Università di Trento
Dipartimento di Matematica

PH.D. SCHOOL IN MATHEMATICS

Geometric realizations of birational maps

Supervisors:
Prof. Luis E. Solá Conde
Prof. Eleonora A. Romano

Candidate:
Lorenzo Barban

XXXVI Cycle
Academic Year 2022-2023

Contents

Abstract

In this thesis we study the relation between algebraic torus actions on complex projective varieties and the birational geometry of their geometric quotients. Given a $\mathbb{C}^*$-action on a normal projective variety X, there exist two unique connected components of the fixed point locus, called the *sink* Y_- and the *source* Y_+, containing the limit at ∞ and 0 of the general orbit. Let $\mathcal{G}X_-$ (resp. $\mathcal{G}X_+$) be the variety parametrizing the orbits converging to the sink (resp. the source). Since there exists an open subset of points converging to $Y_\pm$, we obtain a birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$. By choosing different linearizations of ample line bundles on X, we obtain a factorization of the birational map ψ among inner geometric quotient, parametrizing different open subsets of stable points.

In this setting, we investigate the local analytic geometry of the birational map ψ. On one hand we link certain birational transformations, called *rooftop flips*, with varieties with two projective bundles structures. On the other we study when the birational map ψ can be locally described by a toric flip of *Atiyah type*.

If on one side a $\mathbb{C}^*$-action naturally induces a birational map among geometric quotients, it is meaningful to study the opposite direction: more precisely, given a birational map $\varphi\colon Z_+ \dashrightarrow Z_-$ among normal projective varieties, how can we construct a normal projective variety X, endowed with a $\mathbb{C}^*$-action, such that Z_- is the sink, Z_+ is the source, and the natural birational map ψ constructed above coincide with φ? Such an X is called a *geometric realization* of the birational map φ. We propose a construction of a geometric realization of φ, whose geometry reflects the factorization of the map as a composition of flips, blow-ups and blow-downs. We describe in particular the case in which φ is a small modification *of dream type*, namely a birational map which is an isomorphism in codimension 1 associated to a finitely generated multisection ring. Moreover, we show that the cone of divisors associated to such multisection rings admits a chamber decomposition where the models are the geometric quotients of the $\mathbb{C}^*$-action. If in addition $Z_\pm$ are assumed to be toric varieties, we construct a function in `SageMath` to compute the polytope of the associated toric geometric realization.

Preface

The original results presented in this manuscript are contained in the following papers:

1. Lorenzo Barban, Eleonora A. Romano. *Toric non-equalized flips associated to* $\mathbb{C}^*$*-actions.* Accepted to appear in the volume "Varieties, Polyhedra, Computations"of EMS Series of Congress Reports (2021);
2. Lorenzo Barban, Eleonora A. Romano, Luis E. Solá Conde, Stefano Urbinati. *Mori dream bonds and* $\mathbb{C}^*$*-actions.* arXiv: `https://arxiv.org/abs/2207.09864` (2022);
3. Lorenzo Barban, Alberto Franceschini. *Morelli–Włodarczyk cobordism and examples of rooftop flips.* Collectanea Mathematica (2023);
4. Lorenzo Barban, Gianluca Occhetta, Luis E. Solá Conde. *Geometric realization of toric small modifications.* In preparation.

The function `GeomReal`, written in `SageMath`, is accessible at the following link:
`https://cocalc.com/share/public_paths/a28daa428b12dfde5fec32ce200547f44fa38f4a`

Introduction

Algebraic torus actions and birational geometry

Over the last half century, birational geometry has grown as one of the leading research areas in algebraic geometry, thanks to the pioneering work of many distinguished mathematicians (among others, S. Mori, Y. Kawamata, C. Hacon, S. McKernan, M. Reid) in the context of the *Minimal model program* (MMP for short), whose goal is to classify complex projective varieties up to birational equivalence. The first step towards this program was the birational classification of algebraic surfaces, started by Castelnuovo in the XIX century and carried over by Enriques and Kodaira. In higher dimensions, the problem becomes much more difficult; one of the perhaps most important differences with the surface case is the need of considering a certain class of birational isomorphisms in codimension 1, called *flips*, as one of the building blocks of the theory.

Shortly after their discovery, it was noticed how flips arised naturally in the context of Mumford's *Geometric Invariant Theory*; indeed, thanks to the work of M. Reid and M. Thaddeus (see [56], [60]), it has been showed that there exists a flip among two different geometric quotients of a reductive group action G on a normal projective variety X endowed with an ample G-linearizable line bundle L. If moreover the algebraic group taken in consideration is the 1-dimensional algebraic torus, something more can be said; indeed, years later, the relation between birational geometry and algebraic torus actions was exploited by the work of J. Włodarczyk (see [65]), who proved the *Weak factorization conjecture*, stating that every birational map among smooth projective varieties $\varphi\colon X_- \dashrightarrow X_+$ can be factorized as a sequence of blow-ups and blow-downs along smooth centers. The technique used by J. Włodarczyk relies on constructing, using Hironaka's resolution of singularities, a *cobordism* of φ, namely a quasi-projective variety B, endowed with a $\mathbb{C}^*$-action such that $X_\pm$ are geometric quotients parametrizing different open subsets of stable points of B. A similar construction was already introduced by R. Morelli in the case of toric varieties and used to prove the *Oda Conjecture* (see [44]).

Years later, the existence of a relation between birational geometry and algebraic torus actions has brought to the notion of *Mori dream spaces* (shortly, MDS), introduced by Y. Hu and S. Keel in [25]; MDS's are a class of normal $\mathbb{Q}$-factorial projective varieties, containing for instance toric varieties and Fano varieties, which, on one hand, enjoy very nice properties from the point of MMP, and, on the other, whose birational geometry is determined by the different quotients of the affine variety associated to their *Cox ring*, that is a multisection ring, finitely generated for MDS's, which generalizes the concept of the homogeneous coordinate ring of the variety.

In recent years, the work of G. Occhetta, L. E. Solá Conde, E. A. Romano and J. A. Wiśniewski (see for instance [10, 49, 48]) has brought new light to the aforementioned relation. The idea is the following: consider a $\mathbb{C}^*$-action on a polarized pair (X, L), where X is a normal projective variety and L is an ample line bundle on X. For any connected component Y of the fixed point locus $X^{\mathbb{C}^*}$, we can define the Białynicki-Birula cells

$$X^{\pm}(Y) := \{x \in X \mid \lim_{t \to 0} t^{\pm 1} \cdot x \in Y, t \in \mathbb{C}^*\}.$$

By the Białynicki-Birula Theorem (see [4]) there exists a unique fixed point connected component Y_- (resp. Y_+) such that $X^-(Y_-)$ (resp. $X^+(Y_+)$) is a dense open subset of X. We call the subvariety Y_- (resp. Y_+) the *sink* (resp. the *source*) of the $\mathbb{C}^*$-action. Using [6], one can prove that $X^-(Y_-) \setminus Y_-$, $X^+(Y_+) \setminus Y_+$ are non-empty open subsets of stable points with respect to different linearizations of L, hence there exist two geometric quotients $\mathcal{G}X_\pm := X^\pm(Y_\pm) \setminus Y_\pm/\mathbb{C}^*$. We can naturally define a birational map

$$\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+,$$

defined over the intersection of the set of stable points. Since such a map is intrinsic to the $\mathbb{C}^*$-action on X, we call it the *natural birational map associated to the* $\mathbb{C}^*$*-action* [3, Remark 2.7]. Intuitively, this map takes a point corresponding to a unique orbit converging at ∞, and maps that point to the limit at 0 of the same orbit.

With this in mind, these authors were able to prove new results about the *LeBrun–Salamon conjecture* (see [50]), and also exploiting new aspects of the geometry of $\mathbb{C}^*$-varieties, such as when they are Mori dream spaces (see [48]), or describing their Chow quotient (see [46]). Parallel to this, a natural question arose: on one hand a $\mathbb{C}^*$-action on a polarized pair (X, L) naturally induces a birational map ψ as above; is such birational map ψ enough to encode the information necessary to explicitly reconstruct X? More precisely, given a birational map $\varphi\colon Y_- \dashrightarrow Y_+$ among normal projective varieties, does there exists a normal projective variety X, endowed with a $\mathbb{C}^*$-action, such that Y_- is the sink, Y_+ is the source, and the natural birational map ψ coincides with φ? Such a variety X is called a *geometric realization* of $\varphi\colon Y_- \dashrightarrow Y_+$. Notice that a geometric realization is projective by definition, in contrast to the quasi-projectivity of the cobordism of Morelli–Włodarczyk, and the goal is to construct such geometric realizations explicitly, and not using resolution of singularities.

Main results

In this thesis we study the natural birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ among the geometric quotients of a polarized pairs (X, L) under a $\mathbb{C}^*$-action, and the construction of geometric realizations of birational maps $\varphi\colon Y_- \dashrightarrow Y_+$. We report here a summary of our main results.

Motivated by the fact that the Morelli–Włodarczyk cobordism is a local model for the well known Atiyah flip, we have studied the local models for other known examples of small modifications, such as the Mukai flop. To this end, we have introduced the notion of *rooftop flips* (see Definition 3.2.1,) that is small modifications whose diagram of resolution of singularities resembles, at the level of the exceptional divisors, a variety with two projective bundles structures. In this setting, we prove the following:

Theorem (Theorem 3.2.12)**.** Given a smooth projective variety Λ of Picard number 2 with two projective bundle structures, there exist two quasi-projective varieties and a rooftop flip modeled by Λ among them.

We have then moved our study to the local analytic geometry of the natural birational map in the case of $\mathbb{C}^*$-actions of criticality 2, that is an action whose fixed point locus decomposes as $X^{\mathbb{C}^*} = Y_- \sqcup Y \sqcup Y_+$, with Y a finite collection of fixed point connected components all of the same weight (cf. Definition 2.1.41). Recall that a $\mathbb{C}^*$-action is *equalized* if every non fixed-point has trivial isotropy group (see Definition 2.1.26). As already observed in [48, Lemma 2.14], equalized $\mathbb{C}^*$-actions enjoyed several nice properties, such as the smoothness of the geometric quotients. Moreover, we say that a $\mathbb{C}^*$-action is a *bordism* if $Y_\pm$ are codimension 1 subvarieties, and the closure of every Białynicki-Birula cell of Y, for $Y \neq Y_\pm$, is not a divisor (cf. Definition 2.3.7). In the setting we show that equalized actions can be locally analytically described as toric Atiyah flips, and we present a criterion to understand when this holds:

Theorem (Theorem 4.1.7)**.** Consider a $\mathbb{C}^*$-action on a polarized pair (X, L) of criticality 2 which is a bordism. The natural birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is locally analytically a toric Atiyah flip if and only if the $\mathbb{C}^*$-action on X is equalized at every inner component.

We have also constructed an explicit example of a normal projective variety X, endowed with a $\mathbb{C}^*$-action, whose natural birational map is locally analytically a toric flip which is not Atiyah, and we call it of *non-equalized type* (cf. §4.1.3.2).

We have then focused our study to the construction of geometric realization of small modifications. Motivated by the notion of *Mori dream region*, we introduce the following:

Definition. [Definition 5.0.1] Let $\varphi\colon Y_- \dashrightarrow Y_+$ be a small modification among normal projective varieties. The map φ is *of dream type* if there exist A, F effective Cartier divisor on Y_- such that

- A is ample;
- F is movable and it holds that $Y_+ \simeq \operatorname{Proj} R(Y_-; \mathcal{O}_{Y_-}(F))$;
- the multisection ring

$$R(Y_-; \mathcal{O}_{Y_-}(A), \mathcal{O}_{Y_-}(F)) = \bigoplus_{a,b\geq 0} \mathrm{H}^0(Y_-, \mathcal{O}_{Y_-}(aA + bF))$$

  is a finitely generated $\mathbb{C}$-algebra.

Small modifications of dream type are the counterpart of bordism $\mathbb{C}^*$-actions equalized at the sink and the source, as explained in the following:

Theorem. [Theorems 5.1.1, 5.2.1] Let $\varphi\colon Y_- \dashrightarrow Y_+$ be a small modification among normal projective varieties of dream type. Then there exists a geometric realization X of φ, and the induced $\mathbb{C}^*$-action on X is a bordism equalized at $Y_\pm$. Conversely, given a $\mathbb{C}^*$-action on a polarized pair (X, L) which is a bordism equalized at the sink and the source, then the natural birational map $\psi\colon Y_- \dashrightarrow Y_+$ is a small modification of dream type.

Moreover, we give an explicit construction of such geometric realizations, which yields the observation that geometric realizations are not unique, but nevertheless they are $\mathbb{C}^*$-equivariantly birational. Moreover, we have showed that small modifications of dream type induce a chamber decomposition, where every chamber model is an inner geometric quotient of the $\mathbb{C}^*$-action on the geometric realization. In we assume in addition that $Y_\pm$ are toric varieties, we can prove the following:

Proposition. [§6.2] Let $\varphi\colon Y_- \dashrightarrow Y_+$ be a small $\mathbb{Q}$-factorial modification among normal, $\mathbb{Q}$-factorial projective toric varieties. Then there exists a geometric realization which is toric.

To do so, we produce an algorithm function in `SageMath`, called GeomReal, which computes the polytope associated to the toric geometric realization of a toric small modification (see §6.2).

While the natural birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is intrinsic to the $\mathbb{C}^*$-action on the variety X, the different choices of linearizations of an ample line bundle L induce factorizations of the map ψ

$$\mathcal{G}X_- \dashrightarrow \mathcal{G}X_1 \dashrightarrow \ldots \dashrightarrow \mathcal{G}X_+$$

through inner geometric quotients $\mathcal{G}X_i$, parametrizing different open subsets of stable points of the pair (X, L) (cf. Proposition 2.3.4). With this in mind, we can construct explicit geometric realizations of the birational maps among inner geometric quotients by performing a *pruning* of the variety X, that is a $\mathbb{C}^*$-equivariant birational modification of X whose properties are described in the following:

Theorem (Theorem 2.3.27)**.** Consider a $\mathbb{C}^*$-action on a polarized pair (X, L) of criticality r. Let $\rho_\pm$ be two rationals numbers such that $\rho_- \in (a_h, a_{h+1}), \rho_+ \in (a_j, a_{j+1})$, where we denote by a_i the L-weights of connected components of $X^{\mathbb{C}^*}$ and set $a_h < a_j$. There exists a normal projective variety $\widetilde{X}$, and a $\mathbb{C}^*$-action on X such that the sink of $\widetilde{X}$ is $\mathcal{G}X_h$, the source is $\mathcal{G}X_j$, and there exists a $\mathbb{C}^*$-equivariant birational map $\Phi\colon X \dashrightarrow \widetilde{X}$.

Structure of the thesis

In Chapter 1 we introduce the notation and basic background about algebraic group actions, divisors and their cones, and toric geometry we will use thoroughly in this work.

Chapter 2 presents the theory of $\mathbb{C}^*$-actions on polarized pairs (X, L), where X is a normal projective variety and L is an ample line bundle on X. In this setting we also characterize the geometric and semigeometric quotients, and explain the construction of the *pruning* of a variety, namely a normal projective variety with a $\mathbb{C}^*$-equivariant birational map to X (see Definition 2.3.24). We then recall basic notions regarding Mori dream spaces and Mori dream regions (see §2.4). We conclude this chapter by presenting some examples of varieties with interesting $\mathbb{C}^*$-actions: namely we introduce *rational homogeneous varieties* and study their relation with *smooth drums*, that is smooth projective varieties with a $\mathbb{C}^*$-action of bandwidth 1 (cf. §2.5.1.2, 2.5.2); we also show, using the theory of test configurations, that a normal projective $\mathbb{C}^*$-variety is birational to a weighted projective fibration (see Proposition 2.5.21).

In Chapter 3 we study the local geometry of the natural birational map among the geometric quotients. We first introduce the notion of *rooftop flip* (see Definition 3.2.1), that is a birational map whose resolution of indeterminacies resembles, at the level of exceptional divisors, a variety with two projective bundle structures. We then show that any smooth projective variety with two projective bundle structures induces a rooftop flip (cf. Theorem 3.2.12). We conclude presenting some applications to flips constructed upon rational homogeneous varieties.

Chapter 4 focuses on studying $\mathbb{C}^*$-actions on polarized pairs which are bordisms of criticality 2. In this setting, we study the case in which such birational maps are locally described by toric flips, presenting a criterion to understand if they are either of *Atiyah* or *non-equalized* type (see Theorem 4.1.7). We find explicit examples of rational homogeneous varieties admitting a $\mathbb{C}^*$-action whose natural birational map is locally a toric non-equalized flip (cf. §4.1.3.1, 4.1.3.2).

In Chapter 5 we introduce the notion of *small modification of dream type* (see Definition 5.0.1), that is a birational map $\varphi\colon Y_- \dashrightarrow Y_+$, isomorphism in codimension 1, such that there exists two Cartier divisors A, F such that the multisection ring $R(Y_-; \mathcal{O}_{Y_-}(A), \mathcal{O}_{Y_-}(F))$ is a finitely generated $\mathbb{C}$-algebra. Moreover, we show the correspondence between small modifications of dream type and $\mathbb{C}^*$-actions on polarized pairs which are bordisms (see Theorems 5.1.1, 5.2.1). We conclude studying the induced chamber decomposition of the cone generated by A, F (cf. Theorem 5.2.8).

In Chapter 6 we focus our study on constructing explicit geometric realizations of toric small $\mathbb{Q}$-factorial modifications among normal, $\mathbb{Q}$-factorial projective toric varieties. To this end, we construct a `SageMath` function which computes the polytope of the geometric realization (cf. §6.2). We conclude by showing some examples yielding future research directions.

Acknowledgements

First and foremost, I am grateful to my Ph.D. advisors, Prof. Luis E. Solá Conde and Prof. Eleonora A. Romano. Four years ago, in the second meeting with Luis about my master thesis, I expressed the interest to do a project about birational geometry; coincidentally, at that meeting there was also Eleonora, who appreciated my willingness and offered to help Luis co-advising me. That day, which I think of as one of the most important for my mathematical journey, was the first time I heard about $\mathbb{C}^*$-actions. I am undoubtedly thankful to them for their guidance in these years, their constant encouragement and more generally for believing in me.

My gratitude goes also to Prof. Stefano Urbinati and Dr. Alberto Franceschini, for the work done together and the numerous mathematical discussions.

I would like to express my sincere gratitude to Prof. Gianluca Occhetta, Prof. Roberto Muñoz, Prof. Jarek Wiśniewski and Prof. H. Abban for the numerous discussions we had about mathematics, their advices and their kindness in helping me.

I would like to thank the referees for their careful reading and useful comments.

The research presented in this manuscript was conducted during various stays at the University of Trento, University of Genova, University of Udine, University of Nottingham and Universidad Politécnica de Madrid: I would like to thank all these Departments for their kind hospitality.

Chapter 1

Notation and basic concepts

We work over the field of complex numbers. We will call a *variety* an integral separated scheme of finite type over $\mathbb{C}$. Given V a finite dimensional complex vector space, we use the Grothendieck notation for its projectivization, that is we denote by $\mathbb{P}(V)$ the space of 1-dimensional quotients of $V^{\vee}$. Given M a free abelian group, we will respectively denote by $M_{\mathbb{Q}}, M_{\mathbb{R}}$ the associated vector spaces with rational and real coefficients, that is $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Given two varieties Y_- and Y_+, we use the symbol $Y_{\pm}$ to describe the properties enjoyed by both varieties at the same time.

1.1 Algebraic group actions

An *algebraic group* G is a variety endowed with a group structure, such that the multiplication map and the inverse map are morphism of varieties. The neutral component of an algebraic group is the connected component $G^{\circ} \subset G$ containing the neutral element e of the group.

Given a variety X, a *G-action* on X is a morphism of varieties

$$G \times X \to X, \quad (g, x) \mapsto g \cdot x.$$

Any G-action induces an action on the coordinate ring $\mathbb{C}[X]$, defined as $(g \cdot f)(x) \coloneqq f(g^{-1} \cdot x)$. Given a variety X with a G-action, and given a point $x \in X$, we define the *orbit of* x as $G \cdot x \coloneqq \{g \cdot x \mid g \in G\}$. The *stabilizer of* x is $G_x \coloneqq \{g \in G \mid g \cdot x = x\}$. A point is *fixed by the G-action* if $G_x = G$. The *fixed point locus of* X is the closed set $X^G \coloneqq \{x \in X \mid g \cdot x = x \ \forall g \in G\}$.

Recall that an orbit $G \cdot x$ is a locally closed, smooth subvariety. Moreover, the closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and of orbits of smaller dimension. Any orbit of minimal dimension is closed.

Given a G-action on a variety X, we say that an action is:

- *trivial* if $g \cdot x = x$ for every $g \in G, x \in X$;
- *transitive* if, for any $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$;
- *free* if $G_x = \{e\}$ for any $x \in X$;
- *faithful* if group morphism $G \to \mathrm{Aut}(X)$ is injective, where by $\mathrm{Aut}(X)$ we denote the group of automorphism of X.

Given two varieties X, Z, both endowed with a G-action, a morphism $f \colon X \to Z$ is said to be *G-equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$, $g \in G$.

In our manuscript we will be interested in the *multiplicative group* $(\mathbb{C}^*, \cdot)$, also called *algebraic torus*, which is a smooth algebraic group of dimension 1. The coordinate of $\mathbb{C}^*$ will be denoted by t. For the sake of notation, we will always abbreviate the multiplicative group as $\mathbb{C}^*$.

Given an algebraic torus T, we define:

- the *set of characters* as $\mathrm{M}(T) \coloneqq \mathrm{Hom}(T, \mathbb{C}^*)$;
- the *set of* 1*-parameter subgroups* as $\mathrm{N}(T) \coloneqq \mathrm{M}(T)^\vee = \mathrm{Hom}(\mathbb{C}^*, T)$.

We have for instance that $\mathrm{M}(\mathbb{C}^*) \simeq \mathrm{N}(\mathbb{C}^*) \simeq \mathbb{Z}$. Finally, given a $\mathbb{C}^*$-action on a finite dimensional complex vector space V, there exists a decomposition

$$V = \bigoplus_{a \in \mathrm{M}(\mathbb{C}^*)} V_a,$$

where $V_a = \{v \in V \mid t \cdot v = a(t)v \quad \forall t \in \mathbb{C}^*\}$. The characters appearing in the decomposition are called the *weights* of the module. Given a weight $a \in \mathrm{M}(\mathbb{C}^*)$, we denote by a^k the occurrence of the weight, with k a positive integer.

1.2 Divisors and birational geometry

Let X be a normal projective variety. A curve C is a reduced projective variety of dimension 1. A *rational curve* is a curve whose normalization is isomorphic to $\mathbb{P}^1$.

Weil and Cartier divisors. A *prime divisor* is a subvariety of X of codimension 1. A *Weil divisor* D is a formal integer linear combination $\sum_i a_i D_i$, with $a_i \in \mathbb{Z}$ and D_i prime divisors. We denote by $\mathrm{Div}(X)$ the free abelian group generated by Weil divisors. A Weil divisor is *effective* if $a_i \geq 0$ for every i. The *support* of a Weil divisor is the subvariety $\cup_{a_i \neq 0} D_i$. Since X is normal, for every prime divisor D the local ring $\mathcal{O}_{X,D}$ is a DVR, which defines a discrete valuation map $\nu_D \colon \mathbb{C}(X) \to \mathbb{Z}$. A Weil divisor E is *principal* if $E = \mathrm{div}(f) \coloneqq \sum_i \nu_{D_i}(f) D_i$, for some $f \in \mathbb{C}(X) \setminus \{0\}$. A Weil divisor D is said to be *Cartier* if there exists an open covering $\{U_i\}_i$ of X such that $D \cap U_i$ is a principal divisor on U_i. We denote by $\mathrm{CDiv}(X)$ the free abelian group generated by Cartier divisors. A divisor is $\mathbb{Q}$*-Cartier* if mD is Cartier for some $m \in \mathbb{Z}_{>0}$. A variety is said to be $\mathbb{Q}$*-factorial* if every Weil divisor is $\mathbb{Q}$-Cartier.

Two Weil divisors D, D' are *linearly equivalent*, written $D \sim D'$, if their difference is a principal divisor. The *divisor class group* $\mathrm{Cl}(X)$ and the *Picard group* $\mathrm{Pic}(X)$ are the quotient of respectively $\mathrm{Div}(X)$ and $\mathrm{CDiv}(X)$ by linear equivalence. Recall that $\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^*)$.

Given D a Cartier divisor on X, and C an irreducible curve, we define the *intersection product* between D and C as $D \cdot C = \deg(f^* \mathcal{O}_X(D)|_C)$, where $f \colon C^\nu \to C$ is the normalization of the curve. Two Cartier divisors D, D' are *numerically equivalent*, written $D \equiv D'$, if $D \cdot C = D' \cdot C$ for every irreducible curve $C \subset X$. We denote by $\mathrm{N}^1(X)$ the group of Cartier divisor modulo numerical equivalence, and by $\mathrm{N}^1(X)_\mathbb{R} = \mathrm{N}^1(X) \otimes_\mathbb{Z} \mathbb{R}$ the associated real vector space. The *Picard number* of X is $\rho_X \coloneqq \dim \mathrm{N}^1(X)_\mathbb{R}$.

Canonical divisor. Since X is normal, the singular locus $\mathrm{Sing}(X)$ of X has codimension greater or equal than 2. Its complement U is by definition smooth, and the sheaf of differentials Ω^1_U is a locally free sheaf of rank equal to $\dim X$. The determinant $\omega_U = \det \Omega^1_U$ is an invertible sheaf on U, whose associated divisor is denoted by K_U. The image K_X of K_U under the bijective map $\mathrm{Div}(U) \to \mathrm{Div}(X)$ is called the *canonical divisor* of X.

Positivity and cones of divisors. Given a Cartier divisor D, it holds that

$$\mathrm{H}^0(X, \mathcal{O}_X(D)) \simeq \{f \in \mathbb{C}(X) \mid \mathrm{div}\, f + D \geq 0\}.$$

Given D a Cartier divisor, its *complete linear system* is defined as $|D| = \{E \geq 0 \mid E \sim D\}$. The *base locus* of $|D|$ is $\mathrm{Bs}(D) = \bigcap_{E \in |D|} \mathrm{Supp}(E)$. A Cartier divisors D is *base point free* if $\mathrm{Bs}\,|D| = \emptyset$, that is if it is generated by global sections. A Cartier divisor is *semiample* if there exists a positive integer such that mD is base point free. A Cartier divisor is *nef* if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$. Given D a Cartier divisor on X, consider the morphism

$$\phi = \phi_{|D|} \colon X \setminus \mathrm{Bs}(D) \to \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(D))), \quad x \mapsto \phi(x) = (s_0(x) \colon \ldots \colon s_N(x)),$$

where $s_0, \ldots, s_N$ is a basis of $\mathrm{H}^0(X, \mathcal{O}_X(D))$. A Cartier divisor D is *very ample* if $\phi_{|D|}$ is an embedding. A Cartier divisor is *ample* if mD is very ample for some positive integer m. A Cartier divisor D is *big* if the associated map $\phi_{|D|}$ is birational onto the image. A Cartier divisor D is *movable* if $\mathrm{codim} \bigcap_{m \geq 0} \mathrm{Bs}(mD) \geq 2$.
The set of nef classes in $\mathrm{N}^1(X)_{\mathbb{R}}$ forms a closed cone, which is denoted by $\mathrm{Nef}(X)$. The set of movable divisors modulo numerical equivalence is a convex cone, denote by $\mathrm{Mov}(X)$. The set of effective divisors modulo numerical equivalence is a convex cone, denote by $\mathrm{Eff}(X)$. There are inclusions: $\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}(X)}$.

Maps. Let X, Y be normal projective varieties. A *contraction* is a surjective morphism with connected fibers. It is called *elementary* if $\rho_X - \rho_Y = 1$. If $\dim X > \dim Y$, it is *of fiber type*. If $\dim X = \dim Y$, it is called *birational*. The *exceptional locus of* a contraction f is the set of points where f is not an isomorphism. If f is birational and $\mathrm{codim}\,\mathrm{Exc}(f) = 1$, it is said to be *divisorial*; otherwise it is called *small*.

A birational map $f \colon X \dashrightarrow Y$ induces a map $f_* \colon \mathrm{Div}(X) \to \mathrm{Div}(Y)$, whose image is called the *strict transform* of a divisor, where we set $f_*D = 0$ if $\mathrm{codim}\, f(D) > 1$. A birational map $f \colon X \dashrightarrow Y$ is *isomorphic in codimension* 1 if the induced map $f_* \colon \mathrm{Div}(X) \to \mathrm{Div}(Y)$ is bijective. A *small modification* is a birational map which is an isomorphism in codimension 1. If we assume X, Y are $\mathbb{Q}$-factorial, such map is called a *small $\mathbb{Q}$-factorial modification*, SQM for short. Given a small modification $f \colon X \dashrightarrow Y$, the induced map f_* is bijective and we set $f^* := f_*^{-1}$. Given a small modification $f \colon X \dashrightarrow Y$ and D a Cartier divisor on Y, it holds $\mathrm{H}^0(X, \mathcal{O}_X(f^*D)) \simeq \mathrm{H}^0(Y, \mathcal{O}_Y(D))$. A *$\mathbb{P}^k$-fibration* is a map $f \colon X \to Y$ whose fibers are isomorphic to $\mathbb{P}^k$. Given $\mathcal{E}$ a rank n vector bundle over Y, the associated $\mathbb{P}^{n-1}$-bundle is defined as $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}\,\mathrm{Sym}\,\mathcal{E}$, using the Grothendieck projectivization, where $\mathrm{Sym}\,\mathcal{E} := \bigoplus_{m \geq 0} S^m\mathcal{E}$ is the symmetric algebra of $\mathcal{E}$.

Given a small contraction $X_- \to X_0$ among normal projective varieties, and given D a $\mathbb{Q}$-Cartier divisor on X_- such that $\mathcal{O}_{X_-}(-D)$ is relatively ample, a *flip* is a D-flip as in [60, p. 693], that is a small contraction $X_+ \to X_0$, with X_+ normal projective, such that, if $g \colon X_- \dashrightarrow X_+$ is the induced birational map, the strict transform of D is $\mathbb{Q}$-Cartier and $\mathcal{O}_{X_+}(g_*D)$ is relatively ample.

1.3 Toric varieties

Affine toric varieties. Let T be an n-dimensional torus, and let M (resp. N) be the associated lattice of characters (resp. of 1-parameter subgroups). A *polyhedral cone* σ in $\mathrm{N}_{\mathbb{R}} = \mathrm{N} \otimes_{\mathbb{Z}} \mathbb{R}$ is a convex set of the form $\sigma = \langle p_1, \ldots, p_k \rangle = \{\sum_{i=1}^k a_i p_i \mid a_i \in \mathbb{R}_{\geq 0}\}$, where $\{p_1, \ldots, p_k\}$ is a finite subset of points of $\mathrm{N}_{\mathbb{R}}$. A polyhedral cone σ is said to be *rational* if the points $p_1, \ldots, p_k$ belong to N. The *dimension* of σ is the dimension of the smallest linear space spanned by σ. A polyhedral cone σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. Given a rational polyhedral cone $\sigma \subset \mathrm{N}_{\mathbb{R}}$, we define the *dual cone* $\sigma^\vee := \{m \in \mathrm{M}_{\mathbb{R}} \mid m(v) \geq 0 \ \forall v \in \sigma\}$, which is still rational polyhedral. Given a polyhedral cone σ, a subset $\tau \subset \sigma$ is a *face* of σ, written $\tau \preccurlyeq \sigma$, if $\tau = \sigma \cap H(v)$, where by $H(v)$ we denote the hyperplane of some vector $v \in \sigma^\vee \cap \mathrm{M}$.

An *affine toric variety* is an irreducible affine variety containing a dense open subset isomorphic to a torus T, such that the action T on itself extends to an action of T on X. A normal affine variety X is toric if and only if there exists a rational polyhedral cone $\sigma \subset \mathrm{N}_{\mathbb{R}}$ such that $\mathbb{C}[X] = \mathbb{C}[\sigma^\vee \cap \mathrm{M}]$, and we remark that the latter is a finitely generated semigroup by Gordan's lemma. We denote by X_σ the affine toric variety associated to a rational polyhedral cone σ. Given two affine toric varieties $X_{\sigma_1}, X_{\sigma_2}$, with associated tori T_1, T_2, we say that a morphism $f\colon X_{\sigma_1} \to X_{\sigma_2}$ is *toric* is $f(T_1) \subset T_2$ and the restriction $f|_{T_1}\colon T_1 \to T_2$ is a group homomorphism. An affine toric variety X_σ is smooth if and only if the cone σ is generated by a set of elements contained in a $\mathbb{Z}$-basis of N. An affine toric variety X_σ is $\mathbb{Q}$-factorial if and only if σ is simplicial, i.e. if the minimal generators of the cone are linearly independent over $\mathbb{R}$.

Fans. A *fan* Σ is a finite collection of strongly convex rational polyhedral cones in $\mathrm{N}_{\mathbb{R}}$ such that

- if $\sigma \in \Sigma$ and $\tau \preccurlyeq \sigma$, then $\tau \in \Sigma$;
- if $\sigma, \sigma' \in \Sigma$ then $\sigma \cap \sigma' \in \Sigma$.

The resulting variety X_Σ obtained by gluing the affine toric varieties $X_\sigma, X_{\sigma'}$, for $\sigma, \sigma' \in \Sigma$ along their common open subset $X_{\sigma\cap\sigma'}$, is the *toric variety* associated to the fan Σ. In a natural way one may generalize the definitions given above in the context of fans. Given a fan σ, we define the *support* of Σ as $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma$. Given a fan Σ, a fan Σ' is a *subdivision* of Σ if $|\Sigma| = |\Sigma'|$ and every cone of Σ is a union of cones of Σ'. A subdivision Σ' of Σ induces naturally a birational toric morphism $\phi\colon X_{\Sigma'} \to X_\Sigma$. Given a cone $\sigma \in \Sigma$, we define the *orbit* of σ as $\mathcal{O}(\sigma) = X_\Sigma \setminus \bigcup_{\tau\preccurlyeq\sigma} X_\sigma$. *Orbit–Cone correspondence.* Given a lattice N of rank n and a fan Σ in $\mathrm{N}_{\mathbb{R}}$, there is a bijective correspondence between cones $\sigma \in \Sigma$ and T-orbits in X_Σ, given by associating to every cone $\sigma \in \Sigma$ the orbit $\mathcal{O}(\sigma)$. Moreover it holds that:

1. $\dim(\mathcal{O}(\sigma)) = n - \dim(\sigma)$;
2. $X_\sigma = \bigcup_{\tau\preccurlyeq\sigma} \mathcal{O}(\tau)$;
3. $\overline{\mathcal{O}(\tau)} = \bigcup_{\sigma\succcurlyeq\tau} \mathcal{O}(\sigma)$.

A fan is *complete* if and only if $|\Sigma| = \mathrm{N}_{\mathbb{R}}$. Given a rational polyhedral cone δ, we denote by $\Sigma(\delta)$ the natural fan associated to it. We denote by $\Sigma(k)$ the set of k-dimensional cones of Σ. Given a complete fan Σ of dimension n, the elements $w \in \Sigma(n-1)$ are called *walls*; notice that $\overline{\mathcal{O}(w)} = \mathbb{P}^1$. Given a fan Σ, an element $\rho \in \Sigma(1)$ is called a *ray*, and its orbit closure $D_\rho = \overline{\mathcal{O}(\rho)}$ is a T-invariant prime divisor of X_Σ. The free abelian group of T-invariant Weil divisors (resp. Cartier divisors) is denote by $\mathrm{Div}_T(X_\Sigma)$ (resp. $\mathrm{CDiv}_T(X_\Sigma)$). Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a T-invariant Cartier divisor on a projective toric variety X_Σ, the *Cartier data* $(m_\sigma)_{\sigma\in\Sigma}$ are a collection of characters such that $m_\sigma(u_\rho) = m_{\sigma'}(u_\rho)$ for every common ray $u_\rho \in \sigma, \sigma'$. Let X_Σ be a projective toric variety of dimension n, let D be a T-invariant $\mathbb{Q}$-Cartier divisor, and let $\tau \in \Sigma(n-1)$ be a wall such that $\tau = \sigma \cap \sigma'$, with $\sigma, \sigma' \in \Sigma(n)$. Notice that $C_\tau := \overline{\mathcal{O}(\tau)} \simeq \mathbb{P}^1$. Then $D \cdot C_\tau = \frac{1}{k} kD \cdot C_\tau = (m_\sigma - m_{\sigma'})(u)$, where $u \in \mathrm{N} \cap \sigma$ is such the image $\pi(u)$ generates the lattice $\mathrm{N}/\mathbb{Z}\tau$, $m_\sigma, m_{\sigma'}$ are the Cartier data of D in σ, σ', and k is a positive integer such that kD is Cartier.

Polytopes. A *lattice polytope* P in $\mathrm{M}_{\mathbb{R}}$ is the convex hull of a finite subset of $\mathrm{M} \simeq \mathbb{Z}^n$. A *facet* is a face of P of codimension 1. Any lattice polytope P may be described as $P = \{m \in \mathrm{M}_{\mathbb{R}} \mid m(u_F) + a_F \geq 0 \text{ for all facets } F \in P\}$, where u_F is the primitive normal vector to the facet F. Given a face $\mathcal{F}$ of P, we may define a cone $\sigma_{\mathcal{F}} = \langle u_1, \ldots, u_k \rangle$, where u_i the normal vectors of a facet F_i containing $\mathcal{F}$. The *normal fan* σ_P is a complete fan generated by the cones $\sigma_{\mathcal{F}}$, for

any face $\mathcal{F}$ of P. There exists a one-to-one inclusion reversing correspondence between faces of P and cones of σ_P such that, for any face $\mathcal{F} \in P$, it holds $\dim \sigma_{\mathcal{F}} + \dim \mathcal{F} = n$.
A toric variety X_Σ associated to a complete fan $\Sigma \in \mathrm{N}_\mathbb{R}$ is projective if and only if Σ is the normal fan of a full-dimensional lattice polytope in $\mathrm{M}_\mathbb{R}$.

Chapter 2

Preliminaries

This chapter is meant to be an introduction to $\mathbb{C}^*$-actions on polarized pairs (X, L), that is pairs where X is a normal projective variety and L is an ample line bundle on X. We introduce the necessary background and notation we will use along the rest of the manuscript. We mainly follow [10], [49] and [48], but specific references are provided for the results stated without proof.

2.1 Generalities on $\mathbb{C}^*$-actions

Set-up 2.1.1. Let X be a normal projective variety, and let $\mathbb{C}^*$ act on X as

$$\alpha\colon \mathbb{C}^* \times X \to X, \quad (t, x) \mapsto t \cdot x.$$

We assume that the action is non-trivial and faithful. Moreover, for the sake of simplicity we abuse notation by writing tx and mean $t \cdot x$.

Consider the decomposition of the fixed point locus $X^{\mathbb{C}^*} \subset X$ in connected components

$$X^{\mathbb{C}^*} = \bigsqcup_{Y \in \mathcal{Y}} Y,$$

where we denote by $\mathcal{Y}$ the set of connected components of $X^{\mathbb{C}^*}$.

Lemma 2.1.2. *[28, Theorem 1.1] Suppose that X is smooth. Then Y is smooth, for any $Y \in \mathcal{Y}$. In particular, Y is irreducible.*

Example 2.1.3. [57, Remark 2] Notice that a similar conclusion does not hold in the singular case. Indeed consider the quadric cone $Q = Z(x_1x_2 + x_3x_4) \subset \mathbb{P}^4$, which is singular in $e_0 = (1 : 0 : 0 : 0 : 0)$ and let $\mathbb{C}^*$ act on $\mathbb{P}^4$ as $tx = (x_0 : x_1 : x_2 : tx_3 : t^{-1}x_4)$. The quadric cone Q is $\mathbb{C}^*$-invariant, and $Q^{\mathbb{C}^*} = (\mathbb{P}^4)^{\mathbb{C}^*} \cap Q = e_3 \sqcup Q \cap Z(x_3, x_4) \sqcup e_4$, where $Q \cap Z(x_3, x_4) = Z(x_1, x_3, x_4) \cup Z(x_2, x_3, x_4)$ is the union of two lines.

Lemma 2.1.4. *In the situation of Set-up 2.1.1, given $x \in X$, the orbit map $\mathbb{C}^* \times \{x\} \to X$, $(t, x) \mapsto tx$ can be extended to a morphism $\mathbb{P}^1 \times \{x\} \to X$.*

Proof. Since we can regard the orbit map as a rational map $\overline{\alpha}\colon \mathbb{P}^1 \times \{x\} \dashrightarrow X$, and by [23, Theorem 12.60] we have $\operatorname{codim} \operatorname{Exc}(\overline{\alpha}) \geq 2$, we conclude. ■

In the notation of the previous Lemma, the images by $\overline{\alpha}$ of the boundary points $0, \infty \in \mathbb{P}^1$ are equal to $\lim_{t\to 0} tx, \lim_{t\to\infty} tx := \lim_{t\to 0} t^{-1}x$, where we consider our varieties with the complex analytic topology.

Definition 2.1.5. For every point $x \in X$, we respectively call $x_- := \lim_{t\to\infty} tx$ the *sink* (resp. $x_+ := \lim_{t\to 0} tx$ the *source*) of the orbit $\mathbb{C}^* \cdot x$.

Example 2.1.6. Consider the $\mathbb{C}^*$-action on $\mathbb{P}^2$ given by $tx = (t^{-1}x_0 : x_1 : tx_2)$. Then $(\mathbb{P}^2)^{\mathbb{C}^*} = e_0 \sqcup e_1 \sqcup e_2$. Given $x \in \mathbb{P}^2$ general point, we compute $\lim_{t\to\infty} tx$:

$$x_- = \lim_{t\to\infty} (t^{-1}x_0 : x_1 : tx_2) = \lim_{t\to\infty} (t^{-2}x_0 : t^{-1}x_1 : x_2) = e_2.$$

A similar computation yields $x_+ = \lim_{t\to 0} tx = e_0$.

Remark 2.1.7. In the situation of Set-up 2.1.1, the closure of a 1-dimensional orbit $C = \overline{\mathbb{C}^* \cdot x}$ is a rational curve, whose normalization is the map $\overline{\alpha}$ of Lemma 2.1.4.

Lemma 2.1.8. *In the situation of Set-up 2.1.1, let $y \in X^{\mathbb{C}^*}$. There exists an induced $\mathbb{C}^*$-action on the Zariski tangent space $T_{X,y}$ of X in y.*

Proof. We have an induced $\mathbb{C}^*$-action on the sheaf of regular functions $\mathcal{O}_X$, and in particular on the local ring $\mathcal{O}_{X,y}$, which preserves the order of vanishing on y. We thus obtain a $\mathbb{C}^*$-representation of $\mathfrak{m}_y/\mathfrak{m}_y^n$ for any $n \in \mathbb{Z}$, with $\mathfrak{m}_y$ the maximal ideal of $\mathcal{O}_{X,y}$, and thus for $n = 2$ we conclude. ■

Lemma 2.1.9. *[4, Theorem, §4] In the situation of Set-up 2.1.1, suppose in addition that X is smooth. Then for any $Y \in \mathcal{Y}$ there exists an induced $\mathbb{C}^*$-action on $T_X|_Y$, inducing a decomposition*

$$T_X|_Y = T^-(Y) \oplus T^0(Y) \oplus T^+(Y),$$

where by $T^{\pm}(Y), T^0(Y)$ we denote the vector subspaces of $T_X|_Y$ on which $\mathbb{C}^*$ acts respectively with positive, negative and 0 weights. Moreover it holds that $T^0(Y) \simeq T_Y$.

Corollary 2.1.10. *There exists an induced $\mathbb{C}^*$-action on the normal bundle $\mathcal{N}_{Y|X}$, which decomposes as*

$$\mathcal{N}_{Y|X} = \mathcal{N}^-(Y) \oplus \mathcal{N}^+(Y) = T^-(Y) \oplus T^+(Y),$$

where $\mathcal{N}^{\pm}(Y)$ are the vector subspaces of $\mathcal{N}_{Y|X}$ on which $\mathbb{C}^*$ acts respectively with positive and negative weights.

Notation 2.1.11. If X is smooth, for every $Y \in \mathcal{Y}$ we set $\nu^{\pm}(Y) = \dim \mathcal{N}^{\pm}(Y)$. Obviously for every component $Y \in \mathcal{Y}$ we have $\dim Y + \nu^-(Y) + \nu^+(Y) = \dim X$.

2.1.1 Białynicki-Birula decomposition

We keep the notation and assumptions of Set-up 2.1.1.

Definition 2.1.12. For every $Y \in \mathcal{Y}$ and every subset $U \subset Y$, we define

$$X^+(U) := \{x \in X \mid \lim_{t\to 0} tx \in U\}, \quad X^-(U) := \{x \in X \mid \lim_{t\to\infty} tx \in U\}.$$

In particular, for $U = Y$, the varieties $X^{\pm}(Y)$ are called respectively *plus and minus Białynicki-Birula cells* (also called *BB-cell*) of Y.

The closure of the Białynicki-Birula cells will be denoted by $\overline{X^{\pm}(Y)}$. For every $Y \in \mathcal{Y}$, we can define the *plus and minus morphisms*

$$f_{\pm}\colon X^{\pm}(Y) \to Y, \quad x \mapsto \lim_{t\to 0} t^{\pm 1}x.$$

We now state a fundamental result in the theory of algebraic torus actions: the *Białynicki-Birula theorem.* We refer to [4] for the original exposition.

Theorem 2.1.13. *[5, Theorems 4.2, 4.4] Let X be a smooth projective variety, and let $\mathbb{C}^*$ act on X. The following hold:*

1. *For every $Y \in \mathcal{Y}$ the plus and minus cells $X^{\pm}(Y)$ are locally closed;*
2. *There exist two decompositions of X induced by the plus and minus cells, that is*

$$X = \bigsqcup_{Y\in\mathcal{Y}} X^{+}(Y) = \bigsqcup_{Y\in\mathcal{Y}} X^{-}(Y);$$

3. *The plus and minus morphisms $f_{\pm}\colon X^{\pm}(Y) \to Y$ are $\mathbb{C}^*$-equivariant $\mathbb{C}^{\nu^{\pm}(Y)}$-fibrations;*
4. *For every $Y \in \mathcal{Y}$ and for every $y \in Y$ there exists an open neighborhood U of y in Y and a $\mathbb{C}^*$-equivariant isomorphism*

$$X^{\pm}(U) \simeq \mathcal{N}^{\pm}(U);$$

5. *For every $m \geq 0$, we have*

$$\mathrm{H}_m(X,\mathbb{Z}) \simeq \bigoplus_{Y\in\mathcal{Y}} \mathrm{H}_{m-2\nu^{+}(Y)}(Y,\mathbb{Z}) \simeq \bigoplus_{Y\in\mathcal{Y}} \mathrm{H}_{m-2\nu^{-}(Y)}(Y,\mathbb{Z}).$$

We refer to Property 2 of Theorem 2.1.13 as the *Białynicki-Birula decomposition* (also *BB-decomposition*). Since we will mainly work with normal projective varieties, we also state the generalization done by Konarski in [37] for such varieties.

Theorem 2.1.14. *[37, Theorems 1,2] Let $\mathbb{C}^*$ act on a normal projective variety X. Label by Y_j, for $j = 1, \ldots, d$, the irreducible components of $X^{\mathbb{C}^*}$. Then the following hold:*

1. *The Białynicki-Birula cells $X^{\pm}(Y_j)$, for $j = 1, \ldots, d$, are locally closed;*
2. *There exist two decompositions of X induced by the plus and minus cells, that is*

$$X = \bigcup_{j=1}^{d} X^{-}(Y_j) = \bigcup_{j=1}^{d} X^{+}(Y_j);$$

3. *For every $j = 1, \ldots, d$, the natural maps $f_{\pm}\colon X^{\pm}(Y_j) \to Y_j$ are $\mathbb{C}^*$-equivariant.*

Notice that the BB-decompositions for normal projective varieties may not be a disjoint union (see for instance Example 2.1.3).

We refer to [37, Section 2] for a discussion of the properties preserved in the non-normal case. Notice that there exist generalizations of the Białynicki-Birula theorem for reductive group actions (see [29]), even in positive characteristic (see [30]).

The main ingredient in the proof of Theorem 2.1.14, which was also used to formulate another proof of Theorem 2.1.13 (see [38]), is the *Sumihiro's Theorem*:

Theorem 2.1.15. *[59, Theorem 1] Let $\mathbb{C}^*$ act on a normal variety X. Then there exists an open covering of X consisting of $\mathbb{C}^*$-invariant affine subsets. Moreover, if X is normal and quasi-projective, there exists a projective embedding $f\colon X \to \mathbb{P}^n$, and a representation $\rho\colon \mathbb{C}^* \to \mathrm{PGL}_n$, such that $f(tx) = \rho(t)f(x)$ for any $t \in \mathbb{C}^*$, $x \in X$.*

We introduce another result we will use along the rest of the manuscript which describes the local geometry of a $\mathbb{C}^*$-action in a neighborhood of a fixed point:

Theorem 2.1.16. *[4, Theorem 2.5] Given a $\mathbb{C}^*$-action on a smooth projective variety, and given a point $y \in X^{\mathbb{C}^*}$, there exists a $\mathbb{C}^*$-invariant neighborhood U of y, and a $\mathbb{C}^*$-equivariant isomorphism $U \simeq (U \cap X^{\mathbb{C}^*}) \times V$, where V is a finite-dimensional $\mathbb{C}^*$-module and the $\mathbb{C}^*$-action on $U \simeq (U \cap X^{\mathbb{C}^*}) \times V$ is induced by the trivial $\mathbb{C}^*$-action on $U \cap X^{\mathbb{C}^*}$ and the linear action on V.*

As a corollary of Theorems 2.1.13, 2.1.14, we obtain:

Definition 2.1.17. For every normal projective variety endowed with a $\mathbb{C}^*$-action there exists a unique irreducible component Y_- (resp. Y_+) such that $X^-(Y_-)$ (resp. $X^+(Y_+)$) is a dense open subset of X. We call the variety Y_- (resp. Y_+) the *sink* (resp. the *source*) of the $\mathbb{C}^*$-action.

Remark 2.1.18. The sink Y_- and the source Y_+ are the unique irreducible components containing the limit, for $t \to \infty$ and $t \to 0$, of the general orbit. Indeed given a general point $x \in X$, it holds that $x \in X^+(Y_+) \cap X^-(Y_-)$, that is $\lim_{t\to 0} t^{\pm 1}x \in Y_\pm$.

Definition 2.1.19. A $\mathbb{C}^*$-action on a normal projective variety X is said to have *extremal isolated points* if the sink and the source of the action are isolated points.

The relation between the Picard group of X and of $Y_\pm$ can be described in terms of the $\mathbb{C}^*$-invariant divisors which are closure of BB-cells, as explained in the following:

Lemma 2.1.20. *[11, Theorem 3] Let X be a smooth projective variety with a $\mathbb{C}^*$-action. Then there exist two short exact sequences*

$$0 \to \sum_{Y \in \mathcal{Y}, \nu^{\mp}(Y)=1} \mathbb{Z} \cdot \overline{X^{\pm}(Y)} \to \mathrm{Pic}(X) \to \mathrm{Pic}(Y_\pm) \to 0.$$

We conclude this section by recalling some well-known results about how the birational geometry of X is affected by the $\mathbb{C}^*$-action.

Lemma 2.1.21. *[49, Lemma 2.3] Let X be a smooth projective variety with an action of $\mathbb{C}^*$. Then X is uniruled.*

Lemma 2.1.22. *[49, Lemma 2.6] Let X be a smooth projective variety with an action of $\mathbb{C}^*$. If X is rationally connected, then $Y_\pm$ are rationally connected.*

2.1.1.1 α-fibrations and non-equalized $\mathbb{C}^*$-actions

Property 4 of Theorem 2.1.13 cannot be, in general, extended to a global $\mathbb{C}^*$-equivariant isomorphism, as noted in [29, Example 7.4]; that is, the Białynicki-Birula cells may fail to be vector bundles, since the transition maps of the $\mathbb{C}^{\nu^{\pm}(Y)}$-fibrations $f_\pm\colon X^\pm(Y) \to Y$ are not necessarily linear. We present some hypotheses that guarantee that the BB-cells are vector bundles.

Remark 2.1.23. Let $\mathbb{C}^*$ act on a smooth projective variety, and let $Y \in \mathcal{Y}$. If Y is a point, then $X^\pm(Y) \simeq \mathcal{N}^\pm(Y)$.

Definition 2.1.24. [4, §3] Let $\alpha\colon \mathbb{C}^* \to \mathrm{GL}(V)$ be a homomorphism of algebraic groups, where V is a finite dimensional complex vector space. Given Y a normal projective variety, an α-*fibration* over Y is a variety $\mathcal{E}$ together with a surjective morphism $\pi\colon \mathcal{E} \to Y$, endowed with an action of $\mathbb{C}^* \times Y$ such that there exists an open covering $\{U_i\}_i$ of Y satisfying that, for every i, there exists a $(\mathbb{C}^* \times U_i)$-equivariant isomorphism $\pi^{-1}(U_i) \simeq U_i \times V$, where the latter is endowed with a $(\mathbb{C}^* \times U_i)$-action induced by α.

Proposition 2.1.25. *[4, Theorem (b)] Let $\mathbb{C}^*$ act on a smooth projective variety X. For every $Y \in \mathcal{Y}$, the plus and minus morphisms $f_\pm\colon X^\pm(Y) \to Y$ are $\alpha_\pm$-fibrations, where $\alpha_\pm\colon \mathbb{C}^* \to \mathrm{GL}(\mathcal{N}^\pm(Y)_y)$, with $y \in Y$.*

Definition 2.1.26. Let $\mathbb{C}^*$ act on a normal projective variety X. A $\mathbb{C}^*$-action is said to be *equalized at* Y if for every point $y \in (X^+(Y) \cup X^-(Y)) \setminus Y$ the isotropy group of the $\mathbb{C}^*$-action at the point y is trivial. If the $\mathbb{C}^*$-action is equalized at every fixed point component, we say that the $\mathbb{C}^*$-action is equalized.

Lemma 2.1.27. *[48, Lemma 2.1] A $\mathbb{C}^*$-action on a normal projective variety X is equalized at $Y \in \mathcal{Y}$ if and only if the weights of the induced $\mathbb{C}^*$-action on $\mathcal{N}^\pm(Y)$ are all equal to ±1.*

Lemma 2.1.28. *[4, Remarks] Let $\mathbb{C}^*$ act on a smooth projective variety X, and let $Y \in \mathcal{Y}$. If the action is equalized at Y, then there exists a $\mathbb{C}^*$-equivariant isomorphism $X^\pm(Y) \simeq \mathcal{N}^\pm(Y)$. In particular $f_\pm\colon X^\pm(Y) \to Y$ are vector bundles of rank $\nu^\pm(Y)$.*

2.1.2 Linearization and $\mathbb{C}^*$-actions on polarized pairs

We introduce the notion of *linearization* of a line bundle with respect to the action of an algebraic group G. We then focus our study on the case of a $\mathbb{C}^*$-action, exploiting the relation between $\mathbb{C}^*$-linearizations of ample line bundles and associated weights of the fixed point connected components.

Definition 2.1.29. Let G be an algebraic group acting on a normal projective variety X, and let L be a line bundle on X. A *G-linearization* of the line bundle L is an induced G-action on L such that

- there exists a commutative diagram:

$$\begin{tikzcd} G \times L \arrow[r] \arrow[d, "\mathrm{Id}\times\pi"'] & L \arrow[d, "\pi"] \\ G \times X \arrow[r] & X \end{tikzcd}$$

  with $\pi\colon L \to X$ the natural bundle map;
- The G-action is linear along the fibers, that is for any $g \in G$, $x \in X$, the map $L_x \to L_{g\cdot x}$ is linear.

A line bundle is *G-linearizable* if there exists a G-linearization. A line bundle is *G-linearized* if we have fixed a G-linearization.

Lemma 2.1.30. *The set of G-linearizable line bundles $\mathrm{Pic}^G(X)$ is a group, and there exists a natural forgetful map $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$.*

Lemma 2.1.31. *[9, Proposition 2.10] There exists a short exact sequence of groups*

$$1 \to \mathrm{M}(G) \to \mathrm{Pic}^G(X) \to \mathrm{Pic}(X) \to 1.$$

In particular, two different linearizations differ by a character.

Lemma 2.1.32. *[35, Proposition 2.4, Remark] Let $\mathbb{C}^*$ act on a normal projective variety X. Then every line bundle L on X is $\mathbb{C}^*$-linearizable.*

Definition 2.1.33. Let $\mathbb{C}^*$ act on a normal projective variety X, and let L be a $\mathbb{C}^*$-linearized line bundle on X. Define the *weight map*

$$\mu_L \colon X^{\mathbb{C}^*} \to \mathbb{Z}, \quad y \mapsto \mu_L(y),$$

where by $\mu_L(y)$ we mean the weight of the induced $\mathbb{C}^*$-action on the fiber L_y.

Lemma 2.1.34. *Let $\mathbb{C}^*$ act on a normal projective variety X, and let L_1, L_2 be two $\mathbb{C}^*$-linearized line bundles on X. Then it holds that $\mu_{L_1 \otimes L_2} = \mu_{L_1} + \mu_{L_2}$ and that $\mu_{L^{-1}} = -\mu_L$. In particular, for every $m \geq 0$ it holds $\mu_{mL_1} = m\mu_{L_1}$.*

Lemma 2.1.35. *In the situation of Definition 2.1.33, the weight map is constant on the connected components, that is for any $x, y \in Y \subset \mathcal{Y}$, we have that $\mu_L(x) = \mu_L(y)$.*

The above Lemma suggests the following:

Definition 2.1.36. Let $\mathbb{C}^*$ act on a normal projective variety X, and let L be a $\mathbb{C}^*$-linearizable line bundle on X. For any connected component $Y \in \mathcal{Y}$, we set $\mu_L(Y) \coloneqq \mu_L(y)$, for any $y \in Y$. The set $\{\mu_L(Y) \mid Y \in \mathcal{Y}\}$ is called the *set of critical values* of the $\mathbb{C}^*$-action.

Lemma 2.1.37. *Let $\mathbb{C}^*$ act on a normal projective variety X, and let L be a $\mathbb{C}^*$-linearizable line bundle on X. Then there exists an induced $\mathbb{C}^*$-action on $\mathrm{H}^0(X, mL)$, for every $m \geq 0$.*

Corollary 2.1.38. *[18, §7.3] In the situation of Lemma 2.1.37, suppose that L is ample. Then the embedding $X \hookrightarrow \mathbb{P}(\mathrm{H}^0(X, mL))$, provided by the complete linear system $|mL|$, for $m \gg 0$, is $\mathbb{C}^*$-equivariant.*

Definition 2.1.39. By a $\mathbb{C}^*$-*action on a polarized pair* (X, L) we mean a $\mathbb{C}^*$-action on a normal projective variety X, and a $\mathbb{C}^*$-linearization of the ample line bundle L. By a $\mathbb{C}^*$-*action on a smooth polarized pair* (X, L) we mean a $\mathbb{C}^*$-action on a polarized pair, where we assume that X is smooth.

Example 2.1.40. [10, Example 2.11] Let V a complex vector space of dimension $n+1$, and let $\mathbb{C}^*$ act on V; we obtain a decomposition $V = \bigoplus_{i=0}^r V_{a_i}$. Set $d_i \coloneqq \dim V_{a_i} - 1$.

Notice that, up to a change of coordinates, we may assume that

$$a_0 > a_1 > \ldots > a_r.$$

Consider the associated projective space $\mathbb{P}(V)$, with coordinates $(x_{0,0} : \ldots : x_{0,d_0} : \ldots : x_{r,d_r})$. We may assume that the induced $\mathbb{C}^*$-action on $\mathbb{P}(V)$ is given by

$$t(x_{0,0} : \ldots : x_{0,d_0} : \ldots : x_{r,d_r}) = (t^{a_0} x_{0,0} : \ldots : t^{a_0} x_{0,d_0} : \ldots : t^{a_r} x_{r,d_r}).$$

The fixed point locus is $\mathbb{P}(V) = \bigsqcup_{i=0}^r \mathbb{P}(V_{a_i})$. A computation shows that the sink is $\mathbb{P}(V_{a_0})$ and the source is $\mathbb{P}(V_{a_r})$.

Moreover, for every $i = 0, \dots, r$, we obtain that

$$X^+(\mathbb{P}(V_{a_i})) = \left\{ x \in \mathbb{P}(V) \setminus \bigsqcup_{j \neq i} \mathbb{P}(V_{a_j}) \mid x_{i+1,0} = x_{i+1,1} = \ldots = x_{r,d_r} = 0 \right\},$$

$$X^-(\mathbb{P}(V_{a_i})) = \left\{ x \in \mathbb{P}(V) \setminus \bigsqcup_{j \neq i} \mathbb{P}(V_{a_j}) \mid x_{0,0} = x_{0,1} = \ldots = x_{i-1,d_{i-1}} = 0 \right\}.$$

The tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ carries a natural linearization such that the critical values are $\mu_{\mathcal{O}_{\mathbb{P}(V)}(-1)}(\mathbb{P}(V_{a_i})) = a_i$. Therefore, by considering the $\mathbb{C}^*$-action on the polarized pair $(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$, we obtain that $\mu_{\mathcal{O}_{\mathbb{P}(V)}(1)}(\mathbb{P}(V_{a_i})) = -a_i$. In particular,

$$\mu_{\mathcal{O}_{\mathbb{P}(V)}(1)}(Y_-) = -a_0 < \ldots < -a_r = \mu_{\mathcal{O}_{\mathbb{P}(V)}(1)}(Y_+).$$

Definition 2.1.41. Let $\mathbb{C}^*$ act on a polarized pair (X, L). Rearrange the weights $\mu_L(Y)$, for $Y \in \mathcal{Y}$, in an increasing order, obtaining a chain of the form

$$a_0 < a_1 < \ldots < a_r.$$

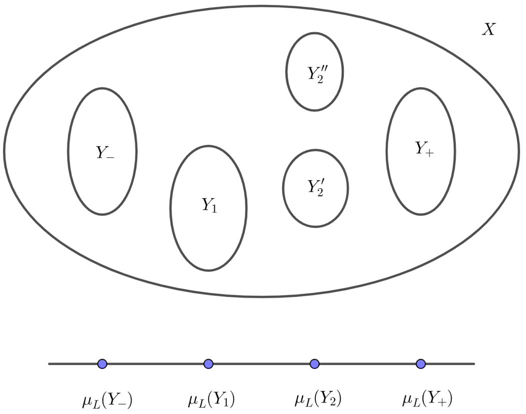
The *criticality* of the $\mathbb{C}^*$-action on (X, L) is the positive integer r.

Notation 2.1.42. Let $\mathbb{C}^*$ act on a polarized pair. For every component $Y \in \mathcal{Y}$, we set $Y_i := \bigsqcup_{Y \in \mathcal{Y}, \mu_L(Y) = a_i} Y$.

Lemma 2.1.43. *[49, Remark 2.12] Let $\mathbb{C}^*$ act on a polarized pair (X, L). Then $\mu_L(Y_-) = \min_{Y \in \mathcal{Y}} \mu_L(Y)$, $\mu_L(Y_+) = \max_{Y \in \mathcal{Y}} \mu_L(Y)$.*

Proof. If the pair is $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, the claim follows by Example 2.1.40. Otherwise, notice that by Corollary 2.1.34 we may assume that L is very ample. Using Corollary 2.1.38, consider a $\mathbb{C}^*$-equivariant embedding of X in $\mathbb{P}(\mathrm{H}^0(X, mL))$, for $m \gg 0$. We may assume that the $\mathbb{C}^*$-action is as in Example 2.1.40. Since X is nondegenerate, the general point x can be written as $\sum_{i=0}^{r} v_i$, where $v_i \in V_{a_i}^\vee$ is non-zero for every $i = 0, \dots, r$. Then $\lim_{t \to \infty} tx \in \mathbb{P}(V_{a_0}), \lim_{t \to 0} tx \in \mathbb{P}(V_{a_r})$; thus we get that $Y_- = X \cap \mathbb{P}(V_{a_0}) \neq \emptyset, Y_+ = X \cap \mathbb{P}(V_{a_r}) \neq \emptyset$, and moreover that the minimal and maximal value of the weight map are attained respectively at the sink and at the source. ∎

We can now represent a $\mathbb{C}^*$-action on a polarized pair (X, L) by mean of the following picture:



Definition 2.1.44. Let $\mathbb{C}^*$ act on a polarized pair (X, L). We will call the sink and the source *extremal fixed point components*, and all other connected components *inner*. We denote by $\mathcal{Y}^\circ$ the set of inner components.

Using Corollary 2.1.34, we immediately get the following:

Lemma 2.1.45. *Let $\mathbb{C}^*$ act on a polarized pair (X, L), and suppose that $\rho_X = 1$. Then the criticality of the action is independent of the choice of the ample line bundle L.*

Definition 2.1.46. Let $\mathbb{C}^*$ act on a polarized pair (X, L). We define the *bandwidth* δ of the $\mathbb{C}^*$-action as

$$\delta := \mu_L(Y_+) - \mu_L(Y_-).$$

Notice that, in the situation of Example 2.1.40, the $\mathbb{C}^*$-action on $(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$ has criticality r and bandwidth $a_r - a_0$. Given a $\mathbb{C}^*$-action on a polarized pair (X, L) with bandwidth δ and criticality r, it holds $r \leq \delta$. The two values may differ: consider for instance the $\mathbb{C}^*$-action on $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ given by $t \cdot [x_0 : x_1 : x_2] = [x_0 : t^2 x_1 : t^2 x_2]$, for $t \in \mathbb{C}^*$. The criticality of the $\mathbb{C}^*$-action is 1, while the bandwidth is 2.

Definition 2.1.47. Let $\mathbb{C}^*$ act on a polarized pair (X, L) with bandwidth δ and criticality r. We say that a linearization is *normalized* is $\mu_L(Y_-) = a_0 = 0$, $\mu_L(Y_+) = a_r = \delta$.

Notice that, thanks to Lemma 2.1.31, we can always assume that $\mathbb{C}^*$-action is normalized.

Lemma 2.1.48. *Let X be a smooth projective variety with a $\mathbb{C}^*$-action. Let $Y \in \mathcal{Y}^\circ$. Then $\nu^{\pm}(Y) \neq 0$.*

Proof. Let us prove it for $\nu^-(Y)$, being the other case similar. By Theorem 2.1.13 there exists a unique component whose minus cell is dense. If by contradiction $X^-(Y)$ is dense, then $Y = Y_-$, which is an absurd since Y is an inner component. ■

We conclude this section by recalling the AMvsFM equality, which has been introduced in [57, Section 3.1], and relates the degree of a line bundle on $\mathbb{P}^1$ with the weights of the action on the fibers of the line bundle over the fixed points.

Lemma 2.1.49. *[57, Lemma 2.2] Let $\mathbb{C}^* \times \mathbb{P}^1 \to \mathbb{P}^1$ be an action with source x_+ and sink x_-. Consider a line bundle L over $\mathbb{P}^1$ with linearization μ_L. Then*

$$\mu_L(x_+) - \mu_L(x_-) = \delta(x_+) \deg L$$

where $\delta(x_+)$ is the weight of the action on the tangent space $T_{\mathbb{P}^1, x_+}$.

As observed in [57, Section 3.1], the above Lemma can be generalized to $\mathbb{C}^*$-actions on polarized pairs as follows:

Lemma 2.1.50. *[57, Corollary 3.2] Let (X, L) be a polarized pair with a $\mathbb{C}^*$-action. Given a point $x \in X$, let C be its orbit closure, with sink x_- and source x_+. Then*

$$\mu_L(x_+) - \mu_L(x_-) = \delta(x_+)(L \cdot C).$$

We present an easy application of the above result to study the intersection product between closures of orbits and the canonical divisor.

Lemma 2.1.51. *Let $\mathbb{C}^*$ act on a smooth polarized pair (X, L). Given a point $x \in X$, let C be its orbit closure, with sink $x_- \in Y_a$ and source $x_+ \in Y_b$. Then*

$$K_X \cdot C = \frac{1}{\delta(x_+)}(w^-(Y_b) - w^+(Y_b) - w^-(Y_a) + w^+(Y_a)),$$

where by $w^+(x)$ (resp. $w^-(x)$) we denote the sum of the positive weights (resp. negative) of the induced $\mathbb{C}^*$-action on $T_{X,x}$.

Proof. Combining Lemma 2.1.50, together with the description of the linearization of T_X done in [10, Lemma 3.11], we conclude. ■

Corollary 2.1.52. *In the situation of Lemma 2.1.51, suppose that the $\mathbb{C}^*$-action is equalized. Then*

$$K_X \cdot C = \nu^-(Y_b) - \nu^+(Y_b) - \nu^-(Y_a) + \nu^+(Y_a).$$

2.2 Geometric invariant theory for $\mathbb{C}^*$-actions

In this section we describe the geometric and semigeometric quotients of a polarized pair under an action of $\mathbb{C}^*$. To this end, we first recall some standard notions regarding geometric invariant theory for reductive group actions, following [18]. We then focus on the case of $\mathbb{C}^*$-actions, giving a complete description of the possible geometric quotients using the theory of sections developed in [6].

Let us first review the various definitions of quotients.

Definition 2.2.1. Let G be an algebraic group acting on a variety X. A *categorical quotient* is a G-invariant morphism $\phi\colon X \to Y$, onto a variety Y, which is universal; that is, every other G-invariant morphism $f\colon X \to Z$ factors uniquely through ϕ so that there exists $h\colon Y \to Z$ such that $f = h \circ \phi$.

Remark 2.2.2. [45, Chap. 0, §2, (2)] If X is normal, then also the categorical quotient Y is normal.

Definition 2.2.3. Let G be an algebraic group acting on a variety X. A morphism $\phi\colon X \to Y$ is a *semigeometric quotient* if

1. ϕ is G-invariant;
2. ϕ is surjective;
3. for every open subset $U \subset Y$, it holds $\mathcal{O}_Y(U) \simeq \mathcal{O}_X(\phi^{-1}(U))^G$;
4. given $W \subset X$ closed and G-invariant, the image $\phi(W)$ is closed;
5. if W_1, W_2 are disjoint closed G-invariant subsets of X, then $\phi(W_1)$ and $\phi(W_2)$ are disjoint;
6. ϕ is affine.

Let us remind that, in this setting, the notion of semigeometric quotient coincides with the one of *good quotient* introduced by Seshadri (see [58, Definition 1.5]).

Definition 2.2.4. Let G be an algebraic group acting on a variety X. A morphism $\phi\colon X \to Y$ is a *geometric quotient* if it is semigeometric and for any point $y \in Y$, the preimage $\phi^{-1}(y)$ is a single orbit.

Lemma 2.2.5. *[18, Proposition 6.1] Semigeometric quotients are categorical.*

Notation 2.2.6. Let G be an algebraic group acting on a variety X. We denote the semigeometric (resp. geometric) quotients of X by G as $X /\!\!/ G$ (resp. X/G). Let us notice that the double slash notation $X /\!\!/ G$ is meant to remind that the semigeometric quotient is not an orbit spaces, that is some orbits may be identified.

Set-up 2.2.7. Let G be a reductive algebraic group acting on a polarized pair (X, L).

As already noticed in Lemma 2.1.37 in the case of $\mathbb{C}^*$-actions, for every $m \geq 0$ there exists an induced G-action on $\mathrm{H}^0(X, mL)$, and thus on the section ring $R(X; L)$. Moreover, as in the case of Corollary 2.1.38 for $\mathbb{C}^*$-actions, up to consider a multiple, assume that L is very ample: then we obtain a G-equivariant embedding $X \hookrightarrow \mathbb{P}(\mathrm{H}^0(X, L))$.

Definition 2.2.8. In the situation of Set-up 2.2.7, we define the *G-invariant section ring of (X, L) under the G-action* as

$$R(X; L)^G := \bigoplus_{m \geq 0} \mathrm{H}^0(X, mL)^G.$$

Definition 2.2.9. In the situation of Set-up 2.2.7, a point $x \in X$ is said to be:

- *semistable* if there exists a G-invariant section $\sigma \in \mathrm{H}^0(X, mL)^G$, for some $m \geq 0$, such that $\sigma(x) \neq 0$ and $X_\sigma = \{y \in X \mid \sigma(y) \neq 0\}$ is affine;
- *stable* if it is semistable, $\dim G_x = \dim G$ and the action of G on X_σ is closed;
- *unstable* if it is not semistable.

We denote by $X^{ss}(L)$ (resp. $X^s(L)$) the set of semistable (resp. stable) points of X under the G-action with the chosen linearization of L.

Theorem 2.2.10. *[18, Theorem 3.3] The G-invariant section ring $R(X; L)^G$ is a finitely generated graded $\mathbb{C}$-algebra.*

Theorem 2.2.11. *In the situation of Set-up 2.2.7, the rational map $\phi\colon X \dashrightarrow \operatorname{Proj} R(X; L)^G$, given by the inclusion $R(X; L)^G \subset R(X; L)$, restricts to a morphism*

$$\phi\colon X^{ss}(L) \to X^{ss}(L) /\!\!/ G := \operatorname{Proj} R(X; L)^G$$

which is a semigeometric quotient, and $X^{ss}(L) /\!\!/ G$ is a normal projective variety. Moreover, there exists an open subset $Y^s \subset X^{ss}(L) /\!\!/ G$ such that $\phi^{-1}(Y^s) = X^s(L)$ and $\phi\colon X^s \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.

We remind that, by construction, the semigeometric quotient $X^{ss}(L) /\!\!/ G$ depends on the choice of a linearization.

Example 2.2.12. Given $q = (q_0, \dots, q_n) \in \mathbb{N}^{n+1}$, consider the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ defined as

$$t \cdot x = (t^{q_0} x_0, \dots, t^{q_n} x_n),$$

for $t \in \mathbb{C}^*, x \in \mathbb{C}^{n+1}$. The open subset $\mathbb{C}^{n+1} \setminus \{0\}$ is $\mathbb{C}^*$-invariant and stable, and we call the geometric quotient $\mathbb{P}_q := \mathbb{P}(q_0, \dots, q_n)$ a *weighted projective space*.

2.2.1 GIT-quotients of $\mathbb{C}^*$-actions and admissible quotients

In this section, following [6], we describe all the geometric and semigeometric quotients of a polarized pair (X, L) under a $\mathbb{C}^*$-action in terms of the ordered set of fixed point components of X.

Set-up 2.2.13. Let $\mathbb{C}^*$ act on a polarized pair (X, L). Suppose that the action is normalized, with bandwidth δ and criticality r.

Definition 2.2.14. In the situation of Set-up 2.2.13, let $Y, Y' \in \mathcal{Y}$. We say that Y *is smaller than* Y', and write $Y \preccurlyeq Y'$, if $X^-(Y) \cap X^+(Y') \neq \emptyset$, that is there exists an orbit converging at Y for $t \to \infty$, and to Y' for $t \to 0$.

Remark 2.2.15. Notice that the order introduced in Definition 2.2.14 is opposite to the one originally defined in [6, Definition 1.1]. The motivation behind our choice lies in the property that, given $Y, Y' \in \mathcal{Y}$ such that $Y \preccurlyeq Y'$, we also have that $\mu_L(Y) \leq \mu_L(Y')$.

Lemma 2.2.16. *[6, Proposition 2.3] For every inner component $Y \in \mathcal{Y}^\circ$, it holds that $Y_- \preccurlyeq Y \preccurlyeq Y_+$.*

Definition 2.2.17. A *semisection* is a partition of $\mathcal{Y}$ in a triple $(\mathcal{Y}_-, \mathcal{Y}_0, \mathcal{Y}_+)$ such that, if $Y \in \mathcal{Y}_- \sqcup \mathcal{Y}_0$, and $Y' \preccurlyeq Y$, then $Y' \in \mathcal{Y}_-$.

A *section* is a semisection such that $\mathcal{Y}_0 = \emptyset$, $\mathcal{Y}_\pm \neq \emptyset$.

Definition 2.2.18. In the situation of Set-up 2.2.13, Let $(\mathcal{Y}_-, \mathcal{Y}_0, \mathcal{Y}_+)$ be a semisection. Then the subset of X

$$U \coloneqq X \setminus (\bigcup_{Y \in \mathcal{Y}_+} X^+(Y) \sqcup \bigcup_{Y \in \mathcal{Y}_-} X^-(Y))$$

is called a *semisectional* set. A subset U associated to a section is called a *sectional* set.

Theorem 2.2.19. *[6, Theorem] Let X be a normal projective variety with an action of $\mathbb{C}^*$. If U is a semisectional set, then U is open, $\mathbb{C}^*$-invariant, and there exists a semigeometric quotient $U \to U /\!\!/ \mathbb{C}^*$. Moreover, if U is sectional, then the quotient $U \to U/\mathbb{C}^*$ is geometric.*

We now present a specific family of sections and semisections whose associated geometric and semigeometric quotients, as we will see in Theorem 2.2.30, are not only complete, but actually projective.

Lemma 2.2.20. *[48, Construction 1] In the situation of Set-up 2.2.13, for any index $i = 0, \dots, r$ consider the following partition of $\mathcal{Y}$:*

$$\begin{aligned} \mathcal{Y}_- &\coloneqq \{Y \in \mathcal{Y} \mid \mu_L(Y) < a_i\}, \\ \mathcal{Y}_0 &\coloneqq \{Y \in \mathcal{Y} \mid \mu_L(Y) = a_i\}, \\ \mathcal{Y}_+ &\coloneqq \{Y \in \mathcal{Y} \mid \mu_L(Y) > a_i\}. \end{aligned}$$

Then $(\mathcal{Y}_-, \mathcal{Y}_0, \mathcal{Y}_+)$ is a semisection. We denote by $X^{ss}(i, i)$ the associated semisectional subset.

Corollary 2.2.21. *[48, Construction 1] For any index $i = 0, \dots, r-1$, consider the following partition of $\mathcal{Y}$:*

$$\begin{aligned} \mathcal{Y}_- &\coloneqq \{Y \in \mathcal{Y} \mid \mu_L(Y) \leq a_i\}, \\ \mathcal{Y}_+ &\coloneqq \{Y \in \mathcal{Y} \mid \mu_L(Y) \geq a_{i+1}\}. \end{aligned}$$

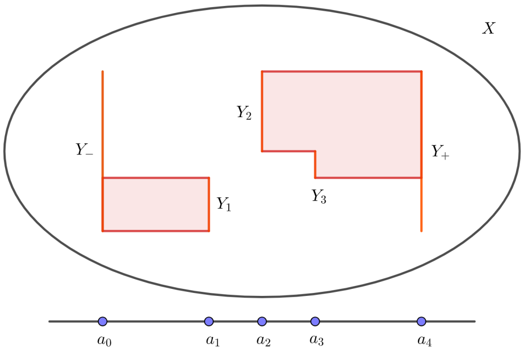
Then $(\mathcal{Y}_-, \mathcal{Y}_+)$ is a section, whose associated sectional open subset will be denoted by $X^s(i, i+1)$.

Let us compute a specific case in the following:

Example 2.2.22. Given a $\mathbb{C}^*$-action on a polarized pair (X, L) with criticality 4, let us compute for example $X^s(1, 2)$: by construction we have that

$$X^s(1,2) = X \setminus (Y_- \cup X^+(Y_1) \cup X^-(Y_2) \cup X^-(Y_3) \cup Y_+),$$

since $X^\pm(Y_\mp) = Y_\mp$. We may intuitively represent $X^s(1,2)$ by means of the following picture, where the colored part is the one removed:



Proposition 2.2.23. *[6, Lemma 2.2] In the situation of Set-up 2.2.13, the set of semistable points $X^{ss}(L)$ is semisectional.*

Thus by Theorem 2.2.19 the subsets $X^{ss}(i,i)$ and $X^s(i,i+1)$ are non-empty, open and $\mathbb{C}^*$-invariant.

Notation 2.2.24. Thanks to Theorem 2.2.19, every semisectional set gives rise to a semigeometric quotient. Therefore, for any $i = 0, \dots, r$, we denote by $\pi_i \colon X^{ss}(i,i) \to \mathcal{S}X_i := X^{ss}(i,i) /\!\!/ \mathbb{C}^*$ the semigeometric quotient. For any $i = 0, \dots, r-1$, we denote by $\pi_i \colon X^s(i,i+1) \to \mathcal{G}X_i := X^s(i,i+1)/\mathbb{C}^*$ the geometric quotient.

The first and the last geometric and semigeometric quotient play a fundamental role in the forthcoming discussion, thus we introduce a special notation which resembles the role of sink and source:

Notation 2.2.25. The geometric quotients

$$\pi_0 \colon X^s(0,1) \to \mathcal{G}X(0,1), \quad \pi_{r-1} \colon X^s(r-1,r) \to \mathcal{G}X(r-1,r)$$

will be respectively also denoted by

$$\pi_- \colon X^s_- \to \mathcal{G}X_-, \quad \pi_+ \colon X^s_+ \to \mathcal{G}X_+.$$

Similarly, the semigeometric quotients

$$\pi_0 \colon X^{ss}(0,0) \to \mathcal{S}X(0,0), \quad \pi_r \colon X^{ss}(r,r) \to \mathcal{S}X(r,r)$$

will be also respectively denoted by

$$\pi_- \colon X^{ss}_- \to \mathcal{S}X_-, \quad \pi_+ \colon X^{ss}_+ \to \mathcal{S}X_+.$$

Definition 2.2.26. The geometric (resp. semigeometric) quotients $\mathcal{G}X_\pm$ (resp. $\mathcal{S}X_\pm$) will be called *extremal* and we will respectively denote them by $\mathcal{G}X_-, \mathcal{G}X_+$ (resp. $\mathcal{S}X_-, \mathcal{S}X_+$). Every geometric (resp. semigeometric) quotient which is not extremal is called *inner*.

Remark 2.2.27. In the situation of Set-up 2.2.13, the extremal semisectional and sectional set can be described as:

$$X_\pm^{ss} = X^\pm(Y_\pm), \quad X_\pm^s = X^\pm(Y_\pm) \setminus Y_\pm.$$

In particular, it holds that $\mathcal{S}X_\pm \simeq Y_\pm$.

Remark 2.2.28. There exist natural morphisms $\mathcal{G}X_\pm \to \mathcal{S}X_\pm \simeq Y_\pm$.

We now show that semigeometric and geometric quotients are projective. To this end, we first introduce the following:

Definition 2.2.29. In the situation of Set-up 2.2.13, for any rational number $\tau \in [0, \delta] \cap \mathbb{Q}$, let I_τ be the homogeneous ideal

$$I_\tau := \bigoplus_{m \geq 0, m\tau \in \mathbb{Z}} \mathrm{H}^0(X, mL)_{m\tau},$$

and let $R(X; L)_\tau$ be the graded subalgebra of $R(X; L)$ defined as

$$R(X; L)_\tau := \bigoplus_{m \geq 0, m\tau \in \mathbb{Z}} \mathrm{H}^0(X, mL)_{m\tau},$$

where we recall that the subindex $m\tau$ denotes the direct summand of $\mathrm{H}^0(X, mL)$ on which $\mathbb{C}^*$ acts with weight equal to $m\tau$.

Theorem 2.2.30. *[48, Proposition 2.11] In the situation of Set-up 2.2.13, the geometric and semigeometric quotients $\mathcal{G}X_i, \mathcal{S}X_i$ are normal projective varieties. In particular:*

- *For every $i = 0, \dots, r-1$, and every $\tau \in (a_i, a_{i+1}) \cap \mathbb{Q}$, it holds that the set of stable points $X^s(i, i+1)$ can be described as*

  $$X^s(i, i+1) = X \setminus Z(I_\tau \otimes_{R(X;L)_\tau} R(X; L))$$

  and the geometric quotient $\mathcal{G}X_i$ can be obtained as

  $$\mathcal{G}X_i = \operatorname{Proj} R(X; L)_\tau = \operatorname{Proj} \bigoplus_{m \geq 0, m\tau \in \mathbb{Z}} \mathrm{H}^0(X, mL)_{m\tau};$$

- *For every $i = 0, \dots, r$, it holds that the semisectional set $X^{ss}(i, i)$ can be described as*

  $$X^s(i, i) = X \setminus Z(I_{a_i} \otimes_{R(X;L)_{a_i}} R(X; L))$$

  and the semigeometric quotient $\mathcal{S}X_i$ can be obtained as

  $$\mathcal{S}X_i = \operatorname{Proj} R(X; L)_{a_i} = \operatorname{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(X, mL)_{ma_i};$$

If we assume that the $\mathbb{C}^*$-action is equalized, using a Corollary of Luna Slice Theorem (see [45, Corollary in p.199]) we may conclude that the geometric quotients are smooth, as explained in the following:

Lemma 2.2.31. *[48, Lemma 2.14] Suppose that the $\mathbb{C}^*$-action on X is equalized. Then the geometric quotients $\pi_i : X^s(i, i+1) \to \mathcal{G}X_i$ are $\mathbb{C}^*$-principal bundles. In particular, if moreover X is smooth, then its geometric quotient $\mathcal{G}X_i$, for $i = 0, \dots, r-1$, are smooth.*

2.3 Birational geometry induced by $\mathbb{C}^*$-actions

In this section we investigate the birational geometry of the geometric quotients of a polarized pair under a $\mathbb{C}^*$-action. We introduce some fundamental notions, such as *B-type actions, bordisms*, and *geometric realization of a birational map* (see respectively Definitions 2.3.2, 2.3.7, 2.3.19). We finish by introducing an algebro-geometric operation, named *pruning*, which allows to easily construct several $\mathbb{C}^*$-equivariant birational modifications of a variety (see Definition 2.3.24).

Set-up 2.3.1. Let $\mathbb{C}^*$ act on a polarized pair (X, L). Suppose that the action is normalized, with bandwidth δ and criticality r.

Definition 2.3.2. In the situation of Set-up 2.3.1, the $\mathbb{C}^*$-action is of *B-type* if the natural maps $\mathcal{G}X_\pm \to \mathcal{S}X_\pm$ are isomorphisms.

As we will see in Remark 2.3.29, the condition of being B-type is not very restrictive, as one can always perform a $\mathbb{C}^*$-equivariant birational modification of X in order to assume the $\mathbb{C}^*$-action is of B-type.

As a corollary, noticing that the set of orbits joining Y_- and Y_+ is open and non-empty (cf. Remark 2.1.18), we may write that:

Lemma 2.3.3. *[49, Lemma 3.4] Let X be a smooth projective variety with an action of $\mathbb{C}^*$ of B-type. Then there exists a birational map*

$$\widetilde{\psi} : Y_- \dashrightarrow Y_+,$$

with $\operatorname{Exc}(\widetilde{\psi}) = \bigsqcup_{Y \neq Y_+ \in \mathcal{Y}} \overline{X^+(Y)} \cap Y_-$*, which associates to every point $y \in Y_- \setminus \operatorname{Exc}(\widetilde{\psi})$ the limit, for $t \to 0$, of the unique orbit having y as limit for $t \to \infty$.*

The above Lemma was the starting point for the investigation of [49]. Indeed the above birational map is the simplest manifestation of the deeper birational equivalence linking all the geometric quotients of a polarized pair under an action of a reductive algebraic group. In our setting, we can generalize the above Lemma as follows:

Proposition 2.3.4. *In the situation of Set-up 2.3.1, there exists a birational map*

$$\psi \colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$$

which factorizes among the inner geometric quotients

$$\mathcal{G}X_- \overset{\psi_1}{\dashrightarrow} \mathcal{G}X_1 \overset{\psi_2}{\dashrightarrow} \dots \overset{\psi_r}{\dashrightarrow} \mathcal{G}X_+.$$

The map ψ is called the natural birational map associated to the $\mathbb{C}^*$-action on (X, L)*.*

Proof. The existence of such birational maps follows by using that the intersection $\bigcap_{i=0}^{r-1} X^s(i, i+1)$ is open and non-empty. ■

Lemma 2.3.5. *In the situation of Set-up 2.3.1, for every $i = 1, \dots, r$, the exceptional locus of the birational map $\psi_i \colon \mathcal{G}X_{i-1} \dashrightarrow \mathcal{G}X_i$ (resp.of ψ_i^{-1}) is contained into $(X^+(Y_i) \setminus Y_i)/\mathbb{C}^*$ (resp. $(X^-(Y_i) \setminus Y_i)/\mathbb{C}^*$).*

Lemma 2.3.6. *[48, Remark 2.13] In the situation of Set-up 2.3.1, the birational maps among the geometric quotients $\mathcal{G}X_i$, for $i = 0, \dots, r-1$, fit in a commutative diagram, whose diagonal arrows are contractions:*

$$\begin{tikzcd}
 & \mathcal{G}X_- \arrow[rr, dashed] \arrow[dl] \arrow[dr] & & \mathcal{G}X_1 \arrow[rr, dashed] \arrow[dl] & & \mathcal{G}X_{r-2} \arrow[rr, dashed] \arrow[dr] & & \mathcal{G}X_+ \arrow[dl] \arrow[dr] & \\
\mathcal{S}X_- & & \mathcal{S}X_1 & & \cdots & & \mathcal{S}X_{r-1} & & \mathcal{S}X_+
\end{tikzcd}$$

In the next chapters, we will construct explicit examples of the natural birational map ψ; before doing so, we aim to find a sufficient criterion which guarantees that ψ is an isomorphism in codimension 1.

Definition 2.3.7. In the situation of Set-up 2.3.1, a $\mathbb{C}^*$-action is called a *bordism* if it is of B-type and, for every inner component $Y \in \mathcal{Y}^\circ$, the closure of the Białynicki-Birula cells $\overline{X^\pm(Y)}$ does not contain codimension one subvarieties.

The notion of bordism has been introduced in [49, Definition 3.8] for smooth projective varieties. In that setting, we have the following characterization (cf. Lemma 2.1.20):

Lemma 2.3.8. *[49, Corollary 3.7] Let $\mathbb{C}^*$ act on a smooth projective variety X. Then the $\mathbb{C}^*$-action is a bordism if and only if the restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(Y_-)$ fits into a short exact sequence*

$$0 \to \mathbb{Z}[Y_+] \to \operatorname{Pic}(X) \to \operatorname{Pic}(Y_-) \to 0.$$

Lemma 2.3.9. *Let (X, L) be a polarized pair with an action of $\mathbb{C}^*$ which is a bordism. Then the natural birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is a small modification.*

Proof. By construction, $\operatorname{Exc}(\psi) = \bigcup_{Y \neq Y_+ \in \mathcal{Y}} (\overline{X^+(Y)} \cap Y_-)/\mathbb{C}^*$. Since by hypothesis for every $Y \in \mathcal{Y}^\circ$ the BB-cell $\overline{X^+(Y)}$ is not a divisor, we conclude. ■

We notice that being a bordism is a global property of a $\mathbb{C}^*$-action. We may define a local version of such notion, by asking that, for a certain index i, the set of stable points $X^s(i, i+1)$ does not contain divisors, as in the following:

Definition 2.3.10. In the situation of Set-up 2.3.1, a geometric quotient $\mathcal{G}X_i$ is *admissible* if $X \setminus (X^s(i, i+1) \cup Y_\pm)$ does not contain codimension one subvarieties.

Remark 2.3.11. Given a $\mathbb{C}^*$-action on (X, L), every geometric quotient is admissible if and only if for every component $Y \in \mathcal{Y}^\circ$ it holds $\operatorname{codim} \overline{X^\pm(Y)} \geq 2$.

Lemma 2.3.12. *If $\mathcal{G}X_i$ is not admissible, then either every $\mathcal{G}X_k$, for $k < i$, or every $\mathcal{G}X_m$, for $m > i$, is not admissible.*

Proof. By assumption, $X \setminus (X^s(i, i+1) \cup Y_\pm)$ contains a divisor. Thus by construction such a divisor will be contained either in the closure of a cell $X^+(Y_j)$, if $j \leq i$, or in the closure of $X^-(Y_j)$, if $j \geq i+1$. Let us prove the statement in the first case, being the other similar. By definition, for any $m > i$ the cell $X^+(Y_j)$ will be contained in $X \setminus (X^s(m, m+1) \cup Y_\pm)$, proving that any other quotient $\mathcal{G}X_m$ will not be admissible. ■

Corollary 2.3.13. *If $\mathcal{G}X_-$ and $\mathcal{G}X_+$ are admissible, then every geometric quotient $\mathcal{G}X_i$, for $i = 1, \ldots, r-2$ is admissible, too.*

Corollary 2.3.14. *A B-type $\mathbb{C}^*$-action is a bordism if and only if every geometric quotient is admissible.*

We conclude this section by introducing a way to lift up the divisors from the geometric quotients $\mathcal{G}X_i$ to the variety X:

Definition 2.3.15. Let the $\mathbb{C}^*$-action on (X, L) be a bordism. For every index $i = 0, \dots, r-1$, we define an *extension map* as $e_i\colon \operatorname{Div}(\mathcal{G}X_i) \to \operatorname{Div}(X)$, $D \mapsto e_i(D) = \overline{\pi_i^{-1}(D)}$, where $\pi_i\colon X^s(i, i+1) \to \mathcal{G}X_i$ is the geometric quotient map.

Lemma 2.3.16. *For any $f \in \mathbb{C}(\mathcal{G}X_i)$, $e_i(\operatorname{div}(f)) = \operatorname{div}(f \circ \pi_i)$.*

Proof. We have $\operatorname{div}(f \circ \pi_i) = \overline{\pi_i^{-1}(\operatorname{div}(f))} + E$, where E is a prime divisor in $X \setminus X^s(i, i+1)$. Since $\mathcal{G}X_i$ is admissible by Corollary 2.3.14, $E = 0$. ■

Lemma 2.3.17. *For any $D, D' \in \operatorname{Div}(\mathcal{G}X_i)$ such that $D \sim D'$, it holds $e_i(D) \sim e_i(D')$.*

Proof. Suppose that $D' = D + \operatorname{div}(f)$. Then, using Lemma 2.3.16, we obtain:

$$\begin{aligned} e_i(D') &= \overline{\pi_i^{-1}(D')} = \overline{\pi_i^{-1}(D + \operatorname{div}(f))} = \\ &= \overline{\pi_i^{-1}(D)} + \overline{\pi_i^{-1}(\operatorname{div} f)} = e_i(D) + e_i(\operatorname{div}(f)). \end{aligned}$$

■

Lemma 2.3.18. *Let $\mathbb{C}^*$ act on the polarized pair (X, L). Then every Cartier divisor in X is linearly equivalent to a $\mathbb{C}^*$-invariant divisor. Moreover, the action of $\mathbb{C}^*$ on X is a bordism if and only if the only $\mathbb{C}^*$-invariant divisors are linear combinations of $Y_\pm$, and the divisors of the form $e_i(E)$, for $E \in \operatorname{Div}(\mathcal{G}X_i)$.*

Proof. Since every Cartier divisor is difference of two very ample divisors, it suffices to show that every very ample divisor is linearly equivalent to a $\mathbb{C}^*$-invariant one. Let us consider the induced $\mathbb{C}^*$-action on the linear system $|A_1|$, with A_1 very ample; such action will have at least a fixed point, which is associated to a $\mathbb{C}^*$-invariant divisor, hence we conclude.

We now show the second part of the statement, noting that the only if part is obvious. Let us then assume that the $\mathbb{C}^*$-action on X is a bordism. Note that the divisors of the form $e_i(E)$, $E \in \operatorname{Div}(\mathcal{G}X_i)$ are clearly $\mathbb{C}^*$-invariant. Now let D be an irreducible $\mathbb{C}^*$-invariant divisor. If D is pointwise fixed by the action, then it is either the sink or the source, by definition of bordism. On the other hand, if D is $\mathbb{C}^*$-invariant but not pointwise fixed, then it contains an $(n-2)$-dimensional family of 1-dimensional orbits (whose union is dense in D). Let $\mathbb{C}^*p$ be the general element of this family, and let Y_1, Y_2 be the fixed point components of the action containing the sink and the source of $\mathbb{C}^*p$, respectively. It follows that $D \subset \overline{X^-(Y_1)} \cap \overline{X^+(Y_2)}$, and from the definition of bordism we conclude that $Y_1 = Y_-$, $Y_2 = Y_+$. It then easily follows that D can be written as divisor of the form $e_i(E)$, for $E \in \operatorname{Div}(\mathcal{G}X_i)$. ■

2.3.1 Geometric realization of a birational map

Let us introduce the notion of *geometric realization of a birational map*, which is the milestone of our discussion.

Definition 2.3.19. Given a birational map $\varphi\colon Z_- \dashrightarrow Z_+$ between normal projective varieties, a *geometric realization of* φ is a normal projective variety X, endowed with a $\mathbb{C}^*$-action of B-type such that the sink and the source are precisely Z_-, Z_+ and the natural birational map ψ among them, defined in Proposition 2.3.4, coincides with φ.

We remark that such definition has been already introduced in [51, Definition 2.10]; however, there is a slight difference, as we ask that the $\mathbb{C}^*$-action is also of B-type. Intuitively, a geometric realization can be thought as a projective compactification of the birational map. As we will see in Chapters 4, 5 and 6, geometric realizations provides a new bridge between algebraic torus actions and birational geometry; more precisely, several properties of the birational map φ, such as its factorizations, can be well understood in terms of VGIT of the geometric quotients of the geometric realization under the $\mathbb{C}^*$-action.

Let us present an example of a geometric realization of the standard Cremona involution:

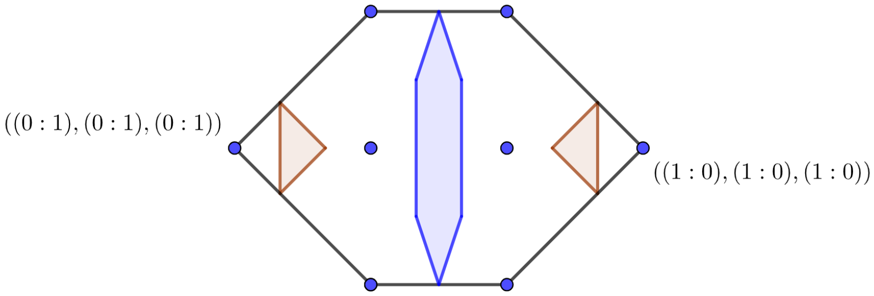
Example 2.3.20. Consider the standard Cremona transformation

$$\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (x : y : z) \mapsto (yz : xz : xy).$$

We aim to construct a geometric realization of φ. To this end, let $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and consider the $\mathbb{C}^*$-action on $(X, \mathcal{O}_X(1,1,1))$ defined as follows:

$$t \cdot ((x_0 : x_1), (y_0 : y_1), (z_0 : z_1)) \to ((x_0 : tx_1), (y_0 : ty_1), (z_0 : tz_1)).$$

The sink and the source of the $\mathbb{C}^*$-action are respectively $y_- = ((0 : 1), (0 : 1), (0 : 1))$ and $y_+ = ((1 : 0), (1 : 0), (1 : 0))$, and the induced $\mathbb{C}^*$-action on $T_{X,y_\pm}$ are $(\pm 1^3)$, where the exponent denote the occurrence of the weight. We may represent the $\mathbb{C}^*$-action on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by means of the following image:



The two red triangles are the polytopes of the extremal geometric quotients, which are isomorphic to $\mathbb{P}^2$, and the blue hexagon is the polytope of the inner geometric quotient, which is the blow-up of $\mathbb{P}^2$ along 3 points. The $\mathbb{C}^*$-action is equalized, thus using Lemma 2.1.28, Remark 2.2.27 and that $\mathbb{P}(\mathcal{N}^\vee_{y_\pm|X}) \simeq \mathbb{P}^2$, one may then easily show that the natural birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ coincides with the standard Cremona transformation.

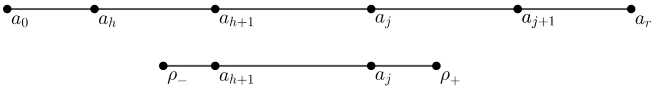
Remark 2.3.21. Geometric realizations are not, in general, unique. For instance, let X be a smooth projective variety which is a geometric realization of a birational map $\varphi\colon Y_- \dashrightarrow Y_+$ among smooth projective varieties, and let Y be an inner component of $X^{\mathbb{C}^*}$. Let $X^\flat$ be the blow-up of X along Y. The blow-up map $X^\flat \to X$ is $\mathbb{C}^*$-equivariant, and the birational map among the extremal geometric quotients of $X^\flat$ coincide with φ. As we will see in Remark 5.1.13, even if the $\mathbb{C}^*$-action on a geometric realization X is a bordism, the uniqueness of X does not hold.

2.3.2 Pruning of a variety

This section is devoted to the construction of the *pruning* of a variety with a $\mathbb{C}^*$-action (see Definition 2.3.24). As we will see, such fundamental procedure will be vastly used in the forthcoming discussion because, as we will see, pruning are geometric realizations of composition of the natural birational maps among the inner geometric quotients. Before stating the main result concerning the pruning (see Theorem 2.3.27), we present an intuitive idea about such construction.

Set-up 2.3.22. Let (Z, E) be a polarized pair with a normalized $\mathbb{C}^*$-action with bandwidth δ and criticality r. Let $\rho_-, \rho_+ \in [a_0, a_r] \cap \mathbb{Q}$, with $\rho_- < \rho_+$. We assume that $\rho_- \in (a_h, a_{h+1}), \rho_+ \in (a_j, a_{j+1})$.

We can picture the critical values of the $\mathbb{C}^*$-action on (Z, E) as the first segment below:



The *pruning of* (Z, E) *with respect to* $\rho_\pm$ is a normal projective variety X, which is denoted by $\mathcal{P}(Z)_{\rho_-}^{\rho_+}$, endowed with a $\mathbb{C}^*$-action such that the sink is $\mathcal{G}Z_h$, the source is $\mathcal{G}Z_j$, and there exists a $\mathbb{C}^*$-equivariant birational map $Z \dashrightarrow X$ which is an isomorphism over $X \setminus (X^-(\mathcal{G}Z_h) \cup X^+(\mathcal{G}Z_j))$. Intuitively, we have cut the segment at the level of $\rho_\pm$, obtaining a $\mathbb{C}^*$-equivariant modification of X, and removing the fixed point components of weights small or equal than a_h (resp. greater or equal than a_{j+1}), as in the second segment above.

The pruning is quite helpful in different contexts. For instance, a pruning along the extremal intervals (see Notation 2.3.25) is a generalization of a blow-up, because it replaces the sink and the source with two codimension 1 subvarieties, namely the extremal geometric quotients of the action. Moreover, if the variety is smooth and the action is equalized, the procedure of blow-up coincides with a pruning along the extremal intervals (see Lemma 2.3.35).

Moreover, as we will see, under certain hypothesis, a pruning allows us to construct $\mathbb{C}^*$-equivariant birational modifications of the $\mathbb{C}^*$-variety which are bordisms (see Proposition 2.3.31).

Lemma 2.3.23. *In the situation of Set-up 2.3.22, assume furthermore that* $\rho_\pm$ *are integers, that* E *is very ample, and that the embedding* $Z \subset \mathbb{P}(\mathrm{H}^0(Z, E))$ *is projectively normal. Then the* $\mathbb{C}$-*algebra*

$$S \coloneqq \bigoplus_{m \geq 0} \bigoplus_{k = m\rho_-}^{m\rho_+} \mathrm{H}^0(Z, mE)_k.$$

is finitely generated.

Proof. We will show that the $\mathbb{C}$-algebra

$$\widetilde{S} \coloneqq \bigoplus_{m \geq 0} \bigoplus_{k = m\rho_-}^{m\rho_+} S^m \, \mathrm{H}^0(Z, E)_k,$$

is finitely generated, and that the natural homomorphism $\widetilde{i^*} \colon \widetilde{S} \to S$ is surjective.

We first prove that the algebra $\widetilde{S}$ is finitely generated. To this end, since E is ample using [10, Lemma 2.4] we may suppose that $\mathrm{H}^0(Z, E)$ is generated by $s_1, \ldots, s_n$, with $s_i \in \mathrm{H}^0(Z, E)_{w_i}$ for every i, where $w_i \in [a_0, a_r] \cap \mathbb{Z}$ are the weights of the induced $\mathbb{C}^*$-action on $\mathrm{H}^0(Z, E)$. The monomials $\prod_i s_i^{m_i}$ in $S^m(\mathrm{H}^0(Z, E))$ belonging to $\widetilde{S}$ are those which satisfy the following system of inequalities:

$$\begin{cases} \sum_{i=1}^n (w_i - \rho_-) m_i \geq 0, \\ \sum_{i=1}^n (\rho_+ - w_i) m_i \geq 0, \\ m_i \geq 0. \end{cases}$$

This is a rational polyhedral cone in $\mathbb{R}^n$, therefore by Gordan's Lemma its intersection with the lattice of monomials is finitely generated.

Finally, in order to prove that $\widetilde{i^*}$ is surjective we simply note that the natural map $i^*\colon \mathrm{Sym}(\mathrm{H}^0(Z,E)) \to \bigoplus_{m\geq 0}\mathrm{H}^0(Z,mE)$ is surjective –thanks to the projective normality of $Z\subset \mathbb{P}(\mathrm{H}^0(Z,E))$– and $\mathbb{C}^*$-equivariant. ■

We remark that the Lemma above holds in a greater generality, but we have presented it in this way for the sake of simplicity.

Definition 2.3.24. In the situation of Set-up 2.3.22, let $d\in\mathbb{Z}_{>0}$ be the minimum positive integer such that $\rho_\pm d\in\mathbb{Z}$. We define the *pruning of* (Z,E) *with respect to* ρ_-,ρ_+ as:

$$\mathcal{P}(Z)_{\rho_-}^{\rho_+} := \mathrm{Proj}\, S^{(nd)}, \quad n\gg 0,$$

where $S^{(nd)}$ is the graded $\mathbb{C}$-algebra $S^{(nd)}=\bigoplus_{m\geq 0} S_m^{(nd)}$ whose graded pieces are defined by

$$S_m^{(nd)} := \bigoplus_{k=mnd\rho_-}^{mnd\rho_+} \mathrm{H}^0(Z,mndE)_k, \quad m\geq 0.$$

Notation 2.3.25. A *pruning with respect to the extremal intervals*, denoted by $\mathcal{P}(Z)_-^+$, is a pruning where $\rho_-\in(a_0,a_1)$, $\rho_+\in(a_{r-1},a_r)$.

Remark 2.3.26. Note that $S^{(nd)}$ is finitely generated for $n\gg 0$ by Lemma 2.3.23, and that $\mathrm{Proj}\,S^{(nd)}=\mathrm{Proj}\,S^{(n'd)}$ for $n,n'\gg 0$, then X is well-defined and depends only on the pair (Z,E) and on the rational numbers ρ_-,ρ_+. Furthermore, the pruning of (Z,E) with respect to ρ_-,ρ_+ is equal to the pruning of (Z,nE) with respect to $n\rho_-,n\rho_+$, for any $n>0$.

Theorem 2.3.27. *In the situation of the Set-up 2.3.22, take* $\rho_-\in(a_h,a_{h+1})\cap\mathbb{Q}$, $\rho_+\in(a_j,a_{j+1})\cap\mathbb{Q}$ *for some* $h,j\in\{0,\dots,r-1\}$. *Then the pruning* $X=\mathcal{P}(Z)_{\rho_-}^{\rho_+}$ *of* (Z,E) *with respect to* ρ_-,ρ_+ *is a normal projective variety, endowed with a B-type* $\mathbb{C}^*$*-action whose sink and source are, respectively,* $\mathcal{G}Z_h,\mathcal{G}Z_j$. *Moreover there exists a* $\mathbb{C}^*$*-equivariant birational map* $\Phi_{\rho_-,\rho_+}\colon Z \dashrightarrow X$.

The proof of Theorem 2.3.27 will be divided in several steps. Without loss of generality, using Remark 2.3.26, we may assume –by exchanging E with a suitable multiple– that $d=n=1$ and $\rho_\pm\in\mathbb{Z}$. Note that, by construction, the action of $\mathbb{C}^*$ on $R(Z;E)$ restricts to an action on

$$S=\bigoplus_{m\geq 0}\bigoplus_{k=m\rho_-}^{m\rho_+}\mathrm{H}^0(Z,mE)_k\subset R(Z;E),$$

providing a $\mathbb{C}^*$-action on $X=\mathrm{Proj}(S)$ such that the natural map $\Phi_{\rho_-,\rho_+}\colon Z\dashrightarrow X$ is $\mathbb{C}^*$-equivariant.

Along the proof, we will use the following notation. For every $m>0$, we will consider the decomposition $S_m=S_m^-\oplus S_m^0\oplus S_m^+$, where

$$S_m^\pm := \mathrm{H}^0(Z,mE)_{m\rho_\pm}, \quad S_m^0 := \bigoplus_{m\rho_-<k<m\rho_+}\mathrm{H}^0(Z,mE)_k.$$

For every homogeneous element $f\in S_m$, $m>0$, we will denote $D^+(f,X):=\mathrm{Spec}\left(S_{(f)}\right)\subset X$. Then we define the following open subsets of X:

$$U_\pm := \bigcup_{m>0}\bigcup_{f\in S_m^\pm} D^+(f,X), \quad U_0 := \bigcup_{m>0}\bigcup_{f\in S_m^0} D^+(f,X),$$

and note that, by construction, $X=U_-\cup U_0\cup U_+$, and that $U_0,U_\pm$ are $\mathbb{C}^*$-invariant.

Step 1. *The variety $X = \operatorname{Proj}(S)$ is normal.*

Proof. We will show that the affine open subsets $D^+(f, X) \subset X$ are normal, for every $f \in S_m^- \cup S_m^0 \cup S_m^+$.

Let us start with the case in which $f \in S_m^0$. We claim that

$$S_{(f)} = R(Z; E)_{(f)} \tag{2.1}$$

The "$\subset$" inclusion is obvious by construction, let us prove the converse. Given an element $\frac{g}{f^a} \in R(Z; E)_{(f)}$, with $g, f^a \in S_{ma}$, we can decompose $g = \sum_{k=0}^{ma\delta} g_k$, with $g_k \in \mathrm{H}^0(Z, mE)_k$. There exists a suitable $l \geq 0$ – it is enough to take $l \geq ma\rho_-, ma(\delta - \rho_+)$– for which

$$f^l g \in S^0_{m(a+l)}, \text{ therefore } \frac{f^l g}{f^{l+a}} \in \left(\bigoplus_{m>0} S_m^0 \right)_{(f)},$$

thus we obtain the other inclusion. This tells us that $D^+(f, X)$ is isomorphic to an open subset $D^+(f, Z) := \operatorname{Spec}\left(R(Z; E)_{(f)}\right)$ of Z, hence normal.

Next we prove that $D^+(f, X)$ is normal for every $f \in S_m^-$ (the proof for $f \in S_m^+$ is analogous). Note that in this case the inclusion $S_{(f)} \subset R(Z; E)_{(f)}$ is not an equality in general, but an argument analogous to the one above tells us that:

$$R(Z, E)_{(f)} = (S')_{(f)}, \qquad \text{where } S' := \bigoplus_{m \geq 0} \bigoplus_{k=0}^{m\rho_+} \mathrm{H}^0(Z, mE)_k.$$

Let us now consider a polynomial ring in one variable $\mathbb{C}[y]$, and consider the $\mathbb{C}^*$-action on it given by $t \cdot (\sum_b c_b y^b) = \sum_b c_b t^{-b} y^b$. We then consider the induced $\mathbb{C}^*$-action on the $\mathbb{C}$-algebra $\overline{S} := S'_{(f)} \otimes_{\mathbb{C}} \mathbb{C}[y] = S'_{(f)}[y]$. Note that the variety $\operatorname{Spec}(\overline{S}) = D^+(f, Z) \times \mathbb{C}$ is normal, and so it is its categorical quotient by the induced $\mathbb{C}^*$-action (cf. Remark 2.2.2), which is $\operatorname{Spec}\left(\overline{S}^{\mathbb{C}^*}\right)$.

We may then conclude by noting that we have an isomorphism $\varphi\colon \overline{S}^{\mathbb{C}^*} \to S_{(f)}$. In fact, every element of $\overline{S}^{\mathbb{C}^*}$ can be written as a finite sum of the form:

$$\sum_{b=0}^{ma(\rho_+ - \rho_-)} \frac{g_b}{f_a} y^b, \text{ where } g_b \in \mathrm{H}^0(Z, maE)_{ma\rho_- + b}.$$

The required isomorphism is then given by $\varphi\left(\sum_{b \geq 0} \frac{g_b}{f_a} y^b\right) = \sum_{b \geq 0} \frac{g_b}{f_a}$. ■

Step 2. *The natural $\mathbb{C}^*$-equivariant map $\Phi_{\rho_-, \rho_+} : Z \dashrightarrow X$ is birational.*

Proof. Using Step 1, it suffices to notice that, as in the proof of the previous step, the inclusion of graded $\mathbb{C}$-algebras $S \subset R(Z; E)$ induces isomorphisms $S_{(f)} \simeq R(Z; E)_{(f)}$ for every $f \in S_m^0$, $m > 0$. In particular the induced rational map $Z \dashrightarrow X$ sends the affine open set $D^+(f, Z) \subset Z$ isomorphically onto $D^+(f, X) \subset X$. Note that this in particular tells us that the open set $U_0 \subset X$ introduced above is the isomorphic image of the open subset $\bigcup_{m>0} \bigcup_{f \in S_m^0} D^+(f, Z) \subset Z$. ■

Since the algebra S is finitely generated by Lemma 2.3.23, there exists a positive integer d' such that $S^{(d')} = \bigoplus_{m \geq 0} S_{d'm}$ is generated in degree 1. Therefore $X \subset \mathbb{P}^N := \mathbb{P}(S_{d'})$, and let us denote by $L = \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$. Since X is normal, L is $\mathbb{C}^*$-linearizable. For the rest of the section we will consider the $\mathbb{C}^*$-action on the polarized pair (X, L).

Step 3. *The sink and the source of the $\mathbb{C}^*$-action on the pruning X of Z are isomorphic to $\mathcal{G}Z_h, \mathcal{G}Z_j$, respectively. The inner fixed components of X are isomorphic to the fixed point components of Z of weights equal to $a_{h+1}, \dots, a_j$. Furthermore, the criticality of the induced $\mathbb{C}^*$-action on (X, L) is equal to $j - h + 1$.*

Proof. Note first that we have a surjective homomorphism of $\mathbb{C}$-algebras:

$$S = \bigoplus_{m \geq 0} \bigoplus_{k = m\rho_-}^{m\rho_+} \mathrm{H}^0(Z, mE)_k \to \bigoplus_{m \geq 0} \mathrm{H}^0(Z, mE)_{m\rho_-},$$

which translates into an inclusion of varieties:

$$\mathcal{G}Z_h = \operatorname{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(Z, mE)_{m\rho_-} \hookrightarrow X.$$

By construction, $\mathcal{G}Z_h$ is fixed by the $\mathbb{C}^*$-action. Moreover, using [49, Remark 2.12], the induced $\mathbb{C}^*$-action on the projective space $\mathbb{P}^N = \mathbb{P}(S_{d'}) \supset X$ defined above as sink $\mathbb{P}(\mathrm{H}^0(Z, d'E)_{d'\rho_-})$. Then we may conclude that $\mathcal{G}Z(h, h+1) \subset X$ is the sink of X by noting that $\mathbb{P}(\mathrm{H}^0(Z, d'E)_{d'\rho_-}) \cap X = \mathcal{G}Z(h, h+1)$. In a similar way, one may prove that the source of X is isomorphic to $\operatorname{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(Z, mE)_{m\rho_+} \simeq \mathcal{G}Z_j$.

In order to compute the inner fixed point components of X, we note first that the complement of the extremal fixed point components of X is the open set $U_0 = \bigcup_{m>0} \bigcup_{f \in S_m^0} D^+(f, X)$, which is $\mathbb{C}^*$-equivariantly isomorphic to an open set $\bigcup_{m>0} \bigcup_{f \in S_m^0} D^+(f, Z) \subset Z$ (see Step 2), whose fixed point components are the fixed point components of Z of L-weight $\mu_L(Y) \in \{a_{h+1}, \dots, a_j\}$.

Finally, considering the embedding $X \subset \mathbb{P}^N$, the inner fixed point components of X are the irreducible components in the intersections $X \cap \mathbb{P}(\mathrm{H}^0(Z, d'E)_{d'a_i})$, $i \in \{h+1, \dots, j\}$. In particular, the criticality of the $\mathbb{C}^*$-action on the polarized pair (X, L) is $j - h + 1$. ■

Step 4. *The $\mathbb{C}^*$-action on (X, L) is of B-type.*

Proof. Using Theorem 2.2.30 and the arguments above one may show that, for every $i = h + 1, \dots, j - 1$, it holds $\mathcal{G}X_{i-h} \simeq \mathcal{G}Z_i$. Therefore since $\mathcal{G}Z_0$ and $\mathcal{G}Z_{r-1}$ are the sink and the source of the $\mathbb{C}^*$-action in X by Step 3, we conclude. ■

Using Steps 1, 2, 3 and 4 we conclude the proof of Theorem 2.3.27.

Corollary 2.3.28. *In the situation of Set-up 2.3.22, the indeterminacy locus of $\Phi_{\rho_-, \rho_+} \colon Z \dashrightarrow \mathcal{P}(Z)_{\rho_-}^{\rho_+}$ is*

$$\operatorname{Ind}(\Phi_{\rho_-, \rho_+}) = \bigcup_{\mu_L(Y) \leq a_h} Z^+(Y) \cup \bigcup_{\mu_L(Y) \geq a_{j+1}} Z^-(Y).$$

We now collect several results about pruning of varieties we will use in the forthcoming chapters.

Remark 2.3.29. Since one can always perform a pruning $\mathcal{P}(Z)_-^+$ along the extremal intervals, we can assume that a $\mathbb{C}^*$-action on a polarized pair (X, L) is of B-type.

Remark 2.3.30. In the case in which $\rho_\pm$ belong to the same open interval (a_i, a_{i+1}), then the resulting variety will be a $\mathbb{P}^1$-fibration over the geometric quotient $\mathcal{G}Z(i, i+1)$, whose fibers are the closures of the 1-dimensional orbits of the induced $\mathbb{C}^*$-action. The sink and the source of the action are two sections of the fibration.

Proposition 2.3.31. *In the situation of Set-up 2.3.22, suppose that* codim $\overline{Z^{\pm}(Y)} \geq 2$ *for every fixed point component* Y *with* $h < \mu_E(Y) < j$*. Then the* $\mathbb{C}^*$*-action on the pruning* X *is a bordism.*

Proof. On one hand, by Step 4, the $\mathbb{C}^*$-action on X is of B-type. On the other, using Steps 2 and 3 we conclude codim $\overline{X^{\pm}(Y)} \geq 2$, that is X is a bordism. ■

Corollary 2.3.32. *In the situation of Set-up 2.3.22, consider the birational map* $\psi_k \colon \mathcal{G}X_k \dashrightarrow \mathcal{G}X_{k+1}$*, with* $k = 0, \dots, r-1$*. Then the pruning* $\mathcal{P}(Z)^{\tau_+}_{\tau_-}$ *of* Z *with respect to* $\tau_- \in (a_k, a_{k+1}) \cap \mathbb{Q}$ *and* $\tau_+ \in (a_{k+1}, a_{k+2}) \cap \mathbb{Q}$ *is a geometric realization of* ψ_k*.*

More generally, every birational map $\psi_{k+w} \circ \ldots \circ \psi_k \colon \mathcal{G}Z_k \dashrightarrow \mathcal{G}Z_{k+w}$, for any $k, w \in \{0, \ldots r-1\}$ such that $k + w \leq r - 1$, admits a geometric realization given by the pruning $\mathcal{P}(Z)^{\tau_+}_{\tau_-}$, with $\tau_- \in (a_k, a_{k+1}), \tau_+ \in (a_{k+w}, a_{k+w+1})$.

Lemma 2.3.33. *In the situation of Set-up 2.3.22, suppose that the action is of B-type. Then the pruning with respect to the extremal intervals is isomorphic to* Z*.*

Proof. It suffices to show that $\operatorname{Ind}(\Phi_{h,j}) = \emptyset$. Indeed by Corollary 2.3.28 it holds $\operatorname{Ind}(\Phi_{-,+}) = Y_- \cup Y_+$. Since the action is of B-type, and thus $Y_{\pm} = \mathcal{G}Z_{\pm} \simeq \mathcal{G}X_{\pm}$, we conclude. ■

Lemma 2.3.34. *Let* (Z, E) *be a smooth polarized pair, with* $\rho_Z = 1$*. Consider an equalized* $\mathbb{C}^*$*-action on* (Z, E)*. Then the pruning along the extremal intervals* $\mathcal{P}(Z)^+_-$ *is a bordism if and only if* $\dim Y_{\pm} > 0$*.*

Proof. Let us prove that if $\dim Y_{\pm} > 0$, then $\mathcal{P}(Z)^+_-$ is a bordism, being the other implication trivial by definition of bordism. By [49, Lemma 2.8 (1)], $\nu^{\pm}(Y) \geq 2$ for every $Y \in \mathcal{Y}^{\circ}$. Since $\mathcal{P}(X)^+_-$ is of B-type by construction (see Step 4), we conclude. ■

Lemma 2.3.35. *[51, Remark 2.7] In the situation of Set-up 2.3.22, suppose that* Z *is smooth and that the* $\mathbb{C}^*$*-action is equalized at* $Y_{\pm}$*. Then the pruning with respect to the extremal intervals coincides with the blow-up of* Z *along* $Y_{\pm}$*.*

Lemma 2.3.36. *In the situation of Set-up 2.3.22, suppose that* Z *is smooth, the* $\mathbb{C}^*$*-action is equalized and a bordism. Then every pruning* $\mathcal{P}(Z)^{\rho_+}_{\rho_-}$ *is smooth. In particular every birational map* $\Phi_{\rho_-,\rho_+} \colon Z \to \mathcal{P}(Z)^{\rho_+}_{\rho_-}$ *is a small* $\mathbb{Q}$*-factorial modification.*

Proof. We argue as in the proof in the Step 1. Indeed, consider the open subsets $U_0, U_{\pm}$ defined in Step 1. The smoothness of U_0 follows since Z is smooth. On the other hand, $U_{\pm}$ are $\mathbb{C}$-principal bundles over $\mathcal{G}Z_{\pm}$, which are smooth by Lemma 2.2.31, hence the smoothness of $U_{\pm}$ follows. Since $X = U_- \cup U_0 \cup U_+$, we conclude. ■

2.4 Mori dream spaces and Mori dream regions

In this section we introduce and discuss the notions of *Mori dream spaces* and *Mori dream regions* (see respectively Definitions 2.4.1, 2.4.12), introduced by Hu and Keel in [25, Definitions 1.10, 2.12]. We then explain the relation between Mori dream spaces and $\mathbb{C}^*$-actions, which has been investigated in [48, Section 4]; the study of the relation between Mori dream regions and $\mathbb{C}^*$-actions will be the content of Chapter 5.

Definition 2.4.1. Let X be a normal, $\mathbb{Q}$-factorial projective variety. We say that X is a *Mori dream space*, MDS for short, if the following properties hold:

(1) The Picard group $\mathrm{Pic}(X)$ is finitely generated;

(2) The nef cone $\mathrm{Nef}(X)$ is generated by finitely many semiample divisors;

(3) There exists a finite number k of small $\mathbb{Q}$-factorial modifications $f_i\colon X \to X_i$, for $i = 0, \dots, k$, such that every X_i satisfies (2) and

$$\mathrm{Mov}(X) = \bigcup_{i=0}^{k} f_i^*(\mathrm{Nef}(X_i)).$$

Notice that the Picard group is finitely generated if and only if $h^1(X, \mathcal{O}_X) = 0$, or equivalently $\mathrm{Pic}(X)_{\mathbb{Q}} \simeq \mathrm{N}^1(X)_{\mathbb{Q}}$.

Let us remark some immediate consequences:

Remark 2.4.2. (1) If X is a Mori dream space, then the nef cone $\mathrm{Nef}(X)$ is rational polyhedral;

(2) In a Mori dream space every Cartier divisor is nef if and only if it is semiample;

(3) If X is a Mori dream space, then every SQM X_i is a Mori dream space as well.

The nef cone and the movable cone are not the only cones of divisors who have a nice geometric behaviour, as explained in the following:

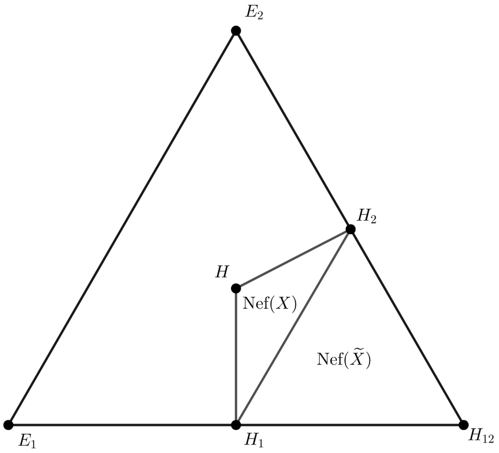
Proposition 2.4.3. *[25, Proposition 1.11 (2)] Let X be a Mori dream space and let D be a prime divisor in X which is not movable. Then there exists a SQM $f_i\colon X \dashrightarrow X_i$ such that the transform D_i in X_i of D is the exceptional divisor of an elementary divisorial contraction. Moreover, let $D_1, \dots, D_s$ be the exceptional divisors of all elementary divisorial rational contractions of X. Then*

$$\mathrm{Eff}(X) = \mathrm{Mov}(X) + \mathbb{R}_+ D_1 + \ldots + \mathbb{R}_+ D_s.$$

In particular $\mathrm{Eff}(X)$ *is a rational polyhedral cone in* $\mathrm{N}^1(X)$*.*

Let us present an example of MDS which will be useful in the forthcoming discussion regarding geometric realizations of birational map among toric varieties (see Chapter 6):

Example 2.4.4. [1, §5.5] Let $\beta\colon X \to \mathbb{P}^3$ the blow-up of $\mathbb{P}^3$ along the points e_1, e_2, with exceptional divisors E_1, E_2. Then a 2-dimensional slice of the effective cone of X can be represented by the following picture:



where we denote by H (resp. by H_1, H_2, H_{12}) the transform of a general hyperplane in $\mathbb{P}^3$ (resp. of a general hyperplane containing x, containing e_2, containing e_1 and e_2), and by $\widetilde{X}$ we denote the variety obtained by the flip of the strict transform of the line passing trough e_1 and e_2.

Mori dream spaces enjoy another key property, namely they can be characterized as those varieties having a finitely generated *Cox ring*. The latter has been introduced by Cox in [15] in the case of toric varieties, and then generalized by Hu and Keel (see [25, Definition 2.6]) for normal $\mathbb{Q}$-factorial projective varieties with finitely generated Picard group. Before stating Theorem 2.4.10, which links Mori dream spaces and Cox rings, we introduce the necessary background regarding multisection rings:

Definition 2.4.5. Let X be a normal projective variety, and let $\mathcal{M} \subset \mathrm{CDiv}(X)_{\mathbb{Q}}$ be a finitely generated monoid. We define the *divisorial $\mathcal{M}$-graded ring* as

$$R(X;\mathcal{M}) \coloneqq \bigoplus_{D\in\mathcal{M}} \mathrm{H}^0(X, \mathcal{O}_X(D)).$$

Definition 2.4.6. Let X be a normal projective variety, and let $\mathcal{C} \subset \mathrm{CDiv}(X)_{\mathbb{Q}}$ be a rational polyhedral cone. Then we define $R(X;\mathcal{C}) \coloneqq R(X;\mathcal{M})$, where $\mathcal{M} = \mathcal{C} \cap \mathrm{CDiv}(X)$ is a finitely generated monoid by Gordan's Lemma.

Definition 2.4.7. Let X be a normal projective variety, and let $D_1 \ldots, D_k \in \mathrm{CDiv}(X)_{\mathbb{Q}}$. The *multisection ring* is

$$R(X; \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_k)) \coloneqq \bigoplus_{(m_1,\ldots,m_k)\in\mathbb{N}^k} \mathrm{H}^0(X, \mathcal{O}_X(m_1D_1 + \ldots + m_kD_k)).$$

Remark 2.4.8. With the notation of the previous Definition, notice that $R(X; \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_k))$ is in principle a complex vector space. We can endow $R(X; \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_k))$ with a ring structure by considering the multiplication of sections. Such operation however needs to be defined by fixing the Cartier divisors $D_1, \ldots, D_k$, and not just their linear equivalence classes (cf. [25, Remark p.341]). More precisely, we identify

$$\mathrm{H}^0(X, \mathcal{O}_X(D_1)^{\otimes m_1}\otimes\ldots\otimes\mathcal{O}_X(D_k)^{\otimes m_k})) = \{f \in \mathbb{C}(X) \mid \mathrm{div}(f)+m_1D_1+\ldots+m_kD_k \geq 0\} \subset \mathbb{C}(X),$$

and then we consider the multiplication of sections induced in $\mathbb{C}(X)$.

Moreover let $\mathcal{C} = \langle D_1, \ldots, D_k\rangle \subset \mathrm{CDiv}(X)_{\mathbb{Q}}$ be the polyhedral cone generated by $D_1, \ldots, D_k$. Notice that $R(X;\mathcal{C}) \simeq R(X; \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_k))$ as $\mathbb{C}$-algebras, but not as $\mathbb{C}$-graded algebras since the grading is different in general.

Definition 2.4.9. [2, Construction 1.4.11] Let X be a normal projective variety with $\mathrm{Cl}(X)$ finitely generated and free. Let K be a subgroup of $\mathrm{Div}(X)$ whose image under the natural projection map $\mathrm{Div}(X) \to \mathrm{Cl}(X)$ generates $\mathrm{Cl}(X)$. The *Cox ring* of X is defined as

$$\mathrm{Cox}(X) \coloneqq \bigoplus_{D\in K} \mathrm{H}^0(X, \mathcal{O}_X(D)) = \bigoplus_{D\in K} \{f \in \mathbb{C}(X) \mid \mathrm{div}(f) + D \geq 0\} \subset \mathbb{C}(X).$$

We remark that, while the above Definition depends the choice of a suitable subgroup K (cf. Remark 2.4.8), the finite generation of $\mathrm{Cox}(X)$ is independent (cf. [2, Lemma 1.4.3.1]).

Theorem 2.4.10. *[25, Proposition 2.9] Let X be a normal $\mathbb{Q}$-factorial projective variety with finitely generated Picard group. Then X is a Mori dream space if and only if* $\mathrm{Cox}(X)$ *is finitely generated. In particular, X is a GIT quotient of* $\mathrm{Spec}(\mathrm{Cox}(X))$ *by the action of* $(\mathbb{C}^*)^{\rho_X}$.

Example 2.4.11. Toric varieties (see [15]), log Fano varieties (see [7, Corollary 3.12]), and the blow-up $\mathrm{Bl}_y \mathbb{P}^n$ for any $n \geq 3$ of $\mathbb{P}^n$ in $y \leq n+3$ points in general position (see [14, Theorem 1.3]) are examples of Mori dream spaces. Moreover, the Cox ring of a smooth projective variety with finitely generated Picard group is a polynomial ring if and only if it is a toric variety (see [25, Corollary 2.10]).

Since the finite generation of the Cox ring is, in some sense, a global property of the variety, it is natural to consider a scenario where a rational polyhedral cone contained in $\mathrm{CDiv}(X)_{\mathbb{Q}}$ is associated to a finitely generated multisection ring; this is precisely the idea behind the notion of Mori dream region:

Definition 2.4.12. Let X be a normal projective variety, and let $\mathcal{C} = \langle D_1, \ldots, D_k \rangle$ be a rational polyhedral cone in $\mathrm{CDiv}(X)_{\mathbb{Q}}$, with D_i effective for every $i = 1, \ldots, k$. The cone $\mathcal{C}$ is a *Mori dream region*, MDR for short, if the multisection ring $R(X; \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_k))$ is a finitely generated $\mathbb{C}$-algebra.

The notion of Mori dream region was introduced in [25, Definition 2.12] as a generalization of the notion of Mori dream space. Over the years, different authors have introduced different notions of Mori dream regions and studied their properties: we refer to [52, §9.2], [31, Theorem 4.2] and [39, §5] for a complete picture.

Lemma 2.4.13. *[13, Lemma 2.7] Let X be a Mori dream space. For any choice of Cartier divisors $D_1, \ldots, D_k$ on X, the rational polyhedral cone $\mathcal{C} = \langle D_1, \ldots, D_k \rangle$ is a Mori dream region.*

Let us recall also the following:

Theorem 2.4.14. *[12, Corollary 2.26] Let X be a normal projective variety. Let $D_1, \ldots, D_k \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ and let $p_1, \ldots, p_k$ be positive rational numbers. Then $R(X; \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_k))$ is finitely generated if and only if $R(X; \mathcal{O}_X(p_1 D_1), \ldots, \mathcal{O}_X(p_k D_k))$ is finitely generated.*

Example 2.4.15. Let X be a normal projective variety, and let $D_1, \ldots, D_k$ be ample Cartier divisors on X. Then $\mathcal{C} = \langle D_1, \ldots, D_k \rangle$ is a Mori dream region. Indeed set $\mathcal{E} := \mathcal{O}_X(D_1) \oplus \ldots \oplus \mathcal{O}_X(D_k)$, and consider the projective bundle $\pi \colon \mathbb{P}(\mathcal{E}) \to X$. For any $m \geq 0$, by [42, Lemma 2.3.2] it holds

$$\mathrm{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)) = \bigoplus_{a_1 + \ldots + a_k = m} \mathrm{H}^0(X, \mathcal{O}_X(a_1 D_1 + \ldots + a_k D_k)).$$

Since for every $i = 1, \ldots, k$ the Cartier divisor D_i is ample, again by [42, Lemma 2.3.2] we have that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample, therefore $R(\mathbb{P}(\mathcal{E}); \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq R(X; \mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_k))$ is a finitely generated $\mathbb{C}$-algebra, hence we conclude.

Example 2.4.16. While every Mori dream space contains infinitely many Mori dream regions by Lemma 2.4.13, it is easy to construct examples of varieties which are not MDS but contain a MDR. Take for instance $\phi \colon X \to \mathbb{P}^n$ the blow-up of $\mathbb{P}^n$ along $n+4$ points in general position. This map factorizes through the blow-up $\pi \colon Y \to \mathbb{P}^n$ along $n+3$ points in general position. The Picard group of X is generated by $\phi^* H$ and $E_1, \ldots, E_{n+4}$, where by E_i we denote the exceptional divisor associated to the blow-up of the point p_i, $i = 1, \ldots, n+4$. The blow-up X is not a Mori dream space (see [14, Theorem 1.3]), but nevertheless it contains several Mori dream regions: indeed by observing that, for any $a_0, \ldots, a_{n+3} \geq 0$, it holds

$$\mathrm{H}^0(X, \mathcal{O}_X(a_0 \phi^* H + a_1 E_1 + \ldots + a_{n+3} E_{n+3})) \simeq \mathrm{H}^0(Y, \mathcal{O}_Y(a_0 \pi^* H + a_1 E_1 + \ldots + a_{n+3} E_{n+3}))$$

and since Y is a Mori dream space, we conclude.

2.4.1 Mori dream spaces and $\mathbb{C}^*$-actions

Set-up 2.4.17. Let (X, L) be a smooth polarized pair, with $\rho_X = 1$. Assume that there exists an equalized and normalized $\mathbb{C}^*$-action on (X, L) with bandwidth δ and criticality r. We assume X is not the projective space with the $\mathbb{C}^*$-action which fixes a point and an hyperplane. Let $\beta\colon X^\flat \to X$ be the blow-up of X along $Y_\pm$, and denote by $Y_\pm^\flat$ the exceptional divisors. For any $0 \leq a \leq b \leq \delta$, set

$$L(a, b) := \beta^* L - aY_-^\flat - (\delta - b)Y_+^\flat.$$

By Remark 2.3.35, $X^\beta \simeq \mathcal{P}(X)_-^+$, and therefore $\mathcal{G}X_\pm = Y_\pm$.

Theorem 2.4.18. *[48, Theorem 1.1] In the situation of Set-up 2.4.17, the blow-up $X^\flat$ is a Mori dream space.*

The strategy used by the authors of [48] to prove the above Theorem consists of giving an explicit description of the movable cone; we will retrace here the major steps.

Proposition 2.4.19. *[48, Proposition 4.7, Remark 4.8] In the situation of Set-up 2.4.17, the movable cone $\mathrm{Mov}(X^\flat)$ is simplicial. Moreover:*

- *If $\dim Y_\pm > 0$, then*

$$\mathrm{Mov}(X^\flat) = \overline{\mathrm{Mov}(X^\flat)} = \langle L(0, \delta), L(0, 0), L(\delta, \delta)\rangle;$$

- *If $\dim(Y_0) = 0$, $\dim(Y_r) > 0$, then*

$$\mathrm{Mov}(X^\flat) = \langle L(0, a_1), L(a_1, a_1), L(\delta, \delta), L(0, \delta)\rangle;$$

- *If $\dim(Y_0) > 0$, $\dim(Y_r) = 0$, then*

$$\mathrm{Mov}(X^\flat) = \langle L(0, 0), L(a_{r-1}, a_{r-1}), L(a_{r-1}, \delta), L(0, \delta)\rangle;$$

- *If $\dim(Y_0) = 0$, $\dim(Y_r) = 0$, then*

$$\mathrm{Mov}(X^\flat) = \langle L(0, a_1), L(a_1, a_1), L(a_{r-1}, a_{r-1}), L(a_{r-1}, \delta), L(0, \delta)\rangle.$$

Proposition 2.4.20. *[48, Corollary 4.9] For every pair of indices (i, j), with $0 \leq i \leq j < r$, set:*

$$N_{i,j} := \{mL(a, b) \mid m \geq 0, 0 \leq a \leq b \leq \delta, a \in (a_i, a_{i+1}), b \in (a_j, a_{j+1})\}.$$

If $Y_\pm$ are not points, the for every (i, j), with $i \leq j$, the chambers $N_{i,j}$ are contained in $\mathrm{Mov}(X^\flat)$.

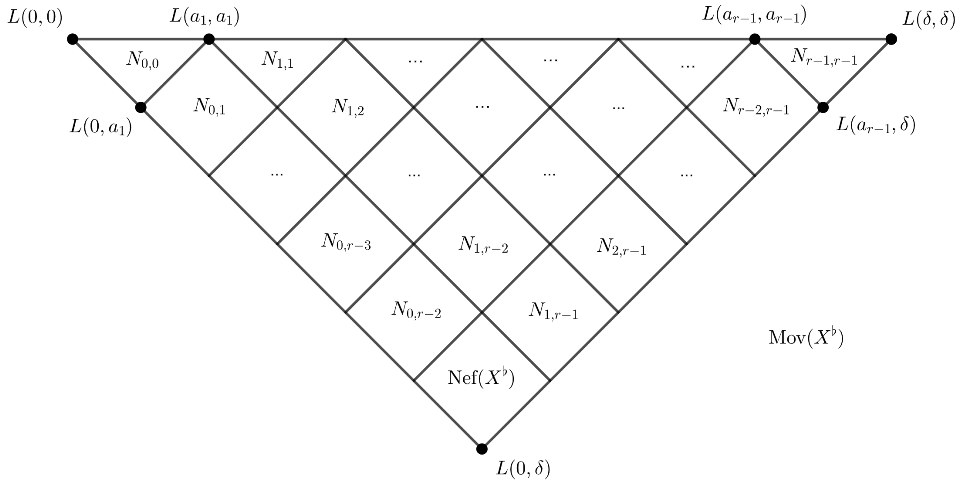
Lemma 2.4.21. *[48, Corollary 4.9] If $\dim Y_- = 0$ (resp. $\dim Y_+ = 0$), then for every (i, j), with $i \leq j$ and $(i, j) \neq (0, 0)$ (resp. $(i, j) \neq (r - 1, r - 1)$), the chambers $N_{i,j}$ are contained in $\mathrm{Mov}(X^\flat)$.*

Theorem 2.4.22. *[48, Proposition 4.11] In the situation of Set-up 2.4.17, if $Y_\pm$ are not points, then for every pair of indices (i, j), with $0 \leq i \leq j < r$, it holds*

$$\mathrm{Mov}(X^\flat) = \bigcup_{(i,j)} N_{i,j}.$$

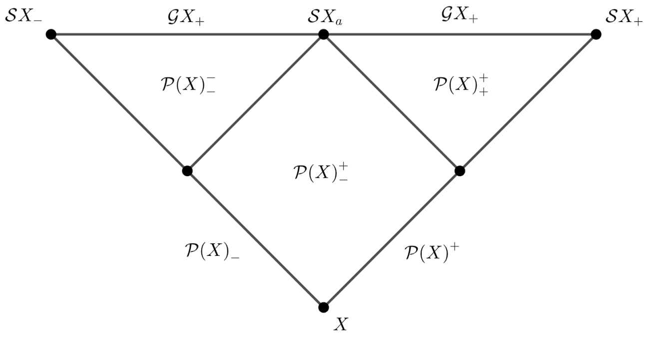
Moreover, $\overline{N_{i,j}} = \phi_{i,j}^* \mathrm{Nef}(\mathcal{P}(X)_{\rho_-}^{\rho_+})$, with $\mathcal{P}(X)_{\rho_-}^{\rho_+}$ a pruning of X and $\rho_- \in (a_i, a_{i+1}) \cap \mathbb{Q}, \rho_+ \in (a_j, a_{j+1}) \cap \mathbb{Q}$.

We may represent an affine slice of $\mathrm{Mov}(X^\flat)$ by means of the following picture:



We remark that a similar description of the movable cone of $X^\flat$ has been obtained in [46, §3] without any assumption on the Picard number of X.

Example 2.4.23. Let (X,L) be a smooth polarized pair, with $\rho_X=1$. Consider a normalized and equalized $\mathbb{C}^*$-action on (X,L) of bandwidth and criticality equal to 2, and let the inner component Y of L-weight a. We may represent an affine slice of $\mathrm{Mov}(\mathcal{P}(X)_-^+)$ by means of the following picture, where we abuse notation by not writing $\mathrm{Nef}(\cdot)$:



2.5 Examples

2.5.1 Rational homogeneous varieties

In this section we briefly recall the construction of *rational homogeneous varieties* (RH, for short), that is smooth projective varieties admitting a transitive action of a semisimple algebraic group. The motivation behind lies on the fact that RH varieties represent a primary source of example where to study $\mathbb{C}^*$-actions (see [5, II, Chapter 3]), and indeed some of them are geometric realization of well-known birational maps (see Example 2.3.20, Proposition 4.1.17, Theorem 4.2.8).

To give a precise introduction to the theory of RH-varieties is beyond the scope of this thesis: we refer the interested reader to [21], while we refer to [27, 26] for the notions we will use about representation theory of semisimple groups.

2.5.1.1 Dynkin diagrams

Let G be a semisimple algebraic group and $H \subset G$ a Cartan subgroup, with associated Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Consider the *Cartan decomposition* of $\mathfrak{g}$ obtained by the adjoint action of $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha := \{g \in \mathfrak{g} \,|\, [h, g] = \alpha(h) g, \text{ for all } h \in \mathfrak{h}\},$$

We define the *root system* of G as $\Phi := \{\alpha \in \mathfrak{h}^\vee \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$; the elements $\alpha \in \Phi$ are called *roots* of G. We considered a root system in the sense of [27, §9.2]. Let E be the n-dimensional real vector space generated by Φ, and endow it with the inner product defined, for $\alpha, \beta \in \Phi$, as

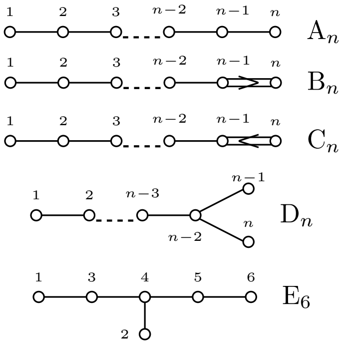
$$\langle \alpha, \beta \rangle := 2\frac{\kappa(\alpha, \beta)}{\kappa(\beta, \beta)},$$

where $\kappa(\cdot, \cdot)$ is the *Killing form* on $\mathfrak{g}$. A *basis of simple roots* $\Delta = \{\alpha_1, \dots, \alpha_n\}$ is a subset of linearly independent elements in Φ such that $\Phi = \Phi^+ \cup \Phi^-$, with $\Phi^+ := \mathbb{Z}_{\geq 0} \cap \Phi$ and $\Phi^- := -\Phi^+$. Given a basis of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$, we define the *Cartan matrix* as $M := (\langle \alpha_i, \alpha_j \rangle)_{i,j}$. The resulting matrix will be such that:

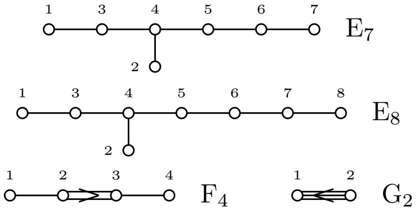
- $\langle \alpha_i, \alpha_i \rangle = 2$ for all i,
- $\langle \alpha_i, \alpha_j \rangle = 0$ if and only if $\langle \alpha_j, \alpha_i \rangle = 0$, and
- if $\langle \alpha_i, \alpha_j \rangle \neq 0$, $i \neq j$, then $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}^-$ and $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle = 1, 2$ or 3.

A *Dynkin diagram* $\mathcal{D}$ is a graph whose set of nodes correspond to the set of indices $D := \{1, \dots, n\}$ and where the nodes i and j are joined by $\langle \alpha_j, \alpha_i \rangle \langle \alpha_i, \alpha_j \rangle$ edges. When two nodes i and j are joined by a double or triple edge, we add to it an arrow, pointing to i if $\langle \alpha_i, \alpha_j \rangle > \langle \alpha_j, \alpha_i \rangle$.

Theorem 2.5.1. *There is a one to one correspondence between isomorphism classes of semisimple Lie algebras and Dynkin diagrams of reduced root systems. Moreover, every reduced root system is a disjoint union of mutually orthogonal irreducible root subsystems, each of them corresponding to one of the connected finite Dynkin diagrams* A_n, B_n, C_n, D_n *(*$n \in \mathbb{N}$*),* E_6, E_7, E_8, F_4, G_2*:*



(2.2)



For the connected Dynkin diagrams we will use the numbering proposed by Bourbaki (cite [8, Planche I–IX]). The connected components of the Dynkin diagram $\mathcal{D}$ determine the simple Lie groups that are factors of the semisimple Lie group G, each of them corresponding to one of the Dynkin diagrams above; in particular the well-known algebraic groups SL_{n+1}, SO_{2n+1}, Sp_{2n} and SO_{2n} correspond to the diagrams A_n, B_n, C_n and D_n, respectively.

2.5.1.2 Construction of rational homogeneous varieties

Definition 2.5.2. A *rational homogeneous* variety (shortly, RH-variety) is a smooth projective variety endowed with a transitive action of a connected algebraic group, that is obtained as a quotient of a connected algebraic group.

By [26, §21.3, Corollary B] every RH-variety is a quotient G/P, where G is a semisimple group and P is a parabolic subgroup of G. The key feature is that parabolic subgroups are described by a set of simple roots of G.

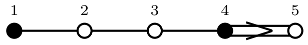
Proposition 2.5.3. *Given a Dynkin diagram $\mathcal{D}$, consider a subset $I \subset D$. Let $\Phi^+(D \setminus I)$ be the subset of Φ^+ generated by the simple roots of $D \setminus I$. Then the subspace*

$$\mathfrak{p}(D \setminus I) := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha \in \Phi^+(I)} \mathfrak{g}_{\alpha} \tag{2.3}$$

is a parabolic subalgebra of $\mathfrak{g}$, determining a parabolic subgroup $P(D \setminus I) \subset G$.

Notation 2.5.4. The RH-variety associated to the quotient of $P(D \setminus I)$ is denoted by $\mathcal{D}(I) := G/P(D \setminus I)$.

Graphically it corresponds to marking the Dynkin diagram $\mathcal{D}$ of G on the indices of the set I. For example, $B_5(1,4)$ can be represented as:



Let us now describe the RH-varieties obtained by associated to the Dynkin diagram of A_n, B_n, C_n, D_n.

Example 2.5.5 (A_n-diagram)**.** The RH-variety $A_n(1)$ obtained by marking the first node is the n-dimensional projective space. By duality, $A_n(n) = (\mathbb{P}^n)^\vee$. More generally, $A_n(k)$ represents the Grassmannian of $(k-1)$-linear subspaces of $\mathbb{P}^n$, and $A_n(k_1, \dots, k_s)$ is the variety of flag of linear subspaces of $\mathbb{P}_n$ with the condition that $\mathbb{P}^{k_1-1} \subset \dots \mathbb{P}^{k_s-1} \subset \mathbb{P}^n$. For instance, the RH variety $A_n(1,n) = \{(p,H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid p \in H\}$, which is associated to the diagram

is isomorphic to $\mathbb{P}(T_{\mathbb{P}^n})$.

Example 2.5.6 (B_n-diagram)**.** The RH-variety $B_n(1)$ is the smooth $(2n{-}1)$-dimensional quadric hypersurface Q^{2n-1} in $\mathbb{P}(V)=\mathbb{P}^{2n}$. RH-varieties of the form $B_n(k)$, for $2\leq k\leq n$, parametrize linear subspaces of $B_n(1)$ (alternatively, they parametrize linear subspaces of $\mathbb{P}(V)$ isotropic with respect to a maximal rank symmetric form on V).

Example 2.5.7 (C_n-diagram)**.** The RH-variety $C_n(k)$, for $k=1,\dots,n$, is called *isotropic Grassmannian* and parametrizes linear subspaces of $\mathbb{P}(V)=\mathbb{P}^{2n-1}$ which are isotropic with respect to a maximal rank skew-symmetric form on V. It holds that $C_n(1)=\mathbb{P}^{2n-1}$.

Example 2.5.8 (D_n-diagram)**.** The RH-variety $D_n(1)$ is the smooth quadric hypersurface of dimension $2n-2$ in $\mathbb{P}(V)=\mathbb{P}^{2n-1}$, and $D_n(k)$, for $2\leq k\leq n-3$, parametrizes linear subspaces of $D_n(1)$. The peculiar form of the diagram reflects geometrically the existence of two disjoint irreducible families of $(n-1)$-dimensional linear spaces (that is, $D_n(n-1)$ and $D_n(n)$), while the family of $(n-2)$-dimensional linear subspaces of $D_n(1)$ is denoted by the RH-variety $D_n(n-1,n)$.

The rational homogeneous varieties obtained by marking nodes on the exceptional cases $E_6, E_7, E_8 F_4, G_2$ still admit a geometric description: we refer to [40] for details.

We conclude this section by recalling an useful result on the geometry of rational homogeneous varieties:

Theorem 2.5.9. *[36, Theorem V.1.4] Let $X=\mathcal{D}(I)$ be a RH-variety obtained by marking a set $I\subset D$. Then X is Fano, and its Picard number is equal to the cardinality of I. Moreover, consider a subset $J\subset I\subset D$. Then the morphisms $\pi_{I,J}\colon\mathcal{D}(I)\to\mathcal{D}(J)$ are proper, surjective, and the fibers are RH-varieties of type $\mathcal{D}$ obtained by removing the nodes of J and marking the nodes of $I\setminus J$. Every contraction of $\mathcal{D}(I)$ is of this form.*

2.5.2 $\mathbb{C}^*$-actions of bandwidth 1

In this section we illustrate one of the simplest examples of $\mathbb{C}^*$-actions on polarized pairs, namely those with bandwidth 1: they are called *drums*. We refer to [49, Section 4] for details.

Set-up 2.5.10. Let Λ be a normal projective variety with $\rho_\Lambda=2$, admitting two elementary contractions

$$\begin{tikzcd} & \Lambda \arrow[dl, "p_-"'] \arrow[dr, "p_+"] & \\ \Lambda_- & & \Lambda_+ \end{tikzcd}$$

Let $L_\pm$ be very ample line bundles respectively on $\Lambda_\pm$, and set $\mathcal{L}_\pm=p_\pm^*(L_\pm)$. Consider $\pi\colon\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)\to\Lambda$ the projective bundle over Λ.

Lemma 2.5.11. *In the situation of Set-up 2.5.10, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)}(1)$ is globally generated, and there exists a contraction, birational onto the image,*

$$\phi=\phi_{\mathcal{O}_{\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)}(1)}\colon\mathbb{P}\left(\mathcal{L}_-\oplus\mathcal{L}_+\right)\longrightarrow\mathbb{P}\left(\mathrm{H}^0\left(\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+),\mathcal{O}_{\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)}(1)\right)\right).$$

Proof. The global generation of $\mathcal{O}_{\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)}(1)$ is immediate. Let us just notice that, by the projection formula, we have an isomorphism

$$\mathrm{H}^0\left(\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+),\mathcal{O}_{\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)}(1)\right)=\mathrm{H}^0(\Lambda_-,L_-)\oplus\mathrm{H}^0(\Lambda_+,L_+). \tag{2.4}$$

Let us prove that the morphism

$$\phi\colon\mathbb{P}(\mathcal{L}_-\oplus\mathcal{L}_+)\to\mathbb{P}(\mathrm{H}^0(\Lambda_-,L_-)\oplus\mathrm{H}^0(\Lambda_+,L_+)),$$

associated to evaluation of sections is a contraction, birational onto the image. Consider the sections $\sigma_\pm\colon \Lambda \to \mathbb{P}\left(\mathcal{L}_- \oplus \mathcal{L}_+\right)$ associated to the quotients $\mathcal{L}_- \oplus \mathcal{L}_+ \to \mathcal{L}_\pm$. The compositions $\phi \circ \sigma_\pm$ coincide with the bundle maps $p_\pm$, in particular they have connected fibers. On the other hand the restriction of ϕ to $\mathbb{P}\left(\mathcal{L}_- \oplus \mathcal{L}_+\right) \setminus (\sigma_-(\Lambda) \cup \sigma_+(\Lambda))$ is an isomorphism onto the image. ■

Definition 2.5.12. The image $X := \phi\left(\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)\right)$ is called the *drum constructed upon the triple* $(\Lambda, \mathcal{L}_-, \mathcal{L}_+)$.

Notice that X comes with a natural ample line bundle L, which is the restriction of the hyperplane class in $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$, such that $\phi^*L = \mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(1)$. We may summarize this construction by means of the following diagram:

$$\begin{tikzcd}
 & \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \arrow[r, "\phi"] \arrow[d, "\pi"] & X \\
L_- \arrow[d] & \Lambda \arrow[dl, "p_-"'] \arrow[dr, "p_+"] & L_+ \arrow[d] \\
\Lambda_- & & \Lambda_+
\end{tikzcd}$$

In general, a drum can be quite singular. However, we may characterize smooth drums thanks to the following:

Theorem 2.5.13. *[49, Lemma 4.4] In the situation of Set-up 2.5.10, a drum X is smooth if and only if the following conditions are satisfied:*

1. $\mathrm{Nef}(\Lambda) = \langle \mathcal{L}_-, \mathcal{L}_+ \rangle$*;*
2. $p_\pm\colon \Lambda \to \Lambda_\pm$ *has a projective bundle structure;*
3. $\deg(\mathcal{L}_\mp|_{F_\pm}) = 1$*, where $F_\pm$ denotes a fiber of $p_\pm$.*

Lemma 2.5.14. *[49, Remark 4.2] In the situation of Set-up 2.5.10, there exists an equalized $\mathbb{C}^*$-action on $(\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+), \mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(1))$ of bandwidth 1, with sink $s_-(\Lambda)$ and source $s_+(\Lambda)$. Moreover the contraction $\phi\colon \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \to X$ is $\mathbb{C}^*$-equivariant, and the induced equalized $\mathbb{C}^*$-action on X has sink Λ_- and source Λ_+.*

Example 2.5.15. [49, Example 4.7] We study the drum associated to $A_m(1) \times A_l(1)$. Consider the following

$$\begin{tikzcd}
 & \mathbb{P}\left(\mathcal{O}_{A_m(1)\times A_l(1)}(1,0) \oplus \mathcal{O}_{A_m(1)\times A_l(1)}(0,1)\right) \arrow[r, "\phi"] \arrow[d, "\pi"] & X \\
\mathcal{O}_{A_m(1)}(1) \arrow[d] & A_m(1) \times A_l(1) \arrow[dl, "p_-"'] \arrow[dr, "p_+"] & \mathcal{O}_{A_l(1)}(1) \arrow[d] \\
A_m(1) & & A_l(1)
\end{tikzcd}$$

It holds that $X \subset \mathbb{P}^{m+l+1}$; since ϕ is surjective, by dimension counting we get that $A_{m+l+1}(1)$ is the drum constructed upon

$$\left(A_m(1) \times A_l(1), \mathcal{O}_{A_m(1)\times A_l(1)}(1,0), \mathcal{O}_{A_m(1)\times A_l(1)}(0,1)\right).$$

Indeed, consider the $\mathbb{C}^*$-action on $(A_{m+l+1}(1), \mathcal{O}_{A_{m+l+1}(1)}(1)$ given by

$$t \cdot [x_0 : \ldots : x_{m+l+1}] = [tx_0 : \ldots : tx_l : x_{l+1} : \ldots : x_{m+l+1}] .$$

This action has bandwidth 1 and the sink and the source are respectively

$$A_m(1) \simeq \{x_{l+1} = \ldots = x_{m+l+1} = 0\} , \qquad A_l(1) \simeq \{x_0 = \ldots = x_l = 0\} .$$

Theorem 2.5.16. *[49, Theorem 4.8] Let X be a smooth projective variety with $\rho_X = 1$ different from the projective space and let L be an ample line bundle on X. Then (X, L) admits a $\mathbb{C}^*$-action of bandwidth 1 if and only if X is a smooth drum.*

Example 2.5.17. [53, Proposition 1.8] Consider the diagram

$$\begin{tikzcd} & A_n(1,n) \arrow[dl] \arrow[dr] & \\ A_n(1) & & A_n(n) \end{tikzcd}$$

The drum associated is is the RH variety $D_{n+1}(1) = Q^{2n} \subset \mathbb{P}^{2n+1}$, endowed with the $\mathbb{C}^*$-action defined as follows:

$$t \cdot [x_0 : \ldots : x_{2n+1}] = [tx_0 : \ldots : tx_n : x_{n+1} : \ldots : x_{2n+1}] .$$

Notice indeed that the sink and the source of the $\mathbb{C}^*$-action on $D_{n+1}(1)$ are respectively

$$A_n(1) \simeq \{x_{n+1} = \ldots = x_{2n+1} = 0\} , \qquad A_n(n) \simeq \{x_0 = \ldots = x_n = 0\} \simeq (\mathbb{P}^n)^\vee .$$

To the best of our knowledge, the only examples of smooth drums are constructed upon a smooth projective variety Λ satisfying the hypothesis of Set-up 2.5.10 such that $\Lambda_\pm$ are RH. Moreover, even if $\Lambda_\pm$ are RH the variety Λ may not be RH, as proven in [32, §2]. However, if Λ is RH, then the resulting drums are precisely the *horospherical varieties* classified by Pasquier in [53, Theorem 0.1].

2.5.3 Test configurations

In this section we show, using the construction of a test configuration, that a variety X endowed with a $\mathbb{C}^*$-action is birational to a fibration in (possibly weighted) projective spaces (see Proposition 2.5.21).

Let X be a normal projective variety of dimension n, endowed with a non-trivial and faithful $\mathbb{C}^*$-action. Consider a co-character $a \colon \mathbb{C}^* \to \mathbb{C}^*$. Let $\mathbb{P}^1 = U_0 \cup U_\infty$, where $U_0 = \{(1 : v) \mid v \in \mathbb{C}\}$ and $U_\infty = \{(u : 1) \mid u \in \mathbb{C}\}$; set $0 = (0 : 1)$, $\infty = (1 : 0)$.

Define $\mathcal{X} \coloneqq (U_0 \times X) \sqcup (U_\infty \times X)/\sim$, glued with the following transition function

$$\begin{aligned} U_{0,\infty} \times X &\to U_{\infty,0} \times X \\ ((1 : v), x) &\mapsto ((v^{-1} : 1), a(v^{-1})x). \end{aligned}$$

We obtain that $\mathcal{X}$ is a normal projective variety with a fibration $f \colon \mathcal{X} \to \mathbb{P}^1$ such that, for every $p \in \mathbb{P}^1$, $\mathcal{X}_p \coloneqq f^{-1}(p) \simeq X$.

Define a $\mathbb{C}^*$-action on $U_0 \times X$ as follows: $t \cdot ((1 : v), x) = ((1 : t^{-1}v), x)$. Using the transition maps above, the $\mathbb{C}^*$-action on $U_\infty \times X$ becomes

$$t((u : 1), x) = t((1 : u^{-1}), a(u^{-1})x) = ((1 : t^{-1}u^{-1}), a(u^{-1})x) = ((tu : 1), a(t)x).$$

We can thus extend the $\mathbb{C}^*$-action over 0 as $t((0 : 1), x) = ((0 : 1), a(t)x)$.

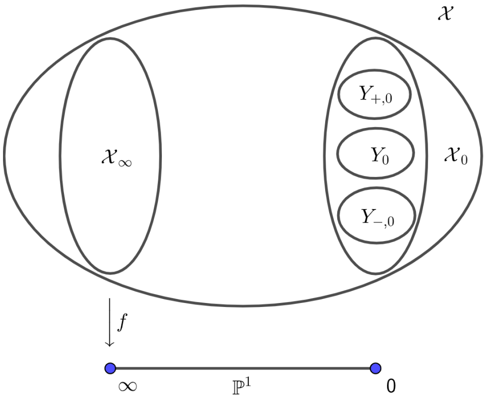
Lemma 2.5.18. *The $\mathbb{C}^*$-action on $\mathcal{X}$ makes the morphism $\mathcal{X} \to \mathbb{P}^1$ equivariant. In particular, $\mathcal{X}_0$ and $\mathcal{X}_\infty$ are $\mathbb{C}^*$-invariant.*

Remark 2.5.19. The variety $\mathcal{X}$ is an example of product test configuration, in the sense of [19, Definition 2.1.1].

Note that while the $\mathbb{C}^*$-action on $\mathcal{X}_\infty$ is trivial, making it a fixed point component, the $\mathbb{C}^*$-action on $\mathcal{X}_0$ is not. The fixed point locus of the $\mathbb{C}^*$-action on $\mathcal{X}$ will be denoted by $\mathcal{Y}(\mathcal{X})$, and the BB-cells by $\mathcal{X}^{\pm}(\cdot)$. The fixed point locus $\mathcal{Y}(\mathcal{X}_0)$ of $\mathcal{X}_0$ is equal to $\mathcal{Y}(\mathcal{X}_0) = \mathcal{Y}(\mathcal{X}) \cap \mathcal{X}_0$. The connected components of $\mathcal{Y}(\mathcal{X}_0)$ will be labeled with a subscript 0, that is for example the sink of the $\mathbb{C}^*$-action on $\mathcal{X}_0$ is $Y_{-,0}$ and the source is $Y_{+,0}$. Similarly, the BB-cells $\mathcal{X}_0^{\pm}$ of $\mathcal{X}_0$ are such that $\mathcal{X}_0^{\pm}(Y_0) = \mathcal{X}^{\pm}(Y_0) \cap \mathcal{X}_0$, for $Y_0 \in \mathcal{Y}(\mathcal{X}_0)$.

Lemma 2.5.20. *The $\mathbb{C}^*$-action on $\mathcal{X}$ has sink $\mathcal{X}_\infty$ and source $Y_{+,0}$. Moreover $\mathcal{Y}(\mathcal{X}_0) \setminus \mathcal{X}_0 = \mathcal{Y}(\mathcal{X})^\circ$, that is the inner components of $\mathcal{X}$ are those contained in $\mathcal{X}_0$.*

Notice that the $\mathbb{C}^*$-action on $\mathcal{X}$ is not a bordism: indeed $\mathcal{X}_0 = \overline{\mathcal{X}_0^-(Y_-)}$ is a $\mathbb{C}^*$-invariant divisor. We may represent the above construction by means of the following picture:



Notice that the $\mathbb{C}^*$-action on $\mathcal{X}$ is equalized if and only if it is equalized in $\mathcal{Y}(\mathcal{X}_0)$: indeed by construction it is equalized on $\mathcal{X}_\infty$.

Let us construct the natural birational map among the extremal geometric quotients $\psi\colon \mathcal{G}\mathcal{X}_- \dashrightarrow \mathcal{G}\mathcal{X}_+$. On one hand, by construction $\mathcal{G}\mathcal{X}_- \simeq \mathcal{X}_\infty$. On the other hand, $\mathcal{G}\mathcal{X}_+ = (\mathcal{X}^+(Y_{+,0}) \setminus Y_{+,0})/\mathbb{C}^*$ is a fibration in weighted projective spaces. We thus obtain that:

Proposition 2.5.21. *The variety X is birational to a fibration in weighted projective spaces.*

In particular, if $Y_{+,0}$ is a point then X is birational to a weighted projective space (or a standard projective space if the action is equalized at $Y_{+,0}$).

Chapter 3

Local models of elementary transformations

In this chapter we investigate the local geometry of the natural birational map ψ among the extremal geometric quotients of a polarized pair by a $\mathbb{C}^*$-action (see Proposition 2.3.4).

We first recall the notion of *Morelli–Włodarczyk cobordism*, which was introduced first by Morelli in the case of toric varieties (see [44]), and then generalized by Włodarczyk for normal projective varieties (see [65, Definition 2]). We then study in detail the Morelli–Włodarczyk cobordism associated to a family of toric flips, which includes for instance the well-known Atiyah flip and the Francia flip. We then introduce the notion of *rooftop flip* (see Definition 3.2.1), namely a small modification whose associated diagram of the exceptional loci is a variety with two projective bundle structures. Examples include the Atiyah flip and the Mukai flop. We conclude the section by proving Theorem 3.2.12, which shows how to construct, given a smooth projective variety Λ with two projective bundle structures, a rooftop flip modeled by Λ, and showing some applications for flips constructed upon rational homogeneous varieties.

3.1 Morelli–Włodarczyk cobordism

Definition 3.1.1. Let X_-, X_+ be birationally equivalent normal varieties. The *Morelli–Włodarczyk cobordism* between X_- and X_+ is a normal variety B, endowed with a $\mathbb{C}^*$-action such that

$$B_+ \coloneqq \{p \in B \mid \lim_{t \to 0} tp \text{ does not exists}\},$$
$$B_- \coloneqq \{p \in B \mid \lim_{t \to \infty} tp \text{ does not exists}\}$$

are non-empty open subsets of B, such that $X_\pm \simeq B_\pm/\mathbb{C}^*$, and the birational equivalence between X_-, X_+ is induced by the inclusions $(B_- \cap B_+)/\mathbb{C}^* \subset B_\pm/\mathbb{C}^* \simeq X_\pm$.

We stress that the notation in the above definition is slightly different from the original one from the point of view of notation (cf. [65, Definition 2]); in particular the role on B_- and B_+ are switched. The reason behind this apparent misleading choice is that in our setting the ±-signs will be coherently related with the sink and source $Y_\pm$.

One key result of Włodarczyk (see [65, Proposition 2.A]) is that given two birationally equivalent normal projective varieties, there exists a quasi-projective variety which is a cobordism among them.

3.1.1 Morelli–Włodarczyk cobordism for toric flips

Set-up 3.1.2. Let $\mathrm{N}_-, \mathrm{N}_0$ and N_+ be lattices of respectively dimension equal to d_-, d_0, d_+, generated by $e_1, \dots, e_{d_-}$, $h_1, \dots, h_{d_0}$, $f_1, \dots, f_{d_+}$. Let $\mathrm{N} \coloneqq \mathrm{N}_- \oplus \mathrm{N}_0 \oplus \mathrm{N}_+$, and set $V \coloneqq \mathrm{N} \otimes_{\mathbb{Z}} \mathbb{R}$. We have that $V \simeq \mathbb{R}^{n+1}$, where $n+1 \coloneqq d_- + d_0 + d_+$.
Consider the simplicial cone $\delta = \langle e_1, \dots, f_{d_+} \rangle$, whose associated affine toric variety is $X_\delta = \mathbb{C}^{n+1}$. Notice that $\mathbb{C}^{n+1} = \mathbb{C}^{d_-} \oplus \mathbb{C}^{d_0} \oplus \mathbb{C}^{d_+}$, and for the sake of notation set $p = (p_-, p_0, p_+)$, where $p_- \in \mathbb{C}^{d_-}, p_0 \in \mathbb{C}^{d_0}, p_+ \in \mathbb{C}^{d_+}$. Consider $d_- + d_+$ positive integers $q_1, \dots, q_{d_-}, w_1, \dots, w_{d_+}$, and without loss of generality assume that they are coprime. Let $q = (-q_1, \dots, -q_{d_-})$, $w = (w_1, \dots, w_{d_+})$ be 1-parameter subgroups respectively in $\mathrm{N}_{\mp}$, and consider $v = (-q_1, \dots, -q_{d_-}, 0, \dots, 0, w_1, \dots, w_{d_+}) \in \mathrm{N}$, which induces the following faithful $\mathbb{C}^*$-action on X_δ:

$$\mathbb{C}^* \times X_\delta \to X_\delta$$
$$(t, p) \mapsto (t^q p_-, p_0, t^w p_+),$$

for $t \in \mathbb{C}^*$ and $p = (p_-, p_0, p_+) \in X_\delta$.

Remark 3.1.3. The fixed point locus equals $X_\delta^{\mathbb{C}^*} = \mathbb{C}^{d_0}$.

Notation 3.1.4. We set $\mathbb{C}[X_\delta] = \mathbb{C}[x_1, \dots, x_{d_-}, z_1, \dots, z_{d_0}, y_1, \dots, y_{d_+}]$. A monomial of $\mathbb{C}[X_\delta]$ will be denoted by $x_1^{j_1} \cdots x_{d_-}^{j_{d_-}} z_1^{l_1} \cdots z_{d_0}^{l_{d_0}} y_1^{m_1} \cdots y_{d_+}^{m_{d_+}}$.

Lemma 3.1.5. *The affine GIT quotient $X_\delta /\!\!/ \mathbb{C}^*$ of X_δ by the $\mathbb{C}^*$-action is an affine toric variety associated to the cone $\overline{\delta} = \pi(\delta)$, where $\pi \colon \mathrm{N} \to \overline{\mathrm{N}} \coloneqq \mathrm{N}/\mathbb{Z}v$.*

Proof. Consider the projection map $\pi \colon \mathrm{N} \to \overline{\mathrm{N}}$, and, dually, the inclusion $\overline{\mathrm{M}} \hookrightarrow \mathrm{M}$. By definition the affine GIT quotient is $X_\delta /\!\!/ \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[X_\delta]^{\mathbb{C}^*}$. One can show that a monomial $x_1^{j_1} \cdots y_{d_+}^{m_{d_+}}$ is $\mathbb{C}^*$-invariant if and only if

$$-j_1 q_1 - \ldots - j_{d_-} q_{d_-} + m_1 w_1 + \ldots + m_{d_+} w_{d_+} = 0.$$

Therefore $\mathbb{C}[X_\delta]^{\mathbb{C}^*} = \mathbb{C}[\overline{\delta}^\vee \cap \overline{\mathrm{M}}]$, hence we conclude. ■

Proposition 3.1.6. *Under the notation of Set-up 3.1.2, the non-empty open subsets $B_\pm$ of Definition 3.1.1 can be described as:*

$$B_- = \{p = (p_-, p_0, p_+) \in X_\delta \mid p_+ \neq 0^{d_+}\} = \mathbb{C}^{n+1} \setminus \{(p_-, p_0, 0)\},$$
$$B_+ = \{p = (p_-, p_0, p_+) \in X_\delta \mid p_- \neq 0^{d_-}\} = \mathbb{C}^{n+1} \setminus \{(0, p_0, p_+)\}.$$

Moreover $B_\pm$ are toric varieties, whose associated fans in N, which we will denote by $\Delta_\pm$, can be described as follows:

$$\Delta_- = \{\tau \in \Sigma(\delta) \mid \tau \not\supset \langle f_i \rangle \text{ for } i = 1, \dots, d_+\},$$

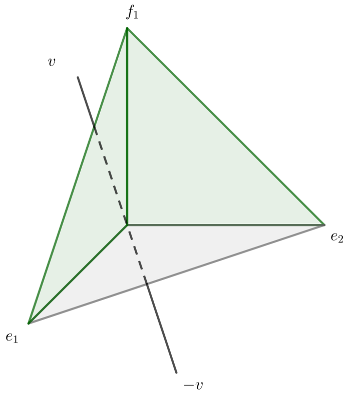
$$\Delta_+ = \{\tau \in \Sigma(\delta) \mid \tau \not\supset \langle e_i \rangle \text{ for } i = 1, \dots, d_-\},$$

where $\Sigma(\delta)$ is the fan of the faces of δ.

Proof. Let us prove only the case of B_-, being the other similar. Considering the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ described in Setup 3.1.2, by definition $B_- = \{p \in X_\delta \mid p_+ \neq 0\} = \mathbb{C}^{n+} \setminus \{(p_-, p_0, 0)\}$. By the Orbit–Cone correspondence (cf. §1.3) the set of points of $\mathbb{C}^{n+1}$ of the form $\{(p_-, p_0, 0)\}$ is the union of orbits associated to the cones of δ which do not contain $\langle f_1, \dots, f_{d_+} \rangle$, therefore the claim. ■

Remark 3.1.7. Suppose $n+1=3$. Then, as noticed in [65, Example 2], the maximal cones of Δ_+ (respectively Δ_-) can be easily detected by looking at the maximal cones visible from v (respectively $-v$). Consider for example the cone $\delta = \langle e_1, e_2, f_1\rangle$ and the action of $\mathbb{C}^*$ on $\mathbb{C}^3$ given by $v=(-2,-1,1)$. Then the corresponding cones of maximal dimension are:

$$\Delta_-(2) = \{\langle e_1, e_2\rangle\}, \quad \Delta_+(2) = \{\langle e_1, f_1\rangle, \langle e_2, f_1\rangle\}.$$



Lemma 3.1.8. *There exist two geometric quotients* $B_\pm \to X_\pm := B_\pm/\mathbb{C}^*$*. Moreover* $X_\pm$ *are toric varieties, and their associated fan is given by* $\overline{\Delta_\pm} := \pi(\Delta_\pm) \subset \overline{\mathrm{N}}$.

Lemma 3.1.9. *There exists a toric flip* $\varphi\colon X_- \dashrightarrow X_+$.

Proof. As noted in [49, §5.2, p. 21] and in [64, p. 265], the fans $\overline{\Delta_\pm}$ determine two simplicial subdivisions of $\overline{\delta}$, that is

$$\overline{\delta} = \bigcup_{i=1}^{d_-} \overline{\delta_i} = \bigcup_{k=1}^{d_+} \overline{\delta_k},$$

where by $\overline{\delta_i}$ (resp. $\overline{\delta_k}$) we mean the image under π of the cone $\delta_i = \langle e_1, \dots, \hat{e}_i, \dots f_{d_+}\rangle$ (resp. $\delta_k = \langle e_1, \dots \hat{f}_k, \dots, f_{d_+}\rangle$) and we abuse notation by denoting with the same name the images of the generators of N under π. It is well known that the map associated with the operation of replacing one subdivision with the other is a flip (see for instance [55, Theorem 3.4] or [64, §3]), hence the claim. ∎

Lemma 3.1.10. *The exceptional locus of the toric flip* $\varphi\colon X_- \dashrightarrow X_+$ *is* $\mathbb{C}^{d_0} \times \mathbb{P}(q_1, \dots, q_{d_-})$.

Remark 3.1.11. By fixing the parameters, we obtain several well-known constructions. We recall some of them (see also [34, Example 4.2]):

- Suppose that $d_- = d_+ = 2, d_0 = 0$, and $v = (-1,-1,1,1)$. The resulting birational map is the well-known *Atiyah flop*;
- Suppose that $d_0 = 0$, and that $v = (-1^{d_-}, 1^{d_+})$. The resulting birational map is called *Atiyah flip*;
- Suppose that $d_- = d_+ = 2, d_0 = 0$, and $v = (-1,-1,1,2)$. The resulting birational map is the *Francia flop*;
- If $d_- = d_0 = 0$ (or $d_+ = d_0 = 0$), the resulting geometric quotient $B_+/\mathbb{C}^*$ (resp. $B_-/\mathbb{C}^*$) is the weighted projective space $\mathbb{P}(w_1, \dots, w_{d_+})$ (resp. $\mathbb{P}(q_1, \dots, q_{d_-})$) (cf. Example 2.2.12).

Corollary 3.1.12. *The affine toric variety* $X_\delta = \mathbb{C}^{n+1}$ *is a cobordism of the birational map* $\varphi\colon X_- \dashrightarrow X_+$*. We may represent the Morelli–Włodarczyk cobordism of* φ *by means of the following diagram:*

$$\begin{tikzcd}
B_- \arrow[r, hook] \arrow[d] & \mathbb{C}^{n+1} \arrow[dd] & B_+ \arrow[l, hook'] \arrow[d] \\
X_- = B_-/\mathbb{C}^* \arrow[rr, dashed] \arrow[dr] & & X_+ = B_+/\mathbb{C}^* \arrow[dl] \\
& \mathbb{C}^{n+1} /\!\!/ \mathbb{C}^* &
\end{tikzcd}$$

Definition 3.1.13. In the situation of Set-up 3.1.2, if all the non-zero weights of the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ are equal to ± 1, then the birational transformation $\varphi\colon X_- \dashrightarrow X_+$ is called *toric Atiyah flip*. Otherwise it will be called *toric non-equalized flip*.

The clean distance between this terminology lies in the fact that, while the former flip is well known in the literature, the latter one has a deep connection on the property of being the $\mathbb{C}^*$-action inducing it non-equalized, as we will see in Chapter 4.

3.2 Rooftop flips: definition and examples

Definition 3.2.1. Consider a normal projective variety Λ with $\rho_\Lambda = 2$ admitting two projective bundle structures:

$$\begin{tikzcd}
& \Lambda \arrow[dl, "p_-"'] \arrow[dr, "p_+"] & \\
\Lambda_- & & \Lambda_+
\end{tikzcd}$$

A small modification $\varphi\colon W_- \dashrightarrow W_+$ between normal quasi-projective varieties is called a *rooftop flip modeled by* Λ if the following hold:

1. There are small contractions $s_\pm\colon W_\pm \to W_0$, with W_0 a normal projective variety,

$$\begin{tikzcd}
W_- \arrow[rr, dashed, "\varphi"] \arrow[dr, "s_-"'] & & W_+, \arrow[dl, "s_+"] \\
& W_0 &
\end{tikzcd}$$

   such that, denoting by $Z_\pm \subset W_\pm$ their exceptional loci, the restrictions $s_\pm|_{Z_\pm}\colon Z_\pm \to Z_0 \subset W_0$ are smooth and the fibers are $\Lambda_\pm$-bundles.

2. There is a resolution

$$\begin{tikzcd}
& W \arrow[dl, "b_-"'] \arrow[dr, "b_+"] & \\
W_- \arrow[rr, dashed, "\varphi"'] & & W_+
\end{tikzcd}$$

   such that $Z \coloneqq b_\pm^{-1}(Z_\pm) \subset W$ is a divisor, and $b_\pm|_Z\colon Z \to Z_\pm$ define projective bundle structures on Z.

3. For any $z_0 \in Z_0$ we have that $b_{\pm}^{-1}|_{s_{\pm}^{-1}(z_0)} = p_{\pm}^{-1}$:

$$\begin{tikzcd}
 & (b_-^{-1} \circ s_-^{-1})(z_0) \simeq \Lambda \simeq (b_+^{-1} \circ s_+^{-1})(z_0) \arrow[dl, "p_-"'] \arrow[dr, "p_+"] & \\
s_-^{-1}(z_0) \simeq \Lambda_- \arrow[dr, "s_-"'] & & \Lambda_+ \simeq s_+^{-1}(z_0) \arrow[dl, "s_+"] \\
 & z_0 &
\end{tikzcd}$$

The reason behind the choice of the name "rooftop" is motivated by the form of the last diagram above. Moreover, the term "roof" has been already used in the literature (see for instance [33, Definition 0.1]) to denote certain varieties with two projective bundle structures.

Remark 3.2.2. A birational map $\chi\colon X_- \dashrightarrow X_+$ between smooth projective varieties is called *K-equivalent simple* if there exists a resolution of indeterminacies

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "f_-"'] \arrow[dr, "f_+"] & \\
X_- \arrow[rr, dashed, "\chi"'] & & X_+
\end{tikzcd}$$

by a smooth projective variety $\widetilde{X}$ such that $f_{\pm}$ are smooth blow-ups and $f_-^* K_{X_-} = f_+^* K_{X_+}$. Let us notice that the notion of rooftop flip is similar to a characterization of K-equivalent simple maps done in [33, Theorem 0.2]. However, in a rooftop flip the fibers of the double projective bundle structures may have different dimensions, in contrast to the case of K-equivalent simple map where by construction they are the same. With this in mind, rooftop flips modeled by $\mathbb{P}^m \times \mathbb{P}^m$ and by $\mathbb{P}(T_{\mathbb{P}^n})$ (see Theorem 3.2.5, Example 3.2.14) are examples of K-equivalent simple maps (see for instance [33, Examples 5.1, 5.2]).

We keep the notation and assumptions of Set-up 3.1.2. Following Remark 3.1.11, we restrict our study to the case of Atiyah flip, that is $d_0 = 0$, $-q = (-1, \dots, -1)$ and $w = (1, \dots, 1)$. We also assume that $d_{\pm} \geq 2$. The reason behind this choice is that the Atiyah flip is the unique toric flip, among the ones constructed above, which is a rooftop flip – modeled respectively by $\mathbb{P}(\mathbb{C}^{d_-}) \times \mathbb{P}(\mathbb{C}^{d_+})$–, as we will show in Theorem 3.2.5. To this end, we first collect some preliminary results:

Lemma 3.2.3. *Under the notation and the assumptions of Set-up 3.1.2, the GIT quotient $X_\delta /\!\!/ \mathbb{C}^*$ has a cone singularity at the origin, which can be resolved by a blow-up $W \to X_\delta /\!\!/ \mathbb{C}^*$. The variety W is toric, and the associated fan is given by the star subdivision of $\pi(\delta)$ with respect to the barycenter of the cone.*

Lemma 3.2.4. *In the situation of Lemma 3.1.8, the geometric quotients $X_{\pm}$ are smooth.*

Proof. Since the non-empty open subsets $B_{\pm}$ are smooth, and $\mathbb{C}^*$ acts freely on them, using [45, Corollary p.199] we obtain that $B_{\pm}$ are $\mathbb{C}^*$-principal bundles over $B_{\pm}/\mathbb{C}^*$, hence they are smooth. ∎

Theorem 3.2.5. *The birational map $\varphi\colon X_- \dashrightarrow X_+$ is a rooftop flip modeled by $\mathbb{P}(\mathbb{C}^{d_-}) \times \mathbb{P}(\mathbb{C}^{d_+})$.*

Proof. We verify that each condition of Definition 3.2.1 is satisfied.
(1). By Lemma 3.1.9 the birational map $\varphi\colon X_- \dashrightarrow X_+$ is a toric flip, and the exceptional loci of $s_\pm\colon X_\pm \to X_\delta /\!\!/ \mathbb{C}^*$ are $\mathbb{P}(\mathbb{C}^{d_\pm})$;
(2). Given the resolution $b_\pm\colon W \to X_\pm$ we have that $\mathbb{P}(\mathbb{C}^{d_-}) \times \mathbb{P}(\mathbb{C}^{d_+}) = b_\pm^{-1}(\mathbb{P}(\mathbb{C}^{d_\pm}))$ is a divisor, and $\mathbb{P}(\mathbb{C}^{d_-}) \times \mathbb{P}(\mathbb{C}^{d_+}) \to \mathbb{P}(\mathbb{C}^{d_\pm})$ clearly defines two projective bundle structures;
(3). In this case Z_0 is the origin, and we know that $s_\pm^{-1}(0) \simeq \mathbb{P}(\mathbb{C}^{d_\pm})$. Moreover $(b_\pm^{-1} \circ s_\pm^{-1})(0) \simeq \mathbb{P}(\mathbb{C}^{d_-}) \times \mathbb{P}(\mathbb{C}^{d_+})$, hence we conclude. ■

3.2.1 Explicit cobordism for rooftop flips

We briefly recall the standard notation and assumptions for the construction of smooth drums (see Section 2.5.2). Consider the triple $(Y, \mathcal{L}_-, \mathcal{L}_+)$, where Y is a smooth projective variety with $\rho_Y = 2$, admitting two projective bundle structures $\pi_\pm\colon Y \to Y_\pm$, and $\mathcal{L}_\pm$ are the pullbacks via $\pi_\pm$ of very ample line bundles $L_\pm$ respectively on $Y_\pm$. Then given the projective bundle $\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)$, the *drum* X is the image of the birational contraction determined by the ring of sections of $\mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(1)$, that is $\operatorname{Proj} R(\mathbb{P}(L_- \oplus L_+); \mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(1))$.

Set-up 3.2.6. Let X be a smooth drum constructed upon a triple $(Y, \mathcal{L}_-, \mathcal{L}_+)$. Let $\hat{X}$ be the affine cone over X, contained in the affine space $V^\vee := V_-^\vee \oplus V_+^\vee$, where

$$V_- := \mathrm{H}^0(Y_-, L_-), \qquad V_+ := \mathrm{H}^0(Y_+, L_+).$$

Consider the $\mathbb{C}^*$-action on $V^\vee$ given by $t \cdot v = \big(tv_-, t^{-1}v_+\big)$, where $v = (v_-, v_+) \in V^\vee$.

By construction, $\hat{X}$ is $\mathbb{C}^*$-invariant, hence we can restrict the action to $\hat{X}$. Let $\hat{X} \to \hat{X} /\!\!/ \mathbb{C}^*$ be the affine GIT quotient, which is singular at the origin, in general. We use the notation of Definition 3.1.1.

Lemma 3.2.7. *The intersections $\hat{X} \cap B_\pm$ between $\hat{X}$ and the open subsets $B_\pm$ of Definition 3.1.1 are non-empty and open, and there exist geometric quotients $\pi_\pm\colon \hat{X} \cap B_\pm \to \hat{X} \cap B_\pm/\mathbb{C}^*$.*

Proposition 3.2.8. *The natural map*

$$\varphi : \hat{X} \cap B_-/\mathbb{C}^* \dashrightarrow \hat{X} \cap B_+/\mathbb{C}^*$$

is a small modification whose exceptional locus is Y_-.

Proof. Consider the restriction to $\hat{X}$ of the diagram of Corollary 3.1.12

$$\begin{tikzcd}
\hat{X} \cap B_-/\mathbb{C}^* \arrow[r, hook] & B_-/\mathbb{C}^* \arrow[rr, dashed] \arrow[dr] & & B_+/\mathbb{C}^* \arrow[dl] & \hat{X} \cap B_+/\mathbb{C}^* \arrow[l, hook'] \\
Y_- \arrow[u, hook] \arrow[r, hook] \arrow[drr] & \mathbb{P}(V_-) \arrow[u, hook] \arrow[dr] & B /\!\!/ \mathbb{C}^* & \mathbb{P}(V_+) \arrow[u, hook] \arrow[dl] & Y_+ \arrow[u, hook] \arrow[l, hook'] \arrow[dll] \\
& & 0 \arrow[u, hook] & &
\end{tikzcd}$$

By the commutativity of the diagram, and the fact that $\hat{X} \cap B_-/\mathbb{C}^* \cap \mathbb{P}(V_-) = Y_-$, we conclude. ■

Remark 3.2.9. Analogously, the exceptional locus of the birational map φ^{-1} is Y_+.

Consider the blow-up $\beta\colon W \to V^\vee /\!\!/ \mathbb{C}^*$ along the vertex of the affine cone $V^\vee /\!\!/ \mathbb{C}^*$ with exceptional divisor $\mathbb{P}(V_-) \times \mathbb{P}(V_+)$.

Notation 3.2.10. Let $R \coloneqq \overline{\beta^{-1}((\hat{X} /\!\!/ \mathbb{C}^*) \setminus 0)}$ be the strict transform of $\hat{X} /\!\!/ \mathbb{C}^*$ under $\beta \colon W \to V^\vee /\!\!/ \mathbb{C}^*$.

We abuse notation by denoting with $b_\pm \colon R \to \hat{X} \cap B_\pm/\mathbb{C}^*$ the restriction of the blow-up $b_\pm \colon W \to B_\pm/\mathbb{C}^*$. Notice that $R \simeq \overline{b_\pm^{-1}((\hat{X} \cap B_\pm/\mathbb{C}^*) \setminus \hat{Y_\pm})}$, where again we abuse notation by denoting with $s_\pm \colon \hat{X} \cap B_\pm \to \hat{X} /\!\!/ \mathbb{C}^*$ the restriction of $s_\pm \colon B_\pm/\mathbb{C}^* \to V^\vee /\!\!/ \mathbb{C}^*$, and by $\hat{Y_\pm}$ the cone over $Y_\pm$. We obtain a diagram:

$$\begin{tikzcd}
& R \arrow[ld, "b_-"'] \arrow[rd, "b_+"] \arrow[dd, "\beta"] & \\
\hat{X} \cap B_-/\mathbb{C}^* \arrow[rr, dashed, "\varphi"] \arrow[rd, "s_-"'] & & \hat{X} \cap B_+/\mathbb{C}^* \arrow[ld, "s_+"] \\
& \hat{X} /\!\!/ \mathbb{C}^* &
\end{tikzcd}$$

Proposition 3.2.11. *It holds that* $b_\pm^{-1}(Y_\pm) \simeq Y$.

Proof. We proceed by steps. First, let us denote by $X^s/\mathbb{C}^*$ the geometric quotient of (X, L) under the $\mathbb{C}^*$-action, defined over the set of stable points $X^s \coloneqq X \setminus (Y_- \cup Y_+)$ (cf. Corollary 2.2.21).

Step 1 We want to prove that $Y \simeq X^s/\mathbb{C}^*$. Thanks to Lemma 2.5.14, the contraction $f \colon \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \to X$ is $\mathbb{C}^*$-equivariant, in particular the geometric quotients of $(\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+), \mathcal{O}_{\mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+)}(1))$ and (X, L) with respect to the $\mathbb{C}^*$-action are isomorphic. Since the former is a $\mathbb{P}^1$-bundle on Y, and therefore its geometric quotient is isomorphic to Y, we conclude.

Step 2 We show that the GIT quotient $\hat{X} /\!\!/ \mathbb{C}^*$ is the affine cone over Y. Let us recall that by $\mathbb{C}_h^*$ we denote the natural $\mathbb{C}^*$-action on the affine space $V^\vee$ given by the homoteties. We claim that

$$\left(\hat{X} /\!\!/ \mathbb{C}^* \setminus 0\right) / \mathbb{C}_h^* \simeq Y.$$

To this end, let us note that the two $\mathbb{C}^*$-actions commute over the open subset of the points stable under both the $\mathbb{C}^*$ and the $\mathbb{C}_h^*$ actions. Therefore we have that

$$\left(\hat{X} /\!\!/ \mathbb{C}^* \setminus 0\right) / \mathbb{C}_h^* \simeq \left(\hat{X} \setminus \left(\hat{Y}_- \cup \hat{Y}_+\right)\right) / \left(\mathbb{C}_h^* \times \mathbb{C}^*\right). \tag{3.1}$$

Notice that

$$\frac{\hat{X} \setminus \left(\hat{Y}_- \cup \hat{Y}_+\right)}{\mathbb{C}_h^*} = \frac{\left(\hat{X} \setminus 0\right) \setminus \left(\left(\hat{Y}_- \setminus 0\right) \cup \left(\hat{Y}_+ \setminus 0\right)\right)}{\mathbb{C}_h^*} \simeq X \setminus (Y_- \cup Y_+)$$

and that

$$\left(X \setminus (Y_- \cup Y_+)\right) / \mathbb{C}^* = X^s/\mathbb{C}^* \simeq Y.$$

Then the right-hand side of (3.1) is isomorphic to Y and we conclude.

Step 3 We want to prove that $\beta^{-1}(0) = Y$. It follows immediately after recalling that we are considering the restriction of the blow-up map to $\hat{X} /\!\!/ \mathbb{C}^*$, which is the affine cover over Y.

Step 4 We show that $s_\pm \circ b_\pm = \beta$ and that $s_\pm^{-1}(0) \simeq Y$. The first claim follows by construction. Since $s_\pm \colon B_\pm/\mathbb{C}^* \to \hat{X} /\!\!/ \mathbb{C}^*$ are small contractions whose exceptional locus is $\mathbb{P}(V_\pm) \cap \hat{X} = Y_\pm$ by Proposition 3.2.8, we conclude. ■

Theorem 3.2.12. *With the notation of Set-up 3.2.6, for any smooth drum X constructed upon $(Y, \mathcal{L}_-, \mathcal{L}_+)$ there exists a rooftop flip $\varphi \colon \hat{X} \cap B_-/\mathbb{C}^* \dashrightarrow \hat{X} \cap B_+/\mathbb{C}^*$ modeled by Y.*

Proof. We verify that each condition of Definition 3.2.1 is satisfied.

1. Easily follow from Proposition 3.2.8.
2. If we consider the resolution $b_\pm \colon R \to \hat{X} \cap B_\pm/\mathbb{C}^*$ we have that $Y = b_\pm^{-1}(Y_\pm)$ is a divisor in R, and $Y \to Y_\pm$ defines two projective bundle structures, by definition of smooth drum.
3. In this case $Z_0 = 0$, and we know that $s_\pm^{-1}(0) \simeq Y_\pm$. Moreover $(b_\pm^{-1} \circ s_\pm^{-1})(0) \simeq Y$ by Proposition 3.2.11, hence we conclude. ■

Corollary 3.2.13. *The geometric quotients $\hat{X} \cap B_\pm/\mathbb{C}^*$ are smooth and in particular the rooftop flip $\varphi \colon \hat{X} \cap B_-/\mathbb{C}^* \dashrightarrow \hat{X} \cap B_+/\mathbb{C}^*$ is a small $\mathbb{Q}$-factorial modification.*

Proof. Since the affine variety $\hat{X}$ has only a singularity at the origin, $\hat{X} \cap B_\pm$ is smooth. Moreover, the $\mathbb{C}^*$-action is free on $\hat{X} \cap B_\pm$, therefore using [45, Corollary p.199] $\hat{X} \cap B_\pm$ is a $\mathbb{C}^*$-principal bundle over $\hat{X} \cap B_\pm/\mathbb{C}^*$, hence they are also smooth. By definition φ is in particular a small $\mathbb{Q}$-factorial modification. ■

We conclude this chapter by using Theorem 3.2.12 to show that some rooftop flips associated to certain smooth drums are well-known birational transformations:

Example 3.2.14 (Mukai flop)**.** Consider the rational homogeneous variety $A_n(1,n)$, which admits two $\mathbb{P}^{n-1}$-bundle structures:

$$\begin{tikzcd}
& A_n(1,n) \simeq \mathbb{P}(T_{\mathbb{P}^n}) \arrow[dl, "\mathbb{P}^{n-1}"'] \arrow[dr, "\mathbb{P}^{n-1}"] & \\
A_n(1) & & A_n(n)
\end{tikzcd}$$

The smooth drum associated to the above diagram is the $2n$-dimensional quadric $D_{n+1}(1)$ (see [53, Theorem 1.7]), and for $n = 2$ the associated rooftop flip modeled by $\mathbb{P}(T_{\mathbb{P}^2})$ is called *Mukai flop* (see [24], [63]).

Example 3.2.15 (Abuaf-Segal flop)**.** Consider the rational homogeneous variety $C_2(1,2)$, which admits two $\mathbb{P}^1$-bundle structures:

$$\begin{tikzcd}
& C_2(1,2) \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\mathbb{P}^1"] & \\
C_2(1) & & C_2(2)
\end{tikzcd}$$

The smooth drum associated to the above diagram is the 5-dimensional symplectic Grassmannian (see [53, Theorem 1.7]), and the associated rooftop flip modeled by $C_2(1,2)$ is called *Abuaf-Segal flop* (see [61], [43, §2.2]).

Example 3.2.16 (Abuaf-Ueda flop)**.** Consider the rational homogeneous variety $G_2(1,2)$, which admits two $\mathbb{P}^1$-bundle structures:

$$\begin{tikzcd}
 & G_2(1,2) \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\mathbb{P}^1"] & \\
G_2(1) & & G_2(2)
\end{tikzcd}$$

The smooth drum associated to the above diagram is a 7-dimensional linear section of $B_4(2)$ (see [53, Theorem 1.7]), that is the Grassmannian of $\mathbb{P}^1$ in the 7-dimensional quadric hypersurface $Q^7 \subset \mathbb{P}^8$, and the associated rooftop flip modeled by $G_2(1,2)$ is called *Abuaf-Ueda flop* (see [62], [43, §2.2]).

Chapter 4

Geometric realizations in small criticality

We investigate the birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ induced by a $\mathbb{C}^*$-action on a polarized pair (X, L) with small criticality, that is $r = 2, 3$. Let us remark that given a $\mathbb{C}^*$-action on (X, L) of criticality 1 it holds that $\mathcal{G}X_- \simeq \mathcal{G}X_+$, hence X is the geometric realization of an isomorphism between normal projective varieties.

If the $\mathbb{C}^*$-action has criticality 2 and is a bordism, we prove in Theorem 4.1.7 that the map ψ is a (locally toric) flip, either of Atiyah or non-equalized type. In this setting we construct a local geometric realization for the toric flip (see Theorem 4.1.3), and also provide a criterion to understand the local geometry of the birational maps, linking the toric Atiyah flip with a rooftop flip of Atiyah type (see Proposition 4.1.8). We conclude by constructing explicit examples of varieties whose induced birational map is locally of type Atiyah or non-equalized (see §4.1.3.1, 4.1.3.2). This section is based on [49, Sections 5,6].

If the $\mathbb{C}^*$-action has criticality 3 and isolated fixed points, we report the results obtained in [49, Section 8], [51], [47] showing that the natural birational map ψ is a Cremona transformation of type $(2, 2)$.

4.1 Criticality 2: Atiyah and non-equalized flips

4.1.1 Geometric realization of locally toric flips

We briefly recall the notation and the assumptions of Section 3.1.1 in the following:

Set-up 4.1.1. Given the affine toric variety $X_\delta \simeq \mathbb{C}^{n+1}$, we consider a $\mathbb{C}^*$-action associated to the 1-parameter subgroup $v = (q, 0^{d_0}, w)$. The non-empty open subsets $B_\pm$ of Definition 3.1.1 produce two toric geometric quotients $X_\pm$, and there exists a toric flip $\varphi\colon X_- \dashrightarrow X_+$ among them.

The aim of this section is to construct a quasi-projective toric variety which realizes geometrically the toric flip. To this end, we will construct X by glueing together X_δ and two fiber bundles $E_\pm$ constructed upon $X_\pm$. We first notice that the geometric quotients $B_\pm \to X_\pm$ are $\mathbb{C}^*$-bundles which are not locally free, since the action is not equalized (see Lemma 2.2.31). Intuitively, we obtain the fiber bundles $E_\pm$ on $X_\pm$ by adding respectively the zero and the infinity section to $B_\pm$.

Proposition 4.1.2. *Consider*

$$E_\pm := B_\pm \times^{\mathbb{C}^*} \mathbb{C} = \frac{B_\pm \times \mathbb{C}}{\sim},$$

where, for $e \in B_\pm$ *and* $\lambda \in \mathbb{C}$, $(e,\lambda) \sim (te, t^{\pm 1}\lambda)$. *Then* $E_\pm$ *are fiber bundles on* $X_\pm$ *with fibers* $\mathbb{C}$. *Moreover,* $E_\pm$ *are toric varieties constructed upon two fans* $\Lambda_\pm$. *The maximal cones of* $\Lambda_\pm$ *can be respectively described as*

$$\begin{aligned}\Lambda_-(n+1) &= \{\langle \delta_j, -v\rangle \mid j = 1, \dots, d_+\},\\ \Lambda_+(n+1) &= \{\langle \delta_i, v\rangle \mid i = 1, \dots, d_-\}.\end{aligned}$$

Proof. Let us consider the case of Λ_+, being the other similar. The fan Λ_+ weakly splits (in the sense of [16, Exercise 3.3.7]) by $\langle -v\rangle$ and Δ_+, hence we conclude it is a fiber bundle with fibers isomorphic to $\mathbb{C}$. ■

Theorem 4.1.3. *Given a toric flip* $\varphi\colon X_- \dashrightarrow X_+$, *there exists a quasi-projective toric variety* X *which realizes geometrically* φ. *Moreover the fan associated to* X *is*

$$\tilde{\Sigma} = \Lambda_+ \cup \Sigma(\delta) \cup \Lambda_- \subset \mathrm{N}_\mathbb{R}.$$

The geometric realization X *admits a* $\mathbb{C}^*$*-action associated to the* 1*-parameter subgroup* v. *The fixed point locus of* X *consists of the sink* X_-, *the source* X_+, *and an inner component isomorphic to* $\mathbb{C}^{d_0}$.

Proof. Let X be the toric variety associated to the fan $\tilde{\Sigma}$. We first notice that X admits a $\mathbb{C}^*$-action associated to the 1-parameter subgroup v. By construction $\mathbb{C}^*$ acts on X_δ; moreover $X_\pm$ are $\mathbb{C}^*$-invariant, and the maps $E_\pm \to X_\pm$ are $\mathbb{C}^*$-equivariant. We study the fixed point locus of the $\mathbb{C}^*$-action separately on the three patches; we have that $X_\delta^{\mathbb{C}^*} = \mathbb{C}^{d_0}$. On the other hand $E_\pm^{\mathbb{C}^*} = s_0(X_\pm) = X_\pm$, with $s_0 : X_\pm \to E_\pm$ the 0-section, and thus they correspond respectively to the sink and the source of the action. ■

So far we have constructed a local geometric realization: it is natural to ask if one can construct a geometric realization of a flip which can be locally described as a toric flip. The positive answer, in the case of toric Atiyah flip, is provided by [49, Theorem 6.3]. Let us first define the global counterpart of the toric Atiyah flip:

Definition 4.1.4. Let $X_\pm$ be smooth projective varieties. A *global Atiyah flip* is a small modification $\varphi\colon X_- \dashrightarrow X_+$ fitting in a commutative diagram

$$\begin{tikzcd} X_- \arrow[rr, dashed, "\varphi"] \arrow[dr, "\pi_-"'] & & X_+ \arrow[dl, "\pi_+"] \\ & X_0 & \end{tikzcd}$$

such that:

- The maps $\pi_\pm$ are small contractions to a normal projective variety X_0;
- The diagram can be locally analytically identified with a toric Atiyah flip.

  In particular, the inderterminacy loci $Z_\pm := \mathrm{Exc}(\pi_\pm)$ are smooth varieties, possibly disconnected: their irreducible components are in one to one correspondence, and we denote them by $Z^j_\pm$, for $j \in J$. For each $j \in J$, the image $\pi_\pm(Z^j_\pm)$ is an irreducible component X^j_0 of the indeterminacy locus $\mathrm{Ind}(\pi^{-1}_\pm)$, and the restrictions $\pi_-: Z^j_- \to X^j_0$, $\pi_+: Z^j_+ \to X^j_0$ are projective bundles.

- There exist $\pi_\pm$-ample line bundles $\mathcal{V}_\pm$ on $X_\pm$ such that
    1. $\operatorname{Pic}(X_\pm) = \pi_\pm^* \operatorname{Pic}(X_0) \oplus \mathbb{Z}\mathcal{V}_\pm$;
    2. The restriction of $\mathcal{V}_\pm$ to every fiber of the projective bundle $\pi_\pm \colon Z_\pm^j \to X_0^j$ is $\mathcal{O}(1)$;
    3. $\varphi_*(\mathcal{V}_-) = -\mathcal{V}_+$.

Theorem 4.1.5. *[49, Theorem 6.3, Corollary 6.6, Theorem 6.7] Let $\varphi \colon X_- \dashrightarrow X_+$ be a global Atiyah flip. There exists a unique geometric realization X of φ. Moreover there exists an ample line bundle L on X such that the induced $\mathbb{C}^*$-action on (X, L) has criticality* 2.

4.1.2 A criterion for locally toric flips among geometric quotients

Set-up 4.1.6. Consider a $\mathbb{C}^*$-action on a smooth polarized pair (X, L) of B-type of criticality 2.

As already noticed in Remark 2.3.29, we recall that the B-type assumption can be always obtained by performing a pruning of X along the extremal intervals.

Theorem 4.1.7. *In the situation of Set-up 4.1.6, assume moreover that the $\mathbb{C}^*$-action on X is a bordism. Then the natural birational map $\psi \colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is locally analytically a toric Atiyah flip if and only if the $\mathbb{C}^*$-action on X is equalized at every inner component.*

Proof. Assume that ψ is locally analytically a toric Atiyah flip. Suppose by contradiction that there exists an irreducible component Y' of $\mathcal{Y}^\circ$ on which the action is non-equalized. Choose a point $p \in Y'$. Using Theorem 2.1.16 there exists an analytic neighborhood $\mathcal{U} \subset X$ of p, which is $\mathbb{C}^*$-invariant and biholomorphic to $\mathcal{N}_{Y'\cap\mathcal{U}|X} \simeq \mathbb{C}^{\dim X}$. Let us consider the following two geometric quotients of $\mathcal{U}$:

$$U_- \colon \{y \in \mathcal{G}X_- \mid y = \lim_{t\to\infty} tx,\ x \in \mathcal{U}\} \quad \text{and} \quad U_+ \colon \{y \in \mathcal{G}X_+ \mid y = \lim_{t\to 0} tx,\ x \in \mathcal{U}\}.$$

Notice that locally in $\mathcal{U}$ the $\mathbb{C}^*$-action is as in Set-up 4.1.1. Since the weights of the $\mathbb{C}^*$-action on $\mathcal{U}$ corresponds to the weights of the $\mathbb{C}^*$-action on $\mathcal{N}_{Y'\cap\mathcal{U}|X^\flat}$ and the action on Y' is non-equalized by assumption, we deduce that $\psi_{|U_-} \colon U_- \dashrightarrow U_+$ is a toric non-equalized flip, hence a contradiction.

Let us prove the converse. Set $Z = \operatorname{Exc}(\psi)$. Consider a point $z \in Z$ and let us prove that there exists an open subset of $U_-(z)$ of z contained in $\mathcal{G}X_-$ such that $\psi_{|U_-(z)}$ is a toric Atiyah flip. By Theorem 2.1.16, there exists a neighborhood V of $z \in Z$ such that $X^-(V) \simeq \mathcal{N}_{V|X}$ it follows that there exists a unique orbit C having sink in z. The source z' of C lies in an inner fixed point component we denote by $\overline{Y}$. Using again Theorem 2.1.16, we may find an analytic neighborhood $\mathcal{U}(z')$ of z' which is $\mathbb{C}^*$-invariant and biholomorphic to $\mathcal{N}_{\bar{Y}\cap\mathcal{U}(z')|X} \simeq \mathbb{C}^{\dim X}$, and we take two geometric quotients $U_\pm(z)$ of $\mathcal{U}(z')$ defined as above. By assumption, the $\mathbb{C}^*$-action is equalized at $\bar{Y}$, then it follows that $\psi_{|U_-(z)} \colon U_-(z) \dashrightarrow U_+(z)$ is a toric Atiyah flip, and we conclude. ■

Proposition 4.1.8. *In the situation on Set-up 4.1.6, assume that the $\mathbb{C}^*$-action is a bordism. Then the natural birational map $\psi \colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is locally analytically a toric Atiyah flip if and only if ψ is a rooftop flip of Atiyah type.*

Proof. By Lemma 2.3.6, there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{G}X_- \arrow[rr, dashed, "\psi"] \arrow[dr, "\pi_-"'] & & \mathcal{G}X_+ \arrow[dl, "\pi_+"] \\
& \mathcal{S}X_1 &
\end{tikzcd}$$

where $\pi_\pm$ are contractions. Let Y be an inner fixed point component of $X^{\mathbb{C}^*}$. If the birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is a rooftop flip of Atiyah type, we claim the $\mathbb{C}^*$-action is equalized at Y. Suppose by contradiction it is not: using Theorem 4.1.7, the birational map ψ among the extremal geometric quotients would be a locally toric non-equalized flip. Therefore, the restriction of $\pi_\pm$ to the exceptional loci are weighted projective fibrations, which is an absurd since ψ is a rooftop flip of Atiyah type.

On the other hand, suppose that ψ is a locally toric flip of Atiyah type. Using Theorem 4.1.7, we obtain that the $\mathbb{C}^*$-action is equalized at $Y \in \mathcal{Y}^\circ$. In particular, by Lemma 2.1.28, we obtain that $(X^\pm(Y) \setminus Y)/\mathbb{C}^* \simeq \mathbb{P}(\mathcal{N}^\pm_{Y|X})$. Therefore, one may show that the diagram

$$\begin{tikzcd}
& & \mathbb{P}(\mathcal{N}^+_{Y|X}) \times \mathbb{P}(\mathcal{N}^-_{Y|X}) \arrow[dll] \arrow[d, hook] \arrow[drr] & & \\
& & \mathcal{G}X_- \times^Y \mathcal{G}X_+ \arrow[dl, "b_-"'] \arrow[dr, "b_+"] & & \\
\mathbb{P}(\mathcal{N}^+_{Y|X}) \arrow[r, hook] \arrow[ddrr] & \mathcal{G}X_- \arrow[rr, dashed, "\psi"] \arrow[dr, "s_-"'] & & \mathcal{G}X_+ \arrow[dl, "s_+"] & \mathbb{P}(\mathcal{N}^-_{Y|X}) \arrow[l, hook'] \arrow[ddll] \\
& & \mathcal{S}X_1 & & \\
& & Y \arrow[u, hook] & &
\end{tikzcd}$$

satisfies the conditions of Definition 3.2.1, making ψ a rooftop flip of Atiyah type. ■

Corollary 4.1.9. *In the situation of Set-up 4.1.6, assume in addition that $\rho_X = 1$. Then ψ is locally a toric Atiyah flip if and only if the $\mathbb{C}^*$-action on X is equalized at every inner component.*

Proof. If $\rho_X = 1$ and the $\mathbb{C}^*$-action is of B-type then [49, Lemma 2.6 (1)] implies that $\nu^\pm(Y) \geq 2$ for every $Y \in \mathcal{Y}^\circ$. Then X is a bordism and the statement follows by Theorem 4.1.7. ■

Theorem 4.1.7 can be generalized to $\mathbb{C}^*$-actions of B-type on polarized pairs (X, L) of higher criticality by requiring that, for every component $Y \in Y^\circ$, there do not exists orbit closures joining Y with other fixed components different from $Y_\pm$. Using the partial order $\preccurlyeq$ introduced in Definition 2.2.14, we can state the following:

Corollary 4.1.10. *In the situation of Set-up 4.1.6, suppose in addition that, for every component $Y \in \mathcal{Y}^\circ$,*

- *the sink is the unique component such that $Y_- \preccurlyeq Y$;*
- *the source is the unique component such that $Y \preccurlyeq Y_+$.*

Then $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is a locally toric Atiyah flip if and only if the $\mathbb{C}^*$-action on X is equalized at every inner component.

4.1.3 Examples

In this section we use Theorem 4.1.7 to construct:

- A family of $\mathbb{C}^*$-actions on $(B_n(1), \mathcal{O}_{B_n(1)}(1))$ of criticality 2 which are not equalized at $Y_\pm$, but equalized at the unique inner component, and thus the natural birational maps are locally toric Atiyah flips (see §4.1.3.1);
- A $\mathbb{C}^*$-action on $(B_n(2), \mathcal{O}_{B_n(2)}(1))$ which is not equalized at the inner component, and thus the natural birational map is a locally toric non-equalized flip (see §4.1.3.2).

Since both varieties are RH, we will extensively use the notation introduced in Examples 2.5.5, 2.5.6. Let us introduce a Set-up which we will use for both examples:

Set-up 4.1.11. Consider the projective space $\mathbb{P}^{2n}$ with homogeneous coordinates $x_0, \dots, x_{2n}$, and a family of $\mathbb{C}^*$-actions on $\mathbb{P}^{2n}$, denoted by α_k, for $k = 1, \dots, n$, defined as follows:

$$\alpha_k \colon \mathbb{C}^* \times \mathbb{P}^{2n} \to \mathbb{P}^{2n}$$
$$(t, p) \to [tp_0 : \dots : tp_{k-1} : p_k : \dots : p_n : t^{-1}p_{n+1} : \dots : t^{-1}p_{n+k} : p_{n+k+1} : \dots : p_{2n}].$$

For the sake of notation, we will denote a point $p \in \mathbb{P}^{2n}$ by

$$p = [p_+ : p_0 : p_-],$$

where p_+, p_0 and p_- represent the coordinates on which α_k acts with respectively positive, zero and negative weights.

Lemma 4.1.12. *For any $k = 1, \dots, n$, the α_k-action on $(\mathbb{P}^{2n}, \mathcal{O}_{\mathbb{P}^{2n}}(1))$ has criticality 2, with sink $\mathbb{P}^{k-1}_- := \{p \in \mathbb{P}^{2n} \mid p = [p_+ : 0 : 0]\}$, source $\mathbb{P}^{k-1}_+ := \{p \in \mathbb{P}^{2n} \mid p = [0 : 0 : p_-]\}$, and inner component $\mathbb{P}^{2n-2k}$.*

4.1.3.1 Non-equalized $\mathbb{C}^*$-actions admitting an Atiyah flip

Notation 4.1.13. Let $X = B_n(1)$ be the smooth quadric hypersurface of $\mathbb{P}^{2n}$. Let us take coordinates such that

$$X = Z(x_0x_{n+1} + \dots + x_{n-1}x_{2n} + x_n^2).$$

By construction, X is invariant under the α_k-actions for any $k = 1, \dots, n$.

Lemma 4.1.14. *The fixed point locus of the induced α_k-actions on $(X, \mathcal{O}_X(1))$ has sink $Y_- = \mathbb{P}^{k-1}_-$, source $Y_+ = \mathbb{P}^{k-1}_+$, and an inner fixed point component equal to $B_{n-k}(1) = Q^{2n-2k-1}$, where we abuse notation by setting $Q^{-1} := \emptyset$ for $k = n$. In particular, the bandwidth is equal to 2 for any $k = 1, \dots, n$, while the criticality of the action is 2 for $k \neq n$ and equal to 1 for $k = n$.*

Proof. The fixed point locus is readily obtained by considering the intersection $X \cap (\mathbb{P}^{2n})^{\alpha_k}$, for any k. To conclude, it suffices to notice that $\mu_{\mathcal{O}_X(1)}(Y_\pm) = \pm 1$ and $\mu_{\mathcal{O}_X(1)}(Q^{2n-2k-1}) = 0$, hence the bandwidth is equal to 2 for any $k = 1, \dots, n$, while the criticality is 2 for $k \neq n$, and it is equal to 1 if $k = n$. ■

Lemma 4.1.15. *The α_k-actions on X are equalized at the sink and the source if and only if $k = 1$, $n > 1$.*

Proof. If $k = 1$, the sink and the source are two isolated points, hence by [57, Theorem 4.1] the action is equalized. Assume that $k \neq 1$. Consider a point $p = [p_+ : 0 : p_-] \in X$, and denote by C the closure of the orbit $\mathbb{C}^* p$. Then C is the line of points of the form $[tp_+ : 0 : t^{-1}p_-]$, for $t \in \mathbb{C}^*$, and applying AMvsFM Lemma 2.1.50 we get $2 = \delta(\tilde{p}) \deg \mathcal{O}_X(1)$, where $\tilde{p}$ is the source of C. Since $\deg \mathcal{O}_X(1) = 1$, one has $\delta(\tilde{p}) = 2$, thus the action is non-equalized. ■

Remark 4.1.16. Using Lemma 4.1.14 and AMvsFM Lemma 2.1.50, we obtain that the α_k-action is equalized at the inner fixed point component $Q^{2n-2k-1}$.

Proposition 4.1.17. *Let $\widetilde{X} = \mathcal{P}(X)^+_-$ be the pruning of X at the extremal intervals. Then the induced α_k-action on $\widetilde{X}$ is a bordism, and and if $k \neq 1$ the natural birational map $\psi \colon \widetilde{\mathcal{G}X_-} \dashrightarrow \widetilde{\mathcal{G}X_+}$ is a locally toric Atiyah flip.*

Proof. By Theorem 2.3.27, the pruning map $X \dashrightarrow \widetilde{X}$ is α_k-equivariant and the induced action on $(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1))$ has criticality 2. By Theorem 2.3.27, $\widetilde{X}$ is of B-type, and since $\operatorname{codim} \overline{X^{\pm}(Q^{2n-2k-1})} \geq 2$, the α_k-action on $\widetilde{X}$ is a bordism. By Remark 4.1.16 the α_k-action is equalized at the inner component, hence, using Theorem 4.1.7, the natural birational map $\psi \colon \widetilde{\mathcal{G}X_-} \dashrightarrow \widetilde{\mathcal{G}X_+}$ is a locally toric Atiyah flip. ■

4.1.3.2 Non-equalized action admitting a non-equalized flip

With the notation of Set-up 4.1.11, set $k = n$ and consider the induced α_n-action on $M := B_n(2)$, obtained as the restriction of the α_n-action on $A_{2n}(2) = \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n+1})$.

Lemma 4.1.18. *The fixed point locus of the induced α_n-action on M equals to $M^{\alpha_n} = A_{n-1}(2)_- \sqcup A_{n-1}(1, n-1) \sqcup A_{n-1}(2)_+$, where $A_{n-1}(2)_\pm$ represent the Grassmannians of lines of $\mathbb{P}^{n-1}_\pm$.*

Proof. By Lemma 4.1.14, $X^{\alpha_n} = \mathbb{P}^{n-1}_- \sqcup \mathbb{P}^{n-1}_+$. The fixed components of M under the induced α_n-action will be the set of α_n-invariant lines in X. We thus immediately obtain that the sink and the source of M are the sets parametrizing lines in the sink and the source of X, that is $A_{n-1}(2)_\pm$. We are left to study the α_n-invariant lines from $\mathbb{P}^{n-1}_-$ to $\mathbb{P}^{n-1}_+$. Note that the intersection between X and the subspace generated by $\mathbb{P}^{n-1}_\pm$ is a $(2n-2)$-dimensional quadric Q^{2n-2}. Consider therefore a point $p_- \in \mathbb{P}^{n-1}_-$, and the set:

$$H(p_-) := \{p_+ \in \mathbb{P}^{n-1}_+ \mid \overline{p_- p_+} \in Q^{2n-2}\}.$$

It is easy to see that $H(p_-)$ is an hyperplane in $\mathbb{P}^{n-1}_+$, and that the map $D : \mathbb{P}^{n-1}_- \to (\mathbb{P}^{n-1}_-)^\vee$, $p_- \mapsto H(p_-)$ is an isomorphism, therefore the α_n-invariant lines from $\mathbb{P}^{n-1}_-$ to $\mathbb{P}^{n-1}_+$ are given by the choice of a point and the associated hyperplane, i.e. the variety $\mathbb{P}(T_{\mathbb{P}^{n-1}}) = A_{n-1}(1, n-1)$. ■

Lemma 4.1.19. *The α_n-action on $(M, \mathcal{O}_M(1))$ has criticality 2 and bandwidth 4. Moreover the sink is $A_{n-1}(2)_-$ and the source is $A_{n-1}(2)_+$.*

Proof. Notice that $\mu_{\mathcal{O}_M(1)}(A_{n-1}(2)_-) = -2$; indeed given $e_0, \dots, e_{2n}$ a basis of $\mathbb{C}^{2n+1}$, α_n acts, via the Plücker embedding, on $e_i \wedge e_j$ with weight 2, for $0 \leq i < j \leq n-1$. Similarly we get $\mu_{\mathcal{O}_M(1)}(A_{n-1}(2)_+) = 2$, and $\mu_{\mathcal{O}_M(1)}(A_{n-1}(1, n-1)) = 0$. Thus the criticality of the action on $(M, \mathcal{O}_M(1))$ is 2 and its bandwidth is 4. Finally, using Lemma 2.1.43 we conclude that $A_{n-1}(2)_-, A_{n-1}(2)_+$ are respectively the sink and the source of the α_n-action on M. ■

We now show that the induced α_n-action on M is non-equalized at the inner component $A_{n-1}(1, n-1)$, and thus by Theorem 4.1.7 the natural birational map $\psi \colon \mathcal{G}M_- \dashrightarrow \mathcal{G}M_+$ is locally a toric non-equalized flip.

Lemma 4.1.20. *The weights of the induced α_n-action on the normal bundle $\mathcal{N}_{A_{n-1}(1,n-1)|M}$ are $(\pm1, \pm2^{n-2})$, where the exponent denotes the occurrence of the weight.*

Proof. For the sake of notation, set $\mathcal{Y}_\pm := A_{n-1}(2)_\pm, Y_1 := \mathbb{P}(T_{\mathbb{P}^{n-1}})$. As in the proof of Lemma 4.1.18, we denote by $H(p)$ the hyperplane in Y_+ corresponding to a point $p \in Y_-$. Let us compute the weights of the α_n-action on $\mathcal{N}^+_{Y_1|M}$. To this end, take a point $s \in Y_1$ and let us denote by p_-, p_+ its intersection with Y_-, Y_+ respectively. Then the family of pencils of lines in X containing s and a line $r \in \mathcal{Y}_+$ is parametrized by the lines passing by p_+ contained in the hyperplane $H(p_-) \subset Y_+$. Since $H(p_-) \simeq \mathbb{P}^{n-2}$ we deduce that such a family is parametrized by a $\mathbb{P}^{n-3}$ in $\mathcal{Y}_+$. This implies that we have $(n-2)$-independent directions from s that correspond to orbits lying in $X^-(Y_1)$. Denote by Γ the closure of one among these orbits. Noticing that Γ has sink at $\mathcal{Y}_-$, and using Lemma 4.1.19 and AMvsFM Lemma 2.1.50, we compute that the weight of the induced α_n-action on the tangent bundle of Γ at p_- is 2.

Moreover, since it is readily seen from the computation of the rank of $\mathcal{N}_{Y_1|M}$ that $\nu^+(Y_1) = n-1$, by Lemma 2.1.50 one may find a α_n-invariant non-singular conic linking s with $\mathcal{Y}_+$, and now the weight of the α_n-action on the tangent bundle of such a conic at p_- is 1. Then, applying Theorem 2.1.13 we deduce that the positive weights of the α_n-action on s are $(1, 2^{n-2})$. Running a symmetric argument replacing $\mathcal{Y}_+$ with $\mathcal{Y}_-$ we conclude that the weights of the α_n-action on $\mathcal{N}^-_{Y_1|M}$ are $(-1, -2^{n-2})$, hence the statement. ■

Proposition 4.1.21. *Let $\widetilde{M} = \mathcal{P}(M)^+_-$ be the pruning of M at the extremal intervals. Then there exists an induced α_n-action on $\widetilde{M}$ which is a bordism, and such that the natural birational map $\psi \colon \widetilde{\mathcal{G}M_-} \dashrightarrow \widetilde{\mathcal{G}M_+}$ is a locally toric non-equalized flip.*

Proof. By Lemma 2.3.27, the pruning map $M \dashrightarrow \widetilde{M}$ is α_n-equivariant and small, and the induced action on $(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(1))$ has criticality 2. By Theorem 2.3.27, $\widetilde{M}$ is of B-type, and moreover $\operatorname{codim} \overline{X^\pm(A_{n-1}(1,n-1))} \geq 2$, hence the α_n-action on $\widetilde{M}$ is a bordism. By Lemma 4.1.20 the induced $\mathbb{C}^*$-action on $\mathcal{N}_{A_{n-1}(1,n-1)|M}$ is non-equalized, hence by Theorem 4.1.7 the natural birational map $\psi \colon \widetilde{\mathcal{G}M_-} \dashrightarrow \widetilde{\mathcal{G}M_+}$ is a locally toric non-equalized flip. ■

4.2 Criticality 3: quadro-quadric Cremona transformations

In this section we study, in the context of a $\mathbb{C}^*$-action on a polarized pair of criticality 3, with isolated fixed points and equalized at $Y_\pm$, the natural birational map ψ among the extremal quotients. Such setting was used by the authors of [50] to study the LeBrun–Salamon conjecture. As we will, in this setting the natural birational map ψ is a *Cremona*:

Definition 4.2.1. A *Cremona map* is a birational map $f \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. A Cremona map is *special* if the base locus of f is smooth and connected.

A Cremona map $f \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ can be described by the choice of $(n+1)$-homogeneous polynomials f_i of the same degree, which we can assume do not share a common factor, such as

$$f \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \quad p \mapsto [f_0(p) : \ldots : f_n(p)].$$

Set $\deg f := \deg f_i$, for $i = 0, \ldots, n$.

Definition 4.2.2. An (a,b)-*Cremona map* is a birational transformation $f \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\deg f = a, \deg f^{-1} = b$.

For example, $(2,2)$-Cremona maps are called *quadro-quadric*. Special quadro-quadric Cremona transformation have been classified in [20]: they are given by system of quadrics through Severi varieties. Such classification was then generalized in [54] allowing reducible fundamental locus.

With this in mind, it is natural to study geometric realization of Cremona maps. Let us recall that the authors of [22] have linked, in the context of equalized $\mathbb{C}^*$-actions on RH-varieties of Picard number 1 with isolated extremal points, Cremona maps among the extremal geometric quotients and Jordan algebras structures on T_{X,Y_-}. We begin by collecting some preliminary results:

Remark 4.2.3. Given an equalized $\mathbb{C}^*$-action on a polarized pair (X,L) with isolated extremal points, the natural birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is a Cremona transformation. Indeed it suffices to notice that, using Remark 2.1.23, we obtain that $\mathcal{G}X_\pm \simeq \mathbb{P}^{\dim X-1}$.

In the situation of the above Remark, if the $\mathbb{C}^*$-action is not equalized, the natural birational map is a *weighted Cremona*, that is a Cremona map between weighted projective spaces, as described in the following:

Example 4.2.4. Let $\mathbb{C}^*$ act on $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ as $t\cdot(x_0:x_1:x_2:x_3) = (x_0 : tx_1 : t^2x_2 : t^3x_3)$. We have that $(\mathbb{P}^3)^{\mathbb{C}^*} = e_3 \sqcup e_2 \sqcup e_1 \sqcup e_0$, with e_3 the sink and e_0 the source. The induced $\mathbb{C}^*$-action on the tangent spaces at the fixed points have weights:

- $(-3,-2,-1)$ on $T_{\mathbb{P}^3,e_3}$;
- $(-2,-1,2)$ on $T_{\mathbb{P}^3,e_2}$;
- $(-1,1,2)$ on $T_{\mathbb{P}^3,e_1}$;
- $(1,2,3)$ on $T_{\mathbb{P}^3,e_0}$.

Therefore, by considering the pruning of $\mathbb{P}^3$ along the extremal intervals (or, equivalently, by performing a $\mathbb{C}^*$-equivariant blow-up along e_0, e_3), we obtain that $\mathcal{G}X_\pm = \mathbb{P}(1,2,3)$. The natural birational map $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is therefore a weighted Cremona map.

Lemma 4.2.5. *Let (X,L) be a smooth polarized pair, and consider a $\mathbb{C}^*$-action on (X,L) with extremal isolated points and equalized at sink and source. Assume that the associated birational map ψ is not an isomorphism. Then the criticality of the action is at least* 3.

Proof. We prove that such an action cannot have criticality equal to $1,2$. To this end, suppose the action has criticality 1: we claim that $X=\mathbb{P}^1$. Indeed if by contradiction X is not the projective line, there exists a positive dimensional family of closures of 1-dimensional orbits linking the sink and the source. By the Bend and Break Lemma (see for instance [17, Proposition 3.2]), either this family breaks, and thus there exists another fixed point component, contradicting the criticality 1 assumption, or this family degenerates to a multiple rational curve, which is an absurd since the $\mathbb{C}^*$-action is equalized.

On the other hand, suppose that the $\mathbb{C}^*$-action has criticality 2: since by hypothesis the $\mathbb{C}^*$-action is equalized at the isolated extremal points, it follows that the action is equalized at the inner components. Therefore, considering the pruning $\mathcal{P}(X)_-^+$ at the extremal intervals we get that the the induced $\mathbb{C}^*$-action on $\mathcal{P}(X)_-^+$ is a bordism, and thus by Theorem 4.1.7 the birational map among the extremal components $\psi\colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ is a locally toric flip of Atiyah type. Since by Remark 4.2.3 it holds $\mathcal{G}X_\pm \simeq \mathbb{P}^{\dim X-1}$, we obtain a contradiction. ■

With this in mind, we can now state the main result for $\mathbb{C}^*$-actions on smooth polarized pairs (X,L) of criticality 3.

Theorem 4.2.6. *[57, Theorem 3.5], [49, Theorem 8.1] Let (X, L) be a smooth polarized pair, with X of dimension $n \geq 3$, endowed with a $\mathbb{C}^*$-action of bandwidth three. Assume that its sink and source are isolated points, and that the action is equalized. Denoting by Y_i the inner components, then one of the following holds:*

(1) $X = \mathbb{P}(\mathcal{V}^\vee)$, with $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(3)$, or $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}$, and $L = \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(1)$. Moreover $(Y_i, L_{|Y_i}) \simeq (\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1))$, $i = 1, 2$.

(2) $X = \mathbb{P}^1 \times Q^{n-1}$, $L = \mathcal{O}(1,1)$, each Y_i is the disjoint union of a smooth quadric Q^{n-3} and a point, and $L_{|Q^{n-3}} \simeq \mathcal{O}_{Q^{n-3}}(1)$.

(3) X is one of the following RH varieties:

$$\mathrm{C}_3(3), \ \ \mathrm{A}_5(3), \ \ \mathrm{D}_6(6), \ \ \mathrm{E}_7(7),$$

L is the ample generator of $\mathrm{Pic}(X)$ *and the varieties Y_i are respectively*

$$\mathbb{P}^2, \ \ \mathbb{P}^2 \times \mathbb{P}^2, \ \ \mathrm{A}_5(2), \ \ \mathrm{E}_6(1).$$

The restriction of L to Y_i is the ample generator of $\mathrm{Pic}(Y_i)$*, except in the case $Y_i \simeq \mathbb{P}^2$, in which $L_{|Y_i} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$.*

We present a sketch of the proof of the above Theorem in the case of $\mathrm{Pic}(X) \simeq \mathbb{Z}$, following [51, Sketch, p.10].

Proof. We start noticing that, using [49, Lemma 2.8 (2)], the $\mathrm{Pic}(X) \simeq \mathbb{Z}$ assumption implies that the inner components Y_1, Y_2 are irreducible. Thanks to Remark 4.2.3, it holds that $\mathcal{G}X_\pm = \mathbb{P}(\Omega_{X,Y_\pm}) \simeq \mathbb{P}^{\dim X - 1}$, and thus the birational map among the extremal geometric quotients is a Cremona.

Let $X' := \mathcal{P}(X)^{\rho_+}_{\rho_-}$ be the pruning with respect to the intervals $\rho_\pm \in (a_1, a_2)$. The authors of [51] prove that the sink (resp. the source) of the induced $\mathbb{C}^*$-action on X' is the blow-up $Y'_- := \mathrm{Bl}_{Y_1} \mathbb{P}(\Omega_{X,Y_-})$ (resp. $Y'_+ := \mathrm{Bl}_{Y_2} \mathbb{P}(\Omega_{X,Y_+})$), where by $Y_1 \subset \mathbb{P}(\Omega_{X,Y_-})$ (resp. $Y_2 \subset \mathbb{P}(\Omega_{X,Y_+})$) we mean the set of $\mathbb{C}^*$-invariant curves between Y_- and Y_1 (resp. Y_+ and Y_2).

By Remark 2.3.30, the induced $\mathbb{C}^*$-action on the pruning X' has criticality 1, thus there exists an unique geometric quotient, that is to say, it holds that $\mathrm{Bl}_{Y_1} \mathbb{P}(\Omega_{X,Y_-}) \simeq \mathrm{Bl}_{Y_2} \mathbb{P}(\Omega_{X,Y_+})$. Therefore the birational map among the geometric quotients can be resolved with a smooth blow-up, that is, ψ is *bispecial* (cf. Definition [51, Definition 4.1])

$$\begin{tikzcd}
 & Y'_- \simeq Y'_+ \arrow[dl] \arrow[dr] & \\
\mathcal{G}X_- = \mathbb{P}(\Omega_{X,Y_-}) & & \mathcal{G}X_+ = \mathbb{P}(\Omega_{X,Y_+})
\end{tikzcd}$$

Using an intersection theoretical argument (see [49, Proof of Theorem 8.4]), one may infer that the birational map ψ is a quadro-quadric Cremona. Therefore, since Y_1, Y_2 are irreducible, thanks to [20, Theorem 2.6] the birational map ψ is one the four $(2,2)$-Cremona transformations defined by the linear system of quadrics containing a Severi variety, that is

$$v_2(\mathbb{P}^2) \subset \mathbb{P}^5, \quad \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, \quad \mathrm{A}_5(2) \subset \mathbb{P}^{14}, \quad \mathrm{E}_6(1) \subset \mathbb{P}^{26}.$$

The authors of [51] then conclude by proving that X is uniquely determined by ψ, and then showing that the varieties $\mathrm{C}_3(3)$, $\mathrm{A}_5(3)$, $\mathrm{D}_6(6)$, $\mathrm{E}_7(7)$ satisfy the properties above. ■

Proposition 4.2.7. *[51, p.11] In the situation of Theorem 4.2.6, let* $\tilde{X} \coloneqq \mathcal{P}(X)_-^+$ *be the pruning of* X *with respect to the extremal intervals. Then* $\mathcal{G}X_\pm \simeq \mathbb{P}^{n-1}$*, and the natural birational map* $\psi \colon \mathcal{G}X_- \dashrightarrow \mathcal{G}X_+$ *is either:*

(1) a linear isomorphism;

(2) a quadro-quadric Cremona transformation whose base locus consists of the union of a point and a quadric Q^{n-3}*;*

(3) a bispecial quadro-quadric Cremona transformation.

Summing up, we obtain the following:

Theorem 4.2.8. *[51, Theorem 3.4] Any quadro-quadric Cremona transformation with smooth nonempty fundamental locus admits a geometric realization, given by an equalized* $\mathbb{C}^*$*-action of criticality* 3*, in one of the following varieties:*

$$\mathbb{P}^1 \times Q^{n-1}, \quad C_3(3), \quad A_5(3), \quad D_6(6), \quad E_7(7).$$

Chapter 5

Geometric realization of small modification of dream type

In this chapter we introduce a new class of small modifications, called *of dream type* (see Definition 5.0.1), and we explicitly construct their geometric realizations. Moreover we show that the natural birational map among the geometric quotients of a polarized pair, endowed with a $\mathbb{C}^*$-action which is a bordism and is equalized at the extremal components, is a small modification of dream type.

Definition 5.0.1. Let $Z_\pm$ be normal projective varieties, and let $\varphi\colon Z_- \dashrightarrow Z_+$ be a small modification. The map φ is *of dream type* if there exist A, F Cartier divisors on Z_- such that:

- A is ample;
- up to consider a multiple, it holds that $Z_+ = \operatorname{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0(Z_-, \mathcal{O}_{Z_-}(mF))$;
- the multisection ring $R(Z_-; \mathcal{O}_{Z_-}(A), \mathcal{O}_{Z_-}(F))$ is a finitely generated $\mathbb{C}$-algebra.

We say that (A, F) is a *dream pair*.

Notice that the third condition of the above Definition is equivalent to ask that $\mathcal{C} = \langle A, F\rangle \subset \mathrm{CDiv}(Z_-)_{\mathbb{Q}}$ is a Mori dream region.

5.1 Construction of the geometric realization of a map of dream type

This section is devoted to the proof of the following:

Theorem 5.1.1. *Let Y_- be a normal projective variety, and let $\varphi\colon Y_- \dashrightarrow Y_+$ be a small modification of dream type with dream pair (A, F). Then there exists a geometric realization of φ, whose induced $\mathbb{C}^*$-action is a bordism, equalized at $Y_\pm$.*

The proof of the above Theorem is divided in various results, namely Lemma 5.1.8, Proposition 5.1.10, Lemma 5.1.11 and Corollary 5.1.12.

Notation 5.1.2. We denote by

$$\mathcal{R} \coloneqq R(Y_-; \mathcal{O}_{Y_-}(A), \mathcal{O}_{Y_-}(F)) = \bigoplus_{m_\pm \geq 0} \mathrm{H}^0(Y_-, \mathcal{O}_{Y_-}(m_-A + m_+F))$$

the finitely generated $\mathbb{C}$-algebra associated to the dream pair (A, F).

Since $\mathcal{R}$ admits a $\mathbb{Z}^2$-grading, there exists an induced action of the 2-dimensional torus $H \coloneqq \operatorname{Hom}(\mathbb{Z}(A, F), \mathbb{C}^*)$ on $\operatorname{Spec}\mathcal{R}$, where by $\mathbb{Z}(A, F)$ we mean the free abelian group generated by A, F. Notice that by construction $\mathrm{M}(H) = \mathbb{Z}(A, F)$.

Definition 5.1.3. Given $\alpha \in \mathrm{M}(H)^\vee$, we say that α is *admissible* if

$$\alpha_- \coloneqq \alpha(A) > 0, \quad \alpha_+ \coloneqq \alpha(F) > 0, \quad \gcd(\alpha_-, \alpha_+) = 1.$$

We denote by H^α the 1-dimensional subtorus of H associated to α.

Remark 5.1.4. For any admissible α, the 1-dimensional torus H^α acts on $\operatorname{Spec}\mathcal{R}$, thus inducing an $\mathbb{N}$-grading of $\mathcal{R}$. The $\mathbb{C}$-algebra $\mathcal{R}$, endowed with such $\mathbb{N}$-grading, will be denoted by $\mathcal{R}^\alpha$. It holds:

$$\mathcal{R}^\alpha \coloneqq \bigoplus_{m \geq 0} \mathcal{R}^\alpha_m, \quad \text{where} \quad \mathcal{R}^\alpha_m \coloneqq \bigoplus_{\substack{m_\pm \in \mathbb{Z}_{\geq 0} \\ \alpha(m_- A + m_+ F) = m}} \mathrm{H}^0(Y_-, \mathcal{O}_{Y_-}(m_- A + m_+ F));$$

Definition 5.1.5. The $\mathbb{N}$-graded algebra $\mathcal{R}^\alpha$ is finitely generated by assumption, so we may define $X^\alpha \coloneqq \operatorname{Proj}\mathcal{R}^\alpha$.

The variety X^α, as we will prove, is a geometric realization of the small modification $\varphi \colon Y_- \dashrightarrow Y_+$.

Remark 5.1.6. For any admissible α, with associated $H^\alpha \subset H$, we may consider the 1-dimensional torus:

$$T \coloneqq H/H^\alpha.$$

Since the H-action on $\mathcal{R}^\alpha$ induces an H-action on X^α, whose kernel is precisely H^α, we obtain that T acts on X^α.

Definition 5.1.7. For any admissible α, let P^α be the $\mathbb{P}^1$-bundle on Y_- defined as

$$P^\alpha \coloneqq \mathbb{P}_{Y_-}(\mathcal{O}_{Y_-}(\alpha_+ A) \oplus \mathcal{O}_{Y_-}(\alpha_- F)).$$

We denote by $s_-(Y_-), s_+(Y_-)$ the sections of P^α over Y_- corresponding respectively to the projections of $\mathcal{O}_{Y_-}(\alpha_+ A) \oplus \mathcal{O}_{Y_-}(\alpha_- F) \to \mathcal{O}_{Y_-}(\alpha_+ A)$ and $\mathcal{O}_{Y_-}(\alpha_+ A) \oplus \mathcal{O}_{Y_-}(\alpha_- F) \to \mathcal{O}_{Y_-}(\alpha_- F)$.

Lemma 5.1.8. *The varieties P^α and X^α are birational. Moreover, X^α is normal.*

Proof. Let us consider the ring of sections of the tautological line bundle $\mathcal{O}_{P^\alpha}(1)$. We have:

$$\begin{aligned} R(P^\alpha; \mathcal{O}_{P^\alpha}(1)) &= \bigoplus_{m \geq 0} \bigoplus_{m_- + m_+ = m} \mathrm{H}^0(Y_-, \mathcal{O}_{Y_-}(m_- \alpha_+ A + m_+ \alpha_- F)) \\ &= \bigoplus_{m \geq 0} \bigoplus_{\alpha(m_- \alpha_+ A + m_+ \alpha_- F) = m\alpha_- \alpha_+} \mathrm{H}^0(Y_-, \mathcal{O}_{Y_-}(m_- \alpha_+ A + m_+ \alpha_- F)) \\ &= \bigoplus_{m \geq 0} \bigoplus_{\alpha(m_- A + m_+ F) = m\alpha_- \alpha_+} \mathrm{H}^0(Y_-, \mathcal{O}_{Y_-}(m_- A + m_+ F)), \end{aligned}$$

where the last equality follows from the fact that α_-, α_+ are coprime. Notice that the latter algebra is the $(\alpha_- \alpha_+)$-Veronese of $\mathcal{R}^\alpha$, which is finitely generated. It thus follows that $R(P^\alpha; \mathcal{O}_{P^\alpha}(1))$ is finitely generated, and that

$$\operatorname{Proj}(R(P^\alpha; \mathcal{O}_{P^\alpha}(1))) \simeq X^\alpha.$$

In particular, by [17, Lemma 7.10 (a)] we obtain that X^α is normal. Furthermore, the line bundle $\mathcal{O}_{P^\alpha}(1)$ is big (cf. [42, Example 6.1.23]), hence the associated map $\Phi := \phi_{|\mathcal{O}_{P^\alpha}(1)|} \colon P^\alpha \dashrightarrow X^\alpha$ is birational. ■

Remark 5.1.9. Notice that the $\mathbb{P}^1$-bundle P^α admits a natural equalized $\mathbb{C}^*$-action with fixed point locus $s_-(Y_-) \sqcup s_+(Y_-)$. Moreover the birational map $\Phi \colon P^\alpha \dashrightarrow X^\alpha$ introduced in Lemma 5.1.8 is $\mathbb{C}^*$-equivariant (cf. [49, Remark 4.2]).

Proposition 5.1.10. *The action of T on X^α is of B-type with sink and source Y_- and Y_+, respectively, and the induced natural birational map $\psi \colon Y_- \dashrightarrow Y_+$ coincides with the small modification $\varphi \colon Y_- \dashrightarrow Y_+$.*

Proof. Consider the birational map $\Phi \colon P^\alpha \dashrightarrow X^\alpha$ introduced in Lemma 5.1.8. We first prove that the indeterminacy locus of Φ is contained in $s_+(Y_-)$. We recall that A is ample on Y_-, thus $\Phi_{|s_-(Y_-)}$ is well defined; since by Remark 5.1.9 we know that Φ is $\mathbb{C}^*$-equivariant, it follows that the indeterminacy locus of Φ is $\mathbb{C}^*$-invariant, hence our claim. Therefore, $\Phi_{|P^\alpha \setminus s_+(Y_-)} \colon P^\alpha \setminus s_+(Y_-) \to \mathcal{U}_- \subset X^\alpha$ is an isomorphism where $\mathcal{U}_-$ is a T-invariant neighborhood of the sink of the action on X^α; it follows that the sink of the T-action on X^α is $s_-(Y_-) \simeq Y_-$ and is isomorphic to the first geometric quotient of such action.

In order to conclude that the T-action on X^α is of B-type, we study a T-invariant neighborhood $\mathcal{U}_+$ of the source of X^α. To do so, we consider the $\mathbb{P}^1$-bundle $\widetilde{P^\alpha} := \mathbb{P}_{Y_+}(\mathcal{O}_{Y_+}(\alpha_+ A) \oplus \mathcal{O}_{Y_+}(\alpha_- F))$ on Y_+ and show, in a similar way as above, that we can find a neighborhood $\mathcal{U}_+$ isomorphic to the complement of a section of $\widetilde{P^\alpha}$. On the other hand, using the arguments above and the construction of the natural birational map ψ among the extremal geometric quotients, it follows that ψ coincides with the small modification φ associated to the dream pair (A, F). ■

Lemma 5.1.11. *The T-action on X^α is a bordism.*

Proof. Thanks to [17, Lemma 7.10], there exists an open subset U of X^α, whose complement has codimension greater or equal than 2, and an open subset V of P^α on which the $\mathbb{C}^*$-equivariant birational map $\Phi_{|V}$ is an isomorphism.

We know that the T-action on X^α is of B-type by Proposition 5.1.10; in order to prove that is a bordism is sufficient to show that the only T-invariant divisors in X^α that are not extensions of divisors in Y_- are the sink and the source of the action (cf. Lemma 2.3.18). Let D be an T-invariant prime divisor on X^α that is not an extension of a divisor in Y_-. Since its intersection with U is nonempty, we may consider its strict transform $\overline{D}$ into P^α, that will be an T-invariant divisor. Then $\overline{D}$ coincides with the sink or the source of P^α, and this implies that D is the sink or the source of X^α. ■

The following concludes the proof of Theorem 5.1.1.

Corollary 5.1.12. *The T-action on X^α is equalized at the sink Y_- and the source Y_+.*

Proof. It suffices to notice that thanks to Remark 5.1.9 the $\mathbb{C}^*$-action on P^α is equalized at the sink and the source and the birational map $\Phi \colon P^\alpha \dashrightarrow X^\alpha$ is $\mathbb{C}^*$-equivariant. ■

Remark 5.1.13. Our construction of a geometric realization depends on the choice of an admissible 1-parameter subgroup $\alpha \in \mathrm{M}(H)^\vee$. Given another admissible $\beta \in \mathrm{M}(H)^\vee$, the geometric realizations X^α and X^β are birational. Indeed it suffices to notice that the $\mathbb{P}^1$-bundles P^α, P^β are by definition $\mathbb{C}^*$-equivariantly birationally equivalent. Moreover the geometric quotients of X^α are independent of the choice of α.

5.2 The natural birational map for bordisms $\mathbb{C}^*$-actions equalized at $Y_\pm$

In this section we aim to characterize the natural birational map induced by a $\mathbb{C}^*$-action which is a bordism equalized at the extremal components. For the sake of simplicity, in this section we will use the same notation for Cartier divisors and their associated invertible sheaves. The main result of this section is the following:

Theorem 5.2.1. *Let (X, L) be a polarized pair, with X a $\mathbb{Q}$-factorial variety, and consider a normalized and faithful $\mathbb{C}^*$-action which is a bordism, equalized at $Y_\pm$. Then the natural birational map $\psi\colon Y_- \dashrightarrow Y_+$ is of dream type, whose dream pair is (L_-, L_+), where $L_- := L|_{Y_-}, L_+ := L|_{Y_-} - \delta Y_-|_{Y_-}$.*

To this end, we first prove some auxiliary results under the following:

Set-up 5.2.2. Let (X, L) be a polarized pair endowed with a faithful $\mathbb{C}^*$-action, where X is a normal and $\mathbb{Q}$-factorial projective variety, and L is an ample line bundle. Suppose that the $\mathbb{C}^*$-action is a bordism and it is equalized at the sink and the source.

Recall that, being X a bordism, it is in particular of B-type hence $Y_\pm \simeq \mathcal{G}X_\pm$. We now prove a slightly different version of a result stated in [48, Lemma 2.5]:

Lemma 5.2.3. *In the situation of Set-up 5.2.2, let $\tau_\pm \in \mathbb{Q}$ be two rational numbers such that $0 \leq \tau_- \leq \tau_+ \leq \delta$, and $m \in \mathbb{Z}_{>0}$ such that $m\tau_\pm \in \mathbb{Z}$. It holds that:*

$$\bigoplus_{k=m\tau_-}^{m\tau_+} \mathrm{H}^0(X, mL)_k = \mathrm{H}^0(X, mL - m\tau_- Y_- - (m\delta - m\tau_+)Y_+).$$

Proof. Let us denote $W := \mathrm{H}^0(X, mL - m\tau_- Y_- - (m\delta - m\tau_+)Y_+) \subset \mathrm{H}^0(X, mL)$. Note first that W is $\mathbb{C}^*$-invariant, therefore, $W = \bigoplus_k (\mathrm{H}^0(X, mL)_k \cap W)$.

We will use [47, Corollary 2.4] (which follows from [10, Lemma 2.17]), which determines the multiplicity of the $\mathbb{C}^*$-invariant sections of $\mathrm{H}^0(X, mL)$ at the extremal fixed point components of the action. We note first that the proof of this result requires only the smoothness of the variety at the general points of $Y_\pm$, and this condition holds in our situation, because X is normal and the action is of B-type. The quoted Corollary tells us that a nonzero section $s \in \mathrm{H}^0(X, mL)_u$ vanishes with multiplicity precisely equal to u at Y_- and $m\delta - u$ at Y_+. This implies that $\mathrm{H}^0(X, mL)_u \subset W$ if $u \in [m\tau_-, m\tau_+]$ and $\mathrm{H}^0(X, mL)_u \cap W = 0$ if $u \notin [m\tau_-, m\tau_+]$, and the claimed equality follows. ■

We will show in Lemma 5.2.5 that there exists an isomorphism between global sections of mL of weight c, for $c = 0, \dots, m\delta$, and global sections of $mL - cY_-$ restricted to Y_-. To this end, we first prove the following:

Lemma 5.2.4. *In the situation of Set-up 5.2.2, the sink Y_- and the source Y_+ are $\mathbb{Q}$-factorial.*

Proof. We prove the result for Y_-; a similar proof works in the case of Y_+. Let D be a prime divisor in Y_-, and consider its extension $e_-(D) \in \operatorname{Div}(X)$ (cf. Definition 2.3.15). By definition it is equal to the closure $\overline{\pi_-^{-1}(D)} \subset X$, where $\pi_- : X_-^s \to Y_-$ is the quotient map (see Notation 2.2.25). The fact that the $\mathbb{C}^*$-action is of B-type implies that $e_-(D)$ can also be written as $\pi_-^{-1}(D)$ with $\pi_- : X^-(Y_-) \to Y_-$ (cf. Remark 2.2.27). Then it follows that $e_-(D) \cap Y_- = D$. Since X is $\mathbb{Q}$-factorial, there exists $m \in \mathbb{Z}_{>0}$ such that $me_-(D) = e_-(mD)$ is Cartier, and so $mD = me_-(D) \cap Y_-$ is Cartier, as well. ■

Lemma 5.2.5. *In the situation of Set-up 5.2.2, there exists a positive integer m_- such that for $m \geq m_-$ and every $c \in [0, \dots, m\delta] \cap \mathbb{Z}$ it holds that:*

$$\mathrm{H}^0(X, mL)_c \simeq \mathrm{H}^0(Y_-, mL_{|Y_-} - cY_{-|Y_-}).$$

Proof. Let us first note that, by Lemma 5.2.3 it follows that, for every $m \in \mathbb{Z}_{>0}$, and every $c \in \mathbb{Z}_{\geq 0}$, $c \leq m\delta$, we have a commutative diagram with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bigoplus_{k=c+1}^{m\delta} \mathrm{H}^0(X, mL)_k \arrow[r, "\simeq"] \arrow[d] & \mathrm{H}^0(X, mL-(c+1)Y_-) \arrow[d] \\
\bigoplus_{k=c}^{m\delta} \mathrm{H}^0(X, mL)_k \arrow[r, "\simeq"] \arrow[d] & \mathrm{H}^0(X, mL-cY_-) \arrow[d] \\
\mathrm{H}^0(X, mL)_c \arrow[r] \arrow[d] & \mathrm{H}^0(Y_-, mL_{|Y_-} - cY_{-|Y_-}) \\
0 &
\end{tikzcd}$$

It is then enough to show that there exists m_- such that for every $m \geq m_-$, and every $c = 0, \dots, m\delta$ the restriction map $\mathrm{H}^0(X, mL - cY_-) \to \mathrm{H}^0(Y_-, (mL - cY_-)_{|Y_-})$ is surjective, or, equivalently, that the rational map

$$|mL - cY_-| \dashrightarrow |(mL - cY_-)_{|Y_-}|$$

is surjective.

We start by claiming that there exists m_- such that for every $m \geq m_-$, $\mathrm{H}^0(X, mL)_c \neq 0$ for every $c \in [0, \dots, m\delta] \cap \mathbb{Z}$. In fact, let C be the closure of the general $\mathbb{C}^*$-orbit in X, which has extremal fixed points in Y_-, Y_+, respectively. The generality assumption implies that the $\mathbb{C}^*$-action on C is faithful, that its extremal points are smooth points of $Y_\pm$ and, by the Białynicki-Birula decomposition, that C is isomorphic to $\mathbb{P}^1$. By Serre vanishing, there exists an integer m_- such that for every $m \geq m_-$ the restriction map:

$$\mathrm{H}^0(X, mL) \to \mathrm{H}^0(C, mL_{|C})$$

is surjective and $\mathbb{C}^*$-equivariant. Since, by [57, Corollary 3.2], $\mathrm{H}^0(C, mL_{|C}) \simeq \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m\delta))$, and since the set of weights of the induced $\mathbb{C}^*$-action on the vector space $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m\delta))$ is $[0, m\delta] \cap \mathbb{Z}$, the claim follows.

Putting it together with the commutative diagram above, the claim implies that the map $\mathrm{H}^0(X, mL-(c+1)Y_-) \to \mathrm{H}^0(X, mL-cY_-)$ is not surjective for $m \geq m_-$ and every $c \in [0, m\delta] \cap \mathbb{Z}$, that is, we have a strict inclusion:

$$|mL - (c+1)Y_-| + Y_- \subsetneq |mL - cY_-|. \tag{5.1}$$

In particular (since by Lemma 2.3.18 the projective space $|mL - cY_-|$ is spanned by $\mathbb{C}^*$-invariant elements), there exists a $\mathbb{C}^*$-invariant effective divisor $D_1 \in |mL - cY_-|$ whose support does not contain Y_-. Using Lemma 2.3.18 It follows that

$$D_1 = e_-(D_1') + aY_+ \text{ for some } a \geq 0 \text{ and some } D_1' \in \mathrm{Div}(Y_-);$$

here we are denoting by $e_-\colon \mathrm{Div}(Y_-)=\mathrm{Div}(\mathcal{G}X_-)\to\mathrm{Div}(X)$ the extension map of divisors introduced in Definition 2.3.15. By restricting the above equality to Y_- we get that $D_{1|Y_-}=D_1'$, hence

$$D_1=e_-(D_{1|Y_-})+aY_+\text{ for some }a\geq 0.$$

Let us now conclude the proof of the statement by showing that the restriction map $|mL-cY_-|\dashrightarrow|(mL-cY_-)_{|Y_-}|$ is surjective. Given $D'\in|(mL-cY_-)_{|Y_-}|$, Lemma 2.3.17 tells us that $e_-(D')\sim e_-(D_{1|Y_-})$ which we have proven to be linearly equivalent to D_1-aY_+, for some $a\geq 0$. It then follows that $(e_-(D')+aY_+)_{|Y_-}=D'$, and $e_-(D')+aY_+\in|D|$.

■

Remark 5.2.6. With a similar proof, one may show that, in the situation of Set-up 5.2.2, there exists a positive integer m_+ such that for any $m\geq m_+$, and for any $c\in[0,\dots,m\delta]$, it holds that

$$\mathrm{H}^0(X,mL)_c\simeq\mathrm{H}^0(Y_+,mL_{|Y_+}-(m\delta-c)Y_{+|Y_+}).$$

Notice that, since the $\mathbb{C}^*$-action a bordism and since $Y_\pm$ are $\mathbb{Q}$-factorial by Lemma 5.2.4, the natural birational map $\psi\colon Y_-\dashrightarrow Y_+$ is an SQM.

We can now prove Theorem 5.2.1:

Proof. (of Theorem 5.2.1). We show each condition of Definition 5.0.1 is satisfied. Notice that $L_\pm$ are effective, and L_- is ample. Using Lemma 5.2.5 and Remark 5.2.6, there exists a positive integer $m_0\geq m_\pm$ such that, for any $m\geq m_0$ and any $c\in[0,\dots,m\delta]\cap\mathbb{Z}$, it holds that

$$\mathrm{H}^0(Y_-,mL_{|Y_-}-cY_{-|Y_-})\simeq\mathrm{H}^0(X,mL)_c\simeq\mathrm{H}^0(Y_+,mL_{|Y_+}-(m\delta-c)Y_{+|Y_+}).$$

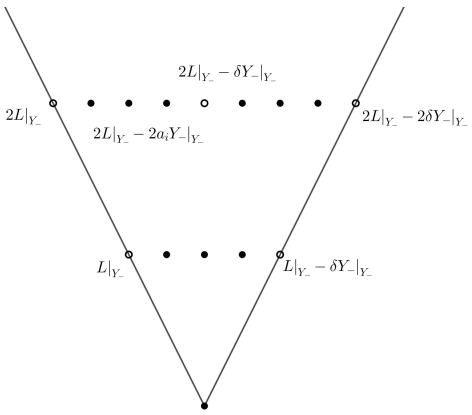
Let $d\geq m_0$ be a positive integer, and consider the d-Veronese algebras $R(Y_\pm,L_{|Y_\pm})^{(d)}$, which are still finitely generated. Using the above identity, it is readily seen that $Y_+=\operatorname{Proj}R(Y_+,L_{|Y_+})^{(d)}\simeq\operatorname{Proj}R(Y_-,L_+)^{(d)}$. It remains to show that $R(Y_-;L_-,L_+)$ is finitely generated. Consider the d-Veronese algebra $R(X;L)^{(d)}$, which is still finitely generated being L ample and X projective. By using Lemma 5.2.5, we know that

$$\bigoplus_{m\geq 0}\bigoplus_{k=0}^{md\delta}\mathrm{H}^0(X,mL)_k\simeq\bigoplus_{m\geq 0}\bigoplus_{k=0}^{md\delta}\mathrm{H}^0(Y_-,mL_{|Y_-}-kY_{-|Y_-})$$

is finitely generated. Notice that we may rewrite the latter algebra using $L_\pm$, thus obtaining that

$$R(X;L)^{(d)}\simeq\bigoplus_{(a,b)\in S}\mathrm{H}^0(Y_-,aL_{|Y_-}+b(L_{|Y_-}-\delta Y_{-|Y_-})),$$

where S denotes the monoid $\frac{1}{d\delta}(\mathbb{Z}_{\geq 0})^{\oplus 2}\subset\mathbb{Q}^{\oplus 2}$. We may represent this situation by means of the following image, where the black dots belong to S, and the empty ones to $\mathbb{N}^{\oplus 2}\subset S$:



Therefore, using [12, Lemma 2.25] (see also [2, Propositions 1.2.2, 1.2.4]), we conclude that the algebra

$$R(Y_-; L_-, L_+) = \bigoplus_{a,b \in \mathbb{N}^{\oplus 2}} \mathrm{H}^0(Y_-, aL_{|Y_-} + b(L_{|Y_-} - \delta Y_{-|Y_-}))$$

associated to the cone $\mathcal{C} = \langle L_-, L_+ \rangle$ is finitely generated. ■

We conclude this section by showing that the Mori dream region $\mathcal{C} = \langle L_-, L_+ \rangle$ obtained in Theorem 5.2.1 admits a chamber decomposition, which is induced by the $\mathbb{C}^*$-action on the polarized pair (X, L). The decomposition of Mori dream regions, which reproduces the behaviour of Mori dream spaces, has been stated by [25, Definition 2.12]: we refer to [31, Theorem 4.3] and [52, Proposition 9.6] for the precise statements.

Definition 5.2.7. Let X be a normal projective variety, and let $\mathcal{C}$ be a rational polyhedral cone in $\mathrm{CDiv}(X)_\mathbb{Q}$. We say that $\mathcal{C}$ is a *chamber* if, for any $D_1, D_2 \in \mathcal{C}$ with finitely generated section ring, it holds that $\operatorname{Proj} R(X; D_1) \simeq \operatorname{Proj} R(X; D_2)$. We call the variety $\operatorname{Proj} R(X; D_1)$ the *chamber model* of $\mathcal{C}$.

Theorem 5.2.8. *In the situation of Set-up 5.2.2, the cone $\mathcal{C} = \langle L_-, L_+ \rangle$ admits a subdivision*

$$\mathcal{C} = \bigcup_{i=0}^{r-1} \mathcal{C}_i, \quad \mathcal{C}_i = \langle L_{|Y_-} - a_iY_{-|Y_-}, L_{|Y_-} - a_{i+1}Y_{-|Y_-} \rangle.$$

Moreover, for every $i = 0, \dots, r-1$ the cone $\mathcal{C}_i$ is a chamber whose model is $\mathcal{G}X_i$.

Proof. The existence of such a subdivision follows immediately by recalling that $a_i < a_{i+1}$ for every $i = 0, \dots, r-1$. In order to conclude, it suffices to show that, for every $i = 0, \dots, r-1$, the cone $\mathcal{C}_i$ is a chamber. Let $D = \beta(L_{|Y_-} - a_iY_{-|Y_-}) + \gamma(L_{|Y_-} - a_{i+1}Y_{-|Y_-})$ be a divisor in $\mathcal{C}_i$, where $\beta, \gamma \in \mathbb{Q}_{>0}$. Let q be a positive integer such that $q\beta, q\gamma \in \mathbb{N}$ and $q \geq m_-$, with m_- as in Lemma 5.2.5. Using again Lemma 5.2.5 we obtain that

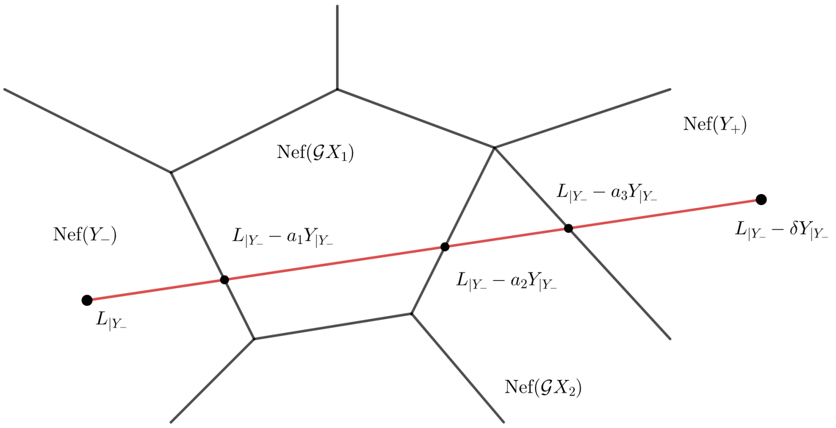
$$\mathrm{H}^0(Y_-, qD) \simeq \mathrm{H}^0(X, q(\beta+\gamma)L)_{q(\beta a_i + \gamma a_{i+1})},$$

and since $q\beta a_i + q\gamma a_{i+1} \in (q(\beta+\gamma)a_i, q(\beta+\gamma)a_{i+1})$, using Theorem 2.2.30 and the above isomorphism, it holds that

$$\operatorname{Proj} R(Y_-; qD) \simeq \operatorname{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(X, mq(\beta+\gamma)L)_{mq(\beta a_i + \gamma a_{i+1})} \simeq \mathcal{G}X_i.$$

■

We may represent Theorem 5.2.8 by means of the following picture, in the case of a $\mathbb{C}^*$-action of criticality 3.



Chapter 6

Geometric realization of toric SQM

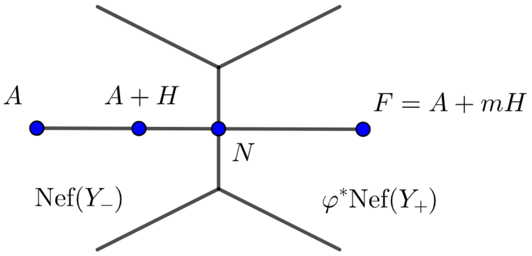
As one could expect, starting with a toric SQM φ among toric normal $\mathbb{Q}$-factorial projective varieties, one may realize φ geometrically with a toric variety endowed with a particular $\mathbb{C}^*$-action. It makes then sense to describe a toric version, written in combinatorial fashion, of our construction of a geometric realization presented in Chapter 5. Moreover, we present a function called `GeomReal`, written in `SageMath`, to compute a polytope associated to such toric geometric realization. We conclude the chapter by presenting some examples of toric geometric realizations of toric SQM's, which highlight some interesting features of this construction.

6.1 Alternative construction of a geometric realization for small modifications of dream type

Set-up 6.1.1. Let $\varphi\colon Y_- \dashrightarrow Y_+$ be a small modification of dream type, whose associated dream pair is (A', F').

We recall that we may identify $\mathrm{Div}(Y_-) \simeq \mathrm{Div}(Y_+)$. Set $H \coloneqq F' - A'$. For $m \gg 0$, the divisor $A' + H/m$ is still ample on Y_-. Set $A \coloneqq mA'$, $F \coloneqq mF'$, so that $A + mH = F$.

Example 6.1.2. For instance, in the case $\mathrm{Nef}(Y_-), \varphi^*\mathrm{Nef}(Y_+)$ share a common $(\rho_{Y_-} - 1)$-dimensional wall, we may represent the situation by means of the following picture



Let $\pi\colon W = \mathbb{P}(\mathcal{E}) \to Y_-$, with $\mathcal{E} = \mathcal{O}_{Y_-}(A) \oplus \mathcal{O}_{Y_-}(A + H)$ be a $\mathbb{P}^1$-bundle over Y_-. Call D_-, D_+ respectively the images of the sections s_-, s_+ associated to the quotients $\mathcal{E} \to \mathcal{O}_{Y_-}(A)$,

$\mathcal{E} \to \mathcal{O}_{Y_-}(A+H)$. Consider the line bundle $\mathcal{O}_W(1) \otimes \mathcal{O}_W((m-1)D_+)$: by [41, Lemma 2.3.2 (ii)], $\mathcal{O}_W(1)$ is ample and $\mathcal{O}_W((m-1)D_+)$ is effective, thus the rational map associated to it

$$\Phi = \Phi_{|\mathcal{O}_W(1) \otimes \mathcal{O}_W((m-1)D_+)|} \colon W \dashrightarrow X$$

is birational onto the image X.

Lemma 6.1.3. *It holds that* $\mathcal{O}_W(1) \otimes \mathcal{O}_W((m-1)D_+) \sim \mathcal{O}_W(m) \otimes \pi^*\mathcal{O}_{Y_-}((1-m)A)$.

Proof. We have that $\mathrm{H}^0(Y_-, \mathcal{E} \otimes \mathcal{O}_{Y_-}(-A)) \simeq \mathrm{H}^0(W, \mathcal{O}_W(1) \otimes \pi^*(\mathcal{O}_{Y_-}(-A)))$. Therefore $\mathcal{O}_W(1) \otimes \mathcal{O}_W((m-1)D_+) \sim \mathcal{O}_W(m) \otimes \pi^*\mathcal{O}_{Y_-}((1-m)A)$. ■

Lemma 6.1.4. *The variety X is normal.*

Proof. Thanks to [17, Lemma 7.10 (a)], it suffices to show that the section ring $R(W; \mathcal{O}_W(1) \otimes (m-1)D_+)$ is finitely generated. Using the projection formula, we obtain that

$$\begin{aligned}
&\bigoplus_{n \geq 0} \mathrm{H}^0\left(W, \mathcal{O}_W(nm) \otimes \pi^*(\mathcal{O}_{Y_-}(n(1-m)A))\right) = \\
&\bigoplus_{n \geq 0} \mathrm{H}^0\left(Y_-, S^{nm}(\mathcal{O}_{Y_-}(A) \oplus \mathcal{O}_{Y_-}(A+H)) \otimes \mathcal{O}_{Y_-}((n-nm)A)\right) = \\
&\bigoplus_{n \geq 0} \mathrm{H}^0\left(Y_-, \mathcal{O}_{Y_-}((n-nm)A) \otimes \bigoplus_{i=0}^{nm} \mathcal{O}_{Y_-}((nm-i)A + i(A+H))\right) = \\
&\bigoplus_{n \geq 0} \mathrm{H}^0\left(Y_-, \mathcal{O}_{Y_-}(nA) \otimes \bigoplus_{i=0}^{nm} \mathcal{O}_{Y_-}(iH)\right).
\end{aligned}$$

Notice that the latter can be described as $Q \coloneqq \bigoplus_{0 \leq b \leq ma} \mathrm{H}^0(Y_-, \mathcal{O}_{Y_-}(aA + bH))$, which is a subalgebra of the multisection ring $R(Y_-; \mathcal{O}_{Y_-}(A), \mathcal{O}_{Y_-}(F))$, which is finitely generated since $\mathcal{C} = \langle A, F \rangle$ is a Mori dream region. Using [12, Lemma 2.25], we obtain that Q is finitely generated, hence we conclude. ■

Proposition 6.1.5. *The variety X is a geometric realization of the SQM $\varphi \colon Y_- \dashrightarrow Y_+$.*

Proof. By Remark 5.1.9, there exists a natural $\mathbb{C}^*$-action on W with sink and source respectively $D_\pm$. Let us study the images $\Phi(D_\pm)$. For the sake of notation, set $E \coloneqq \mathcal{O}_W(1) \otimes \mathcal{O}_W((m-1)D_+) = \mathcal{O}_W(m) \otimes \pi^*\mathcal{O}_{Y_-}((1-m)A)$. We obtain that the image of D_- via Φ is given by $E_{|D_-} = \mathcal{O}_{Y_-}(A)$, and for D_+ is given by $E_{|D_+} = \mathcal{O}_{Y_-}(A+mH) = \mathcal{O}_{Y_-}(F)$. Moreover, the map Φ is $\mathbb{C}^*$-equivariant, with sink $Y_- = \operatorname{Proj} R(Y_-, \mathcal{O}_{Y_-}(A))$ and source $Y_+ \simeq \operatorname{Proj} R(Y_-, \mathcal{O}_{Y_-}(F))$. ■

Let us notice that the construction of geometric realization presented here slightly differs from the one given in Section 5.1, as explained in the following:

Remark 6.1.6. Arguing as in Section 5.1, let $\widetilde{X}$ be the geometric realization constructed as the image of the $\mathbb{P}^1$-bundle $P = \mathbb{P}(\alpha_+\mathcal{O}_{Y_-}(A) \oplus \alpha_-\mathcal{O}_{Y_-}(F))$ under $\Phi' = \Phi'_{|\mathcal{O}_P(1)|}$. Using the proof of Proposition 5.1.10, it holds that $\Phi'_{|P^-(s_-(Y_-))} \simeq \widetilde{X}^-(Y_-), \Phi_{|W^-(s_-(Y_-))} \simeq X^-(Y_-)$, and thus $\mathcal{N}_{\Phi'(Y_-)|\widetilde{X}} \simeq \mathcal{N}_{Y_-|P}, \mathcal{N}_{\Phi(Y_-)|X} \simeq \mathcal{N}_{Y_-|W}$. Since $\mathcal{N}_{Y_-|P} = \mathcal{O}_{Y_-}(\alpha_+A - \alpha_-F)$ and $\mathcal{N}_{\Phi(Y_-)|X} = \mathcal{O}_{Y_-}(H)$, we conclude.

6.2 `SageMath` code

We keep the notation of the previous section. Notice that, if we assume in addition that $\varphi\colon Y_- \dashrightarrow Y_+$ is a toric SQM among toric varieties, then the above construction may be described also in terms of the associated polytopes. Indeed Let P_A, P_{A+H}, P_F and P_W be respectively the polytopes associated to the pairs $(Y_-, \mathcal{O}_{Y_-}(A)), (Y_-, \mathcal{O}_{Y_-}(A+H)), (Y_-, \mathcal{O}_{Y_-}(F))$, and $(W, \mathcal{O}_W(1))$. Then $P_W = P_A \star P_{A+H}$, where by $\star$ we denote the Cayley sum of the two polytopes, that is P_W is the convex hull of $(P_A \times \{0\}) \cup (P_{A+H} \times \{1\})$. Moreover $\mathcal{O}_W(1)$ and $\mathcal{O}_W((m-1)D_+)$ are T-invariant, and so is the linear system $|\mathcal{O}_W(1) \otimes \mathcal{O}_W((m-1)D_+)|$. Hence the geometric realization X, that is the image of W under such linear system, is toric.

In this section we present and explain the code to compute the polytope associated to such toric geometric realization using a `SageMath` function called `GeomReal`.

Remark 6.2.1. In principle, given a toric small modification of dream type $\varphi\colon Y_- \dashrightarrow Y_+$ among normal, $\mathbb{Q}$-factorial toric projective varieties, with dream pair (A, F), it would be natural to construct a toric geometric realization X of φ by considering the variety associated to the polytope of the Cayley sum $P_A \star P_F$ between P_A and P_F. The resulting variety however will be quite singular, while with our construction the fan of the geometric realization is usually simplicial, that is the geometric realization is $\mathbb{Q}$-factorial.

Algorithm (Geometric realization of toric SQM).

- Input: Rays of the fan Σ_{Y_-} of Y_-, an ample Cartier divisor A on Y_-, a Cartier divisor H on Y_- and a positive integer k.
- Output: Polytope of the geometric realization X associated to the SQM $\varphi\colon Y_- \dashrightarrow \operatorname{Proj} R(Y_-; \mathcal{O}_{Y_-}(A + kH))$.

The case we will be interested in is $k = m - 1$, so that $A + mH = F$.

Remark 6.2.2. Let X be a toric variety with $\Sigma(1) = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^h$, and let $D = \sum_{i=1}^n a_i D_{v_i}$ be a T-invariant $\mathbb{Q}$-Cartier divisor on X. In `SageMath`, we may represent D as a string $D = [a_1, \ldots, a_n]$. The polytope P_D associated to the pair $(X, \mathcal{O}_X(D))$ can be described as a set of inequalities of the form, for $m = (m_1, \ldots, m_h)$,

$$\begin{cases} v_1 m + a_1 \geq 0 \\ \vdots \\ v_n m + a_n \geq 0 \end{cases}$$

Let us recall that $\mathrm{H}^0(X, \mathcal{O}_X(D)) \simeq \bigoplus_{m \in P_D \cap \mathrm{M}} x^m$ (see [16, Proposition 4.3.2]).

The function

```
def GeomReal(rays ,A,H,k):
```

is constructed upon several functions; we describe each of them. For the sake of notation, we explain the code using a general toric variety K, and then illustrate how we apply it to our case.

```
def poly(rays2 , D) :
        ieq=transpose(matrix(rays2)).rows()
        ieq[0:0]= matrix(D)
        ieq=matrix(ieq).transpose().rows()
        p = Polyhedron(ieqs = ieq)
        return p
```

The function `poly(rays2,D)` constructs, given a collection of rays `rays2` of a fan Σ_K of a toric variety K and a $|\Sigma_K(1)|$-tuple $D = [d_1, \dots, d_n]$, which essentially corresponds to the string of coefficients of Equation 6.2.2, the polytope associated to the pair $(K, \mathcal{O}_K(D))$. In particular, `poly(rays,A)` returns the ample polytope of $(Y_-, \mathcal{O}_{Z_-}(A))$.

```
def fd(p,a) :
        c_list =[]
        q = p.vertices_list()
        M = matrix(q).transpose().rows()
        for i in range(len(q)) :
                c_list.append(a)
        M.append(c_list)
        M = matrix(M)
        return M
```

Given a polytope P and an integer a, the function `fd(p,a)` constructs an $m \times n$-matrix whose last row is equal to $(a, \dots, a)$, and whose first $(m-1)$-rows are the transpose of the matrix of the vertices of P.

```
def plus(D,E):
        D=vector(D)
        E=vector(E)
        DE=vector(D+E).list()
        return DE
```

The function `plus(D,E)` computes the sum of two lists. We will use it to compute $A + H =$`plus(A,H)`.

```
def pb(rays3,D,E) :
        w=len(D)
        p=poly(rays3,D)
        q=poly(rays3,E)
        P=fd(p,0)
        Q=fd(q,1)
        P1=P.transpose().rows()
        Q1=Q.transpose().rows()
        P1[w:w]=Q1
        C=matrix(P1).transpose()
        M=C.transpose().rows()
        pb=Polyhedron(vertices=M)
        return pb
```

Given two ample divisors D, E on a toric variety K such that $\Sigma_K(1) =$ `rays3`, the function `pb(rays3,D,E)` returns a polytope associated to the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_K(D) \oplus \mathcal{O}_K(E))$, computed as the Cayley sum $D \star E$, that is the convex hull of $(D \times \{0\}) \times (E \times \{1\})$. For our purpose, we will later compute `pb(rays,A,plus(A,H))`.

```
def rapA(P) :
        A_list =[]
        for i in range(len(P.Hrepresentation())):
                Hrep=P.Hrepresentation(i)
                A_list.append(Hrep.A())
        return A_list
def rapb(P) :
        b_list =[]
        for i in range(len(P.Hrepresentation())) :
                Hrep=P.Hrepresentation(i)
                b_list.append(Hrep.b())
        return b_list
```

The functions `rapA(P), rapb(P)` return the lists A, b of coefficients of the inequalities $Ax + b \geq 0$ of the supporting hyperplanes defining a polytope P (cf. Remark 6.2.2).

```
def detect(A_list, b_list, b):
        v=zero_vector(len(A_list[1])-1).list()+[-1]
        w=vector(v)
        s=A_list.index(w)
        b_list[s]=b+1
        return b_list
```

The function `detect(A_list,b_list,a)` searches on `A_list` the position of the element $v = (0, \dots, 0, -1)$, and it adds $a + 1$ to the component of `b_list` which has the same position. We use this function to compute $\mathcal{O}_{Y_-}(A + kH)$.

```
I=rapA(pb(rays,A,plus(A,H)))
J=rapb(pb(rays, A,plus(A,H)))
J1=detect(I,J,k)
return poly(I,J1)
```

Summing up, the function `GeomReal(rays,A,H,k)`, with $k = m - 1$, returns as output the polytope constructed as the birational contraction of the $\mathbb{P}^1$-bundle $W = \mathbb{P}(\mathcal{O}_{Y_-}(A) \otimes \mathcal{O}_{Y_-}(A + H))$ via the morphism associated to the Cartier divisor $\mathcal{O}_W(1) \otimes \pi^*\mathcal{O}_{Y_-}((m-1)A)$.

We conclude this section by introducing another useful function we will use in the rest of the chapter:

```
def check(P) :
        fan=NormalFan(P)
        return fan.is_complete(), fan.is_simplicial(), fan.is_smooth()
```

The function `check(P)` says if the normal fan of a polytope P is respectively complete, simplicial and smooth.

The whole function `GeomReal` can be accessed, and used, through the following link:
`https://cocalc.com/share/public_paths/a28daa428b12dfde5fec32ce200547f44fa38f4a`

6.3 Examples

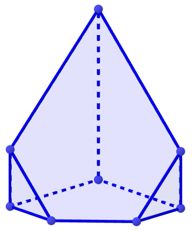
6.3.1 Blow-up of $\mathbb{P}^3$ along two points

Given the 3-dimensional projective space $\mathbb{P}^3$, let $\beta \colon Y_- \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along e_1, e_2, so that the rays of the fan of Y_- are

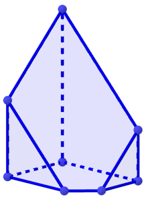
$$\Sigma_{Y_-}(1) = \{e_1, e_2, e_3, e_4 = -e_1 - e_2 - e_3, -e_1, -e_2\}.$$

As we have seen in Example 2.4.11, the variety Y_- is an MDS. Let H be the transform of the hyperplane divisor in $\mathbb{P}^3$, and let E_1, E_2 be the exceptional divisors corresponding to e_1, e_2.

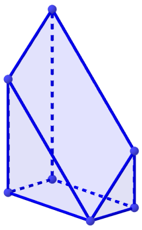
Consider the ample Cartier divisor $A = 6H + 2E_1 + 2E_2$ on Y_-, associated to the string `A=[0,0,0,6,4,4]`. The polytope associated to the pair $(Y_-, \mathcal{O}_{Y_-}(A))$ can be represented as follows:



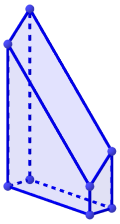
Let H be the Cartier divisor $H = -E_1$. associated to the string `H=[0,0,0,0,-1,0]`. Notice that $A+H$, written `sum(A,H)` is still ample, as one may say by considering the polytope associated to it:



We keep adding H, obtaining that the Cartier divisor $N := A+2H$, written `N=[0,0,0,6,2,4]` is nef.



The Cartier divisor $F := A + 3H$ is movable, and it is associated to the flip of the strict transform l of the line joining e_1, e_2. We set $Y_+ := \operatorname{Proj} R(Y_-; \mathcal{O}_{Y_-}(F))$.



We now construct the polytope P of the geometric realization of the toric SQM

$$\varphi : Y_- \dashrightarrow Y_+$$

Using the function `GeomReal` we obtain:

```
In:  rays=[[1,0,0],[0,1,0],[0,0,1],[-1,-1,-1],[-1,0,0],[0,-1,0]]
     A=[0,0,0,6,4,4]
     H=[0,0,0,0,-1,0]
     P=GeomReal(rays,A,H,2)
     P
Out: A 4-dimensional polyhedron in QQ^4 defined as the convex hull
       of 17 vertices
```

By using the command `check()` we get that the geometric realization X is smooth and projective. Consider the $\mathbb{C}^*$-action on the geometric realization X corresponding to the fourth natural projection of the character lattice. We obtain that, with respect to the embedding determined by the polytope, such action has criticality 2 and bandwidth 3, with sink Y_-, source Y_+ and an inner fixed point Y of weight 2, associated to the vertex `(2, 4, 0, 2)`.

6.3.2 Blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along two points

In the previous example, we have constructed a geometric realization as a birational contraction of a $\mathbb{P}^1$-bundle W over Y_-, whose exceptional locus is contained in the source of W. Alternatively, one may do a similar construction modifying the sink of W, or both extremal fixed point components. This is what we will do in the following example, which has appeared in [46, Example 5.10].

Consider the blow-up of G of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ along the points $(0,\infty,0),(0,0,\infty)$, where we set $0=(1:0),\infty=(0:1)$. Call l_-,l_+ respectively the transform of the lines $\{0\}\times\mathbb{P}^1\times\{0\},\{0\}\times\{0\}\times\mathbb{P}^1$, which meet at the strict transform of the point $(0,0,0)$. The variety G admits two SQM's $\varphi_\pm\colon G\to Y_\pm$ associated to the flips of l_-,l_+. We now construct the geometric realization of

$$\varphi\coloneqq\varphi_+\circ\varphi_-^{-1}\colon Y_-\dashrightarrow Y_+.$$

The rays of the fan of G are

```
rays=[[1,0,0],[-1,0,0],[0,1,0],[0,-1,0],[0,0,1],[0,0,-1],[-1,1,-1],[1,-1,-1]]
```

Consider the Cartier divisors

```
A=[3,0,3,0,3,0,2,2]
H=[1,0,0,0,0,0,0,0]
F1=[2,0,4,0,6,0,1,1]
F2=[6,0,4,0,6,0,1,1]
```

We abuse notation by writing F_1,F_2 and mean `F1,F2`. Notice that $F_2=F_1+4H$. We represent the polytopes associated respectively to F_1,A,F_2 in Figure 6.1.

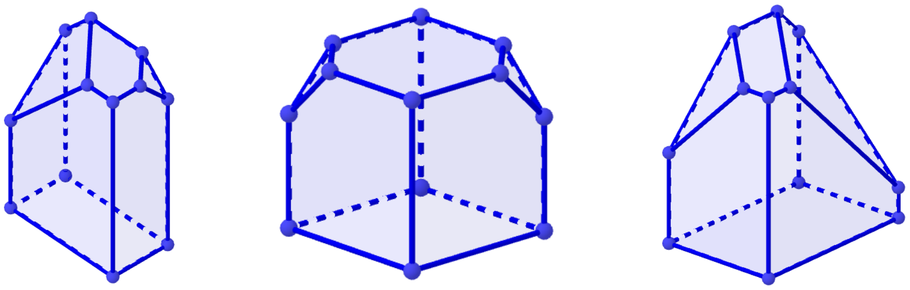


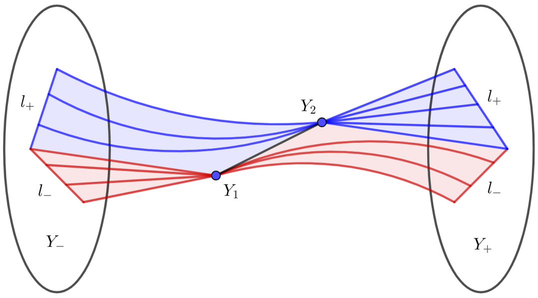
Figure 6.1: The polytopes associated to $(G,\mathcal{O}_G(F_1)),(G,\mathcal{O}_G(A)),(G,\mathcal{O}_G(F_2))$

We can thus compute the geometric realization of $\varphi\colon Y_-\dashrightarrow Y_+$:

```
In: GeomReal(rays,F1,H,3)
Out: A 4-dimensional polyhedron in QQ^4 defined as the convex hull
     of 26 vertices
```

Consider the $\mathbb{C}^*$-action on the geometric realization X corresponding to the fourth natural projection of the character lattice. We obtain that, with respect to the embedding determined by the polytope, such action has criticality 2 and bandwidth 3, with sink Y_-, source Y_+ and two inner fixed points Y_1,Y_2 of respectively weight $1,4$, associated to the vertices `(-3,-4,0,1),(-5,-4,0,3)`.

We may represent the $\mathbb{C}^*$-action on X by means of the following picture:



6.3.3 Weighted blow-up of $\mathbb{P}^3$ along two points

In the construction of a geometric realization presented at the beginning of this chapter, we have assumed that the divisor $H := F' - A'$ is Cartier. With this condition, the resulting geometric realization, as proved in Corollary 5.1.12, is equalized at the sink and the source. Here we present an example where we assume only that H is $\mathbb{Q}$-Cartier, so that, as we will see in Lemma 6.3.1, the resulting action on the geometric realization is not equalized at $Y_{\pm}$.

Notice that, in this setting, the toric variety $W := \operatorname{Proj}\operatorname{Sym}(\mathcal{O}_{Y_-}(A) \oplus \mathcal{O}_{Y_-}(A+H))$, which is constructed using the function `pb(rays, A,sum(A,H))` is not a $\mathbb{P}^1$-bundle over Y_-, but only a $\mathbb{P}^1$-fibration, as it is not locally free.

Given the 3-dimensional projective space $\mathbb{P}^3$, let p, q be two points invariant under the action of the maximal torus T of $\mathbb{P}^3$, and let $\beta \colon Y_- \to \mathbb{P}^3$ be the weighted blow-up of $\mathbb{P}^3$ along p, q with weights corresponding to inserting the rays $e_p = -e_1 - 2e_2$, $e_q = -2e_1 - e_2$, so that the rays of the fan of Y_- are

$$\Sigma_{Y_-}(1) = \{e_1, e_2, e_3, e_4 = -e_1 - e_2 - e_3, e_p, e_q\}.$$

Let H be the transform of the hyperplane divisor in $\mathbb{P}^3$, and let E_p, E_q be the exceptional divisors corresponding to p, q; by construction, $E_p, E_q \simeq \mathbb{P}(1,1,2)$. We aim to construct the geometric realization of the SQM $\varphi \colon Y_- \dashrightarrow Y_+$ associated to the flip of the line passing through e_p, e_q.

To this end, we need to find divisors A, H, F and a positive integer k, where $A, A+H$ are Cartier and ample, H is $\mathbb{Q}$-Cartier, and $F = A + kH$ gives the flip. Consider for instance the following divisors

```
A=[0,0,0,6,10,10]
H=[0,0,0,0,-1,0]
N=[0,0,0,6,8,10]
F=[0,0,0,6,7,10]
```

whose associated polytopes are represented in Figure 6.2.

It is readily seen that A is ample, N is nef, and $F = A + 3H$ is the movable divisor giving the flip. We may thus compute the polytope associated to the geometric realization of $\varphi \colon Y_- \dashrightarrow Y_+$.

```
In:  rays=[[1,0,0],[0,1,0],[0,0,1],[-1,-1,-1],[-1,-2,0,],[-2,-1,0]]
     A=[0,0,0,6,10,10]
     H=[0,0,0,0,-1,0]
     P=GeomReal(rays,A,H,2)
     P
Out: A 4-dimensional polyhedron in QQ^4 defined as the convex hull
       of 17 vertices
```

By looking at the vertices of P, we notice that the divisor associated to the polytope P is not Cartier, thus we consider a multiple to obtain a polytope `P2` with integer vertices:

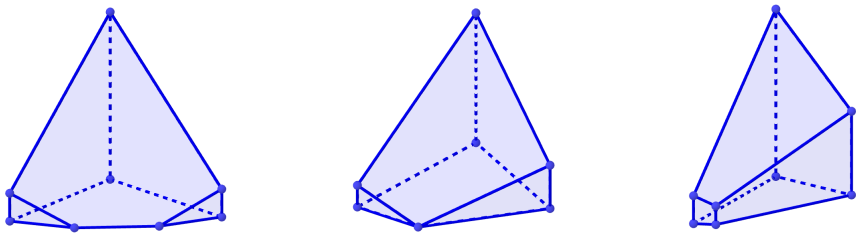
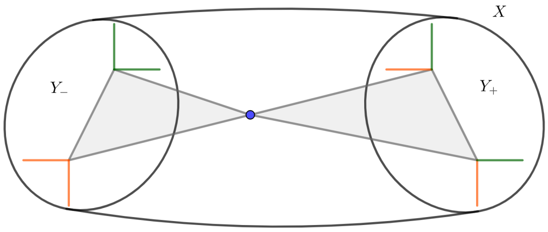


Figure 6.2: The polytopes associated to $(Y_-, \mathcal{O}_{Y_-}(A))$,$(Y_-, \mathcal{O}_{Y_-}(N))$, $(Y_-, \mathcal{O}_{Y_-}(F))$

```
In: J2=prodotto(J1,6)
    P2=poly2(I,J2)
    P2.vertices()

Out:(A vertex at (30, 0, 0, 0),
A vertex at (26, 8, 2, 18),
A vertex at (26, 8, 0, 18),
A vertex at (24, 12, 0, 12),
A vertex at (24, 12, 0, 0),
A vertex at (12, 24, 0, 0),
A vertex at (30, 0, 0, 18),
A vertex at (30, 0, 6, 18),
A vertex at (30, 0, 6, 0),
A vertex at (0, 0, 36, 0),
A vertex at (0, 0, 36, 18),
A vertex at (0, 0, 0, 18),
A vertex at (0, 30, 6, 0),
A vertex at (0, 0, 0, 0),
A vertex at (0, 21, 15, 18),
A vertex at (0, 21, 0, 18),
A vertex at (0, 30, 0, 0))
```

Consider the $\mathbb{C}^*$-action on the geometric realization X corresponding to the fourth natural projection of the character lattice. We obtain that, with respect to the embedding determined by the polytope `P2`, such action has criticality 2 and bandwidth 18, with sink Y_-, source Y_+ and an inner fixed point Y of weight 12, associated to the vertex `(24,12,0,12)`. We may represent X, together with the natural birational map ψ, which coincides by definition with φ, as follows:



We conclude by showing the following:

Lemma 6.3.1. *The $\mathbb{C}^*$-action on X is not equalized at $Y_\pm$.*

Proof. Since the map Φ is $\mathbb{C}^*$-equivariant, it is sufficient to show that the natural $\mathbb{C}^*$-action on the $\mathbb{P}^1$-fibration $W = \operatorname{Proj}\operatorname{Sym}(\mathcal{O}_{Y_-}(A) \oplus \mathcal{O}_{Y_-}(A+H))$ is not equalized at $s_\pm(Y_\pm)$. Consider its vertices

```
In: PB=pb(rays,A,plus(A,H))
    PB.vertices()
Out:    (A vertex at (0, 0, 0, 0),
A vertex at (0, 0, 0, 1),
A vertex at (0, 0, 6, 0),
A vertex at (0, 0, 6, 1),
A vertex at (0, 5, 0, 0),
A vertex at (0, 5, 1, 0),
A vertex at (0, 9/2, 0, 1),
A vertex at (0, 9/2, 3/2, 1),
A vertex at (2, 4, 0, 0),
A vertex at (3, 3, 0, 1),
A vertex at (4, 2, 0, 0),
A vertex at (4, 2, 0, 1),
A vertex at (5, 0, 0, 0),
A vertex at (5, 0, 0, 1),
A vertex at (5, 0, 1, 0),
A vertex at (5, 0, 1, 1))
```

and the supporting hyperplanes defining the polytope

```
In: PB=pb(rays,A,plus(A,H))
    PB.Hrepresentation()
Out:     (An inequality (0, 1, 0, 0) x + 0 >= 0,
An inequality (-1, -1, -1, 0) x + 6 >= 0,
An inequality (0, 0, 0, -1) x + 1 >= 0,
An inequality (-2, -1, 0, 0) x + 10 >= 0,
An inequality (-1, -2, 0, -1) x + 10 >= 0,
An inequality (0, 0, 0, 1) x + 0 >= 0,
An inequality (1, 0, 0, 0) x + 0 >= 0,
An inequality (0, 0, 1, 0) x + 0 >= 0)
```

where the vectors correspond to the primitive generators of ray corresponding to the inward pointing facet. We label such elements by $u_1, \dots, u_8$, and we denote by $F_1, \dots, F_8$ (resp. $D_1, \dots, D_8$) the associated facets (resp. divisors). The ample divisor associated to the polytope `P2` is

$$D = 6D_2 + D_3 + 10D_4 + 10D_5.$$

Consider the T-fixed points $p_- = (0,5,1,0), p_+ = (0, \frac{9}{2}, \frac{3}{2}, 1)$, associated to the cones $\sigma = \langle u_2, u_5, u_6, u_7 \rangle$, $\sigma' = \langle u_2, u_3, u_5, u_7 \rangle$ and let Γ be the T-invariant rational curve joining $p_\pm$, which is associated to the wall $\tau = \sigma \cap \sigma' = \langle u_2, u_5, u_7 \rangle$. By AMvsFM Lemma (see Lemma 2.1.50), it is sufficient to study the intersection product $Y_- \cdot \Gamma = (m_\sigma - m_{\sigma'})(u)$. By construction, $Y_- = D_6$, hence its Cartier data $m_\sigma, m_{\sigma'}$ are such that $m_\sigma(u_i) = 0$ for every $i \neq 6$, $m_\sigma(u_6) = 1$, and $m_{\sigma'}(u_i) = 0$ for every $i = 1, \dots, 8$. We thus obtain that $m_\sigma - m_{\sigma'} = (0, -\frac{1}{2}, \frac{1}{2}, 1)$. Consider $u = (0, -1, 0, 0)$; one may show that the image $\pi(u)$ generates the quotient lattice $\mathrm{N}/\mathbb{Z}\tau$. Hence $(m_\sigma - m_{\sigma'})(u) = \frac{1}{2}$, we conclude. ■

Chapter 7

Glossary of Notations

Chapter 8

Index

Bibliography

[1] Klaus Altmann and Jarosław A. Wiśniewski. Polyhedral divisors of Cox rings. *Michigan Math. J.*, 60(2):463–480, 2011.

[2] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface. *Cox rings*, volume 144. Cambridge: Cambridge University Press, 2015.

[3] Lorenzo Barban, Eleonora A. Romano, Luis E. Solá Conde, and Stefano Urbinati. Mori dream pairs and $\mathbb{C}^*$-actions. Preprint ArXiv:2207.09864, 2022.

[4] Andrzej S. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

[5] Andrzej S. Białynicki-Birula, James B. Carrell, and William M. McGovern. *Algebraic quotients. Torus actioans and cohomology. The adjoint representation and the adjoint action*, volume 131 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin, 2002. Invariant Theory and Algebraic Transformation Groups, II.

[6] Andrzej S. Białynicki-Birula and Joanna Święcicka. Complete quotients by algebraic torus actions. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 10–22. Springer, Berlin, 1982.

[7] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[8] Nicolas Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[9] Michel Brion. On linearization of line bundles. *J. Math. Sci. Univ. Tokyo*, 22(1):113–147, 2015.

[10] Jarosław Buczyński, Jarosław A. Wiśniewski, and Andrzej Weber. Algebraic torus actions on contact manifolds. *J. Differential Geom.*, 121(2):227–289, 2022.

[11] James B. Carrell and Andrew J. Sommese. Some topological aspects of $\mathbb{C}^*$-actions on compact Kaehler manifolds. *Commentarii mathematici Helvetici*, 54:567–582, 1979.

[12] Paolo Cascini and Vladimir Lazić. New outlook on the minimal model program, I. *Duke Math. J.*, 161(12):2415–2467, 2012.

[13] Ana-Maria Castravet. Mori dream spaces and blow-ups. In *Algebraic geometry: Salt Lake City 2015*, volume 97.1 of *Proc. Sympos. Pure Math.*, pages 143–167. Amer. Math. Soc., Providence, RI, 2018.

[14] Ana-Maria Castravet and Jenia Tevelev. Hilbert's 14th problem and Cox rings. *Compositio Mathematica*, 142(6):1479–1498, 2006.

[15] David A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.

[16] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[17] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

[18] Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[19] Simon K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differential Geom.*, 62(2):289–349, 2002.

[20] Lawrence Ein and Nicholas Shepherd-Barron. Some special Cremona transformations. *American Journal of Mathematics*, 111(5):783–800, 1989.

[21] Alberto Franceschini. $\mathbb{C}^*$-action on rational homogeneous varieties and the associated birational maps. 2023. Ph.D. Thesis.

[22] Alberto Franceschini and Luis E. Solá Conde. Inversion maps and torus action on rational homogeneous spaces. Preprint ArXiv: `2208.14216`, 2022.

[23] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010.

[24] Jianxun Hu and Wanchuan Zhang. Mukai flop and Ruan cohomology. *Mathematische Annalen*, 330(3):577–599, 2004.

[25] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000.

[26] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975.

[27] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978.

[28] Birger Iversen. A fixed point formula for action of tori on algebraic varieties. *Inv. Math.*, 16(3):229–236, 1972.

[29] Joachim Jelisiejew and Łukasz Sienkiewicz. Białynicki-Birula decomposition for reductive groups. *Journal de Mathématiques Pures et Appliquées*, 131:290–325, 2019.

[30] Joachim Jelisiejew and Łukasz Sienkiewicz. Białynicki-Birula decomposition for reductive groups in positive characteristic. *Journal de Mathématiques Pures et Appliquées*, 152:189–210, 2021.

[31] Anne-Sophie Kaloghiros, Alex Küronya, and Vladimir Lazić. Finite generation and geography of models. In *Minimal models and extremal rays (Kyoto, 2011)*, volume 70 of *Adv. Stud. Pure Math.*, pages 215–245. Math. Soc. Japan, [Tokyo], 2016.

[32] Akihiro Kanemitsu. Extremal Rays and Nefness of Tangent Bundles. *Michigan Math. J.*, 68(2):301–322, 2019.

[33] Akihiro Kanemitsu. Mukai pairs and simple K-equivalence. *Mathematische Zeitschrift*, 302 (4):2037–2057, 2022.

[34] Yujiro Kawamata. Francia's flip and derived categories. In *Algebraic geometry*, pages 197–215. de Gruyter, Berlin, 2002.

[35] Friedrich Knop, Hanspeter Kraft, Domingo Luna, and Thierry Vust. Local properties of algebraic group actions. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 63–75. Birkhäuser, Basel, 1989.

[36] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[37] Jerzy Konarski. Decompositions of normal algebraic varieties determined by an action of a one-dimensional torus. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 26:295–300, 1978.

[38] Jerzy Konarski. The B-B decomposition via Sumihiro's theorem. *J. Algebra*, 182(1):45–51, 1996.

[39] Alex Küronya and Stefano Urbinati. Geometry of multigraded rings and embeddings of toric varieties. Preprint ArXiv:1912.04374, 2019.

[40] Joseph M. Landsberg and Laurent Manivel. On the projective geometry of rational homogeneous varieties. *Comment. Math. Helv.*, 78(1):65–100, 2003.

[41] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004.

[42] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004.

[43] Laurent Manivel. Topics on the Geometry of Rational Homogeneous Spaces. *Acta. Math. Sin.-English Ser.*, 36:851–872, 2020.

[44] Robert Morelli. The birational geometry of toric varieties. *J. Algebr. Geom.*, 5(4):751–782, 1996.

[45] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[46] Gianluca Occhetta, Eleonora A. Romano, Luis E. Solá Conde, and Jarosław A. Wiśniewski. Chow quotients of $\mathbb{C}^*$-actions. Preprint ArXiv:2207.09864, 2023.

[47] Gianluca Occhetta, Eleonora A. Romano, Luis E. Solá Conde, and Jarosław A. Wiśniewski. Rational homogeneous spaces as geometric realizations of birational transformations. *Rend. Circ. Mat. Palermo, II Ser*, 2023.

[48] Gianluca Occhetta, Eleonora A. Romano, Luis E. Solá Conde, and Jarosław A. Wiśniewski. Small modifications of Mori dream spaces arising from $\mathbb{C}^*$-actions. *Eur. J. Math.*, 8(3):1072–1104, 2022.

[49] Gianluca Occhetta, Eleonora A. Romano, Luis E. Solá Conde, and Jarosław A. Wiśniewski. Small bandwidth $\mathbb{C}^*$-actions and birational geometry. *J. Algebraic Geom.*, 32(1):1–57, 2023.

[50] Gianluca Occhetta, Luis E. Solá Conde, Eleonora A. Romano, and Jarosław A. Wiśniewski. High rank torus actions on contact manifolds. *Selecta Math.*, 27(10), 2021.

[51] Gianluca Occhetta, Luis E. Solá Conde, Eleonora A. Romano, and Jarosław A. Wiśniewski. Geometric realizations of birational transformations via $\mathbb{C}^*$-actions. *Mathematische Zeitschrift*, 304:45, 2023.

[52] Shinnosuke Okawa. On images of Mori dream spaces. *Mathematische Annalen*, 364(3), 2016.

[53] Boris Pasquier. On some smooth projective two-orbit varieties with Picard number 1. *Math. Ann.*, 344(4):963–987, 2009.

[54] Luc Pirio and Francesco Russo. Varieties n-covered by curves of degree δ. *Comment. Math. Helv.*, 88(3):715–757, 2013.

[55] Miles Reid. Decomposition of toric morphisms. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 395–418. Birkhäuser Boston, Boston, MA, 1983.

[56] Miles Reid. What is a flip? http://homepages.warwick.ac.uk/staff/Miles.Reid/3folds, 1992.

[57] Eleonora A. Romano and Jarosław A. Wiśniewski. Adjunction for varieties with a $\mathbb{C}^*$ action. *Transform. Groups*, 27(4):1431–1473, 2022.

[58] Conjeevaram S. Seshadri. Quotient spaces modulo reductive algebraic groups. *Ann. of Math.*, 95:511–556, 1972.

[59] Hideyasu Sumihiro. Equivariant completion. *J. Math. Kyoto Univ.*, 14 (1):1–28, 1974.

[60] Michael Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.

[61] Kazushi Ueda. A new 5-fold flop and derived equivalence. *Bull. Lond. Math. Soc*, 58:533–538, 2016.

[62] Kazushi Ueda. G_2-Grassmannians and derived equivalences. *Manuscripta Math.*, 159:549–559, 2019.

[63] Jan Wierzba and Jarosław A. Wiśniewski. Small contractions of symplectic 4-folds. *Duke Math. J.*, 120(1):65–95, 2003.

[64] Jarosław A. Wiśniewski. Toric Mori theory and Fano manifolds. In *Geometry of toric varieties. Lectures of the summer school, Grenoble, France, June 19–July 7, 2000*, pages 249–272. Paris: Société Mathématique de France, 2002.

[65] Jarosław Włodarczyk. Birational cobordisms and factorization of birational maps. *J. Algebr. Geom.*, 9(3):425–449, 2000.