge\tau} F'(Y_s^n - Y_s)(2\tilde{C} + K|Z_s|^2) \,\mathrm{d}s \quad (6.6)$$

and from the limit

$$\liminf_{m\to\infty}[-\mathbb{E}\int_0^{T\wedge\tau} F'(Y_s^n - Y_s)|Z_s^n - Z_s^m|^2 \,\mathrm{d}s \leq -\mathbb{E}\int_0^{T\wedge\tau} F'(Y_s^n - Y_s)|Z_s^n - Z_s|^2 \,\mathrm{d}s \quad (6.7)$$

we obtain

$$\mathbb{E}F(Y_0^n - Y_0) + \liminf_{m\to\infty} \mathbb{E}\int_0^{T\wedge\tau} [\frac{1}{2}F'' - 2KF'](Y_s^n - Y_s)|Z_s^n - Z_s^m|^2 \,\mathrm{d}s$$
$$\leq \mathbb{E}F(Y_T^n - Y_T) + \mathbb{E}\int_0^{T\wedge\tau} F'(Y_s^n - Y_s)(2\tilde{C} + K|Z_s|^2) \,\mathrm{d}s. \quad (6.8)$$

We now choose F such that $\frac{1}{2}F'' - 2KF' = 1$: in particular, by setting $F'(0) = F(0) = 0$, we obtain $F(x) = \frac{1}{4K}(e^{4x} - 4Kx - 1)$. It is straightforward to check that F is a C^∞ function, increasing on $[0, 2k]$. Noting that by the convexity of the l.s.c. functional

$$\Theta(Z) = \mathbb{E}\int_0^{T\wedge\tau} |Z_s^n - Z_s|^2 \,\mathrm{d}s$$

one has

$$\mathbb{E}\int_0^{T\wedge\tau} |Z_s^n - Z_s|^2 \,\mathrm{d}s \leq \liminf_{m\to\infty, m\in(n_j)} \mathbb{E}\int_0^{T\wedge\tau} |Z_s^n - Z_s^m|^2 \,\mathrm{d}s,$$

we obtain

$$\mathbb{E}F(Y_0^n - Y_0) + \mathbb{E}\int_0^{T\wedge\tau} |Z_s^n - Z_s|^2 \,\mathrm{d}s$$
$$\leq \mathbb{E}F(Y_T^n - Y_T) + \mathbb{E}\int_0^{T\wedge\tau} F'(Y_s^n - Y_s)(2\tilde{C} + K|Z_s|^2)\mathrm{d}s \quad (6.9)$$

By Lebesgue's dominated convergence theorem, the right-hand side of this inequality converges to 0 as $n \to \infty$, as well as the first term of the left-hand side. Now, passing to the limit as $n \to \infty$, we find, for all $T > 0$,

$$\limsup_{n\to\infty} \mathbb{E} \int_0^{T\wedge\tau} |Z_s^n - Z_s|^2 \, \mathrm{d}s = 0.$$

Consequently the whole sequence Z^n converges to Z in $L^2(\Omega \times [0,T], U^*)$.

Step 2. The uniform convergence of a subsequence of Y^n to Y.

At this stage of the proof we know that $(Y_t^n)_n$ converges pointwise: for all $t \in [0,T]$, $\lim_{n\to\infty} Y_t^n = Y_t$; also, the sequence $(Z^n)_n$ converges to Z in $L^2(\Omega \times [0,T], U^*)$.

We proceed proving the following result.

Lemma 6.1. *There exists a subsequence Z^{n_j} of Z^n such that Z^{n_j} converges almost surely to Z and such that $\tilde{Z} = \sup_j Z^{n_j} \in L^2(\Omega \times [0,T], U^*)$.*

Proof. For the reader's convenience we sketch the proof of this lemma. Extracting if necessary a subsequence, we may assume without loss of generality that the sequence $(Z^n)_n$ converges almost surely to Z. Since Z^n is a Cauchy sequence in $L^2(\Omega \times [0,T], U^*)$, we can extract a subsequence Z^{n_j} such that $||Z^{n_{j+1}} - Z^{n_j}||_{L^2} \le \frac{1}{2^j}$ for $j \in \mathbb{N}$. Then we set

$$g = |Z^{n_0}| + \sum_{j=0}^{\infty} |Z^{n_{j+1}} - Z^{n_j}|;$$

from the properties of the sequence Z^{n_j}, we have

$$||g||_{L^2} \le ||Z^{n_0}||_{L^2} + \sum_{j=0}^{\infty} ||Z^{n_{j+1} - Z^{n_j}}||_{L^2} \le ||Z^{n_0}||_{L^2} + \sum_{j=0}^{\infty} \frac{1}{2^j} < \infty.$$

Moreover, for any $p \in \mathbb{N}$, we also have

$$|Z^{n_p}| \le |Z^{n_0}| + \sum_{j=0}^{p} |Z^{n_{j+1} - Z^{n_j}}| \le g;$$

therefore, $\tilde{Z} = \sup_j |Z^{n_j}| \in L^2(\Omega \times [0,T], U^*)$ and the proof is complete.

In order to keep notation simpler, we still denote by (Z^n) the subsequence (Z^{n_j}) given by Lemma 6.1 [resp. $(Y^n)_n$ and $(\psi^n)_n$ the sequences $(Y^{n_j})_j$ and $(\psi^{n_j})_j$] and therefore we have

$$Z^n \to Z \text{ a.s. } \mathrm{d}t \otimes \mathrm{d}P \text{ and } \tilde{Z} = \sup_n |Z^n| \in L^2(\Omega \times [0,T], U^*).$$

Since ψ^n satisfies condition (6.2), we have

$$|\psi_n(t, v_t, Z_t^n)| \le 2\tilde{C}(1 + \sup_n |Z_t^n|^2) = 2\tilde{C}(1 + \tilde{Z}^2).$$

By assumption, for almost all $\omega \in \Omega$ and $t \in [0, \tau]$:

$$\lim_{n\to\infty} \psi_n(t, Y_t^n, Z_t^n) = \psi(t, Y_t, Z_t) \tag{6.10}$$

thus, for almost all $\omega \in \Omega$ and, uniformly in $t \in [0, \tau]$, Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} \int_0^{t\wedge\tau} \psi_n(s, v_s, Z_s^n) = \int_0^{t\wedge\tau} \psi(s, v_s, Z_s)\,\mathrm{d}s.$$

On the other hand, from the continuity properties of stochastic integral, we get

$$\lim_{n\to\infty} \sup_{0\le t\le T} \int_0^{t\wedge\tau} Z_s^n\,\mathrm{d}W_s - \int_0^{t\wedge\tau} Z_s\,\mathrm{d}W_s = 0 \text{ in probability.}$$

Extracting again a subsequence if necessary, we may assume that the last convergence is $\mathbb{P}^\circ$-a.s. Finally,

$$\begin{aligned} |Y_t^n - Y_t^m| \le |Y_T^n - Y_T^m| + \int_0^{t\wedge\tau} |\psi_n(s, v_s, Z_s^n) - \psi_m(s, v_s, Z_s^m)|\,\mathrm{d}s \\ + |\int_0^{t\wedge\tau} Z_s^n\,\mathrm{d}W_s^\circ - \int_0^{t\wedge\tau} Z_s^m\,\mathrm{d}W_s^\circ|. \end{aligned} \tag{6.11}$$

Therefore, taking limits on m and supremum over $t \in [0, T]$, we get, for almost all $\omega \in \Omega$

$$\begin{aligned} \sup_{0\le t\le T} |Y_t^n - Y_t| \le |Y_T^n - Y_T| + \int_0^{t\wedge\tau} |\psi_n(s, v_s, Z_s^n) - \psi(s, v_s, Z_s)|\,\mathrm{d}s \\ + \sup_{0\le t\le T} |\int_0^{t\wedge\tau} Z_s^n\,\mathrm{d}W_s^\circ - \int_0^{t\wedge\tau} Z_s\,\mathrm{d}W_s^\circ| \end{aligned} \tag{6.12}$$

from which we deduce that $(Y^n)_n$ converges to Y uniformly for $t \in [0, T]$ (in particular Y is a continuous process if the Y^n are).

References

[1] Agrawal, O. P. (2002). Solution for a Fractional Diffusion-Wave Equation Defined in a Bounded Domain. *Nonlinear Dynamics* **29**, 145–155.

[2] Ambrosetti, A. and Prodi, G. (1995). *A primer of nonlinear analysis.* Cambridge Studies in Advanced Mathematics, 34. Cambridge University Press, Cambridge.

[3] Bonaccorsi, S. and Desch, W. (2006). Volterra equations perturbed by noise. Technical Report UTM 698, June 2006, Matematica, University of Trento. Available at http://eprints.biblio.unitn.it/archive/00001021/

[4] Bonaccorsi, S. and Mastrogiacomo, E. (2009). An analytic approach to stochastic Volterra equations with completely monotone kernels. *Journal of Evolution Equations* **9**, pp.315–339.

[5] Briand, Ph. and Hu, Y. (2006). BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields* **136**, n. 4, pp. 604–618.

[6] Da Prato, G. and Zabczyk, J. (1996). *Ergodicity for Infinite Dimensional Systems.* Cambridge Univ. Press, Cambridge.

[7] Da Prato, G. and Zabczyk, J. (1992). *Stochastic Equations in Infinite Dimensions.* Cambridge Univ. Press, Cambridge.

[8] Debussche, A., Fuhrman, M. and Tessitore, G. (2007). Optimal Control of a Stochastic Heat Equation with Boundary-noise and Boundary-control. *ESAIM Control Optim. Calc. Var.* **13**, no. 1, 178–205 (electronic).

[9] Desch, W. and Miller, R.K. (1988). Exponential stabilization of Volterra integral equations with singular kernels. *J. Integral Equations Appl.* **1**, pp.397–433.

[10] El Karoui, N., Peng, S. and Quenez, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance*, **7**, n. 1, pp. 1–71.

[11] El Karoui, N. and Hamadène, S. (2003). BSDEs and risk-sensitive control, zero-sum and nonzerosum game problems of stochastic functional differential equations. *Stochastic Proc. Appl.*, **107**, pp. 145–169.

[12] Fuhrman, M. (2003). A class of stochastic optimal control problems in Hilbert spaces: BSDEs and optimal control laws, state constraints, conditioned processes. *Stochastic Process. Appl.* **108**, n. 2, pp. 263–298.

[13] Fuhrman, M., Hu, Y. and Tessitore, G. (2006). On a class of stochastic optimal control problems related to BSDEs with quadratic growth. *SIAM J. Control Optim.* **45**, 1279–1296 (electronic).

[14] Fuhrman, M. and Tessitore, G. (2002). Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *Ann. Probab.* **30**, n.3, pp.1397–1465.

[15] Fuhrman, M. and Tessitore, G. (2004). Existence of optimal stochastic controls and global solutions of forward-backward stochastic differential equations- *SIAM J. Control Optim.*, **43**, pp. 813–830.

[16] Hamadène, S. and Lepeltier, J.-P. (1995). Backward equations, stochastic control and zero-sum stochastic differential games. *Stochastics Stochastics Rep.*, **54**, pp. 221–231.

[17] Homan, K. W. (2003). *An Analytic Semigroup Approach to Convolution Volterra Equations.* Ph.D. Thesis, Delft University Press.

[18] Kobylanski, Y. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* **28**, n. 2, pp. 558–602.

[19] Miller, R.K. (1974). Linear Volterra integrodifferential equations as semigroups. *Funkcial. Ekvac.* **17**, pp.39–55.

[20] Pardoux, E. and Peng, S. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, **14**, pp.55–61.

[21] Prüss, J. (1993). *Evolutionary integral equations and applications.* Monographs in Mathematics, 87. Birkhuser Verlag, Basel.

[22] Yong, J. and Zhou, X. Y. (1999). *Stochastic controls.* Hamiltonian systems and HJB equations. Applications of Mathematics (New York), 43. Springer-Verlag, New York.