European Journal of Operational Research 301 (2022) 667–674

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

An extension of Majority Judgment to non-uniform qualitative scales

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ARTICLE INFO

Article history:
Received 6 May 2021
Accepted 1 November 2021
Available online 6 November 2021

Keywords:
Group decisions and negotiations
Qualitative scales
Majority Judgment
Ordinal proximity measures

ABSTRACT

We consider the context of social choice in which individual evaluations of the alternatives are expressed over an ordered qualitative scale. We propose to extend the framework of Majority Judgment to the case of non-uniform ordered qualitative scales, described by an ordinal proximity measure. The central construct in our model is a weak order defined over multisets of ordinal proximity degrees. On the basis of this weak order, each alternative profile is represented by an ordinal mean which balances the multisets of ordinal proximity degrees associated with the upper and lower ordinal evaluations. The procedure for ranking the alternative profiles consists primarily in comparing means, plus a tie-breaking scheme in which the weak order plays once more an important role.

1. Introduction

In the *Majority Judgment* (MJ) voting system, introduced and analyzed by Balinski & Laraki (2007b, 2011, 2014, 2020), the individual evaluations of the alternatives are expressed by terms of an ordered qualitative scale.[1] The MJ voting system associates to each alternative the lower median of its individual assessments, a natural aggregation function with many interesting properties, see for instance the discussion in Yager & Rybalov (1997). In MJ the alternatives are then ranked according to their lower median evaluations, plus a tie-breaking procedure based on the upper and lower individual assessments with respect to the lower medians.

In this paper we present an extension of the MJ voting system in which the basic ordinal scale of evaluations is possibly non-uniform (the psychological proximities between the pairs of consecutive terms of the scale are not perceived as identical). The notion of ordinal proximity measure for describing non-uniform ordered qualitative scales has been introduced by García-Lapresta & Pérez-Román (2015). The particular case of metrizable ordinal proximity measures has been discussed in García-Lapresta, González del Pozo, & Pérez-Román (2018).

An ordinal proximity measure assigns an ordinal degree of proximity to each pair of terms of the ordered qualitative scale. Given the individual evaluations of the alternatives, in the model introduced in García-Lapresta & Pérez-Román (2018) the alternatives are ranked according to the medians of the ordinal degrees of proximity between their individual assessments and the highest term of the qualitative scale. Since some alternatives may share the same median, an appropriate tie-breaking procedure is introduced.

The model described in this paper combines the MJ voting system with the structure provided by an ordinal proximity measure. In this model we define a weak order between multisets of ordinal proximity degrees and, on this basis, the model associates to each alternative the mean term(s) of the qualitative scale whose corresponding upper and lower multisets of ordinal proximity degrees are balanced.

If two alternatives share the same mean(s), the tie-breaking procedure in our model is based on the weak order between their respective upper and lower multisets of ordinal proximity degrees. The weak order between two multisets of ordinal proximity degrees relies on two hierarchical levels of comparison, an ordinal level which compares sums of ordinal proximity indices and a cardinal level which compares dispersions of ordinal proximity indices by means of a measure based on the absolute Gini index (Gini, 1912).

The paper is organized as follows. We begin by reviewing the framework of ordinal qualitative scales and ordinal proximity measures. We then define the weak order between multisets of ordinal proximity degrees and we describe the procedure in which this

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[1] The authors consider the following 6-term scale for political elections: {To Reject, Poor, Acceptable, Good, Very Good, Excellent}.

https://doi.org/10.1016/j.ejor.2021.11.002
0377-2217/

weak order is used to rank the alternatives in our model. Finally, an illustrative example is presented.

2. Preliminaries

Along the paper we consider an ordered qualitative scale (**OQS**) $\mathcal{L} = \{l_1, \dots, l_g\}$, with $g \geq 3$ and $l_1 \prec l_2 \prec \cdots \prec l_g$. We say that $\mathcal{L}$ is *uniform* if the proximity between each pair of consecutive linguistic terms, l_r and l_{r+1} for $r \in \{1, \dots, g-1\}$, is perceived as identical. In general, however, the proximities between consecutive linguistic terms in the OQS may be perceived to differ. In such case, these differences in the perception of proximity along the OQS can be described in the following way.

2.1. Ordinal proximity measures

In the general case of non-uniform OQSs, the notion of ordinal proximity measure introduced by García-Lapresta & Pérez-Román (2015) provides a formal description of the qualitative differences which may be perceived in the pairwise proximity between linguistic terms of the OQS.

An ordinal proximity measure is a mapping that assigns an ordinal degree of proximity to each pair of linguistic terms of an OQS $\mathcal{L}$. These ordinal degrees of proximity constitute a qualitative proximity scale, that is, a linear order $\Delta = \{\delta_1, \dots, \delta_h\}$, with $\delta_1 \succ \cdots \succ \delta_h$, where δ_1 and δ_h represent the maximum and minimum ordinal degrees of proximity, respectively. It is important to notice that the elements of Δ are not numbers.

Definition 1 (García-Lapresta & Pérez-Román (2015)).

An *ordinal proximity measure (**OPM**) on $\mathcal{L}$ with values in Δ* is a mapping $\pi : \mathcal{L} \times \mathcal{L} \longrightarrow \Delta$, where $\pi(l_r, l_s) = \pi_{rs}$ represents the degree of proximity between l_r and l_s, satisfying the following conditions:

1. *Exhaustiveness*: For every $\delta \in \Delta$, there exist $l_r, l_s \in \mathcal{L}$ such that $\delta = \pi_{rs}$.
2. *Symmetry*: $\pi_{sr} = \pi_{rs}$, for all $r, s \in \{1, \dots, g\}$.
3. *Maximum proximity*: $\pi_{rs} = \delta_1 \Leftrightarrow r = s$, for all $r, s \in \{1, \dots, g\}$.
4. *Monotonicity*: $\pi_{rs} \succ \pi_{rt}$ and $\pi_{st} \succ \pi_{rt}$, for all $r, s, t \in \{1, \dots, g\}$ such that $r < s < t$.

The first condition requires the function π to be exhaustive with respect to its image set, in the sense that all the ordinal degrees of Δ are obtained as proximity degrees. The second condition is a simple symmetry condition on the function π. The third condition states that the maximum degree of proximity is obtained only when comparing a linguistic term with itself. The fourth condition is a monotonicity condition, requiring that the ordinal proximity between any two non consecutive linguistic terms must be lower than any of the ordinal proximities between those linguistic terms and some intermediate one.

Every OPM can be represented by a symmetric square matrix of order g with coefficients in Δ, in which the diagonal elements are $\pi_{rr} = \delta_1,\ r = 1, \dots, g$:

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1s} & \cdots & \pi_{1g} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{r1} & \cdots & \pi_{rs} & \cdots & \pi_{rg} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{g1} & \cdots & \pi_{gs} & \cdots & \pi_{gg} \end{pmatrix}.$$

This matrix is called *proximity matrix* associated with π.

Example 1. Consider the human resources department of a firm has to select an employee for a job and each member of the team evaluates all candidates through an OQS with $g = 4$ evaluation levels, $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$, with the meaning of 'low', 'medium', 'high' and 'excellent', respectively. Suppose that this OQS is equipped with an OPM with $h = 6$ proximity degrees, $\Delta = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$, and the OPM is that associated with the proximity matrix

$$A_{223} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}$$

Fig. 1. Ordinal proximity measure with associated matrix A_{223}.

which can be represented by the diagram in Fig. 1.

As the diagram suggests, the subscripts in the matrix notation A_{223} correspond to the indices of the proximity degrees between consecutive evaluation levels, recalling that higher index values correspond to lower proximity degrees.[2] In this sense, the maximum degree of proximity, δ_1, is only reached when comparing a linguistic term with itself: $\pi_{11} = \pi_{22} = \pi_{33} = \pi_{44} = \delta_1$. The second degree of proximity, δ_2, corresponds to the proximities between l_1 and l_2, and l_2 and l_3, i.e., $\pi_{12} = \pi_{23} = \delta_2$. The third degree of proximity, δ_3, corresponds to the proximity between l_3 and l_4, i.e., $\pi_{34} = \delta_3$. The fourth degree of proximity, δ_4, corresponds to the proximity between l_1 and l_3, i.e., $\pi_{13} = \delta_4$. The fifth degree of proximity, δ_5, corresponds to the proximity between l_2 and l_4, i.e., $\pi_{24} = \delta_5$. And, finally, the minimum degree of proximity, δ_6, corresponds to the proximity between l_1 and l_4, i.e., $\pi_{14} = \delta_6$. Notice that different OPMs may share the same ordinal degrees of proximity between consecutive linguistic terms (see García-Lapresta et al. (2018, Subsect. 2.3)).

2.2. Majority Judgment

In the *Majority Judgment* (MJ) voting system introduced by Balinski & Laraki (2007b, 2011, 2014, 2020), voters evaluate alternatives by means of linguistic assessments in an OQS, and alternatives are ranked through their lower median assessments, called majority grades.

When alternatives share the same majority grade, Balinski and Laraki have in time considered different tie-breaking procedures:

1. In Balinski & Laraki (2007b), the majority-grades of the alternatives that are in a tie are removed from the alternative profiles, and the procedure is repeated until the ties are broken. Alternatives with different individual assessments are never in a final tie.
2. In Balinski & Laraki (2007a, Appendix), the authors propose a tie-breaking procedure that takes into account the number of individual assessments higher and lower than the majority-grade for those alternatives that are in a tie. Again, alternatives with different individual assessments are never in a final tie.
3. In Balinski & Laraki (2011), the authors simplify the previous proposal, as they focus on large electorates where ties are less likely. In this case, alternatives with different individual assessments may be in a final tie.

After some criticisms about MJ by Felsenthal & Machover (2008), some alternative procedures to MJ have been proposed in the literature. Some of them fall in the classical debate between the mean and the median as appropriate central positions.

Range Voting, proposed by Smith, considers a numerical scale and assigns the average of the individual assessments as the majority-grade.

[2] We will use this notation further in what follows.

García-Lapresta & Martínez-Panero (2009) provide a family of voting systems that avoid some drawbacks of MJ and Range Voting through appropriate aggregation functions between the mean and the median.

Falcó & García-Lapresta (2011) propose as majority-grade the linguistic term that minimizes the distance to the individual assessments. In addition, they propose a tie-breaking method based on distances.

Zahid and de Swart (2015) introduce the Borda Majority Count as an alternative voting system to MJ, from a cardinal perspective based on the Borda rule.

Ngoie, Savadogo, & Ulungu (2014) propose the Mean-Median Compromise Method, a voting system that combines the approaches of MJ and the Borda Majority Count (Zahid & de Swart, 2015).

García-Lapresta & Pérez-Román (2018) introduce and analyze a voting system where the alternatives are ranked according to the medians of the ordinal degrees of proximity between the given individual assessments and the highest linguistic term of the scale. In García-Lapresta & González del Pozo (2019) the decision procedure of García-Lapresta & Pérez-Román (2018) is generalized to the case of multiple criteria and the possibility that agents might assign two consecutive linguistic terms to the alternatives, when they hesitate. In García-Lapresta, Moreno-Albadalejo, Pérez-Román, & Temprano-García (2021) agents evaluate the alternatives with respect to different criteria with different ordered qualitative scales, each one equipped with an ordinal proximity measure, through a normalization procedure in the set of ordinal degrees of proximity.

In all these papers a "sum and dispersion" methodology is used, in an intermediate stage, for ranking the alternatives. In García-Lapresta & Pérez-Román (2018) and in García-Lapresta et al. (2021), this methodology is applied to subindices of pairs of ordinal degrees of proximity. In García-Lapresta & González del Pozo (2019), an analogous methodology is applied to pairs of ordinal degrees of proximity and, in this paper, a similar methodology is applied to multisets of ordinal degrees of proximity.

The novelty of the approach in the present paper is that we introduce the notion of mean qualitate evaluation in association with each alternative, as being the term in the qualitative scale whose corresponding upper and lower multisets of ordinal proximity degrees are balanced for that alternative. Moreover, in case two alternatives share the same mean, a tie-breaking procedure is proposed based on a newly defined weak order between the upper and lower multisets of ordinal proximity degrees.

Fabre (2021) proposes and analyzes some tie-breaking procedures in the MJ voting system.

In the following section we propose to extend the framework of MJ to the case of non-uniform OQSs, which are described by an OPM.

3. The procedure

Let $A = \{1, \ldots, m\}$, with $m \geq 2$, be a set of agents and let $X = \{x_1, \ldots, x_n\}$, with $n \geq 2$, be the set of alternatives which have to be evaluated by the agents through the linguistic terms of an OQS $\mathcal{L} = \{l_1, \ldots, l_g\}$. Consider that $\mathcal{L}$ is equipped with an OPM $\pi : \mathcal{L} \times \mathcal{L} \longrightarrow \Delta$, with $\Delta = \{\delta_1, \ldots, \delta_h\}$, $\delta_1 \succ \cdots \succ \delta_h$.

The evaluations of the alternatives provided by the agents are collected in a *profile*, that is a matrix

$$V = \begin{pmatrix} v_1^1 & \cdots & v_i^1 & \cdots & v_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_1^a & \cdots & v_i^a & \cdots & v_n^a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_1^m & \cdots & v_i^m & \cdots & v_n^m \end{pmatrix} \tag{1}$$

that consists of m rows and n columns of linguistic terms, where the element $v_i^a \in \mathcal{L}$ is the linguistic assessment given by the agent $a \in A$ to the alternative $x_i \in X$.

With $\boldsymbol{v}_i = (v_i^1, \ldots, v_i^m)$ we denote the vector that contains the assessments obtained by the alternative $x_i \in X$ (it corresponds to the i-th column of the profile V).

3.1. Ordering multisets of ordinal degrees of proximity

Let $\mathcal{M}(\Delta)$ denote the set of finite multisets[3] of elements of the ordinal proximity scale Δ. The empty multiset is $\emptyset$ and a general multiset in $\mathcal{M}(\Delta)$ of cardinality $t \in \mathbb{N}$ is indicated as $\{\!\{d_1, \ldots, d_t\}\!\} \in \mathcal{M}(\Delta)$.

We now introduce a weak order on $\mathcal{M}(\Delta)$ by means of a primary function $S : \mathcal{M}(\Delta) \longrightarrow \mathbb{N} \cup \{0\}$ and a secondary function $D : \mathcal{M}(\Delta) \longrightarrow [0, \infty)$.

Let $\rho : \Delta \longrightarrow \{1, \ldots, h\}$ be the mapping defined as $\rho(\delta_k) = k$, for every $\delta_k \in \Delta$.

Definition 2. The primary function S sums the indices in a multiset in $\mathcal{M}(\Delta)$,

$$S(\{\!\{d_1, \ldots, d_t\}\!\}) = \sum_{k=1}^{t} \rho(d_k), \tag{2}$$

for all $d_1, \ldots, d_t \in \Delta$, whereas the secondary function D is a dispersion measure defined as

$$D(\{\!\{d_1, \ldots, d_t\}\!\}) = \frac{1}{t^2} \sum_{\substack{r,s=1 \\ r<s}}^{t} |\rho(d_r) - \rho(d_s)|, \tag{3}$$

for all $d_1, \ldots, d_t \in \Delta$.

Notice that $|\rho(d_r) - \rho(d_s)|$ is just the number of steps that are necessary for going from d_r to d_s in the set of ordinal degrees of proximity, Δ. Since Δ is considered as a uniform OQS, the summation in Eq. (3) only contains ordinal information.

Remark 1.

1. The primary and secondary functions are null on the empty multiset: $S(\emptyset) = D(\emptyset) = 0$.
2. The primary and secondary functions have different characters: the former is a sum[4] and the latter is the average of the absolute values of the index differences,[5]
$$\frac{2}{t(t-1)} \sum_{\substack{r,s=1 \\ r<s}}^{t} |\rho(d_r) - \rho(d_s)|, \tag{4}$$
multiplied by $\frac{t-1}{2t}$.
3. $D(\{\!\{d_1, \ldots, d_t\}\!\})$ is based on the absolute Gini index[6] (Gini, 1912) and it is related to the variance of $\rho(d_1), \ldots, \rho(d_t)$,
$$\frac{1}{t^2} \sum_{\substack{r,s=1 \\ r<s}}^{t} (\rho(d_r) - \rho(d_s))^2.$$
4. Under replication, the primary function replicates,
$$S(\{\!\{d_1, \ldots, d_t\}\!\} \uplus \overset{k}{\cdots} \uplus \{\!\{d_1, \ldots, d_t\}\!\}) = k\, S(\{\!\{d_1, \ldots, d_t\}\!\}),$$
for all $k \in \mathbb{N}$ and $d_1, \ldots, d_t \in \Delta$

[3] A multiset is a collection of objects in which elements may occur more than once.
[4] The reason why S is a sum, rather than an average, is explained in Remark 4.
[5] The reason we use Eq. (3) instead of Eq. (4) as secondary function is explained in Subsection 3.3.
[6] Different formulations of the Gini index can be found in Yitzhaki (1998).

5. The secondary function is invariant under replications,

$$D(\{\!\{d_1,\dots,d_t\}\!\} \uplus \overset{k}{\cdots} \uplus \{\!\{d_1,\dots,d_t\}\!\}) = D(\{\!\{d_1,\dots,d_t\}\!\}),$$

for all $k \in \mathbb{N}$ and $d_1,\dots,d_t \in \Delta$.
6. $D(\{\!\{\delta_k\}\!\}) = 0$ for every $\delta_k \in \Delta$, and if $t > 1$, $D(\{\!\{d_1,\dots,d_t\}\!\}) = 0$ if and only if $d_1 = \cdots = d_t$.
7. The secondary function is anti-self-dual,

$$D(\{\!\{N(d_1),\dots,N(d_t)\}\!\}) = D(\{\!\{d_1,\dots,d_t\}\!\}),$$

for all $d_1,\dots,d_t \in \Delta$, where $N : \Delta \longrightarrow \Delta$ is the negation operator, defined as $N(\delta_k) = \delta_{h+1-k}$ for every $k \in \{1,\dots,h\}$.
8. The secondary function is invariant under translations,

$$D(\{\!\{T_r(d_1),\dots,T_r(d_t)\}\!\}) = D(\{\!\{d_1,\dots,d_t\}\!\}),$$

for all $d_1,\dots,d_t \in \Delta$, where $T_r(\delta_k) = \delta_{k+r}$ for every $k \in \{1,\dots,h\}$ such that $k + r \le h$.

On the basis of the two functions S and D, the weak order $\succcurlyeq$ on $\mathcal{M}(\Delta)$ is defined as follows. As usual, $\succ$ and $\sim$ denote the asymmetric and the symmetric parts of $\succcurlyeq$, respectively.

Definition 3. The binary relation $\succcurlyeq$ on $\mathcal{M}(\Delta)$ is defined as

$$\{\!\{d_1,\dots,d_t\}\!\} \succcurlyeq \{\!\{d'_1,\dots,d'_{t'}\}\!\} \Leftrightarrow \begin{cases} S(\{\!\{d_1,\dots,d_t\}\!\}) > S(\{\!\{d'_1,\dots,d'_{t'}\}\!\}) \\ \text{or} \\ S(\{\!\{d_1,\dots,d_t\}\!\}) = S(\{\!\{d'_1,\dots,d'_{t'}\}\!\}) \text{ and} \\ D(\{\!\{d_1,\dots,d_t\}\!\}) \le D(\{\!\{d'_1,\dots,d'_{t'}\}\!\}), \end{cases}$$

for all $\{\!\{d_1,\dots,d_t\}\!\}, \{\!\{d'_1,\dots,d'_{t'}\}\!\} \in \mathcal{M}(\Delta)$.

It is easy to check that $\succcurlyeq$ is a weak order on $\mathcal{M}(\Delta)$.

Remark 2. For all $\{\!\{d_1,\dots,d_t\}\!\}, \{\!\{d'_1,\dots,d'_{t'}\}\!\} \in \mathcal{M}(\Delta)$ we have

$$\{\!\{d_1,\dots,d_t\}\!\} \succ \{\!\{d'_1,\dots,d'_{t'}\}\!\} \Leftrightarrow \begin{cases} S(\{\!\{d_1,\dots,d_t\}\!\}) > S(\{\!\{d'_1,\dots,d'_{t'}\}\!\}) \\ \text{or} \\ S(\{\!\{d_1,\dots,d_t\}\!\}) = S(\{\!\{d'_1,\dots,d'_{t'}\}\!\}) \text{ and} \\ D(\{\!\{d_1,\dots,d_t\}\!\}) < D(\{\!\{d'_1,\dots,d'_{t'}\}\!\}) \end{cases}$$

and

$$\{\!\{d_1,\dots,d_t\}\!\} \sim \{\!\{d'_1,\dots,d'_{t'}\}\!\} \Leftrightarrow \begin{cases} S(\{\!\{d_1,\dots,d_t\}\!\}) = S(\{\!\{d'_1,\dots,d'_{t'}\}\!\}) \\ \text{and} \\ D(\{\!\{d_1,\dots,d_t\}\!\}) = D(\{\!\{d'_1,\dots,d'_{t'}\}\!\}). \end{cases}$$

Example 2. Since

$$S(\{\!\{\delta_4,\delta_5\}\!\}) = S(\{\!\{\delta_2,\delta_2,\delta_5\}\!\}) = S(\{\!\{\delta_2,\delta_3,\delta_4\}\!\}) = S(\{\!\{\delta_3,\delta_3,\delta_3\}\!\}) = 9$$

and

$$D(\{\!\{\delta_3,\delta_3,\delta_3\}\!\}) = 0 < D(\{\!\{\delta_4,\delta_5\}\!\}) = 0.25 < D(\{\!\{\delta_2,\delta_3,\delta_4\}\!\}) = 0.444 < D(\{\!\{\delta_2,\delta_2,\delta_5\}\!\}) = 0.666,$$

we have

$$\{\!\{\delta_3,\delta_3,\delta_3\}\!\} \succ \{\!\{\delta_4,\delta_5\}\!\} \succ \{\!\{\delta_2,\delta_3,\delta_4\}\!\} \succ \{\!\{\delta_2,\delta_2,\delta_5\}\!\}.$$

We now establish an interesting property that provides a sufficient condition for decreasing the dispersion[7]

[7] It is a well-known property in the setting of Welfare Economics (see, for instance, Chakravarty, 1988).

Definition 4. Consider two multisets of the same cardinality, and an appropriate indexing $\{\!\{d_1,\dots,d_t\}\!\}, \{\!\{d'_1,\dots,d'_t\}\!\} \in \mathcal{M}(\Delta)$ with $t \ge 2$, such that there exist $r, s \in \{1,\dots,t\}$ for which $d_k = d'_k$ for every $k \in \{1,\dots,t\} \setminus \{r,s\}$ and $\rho(d_r) + \rho(d_s) = \rho(d'_r) + \rho(d'_s)$. In such a case, we say that the multiset $\{\!\{d'_1,\dots,d'_t\}\!\}$ is obtained from the multiset $\{\!\{d_1,\dots,d_t\}\!\}$ by means of a *progressive transfer* if $\rho(d_r) \le \rho(d'_r) \le \rho(d'_s) \le \rho(d_s)$.

Proposition 1. *Given* $\{\!\{d_1,\dots,d_t\}\!\}, \{\!\{d'_1,\dots,d'_t\}\!\} \in \mathcal{M}(\Delta)$, *if* $\{\!\{d'_1,\dots,d'_t\}\!\}$ *is obtained from* $\{\!\{d_1,\dots,d_t\}\!\}$ *by a progressive transfer, then* $\{\!\{d'_1,\dots,d'_t\}\!\} \succ \{\!\{d_1,\dots,d_t\}\!\}$.

Proof. Obviously, if $\{\!\{d'_1,\dots,d'_t\}\!\}$ is obtained from $\{\!\{d_1,\dots,d_t\}\!\}$ by a progressive transfer, then we have $S(\{\!\{d_1,\dots,d_t\}\!\}) = S(\{\!\{d'_1,\dots,d'_t\}\!\})$. Following Gini (1936), D satisfies the Pigou-Dalton transfer principle (it is Schur-convex in words of Marshall, Olkin, & Arnold (2011)), and thus we have $D(\{\!\{d'_1,\dots,d'_t\}\!\}) < D(\{\!\{d_1,\dots,d_t\}\!\})$, hence $\{\!\{d'_1,\dots,d'_t\}\!\} \succ \{\!\{d_1,\dots,d_t\}\!\}$. □

3.2. Ordering alternatives

Let $\mathcal{L}_2 = \{[l_r, l_s] \mid r, s \in \{1,\dots,g\}, s \in \{r, r+1\}\}$. The elements of $\mathcal{L}_2$ are either subsets of two consecutive linguistic terms, $[l_r, l_{r+1}] = \{l_r, l_{r+1}\}$, or a single linguistic term, $[l_r, l_r] = \{l_r\}$. For practical reasons we identify $[l_r, l_r]$ with l_r. Notice that the cardinality of $\mathcal{L}_2$ is $2g - 1$.

The original linear order on $\mathcal{L}$ can be extended to $\mathcal{L}_2$ in the natural way: $l_r \prec \{l_r, l_{r+1}\} \prec l_{r+1}$, for every $r \in \{1,\dots,g-1\}$.

Given a profile V, for each alternative $x_i \in X$ and each linguistic term $l_r \in \mathcal{L}$, we introduce two multisets of $\mathcal{M}(\Delta)$:

$$E_r^-(x_i) = \left\{\!\!\left\{ \pi\left(v_i^a, l_r\right) \mid v_i^a \prec l_r \right\}\!\!\right\} \quad \text{and} \quad E_r^+(x_i) = \left\{\!\!\left\{ \pi\left(v_i^a, l_r\right) \mid v_i^a \succ l_r \right\}\!\!\right\}.$$

Remark 3. In case all agents assign the same evaluation $l_r \in \mathcal{L}$ to the alternative $x_i \in X$, then $E_r^-(x_i) = E_r^+(x_i) = \emptyset$. Moreover, given that $E_1^-(x_i) = E_g^+(x_i) = \emptyset$, we have $E_1^+(x_i) \succcurlyeq E_1^-(x_i)$ and $E_g^-(x_i) \succcurlyeq E_g^+(x_i)$.

This remark suggests that, for some intermediate evaluation $l_r \in \mathcal{L}$, with $1 < r < g$, the sign of the dominance relation between the multisets $E_r^-(x_i)$ and $E_r^+(x_i)$ changes, from the initial $E_1^+(x_i) \succcurlyeq E_1^-(x_i)$ to the final $E_g^-(x_i) \succcurlyeq E_g^+(x_i)$. We will now prove that this does in fact occur and has a central role in the concept of *mean* which we will introduce later.

Proposition 2. *Given a profile* V *and an alternative* $x_i \in X$, *there are two possibilities:*

1. *There exists a unique evaluation level* $r \in \{1,\dots,g\}$ *for which* $E_r^-(x_i) \sim E_r^+(x_i)$.
2. *There exists a unique pair of consecutive evaluation levels* $r, r+1$, *with* $r \in \{1,\dots,g-1\}$, *such that* $E_r^-(x_i) \prec E_r^+(x_i)$ *and* $E_{r+1}^-(x_i) \succ E_{r+1}^+(x_i)$.

Proof. Consider an alternative $x_i \in X$ and at least two evaluating agents, that is, $m \ge 2$. If $v_i^1 = \cdots = v_i^m = l_r$ for some $r \in \{1,\dots,g\}$, then $E_r^-(x_i) = E_r^+(x_i) = \emptyset$ and, consequently, for this unique evaluation level $r \in \{1,\dots,g\}$, it holds that $E_r^-(x_i) \sim E_r^+(x_i)$.

Now, assume that the ordinal evaluations assigned to alternative x_i are not all the same. In this case we have

$$E_1^-(x_i) = \emptyset \quad E_1^+(x_i) \ne \emptyset \quad \text{and thus} \quad E_1^-(x_i) \prec E_1^+(x_i)$$
$$E_g^-(x_i) \ne \emptyset \quad E_g^+(x_i) = \emptyset \quad \text{and thus} \quad E_g^-(x_i) \succ E_g^+(x_i).$$

Let l_s and l_t be the lowest and highest ordinal evaluations obtained by x_i, respectively, with $1 \le s < t \le g$. Accordingly, we have that

$$\emptyset = E_1^-(x_i) = \cdots = E_s^-(x_i) \prec E_{s+1}^-(x_i) \prec \cdots \prec E_g^-(x_i)$$
$$E_1^+(x_i) \succ \cdots \succ E_{t-1}^+(x_i) \succ E_t^+(x_i) = \cdots = E_g^+(x_i) = \emptyset.$$

Note that the two sequences of strict inequalities, with opposite signs, are due to the monotonicity of the OPM: indeed, if $E_r^-(x_i) \neq \emptyset$, then $E_r^-(x_i) \prec E_{r+1}^-(x_i)$ for any $r \in \{s+1, \ldots, g-1\}$; analogously, if $E_r^+(x_i) \neq \emptyset$, then $E_{r-1}^+(x_i) \succ E_r^+(x_i)$ for any $r \in \{2, \ldots, t-1\}$.

In order to illustrate this crucial point, consider for instance the case $E_r^-(x_i) \neq \emptyset$. This means that the multiset $E_r^-(x_i)$ contains one or more elements of the form $\pi(l_s, l_r) = \pi_{sr}$, with $1 \leq s < r < g$. In correspondence with each of these terms, the multiset $E_{r+1}^-(x_i)$ contains terms of the form $\pi(l_s, l_{r+1}) = \pi_{s,r+1} \succ \pi_{sr}$ by the monotonicity property of the OPM. We therefore conclude that the two sequences $E_1^\pm(x_i), \ldots, E_g^\pm(x_i)$ are strictly monotonic along their non-trivial ranges, where by trival ranges we mean those in which equality holds. Moreover, note that in each of the two sequences at least one strict inequality must be present. As a result of the strict monotonicity ranges present in the two sequences of multisets $E^\pm(x_i)$, we conclude that there must be a unique pair of consecutive evaluation levels $r, r+1$, with $r \in \{1, \ldots, g-1\}$, for which there is a dominance inversion between the corresponding multisets of the two sequences, that is, $E_r^-(x_i) \prec E_r^+(x_i)$ and $E_{r+1}^-(x_i) \succ E_{r+1}^+(x_i)$ as stated in the proposition. □

As anticipated after the introduction of the primary and secondary functions, which together represent the weak order on $\mathcal{M}(\Delta)$, we are now in the position to explain the choice of the primary function S, which has been defined as a sum rather than an average. The reason is that only the former character ensures the existence and uniqueness of the dominance inversion in Proposition 2, as explained in the following remark.

Remark 4. If in Eq. (2) the sum S was changed by the average,

$$S'(\{\!\{d_1, \ldots, d_t\}\!\}) = \frac{1}{t} \sum_{k=1}^{t} \rho(d_k),$$

then Proposition 2 would not be true, in the sense that the sign inversion of the dominance relation between the multisets E_r^- and E_r^+, with $r = 1, \ldots, g$, may not be unique. The following example illustrates this fact.

Suppose an alternative x_i is evaluated by 7 agents through a 4-term OQS $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$ whose assessments are gathered in $\boldsymbol{v}_i = (l_1, l_2, l_3, l_3, l_3, l_3, l_4)$. If $\mathcal{L}$ is equipped with the OPM with associated proximity matrix

$$\begin{pmatrix} \delta_1 & \delta_3 & \delta_5 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_6 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix},$$

then we have $E_1^-(x_i) = \emptyset$, $E_1^+(x_i) = \{\!\{\delta_3, \delta_5, \delta_5, \delta_5, \delta_5, \delta_7\}\!\}$, $E_2^-(x_i) = \{\!\{\delta_3\}\!\}$, $E_2^+(x_i) = \{\!\{\delta_2, \delta_2, \delta_2, \delta_2, \delta_6\}\!\}$, $E_3^-(x_i) = \{\!\{\delta_2, \delta_5\}\!\}$, $E_3^+(x_i) = \{\!\{\delta_4\}\!\}$, $E_4^-(x_i) = \{\!\{\delta_4, \delta_4, \delta_4, \delta_4, \delta_6, \delta_7\}\!\}$ and $E_4^+(x_i) = \emptyset$. Since $S'(E_1^-(x_i)) = 0 < 5 = S'(E_1^+(x_i))$, $S'(E_2^-(x_i)) = 3 > 2.8 = S'(E_2^+(x_i))$, $S'(E_3^-(x_i)) = 3.5 < 4 = S'(E_3^+(x_i))$ and $S'(E_4^-(x_i)) = 4.83 > 0 = S'(E_4^+(x_i))$, we obtain $E_1^-(x_i) \prec E_1^+(x_i)$, $E_2^-(x_i) \succ E_2^+(x_i)$, $E_3^-(x_i) \prec E_3^+(x_i)$ and $E_4^-(x_i) \succ E_4^+(x_i)$.

Taking into account Proposition 2, we now introduce the mean operator.

Definition 5. Given a profile V, the associated *mean operator* is the mapping $M : X \longrightarrow \mathcal{L}_2$ defined as

$$M(x_i) = \begin{cases} l_r, & \text{if } E_r^-(x_i) \sim E_r^+(x_i), \\ \{l_r, l_{r+1}\}, & \text{if } E_r^-(x_i) \prec E_r^+(x_i) \text{ and } E_{r+1}^-(x_i) \succ E_{r+1}^+(x_i). \end{cases} \tag{5}$$

Regarding Eq. (5), we denote $M^-(x_i) = M^+(x_i) = l_r$, if $M(x_i) = l_r$; and $M^-(x_i) = l_r$ and $M^+(x_i) = l_{r+1}$, if $M(x_i) = \{l_r, l_{r+1}\}$.

Different alternatives may share the same mean. In order to rank the alternatives, we introduce two multisets that contain the ordinal degrees of proximity between the mean and the linguistic assessments upper and lower to that mean,

$$N^+(x_i) = \{\!\{\pi\left(v_i^a, M^+(x_i)\right) \mid v_i^a \succ M(x_i)\}\!\},$$

$$N^-(x_i) = \{\!\{\pi\left(v_i^a, M^-(x_i)\right) \mid v_i^a \prec M(x_i)\}\!\},$$

and the index

$$s(x_i) = \begin{cases} 1, & \text{if } N^+(x_i) \succ N^-(x_i), \\ 0, & \text{if } N^+(x_i) \sim N^-(x_i), \\ -1, & \text{if } N^-(x_i) \succ N^+(x_i). \end{cases}$$

The alternatives of X are ranked through the following weak order[8]

Definition 6. Let $\succcurlyeq$ the binary relation on X defined as $x_i \succcurlyeq x_j$ if one of the following conditions holds:

1. $M(x_i) \succ M(x_j)$.
2. $M(x_i) = M(x_j)$ and $s(x_i) > s(x_j)$.
3. $M(x_i) = M(x_j)$, $s(x_i) = s(x_j) = 1$ and $\big(N^+(x_i) \succ N^+(x_j)$ or $\big(N^+(x_i) \sim N^+(x_j)$ and $N^-(x_j) \succcurlyeq N^-(x_i)\big)\big)$.
4. [9]$M(x_i) = M(x_j)$, $s(x_i) = s(x_j) = 0$ and $\big(N^+(x_j) \uplus N^-(x_j)\big) \succcurlyeq \big(N^+(x_i) \uplus N^-(x_i)\big)$.
5. $M(x_i) = M(x_j)$, $s(x_i) = s(x_j) = -1$ and $\big(N^-(x_j) \succ N^-(x_i)$ or $\big(N^-(x_i) \sim N^-(x_j)$ and $N^+(x_i) \succcurlyeq N^+(x_j)\big)\big)$.

Proposition 3. *The binary relation $\succcurlyeq$ introduced in Definition 6 is a weak order.*

Proof. It is easy to check that $\succcurlyeq$ is complete. In order to justify that $\succcurlyeq$ is transitive, consider $x_i \succcurlyeq x_j$ and $x_j \succcurlyeq x_k$. There are 5 possible cases for $x_i \succcurlyeq x_j$:

1. $M(x_i) \succ M(x_j)$. From $x_j \succcurlyeq x_k$, we have $M(x_j) \succcurlyeq M(x_k)$. Thus, $M(x_i) \succ M(x_k)$ and, consequently, $x_i \succ x_k$.
2. $M(x_i) = M(x_j)$ and $s(x_i) > s(x_j)$. From $x_j \succcurlyeq x_k$, there are three possibilities:
   (a) If $M(x_j) \succ M(x_k)$, then $M(x_i) \succ M(x_k)$. Consequently, $x_i \succ x_k$.
   (b) If $M(x_j) = M(x_k)$ and $s(x_j) > s(x_k)$, then $M(x_i) = M(x_k)$ and $s(x_i) > s(x_k)$. Consequently, $x_i \succ x_k$.
   (c) If $M(x_j) = M(x_k)$ and $s(x_j) = s(x_k)$, then $M(x_i) = M(x_k)$ and $s(x_i) > s(x_k)$. Consequently, $x_i \succ x_k$.
3. $M(x_i) = M(x_j)$, $s(x_i) = s(x_j) = 1$ and $\big(N^+(x_i) \succ N^+(x_j)$ or $\big(N^+(x_i) \sim N^+(x_j)$ and $N^-(x_j) \succcurlyeq N^-(x_i)\big)\big)$. From $x_j \succcurlyeq x_k$, there are three possibilities:
   (a) If $M(x_j) \succ M(x_k)$, then $M(x_i) \succ M(x_k)$. Consequently, $x_i \succ x_k$.
   (b) If $M(x_j) = M(x_k)$ and $s(x_j) > s(x_k)$, then $M(x_i) = M(x_k)$ and $s(x_i) > s(x_k)$. Consequently, $x_i \succ x_k$.
   (c) If $M(x_j) = M(x_k)$, $s(x_j) = s(x_k) = 1$, $\big(N^+(x_j) \succ N^+(x_k)$ or $\big(N^+(x_j) \sim N^+(x_k)$ and $N^-(x_k) \succcurlyeq N^-(x_j)\big)\big)$, then $M(x_i) = M(x_k)$, $s(x_i) = s(x_k) = 1$ and one of the following four situations occurs:
      (1) $N^+(x_i) \succ N^+(x_j)$ and $N^+(x_j) \succ N^+(x_k)$. Then, $N^+(x_i) \succ N^+(x_k)$ and, consequently, $x_i \succ x_k$.
      (2) $N^+(x_i) \succ N^+(x_j)$, $N^+(x_j) \sim N^+(x_k)$ and $N^-(x_k) \succcurlyeq N^-(x_j)$. Then, $N^+(x_i) \succ N^+(x_k)$ and, consequently, $x_i \succ x_k$.

[8] It is based on Balinski & Laraki (2007a, Appendix).

[9] With $\uplus$ we denote the union of multisets. For instance, $\{\!\{\delta_2, \delta_3, \delta_3\}\!\} \uplus \{\!\{\delta_1, \delta_2, \delta_2, \delta_3\}\!\} = \{\!\{\delta_1, \delta_2, \delta_2, \delta_2, \delta_3, \delta_3, \delta_3\}\!\}$.

(3) $N^+(x_i) \sim N^+(x_j)$, $N^-(x_j) \succcurlyeq N^-(x_i)$ and $N^+(x_j) \succ N^+(x_k)$. Then, $N^+(x_i) \succ N^+(x_k)$ and, consequently, $x_i \succ x_k$.
(4) $N^+(x_i) \sim N^+(x_j)$, $N^-(x_j) \succcurlyeq N^-(x_i)$, $N^+(x_j) \sim N^+(x_k)$ and $N^-(x_k) \succcurlyeq N^-(x_j)$. Then, $N^+(x_i) \sim N^+(x_k)$ and $N^-(x_k) \succcurlyeq N^-(x_i)$. Consequently, $x_i \succcurlyeq x_k$.

4. $M(x_i) = M(x_j)$, $s(x_i) = s(x_j) = 0$ and $\big(N^+(x_j) \uplus N^-(x_j)\big) \succcurlyeq \big(N^+(x_i) \uplus N^-(x_i)\big)$. From $x_j \succcurlyeq x_k$, there are three possibilities:
   (a) If $M(x_j) \succ M(x_k)$, then $M(x_i) \succ M(x_k)$. Consequently, $x_i \succ x_k$.
   (b) If $M(x_j) = M(x_k)$ and $s(x_j) > s(x_k)$, then $M(x_i) = M(x_k)$ and $s(x_i) > s(x_k)$. Consequently, $x_i \succ x_k$.
   (c) If $M(x_j) = M(x_k)$, $s(x_j) = s(x_k) = 0$ and $\big(N^+(x_k) \uplus N^-(x_k)\big) \succcurlyeq \big(N^+(x_j) \uplus N^-(x_j)\big)$. Then, $M(x_i) = M(x_k)$, $s(x_i) = s(x_k) = 0$ and $\big(N^+(x_k) \uplus N^-(x_k)\big) \succcurlyeq \big(N^+(x_i) \uplus N^-(x_i)\big)$. Consequently, $x_i \succcurlyeq x_k$.
5. $M(x_i) = M(x_j)$, $s(x_i) = s(x_j) = -1$ and $\big(N^-(x_j) \succ N^-(x_i)$ or $\big(N^-(x_i) \sim N^-(x_j)$ and $N^+(x_i) \succcurlyeq N^+(x_j)\big)\big)$. From $x_j \succcurlyeq x_k$, there are only two possibilities:
   (a) If $M(x_j) \succ M(x_k)$, then $M(x_i) \succ M(x_k)$ and, consequently, $x_i \succ x_k$.
   (b) If $M(x_j) = M(x_k)$, $s(x_j) = s(x_k) = -1$ and $\big(N^-(x_k) \succ N^-(x_j)$ or $\big(N^-(x_j) \sim N^-(x_k)$ and $N^+(x_j) \succcurlyeq N^+(x_k)\big)\big)$ and one of the following four situations occurs:
      (1) $N^-(x_j) \succ N^-(x_i)$ and $N^-(x_k) \succ N^-(x_j)$. Then, $N^-(x_k) \succ N^-(x_i)$ and, consequently, $x_i \succ x_k$.
      (2) $N^-(x_j) \succ N^-(x_i)$, $N^-(x_j) \sim N^-(x_k)$ and $N^+(x_j) \succcurlyeq N^+(x_k)$. Then, $N^-(x_k) \succ N^-(x_i)$ and, consequently, $x_i \succ x_k$.
      (3) $N^-(x_i) \sim N^-(x_j)$, $N^+(x_i) \succcurlyeq N^+(x_j)$ and $N^-(x_k) \succ N^-(x_j)$. Then, $N^-(x_k) \succ N^-(x_i)$ and, consequently, $x_i \succ x_k$.
      (4) $N^-(x_i) \sim N^-(x_j)$, $N^+(x_i) \succcurlyeq N^+(x_j)$, $N^-(x_j) \sim N^-(x_k)$ and $N^+(x_j) \succcurlyeq N^+(x_k)$. Then, $N^-(x_i) \sim N^-(x_k)$ and $N^+(x_i) \succcurlyeq N^+(x_k)$. Consequently, $x_i \succcurlyeq x_k$. □

Remark 5. Although ties are unlikely, they may occur. It is easy to check that

$$x_i \sim x_j \;\Leftrightarrow\; \big(M(x_i) = M(x_j),\; N^+(x_i) \sim N^+(x_j) \text{ and } N^-(x_i) \sim N^-(x_j)\big).$$

Example 3. We now consider three examples included in Zahid and de Swart (2015), where two alternatives x_1 and x_2 are evaluated through a 6-term OQS $\mathcal{L} = \{l_1, l_2, l_3, l_4, l_5, l_6\}$. We compare the outcomes generated by MJ and our procedure, under the assumption that $\mathcal{L}$ is uniform.

1. In Zahid and de Swart (2015, Example 1) the assessments of 9 agents are

   $$\boldsymbol{v}_1 = (l_1, l_1, l_1, l_2, l_4, l_4, l_4, l_4, l_5) \text{ and } \boldsymbol{v}_2 = (l_2, l_3, l_3, l_3, l_3, l_5, l_6, l_6, l_6).$$

   With MJ, x_1 defeats x_2 because the medians of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are l_4 and l_3, respectively. However, with our procedure x_2 defeats x_1: $M(x_1) = M(x_2) = \{l_3, l_4\}$ and $s(x_1) = -1 < 1 = s(x_2)$.
2. In Zahid and de Swart (2015, Example 3) the alternatives have obtained 10 assessments:

   $$\boldsymbol{v}_1 = (l_1, l_1, l_1, l_1, l_1, l_4, l_5, l_5, l_6, l_6) \text{ and } \boldsymbol{v}_2 = (l_1, l_1, l_1, l_1, l_2, l_2, l_2, l_2, l_2, l_2).$$

   With MJ, x_2 defeats x_1 because the lower medians of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are l_1 and l_2, respectively. However, with our procedure x_1 defeats x_2: $M(x_1) = \{l_3, l_4\} \succ \{l_1, l_2\} = M(x_2)$.
3. In Zahid and de Swart (2015, Example 4) the assessments of 5 agents are

   $$\boldsymbol{v}_1 = (l_1, l_2, l_4, l_4, l_6) \text{ and } \boldsymbol{v}_2 = (l_2, l_3, l_3, l_6, l_6).$$

   With MJ, x_1 defeats x_2 because the medians of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are l_4 and l_3, respectively. However, with our procedure x_2 defeats x_1: $M(x_1) = M(x_2) = \{l_3, l_4\}$ and $s(x_1) = -1 < 1 = s(x_2)$.

   Zahid and de Swart (2015) also considered that a sixth agent evaluates the alternatives x_1 and x_2 with l_2 and l_1, respectively, i.e.,

   $$\boldsymbol{v}'_1 = (l_1, l_2, l_2, l_4, l_4, l_6) \text{ and } \boldsymbol{v}'_2 = (l_1, l_2, l_3, l_3, l_6, l_6).$$

   Applying MJ to the new profile, now x_2 defeats x_1 because the lower medians of $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$ are l_2 and l_3, respectively. However, with our procedure again x_2 defeats x_1: $M(x_1) = M(x_2) = \{l_3, l_4\}$ and $s(x_1) = -1 < 1 = s(x_2)$.

The criticisms of Zahid and de Swart (2015) on the outcomes generated by MJ are confirmed by our procedure.

3.3. Properties

We examine the main properties of the procedure proposed in our model of extended MJ for computing the mean evaluation of every alternative. In relation to any given profile matrix involving m individuals and n alternatives, as in Eq. (1), the evaluation profile of an individual refers to the corresponding matrix row, and the evaluation profile of an alternative refers to the corresponding matrix column.

1. *Anonymity*: The final ranking of the alternatives is invariant with respect to permutations of the individual evaluations of the alternatives. In other words, all voters are treated equally.
2. *Neutrality*: The final ranking of the alternatives is stable under any relabelling of the alternatives, in the sense that the same relabelling applies equally at the input and output levels. In other words, all alternatives are treated equally.
3. *Independence of irrelevant alternatives*: The final ranking between any two alternatives depends only on their own evaluation profiles, not on the evaluation profiles of any other alternatives.
4. *Monotonicity*: An alternative cannot decrease in evaluation when one voter increases the support for that alternative and the other voters maintain the previous opinions.
5. *Strong Pareto*: If all voters value one alternative better or equal than another and at least one voter values the first alternative better than the second one, then the first alternative will be ahead of the second in the final ranking.
6. *Positive responsiveness*: If two alternatives are indifferent in the final ranking and one voter improves the opinion of the first alternative compared to the second one, and the other voters maintain the previous opinions, now the first alternative will be ahead of the second one in the final ranking.
7. *Continuity/Archimedean*: If an alternative is the only winner for a subset of voters, it remains being the only winner when adding another subset of voters, whenever is allowed to replicate the opinions of the first subset of voters a number enough times.
8. *Replication invariance*: If all agents are replicated a number of times with the same assessments, then the final ranking of the alternatives does not change[10]

3.4. An illustrative example

Consider 5 agents that evaluate four alternatives, x_1, x_2, x_3 and x_4, through an OQS $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$, with the following profile

$$\begin{pmatrix} l_2 & l_4 & l_2 & l_1 \\ l_2 & l_3 & l_4 & l_4 \\ l_4 & l_3 & l_2 & l_4 \\ l_3 & l_1 & l_4 & l_4 \\ l_3 & l_3 & l_2 & l_1 \end{pmatrix},$$

[10] This property would not hold if the form of the secondary function, which measures the dispersion of multisets, was chosen as in Eq. (4), instead of Eq. (3).

Fig. 2. OPM with associated matrix A_{222}.

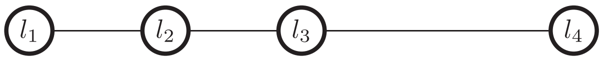

Fig. 3. OPM with associated matrix A_{224}.

i.e., $\boldsymbol{v}_1 = (l_2, l_2, l_4, l_3, l_3)$, $\boldsymbol{v}_2 = (l_4, l_3, l_3, l_1, l_3)$, $\boldsymbol{v}_3 = (l_2, l_4, l_2, l_4, l_2)$ and $\boldsymbol{v}_4 = (l_1, l_4, l_4, l_4, l_1)$.

Notice that with Range Voting the four alternatives are in a tie, because $2+2+4+3+3 = 4+3+3+1+3 = 2+4+2+4+2 = 1+4+4+4+1 = 14$ and they share the same average, $14/5 = 2.8$.

We now apply MJ and our proposal with four different OPMs[11]

1. MJ. The medians of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ are l_3, l_3, l_2 and l_4, respectively. Then, $x_4 \succ x_1 \succ x_3$ and $x_4 \succ x_2 \succ x_3$, but x_1 and x_2 are in a tie. Taking into account the tie-breaking procedure proposed by Balinski and Laraki in Balinski and Laraki (2007a, Appendix), we have $x_2 \succ x_1$: x_1 has one assessment upper the median and two lower the median, and x_2 has one assessment upper the median and one lower the median. Thus, the alternatives are ranked $x_4 \succ x_2 \succ x_1 \succ x_3$. We note that applying the tie-breaking procedures proposed by Balinski and Laraki in Balinski and Laraki (2007b, 2011), we obtain the same ranking.
2. If in our proposal we consider that the OQS is uniform and it is equipped with the OPM with associated proximity matrix.
$$A_{222} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ & \delta_1 & \delta_2 & \delta_3 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}$$
that can be visualized in Fig. 2,
we obtain $M(x_4) = l_3 \succ \{l_2, l_3\} = M(x_1) = M(x_2) = M(x_3)$, $s(x_1) = s(x_2) = s(x_3) = 1$, $N^+(x_1) = \{\!\{\delta_1, \delta_1, \delta_2\}\!\}$, $N^+(x_2) = \{\!\{\delta_1, \delta_1, \delta_1, \delta_2\}\!\}$ and $N^+(x_3) = \{\!\{\delta_2, \delta_2\}\!\}$. Since $N^+(x_2) \succ N^+(x_3) \succ N^+(x_1)$, the alternatives are ranked $x_4 \succ x_2 \succ x_3 \succ x_1$.
3. If in our proposal we consider that the OQS is not uniform and it is equipped with the OPM with associated proximity matrix
$$A_{224} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix}$$
that can be visualized in Fig. 3.
Since $M(x_2) = M(x_3) = M(x_4) = \{l_3, l_4\} \succ l_3 = M(x_1)$, $s(x_2) = s(x_3) = s(x_4) = -1$, $N^-(x_3) = \{\!\{\delta_2, \delta_2, \delta_2\}\!\} \sim \{\!\{\delta_3, \delta_3\}\!\} = N^-(x_4) \succ \{\!\{\delta_1, \delta_1, \delta_1, \delta_3\}\!\} = N^-(x_2)$ and $N^+(x_4) = \{\!\{\delta_1, \delta_1, \delta_1\}\!\} \succ \{\!\{\delta_1, \delta_1\}\!\} = N^+(x_3)$, the alternatives are ranked $x_4 \succ x_3 \succ x_2 \succ x_1$.
4. If in our proposal we consider that the OQS is not uniform and it is equipped with the OPM with associated proximity matrix
$$A_{432} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}$$
that can be visualized in Fig. 4,
we obtain $M(x_1) = M(x_2) = M(x_3) = M(x_4) = \{l_2, l_3\}$, $s(x_1) = s(x_2) = s(x_3) = 1$, $s(x_4) = -1$, $N^+(x_1) = \{\!\{\delta_1, \delta_1, \delta_2\}\!\}$, $N^+(x_2) = \{\!\{\delta_1, \delta_1, \delta_1, \delta_2\}\!\}$ and $N^+(x_3) = \{\!\{\delta_2, \delta_2\}\!\}$. Since $N^+(x_2) \succ N^+(x_3) \succ N^+(x_1)$, the alternatives are ranked $x_2 \succ x_3 \succ x_1 \succ x_4$.
5. If in our proposal we consider that the OQS is not uniform and it is equipped with the OPM with associated proximity matrix
$$A_{325} = \begin{pmatrix} \delta_1 & \delta_3 & \delta_4 & \delta_7 \\ & \delta_1 & \delta_2 & \delta_6 \\ & & \delta_1 & \delta_5 \\ & & & \delta_1 \end{pmatrix}$$
that can be visualized in Fig. 5,
we obtain $M(x_1) = M(x_2) = M(x_3) = M(x_4) = \{l_3, l_4\}$, $s(x_1) = s(x_2) = s(x_3) = s(x_4) = -1$, $N^-(x_1) = \{\!\{\delta_1, \delta_1, \delta_2, \delta_2\}\!\}$, $N^-(x_2) = \{\!\{\delta_1, \delta_1, \delta_1, \delta_4\}\!\}$, $N^-(x_3) = \{\!\{\delta_2, \delta_2, \delta_2\}\!\}$ and $N^-(x_4) = \{\!\{\delta_4, \delta_4\}\!\}$. Since $N^-(x_4) \succ N^-(x_2) \succ N^-(x_3) \succ N^-(x_1)$, the alternatives are ranked $x_1 \succ x_3 \succ x_2 \succ x_4$.

Table 1
Summary.

Model	Ranking
MJ	$x_4 \succ x_2 \succ x_1 \succ x_3$
A_{222}	$x_4 \succ x_2 \succ x_3 \succ x_1$
A_{224}	$x_4 \succ x_3 \succ x_2 \succ x_1$
A_{432}	$x_2 \succ x_3 \succ x_1 \succ x_4$
A_{325}	$x_1 \succ x_3 \succ x_2 \succ x_4$

The outcomes are summarized in Table 1.

Notice that the outcomes depend on how the OQS is perceived, i.e., depending on the OPM associated with the OQS. For instance, x_4 is the winner in the first three cases, while it is the loser in the last two cases; x_1 is ranked in the first, third or fourth position depending on the case; etc.

4. Concluding remarks

The 6-term scale used by Balinski & Laraki (2011) in their MJ voting system for political elections, {To Reject, Poor, Acceptable, Good, Very Good, Excellent}, is not necessarily perceived as being uniform (for instance, if Poor is perceived closer to To Reject than to Acceptable). García-Lapresta & Pérez-Román (2018) propose an alternative voting system to MJ by considering non-uniform OQSs through OPMs, and also in this paper, but from a different perspective.

MJ voting system was devised for large electorates that evaluate alternatives through an OQS. When the OQS is uniform, the lower median of individual assessments can be considered a good statistics, hence appropriate for representing the global assessments of the alternatives.

However, applying MJ to committees can generate some paradoxes (see Felsenthal & Machover, 2008). One of the reasons is that selecting the lower median of the individual assessments as majority-grade produces a loss of information and it could be considered as arbitrary (using the upper median the outcomes may be different, see Felsenthal & Machover, 2008, 3.2 and 3.7).

In evaluating and comparing decisional alternatives, we consider evaluation profiles expressed through the linguistic terms of an OQS $\mathcal{L}$. When the qualitative differences between consecutive linguistic terms in the OQS $\mathcal{L}$ are not perceived as uniform, an OPM $\pi : \mathcal{L} \times \mathcal{L} \longrightarrow \Delta$, expressed in terms of an OQS Δ, provides convenient ordinal information on the qualitative difference between any two linguistic terms in the OQS $\mathcal{L}$.

[11] In a 4-term OQS there are 25 metrizable OPMs (see García-Lapresta et al., 2018, 2.3).

Fig. 4. OPM with associated matrix A_{432}.

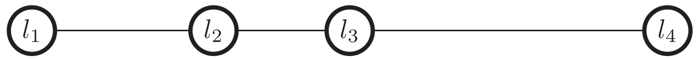

Fig. 5. OPM with associated matrix A_{325}.

On the basis of an OPM $\pi : \mathcal{L} \times \mathcal{L} \longrightarrow \Delta$, we introduce a weak order on the set of finite multisets over Δ. This weak order allows us to define the central notion of *ordinal mean* of an evaluation profile over an alternative, given by a single element or a pair of consecutive elements of $\mathcal{L}$, not necessarily contained in the evaluation profile itself. By means of this weak order, our procedure obtains a weak order between the decisional alternatives, satisfying a number of desirable properties. Various illustrative examples are provided. The procedure of our ordinal evaluation model extends and enhances the standard paradigm of MJ in a broader framework. In this sense, given the established relevance of the MJ model in the context of ranking and choice procedures using qualitate scales, we believe that our proposal might contribute to improving the quality of the solutions in actual problems of that nature.

García-Lapresta & González del Pozo (2019) allow agents to assign two consecutive linguistic terms to the alternatives, when they hesitate. It requires to extend the procedure of García-Lapresta & Pérez-Román (2018) to the new setting. It is also possible to consider the hesitation in the context of this paper, and it would be interesting for further research.

Our procedure has been devised under the assumption that voters provide sincere opinions. On the other hand, when voters act strategically, it is possible to remove extreme opinions, as is the case in swimming, skating, gymnastics, etc.

Our procedure can be extended to multiple criteria (see Greco, Ehrgott, & Figueira, 2016). García-Lapresta & Pérez-Román (2018, Sect. 5) and García-Lapresta et al. (2021) extended the voting system introduced in García-Lapresta and Pérez-Román (2018, Sect. 3) to the case of multiple criteria[12], by replicating the criteria profiles proportionally to the corresponding weights. Since our proposal falls in the same setting of García-Lapresta & Pérez-Román (2018) and García-Lapresta et al. (2021), it would be possible to apply the same replication procedure to our model.

Acknowledgments

The financial support of the Spanish Ministerio de Economía y Competitividad (project ECO2016-77900-P) and ERDF is acknowledged. The authors are grateful to the participants of VI Jornadas de Trabajo sobre Sistemas de Votación (Valdeavellano de Tera, Soria, Spain, May 2019) and The 13th International Conference on Multiple Objective Programming and Goal Programming (Marrakech, Morocco, October 2019), for their comments and suggestions. The authors are also grateful to three anonymous reviewers for their useful comments and suggestions.

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[12] In the first case by using the same OQS represented by a unique OPM for all the criteria, and in the second one allowing the use of different OQSs, represented by the corresponding OPMs, in the criteria.