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Volterra equations perturbed by noise

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We consider a linear abstract Volterra integrodifferential equation in a Hilbert space, forced by a Gaussian process. The equation involves a completely monotone convolution kernel with a singularity at $t = 0$ and a sectorial linear spatial operator. Existence and uniqueness of a weak solution is established. Furthermore we give conditions such that the solution converges to a stationary process. Our method consists in a state space setting so that the corresponding solution process is Markovian, and the tools of linear analytic semigroup theory can be utilized.

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1. Introduction

Consider the following model of heat conduction in a material with memory, as proposed in [11], Section 5.3. Let $\theta(t, x)$ denote the temperature field at time t, for x in a bounded three dimensional body G with smooth boundary ∂G. Let $\varepsilon(t, x)$ the density of stored energy and let $q(t, x)$ denote the heat flux. Suppose that the flux obeys Fourier's law (all physical constants will be normalized to one in this introduction):

$$q(t, x) = \nabla\theta(t, x).$$

The stored energy, however, follows the temperature only with some delay

$$\varepsilon(t, x) = \int_{-\infty}^{t} dm(t - s)\theta(s)$$

where m is a creep function

$$m(t) = m_0 + \int_0^t m_1(s)\, ds$$

with $m_0 \geq 0$ and m_1 locally integrable, nonnegative and decreasing. In our example we let the instantaneous heat capacity $m_0 = 0$, and we choose

$$m_1(t) = \frac{t^{\rho-1}}{\Gamma(\rho)} e^{-t}$$

with some $\rho \in (0, 1/2)$. Assume that there exists a distributed random heat source $g(t, x)$ which we model as a stochastic process. For this purpose we introduce a cylindrical process W defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a separable Hilbert space U, mediated by a Hilbert Schmidt operator $\Phi : U \to H = L^2(G)$, and we let $g(t, x) = [\Phi\dot{W}](t, x)$. Energy conservation implies then

$$\frac{\partial}{\partial t}\varepsilon(t, x) = \operatorname{div} q(t, x) + [\Phi\dot{W}](t, x).$$

Combining all these equations we arrive at

$$\frac{\partial}{\partial t}\int_{-\infty}^{t} m_1(t-s)\theta(s, x) = \Delta\theta(x, s) + [\Phi\dot{W}](t, x)$$

with suitable boundary conditions, e.g. Neumann conditions, if the boundary is heat insulated.

In this paper we shall consider the following abstract stochastic Volterra equation on a Hilbert space H

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s = Au(t) + \Phi(t)\dot{W}(t) & t \geq 0, \\ u(t) = u_0(t) & t \leq 0. \end{cases} \tag{1.1}$$

Although more general assumptions may be given, compare Assumption 4.1, here we focus on the previous example. The kernel $a(t) = m_1(t)$ is completely monotone with $a \in L^1(0, \infty)$ and $a(0+) = +\infty$. Furthermore, A is selfadjoint negative definite, and for some real constants $\gamma \geq 0$, $0 \leq \beta < 1$ the operator $(-A)^{-\gamma-\beta/2}$ is Hilbert-Schmidt on H; we fix $\Phi(t) \equiv \Phi = (-A)^{-\gamma}R$, where $R : U \to H$ is a bounded operator.

Under the assumption of complete monotonicity of the kernel, a semigroup approach to a type of abstract integro-differential equations encountered in linear viscoelasticity was introduced in [5]. The idea to utilize this setting for stochastic equations is due to P. Clément, and a stochastic scalar equation analogous to (1.1) was first investigated in [8]. We extend the relevant machinery to the Hilbert space valued case and show that equation (1.1) is equivalent to an abstract stochastic evolution equation in a (different) Hilbert space X

$$\begin{cases} \mathrm{d}v(t) = Bv(t)\,\mathrm{d}t + (I-B)P\Phi(t)\,\mathrm{d}W(t), & t > 0, \\ v(0) = v_0 \in X. \end{cases} \tag{1.2}$$

To relate this system to Equation (1.1), v_0 is given suitably in terms of u_0, and the solution $u(t)$ is recovered from the state $v(t)$ by an operator $Jv(t) = u(t)$.

Under the special assumptions introduced above, the space operator B is the generator of an analytic semigroup on X, and P is a linear operator from H into the interpolation space X_θ, for arbitrary $\theta < \frac{2-\rho}{2}$. It is now a consequence of the readily available theory of stochastic differential equations with generators of analytic semigroups [4] that there exists a unique solution $v(t)$ to (1.2), and it is a mean square continuous process with values in the interpolation space X_η for any $\eta < \frac{1-\rho}{2}$.

Although J is unbounded on the state space X, it is bounded on interpolation spaces $J : X_\eta \to H_\mu$ with $\frac{1-2\eta}{1-2\mu} < 1-\rho$. Therefore, we arrive at the following result concerning the existence of a solution to problem (1.1), whose statement will be given in more generality in Theorem 3.7.

Theorem 1.1. *Let us fix the above assumptions:*
$a(t) = \frac{t^{\rho-1}}{\Gamma[\rho]} e^{-t}$ *with some* $\rho \in (0, 1/2)$, $\Phi(t) \equiv \Phi = (-A)^{-\gamma} R$, $u_0 \equiv 0$.
If $v(t)$ *denotes the mild solution of (1.2), the process*

$$u(t) = \begin{cases} Jv(t), & t \geq 0, \\ u_0(t), & t \leq 0. \end{cases} \tag{1.3}$$

is a weak solution to problem (1.1).

As a corollary to Lemma 2.29 and Theorem 3.3, we obtain the following result concerning regularity of the solution $u(t)$. In the statement below, $L^2_{\mathcal{F}}(0, \infty; X_\eta; 1)$ denotes the space of adapted processes on $\mathbb{R}^+$ with values in X_η that are mean square integrable with respect to the measure $e^{-t} dt$ on $[0, \infty)$.

Corollary 1.2. *Since* $v(t) \in L^2_{\mathcal{F}}(0, \infty; X_\eta; 1)$ *for* $\eta < \frac{1-\rho}{2}$, *the solution* $u(t)$ *of (1.1) belongs to* $L^2_{\mathcal{F}}(0, \infty; H_\mu; 1)$ *for* $\mu < \frac{1-2\rho}{2-2\rho}$.

Finally we analyze the longtime behaviour of the solution $v(t)$ of (1.2). The following result is proven, again in greater generality, in Theorem 4.5.

Theorem 1.3. *Assume that the assumptions of Theorem 1.1 holds, and let* $0 \leq \beta < 1$ *be such that* $(-A)^{-\gamma-\beta/2}$ *is an Hilbert-Schmidt operator on* H. *Then there exists (at least) one invariant measure for equation (1.2) which is concentrated on the space* X_η *for arbitrary* $\eta < (1-\beta)\frac{1-\rho}{2}$.

It is natural that we read this result in terms of (1.1). Let $\bar{v}$ be a stationary solution to problem (1.2); then we want to apply the operator J to $\bar{v}$ in order to get the corresponding solution $\bar{u}$ of (1.1). It happens that this is possible if $\bar{v}$ is suitably regular, which in turn becomes an assumption on β (but notice that we may control β – compare the assumptions in previous theorem – by means of γ).

Corollary 1.4. *With the notation of the theorem above, assume that* $\beta < \frac{1-2\rho}{1-\rho}$. *Let* $\bar{v}$ *be the stationary solution of* (1.2) *and define the stationary solution* $u(t) \equiv \bar{u} = J\bar{v}$ *as in Theorem 1.1. Then* $\bar{u}$ *is a square integrable random variable in* H_μ *for some* $\mu > 0$.

With the highly developed theory on Volterra equations (e.g. [11], [12]) available, semigroup methods are by no means the only approach to tackle this problem. However, the theory of stochastic processes governed by analytic semigroups is rich and convenient ([4]), and it is interesting to note that this tool can be applied to our problem with only a marginal loss of regularity. From a philosophical viewpoint, it seems satisfactory that the system may be associated with an internal state evolving as a Markov process. More important, we expect that the state space approach will be useful for more detailed investigations on the dynamics of semilinear modifications of the problem.

This paper is organized as follows: Section 2 is purely deterministic and developes the semigroup setting. In particular, all questions of regularity (in terms of interpolation spaces) are concentrated in this section. In Section 3 we prove the existence and uniqueness of solutions to the stochastic equation, while Section 4 is devoted to the investigation of the asymptotic stationary process.

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2. A semigroup setting for integral equations

2.1. **A deterministic integral equation in Hilbert space.** In this section we consider the following Volterra integro-differential equation in a Hilbert space:

$$\begin{cases} \frac{d}{dt}\int_{-\infty}^t a(t-s)u(s)\,ds = Au(t) + f(t), & t \geq 0 \\ u(t) = u_0(t) & t < 0. \end{cases} \tag{2.1}$$

Here u takes values in a separable Hilbert space $(H, |\cdot|_H)$. We make the following assumptions:

Assumption 2.1. *$A : D(A) \subset H \to H$ is a linear operator such that there is a constant $M > 0$ and an angle $\tau \in (0, \frac{\pi}{2})$ such that all $s \in \overline{\Sigma}_{\pi/2+\tau}$ are contained in the resolvent set of A with $\|R(s,A)\| \leq \frac{M}{|s|}$. Here $R(s,A) = (s-A)^{-1}$, and $\overline{\Sigma}_{\pi/2+\tau}$ is the closure of the sector*

$$\Sigma_{\pi/2+\tau} = \left\{ z \in \mathbb{C} \setminus \{0\} \mid |\arg(z)| < \frac{\pi}{2} + \tau \right\}.$$

Thus A generates an analytic semigroup e^{tA}. Interpolation and extrapolation spaces of H will always be constructed by interpolation or extrapolation with respect to A.

Assumption 2.2. *The convolution kernel $a : (0,\infty) \to \mathbb{R}$ is completely monotone, with $a(0+) = \infty$ and $\int_0^1 a(s)\,ds < \infty$.*

In particular, by Bernstein's Theorem there exists a measure ν on $[0,\infty)$ such that

$$a(t) = \int_{[0,\infty)} e^{-\kappa t}\,\nu(d\kappa). \tag{2.2}$$

¿From Assumption 2.2 we infer that $\nu([0,\infty)) = a(0+) = \infty$ while for $s > 0$

$$\hat{a}(s) = \int_{[0,\infty)} \frac{1}{s+\kappa}\,\nu(d\kappa) < \infty.$$

Here $\hat{a}$ denotes the Laplace transform of a.

We will also require assumptions on the singularity of a at $0+$, which will be given in terms of the following quantities:

Definition 2.3.

$$\alpha(a) = \sup\left\{ \rho \in (0,1) \mid \int_c^\infty t^{\rho-2}\frac{1}{\hat{a}(t)}\,dt < \infty \right\},$$
$$\delta(a) = \inf\left\{ \rho \in (0,1) \mid \int_0^c t^{\rho-1}a(t)\,dt < \infty \right\}.$$

Here $c > 0$. The definition is, in fact, independent of the choice of c.

The definition of $\alpha(a)$ is equivalent to Definition 3.2.2. in Homan [8], c.f. Clément and Desch [2]. If $a(t) = t^{\rho-1}$, then $\alpha(a) = \delta(a) = 1 - \rho$. It is always true that $\alpha(a) \leq \delta(a)$. One can, however, construct completely monotone kernels such that a strict inequality holds: $\alpha(a) < \delta(a)$.

Assumption 2.4. $\alpha(a) > \frac{1}{2}$.

Assumption 2.5. *The forcing function f is in $L^\infty(0,\infty; H_{-\sigma})$ with some $\sigma \in (0, \frac{1}{2})$.*

(Further restrictions on σ will follow later. For the meaning of $H_{-\sigma}$ see the following subsection.)

Assumption 2.6. *The initial function $u_0 : (-\infty, 0] \to H$ is measurable and satisfies one or several of the following conditions:*

(a) *There exist some $M, \omega > 0$ such that $|u_0(s)|_H \leq Me^{\omega s}$ for all $s \leq 0$.*
(b) *Moreover, for some (hence all) $c > 0$*

$$\int_0^c a''(t)|u_0(-t) - u_0(0)|_H^2 \, dt < \infty,$$

and $u_0(0) \in H_\gamma$ for some $\gamma \in (0, \frac{1}{2})$.
(c) *Moreover, $u_0(0) \in D(A)$ and*

$$\int_0^\infty (-a'(t))(u_0(-t) - u_0(0)) \, dt + Au_0(0) = 0.$$

(Notice: Condition (c) includes Condition (b), which in turn includes Condition (a).)

(Notice that the positivity of ω in Assumption 2.6(a) means that u decays exponentially as $s \to -\infty$. We remark also that the integral in (c) exists if (b) holds. Further restrictions on γ will follow later. For the meaning of H_γ see the following subsection.)

With these assumptions we can prove existence and uniqueness of a solution to (2.1) in the following sense:

Definition 2.7. *By a weak solution to* (2.1) *we mean a function $u : (-\infty, \infty) \to H$ such that*

(a) *$u(t) = u_0(t)$ for $t \leq 0$.*
(b) *u is continuous on $[0, \infty)$.*
(c) *For all $\zeta \in D(A^*)$ and all $t > 0$, the following equation holds*

$$\int_{-\infty}^t \langle a(t-s)u(s)\, ds, \zeta\rangle_H = \langle \bar{u}, \zeta\rangle_H + \int_0^t \langle u(s), A^*\zeta\rangle_H \, ds + \int_0^t \langle f(s), \zeta\rangle_H \, ds, \tag{2.3}$$

$$\text{where} \qquad \bar{u} = \int_{-\infty}^0 a(-s)u(s)\, ds.$$

With suitable regularity conditions we can establish existence and uniqueness of weak solutions:

Theorem 2.8. *Suppose that Assumptions 2.1, 2.2 hold. Let $\alpha(a)$ and $\delta(a)$ be defined by Definition 2.3. Let $\gamma, \sigma, \mu \in (0, \frac{1}{2})$ and suppose that Assumption 2.5 holds with σ, and Assumption 2.6(b) holds with γ. Moreover, suppose that $\delta(a)(1 - 2\gamma) < \alpha(a)(1 - 2\mu)$.*
Then there exists a unique weak solution $u : \mathbb{R} \to H_\mu$ of (2.1).

Finally we state a theorem about regularity of the solution, dependent of the regularity of the forcing term. For simplicity we state the result with initial function $u_0 = 0$.

Theorem 2.9. *Suppose that Assumptions 2.1, 2.2 hold. Let $u_0 = 0$. Let $\alpha(a)$ and $\delta(a)$ be defined by Definition 2.3. Let $\sigma, \mu \in (0, \frac{1}{2})$ and suppose that $f \in L^p(0, \infty, X_{-\sigma})$ with some $p \in [1, \infty)$. Choose any $\tau < \alpha(a)(1 - \sigma - \mu)$. Let*
Then a mild solution u of (2.1) can be defined by approximation: Let $f_n \in L^\infty(0, \infty, X_{-\sigma})$ and $f_n \to f$ in $L^p(0, \infty, H_{-\sigma})$. Let u_n be the weak solution of (2.1) with f_n instead of f. Then u_n converges to some u in the following function spaces:

(a) *If $p \in [1, \frac{1}{\tau})$, then $u \in L^q(0, T, H_\mu)$ for all $q \in [1, \frac{p}{1-\tau p})$.*
(b) *If $p = \frac{1}{\tau}$, then $u \in L^q(0, T, H_\mu)$ for all $q \in [1, \infty)$.*
(c) *If $p \in (\frac{1}{\tau}, \infty)$, then $u \in \mathcal{C}^{\tau - 1/p}([0, T], H_\mu)$.*

These remainder of this section will be devoted to the technical details for proving the two theorems.

2.2. **Intermediate spaces.** Much of our work relies on the machinery of analytic semigroups and interpolation spaces, therefore we introduce the basic notations:

Let $(X, \|\cdot\|)$ be a Hilbert space and B be an operator, such that for some $\omega > 0$ the operator $B - \omega$ is a sectorial operator on X of negative type. Therefore, $B - \omega$ is invertible with $(B - \omega)^{-1} \in \mathcal{L}(X)$. On the domains $D(B^n)$ we define the n-norm

$$\|x\|_n := \|(B - \omega)^n x\|,$$

and call $X_n := (D(B^n), \|\cdot\|_n)$ the Sobolev space of order n associated to B. We also define the extrapolation space X_{-1} as the completion of X under the norm $\|x\|_{-1} = \|(B - \omega)^{-1} x\|$. The operator $B_{-1} : X \to X_{-1}$ is the unique extension of $B : X_1 \to X$ such that $B_{-1} - \omega$ is an isometry from X onto X_{-1}.

Definition 2.10. *Let $\theta \in (0, 1)$.*

(a) *By X_θ we denote the real interpolation space $(X, X_1)_{(\theta,2)}$ between X and $X_1 = D(B)$.*
(b) *By $X_{\theta-1}$ we denote the real interpolation space $(X_{-1}, X)_{(\theta,2)}$ between the extrapolation space X_{-1} and X.*

By B_θ we denote the restriction of B as an operator from $D(B_\theta) = (X_1, X_2)_{(\theta,2)}$ into X_θ.

Remark 2.11. *Let $x \in X$ and $\theta \in (0, 1)$. Then $x \in X_\theta$ if either one of the following equivalent norms is finite:*

$$[\![x]\!]_\theta^2 = \int_0^\infty t^{1-2\theta} \|(B - \omega) e^{-\omega t} e^{tB} x\|^2 \, dt,$$
$$\|x\|_\theta^2 = \int_c^\infty t^{2\theta - 1} \|(B - \omega) R(t, B) x\|^2 \, dt.$$

(The constant $c > \omega$ in the second integral can be taken arbitrary.)

Remark 2.12. *Let $\theta \in (0, 1)$. The operator B_θ is the generator of an analytic semigroup e^{tB_θ} on X_θ. In fact, e^{tB_θ} is the restriction of e^{tB} to the invariant subspace X_θ.*

(These results, and more information on interpolation spaces and analytic semigroups, can be found in Lunardi [9], Chapter 2.2.)

Lemma 2.13. *Let $\theta \in (0, 1)$. Then*

$$X_\theta = (X_{-1}, X_1)_{((\theta+1)/2,2)},$$
$$X_{\theta-1} = (X_{-1}, X_1)_{(\theta/2,2)}.$$

Proof. Let $x \in D(B)$. Then

$$x = (t-\omega)[1 + (\omega - t)(t-B)^{-1}](\omega - B)^{-1}x + (t-B)^{-1}(\omega - B)x, \tag{2.4}$$

therefore with a suitable constant M,

$$\begin{aligned} \|x\| &\le M\left[(t-\omega)\|(\omega-B)^{-1}x\| + \frac{1}{t-\omega}\|(\omega-B)x\|\right] \\ &= M\left[(t-\omega)\|x\|_{-1} + \frac{1}{t-\omega}\|x\|_1\right]. \end{aligned}$$

Taking $t-\omega = \sqrt{\|x\|_1/\|x\|_{-1}}$ we obtain $\|x\| \le 2M\sqrt{\|x\|_{-1}\|x\|_1}$. This is condition J from the reiteration theorem, and we obtain

$$\begin{aligned} (X, X_1)_{(\theta,2)} &\subset (X_{-1}, X_1)_{(\frac{1}{2}(1-\theta)+1\theta,2)} = (X_{-1}, X_1)_{(\frac{1+\theta}{2},2)}, \\ (X_{-1}, X)_{(\theta,2)} &\subset (X_{-1}, X_1)_{(0(1-\theta)+\frac{1}{2}\theta,2)} = (X_{-1}, X_1)_{(\frac{\theta}{2},2)}. \end{aligned}$$

On the other hand, decompose any $x \in X$ according to (2.4) (replacing B by B_{-1} wherever necessary). Then

$$\begin{aligned} &\|(t-\omega)[1 + (\omega-t)(t-B)^{-1}](\omega-B)^{-1}x\|_1 \le Mt\|(\omega-B)^{-1}x\|_1 = Mt\|x\|, \\ &\|(t-B)^{-1}(\omega-B_{-1})x\|_{-1} \le \frac{M}{t-\omega}\|(\omega-B_{-1})x\|_{-1} = \frac{M}{t-\omega}\|x\|. \end{aligned}$$

Now let $s > (\omega+1)^2$ and put $t = \sqrt{s}$ to see that

$$s^{-1/2}\|(t-\omega)[1 + (\omega-t)(t-B)^{-1}](\omega-B)^{-1}x\|_1 + \|(t-B)^{-1}(\omega-B_{-1})x\|_{-1} \le Ms^{1/2}.$$

Consequently, condition K from the reiteration theorem is satisfied and

$$\begin{aligned} (X, X_1)_{(\theta,2)} &\supset (X_{-1}, X_1)_{(\frac{1}{2}(1-\theta)+1\theta,2)} = (X_{-1}, X_1)_{(\frac{1+\theta}{2},2)}, \\ (X_{-1}, X)_{(\theta,2)} &\supset (X_{-1}, X_1)_{(0(1-\theta)+\frac{1}{2}\theta,2)} = (X_{-1}, X_1)_{(\frac{\theta}{2},2)}. \end{aligned}$$

□

Lemma 2.14. *Let $-1 \le \theta < \mu < \eta \le 1$ with $\theta, \eta, \mu \ne 0$. Then*

$$X_\mu = (X_\theta, X_\eta)_{(\gamma,2)} \quad \textit{with} \quad \gamma = \frac{\mu-\theta}{\eta-\theta}.$$

Proof. Using Lemma 2.13 we see that this is just a straightforward application of the reiteration theorem of interpolation. □

Lemma 2.15. *Let $J_0 : D(B) \to H$ be an operator, relatively bounded with respect to B. (Notice that then $J_0R(s,B)$ is a bounded linear operator from X to H.) Suppose that for some $c > \omega$ and some $\eta \in (0,1)$*

$$\int_c^\infty s^{1-2\eta}\|J_0R(s,B)\|^2_{X\to H}\,ds < \infty.$$

Then J_0 admits a continuous extension as an operator $J : X_\eta \to H$. Moreover, for $x \in X_\eta$ we have $Jx = \lim_{t\to\infty} tJ_0R(t,B)x$.

Proof. We show that $tJ_0R(t,B)$ is uniformly bounded for $t > 2\omega$ as an operator from X_η into H. The remainder follows from standard arguments. Thus, let $x \in X_\eta$. Notice that

$$\frac{d}{dt}(t-\omega)R(t,B)x = R(t,B)x - (t-\omega)R(t,B)^2x = (\omega - B)R(t,B)^2x.$$

Thus, for $t > 2\omega$

$$\begin{aligned}
&\frac{1}{2}|tJ_0R(t,B)x|_H \le |(t-\omega)J_0R(t,B)x|_H \\
=&|(c-\omega)J_0R(c,B)x + \int_c^t J_0R(s,B)(\omega-B)R(s,B)x\,ds|_H \\
\le&(c-\omega)\|J_0R(s,B)\|\,\|x\|_X \\
&+\int_c^\infty s^{1/2-\eta}\|J_0R(s,B)\|_{X\to H}\,s^{\eta-1/2}\|(\omega-B)R(s,B)x\|_X\,ds \\
\le&(c-\omega)\|J_0R(s,B)\|_{X\to H}\,\|x\|_X \\
&+\left[\int_c^\infty s^{1-2\eta}\|J_0R(s,B)\|^2_{X\to H}\,ds\right]^{1/2}\left[\int_c^\infty s^{2\eta-1}\|(\omega-B)R(s,B)x\|^2_X\,ds\right]^{1/2} \\
\le&M(\|x\|_X + \left[\int_c^\infty s^{2\eta-1}\|(\omega-B)R(s,B)x\|^2_X\,ds\right]^{1/2}) \le M_1\|x\|_\eta
\end{aligned}$$

with suitable constants M, M_1. □

2.3. **Setting up the infinitesimal generator.** We return now to Equation (2.1). Using Bernstein's Theorem (2.2) we rewrite formally

$$\int_{-\infty}^t a(t-s)u(s)\,ds = \int_{-\infty}^t \int_{[0,\infty)} e^{-\kappa(t-s)}\,\nu(d\kappa)\,u(s)ds = \int_{[0,\infty)} v(t,\kappa)\,\nu(d\kappa)$$

with

$$v(t,\kappa) = \int_{-\infty}^t e^{-\kappa(t-s)}u(s)\,ds. \tag{2.5}$$

Formal differentiation yields

$$\frac{\partial}{\partial t}v(t,\kappa) = -\kappa v(t,\kappa) + u(t), \tag{2.6}$$

while the integral equation (2.1) can be rewritten

$$\int_{[0,\infty)} (-\kappa v(t,\kappa) + u(t))\,\nu(d\kappa) = Au(t) + f(t). \tag{2.7}$$

As an initial condition we obtain

$$v(0,\kappa) = \int_{-\infty}^0 e^{\kappa s}u_0(s)\,ds. \tag{2.8}$$

In our setting, the function $v(t,\cdot)$ will be considered the state of the system, contained in a suitable function space. Equation (2.6) is a differential equation for the state, so it will be used to set up an infinitesimal generator of a semigroup. For the homogeneous case ($f = 0$), the rewritten integral equation (2.7) will contribute to the definition of the domain of the generator.

Definition 2.16. *For a Borel measurable function $\tilde{x} : [0, \infty) \to H$ we consider the seminorm (finite or infinite):*

$$\|\tilde{x}\|_X^2 := \int_{[0,\infty)} (\kappa + 1)|\tilde{x}(\kappa)|_H^2 \, \nu(d\kappa)$$

and the function space

$$\tilde{X} := \{\tilde{x} : [0, \infty) \to H, \text{measurable} \mid \|\tilde{x}\|_X < \infty\} .$$

The state space X consists of all equivalence classes in $\tilde{X}$ with respect to equality almost everywhere in ν.

Remark 2.17.

(a) *In the sequel, as usual, we will not distinguish between functions and their equivalence classes.*
(b) *For all $x \in X$, the integral $\int_{[0,\infty)} |x(\kappa)|_H \, \nu(d\kappa)$ is finite.*

Proof. We use the Cauchy Schwarz Inequality:

$$\int_{[0,\infty)} |x(\kappa)|_H \, \nu(d\kappa) = \int_{[0,\infty)} (\kappa + 1)^{-1/2}(\kappa + 1)^{1/2}|x(\kappa)|_H \, \nu(d\kappa)$$

$$\leq \left[\int_{[0,\infty)} (\kappa + 1)^{-1} \, \nu(d\kappa)\right]^{1/2} \left[\int_{[0,\infty)} (\kappa + 1)|x(\kappa)|_H^2 \, \nu(d\kappa)\right]^{1/2} .$$

□

Definition 2.18. *We define a linear operator $J_0 : D(J_0) \subset X \to X$ by*

$$\begin{aligned} D(J_0) &= \{x \in X \mid (\exists u \in H) : -\kappa x(\kappa) + u \in X\} , \\ J_0 x &= u \quad \text{as above.} \end{aligned}$$

Remark 2.19. *J_0 is well-defined as a single valued operator.*

Proof. Suppose there are u_1, $u_2 \in H$ such that both, $-\kappa x + u_1$ and $-\kappa x + u_2$ are contained in X. Then the constant function $u_1 - u_2$ is contained in X. However, if $u_1 \neq u_2$, then

$$\|u_1 - u_2\|_X^2 = \int_{[0,\infty)} (\kappa + 1)|u_1 - u_2|_H^2 \, \nu(d\kappa) = \infty$$

since the measure ν is infinite on $[0, \infty)$. □

Definition 2.20. *We define a linear operator $B : D(B) \subset X \to X$ by*

$$\begin{aligned} D(B) &= \left\{ x \in D(J_0) \mid J_0 x \in D(A), \int_{[0,\infty)} (-\kappa x(\kappa) + J_0 x) \, \nu(d\kappa) = AJ_0 x \right\} , \\ (Bx)(\kappa) &= -\kappa x(\kappa) + J_0 x. \end{aligned}$$

By this definition, the problem (2.6), (2.7) for the homogeneous case $f = 0$ is rewritten as

$$\frac{d}{dt} v(t, \cdot) = Bv(t, \cdot).$$

In fact, we will see in the next subsection, that B is the generator of an analytic semigroup, so that a weak solution of the homogeneous problem can be obtained by semigroup methods.

To take care of the inhomogeneous problem, compare the rewritten integral equation (2.7) with the definition of $D(B)$:

$$\int_{[0,\infty)} (-\kappa v(t,\kappa) + u(t))\,\nu(d\kappa) = Au(t) + f(t),$$
$$\int_{[0,\infty)} (-\kappa v(t,\kappa) + J_0 v(t))\,\nu(d\kappa) = Au(t) \text{ if } v(t) \in D(B).$$

We see that the forcing function does not enter simply as an additive perturbation, but it acts by shifting the domain of the generator. To deal with this situation, we use a standard procedure from control theory. Our aim is to rewrite (2.7) in the form

$$\frac{d}{dt} v(t,\cdot) = B(v(t,\cdot) - Pf(t)) + Pf(t). \tag{2.9}$$

This works out formally with the following

Definition 2.21. *We define an operator $P : H_{-1} \to X$ by*

$$(Pf)(\kappa) = \frac{1}{1+\kappa} R(\hat{a}(1), A) f.$$

The following lemma guarantees that P is suitable to rewrite (2.7) in the form (2.9):

Lemma 2.22. *Let $f \in H_{-1}$, $x \in X$. Then*

(a) $Pf \in D(J_0)$ *with* $J_0 Pf = R(\hat{a}(1), A) f$.
(b) $-\kappa (Pf)(\kappa) + J_0 Pf - (Pf)(\kappa) = 0$.
(c) $\int_{[0,\infty)} (-\kappa (Pf)(\kappa) + J_0 Pf)\,\nu(d\kappa) = AJ_0 Pf + f$.
(d) $x - Pf \in D(B)$ *iff* $x \in D(J_0)$ *and* $\int_{[0,\infty)} (-\kappa x(\kappa) + J_0(x))\,\nu(d\kappa) = AJ_0 x + f$. *In this case* $-\kappa x + J_0 x = [B(x - Pf) + Pf](\kappa)$.

Proof. It is easily seen that $\|Pf\|_X < \infty$. Moreover

$$\begin{aligned} &- \kappa (Pf)(\kappa) + R(\hat{a}(1), A) f = -\frac{\kappa}{1+\kappa} R(\hat{a}(1), A) f + R(\hat{a}(1), A) f \\ =&\frac{1}{1+\kappa} R(\hat{a}(1), A) f = (Pf)(\kappa). \end{aligned}$$

Therefore, $J_0 Pf = R(\hat{a}(1), A) f$, which implies (a), and (b) holds. To prove (c) we compute (using (a) and (b))

$$\begin{aligned} &\int_{[0,\infty)} (-\kappa (Pf)(\kappa) + J_0 Pf)\,\nu(d\kappa) - AJ_0 Pf \\ =&\int_{[0,\infty)} Pf(\kappa)\,\nu(d\kappa) - AJ_0 Pf \\ =&\int_{[0,\infty)} \frac{1}{\kappa+1} R(\hat{a}(1), A) f\,\nu(d\kappa) - AR(\hat{a}(1), A) f \\ =&\hat{a}(1) R(\hat{a}(1), A) f - AR(\hat{a}(1), A) f = f. \end{aligned}$$

Finally, (d) is a straightforward application of (a), (b), (c). □

2.4. **Generation of a semigroup and estimates on the resolvent.** The state space and the generator B look very large and complicated. However, the resolvent $(s-B)^{-1}$ can be computed explicitely in terms of the resolvent of A. This will finally allow to prove that B generates a semigroup and to characterize the interpolation spaces X_θ with respect to B.

Lemma 2.23. *Let $s \in \Sigma_{\pi/2+\tau}$. Then $s\hat{a}(s) \in \Sigma_{\pi/2+\tau}$. Moreover, s lies in the resolvent set of B. The resolvent $R(s,B) = (s-B)^{-1}$ is given by*

$$[R(s,B)x](\kappa) = \frac{1}{\kappa+s}\left[x(\kappa)+u\right], \quad \textit{with} \tag{2.10}$$

$$u = J_0 R(s,B)x = R(s\hat{a}(s),A)\int_{[0,\infty)} \frac{\kappa}{\kappa+s} x(\kappa)\,\nu(d\kappa). \tag{2.11}$$

Proof. Let $s \in \Sigma_{\pi/2+\tau}$. Without loss of generality let $\arg(s) \geq 0$. Since $\hat{a}(s) = \int_{[0,\infty)} \frac{1}{s+\kappa}\,\nu(d\kappa)$, we have that $-\arg(s) < \arg(\hat{a}(s)) < 0$. Consequently, $s\hat{a}(s) \in \Sigma_{\pi/2+\tau}$ and $R(s\hat{a}(s),A) = (s\hat{a}(s)-A)^{-1}$ exists as a bounded operator on H. We have to solve $y = R(s,B)x$, i.e., $(s-B)y = x$ and find $u = J_0 y$. The former is equivalent to

$$sy(\kappa) + \kappa y(\kappa) - u = x(\kappa), \tag{2.12}$$

$$\int_{[0,\infty)} (-\kappa y(\kappa) + u)\,\nu(d\kappa) = Au. \tag{2.13}$$

¿From (2.12) we obtain immediately (2.10). Inserting this expression into (2.13) we obtain

$$\begin{aligned} Au &= \int_{[0,\infty)} \left[-\kappa\left(\frac{1}{s+\kappa}(x(\kappa)+u)\right) + u\right]\nu(d\kappa) \\ &= \int_{[0,\infty)} \frac{-\kappa}{s+\kappa} x(\kappa)\,\nu(d\kappa) + \int_{[0,\infty)} \frac{s}{s+\kappa}\,\nu(d\kappa)u \\ &= \int_{[0,\infty)} \frac{-\kappa}{s+\kappa} x(\kappa)\,\nu(d\kappa) + s\hat{a}(s)u. \end{aligned}$$

¿From this we infer (2.11). □

Our next aim is to get estimates on the resolvent.

Lemma 2.24. *For each $\epsilon > 0$ there exists $M_1 > 0$, such that for all $s \in \Sigma_{\pi/2+\tau}$ with $|s| \geq \epsilon$ we have the estimate*

$$\int_{[0,\infty)} \frac{\kappa+1}{|\kappa+s|^2}\,\nu(d\kappa) \leq M_1 |\hat{a}(s)|. \tag{2.14}$$

Proof. Without loss of generality we assume $\epsilon < 1$. Let $s = \rho + i\sigma$. First let $\sigma = 0$, then $s = \rho \geq \epsilon$. This implies that

$$\frac{\kappa+1}{|\kappa+s|^2} \leq \frac{1}{\epsilon}\frac{\kappa+s}{|\kappa+s|^2} = \frac{1}{\epsilon}\frac{1}{\kappa+s}.$$

Taking integrals we obtain (2.14). Now let (without loss of generality) $\sigma > 0$. Notice that

$$\hat{a}(s) = \int_{[0,\infty)} \frac{\kappa+\rho}{|\kappa+s|^2}\,\nu(d\kappa) - i\int_{[0,\infty)} \frac{\sigma}{|\kappa+s|^2}\,\nu(d\kappa).$$

Thus

$$\begin{aligned}\Re\hat{a}(s) &= \int_{[0,\infty)} \frac{\rho - \epsilon/2}{|\kappa+s|^2}\,\nu(d\kappa) + \int_{[0,\infty)} \frac{\kappa + \epsilon/2}{|\kappa+s|^2}\,\nu(d\kappa) \\ &= \frac{\rho-\epsilon/2}{\sigma}|\Im\hat{a}(s)| + \frac{\epsilon}{2}\int_{[0,\infty)} \frac{2\kappa/\epsilon+1}{|\kappa+s|^2}\,\nu(d\kappa).\end{aligned}$$

Thus

$$\Re\hat{a}(s) \geq \frac{\rho-\epsilon/2}{\sigma}|\Im\hat{a}(s)| + \frac{\epsilon}{2}\int_{[0,\infty)} \frac{\kappa+1}{|\kappa+s|^2}\,\nu(d\kappa). \tag{2.15}$$

We distinguish three cases: If $\rho \geq \epsilon/2$, (2.15) yields immediately

$$\frac{\epsilon}{2}\int_{[0,\infty)} \frac{\kappa+1}{|\kappa+s|^2}\,\nu(d\kappa) \leq \Re\hat{a}(s) \leq |\hat{a}(s)|.$$

If $|\rho| \leq \epsilon/2$, then $|\sigma| \geq \epsilon/2$ since $|s| \geq \epsilon$. Thus $\sigma^{-1}|\rho - \epsilon/2| \leq 2$ and (2.15) yields

$$\frac{\epsilon}{2}\int_{[0,\infty)} \frac{\kappa+1}{|\kappa+s|^2}\,\nu(d\kappa) \leq \Re\hat{a}(s) + 2|\Im\hat{a}(s)|.$$

Finally, if $\rho < -\epsilon/2$, then $|\rho - \epsilon/2| \leq 2|\rho|$. We use the fact that $s \in \Sigma_{\pi/2+\tau}$, so that $\sigma > |\rho|\cot(\tau)$. Thus $\sigma^{-1}|\rho-\epsilon/2| \leq 2\tan(\tau)$. (2.15) yields

$$\frac{\epsilon}{2}\int_{[0,\infty)} \frac{\kappa+1}{|\kappa+s|^2}\,\nu(d\kappa) \leq \Re\hat{a}(s) + 2\tan(\tau)|\Im\hat{a}(s)|.$$

□

Lemma 2.25. *For each $\epsilon > 0$ there exists $M_2 > 0$, such that for all $s \in \Sigma_{\pi/2+\tau}$ with $|s| \geq \epsilon$ we have the estimate*

$$\|R(s,B)x\|_X \leq \frac{M_2}{|s|}\|x\|_X.$$

In particular, B is the generator of a holomorphic semigroup $\{e^{tB} \mid t \in \Sigma_\tau\}$. For all $\epsilon > 0$, we have that $\lim_{t\to\infty} e^{-\epsilon t}\|e^{tB}\| = 0$.

Proof. $R(s,B)x$ is given explicitely in Lemma 2.23. Since $\tau < \pi/2$, there exists a constant M_3 such that $|s+\kappa| \geq M_3|s|$ for all $\kappa > 0$ and all $s \in \Sigma_{\pi/2+\tau}$. We infer immediately that

$$\|\frac{1}{\kappa+s}x(\kappa)\|_X \leq \frac{1}{M_3|s|}\|x\|_X.$$

To estimate $\|(\kappa+s)^{-1}u\|_X$, we start with estimating

$$\begin{aligned}&|\int_{[0,\infty)} \frac{\kappa}{\kappa+s}x(\kappa)\,\nu(d\kappa)|_H \leq \int_{0,\infty)} \frac{\kappa}{|\kappa+s|\sqrt{\kappa+1}}\sqrt{\kappa+1}|x(\kappa)|_H\,\nu(d\kappa) \\ &\leq \left[\int_{[0,\infty)} \frac{\kappa^2}{|\kappa+s|^2(\kappa+1)}\,\nu(d\kappa)\right]^{1/2}\left[\int_{[0,\infty)} (\kappa+1)|x(\kappa)|_H^2\,\nu(d\kappa)\right]^{1/2} \\ &\leq \left[\int_{[0,\infty)} \frac{\kappa+1}{|\kappa+s|^2}\,\nu(d\kappa)\right]^{1/2}\|x\|_X \leq M_1|\hat{a}(s)|^{1/2}\|x\|_X.\end{aligned}$$

For the last inequality we have used Lemma 2.24. Using the fact that $s\hat{a}(s) \in \Sigma_{\pi/2+\tau}$ and assumption 2.1 we have

$$\begin{aligned}|u|_H &= |R(s\hat{a}(s), A) \int_{[0,\infty)} \frac{\kappa}{\kappa + s} x(\kappa)\, \nu(d\kappa)|_H \\ &\le \frac{M}{|s\hat{a}(s)|} M_1 |\hat{a}(s)|^{1/2} \|x\|_X = \frac{M_4}{|s||\hat{a}(s)|^{1/2}} \|x\|_X.\end{aligned}$$

We use again Lemma 2.24 to estimate

$$\begin{aligned}\|\frac{1}{\kappa + s} u\|_X^2 &= \int_{[0,\infty)} \frac{\kappa + 1}{|\kappa + s|^2} |u|_H^2\, \nu(d\kappa) \\ &\le M_1 |\hat{a}(s)||u|_H^2 \le M_5 \frac{1}{|s|^2} \|x\|_X^2.\end{aligned}$$

□

Remark 2.26. *For convenience we will use $\epsilon = 1$ as a growth bound for e^{tB}, although Lemma 2.25 above show that arbitrarily small $\epsilon > 0$ could be taken.*

2.5. **Handling the forcing term: Properties of P and interpolation spaces.** We begin with some identities concerning the operator P defined in Definition 2.21.

Lemma 2.27. *Let $s \in \Sigma_{\pi/2+\tau}$, $s \neq 1$, $u \in H$. We have*

$$[(1 - B)R(s, B)Pu](\kappa) = \frac{1}{s + \kappa} R(s\hat{a}(s), A)u, \tag{2.16}$$

$$J_0(1 - B)R(s, B)Pu = R(s\hat{a}(s), A)u, \tag{2.17}$$

Moreover, for each $\epsilon > 0$ there exists M_1, such that

$$\|(1 - B)R(s, B)Pu\|_X^2 \le M_1 |\hat{a}(s)|\, |R(s\hat{a}(s), A)u|_H^2. \tag{2.18}$$

whenever $s \in \Sigma_{\pi/2+\tau}$, $s \neq 1$, $|s| \ge \epsilon$.

Proof. By definition

$$[(s - B)R(s, B)Pu](\kappa) = [Pu](\kappa) = \frac{1}{1 + \kappa} R(\hat{a}(1), A)u.$$

To compute $R(s, B)Pu$ using Lemma 2.23 we start with

$$\begin{aligned}v :=& R(s\hat{a}(s), A) \int_{[0,\infty)} \frac{\kappa}{s + \kappa} Pu(\kappa)\, \nu(d\kappa) \\ =& \int_{[0,\infty)} \frac{\kappa}{(s + \kappa)(1 + \kappa)}\, \nu(d\kappa) R(s\hat{a}(s), A) R(\hat{a}(1), A)u \\ =& \int_{[0,\infty)} \frac{1}{1 - s} \left[\frac{1}{1 + \kappa} - \frac{s}{s + \kappa} \right] \nu(d\kappa) R(s\hat{a}(s), A) R(\hat{a}(1), A)u \\ =& \frac{\hat{a}(1) - s\hat{a}(s)}{1 - s} \frac{1}{\hat{a}(1) - s\hat{a}(s)} \left[R(s\hat{a}(s), A)u - R(\hat{a}(1), A)u) \right] \\ =& \frac{1}{1 - s} \left[R(s\hat{a}(s), A)u - R(\hat{a}(1), A)u) \right].\end{aligned}$$

Therefore,

$$\begin{aligned}[R(s,B)Pu](\kappa) =& \frac{1}{\kappa+s}\left[\frac{1}{1+\kappa}R(\hat a(1),A)u + \frac{1}{1-s}\left(R(s\hat a(s),A)u - R(\hat a(1),A)u\right)\right] \\ =& \frac{1}{(1+\kappa)(s-1)}R(\hat a(1),A)u - \frac{1}{(s+\kappa)(s-1)}R(s\hat a(s),A)u.\end{aligned}$$

Now we take a linear combination

$$\begin{aligned}&[(1-B)R(s,B)Pu](\kappa) = [(s-B)R(s,B)Pu - (s-1)R(s,B)Pu](\kappa) \\ =& \frac{1}{1+\kappa}R(\hat a(1),A)u - \frac{1}{1+\kappa}R(\hat a(1),A)u + \frac{1}{s+\kappa}R(s\hat a(s),A)u \\ =& \frac{1}{s+\kappa}R(s\hat a(s),Au).\end{aligned}$$

This shows (2.16). To prove (2.17), notice that

$$-\kappa[(1-B)R(s,B)Pu](\kappa) + R(s\hat a(s),A)u = \frac{s}{s+\kappa}R(s\hat a(s),A)u$$

which determines a function in X. Finally, (2.18) follows from

$$\begin{aligned}&\|(1-B)R(s,B)Pu\|_X^2 = \int_{[0,\infty)} \frac{\kappa+1}{|\kappa+s|^2}\,\nu(d\kappa)|R(s\hat a(s),A)u|_H^2 \\ \le& |\hat a(s)|\,|R(s\hat a(s),A)u|_H^2\end{aligned}$$

by Lemma 2.24. □

Lemma 2.28. *Consider $\alpha(a)$ given by Definition 2.3.*

(a) *If $\theta \in (1/2,1)$ and $\alpha(a) > 2\theta - 1$, then P is a continuous linear operator from H into X_θ.*
(b) *If $\theta \in (1/2,1)$, $\sigma \in (0,1/2)$, and $\alpha(a) > \frac{2\theta-1}{1-2\sigma}$, then P is a continuous linear operator from the extrapolation space $H_{-\sigma}$ into the interpolation space X_θ.*
(c) *If $\theta \in (0,1/2)$, $\sigma \in (1/2,1)$, and $\delta(a) < \frac{1-2\theta}{2\sigma-1}$, then P is a continuous linear operator from the extrapolation space $H_{-\sigma}$ into the interpolation space X_θ.*

Proof. To prove (a), we pick $c > 1$ and estimate

$$\begin{aligned}&\int_c^\infty s^{2\theta-1}\|(1-B)R(s,B)Pu\|_X^2\,ds \le M\int_c^\infty s^{2\theta-1}\hat a(s)|R(s\hat a(s),A)u|_H^2\,ds \\ \le& M\int_c^\infty s^{2\theta-1}\hat a(s)\left(\frac{M}{s\hat a(s)}\right)^2|u|_H^2 = M\int_c^\infty s^{2\theta-3}\frac{1}{\hat a(s)}\,ds\,|u|_H^2.\end{aligned}$$

The latter integral is finite if $2\theta - 1 > \alpha(a)$.
To prove (b) let $u \in H_{-\sigma}$. Then (by standard theory of analytic semigroups)

$$|R(s\hat a(s),A)u|_H \le M(s\hat a(s))^{\sigma-1}|u|_{-\sigma}.$$

Repeating the same estimates as above we obtain

$$\int_c^\infty s^{2\theta-1}\|(1-B)R(s,B)Pu\|_X^2\,ds \le M\int_c^\infty s^{2\theta-3+2\sigma}\hat a(s)^{2\sigma-1}\,ds\,|u|_{-\sigma}^2. \tag{2.19}$$

Now choose some $\rho \in ((2\theta-1)/(1-2\sigma), \alpha(a))$, put $p = 1/(1-2\sigma)$ and $q = 1/(2\sigma)$, so $p^{-1}+q^{-1} = 1$. The integral in (2.19) can be rewritten and estimated by Hölder's inequality:

$$\begin{aligned}
&\int_c^\infty s^{2\theta-3+2\sigma} \hat{a}(s)^{2\sigma-1}\, ds \\
&= \int_c^\infty s^{2\theta-3+2\sigma+(2-\rho)(1-2\sigma)} \left(\frac{s^{\rho-2}}{\hat{a}(s)}\right)^{1-2\sigma} ds \\
&\le \left[\int_c^\infty s^{[2\theta-3+2\sigma+(2-\rho)(1-2\sigma)]q}\, ds\right]^{1/q} \left[\int_c^\infty \left(\frac{s^{\rho-2}}{\hat{a}(s)}\right)^{(1-2\sigma)p} ds\right]^{1/p}.
\end{aligned}$$

The second integral in this estimate is just

$$\int_c^\infty s^{\rho-2} \frac{1}{\hat{a}(s)}\, ds$$

which is finite since $\rho < \alpha(a)$. The first integral is finite iff

$$\begin{aligned}
&[2\theta-3+2\sigma+(2-\rho)(1-2\sigma)]q < -1, \\
\text{i.e. }\ & 2\theta-3+2\sigma+(2-\rho)(1-2\sigma) < -2\sigma, \\
\text{i.e. }\ & 2\theta-1 < \rho(1-2\sigma),
\end{aligned}$$

which holds by the choice of ρ.
To prove (c) choose $\rho \in (\delta(a), (1-2\theta)/(2\sigma-1))$ and let $p = 1/(2-2\sigma)$, $q = 1/(2\sigma-1)$. We refer again to (2.19) and estimate

$$\begin{aligned}
&\int_c^\infty s^{2\theta-3+2\sigma} \hat{a}(s)^{2\sigma-1}\, ds \\
&\le \int_c^\infty s^{2\theta-3+2\sigma+\rho(2\sigma-1)} (s^{-\rho}\hat{a}(s))^{2\sigma-1}\, ds \\
&\le \left(\int_c^\infty s^{[2\theta-3+2\sigma+\rho(2\sigma-1)]/[2-2\sigma]}\right)^{2-2\sigma} \left(\int_c^\infty s^{-\rho}\hat{a}(s)\, ds\right)^{2\sigma-1}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_c^\infty s^{[2\theta-3+2\sigma+\rho(2\sigma-1)]/[2-2\sigma]} < \infty \\
\text{iff }\ & 2\theta-3+2\sigma+\rho(2\sigma-1) < -(2-2\sigma) \\
\text{iff }\ & \rho(2\sigma-1) < 1-2\theta,
\end{aligned}$$

which holds by the choice of ρ. The other integral is estimated by

$$\int_c^\infty s^{-\rho}\hat{a}(s)\, ds = \int_c^\infty \int_0^\infty s^{-\rho} e^{-st} a(t)\, dt\, ds = \int_0^\infty t^{\rho-1} a(t) \int_{ct}^\infty \sigma^{-\rho} e^{-\sigma}\, d\sigma\, dt.$$

We split the integral into two parts:

$$\begin{aligned}
&\int_1^\infty t^{\rho-1} a(t) \int_{ct}^\infty \sigma^{-\rho} e^{-\sigma}\, d\sigma\, dt\\
\le &\int_1^\infty t^{\rho-1} e^{-ct/2} a(t) \int_{ct}^\infty \sigma^{-\rho} e^{-\sigma/2}\, d\sigma\, dt\\
\le &\left(a(1)\int_1^\infty t^{\rho-1} e^{-ct/2}\, dt\right)\left(\int_0^\infty \sigma^{-\rho} e^{-\sigma/2}\, d\sigma\right).
\end{aligned}$$

On the other hand

$$\int_0^1 t^{\rho-1} a(t) \int_{ct}^\infty \sigma^{-\rho} e^{-\sigma}\, d\sigma\, dt \le \left(\int_0^1 t^{\rho-1} a(t)\, dt\right)\left(\int_0^\infty \sigma^{-\rho} e^{-\sigma}\, d\sigma\right).$$

This is finite since $\rho > \delta(a)$. □

2.6. **Recovering u from the abstract solution: Extending J_0.** The solution of the original Volterra equation (2.1) is obtained from the semigroup solution v by an unbounded operator $u = J_0 v$. In order to gain a weak solution from the semigroup setting, we have to extend the domain of J_0.

Lemma 2.29. *Let $\eta \in (0,1/2)$, $\mu \in (0,1/2)$ and $\alpha(a)$ as in Definition 2.3.*

(a) *If $1-2\eta < \alpha(a)$, then J_0 admits a continuous extension as an operator $J : X_\eta \to H$.*
(b) *If $(1-2\eta)/(1-2\mu) < \alpha(a)$, then J_0 admits a continuous extension as an operator $J : X_\eta \to H_\mu$.*

Proof. We resort to Lemma 2.15. Notice that

$$\begin{aligned}
J_0 R(s,B)x = &R(s\hat{a}(s)) \int_{[0,\infty)} \frac{\kappa}{\kappa+s}\, x(\kappa)\, \nu(d\kappa)\\
= &R(s\hat{a}(s),A) \int_{[0,\infty)} \frac{\kappa}{(\kappa+s)\sqrt{\kappa+1}} \sqrt{\kappa+1}\, x(\kappa)\, \nu(d\kappa).
\end{aligned}$$

Thus $J_0 R(s,B)$, as an operator from X into a subspace $Y \subset H$, is bounded by

$$\begin{aligned}
&\|J_0 R(s,B)\|^2_{X\to Y} \le \|R(s\hat{a}(s),A)\|^2_{H\to Y} \int_{[0,\infty)} \frac{\kappa^2}{|\kappa+s|^2(\kappa+1)}\, \nu(d\kappa)\\
\le &M \|R(s\hat{a}(s),A)\|^2_{H\to Y} |\hat{a}(s)|
\end{aligned}$$

with a suitable constant M by Lemma 2.24.
To prove (a), let $Y = H$, and notice that $\|R(s\hat{a}(s),A)\| \le M(s\hat{a}(s))^{-1}$ for $s > 0$. Then

$$\begin{aligned}
&\int_c^\infty s^{1-2\eta} \|J_0 R(s,B)\|^2_{X\to H}\, ds \le M \int_c^\infty s^{1-2\eta} \left(\frac{1}{s\hat{a}(s)}\right)^2 \hat{a}(s)\, ds\\
\le &M \int_c^\infty s^{1-2\eta-2} \frac{1}{\hat{a}(s)}\, ds,
\end{aligned}$$

which is finite if $1-2\eta < \alpha(a)$.
To prove (b), let $Y = H_\mu$. We have

$$\|R(s\hat{a}(s),A)\|_{H\to H_\mu} \le M(s\hat{a}(s))^{\mu-1}.$$

Choose ρ such that $(1-2\eta)/(1-2\mu) < \rho < \alpha(a)$, let $p = (1-2\mu)^{-1}$, $q = (2\mu)^{-1}$ so that $p^{-1}+q^{-1} = 1$.

$$\begin{aligned}
&\int_c^\infty s^{1-2\eta} \|J_0 R(s,B)\|^2_{X\to H_\mu}\, ds \le M \int_c^\infty s^{1-2\eta} \left(\frac{1}{s\hat{a}(s)}\right)^{2-2\mu} \hat{a}(s)\, ds \\
\le& M \int_c^\infty s^{-1-2\eta+2\mu-(1-2\mu)(\rho-2)} \left(\frac{s^{\rho-2}}{\hat{a}(s)}\right)^{1-2\mu} ds \\
\le& M \left[\int_c^\infty s^{[1-2\eta-2\mu-(1-2\mu)\rho]q}\, ds\right]^{1/q} \left[\int_c^\infty \left(\frac{s^{\rho-2}}{\hat{a}(s)}\right)^{(1-2\mu)p} ds\right]^{1/p}.
\end{aligned}$$

The latter integral is just

$$\int_c^\infty s^{\rho-2}\frac{1}{\hat{a}(s)}\, ds$$

which is finite since $\rho < \alpha(a)$. The former integral is finite iff

$$\begin{aligned}
&(1-2\eta-2\mu-(1-2\mu)\rho)q < -1 \\
\text{i.e. }\ & 1-2\eta-2\mu-(1-2\mu)\rho < -2\mu \\
\text{i.e. }\ & 1-2\eta < \rho(1-2\mu).
\end{aligned}$$

This holds by the choice of ρ.

We have shown that J_0 can be extended continuously from $D(B)$ to X_η. We have to ascertain that the extension J in fact coincides with J_0 on $D(J_0)$. For this purpose we prove that for $x \in D(J_0)$ we have $\lim_{t\to\infty} tJ_0R(t,B)x = J_0x$. Let $u = J_0x$. Then

$$\begin{aligned}
&tJ_0R(t,B)x = tR(t\hat{a}(t),A)\int_{[0,\infty)} \frac{\kappa}{\kappa+t} x(\kappa)\, \nu(d\kappa) \\
=& tR(t\hat{a}(t),A)\left[\int_{[0,\infty)} \frac{1}{\kappa+t}(\kappa x(\kappa)-u)\, \nu(d\kappa) + \int_{[0,\infty)} \frac{1}{\kappa+t}\, \nu(d\kappa) u\right] \\
=& tR(t\hat{a}(t),A)\int_{[0,\infty)} \frac{1}{\kappa+t}(\kappa x(\kappa)-u)\, \nu(d\kappa) + t\hat{a}(t)R(t\hat{a}(t),A)u.
\end{aligned}$$

The second term converges to u as $t\to\infty$, since $t\hat{a}(t)\to\infty$. The first term can be estimated

$$\begin{aligned}
&|tR(t\hat{a}(t),A)\int_{[0,\infty)} \frac{1}{\kappa+t}(\kappa x(\kappa)-u)\, \nu(d\kappa)|_H \\
\le& t\|R(t\hat{a}(t),A)\| \int_{[0,\infty)} \frac{1}{(\kappa+t)\sqrt{\kappa+1}} \sqrt{\kappa+1}\, |\kappa x(\kappa)-u|_H\, \nu(d\kappa) \\
\le& t\frac{M}{t\hat{a}(t)} \left[\int_{[0,\infty)} \frac{1}{(\kappa+t)^2(\kappa+1)}\, \nu(d\kappa)\right]^{1/2} \left[\int_{[0,\infty)} (\kappa+1)|\kappa x(\kappa)-u|_H^2\, \nu(d\kappa)\right]^{1/2} \\
\le& \frac{M}{t}\frac{1}{\hat{a}(t)} \left[\int_{[0,\infty)} \frac{1}{(\kappa+1)}\, \nu(d\kappa)\right]^{1/2} \|\kappa x-u\|_X \\
=& \frac{M\hat{a}(1)^{1/2}}{t\hat{a}(t)} \|\kappa x-u\|_x,
\end{aligned}$$

which converges to 0 as $t\to\infty$. □

Lemma 2.30. *Suppose $\eta \in (0,1)$ is such that J_0 admits a continuous extension $J : X_\eta \to H$. We consider the restriction of $B : X_1 \to X$ to an operator $B_\eta : X_{\eta+1} \to X_\eta$. Let $\zeta \in D(A^*) \subset H$. We define the vector $\xi \in X$ by*

$$\xi(\kappa) = \frac{1}{\kappa+1}\zeta.$$

Then ξ (considered as an element of $X^* = X$) is contained in the domain of B_η^* with

$$[B_\eta^*\xi](x) = \langle A^*\zeta, Jx\rangle_H$$

for all $x \in X_\eta$.

Proof. Let $x \in X_{\eta+1}$. Using the definition of $D(B)$, we have

$$\begin{aligned}\langle \xi, B_\eta x\rangle_X = &\int_{[0,\infty)} (\kappa+1)\langle \frac{1}{\kappa+1}\zeta, (-\kappa x(\kappa) + J_0 x)\rangle_H \, \nu(d\kappa) \\ =&\langle \zeta, \int_{[0,\infty)} (-\kappa x(\kappa) + J_0 x)\, \nu(d\kappa)\rangle_H = \langle \zeta, AJ_0 x\rangle_H = \langle A^*\zeta, J_0 x\rangle_H.\end{aligned}$$

This extends continuously to $x \in X_\eta$. □

2.7. **Handling the initial conditions.** In order to treat the Volterra equation (2.1) by the state space approach (2.9), we need to make sure that the initial vector derived from u_0 will be in the state space.

We will need the following auxiliary result:

Lemma 2.31. *Let a be completely monotone with $a(0+) = \infty$ and $\int_0^1 a(s)\, ds < \infty$. Then*

$$\lim_{t\to 0+} \frac{a''(t)}{-a'(t)} = \infty.$$

Proof. We show that for any $M > 0$ we have

$$\lim_{t\to 0+} [a''(t) + Ma'(t)] = \infty.$$

Notice that

$$\begin{aligned}a''(t) + Ma'(t) = &\int_{[0,\infty)} (\kappa^2 - M\kappa)e^{-\kappa t}\, \nu(d\kappa) \\ = &\int_{[0,2M)} (\kappa^2 - M\kappa)e^{-\kappa t}\, \nu(d\kappa) + \int_{[2M,\infty)} (\kappa^2 - M\kappa)e^{-\kappa t}\, \nu(d\kappa).\end{aligned}$$

The first integral is bounded from below by $-2M^2\nu([0,M))$. The second integral is positive and bounded from below by

$$\int_{[2M,\infty)} 2M^2 e^{-\kappa t}\, \nu(d\kappa)$$

which converges to $2M^2\nu([2M,\infty) = \infty$ by the monotone convergence principle. □

Lemma 2.32. *For $u_0 : (-\infty, 0] \to H$ define*

$$v_0(\kappa) = \int_{-\infty}^0 e^{\kappa s} u_0(s)\, ds.$$

(a) *If u_0 satisfies Assumption 2.6(a), then $v_0 \in X$.*

(b) *Let u_0 satisfy Assumptions 2.6(a) and (b) with some $\gamma \in (0,1/2)$. Let $\eta \in (0,1/2)$ be such that*
$$\delta(a) < \frac{1-2\eta}{1-2\gamma}.$$
(Here $\delta(a)$ is given by Definition 2.3.) Then $v_0 \in X_\eta$.

(c) *If u_0 satisfies Assumption 2.6(a), (b) and (c), then $v_0 \in D(B)$*

Proof. To prove (a) we estimate

$$\begin{aligned}
\|v_0\|_X^2 &= \int_{[0,\infty)} \left| \int_{-\infty}^0 e^{\kappa s} u_0(s)\, ds \right|_H^2 (1+\kappa)\nu(d\kappa) \le \int_{[0,\infty)} \left| \int_{-\infty}^0 M e^{s(\kappa+\omega)}\, ds \right|^2 (1+\kappa)\nu(d\kappa) \\
&\le M^2 \int_{[0,\infty)} \frac{1+\kappa}{(\omega+\kappa)^2}\, \nu(d\kappa) < \frac{M_2}{\omega} \int_{[0,\infty)} \frac{1}{\omega+\kappa}\, \nu(d\kappa) = \frac{M_2}{\omega} \hat{a}(\omega) < \infty.
\end{aligned}$$

Next we show:

$$\text{Assumptions 2.6(a) and (b) imply: } (-\kappa v_0 + u_0(0)) \in X. \tag{2.20}$$

This says that $v_0 \in D(J_0)$ with $J_0 v_0 = u_0(0)$. Notice that

$$-\kappa v_0(\kappa) + u_0(0) = \kappa \int_0^\infty e^{-\kappa t} (u_0(0) - u_0(-t))\, dt.$$

Therefore

$$\begin{aligned}
&\int_{[0,\infty)} (\kappa+1)\, |u_0(0) - \kappa v_0(\kappa)|_H^2\, \nu(d\kappa) \\
&= \int_{[0,\infty)} (\kappa+1)\, |\kappa \int_0^\infty e^{-\kappa t} (u_0(0) - u_0(-t))\, dt|_H^2\, \nu(d\kappa) \\
&\le \int_{[0,\infty)} (\kappa+1) \left(\kappa \int_0^\infty e^{-\kappa t}\, dt \right) \left(\kappa \int_0^\infty e^{-\kappa t} |u_0(0) - u_0(-t)|_H^2\, dt \right) \nu(d\kappa) \\
&= \int_0^\infty |u(0) - u(-t)|_H^2 \int_{[0,\infty)} (\kappa+1)\kappa e^{-\kappa t}\, \nu(d\kappa)\, dt \\
&= \int_0^\infty |u_0(0) - u_0(-t)|_H^2 (a''(t) - a'(t))\, dt.
\end{aligned}$$

Because of Assumption 2.6(a) we can estimate

$$\int_1^\infty |u_0(0) - u_0(-t)|_H^2 (a''(t) - a'(t))\, dt \le 2(-a'(1) + a(1)) \sup_{t \in (-\infty,0]} |u_0(t)|_H < \infty.$$

By Lemma 2.31, we can estimate

$$\int_0^1 |u_0(0) - u_0(-t)|_H^2 (a''(t) - a'(t))\, dt \le M \int_0^1 |u_0(0) - u_0(-t)|_H^2 a''(t)\, dt.$$

We show now that Assumptions 2.6(a) and (b) imply

$$\int_{[0,\infty)} (-\kappa v_0(\kappa) + u_0(0))\, \nu(d\kappa) = \int_0^\infty (-a'(t))(u_0(0) - u_0(-t))\, dt. \tag{2.21}$$

We have

$$\begin{aligned}&\int_{[0,\infty)} (-\kappa v_0(\kappa) + u_0(0))\,\nu(d\kappa) = \int_{[0,\infty)} -\kappa \int_0^\infty e^{-\kappa t}(u_0(0) - u_0(-t))\,dt\,\nu(d\kappa)\\ &= \int_0^\infty (u_0(0) - u_0(-t)) \int_{[0,\infty)} \kappa e^{-\kappa t}\,\nu(d\kappa)\,dt = \int_0^\infty (u_0(0) - u_0(-t))(-a'(t))\,dt.\end{aligned}$$

We are now in the position to prove Part (c) of the lemma: If Assumptions 2.6(a) and (b) are satisfied, then $v_0 \in D(J_0)$ with $J_0 v_0 = u_0(0)$. Assumption 2.6(c) and (2.21) yield

$$\int_{[0,\infty)} (-\kappa v_0 + J_0 v_0) = A J_0 v_0$$

which implies that $v_0 \in D(A)$.

Finally we prove Part (b) of the Lemma. From (2.20) we know that $v_0 \in D(J_0)$ with $J_0 v_0 = u_0(0)$. We construct now a function $u_1 : (-\infty, 0] \to H$ and the corresponding $v_1(\kappa) := \int_0^\infty e^{-\kappa t} u_1(-t)\,dt$ such that $v_1 - v_0 \in D(A)$:

$$\begin{aligned}u_1(t) &= u_1(0) e^t,\\ v_1(\kappa) &= \frac{1}{\kappa+1} u_1(0).\end{aligned}$$

It is easily seen (e.g. by Remark 2.33 below) that u_1 satisfies Assumptions 2.6(a) and (b). In order to have $v_1 - v_0 \in D(A)$ we need therefore to choose $u_1(0)$ such that

$$\int_{[0,\infty)} (-\kappa v_1(\kappa) + \kappa v_0(\kappa) + u_1(0) - u_0(0))\,\nu(d\kappa) = A(u_1(0) - u_0(0)).$$

This can be rewritten

$$\begin{aligned}&\int_{[0,\infty)} [-\kappa v_1(\kappa) + \kappa v_0(\kappa) + u_1(0) - u_0(0)]\;\nu(d\kappa) - A(u_1(0) - u_0(0))\\ =&\int_{[0,\infty)} (-\frac{\kappa}{\kappa+1} + 1)(u_1(0) - u_0(0))\,\nu(d\kappa)\\ &\quad + \int_{[0,\infty)} \left[\kappa v_0(\kappa) - u_0(0) + \frac{1}{1+\kappa} u_0(0)\right]\;\nu(d\kappa) - A(u_1(0) - u_0(0))\\ =&(\hat{a}(1) - A)(u_1(0) - u_0(0)) - g\end{aligned}$$

with

$$g = \int_{[0,\infty)} (-\kappa v_0(\kappa) + u_0(0))\,\nu(d\kappa) + \hat{a}(1) u_0(0).$$

Therefore we put

$$u_1(0) = u_0(0) + R(\hat{a}(1), A) g \in H_\gamma.$$

Since $v_1 - v_0 \in D(A)$, it is sufficient to show that $v_1 \in H_\eta$. Now, by definition,

$$v_1(\kappa) = \frac{1}{\kappa+1} R(\hat{a}(1), A)(\hat{a}(1) - A) u_1(0),$$

thus $v_1 = P(\hat{a}(1) - A) u_1(0)$. Now $(\hat{a}(1) - A) u_1(0) \in H_{-\sigma}$ with $\sigma = 1 - \gamma$. Lemma 2.28(c) implies that $v_1 \in X_\eta$ if

$$\delta(a) < \frac{1 - 2\eta}{2\sigma - 1} = \frac{1 - 2\eta}{1 - 2\gamma}.$$

□

Remark 2.33. *If $u_0 : (-\infty, 0] \to H$ satisfies Assumption 2.6(a) and there is some $\delta > 0$ and $M > 0$ such that for $t \in (0, \delta)$ the following Lipschitz estimate holds:*

$$|u_0(-t) - u_0(0)| \le Mt,$$

then u_0 satisfies Assumption 2.6(b).

Proof.

$$\int_0^1 a''(t)t^2\,dt = \int_0^1 \int_{[0,\infty)} \kappa^2 t^2 e^{-\kappa t}\,\nu(d\kappa)\,dt = \int_{[0,\infty)} \frac{1}{\kappa} \int_0^\kappa \sigma^2 e^{-\sigma}\,d\sigma\,\nu(d\kappa).$$

We split the integral:

$$\int_{[0,1)} \frac{1}{\kappa} \int_0^\kappa \sigma^2 e^{-\sigma}\,d\sigma\,\nu(d\kappa) \le \nu([0,1)) \sup_{\sigma\in[0,1)} (\sigma^2 e^{-\sigma}) < \infty,$$
$$\int_{[1,\infty)} \frac{1}{\kappa} \int_0^\kappa \sigma^2 e^{-\sigma}\,d\sigma\,\nu(d\kappa) \le 2\left(\int_{[0,\infty)} \frac{1}{\kappa+1}\,\nu(d\kappa)\right)\left(\int_0^\infty \sigma^2 e^{-\sigma}\,d\sigma\right) < \infty.$$

□

2.8. **Proofs of the deterministic existence and regularity theorems.** We are now in the position to restate Theorems 2.8 in the language of the semigroup and complete its proof.

Theorem 2.34. *Suppose that Assumptions 2.1, 2.2 hold. Let X, J_0, B, P be defined as in Definitions 2.16, 2.18, 2.20, 2.21. Let $v_0(\kappa) = \int_0^\infty e^{-s\kappa} u_0(-s)\,ds$.*
Let $\alpha(a)$ and $\delta(a)$ be defined by Definition 2.3. Let $\gamma, \sigma, \mu \in (0, \frac{1}{2})$ and suppose that Assumption 2.5 holds with σ, and Assumption 2.6(b) holds with γ. Moreover, suppose that $\delta(a)(1-2\gamma) < \alpha(a)(1-2\mu)$.

a) *We choose η and θ such that*

$$\begin{aligned} 1 &< 2\theta < 1 + \alpha(a)(1-2\sigma), \\ 1 - \alpha(a)(1-2\mu) &< 2\eta < 1 - \delta(a)(1-2\gamma). \end{aligned}$$

Then P is a bounded operator $P : H_{-\sigma} \to X_\theta$, and J_0 admits a bounded extension $J : X_\eta \to H_\mu$. Moreover, $v_0 \in X_\eta$.

b) *If v is the mild solution to $v'(t) = B(v(t) - Pf(t)) + Pf(t)$ given by*

$$v(t) = e^{tB} v_0 + (1 - B) \int_0^t e^{(t-s)B} P f(s)\,ds$$

then v is continuous from $[0,\infty) \to X_\eta$.

c) *$u(t) = Jv(t)$ is the unique weak solution of (2.1).*

d) *$u(t)$ is continuous from $[0,\infty)$ into H_μ.*

Proof. ¿From Lemma 2.28 we infer that P maps $H_{-\sigma}$ continuously into X_θ. On the other hand, from Lemma 2.29 we know that J is continuous from X_η into H_μ. By Lemma 2.32, v_0 is contained in X_η. Now, the function Pf is in $L^2(0,\infty, X_\theta)$, and $\eta < \theta$, so that $v(t)$ is a continuous function with values in X_η. Therefore $u(t) = Jv(t)$ is a continuous function with values in H_μ.

We have to prove that u is a weak solution. Let us first look at the solution $v(t)$ of the abstract problem under the additional assumption that $v_0 \in DB$ and f is continuously differentiable with

$f(0) = 0$. In this case, by standard semigroup theory, $v(t)$ is a strong solution of $v'(t) = B(v(t) - Pf(t)) + Pf(t)$, which implies

$$\frac{d}{dt}v(t,\kappa) = -\kappa v(t,\kappa) + J_0 v(t)$$

for almost all κ and almost all t. We infer

$$v(t,\kappa) = e^{-\kappa t}v_0(\kappa) + \int_0^t e^{-\kappa(t-s)}J_0 v(s)\,ds.$$

Integrating by the measure ν we obtain

$$\int_{[0,\infty)} v(t,\kappa)\,\nu(d\kappa) = \int_{[0,\infty)} e^{-\kappa t}v_0(\kappa)\,\nu(d\kappa) + \int_0^t a(t-s)Jv(s)\,ds. \tag{2.22}$$

By continuous extension, Equation (2.22) holds also if the additional requirements that $v_0 \in DB$ and f is smooth are removed.

¿From semigroup theory it is known that $v(t)$ is the unique weak solution of $v'(t) = B_\eta(v(t) - Pf(t)) + Pf(t)$ on X_η in the sense that for all $\xi \in DB_\eta^*$

$$\langle \xi, v(t)\rangle_{X_\eta^*,X_\eta} = \langle \xi, v_0\rangle_{X_\eta^*,X_\eta} + \int_0^t \langle B_\eta^*\xi, v(s) - Pf(s)\rangle_{X_\eta^*,X_\eta}\,ds + \int_0^t \langle \xi, Pf(s)\rangle_{X_\eta^*,X_\eta}\,ds. \tag{2.23}$$

Here, by an usual convention, we consider X_η^* as an extension of X so that the inner product is reduced to $\langle \xi, x\rangle_{X_\eta^*,X_\eta} = \langle \xi, x\rangle_X$ if $\xi \in X$. Now let $\zeta \in DA^*$. We define ξ by $\xi(\kappa) = \frac{1}{\kappa+1}\zeta$. From Lemma 2.30 we know that $\xi \in DB_\eta^*$ with

$$\langle B_\eta^*\xi, x\rangle_{X_\eta^*,X_\eta} = \langle A^*\zeta, Jx\rangle_H \tag{2.24}$$

for all $x \in X_\eta$. We insert this vector ξ into (2.23) and utilize (2.24), (2.22), and the Definition 2.21 of P:

$$\begin{aligned}
0 &= \langle \xi, v(t)\rangle_{X_\eta^*,X_\eta} - \langle \xi, v_0\rangle_{X_\eta^*,X_\eta} - \int_0^t \langle B_\eta^*\xi, v(s) - Pf(s)\rangle_{X_\eta^*,X_\eta}\,ds + \int_0^t \langle \xi, Pf(s)\rangle_{X_\eta^*,X_\eta}\,ds \\
&= \int_{[0,\infty)} \langle \zeta, v(t,\kappa)\rangle_H\,\nu(d\kappa) - \int_{[0,\infty)} \langle \zeta, v_0(\kappa)\rangle_H\,\nu(d\kappa) \\
&\quad - \int_0^t \langle A^*\zeta, J(v(s) - Pf(s))\rangle_H\,ds - \int_0^t \int_{[0,\infty)} \langle \zeta, \frac{1}{\kappa+1}R(\hat{a}(1),A)f(s)\rangle_H\,\nu(d\kappa)\,ds \\
&= \int_0^t \langle \zeta, a(t-s)Jv(s)\rangle_H\,ds + \int_{[0,\infty)} \langle \zeta, e^{-\kappa t}v_0(\kappa)\rangle_H\,\nu(d\kappa) - \int_{[0,\infty)} \langle \zeta, v_0(\kappa)\rangle_H\,\nu(d\kappa) \\
&\quad - \int_0^t \langle A^*\zeta, Jv(s)\rangle_H\,ds - \int_0^t \langle \zeta, (\hat{a}(1) - A)R(\hat{a}(1),A)f(s)\rangle_H\,ds \\
&= \int_0^t \langle \zeta, a(t-s)u(s)\rangle_H\,ds + \int_{[0,\infty)} \langle \zeta, (e^{-\kappa t} - 1)\int_{-\infty}^0 e^{\kappa s}u_0(s)\rangle_H\,ds\,\nu(d\kappa) \\
&\quad - \int_0^t \langle A^*\zeta, u(s)\rangle_H\,ds - \int_0^t \langle \zeta, f(s)\rangle_H\,ds
\end{aligned}$$

$$\begin{aligned}
&= \int_0^t \langle \zeta, a(t-s)u(s)\rangle_H \, ds + \int_{-\infty}^0 \langle \zeta, (a(t-s)-a(-s))u_0(s)\rangle_H \\
&\quad - \int_0^t \langle A^*\zeta, u(s)\rangle_H \, ds - \int_0^t \langle \zeta, f(s)\rangle_H \, ds \\
&= \int_{-\infty}^t \langle \zeta, a(t-s)u(s)\rangle_H \, ds - \int_{-\infty}^0 \langle \zeta, a(-s)u_0(s)\rangle_H \\
&\quad - \int_0^t \langle A^*\zeta, u(s)\rangle_H \, ds - \int_0^t \langle \zeta, f(s)\rangle_H \, ds.
\end{aligned}$$

Therefore, u is a weak solution of (2.1).

In order to show uniqueness, let u be a nontrivial solution of (2.1) with initial function $u_0 = 0$ and forcing term $f = 0$. By integrating as many times as we need, and multiplying with $R(1, A)$ if necessary, we may assume without loss of generality that u is continuously differentiable with values in H and continuous with values in DA, so that (2.3) is reduced to

$$\frac{d}{dt}\int_0^t a(t-s)u(s)\, ds = Au(t).$$

Now we define

$$v(t,\kappa) = \int_0^t e^{-\kappa(t-s)}u(s)\, ds$$

and obtain by straightforward computation: $v'(t) = Bv(t)$, $v(0) = 0$. Since B generates a semigroup, we infer $v(t) = 0$ for all t. Since $u(t) = v'(t,\kappa) + \kappa v(t,\kappa)$, we infer $u = 0$. □

Proof of Theorem 2.9:
With the notation of Theorem 2.34, we put $g_n = Pf_n$ and

$$v_n(t) = e^{tB}v_0 + (1-B)\int_0^t e^{(t-s)B}g_n(s)\, ds.$$

We choose $\eta < \theta$ such that $\tau < \theta - \eta$. Then $u_n = Jv_n$. Since J is a continuous linear operator from X_η into H_μ (Lemma 2.29) and $g_n \in L^p(0,\infty,X_\theta)$ (Lemma 2.28), we may utilize the corresponding, well-known regularity results for analytic semigroups (for a summary see Proposition A.2 in Clément, Desch and Homan [3]):

(a) If $p \in [1, \frac{1}{\tau})$, then $v_n \in L^q(0,T,X_\eta)$ for all $q \in [1, \frac{p}{1-\tau p})$.
(b) If $p = \frac{1}{\tau}$, then $v_n \in L^q(0,T,X_\eta)$ for all $q \in [1,\infty)$.
(c) If $p \in (\frac{1}{\tau}, \infty)$, then $v_n \in \mathcal{C}^{\tau-1/p}([0,T],X_\eta)$.

Here, v_n in the corresponding function space depends continuously on $g_n \in L^q(0,\infty,X_\theta)$. □

3. Stochastic Volterra equation

In this section we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{F}_t\}$ on it. All Brownian motions in the sequel will be defined on this space.

Let U be a real separable (infinite dimensional) Hilbert space, endowed with a complete orthonormal basis $\{g_k\}$. We define a cylindrical Wiener process $\{W(t)\}$ by the formula

$$W(t) = \sum_{j=1}^{\infty} g_j\, \beta_j(t), \qquad t \geq 0. \tag{3.1}$$

Here, $\{\beta_j\}$ is a family of real standard independent Brownian motions; it is known that the series (3.1) does not converge in U, but we can give a sense to it in a larger space, compare [4]; let us choose a trace class operator Q: then the series $Q^{1/2}W(t)$ converges in U to a centered Gaussian random variable with covariance operator Q. Let U_1 be the Hilbert space defined as the completion of U with respect to the norm $\|x\|_1 = \|Q^{1/2}x\|_U$: we can consider then $W(t) = Q^{-1/2}(Q^{1/2}W(t))$ as a Gaussian random variable with values in U_1 and covariance operator identity.

A central rôle in the construction of the stochastic integral is provided by the space $L_2 = L_2(U,H)$ of Hilbert-Schmidt operators from U into H. This is a separable Hilbert space, endowed with the norm $\|\Phi\|^2_{L_2} = \mathrm{Tr}[\Phi Q \Phi^*]$. It is possible to define the Itô integral for all processes $\Phi = \{\Phi(t)\}$ in the space $L^2_{\mathcal{F}}(0,T;L_2)$ of predictable, square integrable processes such that $\mathbb{E}\int_0^T \|\Phi(t)\|^2_{L_2}\,\mathrm{d}t < \infty$. In order to handle the convergence at infinity, we introduce the following weighted spaces of processes. Let K be a real separable Hilbert space; for any $\omega > 0$, we denote $L^2_{\mathcal{F}}(0,+\infty;K;\omega)$ the space of square integrable predictable processes $\{z(t)\}$ such that

$$\|z\|^2_{L^2_{\mathcal{F}}(0,+\infty;K;\omega)} = \mathbb{E}\int_0^{+\infty} |e^{-\omega t} z(t)|^2_K \,\mathrm{d}t < +\infty.$$

In addition to Assumptions 2.1, 2.2, 2.4, and 2.6 we impose

Assumption 3.1. *The process $\{\Phi(t)\}_{t\ge 0}$ belongs to the space $L^2_{\mathcal{F}}(0,+\infty;L_2(U,H_{-\sigma};1))$ for some $0 \le \sigma < 1 - \frac{1}{2\alpha(a)}$, where $H_{-\sigma}$ is the extrapolation space with respect to the operator A and we use the convention $H_0 = H$.*

With no loss of generality, we assume that the stochastic term can always be rewritten in the form

$$\Phi(t)W(t) = \sum_{j=1}^{\infty} \Phi_j(t)\beta_j(t)$$

where $\Phi_j(t) = \Phi(t)g_j$ are processes with values in $H_{-\sigma}$.

We are concerned with the abstract stochastic Volterra equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s = Au(t) + \Phi(t)\dot{W}(t) & t \ge 0, \\ u(t) = u_0(t) & t \le 0, \end{cases} \tag{3.2}$$

In order to give a meaning to (3.2) we introduce the concept of *weak solution.*

Definition 3.2. *A process $\{u(t)\}_{t\ge 0}$ in $L^2_{\mathcal{F}}(0,+\infty;H;\omega)$ is a weak solution of problem (3.2) if for any $\zeta \in D(A^*)$ and $\mathbb{P}_T$-almost surely it holds*

$$\int_{-\infty}^t \langle a(t-s)u(s),\zeta\rangle_H \,\mathrm{d}s = \langle \bar{u},\zeta\rangle_H + \int_0^t \langle A^*\zeta, u(s)\rangle_H \,\mathrm{d}s + \int_0^t \langle \zeta, \Phi(s)\,\mathrm{d}W(s)\rangle_H, \tag{3.3}$$

where

$$\bar{u} = \int_{-\infty}^0 a(-s)u_0(s)\,ds.$$

In this paper, we follow the semigroup approach in order to solve (3.2). In the next subsection, we introduce and solve the relevant stochastic differential equation; the last part of the section is then devoted to study (3.2).

3.1. **The stochastic differential equation.** In this section we extend the semigroup setting of Section 2 to the stochastic case. In this framework, we can appeal to the well known theory of stochastic evolution equations and we are lead to study the stochastic convolution process associated to the equation.

With the notation of Section 2 we rewrite (3.2) as a stochastic equation

$$\begin{cases} \mathrm{d}v(t) = Bv(t)\,\mathrm{d}t + (I-B)P\Phi(t)\,\mathrm{d}W(t), & t>0, \\ v(0) = v_0 \in X, \end{cases} \tag{3.4}$$

where the initial condition is given by

$$v(0,\kappa) = \int_{[0,\infty)} e^{-\kappa s} u_0(-s)\,\mathrm{d}s. \tag{3.5}$$

In the following we write $\Psi = (I-B)P\Phi$ the (operator valued) process which corresponds to Φ on the spaces X_η, where $\eta \geq -1$.

Theorem 3.3. *For every $v_0 \in X$, problem (3.4) admits a unique weak solution given by the formula*

$$v(t) = e^{tB}v_0 + \int_0^t e^{(t-s)B}\Psi(s)\,\mathrm{d}W(s). \tag{3.6}$$

Moreover, if $\eta \in (0, \theta - \frac{1}{2})$ and $v_0 \in X_\eta$, there exists a version of $v(t)$ with continuous trajectories as a process with values in X_η.

The proof of the above theorem is standard and relies essentially on the following result; notice that (3.8) is given for the seminorms of the interpolation spaces (rather than for the norms on the relevant spaces).

Theorem 3.4. *Let $B-\omega$ be a sectorial operator of negative type and assume that $\Psi(t)$ belongs to $L^2_{\mathcal{F}}(0,\infty;L_2(U_0,X_{\theta-1});\omega)$ for some $\frac{1}{2}<\theta<1$. Then the stochastic convolution process*

$$Z(t) = \int_0^t e^{(t-s)B}\Psi(s)\,\mathrm{d}W(s), \qquad t \geq 0, \tag{3.7}$$

belongs to $L^2_{\mathcal{F}}(0,\infty;X_{\theta-\frac{1}{2}};\omega)$ and

$$[\![Z]\!]^2_{L^2_{\mathcal{F}}(0,\infty;X_{\theta-\frac{1}{2}};\omega)} \leq \frac{1}{3-2\theta}[\![\Psi]\!]^2_{L^2_{\mathcal{F}}(0,\infty;L_2(U_0,X_{\theta-1});\omega)}. \tag{3.8}$$

Proof. We argue as in Da Prato & Zabczyk [4], Theorem 6.12. We consider $\theta \in \left(\frac{1}{2}, \frac{\alpha(a)+1}{2}\right)$; then we have

$$\begin{aligned}
&\mathbb{E}\int_0^\infty [\![e^{-t}Z(t)]\!]^2_{\theta-\frac{1}{2}}\,\mathrm{d}t \\
&\le \sum_j \mathbb{E}\int_0^\infty \int_0^t \int_0^\infty \xi^{1-2(\theta-\frac{1}{2})}\|e^{-\xi}(1-B)e^{\xi B}e^{-t}e^{(t-s)B}\Psi_j(s)\|^2_X\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t \\
&= \sum_j \mathbb{E}\int_0^\infty \int_0^\infty \int_s^\infty \xi^{2-2\theta}\|e^{-\xi-t}(1-B)e^{(t-s+\xi)B}\Psi_j(s)\|^2_X\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}s \\
&= \sum_j \mathbb{E}\int_0^\infty \int_0^\infty \int_\xi^\infty \xi^{2-2\theta}\|e^{-s-t}(1-B)e^{tB}\Psi_j(s)\|^2_X\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}s \\
&= \sum_j \mathbb{E}\int_0^\infty \int_0^\infty \int_0^t \xi^{2-2\theta}\|e^{-s-t}(1-B)e^{tB}\Psi_j(s)\|^2_X\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}s \\
&= \sum_j \mathbb{E}\int_0^\infty \int_0^\infty \frac{t^{3-2\theta}}{3-2\theta}\|e^{-t}(1-B)e^{tB}e^{-s}\Psi_j(s)\|^2_X\,\mathrm{d}t\,\mathrm{d}s \\
&= \sum_j \mathbb{E}\int_0^\infty \frac{1}{3-2\theta}[\![e^{-s}\Psi_j(s)]\!]^2_{\theta-1}\,\mathrm{d}s.
\end{aligned}$$

□

3.2. **Laplace transform for the stochastic process.** In this section we search for a different representation of the solution $v(t)$, as a necessary tool in order to get back to the integral equation. To prepare the result, we state and prove a preliminary lemma.

Before we proceed further, we introduce the following notation. We are given a cylindrical Wiener process $\{W(t)\}_{t\ge 0}$ on U. Let K be a real Hilbert space and assume that $\Psi \in L^2_{\mathcal{F}}(0,\infty; L_2(U,K);\omega)$. We define the process

$$\mathcal{L}[\Psi\,\mathrm{d}W](s) = \int_0^\infty e^{-st}\Psi(t)\,\mathrm{d}W(t), \qquad s \ge \omega.$$

Remark 3.5. *Let $z(t)$ denote the process defined above. $z(t)$ is not adapted to $\{\mathcal{F}_t\}$, and for every $t \ge \omega$ it is $z(t) \in L^2(\Omega,\mathcal{F};K)$. We may wonder if this process is continuous. It holds that*

$$z(t) - z(s) = \int_0^\infty [e^{-tx} - e^{-sx}]\Psi(x)\,\mathrm{d}W(x)$$

therefore

$$\begin{aligned}
\mathbb{E}[z(t)-z(s)]^2 &\le \mathbb{E}\left|\int_0^\infty [e^{-tx}-e^{-sx}]\Psi(x)\,\mathrm{d}W(x)\right|^2 \\
&= \mathbb{E}\int_0^\infty \left|\int_s^t e^{-zx}\,\mathrm{d}z\,x\Psi(x)\right|^2\,\mathrm{d}x \\
&\le \mathbb{E}\int_0^\infty x^2e^{-2sx}(t-s)^2\Psi^2(x)\,\mathrm{d}x \le C(t-s)^2,
\end{aligned}$$

which implies, by Kolmogorov's continuity criterium, the existence of a continuous modification of $z(t)$.

Lemma 3.6. *Let Φ satisfiy Assumption 3.1 and take $v_0 \in X_\eta$ for some $\frac{1-\alpha(a)}{2} < \eta < \frac{1-2\sigma}{2}\alpha(a)$. Consider the processes*

$$v(t) = e^{tB}v_0 + (I-B)\int_0^t e^{(t-s)B}P\Phi(s)\,\mathrm{d}W(s) \tag{3.9}$$

and

$$\tilde{v}(t,\kappa) = e^{-\kappa t}v_0(\kappa) + \int_0^t e^{-\kappa(t-s)}Jv(s)\,\mathrm{d}s. \tag{3.10}$$

Then $\tilde{v}(t)$ is a modification of $v(t)$.

Proof. Notice that by assumption $\frac{1-\alpha(a)}{2} < \eta$, we get from Lemma 2.29 that J_0 admits a continuous extension $J : X_\eta \to H$. We take Laplace transform of our processes:

$$\mathcal{L}[\tilde{v}(\cdot,\kappa)](s) = \frac{1}{\kappa+s}v_0(\kappa) + \frac{1}{\kappa+s}J\mathcal{L}[v](s), \tag{3.11}$$

and

$$\mathcal{L}[v](s) = R(s,B)v_0 + (I-B)R(s,B)P\mathcal{L}[\Phi\,\mathrm{d}W](s). \tag{3.12}$$

Notice further that

$$R(s,B)v_0(\kappa) = \frac{1}{s+\kappa}(v_0(\kappa) + JR(s,B)v_0);$$

therefore, using Lemma 2.27, (2.16) and (2.17), we obtain

$$\begin{aligned}\mathcal{L}[v](s) &= \frac{1}{\kappa+s}(v_0(\kappa) + JR(s,B)v_0) + \frac{1}{\kappa+s}J(I-B)R(s,B)P\mathcal{L}[\Phi\,\mathrm{d}W](s)\\ &= \frac{1}{\kappa+s}\big(v_0(\kappa) + J\mathcal{L}[v](s)\big) = \mathcal{L}[\tilde{v}(\cdot,\kappa)](s).\end{aligned}$$

□

3.3. **The solution to the stochastic Volterra equation.** In this section, we shall combine the results in Theorem 3.3 and Lemma 3.6 in order to prove the existence of the solution to problem (3.2).

Theorem 3.7. *Let Φ satisfy Assumption 3.1 and take $v_0 \in X_\eta$ for some $\frac{1-\alpha(a)}{2} < \eta < \frac{1-2\sigma}{2}\alpha(a)$. Given the process*

$$v(t) = e^{tB}v_0 + \int_0^t e^{(t-s)B}\Psi(s)\,\mathrm{d}W(s) \tag{3.13}$$

we define the process

$$u(t) = \begin{cases} Jv(t), & t \geq 0,\\ u_0(t), & t \leq 0.\end{cases} \tag{3.14}$$

Then $u(t)$ is a weak solution to problem (3.2).

Proof. Again, in our assumption we have by Lemma 2.29 that the operator J_0 can be extended to a bounded operator $J : X_\eta \to H$. For fixed $\zeta \in D(A^*)$, we define the vector $\xi \in X$ by $\xi(\kappa) = \frac{1}{1+\kappa}\zeta$. By Lemma 2.30 we have $\xi \in D(B_\eta^*)$. Moreover, by Lemma 2.28

$$\Psi(t) = (I - B)P\Phi(t) \in L^2_{\mathcal{F}}(0,\infty; L_2(U_0, X_{\theta-1}); 1) \quad \text{for any } \theta < \frac{(1-2\sigma)\alpha(a)+1}{2}. \tag{3.15}$$

Therefore

$$\langle v(t), \xi\rangle_X = \langle v_0, \xi\rangle_X + \int_0^t \langle B_\eta^* \xi, v(s)\rangle\,\mathrm{d}s + \int_0^t \langle \xi, \Psi\,\mathrm{d}W(s)\rangle_X. \tag{3.16}$$

Let us consider separately the four terms. The initial condition yields

$$\begin{aligned}\langle v_0, \xi\rangle_X &= \int_{[0,\infty)} \langle v(0,\kappa), \zeta\rangle_H\,\nu(\mathrm{d}\kappa)\\ &= \langle \int_{[0,\infty)} \int_{[0,\infty)} e^{-\kappa s} u_0(-s)\,\mathrm{d}s\,\nu(\mathrm{d}\kappa), \zeta\rangle_H = \langle \bar{u}, \zeta\rangle_H;\end{aligned}$$

next, the deterministic integral can be evaluated using Lemma 2.30

$$\begin{aligned}\int_0^t \langle B_\eta^* \xi, v(s)\rangle_X\,\mathrm{d}s &= \int_0^t \langle A^*\zeta, Jv(s)\rangle_H\,\mathrm{d}s\\ &= \int_0^t \langle A^*\zeta, u(s)\rangle_H\,\mathrm{d}s;\end{aligned}$$

we then consider the stochastic integral

$$\begin{aligned}\int_0^t \langle \xi, \Psi(s)\,\mathrm{d}W(s)\rangle_X &= \sum_j \int_0^t \langle \xi, \Psi_j(s)\rangle_X\,\mathrm{d}\beta_j(s)\\ &= \sum_j \int_0^t \langle \xi, (I-B)P\Phi_j(s)\rangle_X\,\mathrm{d}\beta_j(s)\\ &= \sum_j \int_0^t \langle \xi, P\Phi_j(s)\rangle_X\,\mathrm{d}\beta_j(s) - \int_0^t \langle B^*\xi, P\Phi_j(s)\rangle_X\,\mathrm{d}\beta_j(s).\end{aligned}$$

The first integrand on the right hand side of the above identity may be written as follows:

$$\begin{aligned}\langle \xi, P\Phi_j(s)\rangle_X &= \int_{[0,\infty)} \langle \frac{1}{\kappa+1}\zeta, \frac{1}{\kappa+1} R(\hat{a}(1), A)\Phi_j(s)\rangle_H (\kappa+1)\,\nu(\mathrm{d}\kappa)\\ &= \int_{[0,\infty)} \frac{1}{\kappa+1}\,\nu(\mathrm{d}\kappa) \langle \zeta, R(\hat{a}(1), A)\Phi_j(s)\rangle_H\\ &= \langle \zeta, \hat{a}(1) R(\hat{a}(1), A)\Phi_j(s)\rangle_H;\end{aligned}$$

while the second gives

$$\begin{aligned}\langle B^*\xi, P\Phi_j(s)\rangle_X &= \langle \zeta, AJ_0 P\Phi_j(s)\rangle_H\\ &= \langle \zeta, AR(\hat{a}(1), A)\Phi_j(s)\rangle_H.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^t \langle \xi, \Psi(s)\,\mathrm{d}W(s)\rangle_X &= \sum_j \int_0^t \langle \zeta, (\hat{a}(1)-A)R(\hat{a}(1),A)\Phi_j(s)\rangle_H \,\mathrm{d}\beta_j(s)\\ &= \sum_j \int_0^t \langle \zeta, \Phi_j(s)\rangle_H \,\mathrm{d}\beta_j(s) = \int_0^t \langle \zeta, \Phi(s)\,\mathrm{d}W(s)\rangle_H.\end{aligned}$$

We have proved so far that

$$\langle v(t),\xi\rangle_X = \langle \bar{u},\zeta\rangle_H + \int_0^t \langle A^*\zeta, u(s)\rangle_H \,\mathrm{d}s + \int_0^t \langle \zeta, \Phi(s)\,\mathrm{d}W(s)\rangle_H. \tag{3.17}$$

It only remains to prove

$$\int_{[0,\infty)} \langle v(t,\kappa),\zeta\rangle_H \,\nu(\mathrm{d}\kappa) = \int_{-\infty}^t \langle a(t-s)u(s),\zeta\rangle_H \,\mathrm{d}s.$$

If we recall the definition of $u(t) = \begin{cases} Jv(t), & t>0\\ u_0(t), & t\le 0,\end{cases}$ we obtain

$$\int_{-\infty}^t \langle a(t-s)u(s),\zeta\rangle_H \,\mathrm{d}s = \int_{-\infty}^0 \langle a(t-s)u_0(s),\zeta\rangle_H \,\mathrm{d}s + \int_0^t \langle a(t-s)Jv(s),\zeta\rangle_H \,\mathrm{d}s.$$

We then exploit the definition of $a(t)$; the first term becomes

$$\begin{aligned}\int_{-\infty}^0 \langle a(t-s)u_0(s),\zeta\rangle_H \,\mathrm{d}s &= \langle \int_{-\infty}^0 \int_{[0,\infty)} e^{-\kappa(t-s)}\,\nu(\mathrm{d}\kappa)u_0(s)\,\mathrm{d}s,\zeta\rangle_H\\ &= \langle \int_{[0,\infty)} e^{-\kappa t} \int_{-\infty}^0 e^{-\kappa(-s)}u_0(s)\,\mathrm{d}s\,\nu(\mathrm{d}\kappa),\zeta\rangle_H\\ &= \langle \int_{[0,\infty)} e^{-\kappa t} v(0,\kappa)\,\nu(\mathrm{d}\kappa),\zeta\rangle_H,\end{aligned}$$

the second term becomes

$$\begin{aligned}\int_0^t \langle a(t-s)Jv(s),\zeta\rangle_H \,\mathrm{d}s &= \langle \int_0^t \int_{[0,\infty)} e^{-\kappa(t-s)}\,\nu(\mathrm{d}\kappa)Jv(s)\,\mathrm{d}s,\zeta\rangle_H\\ &= \langle \int_{[0,\infty)} \int_0^t e^{-\kappa(t-s)}Jv(s)\,\mathrm{d}s\,\nu(\mathrm{d}\kappa),\zeta\rangle_H,\end{aligned}$$

and the thesis follows from the representation (3.10). □

4. Long time behavior of the solution

We introduce the dynamical system $P_t(\cdot)$ determined by the stochastic problem (3.4) on the state space X. Let $v = v(t,x)$ be the process solution to (3.4) with initial condition x. Then, for $\phi \in B_b(X)$, $t \ge 0$ and $x \in X$, we define

$$P_t\phi(x) = \mathbb{E}[v(t,x)].$$

It is interesting to note that this system is closely related to a stochastic Volterra equation; despite of this, it is a Markov process.

Our aim is to show that, under some additional assumptions, the stochastic problem (3.4) shows some asymptotic properties and, in particular, it admits an invariant measure.

Assumption 4.1. *The operator function $\Phi(t)$ is identical to an operator Φ in $L_2(U,H)$ and there exists $\gamma \ge 0$ such that $(A^\gamma \Phi)^*$ is a bounded operator $H \to U$.*

There exists a $\beta < 1$ such that $A^{-(\gamma+\frac{\beta}{2})}$ (where γ is the constant from the previous assumption) is an Hilbert-Schmidt operator on H.

As far as the kernel $a(t)$, we require an integrability condition $a(t) \in L^1(0,\infty)$. Notice that this is equivalent to require $\hat{\alpha}(0) < \infty$.

Theorem 4.2. *There exists (at least) an invariant measure for equation (3.4).*

Proof. It is known, cf. Da Prato & Zabczyk [4], that in order to get the thesis it suffices to prove that

$$\sup_{t\ge 0} \operatorname{Tr} \int_0^t e^{sB}\Psi\Psi^* e^{sB^*}\,\mathrm{d}s < \infty. \tag{4.1}$$

Let $\{f_j\}$ be an orthonormal basis in X; we have

$$\begin{aligned}
\operatorname{Tr} \int_0^\infty e^{sB}\Psi\Psi^* e^{sB^*}\,\mathrm{d}s &= \sum_{j=1}^\infty \int_0^\infty \langle e^{sB}\Psi\Psi^* e^{sB^*} f_j, f_j\rangle_X \,\mathrm{d}s \\
&= \sum_{j=1}^\infty \int_0^\infty \langle \Psi^* e^{sB^*} f_j, \Psi^* e^{sB^*} f_j\rangle_U \,\mathrm{d}s \\
&= \sum_{j=1}^\infty \int_0^\infty \|\Phi^* P^*(1-B)^* e^{sB^*} f_j\|_U^2 \,\mathrm{d}s
\end{aligned}$$

by Plancherel's theorem

$$\begin{aligned}
\operatorname{Tr} \int_0^\infty e^{sB}\Psi\Psi^* e^{sB^*}\,\mathrm{d}s &= \sum_{j=1}^\infty \int_{\mathbb{R}} \|\mathcal{F}(\Phi^* P^*(1-B)^* e^{\cdot B^*} f_j)(s)\|_U^2 \,\mathrm{d}s \\
&= \sum_{j=1}^\infty \int_{\mathbb{R}} \|\Phi^* P^*(1-B)^*(is - B^*)^{-1} f_j\|_U^2 \,\mathrm{d}s \\
&= \sum_{j=1}^\infty \int_{\mathbb{R}} \|\Phi^*[(-is - B)(1-B)P]^* f_j\|_U^2 \,\mathrm{d}s
\end{aligned}$$

which is equal to

$$\int_{\mathbb{R}} \|[(-is-B)^{-1}(1-B)P]\Phi\|_{HS}^2 \,\mathrm{d}s \tag{4.2}$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm of operators from U to X. In order to prove convergence of the integral, we need to consider the singularity in zero and the convergence at infinity. By the symmetry of the estimates, we may consider the integrals in $(0,1)$ and in $(1,\infty)$, respectively; as for the first, we have, from Lemma 2.27,

$$\int_0^1 \|[(-is-B)^{-1}(1-B)P]\Phi\|_{HS}^2 \,\mathrm{d}s = \sum_{j=1}^\infty \int_0^1 \|[(-is-B)^{-1}(1-B)P]\Phi g_j\|_X^2 \,\mathrm{d}s$$

$$= \sum_{j=1}^\infty \int_0^1 \int_{[0,\infty)} (\kappa+1) \left\| \frac{1}{\kappa+is} R(is\hat{a}(is),A)A^{\beta/2}A^{-(\gamma+\beta/2)}A^\gamma \Phi g_j \right\|_H^2 \nu(\mathrm{d}\kappa)\,\mathrm{d}s$$

$$\leq \int_0^1 \int_{[0,\infty)} \frac{\kappa+1}{\kappa^2+s^2} \|R(is\hat{a}(is),A)A^{\beta/2}\|_{L(H,H)}^2$$

$$\|A^{-(\gamma+\beta/2)}\|_{HS}^2 \|A^\gamma\Phi\|_{L(U,H)}^2 \,\nu(\mathrm{d}\kappa)\,\mathrm{d}s$$

$$\leq C \int_0^1 \int_{[0,\infty)} \frac{\kappa+1}{\kappa^2+s^2} \,\nu(\mathrm{d}\kappa)\,\mathrm{d}s$$

using Fubini's theorem, after a rescaling we get

$$\leq C \int_{[0,\infty)} (1+\frac{1}{\kappa}) \int_0^{1/\kappa} \frac{1}{1+s^2} \,\mathrm{d}s\,\nu(\mathrm{d}\kappa)$$

$$\leq C \int_{[0,\infty)} (1+\frac{1}{\kappa}) \min(1,\frac{1}{\kappa})\,\nu(\mathrm{d}\kappa)$$

$$\leq C \int_{[0,1)} \nu(\mathrm{d}\kappa) + C \int_{[0,\infty)} \frac{1}{\kappa}\,\nu(\mathrm{d}\kappa) + C \int_{[1,\infty)} \frac{1}{\kappa^2}\,\nu(\mathrm{d}\kappa)$$

which is finite due to the properties of ν and the assumption on the finiteness of

$$\hat{a}(0) = \int_{[0,\infty)} \frac{1}{\kappa}\,\nu(\mathrm{d}\kappa) < \infty.$$

We now turn to consider the behavior at infinity. We start again from Lemma 2.24, Lemma 2.27 and the assumptions on A and Φ

$$\int_1^\infty \|[(-is-B)^{-1}(1-B)P]\Phi\|_{HS}^2 \,\mathrm{d}s = \sum_{j=1}^\infty \int_1^\infty \|[(-is-B)^{-1}(1-B)P]\Phi g_j\|_X^2 \,\mathrm{d}s$$

$$= \int_1^\infty |\hat{a}(is)| \|R(is\hat{a}(is),A)A^{\beta/2}\|_{L(H,H)}^2 \|A^{-(\gamma+\beta/2)}\|_{HS}^2 \|A^\gamma\Phi\|_{L(U,H)}^2 \,\mathrm{d}s$$

$$\leq C \int_1^\infty s^{\beta-2} \frac{1}{|\hat{a}(s)|^{1-\beta}} \,\mathrm{d}s$$

$$\leq C \left(\int_1^\infty s^{-1-\varepsilon/\beta} \,\mathrm{d}s \right)^\beta \left(\int_1^\infty s^{(2\beta+\varepsilon-2)/(1-\beta)} \frac{1}{|\hat{a}(s)|} \,\mathrm{d}s \right)^{1-\beta}$$

and the last term is bounded provided we choose ε such that $0 < \frac{\varepsilon}{(1-\beta)} < \alpha(a)$, which is possible since $\beta < 1$ and $\alpha(a)$ is positive due to Assumption 2.4. □

We further search for an estimate like (4.1) in interpolation spaces X_η, $\eta > 0$. Let us recall some tools that we shall use in the following.

The norm in the space X_η is given by the following integral, where $c > 1$ is arbitrary

$$\|x\|_\eta^2 = \int_c^\infty t^{2\eta-1} \|(1-B)R(t,B)x\|_X^2 \,\mathrm{d}t;$$

the identity in Lemma 2.27 reads

$$[R(s,B)(1-B)P]u(\kappa) = \frac{1}{s+\kappa} R(s\hat{a}(s),A)u;$$

Starting from (4.2), assume that g_j is an orthonormal basis in U; hence we are concerned with the quantity

$$\begin{aligned}
&\sum_{j=1}^\infty \int_{\mathbb{R}} \|[R(-is,B)(1-B)P]\Phi g_j\|_\eta^2 \,\mathrm{d}s \\
&= \sum_{j=1}^\infty \int_{\mathbb{R}} \int_c^\infty t^{2\eta-1} \|(1-B)R(t,B)[R(-is,B)(1-B)P]\Phi g_j\|_X^2 \,\mathrm{d}t \,\mathrm{d}s
\end{aligned}$$

In the above expression, we can compute

$$\begin{aligned}
(1-B)&R(t,B)R(-is,B)(1-B)P \\
&=(1-t)R(t,B)R(-is,B)(1-B)P + R(-is,B)(1-B)P \\
&=\frac{1-t}{t+is}[R(-is,B) - R(t,B)](1-B)P + R(-is,B)(1-B)P \\
&= \left[\frac{t-1}{t+is} R(t,B) + \frac{1+is}{t+is} R(-is,B) \right] (1-B)P.
\end{aligned}$$

Now we use Lemma 2.27 to obtain

$$\begin{aligned}
[(1-B)&R(t,B)R(-is,B)(1-B)P\Phi g_j](\kappa) \\
&=\frac{1+is}{t+is}\,\frac{1}{\kappa-is} R(-is\hat{a}(-is),A)\Phi g_j + \frac{t-1}{t+is}\,\frac{1}{\kappa+t} R(t\hat{a}(t),A)\Phi g_j.
\end{aligned}$$

We use this formula in the previous computation to get

$$\begin{aligned}
&\sum_{j=1}^\infty \int_{\mathbb{R}} \|[R(-is,B)(1-B)P]\Phi g_j\|_\eta^2 \,\mathrm{d}s \\
&\quad= \sum_{j=1}^\infty \int_{\mathbb{R}} \int_c^\infty \int_{[0,\infty)} t^{2\eta-1} \Big\| \left(\frac{1+is}{(\kappa-is)(t+is)} \right) R(-is\hat{a}(-is),A)\Phi g_j \\
&\qquad\qquad + \frac{1}{\kappa+t}\,\frac{t-1}{t+is} R(t\hat{a}(t),A)\Phi g_j \Big\|_H^2 \nu(\mathrm{d}\kappa)\,\mathrm{d}t\,\mathrm{d}s
\end{aligned} \tag{4.3}$$

We shall consider the two terms separately. The second term is handled by Lemma 4.4; the first term is studied below.

Lemma 4.3. *The first term in the estimate (4.3) is bounded, provided that the following condition holds:*

$$\int_c^\infty t^{2\eta+\beta-2} \frac{1}{|\hat{a}(t)|^{1-\beta}} \,\mathrm{d}t < \infty. \tag{4.4}$$

Proof. We have to estimate

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{c}^{\infty} t^{2\eta-1}\left\|\frac{1+is}{t+is}\,\frac{1}{\kappa-is}R(-is\hat{a}(-is),A)\Phi g_j\right\|^2 \mathrm{d}s\,\mathrm{d}t$$
$$= \sum_{j=0}^{\infty}\int_{-\infty}^{\infty}\int_{[0,\infty)}\int_{c}^{\infty} t^{2\eta-1}\frac{1+s^2}{t^2+s^2}\,\frac{\kappa+1}{\kappa^2+s^2}\,\|R(-is\hat{a}(-is)A)A^{\beta/2}\|^2\,\|A^{-\beta/2}\Phi g_j\|^2\,\nu(d\kappa)\,\mathrm{d}t\,\mathrm{d}s$$
$$\leq C\int_{-\infty}^{\infty}\left[\int_{c}^{\infty}\frac{t^{2\eta-1}}{t^2+s^2}\,\mathrm{d}t\right]\left[\int_{[0,\infty)}\frac{\kappa+1}{\kappa^2+s^2}\,\nu(d\kappa)\right](1+s^2)\|R(-is\hat{a}(-is),A)A^{\beta/2}\|^2\,\mathrm{d}s.$$

We split the integral with respect to s in the intervals $[0,1]$ and $[1,\infty)$; of course, the negative part is estimated the same way.

Since 0 is in the resolvent set of A, we infer that for $s\in[0,1]$, the term

$$(1+s^2)\|R(-is\hat{a}(-is),A)A^{\beta/2}\|^2$$

is bounded. We estimate the integral with respect to t:

$$\int_{c}^{\infty}\frac{t^{2\eta-1}}{t^2+s^2}\,\mathrm{d}t = s^{2\eta-1-2+1}\int_{c/s}^{\infty}\frac{\tau^{2\eta-1}}{1+\tau^2}\,d\tau \leq s^{2\eta-2}\int_{c/s}^{\infty}\tau^{2\eta-3}\,d\tau = s^{2\eta-2}\frac{1}{2-2\eta}\left(\frac{c}{s}\right)^{2\eta-2} = C.$$

We know already from the proof of Theorem 4.2 that

$$\int_0^t\int_{[0,\infty)}\frac{\kappa+1}{\kappa^2+s^2}\,\nu(d\kappa)\leq C.$$

Combining all these estimates we find that

$$\int_0^1\left[\int_{c}^{\infty}\frac{t^{2\eta-1}}{t^2+s^2}\,\mathrm{d}t\right]\left[\int_{[0,\infty)}\frac{\kappa+1}{\kappa^2+s^2}\,\nu(d\kappa)\right](1+s^2)\|R(-is\hat{a}(-is),A)A^{\beta/2}\|^2\,\mathrm{d}s\leq C<\infty.$$

For the interval $s\in[1,\infty)$ we estimate

$$\|R(-is\hat{a}(is),A)A^{\beta/2}\|^2\leq C(s|\hat{a}(is)|)^{\beta-2}.$$

The integral with respect to t is estimated by

$$\int_{c}^{\infty}\frac{t^{2\eta-1}}{t^2+s^2}\,\mathrm{d}t = s^{2\eta-2}\int_0^{\infty}\frac{\tau^{2\eta-1}}{1+\tau^2}\,d\tau\leq Cs^{2\eta-2}.$$

The integral with respect to κ is estimated by

$$\int_{[0,\infty)}\frac{\kappa+1}{\kappa^2+s^2}\,\nu(d\kappa)=\Re(\hat{a}(is))-\frac{1}{s}\Im(\hat{a}(is))\leq C|\hat{a}(is)|.$$

Combining these estimates we obtain

$$\begin{aligned}
&\int_1^\infty \left[\int_c^\infty \frac{t^{2\eta-1}}{t^2+s^2}\,\mathrm{d}t\right]\left[\int_{[0,\infty)} \frac{\kappa+1}{\kappa^2+s^2}\,\nu(d\kappa)\right](1+s^2)\|R(-is\hat{a}(-is),A)A^{\beta/2}\|^2\,\mathrm{d}s\\
&\quad\le C\int_1^\infty s^{2\eta-2}|\hat{a}(is)|s^2 s^{\beta-2}|\hat{a}(is)|^{\beta-2}\,\mathrm{d}s\\
&\quad= C\int_1^\infty s^{2\eta+\beta-2}|\hat{a}(is)|^{\beta-1}\,\mathrm{d}s\\
&\quad\le C\int_1^\infty s^{2\eta+\beta-2}|\hat{a}(s)|^{\beta-1}\,\mathrm{d}s
\end{aligned}$$

and the latter integral is bounded by assumption (4.4). □

Lemma 4.4. *The second term in the estimate (4.3) is bounded, provided that condition (4.4) is satisfied.*

Proof. We are concerned with the quantity

$$I := \sum_{j=1}^\infty \int_{\mathbb{R}}\int_c^\infty \int_{[0,\infty)} t^{2\eta-1}(\kappa+1)\left\|\frac{1}{\kappa+t}\frac{1-t}{t+is}R(t\hat{a}(t),A)\Phi g_j\right\|_H^2 \nu(\mathrm{d}\kappa)\,\mathrm{d}t\,\mathrm{d}s$$

With the usual assumptions on A and Φ we obtain

$$\begin{aligned}
I &\le \int_{\mathbb{R}}\int_c^\infty t^{2\eta-1}\left(\int_{[0,\infty)}\frac{(\kappa+1)}{(\kappa+t)^2}\,\nu(\mathrm{d}\kappa)\right)\left|\frac{1-t}{t+is}\right|^2 \|R(t\hat{a}(t),A)A^{\beta/2}\|^2\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\int_{\mathbb{R}}\int_c^\infty t^{2\eta-1}\hat{a}(t)\frac{t^2}{t^2+s^2}|t\hat{a}(t)|^{\beta-2}\,\mathrm{d}t\,\mathrm{d}s\\
&\le C\int_c^\infty t^{2\eta-1}\hat{a}(t)t|t\hat{a}(t)|^{\beta-2}\,\mathrm{d}t = C\int_c^\infty t^{2\eta+\beta-2}\frac{1}{|\hat{a}(t)|^{1-\beta}}\,\mathrm{d}t
\end{aligned}$$

which is bounded provided that (4.4) holds. □

We first remark that it is possible to give a sufficient condition on η and β, in terms of $\alpha(a)$, which replace (4.4):

$$2\eta < (1-\beta)\alpha(a). \tag{4.5}$$

Notice moreover that a solution to (3.4) exists in the space X_η for arbitrary $\eta < \frac{1}{2}\alpha(a)$, provided the initial condition is suitably regular.

Theorem 4.5. *Assume that Assumption 4.1 holds, and further choose η such that (4.4) holds (compare also (4.5)). Then there exists (at least) one invariant measure for equation (3.4) which is concentrated on the space X_η.*

The above theorem implies the existence of a stationary solution $\bar{v}$ of (3.4); we may wonder if this result implies the existence of a stationary solution also of (3.2). This is the case if the solution $\bar{v}$ is suitably regular; actually, in order to apply Lemma 2.29, it must be $\eta > \frac{1-\alpha(a)}{2}$, hence we write the following condition on β

$$\beta < \frac{2\alpha(a) - 1}{\alpha(a)}. \tag{4.6}$$

Corollary 4.6. *Assume that Assumption 4.1 holds, with a constant β that satisfies (4.6). Define the solution $u(t)$ as in Theorem 1.1 in terms of the solution $v(t) \equiv \bar{v}$ of (1.2), so that $u(t)$ is a stationary solution, in the sense that $u(t) \equiv \bar{u}$ for every $t > 0$, is a square integrable random variable in H_μ for $\mu < 1 - \frac{1}{2\alpha} - \frac{\beta}{2}$.*

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