A RESULT ABOUT C^2-RECTIFIABILITY OF ONE-DIMENSIONAL RECTIFIABLE SETS. APPLICATION TO A CLASS OF ONE-DIMENSIONAL INTEGRAL CURRENTS.

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ABSTRACT. Let $\gamma, \tau : [a,b] \to \mathbf{R}^{k+1}$ be a couple of Lipschitz maps such that $\gamma' = |\gamma'|\tau$ almost everywhere in $[a,b]$. Then $\gamma([a,b])$ is a C^2-rectifiable set, namely it may be covered by countably many curves of class C^2 embedded in $\mathbf{R}^{k+1}$. As a consequence, projecting the rectifiable carrier of a one-dimensional generalized Gauss graph provides a C^2-rectifiable set.

1. Introduction

Let $f \in C^1(\mathbf{R}^n)$, $F \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ and consider the closed set

$$K := \{x \in \mathbf{R}^n \mid \nabla f(x) = F(x)\}.$$

Observe that if x_0 is an internal point of K then f is of class C^2 within a small ball centered at x_0. Thus, in particular, the graph of $f|K^\circ$ is C^2-rectifiable. Recall that a subset of a Euclidean space is said to be C^2-rectifiable if $\mathcal{H}^n$-almost all of it may be covered by countably many n-dimensional submanifolds of class C^2, [1]. As an obvious consequence, the graph of $f|K$ has to be C^2-rectifiable provided

$$\mathcal{L}^n(K \backslash K^\circ) = 0. \tag{1.1}$$

Quite surprisingly, this fact does not necessarily occur without assuming condition (1.1). For the convenience of the reader, we shall now present a counterexample and retrace some steps from [1, Appendix], where such a counterexample is given. Let $n = 1$ and

$$F(x) := 0, \quad f(x) := \int_0^x \operatorname{dist}(t, E)^{1/2}\, dt \qquad (x \in \mathbf{R}) \tag{1.2}$$

where E is a certain Cantor-like set of positive measure. Then $K = E$ (thus $K \backslash K^\circ = K$ has positive measure) and the graph of $f|K = f|E$ is not C^2-rectifiable, as it follows at once from the following result:

For every $\varphi \in C^2(\mathbf{R})$, the closed set

$$C_\varphi := \{\varphi = f|E\} = \{x \in E \mid \varphi(x) = f(x)\}$$

does not contain points of density. Thus $\mathcal{L}^1(C_\varphi) = 0$.

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In order to verify such a statement, assume by absurd that a point x_0 of density of C_φ exists. Then an easy argument shows that $\varphi'(x_0) = 0$, compare [7, Proof of Lemma 3.1], hence a positive constant c has to exist such that

$$|f(x) - f(x_0)| = |\varphi(x) - \varphi(x_0) - \varphi'(x_0)(x - x_0)| < c|x - x_0|^2$$

for all $x \in C_\varphi$, with $|x - x_0| \leq 1$. If $\beta := 1$ and F_{x_0} denotes the set which corresponds to the constant c according to [1, Proposition 4.5], it follows that

$$F_{x_0} \cap [x_0 - 1, x_0 + 1] \subset [x_0 - 1, x_0 + 1] \backslash C_\varphi$$

by (a) in [1, Proposition 4.5]. As a consequence F_{x_0} has density zero at x_0, which contradicts (b) in [1, Proposition 4.5].

By invoking [5, Remark 4.1], one can immediately get convinced that the arguments above may be restated in the context of countably n-rectifiable sets G generalizing the notion of Gauss map graph, namely such that

$$G \subset \mathbf{R}^{n+1} \times \mathbf{S}^n$$

and

$$\nu \perp T_P(\pi G) \tag{1.3}$$

is satisfied at a.e. $(P, \nu) \in G$ such that $d(\pi|G)_P$ exists and has rank n, where π denotes the projection on the first component, i.e. $\pi(P, \nu) := P$. In particular, the set πG has not necessarily to be C^2-rectifiable and the function f defined in (1.2) provides again an easy counterexample. Indeed, the set

$$G := \{((x, f(x)); (0, 1)) \,|\, x \in E\} \subset \mathbf{R}^2 \times \mathbf{S}^1 \tag{1.4}$$

is obviously 1-rectifiable and condition (1.3) is satisfied, in that $T_P(\pi G)$ coincides with the x-axis at every $P \in \pi G$. But πG is just the graph of $f|E$ which, as we have recalled above, is not C^2-rectifiable.

Incredibly enough, for quite a long time after [1] appeared, the people working on this subject continued to try to prove that the image by π of a set G as above had to be countably n-rectifiable of class C^2. Such a (false) statement seemed actually to be proved by Fu in the paper [9], which was followed by [5] where a simpler proof was presented. Subsequently, a mistake was found in the Fu's argument, see [6] and its successive generalization [7] (see also [10] for further details). Eventually, in a recent private comunication, Fu brought the counterexample in [1] to our attention.

As for the bug consequently affecting the proof in [5]), one can easily verify that it is necessarily due to the preliminary result [1, Lemma 3.6]. Indeed such a result erroneously asserts that a certain number a, involved in its statement, depends on only the Lipschitz constant A of the tangent spaces field and more precisely that

$$a(A) = \frac{1}{2(8A + 1)}$$

e.g. for the set R defined in (1.4), one has $a(A) = a(0) = 1/2$. Actually, if this were true, the simple Proposition 4.6 in [5] would imply (via the Whitney extension Theorem, invoked in [5, Theorem 3.1]) the C^2-rectifiability of πG. On the other hand, the falseness of [5, Proposition 4.6] can also be proved by a direct computation based on the example exhibited in [6, §4.2]. In fact, if $f \in C^1(\mathbf{R})$ is

the function considered in such an example, [5, Proposition 4.6] would yield the existence of $\lambda > 0$ such that
$$f(\varepsilon_j) < \lambda \varepsilon_j^2, \qquad \varepsilon_j := 2^{-j}$$
provided j is big enough. But this inequality contradicts [6, Proposition 4.3], according to which one has
$$\lim_j \frac{f(\varepsilon_j)}{\varepsilon_j^2} = \lim_j f_{2,\varepsilon_j}(1) = +\infty.$$

The main achievement of the present work is the following result:

Theorem 1.1. *Let be given a couple of Lipschitz maps*
$$\gamma, \tau : [a, b] \to \mathbf{R}^{k+1}$$
satisfying the equality
$$\gamma' = |\gamma'|\tau \tag{1.5}$$
almost everywhere in $[a, b]$*. Then* $\gamma([a, b])$ *is a* C^2*-rectifiable set.*

As a consequence of Theorem 1.1, we finally get a sufficient condition for the C^2-rectifiability of πG, where G is a one-dimensional set of the type described above. More precisely, in Theorem 5.1 we state that πG is C^2-rectifiable, provided G carries a one-dimensional generalized Gauss graph in $\mathbf{R}^{k+1}$, [2, 4].

From our point of view, the context of generalized Gauss graphs is the one where the fallacious arguments discussed above emerged and where we expect to find interesting applications to geometric variational problems. Theorem 5.1 represents the first step of a long term program. Future work will naturally be devoted to extend such a result in two different directions, corresponding respectively to higher dimension and higher order of rectifiability.

2. Preliminaries I (general dimension)

Let n, k be a couple of positive integers and denote by $G(n + k, n)$ the Grassmannian manifold of all n-dimensional linear subspaces of $\mathbf{R}^{n+k}$. If V is a linear subspace of $\mathbf{R}^{n+k}$ then its orthogonal complement is indicated with $V^\perp$, while P_V is the orthogonal projection mapping $\mathbf{R}^{n+k}$ onto V. If $V, W \in G(n + k, n)$, then $\mathcal{L}(V, W)$ is the vector space of linear operators from V to W. The graph of $L \in \mathcal{L}(V, W)$ is denoted by G_L.

Definition 2.1. *Given an integer* $i \geq 1$*, a real number* $\delta > 0$ *and* $V \in G(n + k, n)$*, let us define the set*
$$\Gamma_i(V, \delta) := \left\{ Q \in \mathbf{R}^{n+k} \,:\, |Q - P_V Q| \geq \delta |P_V Q|^i \right\}.$$

Remark 2.1. The space $V^\perp$ is a subset of $\Gamma_i(V, \delta)$, for all δ and i.

The following simple result characterizes the n-dimensional subspaces not intersecting (except for the origin!) the cone $\Gamma_1(V, \delta)$.

Proposition 2.1. *Let $W \in G(n+k,n)$. Then*

$$W \cap \Gamma_1(V,\delta) = \{0\} \tag{2.1}$$

if and only if there exists $L \in \mathcal{L}(V, V^\perp)$ such that

$$W = G_L, \qquad \|L\| < \delta.$$

Proof. Assume that (2.1) holds. Let $\{w_1, \dots, w_n\}$ be a basis of W and set

$$v_i := P_V w_i \qquad (i = 1, \dots, n).$$

Let us prove that the v_i are linearly independent (hence a basis of V). Indeed, if c_i $(i = 1, \dots, n)$ are real constants such that

$$\sum_{i=1}^n c_i v_i = 0 \qquad \text{i.e.} \qquad P_V\left(\sum_{i=1}^n c_i w_i\right) = 0,$$

then, by recalling Remark 2.1, we find

$$\sum_{i=1}^n c_i w_i \in V^\perp \cap W \subset \Gamma_1(V,\delta) \cap W = \{0\}.$$

It follows that $c_i = 0$ (for $i = 1, \dots, n$), hence $\{v_1, \dots, v_n\}$ is a basis of V. Then W is just the graph of the linear operator $L : V \to V^\perp$ such that

$$L(v_i) = w_i - P_V w_i = w_i - v_i \qquad (i = 1, \dots, n).$$

In order to prove that $\|L\| < \delta$, consider $v \in V \backslash \{0\}$ and observe that, since $v + Lv \in W \backslash \{0\}$ and (2.1) holds, then

$$v + Lv \notin \Gamma_1(V,\delta).$$

It follows that

$$|v + Lv - P_V(v + Lv)| < \delta |P_V(v + Lv)|$$

i.e.

$$|Lv| < \delta |v|.$$

Vice versa, suppose that W coincides with the graph of a linear operator $L : V \to V^\perp$ such that $\|L\| < \delta$. Then consider $Q \in W \backslash \{0\}$, i.e. $Q = v + Lv$ for some $v \in V \backslash \{0\}$. One has

$$|Q - P_V Q| = |Lv| \le \|L\| \, |v| < \delta |v| = \delta |P_V Q|$$

namely $Q \notin \Gamma_1(V,\delta)$. Hence $W \cap \Gamma_1(V,\delta) = \{0\}$. $\square$

Now we shall prove the following expected result.

Proposition 2.2. *Let λ, μ be positive real numbers and $V_0, V \in G(n+k,n)$ satisfy*

$$V \cap \Gamma_1(V_0, \lambda) = \{0\}.$$

Then there exists $\bar{r} = \bar{r}(\lambda, \mu, V_0, V)$ positive, continuous in its arguments and such that

$$\Gamma_1(V_0,\lambda) \cap B_{\bar{r}}(0) \subset \Gamma_2(V,\mu).$$

Proof. Let $e \in \mathbf{S}^{n-1}(V)$, $\varepsilon \in \mathbf{S}^{k-1}(V^\perp)$ and define

$$e_0 := P_{V_0} e, \qquad \varepsilon_0 := P_{V_0}\varepsilon, \qquad e_\perp := e - e_0 = P_{V_0^\perp} e, \qquad \varepsilon_\perp := \varepsilon - \varepsilon_0 = P_{V_0^\perp}\varepsilon$$

$$A := \lambda^2 |e_0|^2 - |e_\perp|^2, \qquad B := \lambda\, e_0 \cdot \varepsilon_0 - e_\perp \cdot \varepsilon_\perp, \qquad C := \lambda^2 |\varepsilon_0|^2 - |\varepsilon_\perp|^2.$$

Then one has

$$se + t\varepsilon \in \Gamma_1(V_0, \lambda)$$

if and only if

$$|se_\perp + t\varepsilon_\perp| \ge \lambda |se_0 + t\varepsilon_0|$$

i.e.

$$As^2 + 2Bst + Ct^2 \le 0.$$

Since $A > 0$, in that $e \notin \Gamma_1(V_0, \lambda)$, this inequality is equivalent to

$$(As + Bt)^2 - (B^2 - AC)t^2 \le 0.$$

It follows that

(i) if $B^2 - AC < 0$ then $\Gamma_1(V_0, \lambda) \cap [e, \varepsilon] = \{0\}$;
(ii) if $B^2 - AC \ge 0$ then $\Gamma_1(V_0, \lambda) \cap [e, \varepsilon]$ is the cone included in $[e, \varepsilon]$, bounded by the lines

$$As + \left(B + (B^2 - AC)^{1/2}\right) t = 0, \qquad As + \left(B - (B^2 - AC)^{1/2}\right) t = 0$$

and not containing e (indeed $e \in V$ and $V \cap \Gamma_1(V_0, \lambda) = \{0\}$).

On the other hand, one has

$$\Gamma_2(V, \mu) \cap [e, \varepsilon] = \left\{ se + t\varepsilon \,\middle|\, |t| \ge \mu s^2 \right\}$$

hence the set

$$I(e, \varepsilon) := \{ r \ge 0 \,|\, \Gamma_1(V_0, \lambda) \cap [e, \varepsilon] \cap B_r(0) \subset \Gamma_2(V, \mu) \}$$

is a compact interval, with

$$r(e, \varepsilon) := \max I(e, \varepsilon) > 0$$

for all (e, ε) belonging to

$$K := \left\{ (e, \varepsilon) \in \mathbf{S}^{n-1}(V) \times \mathbf{S}^{k-1}(V^\perp) \,|\, B^2 - AC \ge 0 \right\}.$$

Since $r : K \to \mathbf{R}$ is continuous and K is compact, there exists $(\bar{e}, \bar{\varepsilon}) \in K$ such that

$$\bar{r} := r(\bar{e}, \bar{\varepsilon}) = \min_K r > 0.$$

Then

$$\Gamma_1(V_0, \lambda) \cap [e, \varepsilon] \cap B_{\bar{r}}(0) \subset \Gamma_2(V, \mu)$$

for all $(e, \varepsilon) \in K$. Now the conclusion follows by observing that

$$\Gamma_1(V_0, \lambda) = \bigcup_{(e,\varepsilon) \in \mathbf{S}^{n-1}(V) \times \mathbf{S}^{k-1}(V^\perp)} \Gamma_1(V_0, \lambda) \cap [e, \varepsilon]$$

□

Corollary 2.1. *Let $\lambda, \bar{\lambda}, \mu$ be positive real numbers with $\bar{\lambda} > \lambda$ and let $V_0 \in G(n+k, n)$. Then there exists $\rho = \rho(\lambda, \bar{\lambda}, \mu, V_0) > 0$ such that*

$$\Gamma_1(V_0, \bar{\lambda}) \cap B_\rho(0) \subset \Gamma_2(V, \mu)$$

for all $V \in G(n+k, n)$ such that $V \cap \Gamma_1(V_0, \lambda) = \{0\}$.

Proof. The image of the compact ball

$$\mathcal{B} := \left\{ L \in \mathcal{L}(V_0, V_0^\perp) \mid \|L\| \le \lambda \right\}$$

through the continuous map

$$G : \mathcal{L}(V_0, V_0^\perp) \to G(n+k, n), \quad L \mapsto G_L$$

has to be compact. Then the function

$$G(\mathcal{B}) \ni V \mapsto \bar{r}(\bar{\lambda}, \mu, V_0, V)$$

has a minimizer, by Poposition 2.2. If $\rho = \rho(\lambda, \bar{\lambda}, \mu, V_0)$ denotes the corresponding minimum value, then one obviously has

$$\Gamma_1(V_0, \bar{\lambda}) \cap B_\rho(0) \subset \Gamma_2(G_L, \mu)$$

for all $L \in \mathcal{B}$. Finally, Proposition 2.1 completes the proof. $\square$

3. Preliminaries II (dimension one)

In this section we will deal with the special case $n = 1$. We begin by stating a very simple preliminary result.

Lemma 3.1. *Let A be a closed subset of $\mathbf{R}$ and*

$$f, d : A \to \mathbf{R}$$

be a couple of bounded functions such that

$$|f(y) - f(x) - d(x)(y - x)| \le C|y - x|^2 \tag{3.1}$$

for all $x, y \in A$, where C is a constant. Then there exists $F \in C^{1,1}(\mathbf{R})$ such that $F|A = f$.

Proof. The inequality (3.1) yields

$$\begin{aligned} |(d(x) - d(y))\,(y - x)| &= |d(x)(y - x) + f(x) - f(y) + d(y)(x - y) + f(y) - f(x)| \\ &\le |d(x)(y - x) + f(x) - f(y)| \,+\, |d(y)(x - y) + f(y) - f(x)| \\ &\le 2C|y - x|^2 \end{aligned}$$

i.e.

$$|d(x) - d(y)| \le 2C|y - x|$$

for all $x, y \in A$. The conclusion follows at once from the Whitney extension Theorem [14, §2.3]. $\square$

Actually the result we will need to invoke below is the following immediate corollary of Lemma 3.1.

Proposition 3.1. *Let $f \in C^1(\mathbf{R})$ be such that*

$$|f(y) - f(x) - f'(x)(y - x)| \le C|y - x|^2 \tag{3.2}$$

for all x, y in a bounded set $M \subset \mathbf{R}$, where C is a constant. Then there exists $F \in C^{1,1}(\mathbf{R})$ such that $F|\overline{M} = f|\overline{M}$.

Proof. By continuity, the inequality (3.2) holds for all $x, y \in \overline{M}$. Then apply Lemma 3.1 with $A := \overline{M}$ and $d := f'|\overline{M}$. $\square$

Now we are ready to prove the following theorem.

Theorem 3.1. *Let E be a $\mathcal{H}^1$-measurable subset of $\mathbf{R}^{k+1}$. Assume that there exist*

$$V_0 \in G(k+1,1), \qquad \lambda, \mu, r > 0$$

and a field of lines

$$E \to G(k+1,1), \quad a \mapsto W_a$$

such that

$$W_a \cap \Gamma_1(V_0,\lambda) = \{0\}, \qquad E \cap (a + \Gamma_2(W_a,\mu)^\circ) \cap B_r(a) = \emptyset$$

for all $a \in E$. Then E is C^2-rectifiable.

Proof. Let $\bar\lambda > \lambda$. Then, by Corollary 2.1, there exists a positive real number $\rho = \rho(\lambda, \bar\lambda, \mu, V_0)$ such that

$$\Gamma_1(V_0,\bar\lambda) \cap B_\rho(0) \subset \Gamma_2(W_a,\mu)$$

for all $a \in E$. Recalling the assumed condition, we get

$$E \cap (a + \Gamma_1(V_0,\bar\lambda)^\circ) \cap B_{\bar\rho}(a) \subset E \cap (a + \Gamma_2(W_a,\mu)^\circ) \cap B_{\bar\rho}(a) = \emptyset \tag{3.3}$$

for all $a \in E$, where

$$\bar\rho := \min\{\rho, r\}.$$

It follows that E is a C^1-rectifiable set, by [11, Lemma 15.13]. Then, without affecting the generality of our argument, we may assume that there exist a function

$$f \in C^1(V_0, \mathbf{R}^k) \cong C^1(\mathbf{R}, \mathbf{R}^k)$$

and a measurable set

$$M \subset V_0 \cong \mathbf{R}$$

such that

$$E = G_{f|M}$$

and M (hence E) has density one at all of its points. Since E is a countable union of measurable sets of diameter less than $\bar\rho$, we can also suppose that

$$E \subset B_{\bar\rho}(a) \subset B_r(a)$$

for all $a \in E$. Hence

$$E \cap (a + \Gamma_1(V_0,\bar\lambda)^\circ) = E \cap (a + \Gamma_2(W_a,\mu)^\circ) = \emptyset \tag{3.4}$$

for all $a \in E$, by (3.3).

Now, let us consider $x_0, x \in M$ and define

$$P_0 := (x_0, f(x_0)), \qquad P := (x, f(x)), \qquad \tau := (1, f'(x_0)), \qquad \hat\tau := \frac{\tau}{|\tau|}$$

and

$$\Delta := P - (P_0 + (x - x_0)\tau) = (0, f(x) - f(x_0) - f'(x_0)(x - x_0)).$$

Observe that one has

$$|f'(x_0)| < \lambda < \bar\lambda \tag{3.5}$$

by assumption and Proposition 2.1.

From (3.4), we obtain

$$\begin{aligned} |\Delta - (\Delta \cdot \hat{\tau})\hat{\tau}| &\leq \mu |(P - P_0) \cdot \hat{\tau}|^2 \\ &\leq \mu |P - P_0|^2 \\ &= \mu (|x - x_0|^2 + |f(x) - f(x_0)|^2) \\ &\leq \mu (1 + \bar{\lambda}^2) |x - x_0|^2 \end{aligned}$$

i.e.

$$|\Delta|^2 - |\Delta \cdot \hat{\tau}|^2 \leq \mu^2 \left(1 + \bar{\lambda}^2\right)^2 |x - x_0|^4. \tag{3.6}$$

Since

$$|\Delta \cdot \hat{\tau}|^2 = \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|^2 |f'(x_0)|^2}{1 + |f'(x_0)|^2} = \frac{|f'(x_0)|^2}{1 + |f'(x_0)|^2} |\Delta|^2$$

and recalling (3.5), we find

$$|\Delta|^2 - |\Delta \cdot \hat{\tau}|^2 = \frac{|\Delta|^2}{1 + |f'(x_0)|^2} \geq \frac{|\Delta|^2}{1 + \bar{\lambda}^2}.$$

Then (3.6) yields the inequality

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \mu \left(1 + \bar{\lambda}^2\right)^{3/2} |x - x_0|^2.$$

From Proposition 3.1 it follows that there exists $F \in C^{1,1}(\mathbf{R}, \mathbf{R}^k)$ such that

$$F|M = f|M.$$

Now the conclusion readily follows from [8, Theorem 3.1.15], according to which F has to coincide with a map $C^2(\mathbf{R}, \mathbf{R}^k)$ except for a measurable set of arbitrarily small measure. □

4. The proof of Theorem 1.1

First of all, define the set

$$J := \{t \in [a, b] \,|\, \gamma'(t), \tau'(t) \text{ exist}, \gamma'(t) \neq 0\}.$$

and observe that

$$\mathcal{H}^1 (\gamma([a, b]) \backslash \gamma(J)) \leq \mathcal{H}^1 (\gamma([a, b] \backslash J)) = \int_{[a,b] \backslash J} |\gamma'| = 0.$$

Let $\varepsilon > 0$ be fixed arbitrarily. Then, by the Lusin Theorem, a closed subset J_ε of J has to exist such that

$$\gamma' | J_\varepsilon \text{ is continuous and } \mathcal{L}^1(J \backslash J_\varepsilon) \leq \varepsilon.$$

If A denotes the Lipschitz constant of the map γ, we obtain

$$\mathcal{H}^1 (\gamma(J) \backslash \gamma(J_\varepsilon)) \leq \mathcal{H}^1 (\gamma(J \backslash J_\varepsilon)) = \int_{J \backslash J_\varepsilon} |\gamma'| \leq A\varepsilon.$$

Now let J_ε^* be the set of points of density of J_ε. Since J_ε is closed, one has

$$J_\varepsilon^* \subset J_\varepsilon.$$

Moreover

$$\mathcal{L}^1(J_\varepsilon \backslash J_\varepsilon^*) = 0$$

by a well known Lebesgue's result, hence

$$\mathcal{H}^1\left(\gamma(J_\varepsilon)\backslash\gamma(J_\varepsilon^*)\right) \le \mathcal{H}^1\left(\gamma(J_\varepsilon\backslash J_\varepsilon^*)\right) = 0.$$

Due to the remarks above and by the arbitrariness of ε, we are reduced to prove that

$$\gamma(J_\varepsilon^*) \text{ is } C^2\text{-rectifiable.} \tag{4.1}$$

The main step in proving the assertion (4.1) will be to show that if $t_0 \in J_\varepsilon^*$ satisfies

$$\gamma'(t_0) = |\gamma'(t_0)|\tau(t_0) \qquad (\text{hence } |\tau(t_0)| = 1) \tag{4.2}$$

then the ratio

$$R(t_0,t) := \frac{|\gamma(t)-\gamma(t_0)\ -\ ((\gamma(t)-\gamma(t_0))\cdot\tau(t_0))\tau(t_0)|}{\left|(\gamma(t)-\gamma(t_0))\cdot\tau(t_0)\right|^2}$$

exists and converges as $t \to t_0$. Finally, we will complete the proof by an easy argument based on Theorem 3.1.

Consider $t_0 \in J_\varepsilon^*$ satisfying (4.2). We can also assume

$$\gamma(t_0) = 0$$

without affecting the generality of our argument. Observe that, for a.e. $t \in J$, one has

$$0 = \frac{\tau(t)\cdot\tau(t)-\tau(t_0)\cdot\tau(t_0)}{t-t_0} = (\tau(t)+\tau(t_0))\cdot\frac{\tau(t)-\tau(t_0)}{t-t_0}.$$

Letting t tend to t_0, at which J has density one, it follows that

$$\tau(t_0)\cdot\tau'(t_0) = 0. \tag{4.3}$$

For $s \in J$, define

$$\rho_1(s) := |\gamma'(s)| - |\gamma'(t_0)|, \qquad \rho_2(s) := \frac{\tau(s)-\tau(t_0)}{s-t_0} - \tau'(t_0)$$

which satisfy

$$\lim_{\substack{s\to t_0\\ s\in J_\varepsilon}} \rho_1(s) = 0, \qquad \lim_{\substack{s\to t_0\\ s\in J}} \rho_2(s) = 0. \tag{4.4}$$

For $t \in [a,b]$ one has

$$\gamma(t) = \int_{[t_0,t]} \gamma' = \int_{[t_0,t]} |\gamma'|\tau = \int_{[t_0,t]\cap J_\varepsilon} |\gamma'|\tau + \int_{[t_0,t]\backslash J_\varepsilon} |\gamma'|\tau \tag{4.5}$$

by assumption. The first integral in the right hand side of (4.5) can be written as follows

$$\begin{aligned} \int_{[t_0,t]\cap J_\varepsilon} |\gamma'|\tau &= \int_{[t_0,t]\cap J_\varepsilon} (|\gamma'(t_0)|+\rho_1(s))\left(\tau(t_0)+(s-t_0)\tau'(t_0)+(s-t_0)\rho_2(s)\right)ds \\ &= (|\gamma'(t_0)|(t-t_0)+\sigma_1(t))\,\tau(t_0)+\left(\frac{|\gamma'(t_0)|}{2}(t-t_0)^2+\sigma_2(t)\right)\tau'(t_0)+\sigma_3(t) \end{aligned} \tag{4.6}$$

where

$$\sigma_1(t) := -|\gamma'(t_0)|\left(t-t_0-|J_\varepsilon\cap[t_0,t]|\right) + \int_{[t_0,t]\cap J_\varepsilon} \rho_1$$

$$\begin{aligned} \sigma_2(t) &:= -|\gamma'(t_0)|\left(\frac{(t-t_0)^2}{2} - \int_{[t_0,t]\cap J_\varepsilon} s-t_0\,ds\right) + \int_{[t_0,t]\cap J_\varepsilon} (s-t_0)\rho_1(s)\,ds \\ &= -|\gamma'(t_0)|\int_{[t_0,t]\backslash J_\varepsilon} s-t_0\,ds + \int_{[t_0,t]\cap J_\varepsilon} (s-t_0)\rho_1(s)\,ds \end{aligned}$$

and

$$\sigma_3(t) := |\gamma'(t_0)| \int_{[t_0,t]\cap J_\varepsilon} (s-t_0)\rho_2(s)\, ds + \int_{[t_0,t]\cap J_\varepsilon} (s-t_0)\rho_1(s)\rho_2(s)\, ds.$$

Observe that

$$\sigma_1(t) = o(t-t_0), \qquad \sigma_2(t) = o(t-t_0)^2, \qquad \sigma_3(t) = o(t-t_0)^2 \tag{4.7}$$

by (4.4) and recalling that t_0 is a point of density of J_ε.

As for the second integral in the right hand side of (4.5), notice that

$$\sigma_4(t) := \int_{[t_0,t]\setminus J_\varepsilon} |\gamma'|\tau = o(t-t_0) \tag{4.8}$$

by the assumptions. It satisfies

$$\sigma_4(t) - (\sigma_4(t)\cdot\tau(t_0))\,\tau(t_0) = \int_{[t_0,t]\setminus J_\varepsilon} |\gamma'(s)|\,(\tau(s) - (\tau(s)\cdot\tau(t_0))\tau(t_0))\,. \tag{4.9}$$

Since $|\tau(t_0)| = 1$, one has

$$\begin{aligned} \tau(s) - (\tau(s)\cdot\tau(t_0))\tau(t_0) &= \tau(s) - \tau(t_0) + (1 - \tau(s)\cdot\tau(t_0))\tau(t_0) \\ &= \tau(s) - \tau(t_0) + ((\tau(t_0) - \tau(s))\cdot\tau(t_0))\,\tau(t_0) \end{aligned}$$

hence

$$|\tau(s) - (\tau(s)\cdot\tau(t_0))\tau(t_0)| \le 2B|s-t_0|$$

where B is the Lipschitz constant of τ. Then (4.9) yields

$$|\sigma_4(t) - (\sigma_4(t)\cdot\tau(t_0))\,\tau(t_0)| \le 2AB \int_{[t_0,t]\setminus J_\varepsilon} |s-t_0|\, ds = o(t-t_0)^2. \tag{4.10}$$

Now we are ready to compute the limit of $R(t_0,t)$, as $t\to t_0$. To this aim, observe that the formulas (4.3), (4.5), (4.6), (4.7), (4.8) and (4.10) obtained above imply

$$\gamma(t)\cdot\tau(t_0) = |\gamma'(t_0)|\,(t-t_0) + o(t-t_0)$$

and

$$\begin{aligned} \gamma(t) - (\gamma(t)\cdot\tau(t_0))\tau(t_0) &= \left(\frac{|\gamma'(t_0)|}{2}(t-t_0)^2 + \sigma_2(t)\right)\tau'(t_0) + \sigma_3(t) + \sigma_4(t) + \\ &\quad - ((\sigma_3(t)+\sigma_4(t))\cdot\tau(t_0))\,\tau(t_0) \\ &= \frac{|\gamma'(t_0)|}{2}(t-t_0)^2\tau'(t_0) + \sigma_2(t)\tau'(t_0) + \sigma_3(t) - (\sigma_3(t)\cdot\tau(t_0))\,\tau(t_0) + \\ &\quad + \sigma_4(t) - (\sigma_4(t)\cdot\tau(t_0))\,\tau(t_0) \\ &= \frac{|\gamma'(t_0)|}{2}(t-t_0)^2\tau'(t_0) + o(t-t_0)^2. \end{aligned}$$

Thus $R(t_0,t)$ exists for t in a neighborhood of t_0 and one has

$$R(t_0,t) = \frac{\frac{|\gamma'(t_0)|}{2}(t-t_0)^2|\tau'(t_0)| + o(t-t_0)^2}{|\gamma'(t_0)|^2(t-t_0)^2 + o(t-t_0)^2} \to \frac{|\tau'(t_0)|}{2|\gamma'(t_0)|} < +\infty \tag{4.11}$$

as $t\to t_0$.

In order to complete the proof of the statement (4.1), let us consider (for $i\in\mathbf{N}$) the set $\Sigma^{(i)}$ of all $t_0\in J^*_\varepsilon$ such that:

- one has $\gamma'(t_0) = |\gamma'(t_0)|\tau(t_0)$;
- the estimates
$$|\tau(t) - \tau(t_0)| < 1, \qquad R(t_0, t) \leq i$$
hold whenever $|t - t_0| \leq (b-a)/i$ (note: the first one is verified provided $|t - t_0| < 1/B$).

Observe that
$$\Sigma^{(i)} \subset \Sigma^{(i+1)} (\subset J_\varepsilon^*)$$
for all $i \in \mathbf{N}$. Moreover
$$\cup_{i\in\mathbf{N}} \Sigma^{(i)} = \{t_0 \in J_\varepsilon^* \mid \gamma'(t_0) = |\gamma'(t_0)|\tau(t_0)\}$$
by (4.11), hence
$$\mathcal{L}^1 \left(J_\varepsilon^* \backslash \cup_{i\in\mathbf{N}} \Sigma^{(i)} \right) = 0. \tag{4.12}$$
Then, given $i \in \mathbf{N}$, consider the uniform partition of $[a, b]$
$$a_j^{(i)} := a + \frac{(b-a)j}{i} \qquad (j = 0, 1, \ldots, i)$$
and define
$$\Sigma_j^{(i)} := \Sigma^{(i)} \cap [a_j^{(i)}, a_{j+1}^{(i)}] \qquad (j = 0, \ldots, i-1).$$
Then the C^2-rectifiability of
$$E := \gamma(\Sigma_j^{(i)})$$
follows from Theorem 3.1, where

$$V_0 \text{ is the line generated by } \tau(\bar{t}_0), \text{ with } \bar{t}_0 \in \Sigma_j^{(i)} \text{ fixed arbitrarily}$$

$$W_{\gamma(t_0)} \text{ is the tangent line to } \gamma \text{ at } \gamma(t_0),\ t_0 \in \Sigma_j^{(i)} \text{ (note: it is generated by } \tau(t_0))$$

$$\lambda := \frac{\sqrt{3}}{2}, \qquad \mu := i$$
and r is positive and chosen arbitrarily. By recalling (4.12), we finally end the proof of the assertion (4.1).

5. Application to one-dimensional generalized Gauss graphs

In this section it is proved the result about the carrier of a one-dimensional generalized Gauss graph announced in the Introduction. Let us recall from [2, 4] that a "one-dimensional generalized Gauss graph (in $\mathbf{R}^{k+1}$)" is an integral current (see [8, 12, 13])
$$T \in \mathbf{I}_1(\mathbf{R}^{k+1} \times \mathbf{R}^{k+1})$$
such that:

(i) The carrier G of T is equivalent in measure to a subset of $\mathbf{R}^{k+1} \times \mathbf{S}^k$, i.e.
$$\mathcal{H}^1(G \backslash (\mathbf{R}^{k+1} \times \mathbf{S}^k)) = 0;$$

(ii) If φ denotes the following 1-form in $\mathbf{R}^{k+1} \times \mathbf{R}^{k+1}$

$$(x,y) \mapsto \sum_{j=1}^{k+1} y_j dx_j$$

and $*$ is the usual Hodge star operator in $\mathbf{R}^{k+1}$, then one has:

- $T(*\varphi \llcorner \omega) = 0$ for all smooth $(k-1)$-forms with compact support in $\mathbf{R}^{k+1} \times \mathbf{R}^{k+1}$;
- $T(g\varphi) \geq 0$ for all nonnegative continuous functions with compact support in $\mathbf{R}^{k+1} \times \mathbf{R}^{k+1}$.

Incidentally, we can observe that a one-dimensional generalized Gauss graph T can have only finitely many indecomposable components. Indeed, if Σ is one of such components, then the normal mass of Σ is at least 2 or 2π according to whether $\partial\Sigma \neq 0$ or $\partial\Sigma = 0$, see [8, 4.2.25] and [3, Theorem 4.1].

We are finally ready to state and prove the result.

Theorem 5.1. *Let $T = [\![G, \eta, \theta]\!]$ be a one-dimensional generalized Gauss graph and π indicate the orthogonal projection*

$$\mathbf{R}^{k+1} \times \mathbf{R}^{k+1} \to \mathbf{R}^{k+1}, \quad (x,y) \mapsto x.$$

Then the set πG is C^2-rectifiable.

Proof. Without loss of generality, we can restrict our attention to the case when T is indecomposable. Then, by [8, 4.2.25], there exists an injective Lipschitz map

$$\Gamma : [0, \mathbf{M}(T)] \to \mathbf{R}^{k+1} \times \mathbf{S}^k$$

such that $\Gamma_\#[0, \mathbf{M}(T)] = T$. In particular G is parametrized by Γ and one has

$$\Gamma' = \eta \circ \Gamma$$

a.e. in $[0, \mathbf{M}(T)]$. If

$$\gamma : [0, \mathbf{M}(T)] \to \mathbf{R}^{k+1}, \quad \tau : [0, \mathbf{M}(T)] \to \mathbf{S}^k$$

denote the components of Γ in $\mathbf{R}^{k+1}$ and $\mathbf{S}^k$ respectively, then we easily obtain the equality

$$\gamma' = |\gamma'|\, \tau$$

a.e. in $[0, \mathbf{M}(T)]$, compare [4, Proposition 4.1]. Now the conclusion follows from Theorem 1.1. $\square$

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