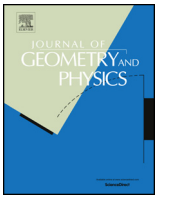
Journal of Geometry and Physics 212 (2025) 105456



Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys



Infinitesimal 2-braidings from 2-shifted Poisson structures



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ARTICLE INFO

Article history:
Received 9 November 2024
Received in revised form 5 February 2025
Accepted 16 February 2025
Available online 18 February 2025

MSC:
14A30
17B37
18N10
53D55

Keywords:
Derived algebraic geometry
Shifted Poisson structures
Lie N-algebras
Deformation quantization
Braided monoidal 2-categories

ABSTRACT

It is shown that every 2-shifted Poisson structure on a finitely generated semi-free commutative differential graded algebra A defines a very explicit infinitesimal 2-braiding on the homotopy 2-category of the symmetric monoidal dg-category of finitely generated semi-free A-dg-modules. This provides a concrete realization, to first order in the deformation parameter $\hbar$, of the abstract deformation quantization results in derived algebraic geometry due to Calaque, Pantev, Toën, Vaquié and Vezzosi. Of particular interest is the case when A is the Chevalley-Eilenberg algebra of a Lie N-algebra, where the braided monoidal deformations developed in this paper may be interpreted as candidates for representation categories of 'higher quantum groups'.

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https://doi.org/10.1016/j.geomphys.2025.105456
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1. Introduction and summary

Quantum groups are interesting and important mathematical objects which enjoy a wide range of applications in representation theory, topological quantum field theory and integrability, see e.g. [25,20,16] for surveys. There exist two complementary main approaches to the study of quantum groups: The algebraic approach based on quasi-triangular Hopf algebras and the categorical approach studying linear braided monoidal categories. The relationship between these approaches is given by the observation that a quasi-triangular Hopf algebra has a linear braided monoidal category of representations from which it can be recovered by Tannakian reconstruction. An important class of examples of quantum groups are the quantized universal enveloping algebras $U_q(\mathfrak{g})$ which are defined as quasi-triangular Hopf algebra deformations of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In the case of formal deformations $q = e^{\hbar}$, their linear braided monoidal representation categories $\mathbf{Rep}_{\hbar}(\mathfrak{g})$ can also be obtained directly as braided monoidal deformations of the linear symmetric monoidal representation category $\mathbf{Rep}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, see e.g. [6] and [20, Chapter XX].

Given the importance of quantum groups, it is natural to search for a generalization of such structures to higher category theory. Such 'higher quantum groups' are then expected to have applications in higher-categorical representation theory, higher-dimensional topological quantum field theory and higher-dimensional integrability. To the best of our knowledge, such applications have still not yet materialized fully, but there are recent research programs in this direction, see e.g. [35,9] for applications in the context of higher gauge theory and higher-dimensional integrability and also [7] for an informative overview. From a mathematical perspective, there have been various proposals to develop a theory of 'higher quantum groups' based on 'higher Lie algebras' such as Lie 2-algebras, reaching from algebraic approaches, see e.g. [26,2,3,8], to more categorical ones, see e.g. [11–13]. These attempts are distinct from the approach to 'higher quantum groups' via 'categorification' in the sense of [14,10,33,22] where 'higher quantum groups' are constructed based on classical Lie algebras.

In the present paper, we initiate a new approach to 'higher quantum groups' which is rooted in recent developments in derived algebraic geometry, in particular the theory of shifted symplectic and shifted Poisson structures [27,5,28]. Our inspiration comes from Safronov's results in [34] which show that the semi-classical data associated with quantum groups, namely Lie bialgebra structures and invariant symmetric tensors, arise naturally as the 1-shifted and 2-shifted Poisson structures on the classifying stack $\mathrm{B}G = [\mathrm{pt}/G]$ of an algebraic group G, or its infinitesimal analogue $\mathrm{B}\mathfrak{g} = [\mathrm{pt}/\mathfrak{g}]$ where $\mathfrak{g}$ denotes the Lie algebra of G.

Given an n-shifted Poisson structure on a derived stack X, for an integer $n \geq 1$, the symmetric monoidal dg-category $\mathbf{Perf}(X)$ of perfect modules over X can be quantized using the $\mathbb{E}_n$-monoidal deformation quantization results from [5]. In the case where $X = \mathrm{B}\mathfrak{g}$, one has an equivalence of symmetric monoidal dg-categories $\mathbf{Perf}(\mathrm{B}\mathfrak{g}) \simeq \mathbf{dgRep}(\mathfrak{g})$ identifying perfect modules over $\mathrm{B}\mathfrak{g}$ with (homotopy-coherent) representations of $\mathfrak{g}$ on cochain complexes. Hence, every 2-shifted Poisson structure on $\mathrm{B}\mathfrak{g}$, which by [34] is an invariant symmetric tensor in $(\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$, defines a braided monoidal deformation $\mathbf{Perf}_{\hbar}(\mathrm{B}\mathfrak{g}) \simeq \mathbf{dgRep}_{\hbar}(\mathfrak{g})$ which is expected to generalize the constructions in [6] and [20, Chapter XX] to representations of $\mathfrak{g}$ on cochain complexes. In Corollary 4.6 we provide a concrete realization of this braided monoidal deformation to all orders of $\hbar$.

Our working hypothesis is that the above picture should generalize to the case where the Lie algebra $\mathfrak{g}$ is replaced by a 'higher Lie algebra', i.e. a Lie N-algebra L for some $N \geq 2$. Inspired by [34], we will *define* the semi-classical 'higher quantum group' data on L to be given by the 2-shifted Poisson structures on the infinitesimal classifying stack $\mathrm{B}L = [\mathrm{pt}/L]$, and *propose* the braided monoidal deformation $\mathbf{Perf}_{\hbar}(\mathrm{B}L) \simeq \mathbf{dgRep}_{\hbar}(L)$ from [5] as a model for the representation category of the 'higher quantum group $U_q(L)$' with $q = e^{\hbar}$.

Instead of making use of the abstract deformation quantization results from [5], we will approach this deformation problem similarly to the case of ordinary Lie algebras in [6] and [20, Chapter XX] by breaking it into two steps: The first step is to consider only first-order deformations in $\hbar$ of the braiding, which will be controlled by a higher-categorical analogue of an infinitesimal braiding. We show that this step is rather explicit and computationally accessible. The second step consists of lifting these first-order deformations to all orders in $\hbar$, which is more difficult and points towards the necessity to develop a homotopy-coherent generalization of the concept of Drinfeld associators. Our paper provides a detailed solution for the first step and an exploratory study of the second step. The problem of developing explicit categorical deformation constructions was also addressed in the recent paper [19] by using complementary methods from factorization homology.

We will now explain our results by outlining the content of this paper. In Section 2, we recall some basic aspects of differential graded algebraic geometry that are needed in our work. In particular, we provide a brief discussion of the crucial concept of an n-shifted Poisson structure (Definition 2.6) on a finitely generated semi-free commutative differential graded algebra (CDGA) A. Since we are mainly interested in the Chevalley-Eilenberg algebras of Lie N-algebras, which carry only non-trivial stacky structure but a trivial derived structure, we can restrict ourselves to the so-called 'stacky affine' context from [30,28,29]. Hence, our focus will be on singly-graded CDGAs, which in our applications are always concentrated

in non-negative cohomological degrees. The technically more involved concepts of the doubly-graded 'stacky CDGAs' from [30,28,29] or the graded mixed CDGAs from [5] will not be needed in our present work.

In Section 3, we study deformations of the symmetric monoidal dg-category ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ of finitely generated semi-free A-dg-modules into a braided monoidal category. More precisely, we restrict our attention to the simpler problem of deforming only the homotopy 2-category

$$ {}_A\mathbf{C} := \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) $$

of this symmetric monoidal dg-category. The homotopy 2-category ${}_A\mathbf{C}$ has the same underlying objects, i.e. finitely generated semi-free A-dg-modules, and its morphisms are given by the good truncations to cohomological degrees $\{-1,0\}$ of the morphism complexes in ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$, see Appendix A.3. Our motivation behind this truncation of dg-categories to their homotopy 2-categories is that the latter are 2-categorical objects (in contrast to ∞-categorical ones) whose deformations can be described very explicitly in the context of the 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ from Proposition A.4. Most importantly, in this context every tower of homotopy coherence data is automatically truncated, therefore making it accessible with our rather direct computational techniques. A further motivation is that our main examples of interest arise from Lie 2-algebras and their representations (see Section 4), which can be considered naturally in this truncated 2-categorical context. We show in Proposition 3.1 that, to first order in the deformation parameter $\hbar$, the braided monoidal deformations of the homotopy 2-category ${}_A\mathbf{C}$ are governed by infinitesimal 2-braidings as in Definition 3.2. Our concept of infinitesimal 2-braidings is compatible with the one proposed by Cirio and Faria Martins [12] in the context of categories enriched over 2-vector spaces, see Remark 3.5.

Our main results are presented in Subsection 3.2, where we show that every 2-shifted Poisson structure on a finitely generated semi-free CDGA A defines, by a very explicit construction, a γ-equivariant infinitesimal 2-braiding on ${}_A\mathbf{C}$, see in particular Theorem 3.10. This means that every 2-shifted Poisson structure on A defines an explicit first-order braided monoidal deformation of the homotopy 2-category of the symmetric monoidal dg-category ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$, which realizes concretely the abstract deformation results from [5] to first order in $\hbar$.

In Subsection 3.3, we explore the behavior of these first-order deformations at higher orders in the deformation parameter $\hbar$. Following the conventional theory of infinitesimal braidings from [6] and [20, Chapter XX], we observe in Proposition 3.13 that a γ-equivariant infinitesimal 2-braiding defines an all-order in $\hbar$ braided monoidal deformation of the homotopy 2-category ${}_A\mathbf{C}$ provided that certain identities which are similar to Drinfeld associators hold true strictly. We observe in Proposition 3.15 that, in general, these identities do not hold true strictly, but only up to homotopies which we construct explicitly up to order $\hbar^2$. This suggests that lifting our first-order deformation construction to all orders in $\hbar$ requires a homotopy-coherent generalization of the concept of Drinfeld associators, involving suitable hexagonators and pentagonators, which we however currently do not know how to develop. We believe that combining the construction of Drinfeld associators from parallel transports (see e.g. [4]) with the categorified Knizhnik-Zamolodchikov connections from [11,13] will provide a suitable framework to attack this problem in future work.

In Section 4, we apply our deformation quantization construction from Section 3 to explicit examples associated with Lie N-algebras. Following [28,29], we describe the infinitesimal classifying stack $\mathrm{B}L = [\mathrm{pt}/L]$ of a Lie N-algebra L by its associated Chevalley-Eilenberg algebra $\mathrm{CE}^\bullet(L)$. For finite-dimensional L, this is a finitely generated semi-free CDGA, hence our techniques from the previous sections apply to this case. In particular, our main Theorem 3.10 shows that every 2-shifted Poisson structure on $\mathrm{CE}^\bullet(L)$ defines an explicit first-order braided monoidal deformation of the homotopy 2-category

$$ {}_{\mathrm{CE}^\bullet(L)}\mathbf{C} \simeq \mathsf{Ho}_2\big(\mathbf{dgRep}(L)\big) $$

of the symmetric monoidal dg-category of representations of the Lie N-algebra L and their ∞-morphisms. In Subsection 4.1, we will specialize this result to the case where $L = \mathfrak{g}$ is an ordinary Lie algebra, which leads to a homotopical generalization of the traditional constructions in [6] and [20, Chapter XX], and in Subsection 4.2 to the case where $L = \mathfrak{g}_\kappa$ is the string Lie 2-algebra obtained from an ordinary Lie algebra $\mathfrak{g}$ and a 3-cocycle $\kappa : \bigwedge^3 \mathfrak{g} \to \mathbb{K}$.

In the technical Appendix A, we develop and present a suitable higher-categorical framework in which our deformation constructions in this paper are carried out. The main concept defined here is the symmetric monoidal 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ from Proposition A.4 whose objects are categories enriched over 2-term cochain complexes concentrated in degrees -1 and 0, morphisms are enriched functors, 2-morphisms are enriched *pseudo*-natural transformations and 3-morphisms are enriched modifications. Working with pseudo-natural transformations provides the required flexibility for the deformation constructions of this paper. Indeed, the homotopy 2-categories ${}_A\mathbf{C} := \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ and their braided monoidal deformations from Section 3 are symmetric or, respectively, braided monoidal category objects in the symmetric monoidal 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$.

2. Differential graded algebraic geometry

2.1. Cochain complexes of modules

Throughout this work, we fix a field $\mathbb{K}$ of characteristic 0. Let R be a commutative, associative and unital $\mathbb{K}$-algebra. We denote the category of (not necessarily bounded) cochain complexes of R-modules by $\mathbf{Ch}_R$. Concretely, an object $V =$

$(V, \mathrm{d}_V) \in \mathbf{Ch}_R$ is a pair consisting of a family $V = \{V^i\}_{i\in\mathbb{Z}}$ of R-modules labeled by their degrees $i \in \mathbb{Z}$ and a family $\mathrm{d}_V = \{\mathrm{d}_V^i : V^i \to V^{i+1}\}_{i\in\mathbb{Z}}$ of degree-increasing R-linear maps (called the differential) which squares to zero, i.e. $\mathrm{d}_V\,\mathrm{d}_V = 0$. A morphism $f : V \to W$ in $\mathbf{Ch}_R$ is a family $f = \{f^i : V^i \to W^i\}_{i\in\mathbb{Z}}$ of degree-preserving R-linear maps which commutes with the differentials, i.e. $\mathrm{d}_W\, f = f\,\mathrm{d}_V$.

The category $\mathbf{Ch}_R$ is closed symmetric monoidal with respect to the following structures: The tensor product $V \otimes_R W \in \mathbf{Ch}_R$ of two cochain complexes $V, W \in \mathbf{Ch}_R$ is defined by the relative tensor product of R-modules

$$(V \otimes_R W)^i := \bigoplus_{j\in\mathbb{Z}} \big(V^j \otimes_R W^{i-j}\big) \quad , \tag{2.1a}$$

for all $i \in \mathbb{Z}$, and the differential

$$\mathrm{d}_{V\otimes_R W}(v \otimes_R w) := \mathrm{d}_V(v) \otimes_R w + (-1)^{|v|}\, v \otimes_R \mathrm{d}_W(w) \quad , \tag{2.1b}$$

for all homogeneous $v \in V$ and $w \in W$, where $|v| \in \mathbb{Z}$ denotes the degree of v. The monoidal unit is $R \in \mathbf{Ch}_R$ concentrated in degree 0 with trivial differential $\mathrm{d}_R = 0$. The symmetric braiding is defined by the Koszul sign rule

$$\gamma_{V,W} : V \otimes_R W \longrightarrow W \otimes_R V \ , \quad v \otimes_R w \longmapsto (-1)^{|v|\,|w|}\, w \otimes_R v \quad , \tag{2.2}$$

for all homogeneous $v \in V$ and $w \in W$. The internal hom $\underline{\hom}_R(V, W) \in \mathbf{Ch}_R$ between two cochain complexes $V, W \in \mathbf{Ch}_R$ is defined by the internal hom of R-modules

$$\underline{\hom}_R(V, W)^i := \prod_{j\in\mathbb{Z}} \hom_R\big(V^j, W^{j+i}\big) \quad , \tag{2.3a}$$

for all $i \in \mathbb{Z}$, and the differential

$$\mathrm{d}_{\underline{\hom}_R(V,W)}(f) := \mathrm{d}_W\, f - (-1)^{|f|}\, f\, \mathrm{d}_V \quad , \tag{2.3b}$$

for all homogeneous $f \in \underline{\hom}_R(V, W)$.

In the special case where $R = \mathbb{K}$ is the base field, we shall simply write $(\mathbf{Ch}_\mathbb{K}, \otimes, \mathbb{K}, \underline{\hom})$ for the closed symmetric monoidal structure, suppressing the subscript $_\mathbb{K}$ from our notations.

Given any integer $n \in \mathbb{Z}$ and any cochain complex $V \in \mathbf{Ch}_R$, we define the n-shifted cochain complex $V[n] \in \mathbf{Ch}_R$ by $V[n]^i := V^{i+n}$, for all $i \in \mathbb{Z}$, and $\mathrm{d}_{V[n]} := (-1)^n\, \mathrm{d}_V$. Using the identification $V[n] \cong R[n] \otimes_R V$, we shall write elements of the n-shifted cochain complex as $s^{-n}\, v \in V[n]$, where $s^{-n} \in R[n]$ denotes the degree $-n$ element represented by the unit of R.

2.2. Semi-free CDGAs and dg-modules

A commutative differential graded algebra over the field $\mathbb{K}$ (in short, a CDGA) is a commutative algebra object $A = (A, \mathrm{d}_A, \mu_A, \eta_A)$ in the closed symmetric monoidal category $\mathbf{Ch}_\mathbb{K}$. More explicitly, a CDGA is a cochain complex $A = (A, \mathrm{d}_A) \in \mathbf{Ch}_\mathbb{K}$ endowed with $\mathbf{Ch}_\mathbb{K}$-morphisms $\mu_A : A \otimes A \to A$ (called multiplication) and $\eta_A : \mathbb{K} \to A$ (called unit) which satisfy the usual associativity, unitality and commutativity axioms internal to $\mathbf{Ch}_\mathbb{K}$.

Definition 2.1. A CDGA $A = (A, \mathrm{d}_A, \mu_A, \eta_A)$ is called *semi-free* if its underlying $\mathbb{Z}$-graded commutative algebra $A^\sharp = (A, \mu_A, \eta_A)$ is free, i.e.

$$A^\sharp \cong \mathrm{Sym}(V^\sharp) \tag{2.4}$$

is isomorphic to the symmetric algebra of a $\mathbb{Z}$-graded vector space $V^\sharp$. A semi-free CDGA is called *finitely generated* if the total dimension $\dim(V^\sharp) := \sum_{i\in\mathbb{Z}} \dim(V^i) < \infty$ of the $\mathbb{Z}$-graded vector space $V^\sharp$ is finite.

Associated with any CDGA A is its dg-category ${}_A\mathbf{dgMod}$ of (left) A-dg-modules. More explicitly, an object $M = (M, \mathrm{d}_M, \ell_M)$ in this category is a cochain complex $M = (M, \mathrm{d}_M) \in \mathbf{Ch}_\mathbb{K}$ endowed with a $\mathbf{Ch}_\mathbb{K}$-morphism $\ell_M : A \otimes M \to M$ (called left action) which satisfies the usual left module axioms internal to $\mathbf{Ch}_\mathbb{K}$. The cochain complex of morphisms between two objects $M, N \in {}_A\mathbf{dgMod}$ is defined as the subcomplex

$$\underline{\hom}_A(M, N) \subseteq \underline{\hom}(M, N) \tag{2.5a}$$

of the internal hom in $\mathbf{Ch}_\mathbb{K}$ containing the homogeneous elements $f \in \underline{\hom}(M, N)$ which commute with the left actions, i.e.

$$f\big(\ell_M(a \otimes m)\big) = (-1)^{|f|\,|a|}\, \ell_N\big(a \otimes f(m)\big) \quad , \tag{2.5b}$$

for all $a \in A$ and $m \in M$. The dg-category ${}_A\mathbf{dgMod}$ is closed symmetric monoidal with respect to the relative tensor product

$$M \otimes_A N := \operatorname{colim}\left(M \otimes N \underset{\mathrm{id}_M \otimes \ell_N}{\overset{(\ell_M \otimes \mathrm{id}_N)\,(\gamma_{M,A} \otimes \mathrm{id}_N)}{\leftleftarrows}} M \otimes A \otimes N \right) \in {}_A\mathbf{dgMod} \quad , \tag{2.6}$$

the monoidal unit $A = (A, \mathrm{d}_A, \mu_A) \in {}_A\mathbf{dgMod}$, the symmetric braiding

$$\gamma_{M,N} : M \otimes_A N \longrightarrow N \otimes_A M \,, \quad m \otimes_A n \longmapsto (-1)^{|m|\,|n|}\, n \otimes_A m \quad , \tag{2.7}$$

for all homogeneous $m \in M$ and $n \in N$, and the internal hom

$$\underline{\hom}_A(M, N) \in {}_A\mathbf{dgMod} \tag{2.8}$$

given by the cochain complex (2.5) and the left action $A \otimes \underline{\hom}_A(M,N) \to \underline{\hom}_A(M,N)\,,\ a \otimes f \mapsto a \cdot f$ defined by $(a \cdot f)(m) := \ell_N(a \otimes f(m))$, for all $a \in A$, $f \in \underline{\hom}_A(M,N)$ and $m \in M$.

Definition 2.2. An A-dg-module $M = (M, \mathrm{d}_M, \ell_M)$ is called *semi-free* if its underlying $A^\sharp$-module $M^\sharp = (M, \ell_M)$ is free, i.e.

$$M^\sharp \cong A^\sharp \otimes W^\sharp \tag{2.9}$$

for a $\mathbb{Z}$-graded vector space $W^\sharp$. A semi-free A-dg-module is called *finitely generated* if the total dimension $\dim(W^\sharp) := \sum_{i\in\mathbb{Z}} \dim(W^i) < \infty$ of the $\mathbb{Z}$-graded vector space $W^\sharp$ is finite. We denote by

$${}_A\mathbf{dgMod}^{\mathrm{fgsf}} \subseteq {}_A\mathbf{dgMod} \tag{2.10}$$

the full dg-subcategory whose objects are all finitely generated semi-free A-dg-modules.

Remark 2.3. The closed symmetric monoidal structure on ${}_A\mathbf{dgMod}$ restricts to a closed symmetric monoidal structure on the full dg-subcategory ${}_A\mathbf{dgMod}^{\mathrm{fgsf}} \subseteq {}_A\mathbf{dgMod}$. Indeed, the monoidal unit $A \in {}_A\mathbf{dgMod}$ is finitely generated semi-free and, given any two finitely generated semi-free A-dg-modules $M, N \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ with $M^\sharp \cong A^\sharp \otimes W^\sharp$ and $N^\sharp \cong A^\sharp \otimes U^\sharp$, we have isomorphisms

$$\big(M \otimes_A N\big)^\sharp \cong A^\sharp \otimes \big(W^\sharp \otimes U^\sharp\big) \tag{2.11a}$$

and

$$\underline{\hom}_A(M,N)^\sharp \cong A^\sharp \otimes \big(U^\sharp \otimes (W^\sharp)^*\big) \quad , \tag{2.11b}$$

which imply that $M \otimes_A N \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ and $\underline{\hom}_A(M,N) \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ are finitely generated semi-free. Here $(W^\sharp)^*$ denotes the dual of the $\mathbb{Z}$-graded vector space $W^\sharp$, which exists since $W^\sharp$ is, by hypothesis, of finite total dimension. △

2.3. Derivations, Kähler differentials and shifted Poisson structures

We recall some basic aspects of the geometry of a finitely generated semi-free CDGA A, i.e. $A^\sharp \cong \mathrm{Sym}(V^\sharp)$ for some $\mathbb{Z}$-graded vector space $V^\sharp$ of finite total dimension. The dg-module of derivations $\mathsf{T}_A \in {}_A\mathbf{dgMod}$ of A is defined as the subcomplex

$$\mathsf{T}_A \subseteq \underline{\hom}(A, A) \tag{2.12a}$$

of the internal hom in $\mathbf{Ch}_{\mathbb{K}}$ containing the homogeneous elements $D \in \underline{\hom}(A,A)$ which satisfy the Leibniz rule

$$D(a\,a') = D(a)\,a' + (-1)^{|D|\,|a|}\, a\, D(a') \quad , \tag{2.12b}$$

for all homogeneous $a, a' \in A$. The A-dg-module structure $A \otimes \mathsf{T}_A \to \mathsf{T}_A\,,\ a \otimes D \mapsto a \cdot D$ is given by $(a \cdot D)(a') := a\, D(a')$, for all $a, a' \in A$ and $D \in \mathsf{T}_A$. Since A is finitely generated semi-free, each derivation is completely specified by its action on the generators $V^\sharp$, which yields an isomorphism

$$(\mathsf{T}_A)^\sharp \cong A^\sharp \otimes (V^\sharp)^* \quad , \tag{2.13}$$

where $(V^\sharp)^*$ denotes the dual of the $\mathbb{Z}$-graded vector space $V^\sharp$. Hence, $\mathsf{T}_A \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ is a finitely generated semi-free A-dg-module.

The underlying cochain complex of T_A can be endowed with the Lie algebra structure given by the commutator

$$[\,\cdot\,,\,\cdot\,] : \mathsf{T}_A \otimes \mathsf{T}_A \longrightarrow \mathsf{T}_A \,, \quad D \otimes D' \longmapsto [D, D'] := D\,D' - (-1)^{|D|\,|D'|}\, D'\,D \quad , \tag{2.14}$$

for all homogeneous $D, D' \in \mathsf{T}_A$. The Lie algebra and A-dg-module structures on T_A satisfy the compatibility condition

$$[D, a \cdot D'] = D(a) \cdot D' + (-1)^{|D||a|}\, a \cdot [D, D'] \quad , \tag{2.15}$$

for all homogeneous $D, D' \in \mathsf{T}_A$ and $a \in A$.

As a consequence of our hypothesis that A is a finitely generated semi-free CDGA, the dg-module of derivations $\mathsf{T}_A \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ is dualizable. We define the dg-module of Kähler differentials by the internal hom

$$\Omega_A := \underline{\hom}_A(\mathsf{T}_A, A) \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}} \quad , \tag{2.16a}$$

which is finitely generated semi-free as a consequence of Remark 2.3 and

$$(\Omega_A)^\sharp \cong A^\sharp \otimes (V^\sharp)^{**} \cong A^\sharp \otimes V^\sharp \quad . \tag{2.16b}$$

We denote the duality pairings between Ω_A and T_A by

$$\langle \cdot\,, \cdot \rangle : \Omega_A \otimes_A \mathsf{T}_A \longrightarrow A\,, \quad \omega \otimes_A D \longmapsto \langle \omega, D \rangle := \omega(D) \tag{2.17a}$$

and

$$\langle \cdot\,, \cdot \rangle : \mathsf{T}_A \otimes_A \Omega_A \longrightarrow A\,, \quad D \otimes_A \omega \longmapsto \langle D, \omega \rangle := (-1)^{|D||\omega|}\, \omega(D) \quad , \tag{2.17b}$$

for all homogeneous $\omega \in \Omega_A$ and $D \in \mathsf{T}_A$. The de Rham differential $\mathrm{d}_{\mathrm{dR}} : A \to \Omega_A$ is defined by

$$\big\langle D, \mathrm{d}_{\mathrm{dR}}(a) \big\rangle := D(a) \quad , \tag{2.18}$$

for all $a \in A$ and $D \in \mathsf{T}_A$. From this definition one directly checks that $\mathrm{d}_{\mathrm{dR}}(a\,a') = \mathrm{d}_{\mathrm{dR}}(a) \cdot a' + a \cdot \mathrm{d}_{\mathrm{dR}}(a')$, for all $a, a' \in A$, and that $\mathrm{d}_{\Omega_A}\, \mathrm{d}_{\mathrm{dR}} = \mathrm{d}_{\mathrm{dR}}\, \mathrm{d}_A$.

A crucial preliminary concept for the definition of shifted Poisson structures is that of shifted polyvectors.

Definition 2.4. Let $n \in \mathbb{Z}$ be an integer. The CDGA of *n-shifted polyvectors* on a finitely generated semi-free CDGA A is defined as the relative symmetric algebra

$$\mathrm{Pol}(A, n) := \mathrm{Sym}_A\big(\mathsf{T}_A[-n-1]\big) = \bigoplus_{m \geq 0} \mathrm{Sym}^m_A\big(\mathsf{T}_A[-n-1]\big) \tag{2.19}$$

of the $(-n-1)$-shift of the A-dg-module of derivations T_A. The non-negative integer $m \in \mathbb{Z}^{\geq 0}$ in the direct sum decomposition in terms of symmetric powers is called the *weight* of polyvectors.

The CDGA of n-shifted polyvectors $\mathrm{Pol}(A, n)$ can be endowed with a canonical $\mathbb{P}_{n+2}$-algebra structure (called Schouten-Nijenhuis bracket)

$$\{\cdot\,, \cdot\} \in \underline{\hom}\Big(\mathrm{Pol}(A, n) \otimes \mathrm{Pol}(A, n), \mathrm{Pol}(A, n)\Big)^{-n-1} \tag{2.20}$$

which is defined on the generators by

$$\{a, a'\} := 0 \quad , \tag{2.21a}$$

$$\{s^{n+1} D, a\} := D(a) \quad , \tag{2.21b}$$

$$\{s^{n+1} D, s^{n+1} D'\} := (-1)^{|D|\,(n+1)}\, s^{n+1}[D, D'] \quad , \tag{2.21c}$$

for all homogeneous $a, a' \in A$ and $D, D' \in \mathsf{T}_A$, where we recall that $D(a) \in A$ denotes the evaluation of the derivation D on $a \in A$ and $[D, D'] \in \mathsf{T}_A$ denotes the Lie bracket (2.14) on T_A. The bracket $\{\cdot\,, \cdot\}$ is then extended to $\mathrm{Pol}(A, n)$ by the defining properties of a $\mathbb{P}_{n+2}$-algebra structure, i.e.

(i) Antisymmetry: For all homogeneous $P, Q \in \mathrm{Pol}(A, n)$,

$$\{P, Q\} = -(-1)^{(n+1)}\, (-1)^{|P||Q|}\, \{Q, P\} \quad . \tag{2.22a}$$

(ii) Derivation property: For all homogeneous $P, Q, R \in \mathrm{Pol}(A, n)$,

$$\{P, Q\, R\} = \{P, Q\}\, R + (-1)^{(|P|-n-1)\,|Q|}\, Q\, \{P, R\} \quad . \tag{2.22b}$$

(iii) Jacobi identity: For all homogeneous $P, Q, R \in \mathrm{Pol}(A, n)$,

$$\big\{P, \{Q, R\}\big\} = (-1)^{(|P|-n-1)\,(n+1)}\, \big\{\{P, Q\}, R\big\} + (-1)^{(|P|-n-1)\,(|Q|-n-1)}\, \big\{Q, \{P, R\}\big\} \quad . \tag{2.22c}$$

Note that the Schouten-Nijenhuis bracket decreases the polyvector weight by 1, i.e. for $P, Q \in \mathrm{Pol}(A,n)$ of weight m_P and m_Q, the weight of $\{P,Q\} \in \mathrm{Pol}(A,n)$ is $m_P + m_Q - 1$.

The definition of n-shifted Poisson structures in [5] and [28,29] uses a completion of the $\mathbb{P}_{n+2}$-algebra $\mathrm{Pol}(A,n)$ in order to avoid bounds on the weights of polyvectors. The analogous definition in our context is as follows.

Definition 2.5. The $\mathbb{P}_{n+2}$-algebra of *completed n-shifted polyvectors* on a finitely generated semi-free CDGA A is defined by

$$\widehat{\mathrm{Pol}}(A,n) := \prod_{m\geq 0} \mathrm{Sym}^m_A\big(\mathsf{T}_A[-n-1]\big) \quad . \tag{2.23}$$

A completed n-shifted polyvector is thus a formal sum $P = \sum_{m\geq 0} P^{(m)}$ of homogeneous weight components $P^{(m)} \in \mathrm{Sym}^m_A\big(\mathsf{T}_A[-n-1]\big)$. The $\mathbb{P}_{n+2}$-algebra structure on $\widehat{\mathrm{Pol}}(A,n)$ is given by

$$P\,Q := \sum_{m\geq 0}\Big(\sum_{k+l=m} P^{(k)}\,Q^{(l)}\Big) \quad , \tag{2.24a}$$

$$\{P,Q\} := \sum_{m\geq 0}\Big(\sum_{k+l-1=m} \big\{P^{(k)}, Q^{(l)}\big\}\Big) \quad , \tag{2.24b}$$

which is well-defined because the weight is bounded from below and hence the sums in the parentheses are finite.

We now define the key concept of an n-shifted Poisson structure following [28, Definition 1.5], see also [29, Definition 2.5].

Definition 2.6. An *n-shifted Poisson structure* on a finitely generated semi-free CDGA A is a completed n-shifted polyvector

$$\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}(A,n)^{n+2} \tag{2.25}$$

of degree $n+2$ and weight ≥ 2 which satisfies the Maurer-Cartan equation

$$\mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi) + \tfrac{1}{2}\{\pi,\pi\} = 0 \quad . \tag{2.26}$$

Remark 2.7. Decomposing the Maurer-Cartan equation into homogeneous weight components one obtains a tower of conditions

$$\begin{aligned} \mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi^{(2)}) &= 0 \quad , \\ \mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi^{(3)}) + \tfrac{1}{2}\{\pi^{(2)},\pi^{(2)}\} &= 0 \quad , \\ &\vdots \\ \mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi^{(m)}) + \tfrac{1}{2}\sum_{k+l-1=m}\{\pi^{(k)},\pi^{(l)}\} &= 0 \quad , \end{aligned} \tag{2.27}$$

for all $m \geq 3$. The first condition states that the bivector component $\pi^{(2)}$ is closed with respect to the differential $\mathrm{d}_{\widehat{\mathrm{Pol}}}$, i.e. it defines an $(n+2)$-cocycle in $\mathrm{Sym}^2_A\big(\mathsf{T}_A[-n-1]\big)$. The second condition provides a homotopical relaxation of the usual condition that a Poisson bivector has a trivial Schouten-Nijenhuis bracket with itself, with the trivector $\pi^{(3)}$ playing the role of a homotopy witnessing this condition. The higher weight components $\pi^{(m)}$ for $m \geq 4$ describe a coherent tower of higher homotopies for this relaxation. △

3. Infinitesimal 2-braidings

The aim of this section is to study a class of first-order deformations, into a braided monoidal category, of the symmetric monoidal category ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ of finitely generated semi-free dg-modules over a finitely generated semi-free CDGA A over the field $\mathbb{K}$. The precise context in which our deformation constructions take place is given by the symmetric monoidal 3-category $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}_R$ from Proposition A.4, consisting of categories, functors, *pseudo-natural* transformations and modifications that are suitably enriched over the symmetric monoidal category $\mathbf{Ch}^{[-1,0]}_R$ of 2-term cochain complexes concentrated in degrees -1 and 0. Our starting point is thus given by the homotopy 2-category

$${}_A\mathbf{C} := \mathsf{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \in \mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}} \tag{3.1}$$

from Appendix A.3, which we endow with the symmetric monoidal structure that is induced from the standard one on ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ given in (2.6) and (2.7).

The effect of forming the homotopy 2-category is that the cochain complex $\underline{\hom}_A(M,N) \in \mathbf{Ch}_{\mathbb{K}}$ of morphisms between any two dg-modules $M, N \in {}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ from (2.5) gets replaced by its good truncation ${}_A\mathbf{C}(M,N) := \tau^{[-1,0]}\big(\underline{\hom}_A(M,N)\big) \in \mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$ to degrees -1 and 0. Concretely, this means that the homotopy 2-category ${}_A\mathbf{C} = \mathrm{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ describes only degree 0 morphisms between A-dg-modules which commute with the differentials and 1-homotopies (modulo 2-homotopies) between such morphisms, but it does not include information about higher homotopies. Our motivation behind this truncation of the dg-category ${}_A\mathbf{dgMod}^{\mathrm{fgsf}}$ is that the homotopy 2-category ${}_A\mathbf{C} = \mathrm{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ is a 2-categorical object whose deformations can be described very explicitly in the context of the 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ from Proposition A.4. Most importantly, in this context every tower of homotopy coherence data is automatically truncated, therefore making it accessible with our more direct computational techniques. A further motivation is that our main examples of interest arise from Lie 2-algebras and their representations (see Section 4), which can be considered naturally in this truncated 2-categorical context. Our main result is that every 2-shifted Poisson structure on the CDGA A as in Definition 2.6 defines a very explicit first-order deformation of the braiding on ${}_A\mathbf{C}$.

3.1. Setup and definitions

To set up a framework for studying first-order deformations of the symmetric monoidal category object ${}_A\mathbf{C}$ from (3.1) in the symmetric monoidal 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ from Proposition A.4, we introduce a deformation parameter $\hbar$ which squares to zero $\hbar^2 = 0$. Performing a change of base along the commutative $\mathbb{K}$-algebra morphism $\mathbb{K} \to \mathbb{K}[\hbar]/(\hbar^2)$ defines an object

$$ {}_A\mathbf{C}^{\hbar} := {}_A\mathbf{C} \otimes \mathbb{K}[\hbar]/(\hbar^2) \in \mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}[\hbar]/(\hbar^2)} \quad , \tag{3.2} $$

which we endow with the symmetric monoidal structure that is induced from the one on ${}_A\mathbf{C}$. More concretely, ${}_A\mathbf{C}^{\hbar}$ is the $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[\hbar]/(\hbar^2)}$-enriched category with the same objects as ${}_A\mathbf{C}$, i.e. finitely generated semi-free A-dg-modules, and whose morphisms are given by ${}_A\mathbf{C}^{\hbar}(M,N) = {}_A\mathbf{C}(M,N) \otimes \mathbb{K}[\hbar]/(\hbar^2) \in \mathbf{Ch}^{[-1,0]}_{\mathbb{K}[\hbar]/(\hbar^2)}$, for all $M, N \in {}_A\mathbf{C}^{\hbar}$. Following the notation from Appendix A.2, we shall denote the symmetric braiding on ${}_A\mathbf{C}^{\hbar}$ by

$$ \begin{array}{ccc} {}_A\mathbf{C}^{\hbar} \boxtimes {}_A\mathbf{C}^{\hbar} & \xrightarrow{\otimes_A} & {}_A\mathbf{C}^{\hbar} \\ & {\scriptstyle\tau} \searrow \quad \Downarrow \gamma \quad \nearrow {\scriptstyle\otimes_A} & \\ & {}_A\mathbf{C}^{\hbar} \boxtimes {}_A\mathbf{C}^{\hbar} & \end{array} \quad , \tag{3.3} $$

where τ is an abbreviation for the component $\tau_{{}_A\mathbf{C}^{\hbar},{}_A\mathbf{C}^{\hbar}}$ of the symmetric braiding in (A.30) of the ambient 3-category $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}[\hbar]/(\hbar^2)}$. When written in terms of the 3-categorical composition operations from Appendix A.2, the symmetric braiding reads as $\gamma : \otimes_A \Rightarrow \otimes_A * \tau$. Note that, by construction, this is a *strict* $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[\hbar]/(\hbar^2)}$-enriched natural transformation.

In the context of (semi-strict) braided monoidal category objects in $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}[\hbar]/(\hbar^2)}$ from Definition A.8, we would like to study deformations of the braiding $\gamma : \otimes_A \Rightarrow \otimes_A * \tau$ of ${}_A\mathbf{C}^{\hbar}$ which keep all the other structures (i.e. the tensor product, monoidal unit, associator and unitors) fixed. The general ansatz for a first-order deformation of the braiding $\gamma : \otimes_A \Rightarrow \otimes_A * \tau$ of ${}_A\mathbf{C}^{\hbar}$ is obtained by considering a $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[\hbar]/(\hbar^2)}$-enriched *pseudo-natural* transformation[1] of the form

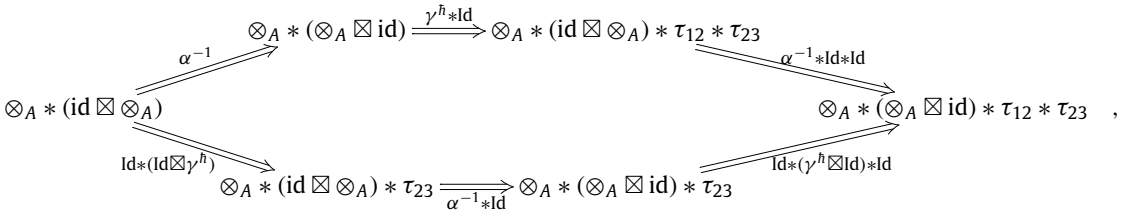
$$ \gamma^{\hbar} := \gamma \circ \big(\mathrm{Id} + \tfrac{\hbar}{2}\, t\big) : \otimes_A \Longrightarrow \otimes_A * \tau : {}_A\mathbf{C}^{\hbar} \boxtimes {}_A\mathbf{C}^{\hbar} \longrightarrow {}_A\mathbf{C}^{\hbar} \tag{3.4} $$

which satisfies the two hexagon identities

$$ \begin{array}{ccccc} & & \otimes_A * (\mathrm{id} \boxtimes \otimes_A) & \xRightarrow{\gamma^{\hbar} * \mathrm{Id}} & \otimes_A * (\otimes_A \boxtimes \mathrm{id}) * \tau_{23} * \tau_{12} \\ & {\scriptstyle\alpha} \nearrow & & & \searrow {\scriptstyle\alpha * \mathrm{Id} * \mathrm{Id}} \\ \otimes_A * (\otimes_A \boxtimes \mathrm{id}) & & & & \otimes_A * (\mathrm{id} \boxtimes \otimes_A) * \tau_{23} * \tau_{12} \\ & {\scriptstyle \mathrm{Id} * (\gamma^{\hbar} \boxtimes \mathrm{Id})} \searrow & & & \nearrow {\scriptstyle \mathrm{Id} * (\mathrm{Id} \boxtimes \gamma^{\hbar}) * \mathrm{Id}} \\ & & \otimes_A * (\otimes_A \boxtimes \mathrm{id}) * \tau_{12} & \xRightarrow[\alpha * \mathrm{Id}]{} & \otimes_A * (\mathrm{id} \boxtimes \otimes_A) * \tau_{12} \end{array} \tag{3.5a} $$

[1] We would like to emphasize that pseudo-naturality, in contrast to strict naturality, is crucial here in order to accommodate our deformation quantization constructions in Subsection 3.2 below.

and



(3.5b)

where α denotes the associator and we use the typical short-hand notations $\tau_{12} := \tau \boxtimes \mathrm{id}$ and $\tau_{23} := \mathrm{id} \boxtimes \tau$. Note that in the top horizontal arrows of these diagrams we applied the properties (A.31) and (A.32) of the symmetric braiding τ. The composition of the $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[\hbar]/(\hbar^2)}$-enriched pseudo-natural transformations along the hexagons (3.5) is given by the $\circ$-composition from Appendix A.2 and these diagrams are required to commute strictly, i.e. up to equality of enriched pseudo-natural transformations. Strictness of the hexagon identities turns out to be sufficient for our first-order deformations in Subsection 3.2, but we will illustrate in Subsection 3.3 that higher-order deformations require in general non-trivial hexagonator modifications for the diagrams (3.5).

Leveraging the fact that $\hbar^2 = 0$, the hexagon identities are equivalent to two linear relations for the deformation t, which can be regarded as the datum of a $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}}$-enriched pseudo-natural transformation (without change of base along $\mathbb{K} \to \mathbb{K}[\hbar]/(\hbar^2)$) of the form $t : \otimes_A \Rightarrow \otimes_A$ for the relative tensor product on ${}_A\mathbf{C}$.

Proposition 3.1. *The first hexagon identity* (3.5a) *for the deformed braiding* (3.4) *is equivalent to the identity*

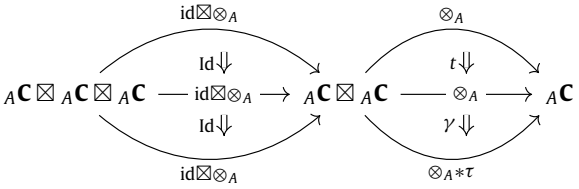
$$\alpha^{-1} \circ (t * \mathrm{Id}) \circ \alpha \;=\; \mathrm{Id} * (t \boxtimes \mathrm{Id}) + \big(\mathrm{Id} * (\gamma \boxtimes \mathrm{Id})^{-1}\big) \circ \Big(\big(\alpha^{-1} \circ \big(\mathrm{Id} * (\mathrm{Id} \boxtimes t)\big) \circ \alpha\big) * \mathrm{Id}\Big) \circ \big(\mathrm{Id} * (\gamma \boxtimes \mathrm{Id})\big) \tag{3.6a}$$

of $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}}$*-enriched pseudo-natural transformations* $\otimes_A * (\otimes_A \boxtimes \mathrm{id}) \Rightarrow \otimes_A * (\otimes_A \boxtimes \mathrm{id})$. *The second hexagon identity* (3.5b) *for the deformed braiding* (3.4) *is equivalent to the identity*

$$\alpha \circ (t * \mathrm{Id}) \circ \alpha^{-1} \;=\; \mathrm{Id} * (\mathrm{Id} \boxtimes t) + \big(\mathrm{Id} * (\mathrm{Id} \boxtimes \gamma)^{-1}\big) \circ \Big(\big(\alpha \circ \big(\mathrm{Id} * (t \boxtimes \mathrm{Id})\big) \circ \alpha^{-1}\big) * \mathrm{Id}\Big) \circ \big(\mathrm{Id} * (\mathrm{Id} \boxtimes \gamma)\big) \tag{3.6b}$$

of $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}}$*-enriched pseudo-natural transformations* $\otimes_A * (\mathrm{id} \boxtimes \otimes_A) \Rightarrow \otimes_A * (\mathrm{id} \boxtimes \otimes_A)$.

Proof. The proof follows directly by computing and simplifying the order $\hbar$ terms of the hexagon identities. The key observation is that the pseudo-functor coherences (A.25) which describe the interchange law between the $*$ and $\circ$ compositions are trivial in all instances appearing in this calculation. Let us illustrate this through an example: The order $\hbar$ term corresponding to the upper path of (3.5a) involves the pseudo-natural transformation $(\gamma \circ t) * \mathrm{Id}$, which arises from a particular composition of the pasting diagram



(3.7)

The pseudo-functor coherences (A.25) for this pasting diagram are trivial because of the identity $t_{(M,N\otimes_A L),(M,N\otimes_A L)}\big(\mathrm{id}_{(M,N\otimes_A L)}\big) = 0$, for all $(M,N,L) \in {}_A\mathbf{C} \boxtimes {}_A\mathbf{C} \boxtimes {}_A\mathbf{C}$, given by property (2) of the Definition A.5 of $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}}$-enriched pseudo-natural transformations. It then follows that

$$(\gamma \circ t) * \mathrm{Id} \;=\; (\gamma \circ t) * (\mathrm{Id} \circ \mathrm{Id}) \;=\; (\gamma * \mathrm{Id}) \circ (t * \mathrm{Id}) \quad , \tag{3.8}$$

which is the key step to identify the left-hand side of (3.6a). In addition to arguments of this form, one also has to use that the pseudo-functor coherences (A.25) are trivial whenever ζ' is strictly natural, i.e. $\zeta'_{d,d'} = 0$ for all $d, d' \in \mathbf{D}$. □

This result justifies the following

Definition 3.2. A (semi-strict) *infinitesimal* 2*-braiding* on the symmetric monoidal category object ${}_A\mathbf{C} = \mathrm{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ in the symmetric monoidal 3-category $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}}$ from Proposition A.4 is a $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}}$-enriched pseudo-natural transformation $t : \otimes_A \Rightarrow \otimes_A : {}_A\mathbf{C} \boxtimes {}_A\mathbf{C} \to {}_A\mathbf{C}$ which satisfies the two infinitesimal hexagon identities in (3.6).

Remark 3.3. As a consequence of Proposition 3.1, the datum of an infinitesimal 2-braiding t is equivalent to a first-order deformation $\gamma^\hbar = \gamma \circ (\mathrm{Id} + \frac{\hbar}{2} t) : \otimes_A \Rightarrow \otimes_A * \tau$ of the symmetric braiding on ${}_A\mathbf{C}$ to a braiding on ${}_A\mathbf{C}^\hbar = {}_A\mathbf{C} \otimes \mathbb{K}[\hbar]/(\hbar^2)$. Note that the deformed braiding is not necessarily symmetric because

$$(\gamma^\hbar * \mathrm{Id}) \circ \gamma^\hbar = \mathrm{Id} + \tfrac{\hbar}{2} \Big(t + (\gamma * \mathrm{Id}) \circ (t * \mathrm{Id}) \circ \gamma \Big) : \otimes_A \Longrightarrow \otimes_A \tag{3.9}$$

is in general different from Id. △

Remark 3.4. Using the explicit composition formulas from Appendix A.2, we can express the infinitesimal hexagon identities (3.6) in terms of the components of the $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformations involved. (Recall also Definition A.5.) In this calculation, various simplifications arise from the fact that all transformations but t are strictly natural, i.e. their homotopy data given by the double-indexed components (A.12) is trivial. To ease notations, we suppress the components of the associator α in what follows. One then finds that the first infinitesimal hexagon identity (3.6a) is equivalent to

$$t_{M,N\otimes_A L} = t_{M,N} \otimes_A \mathrm{id}_L + \big(\gamma_{N,M} \otimes_A \mathrm{id}_L\big)\, \big(\mathrm{id}_N \otimes_A t_{M,L}\big)\, \big(\gamma_{M,N} \otimes_A \mathrm{id}_L\big) \quad , \tag{3.10a}$$

for all $M, N, L \in {}_A\mathbf{C}$, and

$$\begin{aligned} t_{(M,N\otimes_A L),(M',N'\otimes_A L')}\big(h\widetilde{\otimes}(k\otimes_A l)\big) &= t_{(M,N),(M',N')}\big(h\widetilde{\otimes}k\big) \otimes_A l \\ &\quad + \big(\gamma_{N',M'} \otimes_A \mathrm{id}_{L'}\big)\, \big(k \otimes_A t_{(M,L),(M',L')}\big(h\widetilde{\otimes}l\big)\big)\, \big(\gamma_{M,N} \otimes_A \mathrm{id}_L\big) \quad , \end{aligned} \tag{3.10b}$$

for all $M, N, L, M', N', L' \in {}_A\mathbf{C}$, $h \in {}_A\mathbf{C}(M, M')^0$, $k \in {}_A\mathbf{C}(N, N')^0$ and $l \in {}_A\mathbf{C}(L, L')^0$. The second infinitesimal hexagon identity (3.6b) is equivalent to

$$t_{M\otimes_A N,L} = \mathrm{id}_M \otimes_A t_{N,L} + \big(\mathrm{id}_M \otimes_A \gamma_{L,N}\big)\, \big(t_{M,L} \otimes_A \mathrm{id}_N\big)\, \big(\mathrm{id}_M \otimes_A \gamma_{N,L}\big) \quad , \tag{3.11a}$$

for all $M, N, L \in {}_A\mathbf{C}$, and

$$\begin{aligned} t_{(M\otimes_A N,L),(M'\otimes_A N',L')}\big((h\otimes_A k)\widetilde{\otimes}l\big) &= h \otimes_A t_{(N,L),(N',L')}\big(k\widetilde{\otimes}l\big) \\ &\quad + \big(\mathrm{id}_{M'} \otimes_A \gamma_{L',N'}\big)\, \big(t_{(M,L),(M',L')}\big(h\widetilde{\otimes}l\big) \otimes_A k\big)\, \big(\mathrm{id}_M \otimes_A \gamma_{N,L}\big) \quad , \end{aligned} \tag{3.11b}$$

for all $M, N, L, M', N', L' \in {}_A\mathbf{C}$, $h \in {}_A\mathbf{C}(M, M')^0$, $k \in {}_A\mathbf{C}(N, N')^0$ and $l \in {}_A\mathbf{C}(L, L')^0$. △

Remark 3.5. Our Definition 3.2 of an infinitesimal 2-braiding is compatible with the one proposed in [12, Definition 14] in the context of categories enriched over 2-vector spaces, in contrast to our $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched categories from Appendix A. Indeed, the datum (r, T) of an infinitesimal 2-braiding in [12, Definition 14] consists of a family of 1-morphisms $r_{M,N}$ and a family of 2-morphisms $T_{h,k}$, which in our context correspond to the single-indexed components $t_{M,N}$ and the double-indexed components $t_{(M,N),(M',N')}(h\widetilde{\otimes}k)$ of the pseudo-natural transformation $t : \otimes_A \Rightarrow \otimes_A : {}_A\mathbf{C} \boxtimes {}_A\mathbf{C} \to {}_A\mathbf{C}$, see also Definition A.5. Our 3-categorical calculus from Proposition A.4 allows us to treat both of these components simultaneously by working with the various composition operations of pseudo-natural transformations. △

Our examples of infinitesimal 2-braidings in Subsection 3.2 will satisfy a useful additional property that simplifies their analysis.

Definition 3.6. An infinitesimal 2-braiding $t : \otimes_A \Rightarrow \otimes_A$ is called *γ-equivariant* if it satisfies the identity

$$\gamma \circ t = (t * \mathrm{Id}) \circ \gamma \tag{3.12}$$

of $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformations $\otimes_A \Rightarrow \otimes_A * \tau$.

Remark 3.7. It is important to emphasize that γ-equivariance does *not* imply that the associated deformed braiding $\gamma^\hbar$ is symmetric. Indeed, recalling the calculation in Remark 3.3, we find that

$$(\gamma^\hbar * \mathrm{Id}) \circ \gamma^\hbar = \mathrm{Id} + \hbar t \quad , \tag{3.13}$$

for every γ-equivariant infinitesimal 2-braiding t. △

Proposition 3.8. *Let $t : \otimes_A \Rightarrow \otimes_A$ be any $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformation which satisfies the γ-equivariance property (3.12). Then the two infinitesimal hexagon identities (3.6) for t are equivalent.*

Proof. This can be verified directly by using the equivalent component-wise description of the infinitesimal hexagon identities from Remark 3.4 and the fact that the γ-equivariance property reads in components as

$$\gamma_{M,N}\, t_{M,N} = t_{N,M}\, \gamma_{M,N} \quad , \tag{3.14a}$$

for all $M, N \in {}_A\mathbf{C}$, and

$$\gamma_{M',N'}\, t_{(M,N),(M',N')}(h \widetilde{\otimes} k) = t_{(N,M),(N',M')}(k \widetilde{\otimes} h)\, \gamma_{M,N} \quad , \tag{3.14b}$$

for all $M, N, M', N' \in {}_A\mathbf{C}$, $h \in {}_A\mathbf{C}(M, M')^0$ and $k \in {}_A\mathbf{C}(N, N')^0$. □

3.2. Construction from 2-shifted Poisson structures

Suppose that $\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}(A,2)^4$ is a 2-shifted Poisson structure (recall Definition 2.6) on the finitely generated semi-free CDGA A. In this subsection, we will present an explicit construction which assigns to this datum a γ-equivariant infinitesimal 2-braiding $t : \otimes_A \Rightarrow \otimes_A$ in the sense of Definitions 3.2 and 3.6 on the homotopy 2-category ${}_A\mathbf{C} = \mathrm{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \in \mathbf{dgCat}_{\mathbb{K}}^{[-1,0],\mathrm{ps}}$ of the dg-category of finitely generated semi-free A-dg-modules. We will illustrate later in Section 4 that our construction is a non-trivial homotopical generalization of the standard construction of an infinitesimal braiding on the representation category $\mathbf{Rep}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ from an invariant symmetric tensor in $(\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$, see e.g. [6] and [20, Chapter XX].

Our infinitesimal 2-braiding t is built as a composite of more primitive $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformations which we are going to define now. Let us denote by

$$\Omega_A[1] \otimes_A (\,\cdot\,) : {}_A\mathbf{C} \longrightarrow {}_A\mathbf{C} \tag{3.15}$$

the $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched functor of taking the relative tensor product with the 1-shift $\Omega_A[1] \in {}_A\mathbf{C}$ of the dg-module of Kähler differentials from (2.16). We define a $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformation

$$\xi : \mathrm{id} \Longrightarrow \Omega_A[1] \otimes_A (\,\cdot\,) : {}_A\mathbf{C} \longrightarrow {}_A\mathbf{C} \tag{3.16a}$$

from the $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched identity functor $\mathrm{id} : {}_A\mathbf{C} \to {}_A\mathbf{C}$ by specifying its components as in Definition A.5: For each object $M \in {}_A\mathbf{C}$, the component

$$\big(\xi_M : M \to \Omega_A[1] \otimes_A M\big) \in {}_A\mathbf{C}\big(M, \Omega_A[1] \otimes_A M\big)^0 \tag{3.16b}$$

is the A-dg-module map which is defined on the generators $\mathbb{1} \otimes w \in M^\sharp \cong A^\sharp \otimes W^\sharp$ of M by

$$\xi_M\big(\mathbb{1} \otimes w\big) := s^{-1}\, (\mathrm{d}_{\mathrm{dR}} \otimes \mathrm{id})\big(\mathrm{d}_M(\mathbb{1} \otimes w)\big) \quad , \tag{3.16c}$$

where we recall our notation from Subsection 2.1 that $s^{-1} \in \mathbb{K}[1]$ denotes the degree -1 element represented by the unit of $\mathbb{K}$. For each pair of objects $M, M' \in {}_A\mathbf{C}$ and each A-dg-module map $(h : M \to M') \in {}_A\mathbf{C}(M, M')^0$, the component

$$\big(\xi_{M,M'}(h) : M \to \Omega_A[1] \otimes_A M'\big) \in {}_A\mathbf{C}\big(M, \Omega_A[1] \otimes_A M'\big)^{-1} \tag{3.16d}$$

is the homotopy which is defined on the generators $\mathbb{1} \otimes w \in M^\sharp \cong A^\sharp \otimes W^\sharp$ of M by

$$\xi_{M,M'}(h)\big(\mathbb{1} \otimes w\big) := -s^{-1}\, (\mathrm{d}_{\mathrm{dR}} \otimes \mathrm{id})\big(h(\mathbb{1} \otimes w)\big) \quad . \tag{3.16e}$$

Lemma 3.9. *The components in (3.16) define a $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformation $\xi : \mathrm{id} \Rightarrow \Omega_A[1] \otimes_A (\,\cdot\,)$.*

Proof. Before we verify the three properties (1), (2) and (3) from Definition A.5, let us first comment on the well-definedness of the components of ξ. Concerning (i) from Definition A.5, we have to show that $\mathrm{d}_{\underline{\mathrm{hom}}_A}(\xi_M) := \mathrm{d}_{\Omega_A[1]\otimes_A M}\, \xi_M - \xi_M\, \mathrm{d}_M = 0$, i.e. that ξ_M commutes with the differentials, which can be verified by picking a basis $\{w_i \in W^\sharp\}$ and evaluating on the associated generators $\mathbb{1} \otimes w_i \in M$. This allows us to write $\mathrm{d}_M(\mathbb{1} \otimes w_i) =: \underline{M_i}^j \otimes w_j$ and hence $\xi_M(\mathbb{1} \otimes w_i) = s^{-1}\, \mathrm{d}_{\mathrm{dR}}(\underline{M_i}^j) \otimes w_j$, with $\underline{M_i}^j \in A$ and summations over repeated indices understood. We then compute

$$\begin{aligned}
\mathrm{d}_{\underline{\mathrm{hom}}_A}(\xi_M)\big(\mathbb{1} \otimes w_i\big) &= \mathrm{d}_{\Omega_A[1]\otimes_A M}\big(\xi_M(\mathbb{1} \otimes w_i)\big) - \xi_M\big(\mathrm{d}_M(\mathbb{1} \otimes w_i)\big) \\
&= -s^{-1} \mathrm{d}_{\Omega_A \otimes_A M}\big(\mathrm{d}_{\mathrm{dR}}(\underline{M_i}^j) \otimes w_j\big) - \underline{M_i}^j\, s^{-1}\, \mathrm{d}_{\mathrm{dR}}(\underline{M_j}^k) \otimes w_k \\
&= -s^{-1} \Big(\mathrm{d}_{\mathrm{dR}}\big(\mathrm{d}_A(\underline{M_i}^j)\big) \otimes w_j + (-1)^{|\underline{M_i}^j|}\, \mathrm{d}_{\mathrm{dR}}\big(\underline{M_i}^j\, \underline{M_j}^k\big) \otimes w_k\Big) \\
&= 0 \quad ,
\end{aligned} \tag{3.17}$$

where the last identity follows from expressing the square-zero condition $\mathrm{d}_M\mathrm{d}_M(\mathbb{1}\otimes w_i)=0$ in our index notation. Concerning (ii) from Definition A.5, we note that $\xi_{M,M'}(h)$ as defined in (3.16) is clearly $\mathbb{K}$-linear in the argument h.

Property (2) from Definition A.5 follows immediately from $\mathrm{d}_{\mathrm{dR}}\mathbb{1}=0$. To verify the first property in (1), we use also the index notations $h(\mathbb{1}\otimes w_i)=:\underline{h_i}^j\otimes w'_j$ and $\mathrm{d}_{M'}(\mathbb{1}\otimes w'_i)=:\underline{M'_i}^j\otimes w'_j$, where $\{w'_i\in W'^\sharp\}$ is a choice of basis for the generators of M', and compute

$$\begin{aligned}
\mathrm{d}_{\underline{\hom}_A}\big(\xi_{M,M'}(h)\big)\big(\mathbb{1}\otimes w_i\big) &= \mathrm{d}_{\Omega_A[1]\otimes_A M'}\big(\xi_{M,M'}(h)\big(\mathbb{1}\otimes w_i\big)\big)+\xi_{M,M'}(h)\big(\mathrm{d}_M(\mathbb{1}\otimes w_i)\big)\\
&= s^{-1}\Big(\mathrm{d}_{\Omega_A\otimes_A M'}\big(\mathrm{d}_{\mathrm{dR}}(\underline{h_i}^j)\otimes w'_j\big)-\underline{M_i}^j\,\mathrm{d}_{\mathrm{dR}}(\underline{h_j}^k)\otimes w'_k\Big)\\
&= s^{-1}\Big(\mathrm{d}_{\mathrm{dR}}\big(\mathrm{d}_A(\underline{h_i}^j)\big)\otimes w'_j+(-1)^{|\underline{h_i}^j|}\,\mathrm{d}_{\mathrm{dR}}(\underline{h_i}^j)\,\underline{M'_j}^k\otimes w'_k\\
&\qquad -\underline{M_i}^j\,\mathrm{d}_{\mathrm{dR}}(\underline{h_j}^k)\otimes w'_k\Big)\\
&= s^{-1}\Big(-(-1)^{|\underline{h_i}^j|}\,\underline{h_i}^j\,\mathrm{d}_{\mathrm{dR}}(\underline{M'_j}^k)\otimes w'_k+\mathrm{d}_{\mathrm{dR}}(\underline{M_i}^j)\,\underline{h_j}^k\otimes w'_k\Big)\\
&= (\mathrm{id}_{\Omega_A[1]}\otimes_A h)\big(\xi_M(\mathbb{1}\otimes w_i)\big)-\xi_{M'}\big(h(\mathbb{1}\otimes w_i)\big)\quad ,
\end{aligned}\tag{3.18}$$

where the fourth step follows from expressing the cocycle condition $\mathrm{d}_{\underline{\hom}_A}(h)\big(\mathbb{1}\otimes w_i\big)=0$ in our index notation. To verify the second property in (1), one performs a similar calculation for $(k:M\to M')\in {}_A\mathbf{C}(M,M')^{-1}$ of degree -1 and finds that

$$\begin{aligned}
&(\mathrm{id}_{\Omega_A[1]}\otimes_A k)\big(\xi_M(\mathbb{1}\otimes w_i)\big)-\xi_{M'}\big(k(\mathbb{1}\otimes w_i)\big)-\xi_{M,M'}\big(\mathrm{d}_{\underline{\hom}_A}(k)\big)\big(\mathbb{1}\otimes w_i\big)\\
&\quad= s^{-1}\Big(\mathrm{d}_{\mathrm{dR}}\big(\mathrm{d}_A(\underline{k_i}^j)\big)\otimes w'_j+(-1)^{|\underline{k_i}^j|}\,\mathrm{d}_{\mathrm{dR}}(\underline{k_i}^j)\,\underline{M'_j}^k\otimes w'_k+(-1)^{|\underline{M_i}^j|}\,\underline{M_i}^j\,\mathrm{d}_{\mathrm{dR}}(\underline{k_j}^k)\otimes w'_k\Big)\quad .
\end{aligned}\tag{3.19}$$

The right-hand side can be identified as the differential $\mathrm{d}_{\underline{\hom}_A}(q)(\mathbb{1}\otimes w_i)$ of the 2-homotopy $q\in\underline{\hom}_A(M,\Omega_A[1]\otimes_A M')^{-2}$ which is defined on generators by $q(\mathbb{1}\otimes w)=-s^{-1}\,(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(k(\mathbb{1}\otimes w)\big)$, hence the right-hand side is zero in our truncated cochain complex ${}_A\mathbf{C}\big(M,\Omega_A[1]\otimes_A M'\big)$.

It remains to verify property (3) from Definition A.5. Given $(h:M\to M')\in {}_A\mathbf{C}(M,M')^0$ and $(h':M'\to M'')\in {}_A\mathbf{C}(M',M'')^0$, we introduce one more index notation $h'(\mathbb{1}\otimes w'_i)=:\underline{h'_i}^j\otimes w''_j$, where $\{w''_i\in W''^\sharp\}$ is a choice of basis for the generators of M'', and compute

$$\begin{aligned}
\xi_{M,M''}(h'h)\big(\mathbb{1}\otimes w_i\big) &= -s^{-1}\,\mathrm{d}_{\mathrm{dR}}\big(\underline{h_i}^j\,\underline{h'_j}^k\big)\otimes w''_k\\
&= -s^{-1}\Big(\mathrm{d}_{\mathrm{dR}}(\underline{h_i}^j)\,\underline{h'_j}^k\otimes w''_k+\underline{h_i}^j\,\mathrm{d}_{\mathrm{dR}}(\underline{h'_j}^k)\otimes w''_k\Big)\\
&= \xi_{M',M''}(h')\big(h(\mathbb{1}\otimes w_i)\big)+(\mathrm{id}_{\Omega_A[1]}\otimes_A h')\big(\xi_{M,M'}(h)(\mathbb{1}\otimes w_i)\big)\quad .
\end{aligned}\tag{3.20}$$

This concludes the proof. □

In addition to the $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformation ξ from (3.16), we require two more *strict* $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched natural transformations to build our infinitesimal 2-braiding. First, using the original symmetric braiding γ, we obtain a family of isomorphisms

$$\mathrm{id}\otimes_A\gamma_{M,\Omega_A[1]}\otimes_A\mathrm{id}:\Omega_A[1]\otimes_A M\otimes_A\Omega_A[1]\otimes_A N\longrightarrow\Omega_A[1]\otimes_A\Omega_A[1]\otimes_A M\otimes_A N\tag{3.21a}$$

in ${}_A\mathbf{C}$, for all $M,N\in{}_A\mathbf{C}$, which defines a strict $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched natural isomorphism

$$\otimes_A*\Big(\big(\Omega_A[1]\otimes_A(\cdot)\big)\boxtimes\big(\Omega_A[1]\otimes_A(\cdot)\big)\Big)\xRightarrow{\cong}\Big(\Omega_A[1]\otimes_A\Omega_A[1]\otimes_A(\cdot)\Big)*\otimes_A\quad .\tag{3.21b}$$

Second, consider the extension $\langle\cdot,\cdot\rangle:\mathrm{Sym}_A^2\big(\mathsf{T}_A[-1]\big)\otimes_A\Omega_A[1]\otimes_A\Omega_A[1]\to A$ of the duality pairing (2.17) defined by

$$\big\langle sD\,sD',s^{-1}\omega\otimes_A s^{-1}\omega'\big\rangle:=(-1)^{|\omega|+|D|}\big\langle D,\langle D',\omega\rangle\cdot\omega'\big\rangle+(-1)^{(|D|+1)(|D'|+1)}\,(-1)^{|\omega|+|D'|}\big\langle D',\langle D,\omega\rangle\cdot\omega'\big\rangle\quad ,\tag{3.22}$$

for all homogeneous $D,D'\in\mathsf{T}_A$ and $\omega,\omega'\in\Omega_A$. Considering the weight 2 component $\pi^{(2)}\in\mathrm{Sym}_A^2\big(\mathsf{T}_A[-3]\big)^4\cong\mathrm{Sym}_A^2\big(\mathsf{T}_A[-1]\big)^0$ of the 2-shifted Poisson structure (recall that this is a cocycle, see Remark 2.7), we obtain a family of morphisms

$$\langle\pi^{(2)},\cdot\rangle\otimes_A\mathrm{id}\otimes_A\mathrm{id}:\Omega_A[1]\otimes_A\Omega_A[1]\otimes_A M\otimes_A N\longrightarrow A\otimes_A M\otimes_A N\cong M\otimes_A N\tag{3.23a}$$

in ${}_A\mathbf{C}$, for all $M,N\in{}_A\mathbf{C}$, which defines a strict $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched natural transformation

$$\iota_\pi \,:\, \Big(\Omega_A[1] \otimes_A \Omega_A[1] \otimes_A (\,\cdot\,)\Big) * \otimes_A \Longrightarrow \otimes_A \quad . \tag{3.23b}$$

Using the $\circ$-composition of $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformations from Appendix A.2, we define the $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformation

$$\begin{array}{ccccc}
\otimes_A & & \overset{t}{\Longrightarrow} & & \otimes_A \\
\| & & & & \Uparrow{\scriptstyle \iota_\pi} \\
\otimes_A * (\mathrm{id} \boxtimes \mathrm{id}) & & & & \Big(\Omega_A[1] \otimes_A \Omega_A[1] \otimes_A (\,\cdot\,)\Big) * \otimes_A \\
& {\scriptstyle \mathrm{Id}*(\xi\boxtimes\xi)} \searrow & & \nearrow {\scriptstyle \cong} & \\
& & \otimes_A * \Big(\big(\Omega_A[1] \otimes_A (\,\cdot\,)\big) \boxtimes \big(\Omega_A[1] \otimes_A (\,\cdot\,)\big)\Big) & &
\end{array} \tag{3.24}$$

as our candidate for an infinitesimal 2-braiding on ${}_A\mathbf{C}$.

Theorem 3.10. *Given any* 2*-shifted Poisson structure* $\pi \in \widehat{\mathrm{Pol}}(A,2)^4$ *on a finitely generated semi-free CDGA* A *over the field* $\mathbb{K}$*, the* $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$*-enriched pseudo-natural transformation* t *from* (3.24) *defines a* γ*-equivariant infinitesimal* 2*-braiding in the sense of Definitions* 3.2 *and* 3.6 *on the homotopy* 2*-category* ${}_A\mathbf{C} = \mathrm{Ho}_2\big({}_A\mathbf{dgMod}^{\mathrm{fgsf}}\big) \in \mathbf{dgCat}_{\mathbb{K}}^{[-1,0],\mathrm{ps}}$ *of the dg-category of finitely generated semi-free* A*-dg-modules.*

Remark 3.11. We would like to highlight that the infinitesimal 2-braiding (3.24) depends only on the bivector component $\pi^{(2)}$ of the 2-shifted Poisson structure $\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}(A,2)^4$. This is to be expected since infinitesimal 2-braidings are characterized by the infinitesimal hexagon identities from Definition 3.2, which are linear in t. Hence, the higher-weight components $\pi^{(3)}, \pi^{(4)}, \dots$ of the 2-shifted Poisson structure are not expected to enter at this linear level, since they control non-linear aspects through the Maurer-Cartan equations from Remark 2.7, which are non-linear identities of the form $\mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi^{(3)}) + \frac{1}{2}\{\pi^{(2)}, \pi^{(2)}\} = 0$. However, we expect that these higher-weight components and their Maurer-Cartan identities will play an important role for deformation quantizations at higher orders in $\hbar$. $\triangle$

Proof of Theorem 3.10. The most straightforward proof of this result is to compute the components of the $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformation t from (3.24), and then to verify directly that these components satisfy the γ-equivariance property from (3.14) and the infinitesimal hexagons from Remark 3.4. Using the composition operations from Appendix A.2, one finds by a straightforward computation that

$$\begin{array}{ccc}
M \otimes_A N & \xrightarrow{\quad t_{M,N} \quad} & M \otimes_A N \\
{\scriptstyle \xi_M \otimes_A \xi_N} \downarrow & & \uparrow {\scriptstyle \langle \pi^{(2)}, \cdot\rangle \otimes_A \mathrm{id} \otimes_A \mathrm{id}} \\
\Omega_A[1] \otimes_A M \otimes_A \Omega_A[1] \otimes_A N & \xrightarrow[\mathrm{id} \otimes_A \gamma_{M,\Omega_A[1]} \otimes_A \mathrm{id}]{} & \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A M \otimes_A N
\end{array} \quad , \tag{3.25a}$$

for all $M, N \in {}_A\mathbf{C}$, and

$$\begin{array}{ccc}
M \otimes_A N & \xrightarrow{\quad t_{(M,N),(M',N')}(h \widetilde{\otimes} k) \quad} & M' \otimes_A N' \\
{\scriptstyle \xi_{M,M'}(h) \otimes_A (\mathrm{id} \otimes_A k)\xi_N + \xi_{M'} h \otimes_A \xi_{N,N'}(k)} \downarrow & & \uparrow {\scriptstyle \langle \pi^{(2)}, \cdot\rangle \otimes_A \mathrm{id} \otimes_A \mathrm{id}} \\
\Omega_A[1] \otimes_A M' \otimes_A \Omega_A[1] \otimes_A N' & \xrightarrow[\mathrm{id} \otimes_A \gamma_{M',\Omega_A[1]} \otimes_A \mathrm{id}]{} & \Omega_A[1] \otimes_A \Omega_A[1] \otimes_A M' \otimes_A N'
\end{array} \quad , \tag{3.25b}$$

for all $M, N, M', N' \in {}_A\mathbf{C}$, $h \in {}_A\mathbf{C}(M, M')^0$ and $k \in {}_A\mathbf{C}(N, N')^0$.

The γ-equivariance property (3.14a) for the degree 0 components $t_{M,N}$ follows immediately by using symmetry of the extended pairing (3.22), i.e. $\langle \pi^{(2)}, \cdot\rangle\, \gamma = \langle \pi^{(2)}, \cdot\rangle$, and the hexagon identities for the original symmetric braiding γ. To prove the γ-equivariance property (3.14b) for the degree -1 components $t_{(M,N),(M',N')}(h \widetilde{\otimes} k)$, one has to use additionally that

$$\begin{aligned}
&\gamma_{\Omega_A[1]\otimes_A M', \Omega_A[1] \otimes_A N'} \Big(\xi_{M,M'}(h) \otimes_A (\mathrm{id} \otimes_A k)\xi_N + \xi_{M'} h \otimes_A \xi_{N,N'}(k)\Big)\, \gamma_{N,M} \\
&\quad = (\mathrm{id} \otimes_A k)\xi_N \otimes_A \xi_{M,M'}(h) + \xi_{N,N'}(k) \otimes_A \xi_{M'} h \\
&\quad = \xi_{N'} k \otimes_A \xi_{M,M'}(h) + \xi_{N,N'}(k) \otimes_A (\mathrm{id} \otimes_A h)\xi_M + \mathrm{d}_{\underline{\mathrm{hom}}_A}\big(\xi_{N,N'}(k) \otimes_A \xi_{M,M'}(h)\big)
\end{aligned}$$

$$= \xi_{N'k} \otimes_A \xi_{M,M'}(h) + \xi_{N,N'}(k) \otimes_A (\mathrm{id} \otimes_A h)\xi_M \quad , \tag{3.26}$$

where in the second step we used property (1) of the Definition A.5 of $\mathbf{Ch}_{\mathbb{K}}^{[-1,0]}$-enriched pseudo-natural transformations and in the last step we used that the morphism complexes of ${}_A\mathbf{C}$ are by definition truncated to degrees $\{-1,0\}$, hence the 2-homotopy in the last term gets identified with 0.

It remains to verify the first infinitesimal hexagon identity (3.10), since the second one (3.11) then follow from Proposition 3.8 and γ-equivariance. For this we recall the definition of the components of ξ from (3.16) and observe that

$$\xi_{N\otimes_A L} = \xi_N \otimes_A \mathrm{id} + (\gamma_{N,\Omega_A[1]} \otimes_A \mathrm{id})\,(\mathrm{id} \otimes_A \xi_L) \quad , \tag{3.27a}$$

for all $N, L \in {}_A\mathbf{C}$, and

$$\xi_{N\otimes_A L, N'\otimes_A L'}(k \otimes_A l) = \xi_{N,N'}(k) \otimes_A l + (\gamma_{N',\Omega_A[1]} \otimes_A \mathrm{id})\,(k \otimes_A \xi_{L,L'}(l)) \quad , \tag{3.27b}$$

for all $N, L, N', L' \in {}_A\mathbf{C}$, $k \in {}_A\mathbf{C}(N,N')^0$ and $l \in {}_A\mathbf{C}(L,L')^0$. Using also the hexagon identities for the original symmetric braiding γ, one verifies the infinitesimal hexagon identity (3.10). □

Remark 3.12. For later use in our discussion of examples in Section 4, let us expand the components $t_{M,N}$ and $t_{(M,N),(M',N')}(h\widetilde{\otimes}k)$ of the infinitesimal 2-braiding (3.24) in terms of a choice of bases for the finitely generated semi-free A-dg-modules $M, N, M', N' \in {}_A\mathbf{C}$. Using similar notations as in the proof of Lemma 3.9, we denote by $\{w_i \in W^\sharp\}$ a basis for $M^\sharp \cong A^\sharp \otimes W^\sharp$, by $\{u_q \in U^\sharp\}$ a basis for $N^\sharp \cong A^\sharp \otimes U^\sharp$, by $\{w'_i \in W'^\sharp\}$ a basis for $M'^\sharp \cong A^\sharp \otimes W'^\sharp$, and by $\{u'_q \in U'^\sharp\}$ a basis for $N'^\sharp \cong A^\sharp \otimes U'^\sharp$. Making use of these bases, we write

$$\mathrm{d}_M(\mathbb{1}\otimes w_i) = \underline{M_i}^j \otimes w_j \quad , \qquad \mathrm{d}_N(\mathbb{1}\otimes u_q) = \underline{N_q}^r \otimes u_r \quad , \tag{3.28a}$$

$$\mathrm{d}_{M'}(\mathbb{1}\otimes w'_i) = \underline{M'_i}^j \otimes w'_j \quad , \qquad \mathrm{d}_{N'}(\mathbb{1}\otimes u'_q) = \underline{N'_q}^r \otimes u'_r \quad , \tag{3.28b}$$

for the differentials, and for the A-linear maps $h \in {}_A\mathbf{C}(M,M')^0$ and $k \in {}_A\mathbf{C}(N,N')^0$ we write

$$h(\mathbb{1}\otimes w_i) = \underline{h_i}^j \otimes w'_j \quad , \qquad k(\mathbb{1}\otimes u_q) = \underline{k_q}^r \otimes u'_r \quad , \tag{3.28c}$$

where again all summations over repeated indices are suppressed. Evaluating the diagram (3.25a), we then find the explicit basis expression

$$\begin{aligned} t_{M,N}\big(\mathbb{1}\otimes w_i\otimes u_q\big) &= (t_{M,N})_{iq}{}^{jr} \otimes w_j \otimes u_r \\ &= (-1)^{|w_j|\,(|\underline{N_q}^r|-1)} \big\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{M_i}^j) \otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{N_q}^r)\big\rangle \otimes w_j \otimes u_r \quad . \end{aligned} \tag{3.29a}$$

Furthermore, evaluating the diagram (3.25b), we find

$$\begin{aligned} t_{(M,N),(M',N')}(h\widetilde{\otimes}k)(\mathbb{1}\otimes w_i\otimes u_q) &= \big(t_{(M,N),(M',N')}(h\widetilde{\otimes}k)\big)_{iq}{}^{jr} \otimes w'_j \otimes u'_r \\ &= -(-1)^{|w'_j|\,(|\underline{N_q}^s|+|\underline{k_s}^r|-1)} \big\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{h_i}^j) \otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{N_q}^s)\,\underline{k_s}^r\big\rangle \otimes w'_j \otimes u'_r \\ &\quad - (-1)^{|\underline{h_i}^k|+|w'_j|\,(|\underline{k_q}^r|-1)} \big\langle \pi^{(2)}, s^{-1}\underline{h_i}^k\,\mathrm{d}_{\mathrm{dR}}(\underline{M'_k}^j) \otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{k_q}^r)\big\rangle \otimes w'_j \otimes u'_r \quad . \end{aligned} \tag{3.29b}$$

Given a concrete example for the bivector $\pi^{(2)} \in \mathrm{Sym}^2_A\big(\mathsf{T}_A[-3]\big)^4 \cong \mathrm{Sym}^2_A\big(\mathsf{T}_A[-1]\big)^0$, one can evaluate these expressions further by using the definition (3.22) of the duality pairing. △

3.3. Towards higher-order deformations

A powerful result in the 1-categorical theory of infinitesimal braidings is that, upon the choice of a Drinfeld associator [15], every infinitesimal braiding lifts to a formal deformation (i.e. to all orders in $\hbar$) of the given symmetric monoidal category into a braided monoidal one. See e.g. [6] and [20, Chapters XIX–XX] for the details. The aim of this subsection is to explore which aspects of this construction admit a direct generalization to our concept of infinitesimal 2-braidings on ${}_A\mathbf{C}$. We observe that a direct generalization is only possible under rather restrictive assumptions (see Proposition 3.13 and Remark 3.14), while the general case points towards the necessity to develop a homotopy-coherent variant of Drinfeld associators which also describe pentagonator and hexagonator modifications. Our study in this subsection has some parallels with [12, Section 2.2], however we believe that the relations for the modifications in (3.44) entering our explicit attempt towards a deformation construction at order $\hbar^2$ are new.

To study formal deformations to all orders in $\hbar$, we introduce the commutative $\mathbb{K}$-algebra $\mathbb{K}[[\hbar]]$ of formal power series and perform a base change

$${}_A\mathbf{C}[[\hbar]] := {}_A\mathbf{C} \otimes \mathbb{K}[[\hbar]] \in \mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}[[\hbar]]} \tag{3.30}$$

along the commutative $\mathbb{K}$-algebra morphism $\mathbb{K} \to \mathbb{K}[[\hbar]]$. The goal is then to deform the symmetric monoidal structure on ${}_A\mathbf{C}[[\hbar]]$ into a braided monoidal structure. In order to simplify notations, we suppress in what follows the original associator α and simply write

$$\otimes_A^2 := \otimes_A * (\otimes_A \boxtimes \mathrm{id}) \cong \otimes_A * (\mathrm{id} \boxtimes \otimes_A) : {}_A\mathbf{C}[[\hbar]] \boxtimes {}_A\mathbf{C}[[\hbar]] \boxtimes {}_A\mathbf{C}[[\hbar]] \longrightarrow {}_A\mathbf{C}[[\hbar]] \tag{3.31}$$

for the 2-times iterated tensor product, and similarly $\otimes_A^n : {}_A\mathbf{C}[[\hbar]]^{\boxtimes(n+1)} \to {}_A\mathbf{C}[[\hbar]]$ for the n-times iterated tensor product. Given any γ-equivariant infinitesimal 2-braiding $t : \otimes_A \Rightarrow \otimes_A$, we denote by

$$t_{ij} : \otimes_A^n \Longrightarrow \otimes_A^n : {}_A\mathbf{C}[[\hbar]]^{\boxtimes(n+1)} \longrightarrow {}_A\mathbf{C}[[\hbar]] \tag{3.32a}$$

the $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched pseudo-natural transformation which evaluates t on the i-th and j-th tensor factor, for all $i, j \in \{1, \dots, n+1\}$ with $i \neq j$. Explicitly, the components of t_{ij} read as

$$\begin{array}{ccc} \bigotimes_{k=1}^{n+1} M_k & \xrightarrow{(t_{ij})_{M_1,\dots,M_{n+1}}} & \bigotimes_{k=1}^{n+1} M_k \\ \cong \downarrow & & \uparrow \cong \\ M_i \otimes_A M_j \otimes_A \bigotimes_{\substack{k=1 \\ k\neq i,j}}^{n+1} M_k & \xrightarrow[t_{M_i,M_j}\otimes_A \bigotimes_k \mathrm{id}_{M_k}]{} & M_i \otimes_A M_j \otimes_A \bigotimes_{\substack{k=1 \\ k\neq i,j}}^{n+1} M_k \end{array} , \tag{3.32b}$$

for all $M_1, \dots, M_{n+1} \in {}_A\mathbf{C}[[\hbar]]$, and

$$\begin{array}{ccc} \bigotimes_{k=1}^{n+1} M_k & \xrightarrow{(t_{ij})_{(M_1,\dots,M_{n+1}),(M'_1,\dots,M'_{n+1})}(h_1 \widetilde{\otimes} \cdots \widetilde{\otimes} h_{n+1})} & \bigotimes_{k=1}^{n+1} M'_k \\ \cong \downarrow & & \uparrow \cong \\ M_i \otimes_A M_j \otimes_A \bigotimes_{\substack{k=1 \\ k\neq i,j}}^{n+1} M_k & \xrightarrow[t_{(M_i,M_j),(M'_i,M'_j)}(h_i \widetilde{\otimes} h_j)\otimes_A \bigotimes_k h_k]{} & M'_i \otimes_A M'_j \otimes_A \bigotimes_{\substack{k=1 \\ k\neq i,j}}^{n+1} M'_k \end{array} , \tag{3.32c}$$

for all $M_1, \dots, M_{n+1}, M'_1, \dots, M'_{n+1} \in {}_A\mathbf{C}[[\hbar]]$ and $h_k \in {}_A\mathbf{C}[[\hbar]](M_k, M'_k)^0$, for $k = 1, \dots, n+1$, where the unlabeled vertical isomorphisms are given by the original symmetric braiding γ. Let us note that there is an obvious generalization of the above construction which assigns to every pair $I = (i_1 \cdots i_I)$ and $J = (j_1 \cdots j_J)$ of tuples of pairwise distinct indices in $\{1, \dots, n+1\}$ the $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched pseudo-natural transformation $t_{IJ} : \otimes_A^n \Longrightarrow \otimes_A^n$ which evaluates t on the tensor product of the I-factors and the tensor product of the J-factors. In this compact notation, the infinitesimal hexagons from (3.6) and the γ-equivariance property from Definition 3.6 read as

$$t_{i(jk)} = t_{ij} + t_{ik} \quad , \qquad t_{(ij)k} = t_{ik} + t_{jk} \quad , \qquad t_{ij} = t_{ji} \quad , \tag{3.33}$$

for all pairwise distinct $i, j, k \in \{1, \dots, n+1\}$. Furthermore, one verifies that

$$[t_{ij}, t_{kl}] := t_{ij} \circ t_{kl} - t_{kl} \circ t_{ij} = 0 \quad , \tag{3.34}$$

for all pairwise distinct $i, j, k, l \in \{1, \dots, n+1\}$, by explicitly computing the components.

Following the same strategy as in [6] and [20, Chapters XIX–XX], we consider a candidate for a deformed associator and braiding on ${}_A\mathbf{C}[[\hbar]]$ which is given by the ansatz

$$\alpha^\hbar := \Phi(t_{12}, t_{23}) : \otimes_A^2 \Longrightarrow \otimes_A^2 : {}_A\mathbf{C}[[\hbar]] \boxtimes {}_A\mathbf{C}[[\hbar]] \boxtimes {}_A\mathbf{C}[[\hbar]] \longrightarrow {}_A\mathbf{C}[[\hbar]] \tag{3.35a}$$

and

$$\gamma^\hbar := \gamma \circ e^{\frac{\hbar}{2}t} : \otimes_A \Longrightarrow \otimes_A * \tau : {}_A\mathbf{C}[[\hbar]] \boxtimes {}_A\mathbf{C}[[\hbar]] \longrightarrow {}_A\mathbf{C}[[\hbar]] \quad , \tag{3.35b}$$

where $\Phi(\mathfrak{a}, \mathfrak{b})$ is some formal power series in two non-commutative variables $\mathfrak{a}$ and $\mathfrak{b}$ satisfying $\Phi(0,0) = 1$. In these expressions, both Φ and the exponential function e, which we describe by its formal Taylor series, are evaluated by using the $\circ$-composition of pseudo-natural transformations from Appendix A.2, hence $\alpha^\hbar$ and $\gamma^\hbar$ are by construction $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched pseudo-natural transformations.

Proposition 3.13. *The tuple* $\big({}_A\mathbf{C}[[\hbar]], \otimes_A, A, \alpha^\hbar, \lambda, \rho, \gamma^\hbar\big)$*, with deformed associator* $\alpha^\hbar$ *and braiding* $\gamma^\hbar$ *from* (3.35)*, defines a semi-strict braided monoidal category object in* $\mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}[[\hbar]]}$ *in the sense of Definition A.8 if and only if all of the following hold true:*

(1) *The identity*

$$\Phi(t_{12}, t_{23}+t_{24}) \circ \Phi(t_{13}+t_{23}, t_{34}) \;=\; \Phi(t_{23}, t_{34}) \circ \Phi(t_{12}+t_{13}, t_{24}+t_{34}) \circ \Phi(t_{12}, t_{23}) \tag{3.36}$$

holds true strictly in the algebra $\big(\mathrm{PsNat}(\otimes_A^3, \otimes_A^3), \circ, \mathrm{Id}_{\otimes_A^3}\big)$ *of* $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$*-enriched pseudo-natural transformations with multiplication* $\circ$*.*

(2) *The identities*

$$\Phi(t_{23}, t_{13}) \circ e^{\frac{\hbar}{2}(t_{12}+t_{13})} \circ \Phi(t_{12}, t_{23}) \;=\; e^{\frac{\hbar}{2}t_{13}} \circ \Phi(t_{12}, t_{13}) \circ e^{\frac{\hbar}{2}t_{12}} \tag{3.37a}$$

and

$$\Phi(t_{13}, t_{12})^{-1} \circ e^{\frac{\hbar}{2}(t_{13}+t_{23})} \circ \Phi(t_{12}, t_{23})^{-1} \;=\; e^{\frac{\hbar}{2}t_{13}} \circ \Phi(t_{13}, t_{23})^{-1} \circ e^{\frac{\hbar}{2}t_{23}} \tag{3.37b}$$

hold true strictly in the algebra $\big(\mathrm{PsNat}(\otimes_A^2, \otimes_A^2), \circ, \mathrm{Id}_{\otimes_A^2}\big)$ *of* $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$*-enriched pseudo-natural transformations with multiplication* $\circ$*.*

Assuming additionally that Φ *satisfies the inversion property* $\Phi(\mathfrak{a}, \mathfrak{b})^{-1} = \Phi(\mathfrak{b}, \mathfrak{a})$*, then the two identities in* (3.37) *are equivalent.*

Proof. The triangle identity holds true automatically as a consequence of $\Phi(0,0) = 1$ and the fact that all components of t involving the monoidal unit $A \in {}_A\mathbf{C}[[\hbar]]$ at least once are necessarily trivial. (The latter is a consequence of (3.33).) Using (3.33) and arguments for strictness of the interchange law for $\circ$ and $*$ which are similar to the ones in the proof of Proposition 3.1, one shows that (1) is equivalent to the pentagon identity for $\alpha^\hbar$. With the same kind of computation, one shows that (2) is equivalent to the two hexagon identities for $\gamma^\hbar$ and $\alpha^\hbar$. Finally, under the additional assumption $\Phi(\mathfrak{a}, \mathfrak{b})^{-1} = \Phi(\mathfrak{b}, \mathfrak{a})$, one easily shows that (3.37a) and (3.37b) are identified by permuting the indices according to $123 \mapsto 321$ and using once again (3.33). □

Remark 3.14. From a superficial perspective, the identities (1) and (2) in Proposition 3.13 take exactly the same form as the defining properties of a Drinfeld associator Φ, see e.g. [6, Eqns. (24) and (25)] or [20, Chapter XIX, Eqns. (8.27)–(8.29)]. However, there is the following caveat: The analogous identities for a Drinfeld associator are formulated in the universal enveloping algebra of the appropriate Drinfeld-Kohno Lie algebra where the symbols t_{ij} satisfy the infinitesimal braid relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{kl}] = 0$ and $[t_{ij}, t_{ik} + t_{jk}] = 0$, for all pairwise distinct indices i, j, k, l. In contrast to this, our identities (1) and (2) must hold true in the appropriate algebra of $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched pseudo-natural transformations with multiplication $\circ$. From (3.33) and (3.34), we know that our t_{ij}'s satisfy the first two types of infinitesimal braid relations, i.e. $t_{ij} = t_{ji}$ and $[t_{ij}, t_{kl}] = 0$ for all pairwise distinct indices i, j, k, l, but as we shall illustrate below they do *not* necessarily satisfy the third type of relation $[t_{ij}, t_{ik} + t_{jk}] = 0$ strictly.

So the conclusion at this moment is as follows: Assuming that our γ-equivariant infinitesimal 2-braiding t satisfies the remaining infinitesimal braid relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ strictly, then the choice of a Drinfeld associator Φ (in the traditional sense) provides via Proposition 3.13 a semi-strict braided monoidal deformation (to all orders in $\hbar$) of the symmetric monoidal category object $\big({}_A\mathbf{C}, \otimes_A, A, \alpha, \lambda, \rho, \gamma\big)$. △

In light of the previous remark, we make the following observation.

Proposition 3.15. *The pseudo-naturality structure of the* γ*-equivariant infinitesimal 2-braiding* $t : \otimes_A \Rightarrow \otimes_A$ *induces* $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$*-enriched modifications*

$$\Gamma_{ijk} : [t_{ij}, t_{ik}+t_{jk}] \;\Rrightarrow\; 0 : \otimes_A^n \;\Longrightarrow\; \otimes_A^n : {}_A\mathbf{C}[[\hbar]]^{\boxtimes(n+1)} \longrightarrow {}_A\mathbf{C}[[\hbar]] \quad , \tag{3.38a}$$

for all pairwise distinct $i, j, k \in \{1, \dots, n+1\}$*, which enforce weakly the remaining infinitesimal braid relations. The components (recall Definition A.7) of these modifications are given explicitly by*

$$\begin{array}{ccc} \bigotimes_{l=1}^{n+1} M_l & \xrightarrow{\;(\Gamma_{ijk})_{M_1,\dots,M_{n+1}}\;} & \bigotimes_{l=1}^{n+1} M_l \\ \big\downarrow \cong & & \big\uparrow \cong \\ M_i \otimes_A M_j \otimes_A M_k \otimes_A \bigotimes_{\substack{l=1 \\ l\neq i,j,k}}^{n+1} M_l & \xrightarrow{\;t_{(M_i\otimes_A M_j, M_k),(M_i\otimes_A M_j, M_k)}(t_{M_i,M_j}\widetilde{\otimes}\mathrm{id})\otimes_A \mathrm{id}\;} & M_i \otimes_A M_j \otimes_A M_k \otimes_A \bigotimes_{\substack{l=1 \\ l\neq i,j,k}}^{n+1} M_l \end{array} \quad , \tag{3.38b}$$

for all $M_1, \dots, M_{n+1} \in {}_A\mathbf{C}[[\hbar]]$.

Proof. Recalling (3.33), one observes that

$$[t_{ij}, t_{ik} + t_{jk}] = [t_{ij}, t_{(ij)k}] = t_{ij} \circ t_{(ij)k} - t_{(ij)k} \circ t_{ij} \quad . \tag{3.39}$$

Using the composition formulas from Appendix A.2, the component expressions (3.32) and pseudo-naturality of $t_{(ij)k}$ in the left entry, one computes the components of this $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched pseudo-natural transformation and finds that

$$\big(t_{ij} \circ t_{(ij)k} - t_{(ij)k} \circ t_{ij}\big)_{M_1,\dots,M_{n+1}} = \mathrm{d}_{\underline{\hom}_A}\big((\Gamma_{ijk})_{M_1,\dots,M_{n+1}}\big) \tag{3.40}$$

with Γ_{ijk} defined in (3.38). By a routine calculation one then shows that Γ_{ijk} satisfies the properties of a $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched modification from Definition A.7. □

Remark 3.16. The reader might have noticed that there are potential ambiguities in defining the $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched modifications Γ_{ijk}. These arise from the multiple equivalent ways to write the expression $[t_{ij}, t_{ik} + t_{jk}]$ by using the identities in (3.33). Such ambiguities do not seem to be present for our γ-equivariant infinitesimal 2-braidings. Indeed, one observes that

$$\Gamma_{ijk} = \Gamma_{jik} \tag{3.41}$$

is symmetric in the first two indices, hence there are no ambiguities arising from rewriting $[t_{ij}, t_{ik} + t_{jk}] = [t_{ji}, t_{jk} + t_{ik}]$ by using the symmetry property $t_{ij} = t_{ji}$. Furthermore, rewriting $[t_{ij}, t_{ik} + t_{jk}] = [t_{ij}, t_{(ij)k}] = [t_{ij}, t_{k(ij)}]$ by using the identities (3.33), one finds from the pseudo-naturality structure of $t_{k(ij)}$ in the right entry a modification which agrees with Γ_{ijk}. △

Remark 3.17. For the complete description of a categorified version of the infinitesimal braid relations, one would also require suitable coherence conditions for the modifications Γ_{ijk}. One could then attempt to weaken the strict identities (1) and (2) from Proposition 3.13 to identities which hold true up to suitable modifications, playing the role of a pentagonator and hexagonators for a weakened variant of the semi-strict braided monoidal category objects from Definition A.8. To the best of our knowledge, this is relatively unexplored territory, with the notable exception given by exploratory works of Cirio and Faria Martins [12]. We will not attempt to address and solve these issues in the present paper. △

Example 3.18. For our infinitesimal 2-braidings t in Theorem 3.10, which are constructed from 2-shifted Poisson structures π on the CDGA A, we can use the formulas from Remark 3.12 in order to provide a more explicit basis expression for the modifications Γ_{ijk} in Proposition 3.15. Denoting $M = M_i$, $N = M_j$ and $L = M_k$, we then find for the bottom horizontal arrow in (3.38b)

$$\begin{aligned} &t_{(M\otimes_A N,L),(M\otimes_A N,L)}(t_{M,N}\widetilde{\otimes}\mathrm{id})(\mathbb{1}\otimes w_i\otimes u_q\otimes v_s) \\ &\quad = -(-1)^{|w_j|\,(|\underline{L}_s{}^t|-1)}\,\big\langle \pi^{(2)}, s^{-1}\mathrm{d}_{\mathrm{dR}}\big((t_{M,N})_{iq}{}^{jr}\big)\otimes_A s^{-1}\mathrm{d}_{\mathrm{dR}}(\underline{L}_s{}^t)\big\rangle\otimes w_j\otimes u_r\otimes v_t \quad , \end{aligned} \tag{3.42}$$

where the components $(t_{M,N})_{iq}{}^{jr}$ are defined in (3.29a) and the second term in (3.29b) vanishes because $k = \mathrm{id} : L \to L$ is the identity in the present case. ▽

To conclude this subsection, we carry out briefly an explicit study of the identities (1) and (2) in Proposition 3.13 modulo $\hbar^3$ by taking as ansatz the traditional Drinfeld associator

$$\Phi(\mathfrak{a},\mathfrak{b}) = 1 + \tfrac{\hbar^2}{24}[\mathfrak{a},\mathfrak{b}] \mod \hbar^3 \quad , \tag{3.43}$$

see e.g. [20, Corollary XIX.6.5]. Using the identity (3.34), one immediately checks that the identity (3.36) holds true strictly modulo $\hbar^3$, i.e. without the need for any modifications. In contrast to this, the identities (3.37) do not hold true strictly modulo $\hbar^3$. Concretely, bringing all terms in (3.37a) to the left-hand side, one finds that

$$\tfrac{\hbar^2}{24}\big([t_{12}, t_{13}+t_{23}] - [t_{13}, t_{12}+t_{32}]\big) \overset{\frac{\hbar^2}{24}(\Gamma_{123}-\Gamma_{132})}{\Longrightarrow} 0 \tag{3.44a}$$

only holds true weakly up to a certain linear combination of the $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched modifications from Proposition 3.15. Similarly, bringing all terms in (3.37b) to the left-hand side, one finds that

$$\tfrac{\hbar^2}{24}\big([t_{23}, t_{21}+t_{31}] - [t_{13}, t_{12}+t_{32}]\big) \overset{\frac{\hbar^2}{24}(\Gamma_{231}-\Gamma_{132})}{\Longrightarrow} 0 \tag{3.44b}$$

only holds true weakly up to a certain linear combination of the $\mathbf{Ch}^{[-1,0]}_{\mathbb{K}[[\hbar]]}$-enriched modifications from Proposition 3.15. This illustrates that the strict identities from Proposition 3.13, and the associated concept of semi-strict braided monoidal category objects from Definition A.8, is highly restrictive even at order $\hbar^2$ and hence they should be relaxed to weaker concepts by allowing for non-trivial pentagonator and hexagonator modifications.

4. Examples

The aim of this section is to study explicit examples of our infinitesimal 2-braidings from Theorem 3.10. The examples we consider are based on Lie N-algebras, for some natural number $N \geq 1$, which are a special class of L_∞-algebras whose underlying cochain complex

$$L = \Big(L^{-N+1} \xrightarrow{\mathrm{d}_L} L^{-N+2} \xrightarrow{\mathrm{d}_L} \cdots \xrightarrow{\mathrm{d}_L} L^{-1} \xrightarrow{\mathrm{d}_L} L^0 \Big) \in \mathbf{Ch}_{\mathbb{K}} \tag{4.1}$$

is concentrated in non-positive cohomological degrees. We shall always assume that the vector spaces L^i are finite-dimensional, for all i. The Lie N-algebra structure on L is given by a family of brackets $\ell_n \in \underline{\hom}(\wedge^n L, L)^{2-n}$ of degree $2-n$, for all $n \geq 2$, which have to satisfy the homotopy Jacobi identities, see e.g. [23] for an excellent review. By a simple degree counting argument, one observes that $\ell_n = 0$ must necessarily vanish for all $n > 1+N$, hence the family of brackets $\{\ell_n\}_{n\geq 2}$ is bounded for every Lie N-algebra.

Associated to a Lie N-algebra $L = (L, \{\ell_n\})$ is its Chevalley-Eilenberg CDGA $\mathrm{CE}^\bullet(L)$. The underlying $\mathbb{Z}$-graded commutative algebra of $\mathrm{CE}^\bullet(L)$ is given by

$$\mathrm{CE}^\sharp(L) := \mathrm{Sym}\big(L^{\sharp *}[-1]\big) \tag{4.2}$$

and the differential $\mathrm{d}_{\mathrm{CE}} := \mathrm{d}_{L^*[-1]} + \sum_{n\geq 2} \ell_n^{*[-1]}$ is defined on the generators $L^{\sharp *}[-1]$ by dualizing and shifting the brackets $\ell_n : L^{\sharp\otimes n} \to L^\sharp$. The Chevalley-Eilenberg CDGA $\mathrm{CE}^\bullet(L)$ of a finite-dimensional Lie N-algebra is finitely generated and semi-free, hence it satisfies our hypotheses from Section 2. Note that this CDGA is concentrated in non-negative cohomological degrees, which allows us to interpret $\mathrm{CE}^\bullet(L)$ geometrically as a stacky affine in the sense of [30, Section 1], see also [28,29]. Such stacky affines carry only a non-trivial stacky structure, but a trivial derived structure, which implies that one can describe them by singly-graded objects, in contrast to the more general doubly-graded stacky CDGAs from [30,28,29] or the graded mixed CDGAs from [5] which carry both stacky and derived structures. In the framework of stacky affines, ones interprets $\mathrm{CE}^\bullet(L)$ as the CDGA of functions on the infinitesimal quotient stack $\mathrm{B}L := [\mathrm{pt}/L]$, which is an infinitesimal analogue of the classifying stack $\mathrm{B}G := [\mathrm{pt}/G]$ of a (higher) group G.

The symmetric monoidal dg-category ${}_{\mathrm{CE}^\bullet(L)}\mathbf{dgMod}^{\mathrm{fgsf}}$ of finitely generated semi-free $\mathrm{CE}^\bullet(L)$-dg-modules describes representations of the Lie N-algebra L and their ∞-morphisms. Our result in Theorem 3.10 then implies that, given any 2-shifted Poisson structure on the CDGA $\mathrm{CE}^\bullet(L)$, one obtains a first-order deformation of the homotopy 2-category $\mathsf{Ho}_2\big({}_{\mathrm{CE}^\bullet(L)}\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ of the representation category of L into a braided monoidal category. Drawing analogies with the standard deformation theory of the (strict) representation category $\mathbf{Rep}(\mathfrak{g})$ of an ordinary Lie algebra $\mathfrak{g}$ and its relationship to the quantum group $U_q\mathfrak{g}$, see e.g. [6] and [20, Chapter XX], one may interpret our deformation constructions as a first step towards a Lie N-algebraic generalization of quantum groups.

Remark 4.1. From the point of view of formal moduli problems [24] and Koszul duality in the form reviewed for instance in [17], it may also seem natural to consider the completed Chevalley-Eilenberg CDGA $\widehat{\mathrm{CE}}^\bullet(L)$ whose underlying $\mathbb{Z}$-graded commutative algebra is given by

$$\widehat{\mathrm{CE}}^\sharp(L) := \widehat{\mathrm{Sym}}\big(L^{\sharp *}[-1]\big) := \prod_{l\geq 0} \mathrm{Sym}^l\big(L^{\sharp *}[-1]\big) \quad . \tag{4.3}$$

We would like to note that this completion of the Chevalley-Eilenberg CDGA in (4.2) to formal power series in the generators does not alter the structure of our results in this section. The key reason for this is that the 2-shifted Poisson structures on the completed $\widehat{\mathrm{CE}}^\bullet(L)$ and on the uncompleted $\mathrm{CE}^\bullet(L)$ Chevalley-Eilenberg CDGA of a Lie N-algebra L are naturally identified with each other: The completed 2-shifted polyvectors on $\widehat{\mathrm{CE}}^\bullet(L)$ are given by

$$\widehat{\mathrm{Pol}}^\sharp\big(\widehat{\mathrm{CE}}^\bullet(L), 2\big) \cong \prod_{m,l\geq 0} \mathrm{Sym}^m\big(L^\sharp[-2]\big) \otimes \mathrm{Sym}^l\big(L^{\sharp *}[-1]\big) \tag{4.4}$$

and they differ from $\widehat{\mathrm{Pol}}^\sharp\big(\mathrm{CE}^\bullet(L), 2\big)$ in Definition 2.5 by completing the direct sum $\bigoplus_{l\geq 0}$ to the direct product $\prod_{l\geq 0}$. A 2-shifted Poisson structure on the completed $\widehat{\mathrm{CE}}^\bullet(L)$ is then given by a formal sum $\pi = \sum_{m\geq 2}\sum_{l\geq 0} \pi^{(m,l)} \in \widehat{\mathrm{Pol}}\big(\widehat{\mathrm{CE}}^\bullet(L), 2\big)^4$, which is also formal in $l \geq 0$, satisfying the Maurer-Cartan equation from Definition 2.6. By a simple degree counting argument, taking into account the specific form of the underlying N-term complex (4.1) of L, one observes that the component $\pi^{(m,l)}$ can only be non-trivial if $(-N+1)\,m \leq 4 - 2m - l \leq (N-1)\,l$. Hence, for every fixed $m \geq 2$, there are only finitely many $l \geq 0$ satisfying these bounds, which implies that $\pi = \sum_{m\geq 2}\sum_{l\geq 0} \pi^{(m,l)} \in \widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(L), 2\big)^4$ also defines a 2-shifted Poisson structure on the uncompleted $\mathrm{CE}^\bullet(L)$. △

4.1. Ordinary Lie algebras

Let us start with the simplest example which is given by the case where the Lie N-algebra L is an ordinary (finite-dimensional) Lie algebra $\mathfrak{g} = (\mathfrak{g}, [\cdot,\cdot])$, i.e. $N = 1$. This means that the underlying $\mathbb{Z}$-graded vector space is $\mathfrak{g}$, concentrated in degree 0, and all brackets but the Lie bracket $\ell_2 := [\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ are trivial. To simplify our calculations below, we shall choose a basis $\{x_a \in \mathfrak{g} : a = 1,\dots,\dim(\mathfrak{g})\}$ and denote the structure constants by $[x_a, x_b] = f_{ab}^{\ c}\, x_c$. The Chevalley-Eilenberg CDGA $\mathrm{CE}^\bullet(\mathfrak{g})$ is then given by the $\mathbb{Z}$-graded commutative algebra

$$\mathrm{CE}^\sharp(\mathfrak{g}) = \mathbb{K}[\{\theta^a\}] \tag{4.5a}$$

which is freely generated by $\dim(\mathfrak{g})$-many generators θ^a of degree 1, for $a = 1,\dots,\dim(\mathfrak{g})$, together with the differential

$$\mathrm{d}_{\mathrm{CE}}(\theta^a) = -\tfrac{1}{2}\, f_{bc}^{\ a}\, \theta^b\, \theta^c \tag{4.5b}$$

which is determined by the structure constants. Here and in what follows we shall always use the standard summation convention according to which summations over repeated indices are suppressed. Note that the square-zero condition $\mathrm{d}_{\mathrm{CE}}\,\mathrm{d}_{\mathrm{CE}}(\theta^a) = 0$ follows from the Jacobi identity of the Lie bracket after expressing the latter in terms of the structure constants.

Since the input datum for Theorem 3.10 is a 2-shifted Poisson structure on $\mathrm{CE}^\bullet(\mathfrak{g})$ in the sense of Definition 2.6, we first have to understand and characterize the possible choices for such data. For the $\mathrm{CE}^\bullet(\mathfrak{g})$-dg-module of derivations $\mathsf{T}_{\mathrm{CE}^\bullet(\mathfrak{g})}$ we choose the basis $\{\partial_a \in \mathsf{T}_{\mathrm{CE}^\bullet(\mathfrak{g})} : a = 1,\dots,\dim(\mathfrak{g})\}$ which is defined by $\partial_a(\theta^b) = \delta_a^b$, for all a, b. Note that each basis element ∂_a has degree -1 since all θ^b are of degree 1. The underlying $\mathbb{Z}$-graded commutative algebra of the completed 2-shifted polyvectors from Definition 2.5 is then given by

$$\widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(\mathfrak{g}), 2\big)^\sharp = \mathbb{K}[\{\theta^a\}][[\{s^3\partial_a\}]] \quad , \tag{4.6}$$

where $[[\cdot]]$ denotes formal power series. Note that the generators $s^3\partial_a$ are of degree $3 - 1 = 2$ in $\widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(\mathfrak{g}), 2\big)$.

Proposition 4.2. *Any 2-shifted Poisson structure $\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(\mathfrak{g}), 2\big)^4$ on $\mathrm{CE}^\bullet(\mathfrak{g})$ is of the form*

$$\pi = \pi^{(2)} = \tfrac{1}{2}\, \pi^{ab}\, s^3\partial_a\, s^3\partial_b \quad , \tag{4.7a}$$

where $\pi^{ab} \in \mathbb{K}$ are coefficients which must satisfy the $\mathfrak{g}$-invariance property

$$f_{cd}^{\ a}\, \pi^{db} + f_{cd}^{\ b}\, \pi^{ad} = 0 \quad . \tag{4.7b}$$

Proof. Using that θ^a is of degree 1 and that $s^3\partial_a$ is of degree 2, one finds by a simple degree-counting argument that (4.7a) is the only possible form for a completed 2-shifted polyvector of degree 4 and weight ≥ 2. The Maurer-Cartan equation from Definition 2.6 for such π, see also Remark 2.7, is equivalent to the two individual equations $\mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi) = 0$ and $\{\pi,\pi\} = 0$. The second equation holds true automatically because (4.7a) has constant coefficients, while the first equation is equivalent to the $\mathfrak{g}$-invariance property (4.7b). To see the latter point explicitly, one uses the Leibniz rule for the differential $\mathrm{d}_{\widehat{\mathrm{Pol}}}$ and $\mathrm{d}_{\widehat{\mathrm{Pol}}}(s^3\partial_a) = -f_{ab}^{\ c}\, \theta^b\, s^3\partial_c$. □

Remark 4.3. Note that the 2-shifted Poisson structures π from Proposition 4.2 are canonically identified with invariant symmetric tensors $\pi = \frac{1}{2}\pi^{ab}\, x_a\, x_b \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$. Hence, our results are compatible with the results of Safronov [34] who has computed the 2-shifted (and also the 1-shifted) Poisson structures on the classifying stack $\mathrm{B}G = [\mathrm{pt}/G]$ of an algebraic group G. △

Consider now the homotopy 2-category ${}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{C} = \mathsf{Ho}_2\big({}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ of the symmetric monoidal dg-category of finitely generated semi-free $\mathrm{CE}^\bullet(\mathfrak{g})$-dg-modules, which is the object of focus in our deformation construction from Theorem 3.10. Let us first illustrate the sense in which this category describes weak representations of the Lie algebra $\mathfrak{g}$. By definition, given any object $M \in {}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{C}$, one has a $\mathbb{Z}$-graded vector space of generators $W^\sharp$ such that $M^\sharp \cong \mathrm{CE}^\sharp(\mathfrak{g}) \otimes W^\sharp$. Choosing any basis $\{w_i \in W^\sharp\}$ for the generators, the differential on M is completely characterized by specifying $\mathrm{d}_M(\mathbb{1}\otimes w_i) \in M$, for all i, which when expanded in terms of the chosen bases reads as

$$\mathrm{d}_M(\mathbb{1}\otimes w_i) = \underline{M}_i^{\ j} \otimes w_j = \sum_{n=0}^{\infty} \tfrac{1}{n!}\, M_{a_1\dots a_n i}^{\ \ \ \ \ j}\, \theta^{a_1}\cdots\theta^{a_n} \otimes w_j \quad , \tag{4.8}$$

where $M_{a_1\dots a_n i}^{\ \ \ \ \ j} \in \mathbb{K}$ are coefficients. The square-zero condition $\mathrm{d}_M\mathrm{d}_M(\mathbb{1}\otimes w_i) = 0$ gives a tower of compatibility conditions between the coefficients $M_{a_1\dots a_n i}^{\ \ \ \ \ j}$ and the structure constants $f_{bc}^{\ a}$ of the Lie algebra $\mathfrak{g}$, which express a concept of Lie

algebra representation up to coherent homotopies. Strict representations of $\mathfrak{g}$ on a cochain complex W are recovered in the special case where $\mathrm{d}_M(\mathbb{1}\otimes w_i) = M_i{}^j\otimes w_j + M_{ai}{}^j\theta^a\otimes w_j$ is of polynomial degree ≤ 1 in θ^a. The first term in this expression defines a differential d_W on $W^\sharp$ and the second term defines a strict Lie algebra action map $\mathfrak{g}\otimes W\to W$ which is a cochain map for the differential of $W=(W^\sharp,\mathrm{d}_W)$.

In addition to weak representations of the Lie algebra $\mathfrak{g}$, the homotopy 2-category ${}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{C} = \mathsf{Ho}_2\big({}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{dgMod}^{\mathrm{fgsf}}\big)$ also describes a concept of ∞-morphisms between such weak representations, i.e. weakly $\mathfrak{g}$-equivariant maps. (The 2-morphisms in ${}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{C}$ are homotopies between such ∞-morphisms.) Indeed, a morphism $h:M\to M'$ in ${}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{C}$ is by definition a cochain map which commutes with the left $\mathrm{CE}^\bullet(\mathfrak{g})$-actions. Hence, by choosing a basis $\{w_i\in W^\sharp\}$ for $M^\sharp\cong\mathrm{CE}^\sharp(\mathfrak{g})\otimes W^\sharp$ and a basis $\{w'_i\in W'^\sharp\}$ for $M'^\sharp\cong\mathrm{CE}^\sharp(\mathfrak{g})\otimes W'^\sharp$, we can expand

$$h(\mathbb{1}\otimes w_i) = \underline{h}_i{}^j\otimes w'_j = \sum_{n=0}^{\infty}\tfrac{1}{n!}\,h_{a_1\dots a_n i}{}^j\,\theta^{a_1}\cdots\theta^{a_n}\otimes w'_j \quad , \tag{4.9}$$

where $h_{a_1\dots a_n i}{}^j\in\mathbb{K}$ are coefficients. The cochain map condition $\mathrm{d}_{M'}\big(h(\mathbb{1}\otimes w_i)\big) = h\big(\mathrm{d}_M(\mathbb{1}\otimes w_i)\big)$ gives a tower of compatibility conditions between the coefficients $h_{a_1\dots a_n i}{}^j$ and the structure constants f^a_{bc} of the Lie algebra $\mathfrak{g}$, which express a concept of $\mathfrak{g}$-equivariance up to coherent homotopies. Strict morphisms $h:M\to M'$ between strict $\mathfrak{g}$-representations on cochain complexes W and W' are recovered in the special case where $h(\mathbb{1}\otimes w_i) = h_i{}^j\otimes w'_j$ is constant in θ^a.

The following result is a direct consequence of Theorem 3.10 and Proposition 4.2.

Corollary 4.4. *Every invariant symmetric tensor $\pi=\frac{1}{2}\pi^{ab}x_a x_b\in(\mathrm{Sym}^2\mathfrak{g})^\mathfrak{g}$ defines via Proposition 4.2 and Theorem 3.10 a γ-equivariant infinitesimal 2-braiding t on the homotopy 2-category ${}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{C}$ of the symmetric monoidal dg-category of finitely generated semi-free $\mathrm{CE}^\bullet(\mathfrak{g})$-dg-modules.*

Remark 4.5. Let us recall that explicit formulas for the components of the γ-equivariant infinitesimal 2-braiding t are given in (3.29). Note that the morphism components $t_{M,N}$ in (3.29a) involve the de Rham differential

$$(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(\mathrm{d}_M(\mathbb{1}\otimes w_i)\big) = \sum_{n=1}^{\infty}\tfrac{1}{(n-1)!}\,M_{a_1\dots a_n i}{}^j\,\theta^{a_1}\cdots\theta^{a_{n-1}}\mathrm{d}_{\mathrm{dR}}(\theta^{a_n})\otimes w_j \tag{4.10}$$

of (4.8), as well as $(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(\mathrm{d}_N(\mathbb{1}\otimes u_q)\big)$. Hence, the components $t_{M,N}$ in (3.29a) are in general non-constant polynomials in θ^a, unless both M and N are strict $\mathfrak{g}$-representations. Similar, the homotopy components $t_{(M,N),(M',N')}(h\widetilde{\otimes}k)$ in (3.29b) involve the de Rham differential

$$(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(h(\mathbb{1}\otimes w_i)\big) = \sum_{n=1}^{\infty}\tfrac{1}{(n-1)!}\,h_{a_1\dots a_n i}{}^j\,\theta^{a_1}\cdots\theta^{a_{n-1}}\mathrm{d}_{\mathrm{dR}}(\theta^{a_n})\otimes w'_j \tag{4.11}$$

of (4.9), as well as $(\mathrm{d}_{\mathrm{dR}}\otimes\mathrm{id})\big(k(\mathbb{1}\otimes u_q)\big)$. Hence, the homotopies $t_{(M,N),(M',N')}(h\widetilde{\otimes}k)$ in (3.29b) are in general non-trivial, unless both h and k are strict morphisms. △

As a direct consequence of Proposition 3.13 and Proposition 3.15, we obtain the following result.

Corollary 4.6. *Every invariant symmetric tensor $\pi=\frac{1}{2}\pi^{ab}x_a x_b\in(\mathrm{Sym}^2\mathfrak{g})^\mathfrak{g}$ defines via Corollary 4.4 and Proposition 3.13 a braided monoidal deformation to all orders in $\hbar$ of the full subcategory of ${}_{\mathrm{CE}^\bullet(\mathfrak{g})}\mathbf{C}\in\mathbf{dgCat}^{[-1,0],\mathrm{ps}}_{\mathbb{K}[[\hbar]]}$ which consists of all $\mathrm{CE}^\bullet(\mathfrak{g})$-dg-modules corresponding to strict $\mathfrak{g}$-representations.*

Proof. Combining our observations from Example 3.18 and Remark 4.5, we obtain that the modifications Γ_{ijk} in Proposition 3.15 are all trivial for strict $\mathfrak{g}$-representations because in this case $(t_{M,N})_{iq}{}^{jr}$ are constant polynomials in θ^a, hence $\mathrm{d}_{\mathrm{dR}}\big((t_{M,N})_{iq}{}^{jr}\big)=0$. This implies that the hypotheses of Proposition 3.13 are satisfied, which yields the desired braided monoidal deformation to all orders in $\hbar$. □

Remark 4.7. Our result in Corollary 4.6 is a homotopical generalization of the deformation constructions in [6] and [20, Chapter XX]. We would like to stress that our generalization is non-trivial because, even though we restrict ourselves to strict $\mathfrak{g}$-representations, the morphisms in the *full* subcategory from Corollary 4.6 are general (i.e. non-strict) ∞-morphisms between strict $\mathfrak{g}$-representations. Recalling our observations from Remark 4.5, this implies in particular that the infinitesimal braiding t is a genuine pseudo-natural transformation, i.e. it has non-trivial homotopy components $t_{(M,N),(M',N')}(h\widetilde{\otimes}k)$, and so are the associator and braiding of the deformed category from Proposition 3.13 and Corollary 4.6. △

4.2. String Lie 2-algebras

String Lie 2-algebras are simple examples of Lie 2-algebras which arise as shifted central extensions determined by an ordinary (finite-dimensional) Lie algebra $\mathfrak{g} = (\mathfrak{g}, [\cdot,\cdot])$, concentrated in degree 0, and a 3-cocycle $\kappa : \bigwedge^3 \mathfrak{g} \to \mathbb{K}$ on $\mathfrak{g}$. Such Lie 2-algebras appeared first in [1, Example 6.10], however the name "string Lie 2-algebras" has only been established later. The resulting Lie 2-algebra $\mathfrak{g}_\kappa$ is given by the $\mathbb{Z}$-graded vector space defined by $\mathfrak{g}_\kappa^0 := \mathfrak{g}$, $\mathfrak{g}_\kappa^{-1} := \mathbb{K}$ and $\mathfrak{g}_\kappa^i := 0$, for all $i \in \mathbb{Z} \setminus \{-1, 0\}$, and the non-trivial brackets read as

$$\ell_2(x,y) := [x,y] \quad , \qquad \ell_3(x,y,z) := \kappa(x,y,z) \quad , \tag{4.12}$$

for all $x, y, z \in \mathfrak{g}$. Let us choose again a basis $\{x_a \in \mathfrak{g}\}$ and denote the structure constants by $[x_a, x_b] = f_{ab}^c\, x_c$ and $\kappa(x_a, x_b, x_c) = \kappa_{abc}$.

The Chevalley-Eilenberg CDGA $\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)$ of the string Lie 2-algebra $\mathfrak{g}_\kappa$ is then given by the $\mathbb{Z}$-graded commutative algebra

$$\mathrm{CE}^\sharp(\mathfrak{g}_\kappa) = \mathbb{K}[\{\theta^a\}, \nu] \tag{4.13a}$$

which is freely generated by $\dim(\mathfrak{g})$-many generators θ^a of degree 1, for $a = 1, \dots, \dim(\mathfrak{g})$, and a single generator ν of degree 2, together with the differential

$$\mathrm{d}_{\mathrm{CE}}(\theta^a) = -\tfrac{1}{2}\, f_{bc}^a\, \theta^b\, \theta^c \quad , \qquad \mathrm{d}_{\mathrm{CE}}(\nu) = -\tfrac{1}{3!}\, \kappa_{abc}\, \theta^a\, \theta^b\, \theta^c \tag{4.13b}$$

which is determined by the structure constants. Note that the square-zero condition $\mathrm{d}_{\mathrm{CE}}\, \mathrm{d}_{\mathrm{CE}}(\theta^a) = 0$ follows from the Jacobi identity of the Lie bracket and $\mathrm{d}_{\mathrm{CE}}\, \mathrm{d}_{\mathrm{CE}}(\nu) = 0$ follows from the 3-cocycle condition of κ.

Let us now characterize the 2-shifted Poisson structures on $\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)$ in the sense of Definition 2.6. For the $\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)$-dg-module of derivations $\mathsf{T}_{\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)}$ we choose the basis $\{\{\partial_a\}, \partial_\nu\}$ which is defined by $\partial_a(\theta^b) = \delta_a^b$, $\partial_a(\nu) = 0$, $\partial_\nu(\theta^a) = 0$ and $\partial_\nu(\nu) = 1$, for all a, b. Note that each basis element ∂_a has degree -1 and ∂_ν has degree -2. The underlying $\mathbb{Z}$-graded commutative algebra of the completed 2-shifted polyvectors from Definition 2.5 is then given by

$$\widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(\mathfrak{g}_\kappa), 2\big)^\sharp = \mathbb{K}[\{\theta^a\}, \nu][[\{s^3\partial_a\}, s^3\partial_\nu]] \quad , \tag{4.14}$$

where $[[\cdot]]$ denotes formal power series. Note that the generators $s^3\partial_a$ are of degree $3-1=2$ and that $s^3\partial_\nu$ is of degree $3-2=1$ in $\widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(\mathfrak{g}_\kappa), 2\big)$.

Proposition 4.8. *Any 2-shifted Poisson structure $\pi = \sum_{m\geq 2} \pi^{(m)} \in \widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(\mathfrak{g}_\kappa), 2\big)^4$ on $\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)$ is of the form*

$$\pi \;=\; \pi^{(2)} \;=\; \tfrac{1}{2}\, \pi^{ab}\, s^3\partial_a\, s^3\partial_b + \tilde{\pi}_b^a\, \theta^b\, s^3\partial_a\, s^3\partial_\nu \quad , \tag{4.15a}$$

where $\pi^{ab} \in \mathbb{K}$ and $\tilde{\pi}_b^a \in \mathbb{K}$ are coefficients which must satisfy the properties

$$f_{cd}^a\, \pi^{db} + f_{cd}^b\, \pi^{ad} \;=\; 0 \quad , \tag{4.15b}$$

$$f_{bc}^d\, \tilde{\pi}_d^a - f_{bd}^a\, \tilde{\pi}_c^d + f_{cd}^a\, \tilde{\pi}_b^d - \kappa_{bcd}\, \pi^{da} \;=\; 0 \quad , \tag{4.15c}$$

$$\pi^{ac}\, \tilde{\pi}_c^b + \pi^{bc}\, \tilde{\pi}_c^a \;=\; 0 \quad . \tag{4.15d}$$

Proof. The general form (4.15a) for a completed 2-shifted polyvector of degree 4 and weight ≥ 2 follows from a simple degree counting argument. (Note that products of the form $s^3\partial_\nu\, s^3\partial_\nu$ vanish in the CDGA $\widehat{\mathrm{Pol}}\big(\mathrm{CE}^\bullet(\mathfrak{g}_\kappa), 2\big)$ because $s^3\partial_\nu$ has degree 1, hence $s^3\partial_\nu\, s^3\partial_\nu = -s^3\partial_\nu\, s^3\partial_\nu = 0$.) The Maurer-Cartan equation from Definition 2.6 for such π, see also Remark 2.7, is equivalent to the two individual equations $\mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi) = 0$ and $\{\pi, \pi\} = 0$. The first equation $\mathrm{d}_{\widehat{\mathrm{Pol}}}(\pi) = 0$ can be evaluated using the Leibniz rule for the differential $\mathrm{d}_{\widehat{\mathrm{Pol}}}$ and

$$\mathrm{d}_{\widehat{\mathrm{Pol}}}(s^3\partial_a) = -f_{ab}^c\, \theta^b\, s^3\partial_c + \tfrac{1}{2}\, \kappa_{abc}\, \theta^b\, \theta^c\, s^3\partial_\nu \quad , \qquad \mathrm{d}_{\widehat{\mathrm{Pol}}}(s^3\partial_\nu) = 0 \quad . \tag{4.16}$$

From this one finds the first two properties (4.15b) and (4.15c). The second equation $\{\pi, \pi\} = 0$ can be evaluated using the definition (2.21) and properties (2.22) of the Schouten-Nijenhuis bracket, which leads to the third property (4.15d). □

Remark 4.9. As a consequence of (4.15b), the first term in (4.15a) is equivalent to the datum of an invariant symmetric tensor $\pi = \frac{1}{2}\, \pi^{ab}\, x_a\, x_b \in (\mathrm{Sym}^2\mathfrak{g})^\mathfrak{g}$ on the underlying Lie algebra $\mathfrak{g}$ of the string Lie 2-algebra $\mathfrak{g}_\kappa$. The second term in (4.15a) is equivalent to a linear map $\tilde{\pi} : \mathfrak{g} \to \mathfrak{g}$, $x_a \mapsto \tilde{\pi}_a^b\, x_b$ on the underlying Lie algebra. Property (4.15d) then demands that the linear map $\tilde{\pi}$ annihilates the invariant symmetric tensor π according to

$$(\tilde{\pi} \otimes \mathrm{id}_\mathfrak{g})(\pi) + (\mathrm{id}_\mathfrak{g} \otimes \tilde{\pi})(\pi) \;=\; 0 \quad . \tag{4.17a}$$

The remaining property (4.15c) can be written equivalently as

$$\tilde{\pi}\big([x,y]\big) = \big[\tilde{\pi}(x),y\big] + \big[x,\tilde{\pi}(y)\big] + \big(\kappa(x,y,-)\otimes \mathrm{id}_{\mathfrak{g}}\big)(\pi) + \big(\mathrm{id}_{\mathfrak{g}}\otimes\kappa(x,y,-)\big)(\pi) \quad , \tag{4.17b}$$

for all $x,y\in\mathfrak{g}$, which means that the linear map $\tilde{\pi}:\mathfrak{g}\to\mathfrak{g}$ is a derivation of the underlying Lie algebra $\mathfrak{g}$, up to corrections arising from the 3-cocycle $\kappa:\bigwedge\mathfrak{g}^3\to\mathbb{K}$ and the invariant symmetric tensor $\pi\in(\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$.

Our 2-shifted Poisson structures on $\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)$ are similar, but as it seems not identical, to the quasi-invariant symmetric tensors on crossed modules of Lie algebras from [12, Definition 30]. A direct comparison between these two structures is difficult because our preferred model for the string Lie 2-algebra $\mathfrak{g}_\kappa$ is given by a finite-dimensional Lie 2-algebra with a non-trivial ℓ_3 bracket (4.12), while [12] use a different model which is given by an infinite-dimensional crossed module of Lie algebras that encodes the cocycle κ less directly. Our suspicion that these two concepts are slightly different will be strengthened by the following example. △

Example 4.10. In order to illustrate Proposition 4.8, let us consider the example where $\mathfrak{g}=\mathfrak{sl}_2$ is the special linear Lie algebra of order 2. We work with the usual basis $\{x_+,x_-,x_3\in\mathfrak{sl}_2\}$ whose Lie bracket relations are

$$[x_+,x_-]=x_3 \quad , \qquad [x_3,x_\pm]=\pm 2x_\pm \quad . \tag{4.18}$$

Using the rescaled Killing form $(\cdot,\cdot):\mathfrak{sl}_2\otimes\mathfrak{sl}_2\to\mathbb{K}$ whose non-vanishing values are given by $(x_3,x_3)=1$ and $(x_+,x_-)=(x_-,x_+)=\frac{1}{2}$, we define the 3-cocycle $\kappa(x,y,z):=(x,[y,z])$. In terms of our basis, this 3-cocycle is given by

$$\kappa(x_+,x_-,x_3) = 1 \tag{4.19}$$

and total antisymmetry in its arguments. This defines the string Lie 2-algebra $(\mathfrak{sl}_2)_\kappa$.

Since $\mathfrak{sl}_2$ is a simple Lie algebra, there exists a unique (up to scale) invariant symmetric tensor, which is concretely given by

$$\pi = \lambda\left(x_+x_- + \tfrac{1}{4}x_3^2\right) \in (\mathrm{Sym}^2\mathfrak{sl}_2)^{\mathfrak{sl}_2} \quad , \tag{4.20}$$

where $\lambda\in\mathbb{K}$ is any parameter. Assuming for the moment that $\lambda\neq 0$ is non-zero, the general solution for the linear map $\tilde{\pi}:\mathfrak{sl}_2\to\mathfrak{sl}_2$ to the equation (4.15d), or the equivalent equation (4.17a), is given by

$$\tilde{\pi}(x_+) = a\,x_+ + b\,x_3 \ , \quad \tilde{\pi}(x_-) = -a\,x_- + c\,x_3 \ , \quad \tilde{\pi}(x_3) = -2\,c\,x_+ - 2\,b\,x_- \quad , \tag{4.21}$$

where $a,b,c\in\mathbb{K}$ are arbitrary parameters. Evaluating the equivalent form (4.17b) of the remaining equation (4.15c) on $x=x_+$ and $y=x_-$, one finds that it does not admit any solutions for $\lambda\neq 0$. This implies that, for every 2-shifted Poisson structure on $\mathrm{CE}^\bullet((\mathfrak{sl}_2)_\kappa)$ as in Proposition 4.8, the invariant symmetric tensor (4.20) must necessarily be trivial $\pi=0$. In this case the equations in (4.15), see also (4.17) for their equivalent forms, simplify to the single condition that $\tilde{\pi}:\mathfrak{sl}_2\to\mathfrak{sl}_2$ is a Lie algebra derivation, i.e. $\tilde{\pi}([x,y])=[\tilde{\pi}(x),y]+[x,\tilde{\pi}(y)]$, for all $x,y\in\mathfrak{sl}_2$. Hence, the 2-shifted Poisson structures on $\mathrm{CE}^\bullet((\mathfrak{sl}_2)_\kappa)$ are given by the Lie algebra derivations $\tilde{\pi}$ on $\mathfrak{sl}_2$. Taking into account the natural concept of homotopies between shifted Poisson structures, see e.g. [34], it follows that every 2-shifted Poisson structure on $\mathrm{CE}^\bullet((\mathfrak{sl}_2)_\kappa)$ is homotopic to the trivial one because every Lie algebra derivation of $\mathfrak{sl}_2$ is inner. We would like to thank the anonymous reviewer for pointing out to us this important observation.

This behavior is drastically different to the case of ordinary Lie algebras $\mathfrak{g}$ from Proposition 4.2 and the quasi-invariant symmetric tensors on crossed modules of Lie algebras from [12, Theorem 36] and [32, Section 5.2]. ▽

Example 4.11. The triviality result for 2-shifted Poisson structures from Example 4.10 generalizes easily to the case of the string Lie 2-algebra of *any* simple Lie algebra $\mathfrak{g}$. This motivates us to search for non-trivial 2-shifted Poisson structures in the case where the underlying Lie algebra $\mathfrak{g}$ is not simple. Let us consider as a concrete example the Heisenberg Lie algebra, which is the 3-dimensional Lie algebra $\mathfrak{heis}=\mathbb{K}[x,y,z]$ with the only non-trivial Lie bracket relations among the generators given by $[x,y]=-[y,x]=z$. Choosing the 3-cocycle $\kappa(x,y,z)=1$, we obtain the associated Heisenberg string Lie 2-algebra $\mathfrak{heis}_\kappa$.

The most general invariant symmetric tensor on the Heisenberg Lie algebra is given by

$$\pi = \tfrac{\lambda}{2}z^2 \in (\mathrm{Sym}^2\mathfrak{heis})^{\mathfrak{heis}} \quad , \tag{4.22a}$$

where $\lambda\in\mathbb{K}$ is any parameter which we choose to be non-zero $\lambda\neq 0$. The linear map $\tilde{\pi}:\mathfrak{heis}\to\mathfrak{heis}$ entering the 2-shifted Poisson structure then satisfies the required equation (4.17a) if and only if $\tilde{\pi}(z)=0$. Analyzing the remaining equation (4.17b), one finds that $\tilde{\pi}$ must be of the form

$$\tilde{\pi}(x) = a\,x + b\,y + c\,z \ , \quad \tilde{\pi}(y) = d\,x - (a+\lambda)\,y + e\,z \ , \quad \tilde{\pi}(z) = 0 \quad , \tag{4.22b}$$

where $a,b,c,d,e\in\mathbb{K}$ are free parameters. This provides a multiparameter family of 2-shifted Poisson structures on the Chevalley-Eilenberg CDGA $\mathrm{CE}^\bullet(\mathfrak{heis}_\kappa)$ of the Heisenberg string Lie 2-algebra $\mathfrak{heis}_\kappa$. ▽

The following result is a direct consequence of Theorem 3.10, Proposition 4.8 and Remark 4.9.

Corollary 4.12. *Every pair $(\pi, \tilde{\pi})$ consisting of a invariant symmetric tensor $\pi = \frac{1}{2}\pi^{ab}\, x_a\, x_b \in (\mathrm{Sym}^2\mathfrak{g})^{\mathfrak{g}}$ and a linear map $\tilde{\pi} : \mathfrak{g} \to \mathfrak{g}$, $x_a \mapsto \tilde{\pi}_a^b\, x_b$ which satisfies the properties in (4.17) defines via Proposition 4.8 and Theorem 3.10 a γ-equivariant infinitesimal 2-braiding t on the homotopy 2-category ${}_{\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)}\mathbf{C}$ of the symmetric monoidal dg-category of finitely generated semi-free $\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)$-dg-modules.*

Remark 4.13. Recalling the explicit formulas for the components of the γ-equivariant infinitesimal 2-braiding t from (3.29), one immediately observes by using also (4.15a) that the morphism components $t_{M,N}$ are in general non-constant polynomials in θ^a in the case where $\tilde{\pi}_b^a \neq 0$ is non-trivial. (As illustrated in Example 4.11, this seems to be the most relevant case.) Hence, the modifications Γ_{ijk} from Proposition 3.15 and Example 3.18 are in general non-trivial. This even holds true for representation of the string Lie 2-algebra $\mathfrak{g}_\kappa$ which are strict in the sense that the expansions as in (4.8) of the differential d_M are polynomials in θ^a and ν of degree ≤ 1. As a consequence, we currently do not have a meaningful generalization of the all-order deformation result from Corollary 4.6 to the case of $\mathrm{CE}^\bullet(\mathfrak{g}_\kappa)$. Hence, further work as outlined in Remark 3.17 is needed in order to extend our result in Corollary 4.12 to a braided monoidal deformation up to all orders in $\hbar$. △

Acknowledgements

We would like to thank Jon Pridham and the anonymous reviewer for useful comments. C.K. is supported by an EPSRC doctoral studentship (2747173). A.S. gratefully acknowledges the support of the Royal Society (UK) through a Royal Society University Research Fellowship (URF\R\211015) and Enhancement Grants (RF\ERE\210053 and RF\ERE\231077).

Appendix A. Higher-categorical framework

In this appendix, we introduce a suitable higher-categorical framework in which the deformation constructions from Section 3 take place. The basic objects of our interest are categories which are enriched over the symmetric monoidal category $\mathbf{Ch}_R^{[-1,0]}$ of 2-term cochain complexes (concentrated in degrees -1 and 0) of modules over a commutative $\mathbb{K}$-algebra R. We will show below that these objects assemble into a suitably weakened kind of 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ whose 1-morphisms are $\mathbf{Ch}_R^{[-1,0]}$-enriched functors, 2-morphisms are $\mathbf{Ch}_R^{[-1,0]}$-enriched *pseudo*-natural transformations and 3-morphisms are $\mathbf{Ch}_R^{[-1,0]}$-enriched modifications. The crucial point here is that we allow for pseudo-natural transformations, in contrast to only strictly natural ones, as these are required in our study of infinitesimal 2-braidings in Section 3. This 3-category carries a symmetric monoidal structure, which allows one to define a concept of $\mathbf{Ch}_R^{[-1,0]}$-enriched braided monoidal categories whose associators, unitors and braidings are enriched *pseudo*-natural transformations, as required in Section 3. We assume that the reader has some familiarity with enriched category theory, see e.g. [21] for a comprehensive treatment, or [18, Chapter 1.3] and [31, Chapter 3] for more concise introductions.

A.1. 3*-categories with pseudo-functorial composition*

Let us recall that a strict 3-category can be defined as a category which is enriched over the category $\mathbf{2Cat}^{\mathrm{str}}$ of strict 2-categories and strict 2-functors. We require a slightly weakened variant of this concept where the enrichment category is generalized to the category $\mathbf{2Cat}^{\mathrm{ps}}$ of strict 2-categories and pseudo-functors. See e.g. [18, Theorem 4.1.30] for a detailed description of this category and, in particular, of its strictly associative and unital composition. The category $\mathbf{2Cat}^{\mathrm{ps}}$ admits finite products and a terminal object, given by the 2-category $\mathbf{1} \in \mathbf{2Cat}^{\mathrm{ps}}$ consisting of a single 0-cell and its identity 1-cell and 2-cell, hence it is symmetric monoidal with respect to the Cartesian symmetric monoidal structure.

Definition A.1. We denote by

$$\mathbf{3Cat} := \mathbf{2Cat}^{\mathrm{ps}}\text{-}\mathbf{Cat} \tag{A.1}$$

the 2-category whose

- objects are $\mathbf{2Cat}^{\mathrm{ps}}$-enriched categories,
- 1-morphisms are $\mathbf{2Cat}^{\mathrm{ps}}$-enriched functors, and
- 2-morphisms are $\mathbf{2Cat}^{\mathrm{ps}}$-enriched natural transformations.

To illustrate how an object $\mathcal{C} \in \mathbf{3Cat}$ admits an interpretation in terms of a weakened kind of 3-category, let us spell out, in some detail, the data defining $\mathcal{C}$. By definition of an enriched category, $\mathcal{C}$ has a class of objects and, for each pair of objects $c, c' \in \mathcal{C}$, a strict 2-category $\mathcal{C}(c,c') \in \mathbf{2Cat}^{\mathrm{ps}}$ of morphisms. Since the 2-category $\mathcal{C}(c,c')$ consists of 0-cells, 1-cells and 2-cells, there are three layers of morphisms in $\mathcal{C} \in \mathbf{3Cat}$, as required for a 3-category. The enriched category $\mathcal{C}$ comes further with identity morphisms $\mathrm{id}_c : \mathbf{1} \to \mathcal{C}(c,c)$, or equivalently 0-cells $\mathrm{id}_c \in \mathcal{C}(c,c)$, and composition morphisms

$$* : \mathcal{C}(c', c'') \times \mathcal{C}(c, c') \longrightarrow \mathcal{C}(c, c'') \tag{A.2}$$

in $\mathbf{2Cat}^{\mathrm{ps}}$, which in our case are pseudo-functors between strict 2-categories. The compositions $*$ must be strictly associative and strictly unital with respect to the identities id_c. Note that the relaxation to pseudo-functors makes our variant of 3-categories from Definition A.1 more general than strict 3-categories, for which the analogous composition morphisms $*$ must be strict 2-functors.

The 2-category **3Cat** carries a symmetric monoidal structure which is determined by the Cartesian symmetric monoidal structure of the enrichment category $\mathbf{2Cat}^{\mathrm{ps}}$, see e.g. [21, Section 1.4] for the details. The monoidal product $\mathcal{C} \times \mathcal{D} \in \mathbf{3Cat}$ of $\mathcal{C}, \mathcal{D} \in \mathbf{3Cat}$ is the $\mathbf{2Cat}^{\mathrm{ps}}$-enriched category whose objects are pairs $(c, d) \in \mathcal{C} \times \mathcal{D}$ of objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ and whose morphisms $(\mathcal{C} \times \mathcal{D})((c, d), (c', d')) := \mathcal{C}(c, c') \times \mathcal{D}(d, d') \in \mathbf{2Cat}^{\mathrm{ps}}$ are given by the Cartesian symmetric monoidal structure on $\mathbf{2Cat}^{\mathrm{ps}}$. The monoidal unit $\mathcal{I} \in \mathbf{3Cat}$ is the $\mathbf{2Cat}^{\mathrm{ps}}$-enriched category with a single object, say $\star \in \mathcal{I}$, and morphisms $\mathcal{I}(\star, \star) := \mathbf{1} \in \mathbf{2Cat}^{\mathrm{ps}}$ the terminal object. The symmetric braiding

$$\mathrm{flip}_{\mathcal{C},\mathcal{D}} : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D} \times \mathcal{C} \tag{A.3a}$$

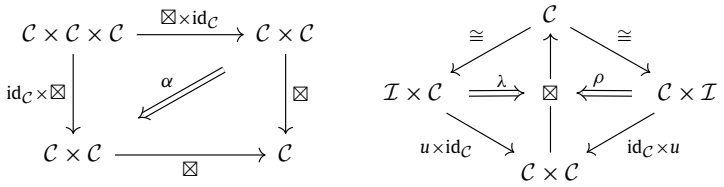
is the $\mathbf{2Cat}^{\mathrm{ps}}$-enriched functor which acts on objects by flipping $(c, d) \mapsto (d, c)$ and on morphisms through the symmetric braiding on $\mathbf{2Cat}^{\mathrm{ps}}$

$$(\mathcal{C} \times \mathcal{D})\big((c, d), (c', d')\big) = \mathcal{C}(c, c') \times \mathcal{D}(d, d') \xrightarrow{\cong} \mathcal{D}(d, d') \times \mathcal{C}(c, c') = (\mathcal{D} \times \mathcal{C})\big((d, c), (d', c')\big) \quad . \tag{A.3b}$$

Making use of this symmetric monoidal structure on **3Cat**, one can define a concept of symmetric monoidal category objects in **3Cat**.

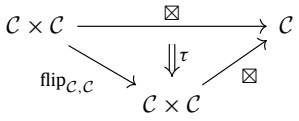
Definition A.2. A *symmetric monoidal category object in* **3Cat** is a tuple $(\mathcal{C}, \boxtimes, u, \alpha, \lambda, \rho, \tau)$ consisting of the following data:

(i) an object $\mathcal{C} \in \mathbf{3Cat}$,
(ii) a 1-morphism $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ in **3Cat** (called monoidal product),
(iii) a 1-morphism $u : \mathcal{I} \to \mathcal{C}$ in **3Cat** (called monoidal unit),
(iv) invertible 2-morphisms



(A.4)

in **3Cat** (called associator, left unitor and right unitor),
(v) an invertible 2-morphism



(A.5)

in **3Cat** (called symmetric braiding).

These data have to satisfy the usual coherence conditions for symmetric monoidal categories internal to the 2-category **3Cat**.

A.2. A symmetric monoidal 3*-category of* 2*-term truncated dg-categories*

Let R be a commutative, associative and unital $\mathbb{K}$-algebra and consider the symmetric monoidal category $\mathbf{Ch}_R$ of cochain complexes of R-modules from Subsection 2.1. We denote by

$$\mathbf{Ch}_R^{[-1,0]} \subseteq \mathbf{Ch}_R \tag{A.6}$$

the full subcategory whose objects are all cochain complexes $V \in \mathbf{Ch}_R$ concentrated in degrees $\{-1, 0\}$, i.e. $V^i = 0$ for all $i \in \mathbb{Z} \setminus \{-1, 0\}$. We shall often denote such objects by

$$V = \Big(V^{-1} \xrightarrow{\mathrm{d}_V} V^0 \Big) \in \mathbf{Ch}_R^{[-1,0]} \quad . \tag{A.7}$$

Note that the tensor product $\otimes_R$ on $\mathbf{Ch}_R$ does not directly restrict to the full subcategory $\mathbf{Ch}_R^{[-1,0]} \subseteq \mathbf{Ch}_R$ because $V \otimes_R W \in \mathbf{Ch}_R$ is, in general, concentrated in degrees $\{-2,-1,0\}$, for $V, W \in \mathbf{Ch}_R^{[-1,0]}$. This issue can be resolved by composing $\otimes_R$ with the good truncation functor $\tau^{[-1,0]} : \mathbf{Ch}_R \to \mathbf{Ch}_R^{[-1,0]}$ to define the truncated tensor product

$$\widetilde{\otimes}_R \;:\; \mathbf{Ch}_R^{[-1,0]} \times \mathbf{Ch}_R^{[-1,0]} \xrightarrow{\subseteq} \mathbf{Ch}_R \times \mathbf{Ch}_R \xrightarrow{\otimes_R} \mathbf{Ch}_R \xrightarrow{\tau^{[-1,0]}} \mathbf{Ch}_R^{[-1,0]} \quad . \tag{A.8a}$$

Explicitly, given two objects $V, W \in \mathbf{Ch}_R^{[-1,0]}$, the truncated tensor product reads as

$$V \widetilde{\otimes}_R W \;=\; \left(\frac{(V^{-1}\otimes_R W^0)\oplus(V^0\otimes_R W^{-1})}{\mathrm{d}_{V\otimes_R W}(V^{-1}\otimes_R W^{-1})} \xrightarrow{\mathrm{d}_{V\otimes_R W}} V^0 \otimes_R W^0 \right) \quad . \tag{A.8b}$$

One directly checks that $\mathbf{Ch}_R^{[-1,0]}$ forms a symmetric monoidal category with respect to the truncated tensor product $\widetilde{\otimes}_R$, the monoidal unit $R \in \mathbf{Ch}_R^{[-1,0]}$ and the symmetric braiding

$$\widetilde{\gamma}_{V,W} \;:\; V \widetilde{\otimes}_R W \;\longrightarrow\; W \widetilde{\otimes}_R V \;,\quad v \otimes_R w \;\longmapsto\; w \otimes_R v \quad , \tag{A.9}$$

where all Koszul signs are trivial because there are no odd-odd degree combinations in the truncated tensor product (A.8). For later use, let us observe that the symmetric monoidal categories $(\mathbf{Ch}_R, \otimes_R, R, \gamma)$ and $(\mathbf{Ch}_R^{[-1,0]}, \widetilde{\otimes}_R, R, \widetilde{\gamma})$ are related by a lax symmetric monoidal functor.

Lemma A.3. *The truncation functor $\tau^{[-1,0]} : \mathbf{Ch}_R \to \mathbf{Ch}_R^{[-1,0]}$ is canonically lax symmetric monoidal.*

Proof. The functor $\tau^{[-1,0]}$ preserves the monoidal units strictly, i.e. $\tau^{[-1,0]}(R) = R$, and the lax structure $\tau^{[-1,0]}(V) \widetilde{\otimes}_R \tau^{[-1,0]}(W) \to \tau^{[-1,0]}\big(V \otimes_R W\big)$ for the tensor product of $V, W \in \mathbf{Ch}_R$ is given by the obvious cochain map (drawn vertically)

$$\begin{array}{ccc} \frac{(\frac{V^{-1}}{\mathrm{d}_V V^{-2}}\otimes_R Z^0(W))\oplus(Z^0(V)\otimes_R \frac{W^{-1}}{\mathrm{d}_W W^{-2}})}{\mathrm{d}_{V\otimes_R W}(V^{-1}\otimes_R W^{-1})} & \xrightarrow{\mathrm{d}_{V\otimes_R W}} & Z^0(V)\otimes_R Z^0(W) \\ \downarrow & & \downarrow \\ \frac{(V\otimes_R W)^{-1}}{\mathrm{d}_{V\otimes_R W}(V\otimes_R W)^{-2}} & \xrightarrow[\mathrm{d}_{V\otimes_R W}]{} & Z^0(V\otimes_R W) \end{array} \quad , \tag{A.10}$$

where $Z^0(U) := \ker(\mathrm{d}_U : U^0 \to U^1)$ denotes the submodule of 0-cocycles associated with a cochain complex $U \in \mathbf{Ch}_R$. □

The goal of the remaining part of this subsection is to provide details for the following result.

Proposition A.4. *There exists a* 3*-category*

$$\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \;\in\; \mathbf{3Cat} \tag{A.11}$$

in the sense of Definition A.1 whose

- *objects are* $\mathbf{Ch}_R^{[-1,0]}$*-enriched categories,*
- 1*-morphisms are* $\mathbf{Ch}_R^{[-1,0]}$*-enriched functors,*
- 2*-morphisms are* $\mathbf{Ch}_R^{[-1,0]}$*-enriched pseudo-natural transformations, and*
- 3*-morphisms are* $\mathbf{Ch}_R^{[-1,0]}$*-enriched modifications.*

This 3*-category admits a symmetric monoidal structure in the sense of Definition A.2 which agrees on objects,* 1*-morphisms and strictly natural* 2*-morphisms with the usual one from enriched category theory, see e.g. [21, Section 1.4].*

The objects and 1-morphisms in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ are defined according to the standard definitions of enriched category theory. The precise definitions of the 2-morphisms and 3-morphisms are as follows.

Definition A.5. Let $F, G : \mathbf{C} \to \mathbf{D}$ be two $\mathbf{Ch}_R^{[-1,0]}$-enriched functors between two $\mathbf{Ch}_R^{[-1,0]}$-enriched categories $\mathbf{C}$ and $\mathbf{D}$. A $\mathbf{Ch}_R^{[-1,0]}$*-enriched pseudo-natural transformation* $\zeta : F \Rightarrow G$ consists of the following data:

(i) For each object $c \in \mathbf{C}$, an element $\zeta_c \in \mathbf{D}(F(c), G(c))^0$ of degree 0 in the Hom-object $\mathbf{D}(F(c), G(c)) \in \mathbf{Ch}_R^{[-1,0]}$.

(ii) For each pair of objects $c, c' \in \mathbf{C}$, an R-linear map

$$\zeta_{c,c'} : \mathbf{C}(c,c')^0 \longrightarrow \mathbf{D}(F(c),G(c'))^{-1} \quad . \tag{A.12}$$

These data have to satisfy the following properties:

(1) For all $h \in \mathbf{C}(c,c')^0$,

$$G(h)\,\zeta_c - \zeta_{c'}\,F(h) = \mathrm{d}_{\mathbf{D}}\big(\zeta_{c,c'}(h)\big) \quad , \tag{A.13a}$$

and for all $k \in \mathbf{C}(c,c')^{-1}$,

$$G(k)\,\zeta_c - \zeta_{c'}\,F(k) = \zeta_{c,c'}\big(\mathrm{d}_{\mathbf{C}}(k)\big) \quad . \tag{A.13b}$$

(2) For all $c \in \mathbf{C}$,

$$\zeta_{c,c}(\mathrm{id}_c) = 0 \quad . \tag{A.14}$$

(3) For all $h \in \mathbf{C}(c,c')^0$ and $h' \in \mathbf{C}(c',c'')^0$,

$$\zeta_{c,c''}\big(h'\,h\big) = \zeta_{c',c''}(h')\,F(h) + G(h')\,\zeta_{c,c'}(h) \quad . \tag{A.15}$$

Remark A.6. Note that in the special case where $\zeta_{c,c'} = 0$, for all $c, c' \in \mathbf{C}$, Definition A.5 reduces to the usual definition of a $\mathbf{Ch}_R^{[-1,0]}$-enriched (strict) natural transformation. △

Definition A.7. Let $\zeta, \eta : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ be two $\mathbf{Ch}_R^{[-1,0]}$-enriched pseudo-natural transformations. A $\mathbf{Ch}_R^{[-1,0]}$-*enriched modification* $\Gamma : \zeta \Rrightarrow \eta$ consists of the following data:

(i) For each object $c \in \mathbf{C}$, an element $\Gamma_c \in \mathbf{D}(F(c),G(c))^{-1}$ of degree -1 in the Hom-object $\mathbf{D}(F(c),G(c)) \in \mathbf{Ch}_R^{[-1,0]}$.

These data have to satisfy the following properties:

(1) For all $c \in \mathbf{C}$,

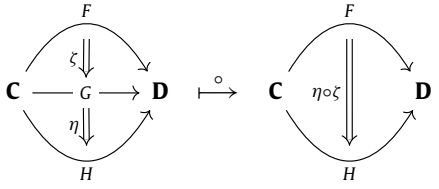
$$\zeta_c - \eta_c = \mathrm{d}_{\mathbf{D}}(\Gamma_c) \quad . \tag{A.16}$$

(2) For all $h \in \mathbf{C}(c,c')^0$,

$$\Gamma_{c'}\,F(h) + \zeta_{c,c'}(h) = \eta_{c,c'}(h) + G(h)\,\Gamma_c \quad . \tag{A.17}$$

In the following paragraphs we describe explicitly the symmetric monoidal 3-category structure of $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ in the sense of Definitions A.1 and A.2. To avoid overloading this appendix, we shall only spell out the relevant data but not the verification of their properties, which are mostly straightforward calculations.

The homomorphism 2-categories: Given two $\mathbf{Ch}_R^{[-1,0]}$-enriched categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{C},\mathbf{D}) \in \mathbf{2Cat}^{\mathrm{ps}}$ the strict 2-category whose objects are $\mathbf{Ch}_R^{[-1,0]}$-enriched functors, 1-morphisms are $\mathbf{Ch}_R^{[-1,0]}$-enriched pseudo-natural transformations and 2-morphisms are $\mathbf{Ch}_R^{[-1,0]}$-enriched modifications. The composition of two 1-morphisms



(A.18a)

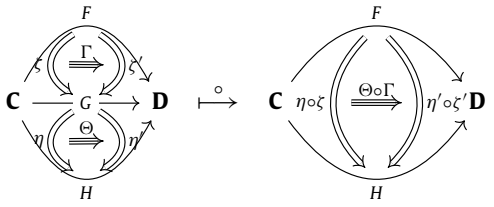
in the 2-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{C},\mathbf{D})$ is defined by

$$(\eta \circ \zeta)_c := \eta_c\,\zeta_c \quad , \tag{A.18b}$$

$$(\eta \circ \zeta)_{c,c'}(h) := \eta_{c,c'}(h)\,\zeta_c + \eta_{c'}\,\zeta_{c,c'}(h) \quad , \tag{A.18c}$$

for all $c, c' \in \mathbf{C}$ and $h \in \mathbf{C}(c,c')^0$. This composition is strictly associative and strictly unital with respect to the identity 1-morphisms $\mathrm{Id}_F : F \Rightarrow F : \mathbf{C} \to \mathbf{D}$ defined by $(\mathrm{Id}_F)_c = \mathrm{id}_{F(c)}$ and $(\mathrm{Id}_F)_{c,c'} = 0$.

The horizontal composition of two 2-morphisms



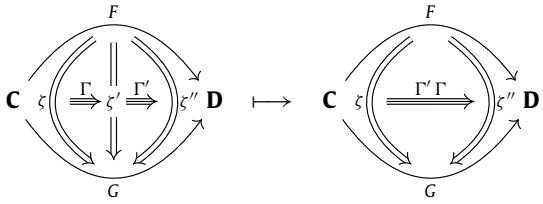
(A.19a)

in the 2-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{C},\mathbf{D})$ is defined by

$$(\Theta \circ \Gamma)_c := \Theta_c \, \zeta_c + \eta'_c \, \Gamma_c \quad , \tag{A.19b}$$

for all $c \in \mathbf{C}$. This composition is strictly associative and strictly unital with respect to the identity 2-morphisms $\mathrm{ID}_{\mathrm{Id}_F} : \mathrm{Id}_F \Rrightarrow \mathrm{Id}_F : F \Rightarrow F : \mathbf{C} \to \mathbf{D}$ defined by $(\mathrm{ID}_{\mathrm{Id}_F})_c = 0$.

The vertical composition of two 2-morphisms



(A.20a)

in the 2-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{C},\mathbf{D})$ is defined by

$$(\Gamma' \, \Gamma)_c := \Gamma'_c + \Gamma_c \quad , \tag{A.20b}$$

for all $c \in \mathbf{C}$. This composition is strictly associative and strictly unital with respect to the identity 2-morphisms $\mathrm{ID}_\zeta : \zeta \Rrightarrow \zeta : F \Rightarrow F : \mathbf{C} \to \mathbf{D}$ defined by $(\mathrm{ID}_\zeta)_c = 0$.

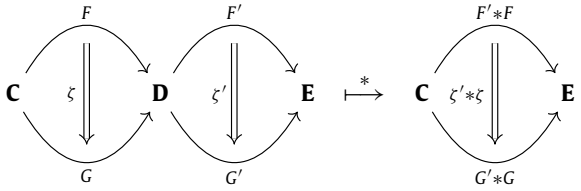
The composition pseudo-functors and identities: Given three $\mathbf{Ch}_R^{[-1,0]}$-enriched categories $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$, we denote by

$$* : \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{D},\mathbf{E}) \times \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{C},\mathbf{D}) \longrightarrow \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{C},\mathbf{E}) \tag{A.21}$$

the composition pseudo-functor between the homomorphism 2-categories from the previous paragraph. On objects in the homomorphism 2-categories, this pseudo-functor acts in terms of the usual composition of $\mathbf{Ch}_R^{[-1,0]}$-enriched functors, which we will denote by

$$\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{F'} \mathbf{E} \quad \overset{*}{\longmapsto} \quad \mathbf{C} \xrightarrow{F' * F} \mathbf{E} \quad . \tag{A.22}$$

On 1-morphisms in the homomorphism 2-categories, the pseudo-functor $*$ acts as



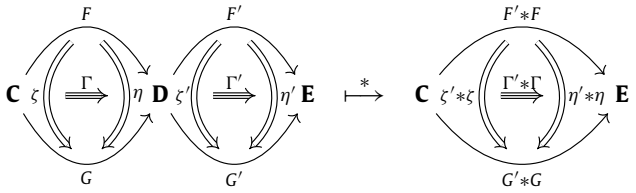
(A.23a)

with

$$(\zeta' * \zeta)_c := \zeta'_{G(c)} \, F'(\zeta_c) \quad , \tag{A.23b}$$

$$(\zeta' * \zeta)_{c,c'}(h) := \zeta'_{G(c),G(c')}\big(G(h)\big) \, F'(\zeta_c) + \zeta'_{G(c')} \, F'\big(\zeta_{c,c'}(h)\big) \quad , \tag{A.23c}$$

for all $c, c' \in \mathbf{C}$ and $h \in \mathbf{C}(c,c')^0$. On 2-morphisms in the homomorphism 2-categories, the pseudo-functor $*$ acts as



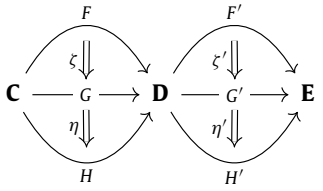
(A.24a)

with

$$(\Gamma' * \Gamma)_c := \Gamma'_{G(c)}\, F'(\zeta_c) + \eta'_{G(c)}\, F'(\Gamma_c) \quad , \tag{A.24b}$$

for all $c \in \mathbf{C}$.

The pseudo-functor coherences for the identities are trivial, i.e. $\mathrm{Id}_{F'*F} = \mathrm{Id}_{F'} * \mathrm{Id}_F$, but those for the compositions are not. Given any composable diagram of the form



(A.25a)

the corresponding pseudo-functor coherence

$$*_{(\eta',\eta),(\zeta',\zeta)} : (\eta' * \eta) \circ (\zeta' * \zeta) \Rrightarrow (\eta' \circ \zeta') * (\eta \circ \zeta) \tag{A.25b}$$

is given by the $\mathbf{Ch}_R^{[-1,0]}$-enriched modification whose components are defined by

$$\big(*_{(\eta',\eta),(\zeta',\zeta)}\big)_c := \eta'_{H(c)}\, \zeta'_{G(c),H(c)}(\eta_c)\, F'(\zeta_c) \quad , \tag{A.25c}$$

for all $c \in \mathbf{C}$.

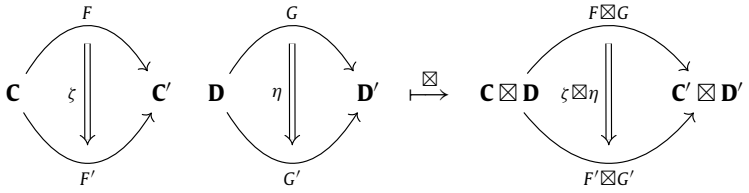
The composition pseudo-functors $*$ are strictly associative and strictly unital with respect to the identities $\mathrm{id}_{\mathbf{C}} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}(\mathbf{C},\mathbf{C})$ given by the identity $\mathbf{Ch}_R^{[-1,0]}$-enriched functors $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$.

The symmetric monoidal structure: The monoidal product

$$\boxtimes : \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \times \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \longrightarrow \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \tag{A.26}$$

is given by the $\mathbf{2Cat}^{\mathrm{ps}}$-enriched functor which assigns to a pair of objects $\mathbf{C},\mathbf{D} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ the usual $\mathbf{Ch}_R^{[-1,0]}$-enriched tensor product category $\mathbf{C}\boxtimes\mathbf{D} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ and to a pair of 1-morphisms $F : \mathbf{C} \to \mathbf{C}'$ and $G : \mathbf{D} \to \mathbf{D}'$ in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ the usual $\mathbf{Ch}_R^{[-1,0]}$-enriched tensor product functor $F \boxtimes G : \mathbf{C}\boxtimes\mathbf{D} \to \mathbf{C}'\boxtimes\mathbf{D}'$ from [21, Section 1.4].

The monoidal product of a pair of 2-morphisms



(A.27a)

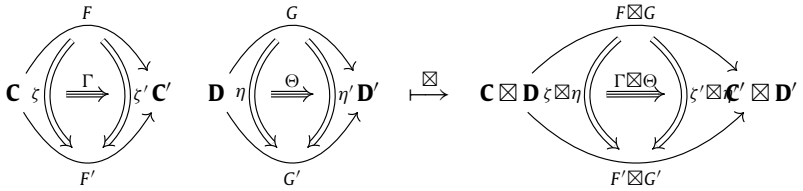
in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is defined by

$$(\zeta \boxtimes \eta)_{(c,d)} := \zeta_c \widetilde{\otimes}_R \eta_d \quad , \tag{A.27b}$$

$$(\zeta \boxtimes \eta)_{(c,d),(c',d')}\big(h \widetilde{\otimes}_R k\big) := \zeta_{c,c'}(h) \widetilde{\otimes}_R G'(k)\, \eta_d + \zeta_{c'}\, F(h) \widetilde{\otimes}_R \eta_{d,d'}(k) \quad , \tag{A.27c}$$

for all $c, c' \in \mathbf{C}$, $d, d' \in \mathbf{D}$, $h \in \mathbf{C}(c,c')^0$ and $k \in \mathbf{D}(d,d')^0$. Note that for two $\mathbf{Ch}_R^{[-1,0]}$-enriched strictly natural transformations, i.e. $\zeta_{c,c'} = 0$ and $\eta_{d,d'} = 0$ for all objects $c, c' \in \mathbf{C}$ and $d, d' \in \mathbf{D}$, the tensor product $\zeta \boxtimes \eta$ agrees with the usual one from enriched category theory.

The monoidal product of a pair of 3-morphisms



(A.28a)

in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is defined by

$$(\Gamma \boxtimes \Theta)_{(c,d)} := \Gamma_c \widetilde{\otimes}_R \eta_d + \zeta'_c \widetilde{\otimes}_R \Theta_d \quad , \tag{A.28b}$$

for all $c \in \mathbf{C}$ and $d \in \mathbf{D}$.

The monoidal unit $u : \mathcal{I} \to \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is the $\mathbf{Ch}_R^{[-1,0]}$-enriched category

$$\mathbf{R} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \tag{A.29}$$

consisting of a single object, say $\star$, with morphisms $\mathbf{R}(\star,\star) := R \in \mathbf{Ch}_R^{[-1,0]}$ given by the monoidal unit of the enrichment category.

The associator and unitors for this monoidal structure are canonically given by the ones of the enrichment symmetric monoidal category $\mathbf{Ch}_R^{[-1,0]}$. The component at $\mathbf{C}, \mathbf{D} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ of the symmetric braiding

$$\tau_{\mathbf{C},\mathbf{D}} : \mathbf{C} \boxtimes \mathbf{D} \longrightarrow \mathbf{D} \boxtimes \mathbf{C} \tag{A.30a}$$

is the $\mathbf{Ch}_R^{[-1,0]}$-enriched functor which sends objects $(c,d) \in \mathbf{C} \boxtimes \mathbf{D}$ to $(d,c) \in \mathbf{D} \boxtimes \mathbf{C}$ and acts on morphisms via the symmetric braiding of the enrichment symmetric monoidal category $\mathbf{Ch}_R^{[-1,0]}$

$$\begin{array}{ccc} (\mathbf{C} \boxtimes \mathbf{D})\big((c,d),(c',d')\big) & \dashrightarrow & (\mathbf{D} \boxtimes \mathbf{C})\big((d,c),(d',c')\big) \\ \| & & \| \\ \mathbf{C}(c,c') \widetilde{\otimes}_R \mathbf{D}(d,d') & \xrightarrow[\widetilde{\gamma}_{\mathbf{C}(c,c'),\mathbf{D}(d,d')}]{} & \mathbf{D}(d,d') \widetilde{\otimes}_R \mathbf{C}(c,c') \end{array} \quad . \tag{A.30b}$$

Note that the symmetric braiding satisfies the strict hexagon identities, i.e.

$$\tau_{\mathbf{C}\boxtimes\mathbf{D},\mathbf{E}} = (\tau_{\mathbf{C},\mathbf{E}} \boxtimes \mathrm{id}_{\mathbf{D}}) * (\mathrm{id}_{\mathbf{C}} \boxtimes \tau_{\mathbf{D},\mathbf{E}}) \quad , \tag{A.31a}$$

$$\tau_{\mathbf{C},\mathbf{D}\boxtimes\mathbf{E}} = (\mathrm{id}_{\mathbf{D}} \boxtimes \tau_{\mathbf{C},\mathbf{E}}) * (\tau_{\mathbf{C},\mathbf{D}} \boxtimes \mathrm{id}_{\mathbf{E}}) \quad , \tag{A.31b}$$

for all $\mathbf{C}, \mathbf{D}, \mathbf{E} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$. Moreover, the symmetric braiding τ is a $\mathbf{2Cat}^{\mathrm{ps}}$-enriched natural transformation, which in particular implies that it is strictly natural with respect to $\mathbf{Ch}_R^{[-1,0]}$-enriched functors $F : \mathbf{C} \to \mathbf{C}'$, i.e.

$$(\mathrm{id}_{\mathbf{D}} \boxtimes F) * \tau_{\mathbf{C},\mathbf{D}} = \tau_{\mathbf{C}',\mathbf{D}} * (F \boxtimes \mathrm{id}_{\mathbf{D}}) \quad , \tag{A.32}$$

for all $\mathbf{D} \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$.

To conclude this subsection, we provide a convenient definition of braided monoidal category objects in the symmetric monoidal 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$. Such objects can be defined with various levels of weakness since our ambient category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is a 3-category. For the purpose of our first-order deformation quantizations in Section 3, it suffices to consider the following semi-strict variant.[2]

Definition A.8. A (semi-strict) *braided monoidal category object in* $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is a tuple $(\mathbf{C}, \otimes, 1, \alpha, \lambda, \rho, \gamma)$ consisting of the following data:

(i) a $\mathbf{Ch}_R^{[-1,0]}$-enriched category $\mathbf{C}$,
(ii) a $\mathbf{Ch}_R^{[-1,0]}$-enriched functor $\otimes : \mathbf{C} \boxtimes \mathbf{C} \to \mathbf{C}$ (called monoidal product),
(iii) a $\mathbf{Ch}_R^{[-1,0]}$-enriched functor $1 : \mathbf{R} \to \mathbf{C}$ (called monoidal unit),
(iv) invertible $\mathbf{Ch}_R^{[-1,0]}$-enriched pseudo-natural transformations $\alpha : \otimes * (\otimes \boxtimes \mathrm{id}_{\mathbf{C}}) \Rightarrow \otimes * (\mathrm{id}_{\mathbf{C}} \boxtimes \otimes)$ (called associator), $\lambda : \otimes * (1 \boxtimes \mathrm{id}_{\mathbf{C}}) \Rightarrow \mathrm{id}_{\mathbf{C}}$ (called left unitor) and $\rho : \otimes * (\mathrm{id}_{\mathbf{C}} \boxtimes 1) \Rightarrow \mathrm{id}_{\mathbf{C}}$ (called right unitor),

[2] A weaker variant of braided monoidal category objects in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$, allowing for non-trivial pentagonator and hexagonator modifications, however seems to be required for the higher-order deformations explored in Subsection 3.3.

(v) an invertible $\mathbf{Ch}_R^{[-1,0]}$-enriched pseudo-natural transformation $\gamma : \otimes \Rightarrow \otimes * \tau_{\mathbf{C},\mathbf{C}}$ (called braiding).

These data have to satisfy the usual coherence conditions for braided monoidal categories internal to the 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ strictly, i.e. the pentagon, triangle and hexagon diagrams commute strictly up to equality of $\mathbf{Ch}_R^{[-1,0]}$-enriched pseudo-natural transformations.

A (semi-strict) *symmetric monoidal category object in* $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is a braided monoidal category object $(\mathbf{C}, \otimes, 1, \alpha, \lambda, \rho, \gamma)$ with the property that the braiding γ squares strictly to the identity, i.e. the $\circ$-composition of

$$\otimes \xRightarrow{\gamma} \otimes * \tau_{\mathbf{C},\mathbf{C}} \xRightarrow{\gamma * \mathrm{Id}_{\tau_{\mathbf{C},\mathbf{C}}}} \otimes * \tau_{\mathbf{C},\mathbf{C}} * \tau_{\mathbf{C},\mathbf{C}} = \otimes \tag{A.33}$$

is equal to the identity $\mathbf{Ch}_R^{[-1,0]}$-enriched pseudo-natural transformation $\mathrm{Id}_\otimes$.

A.3. Homotopy 2-categories of symmetric monoidal dg-categories

Let R be a commutative $\mathbb{K}$-algebra and denote by $\mathbf{dgCat}_R$ the 2-category of $\mathbf{Ch}_R$-enriched categories (dg-categories), $\mathbf{Ch}_R$-enriched functors (dg-functors) and $\mathbf{Ch}_R$-enriched strict natural transformations (dg-natural transformations). We shall regard $\mathbf{dgCat}_R$ as a 3-category (with trivial 3-morphisms) in the sense of Definition A.1. The lax symmetric monoidal truncation functor $\tau^{[-1,0]} : \mathbf{Ch}_R \to \mathbf{Ch}_R^{[-1,0]}$ from Lemma A.3 induces a change of base 3-functor

$$\mathsf{Ho}_2 : \mathbf{dgCat}_R \longrightarrow \mathbf{dgCat}_R^{[-1,0]} \subseteq \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}} \quad , \tag{A.34}$$

which sends dg-natural transformations to strict $\mathbf{Ch}_R^{[-1,0]}$-enriched pseudo-natural transformations. Observe that, at the level of objects, this 3-functor assigns to a dg-category $\mathbf{C} \in \mathbf{dgCat}_R$, with objects $\mathbf{C}_0$ and morphism cochain complexes $\mathbf{C}(c,c') \in \mathbf{Ch}_R$, its homotopy 2-category $\mathsf{Ho}_2(\mathbf{C}) \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$, i.e. the $\mathbf{Ch}_R^{[-1,0]}$-enriched category with the same objects $\mathbf{C}_0$ and morphisms given by the truncated cochain complexes $\tau^{[-1,0]}\big(\mathbf{C}(c,c')\big) \in \mathbf{Ch}_R^{[-1,0]}$.

Since the enrichment category $\mathbf{Ch}_R$ of cochain complexes of R-modules is symmetric monoidal with respect to the relative tensor product $\otimes_R$, there exists an induced symmetric monoidal structure on $\mathbf{dgCat}_R$, see e.g. [21, Section 1.4]. The 3-functor (A.34) assigning homotopy 2-categories is lax symmetric monoidal with the lax structure $\mathsf{Ho}_2(\mathbf{C}) \boxtimes \mathsf{Ho}_2(\mathbf{D}) \to \mathsf{Ho}_2(\mathbf{C} \boxtimes \mathbf{D})$ induced from the lax structure of the good truncation functor in Lemma A.3, for all $\mathbf{C}, \mathbf{D} \in \mathbf{dgCat}_R$. For later use, we record the following direct consequence of this result.

Corollary A.9. *Let $(\mathbf{C}, \otimes, 1, \gamma)$ be a braided (or symmetric) monoidal dg-category. Then the homotopy 2-category $\mathsf{Ho}_2(\mathbf{C}) \in \mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ is canonically a braided (or symmetric) monoidal category object in the symmetric monoidal 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ from Proposition A.4.*

Remark A.10. Note that the braided (or symmetric) monoidal category objects in $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ which are obtained from this construction always have strictly natural associators, unitors and braidings. Our symmetric monoidal 3-category $\mathbf{dgCat}_R^{[-1,0],\mathrm{ps}}$ from Proposition A.4 is designed to provide more flexibility through the concept of enriched pseudo-natural transformations, see also Definition A.8, which will be relevant for our deformation constructions in Section 3. △

Data availability

No data was used for the research described in the article.

References

[1] J.C. Baez, A.S. Crans, Higher-dimensional algebra VI: Lie 2-algebras, Theory Appl. Categ. 12 (2004) 492–538, arXiv:math/0307263 [math.QA].
[2] C. Bai, Y. Sheng, C. Zhu, Lie 2-bialgebras, Commun. Math. Phys. 320 (2013) 149–172, arXiv:1109.1344 [math-ph].
[3] D. Bashkirov, A.A. Voronov, r_∞-matrices, triangular L_∞-bialgebras and quantum ∞-groups, in: P. Kielanowski, P. Bieliavsky, A. Odzijewicz, M. Schlichenmaier, T. Voronov (Eds.), Geometric Methods in Physics, in: Trends in Mathematics, Birkhäuser, Cham, 2015, arXiv:1412.2413 [math.QA].
[4] M. Bordemann, A. Rivezzi, T. Weigel, A gentle introduction to Drinfel'd associators, Rev. Math. Phys. (2025), https://doi.org/10.1142/S0129055X24300103, in press, arXiv:2304.07012 [math.QA].
[5] D. Calaque, T. Pantev, B. Toën, M. Vaquié, G. Vezzosi, Shifted Poisson structures and deformation quantization, J. Topol. 10 (2) (2017) 483–584, arXiv:1506.03699 [math.AG].
[6] P. Cartier, Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds, C. R. Acad. Sci., Sér. 1 Math. 316 (11) (1993) 1205–1210.
[7] H. Chen, Hopf 2-algebras: Homotopy higher symmetries in physics, PhD thesis, University of Waterloo, 2024, http://hdl.handle.net/10012/20681.
[8] H. Chen, F. Girelli, Categorical quantum groups and braided monoidal 2-categories, arXiv:2304.07398 [math.QA].
[9] H. Chen, J. Liniado, Higher gauge theory and integrability, Phys. Rev. D 110 (8) (2024) 086017, arXiv:2405.18625 [hep-th].
[10] J. Chuang, R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, Ann. Math. 167 (1) (2008) 245–298, arXiv:math.RT/0407205.
[11] L.S. Cirio, J. Faria Martins, Categorifying the Knizhnik-Zamolodchikov connection, Differ. Geom. Appl. 30 (3) (2012) 238–261, arXiv:1106.0042 [hep-th].
[12] L.S. Cirio, J. Faria Martins, Infinitesimal 2-braidings and differential crossed modules, Adv. Math. 277 (2015) 426–491, arXiv:1309.4070 [math.CT].

[13] L.S. Cirio, J. Faria Martins, Categorifying the $\mathfrak{sl}(2,\mathbb{C})$ Knizhnik-Zamolodchikov connection via an infinitesimal 2-Yang-Baxter operator in the string Lie-2-algebra, Adv. Theor. Math. Phys. 21 (2017) 147–229, arXiv:1207.1132 [hep-th].
[14] L. Crane, I.B. Frenkel, Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases, J. Math. Phys. 35 (10) (1994) 5136–5154, arXiv:hep-th/9405183.
[15] V.G. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Algebra Anal. 2 (4) (1990) 149–181, Leningr. Math. J. 2 (4) (1991) 829–860.
[16] P. Etingof, O. Schiffmann, Lectures on Quantum Groups, Lectures in Mathematical Physics, International Press, Somerville, MA, 2002.
[17] A. Guan, A. Lazarev, Y. Sheng, R. Tang, Review of deformation theory II: a homotopical approach, Adv. Math. (China) 49 (3) (2020) 278–298, arXiv:1912.04028 [math.AT].
[18] N. Johnson, D. Yau, 2-Dimensional Categories, Oxford University Press, Oxford, 2021, arXiv:2002.06055 [math.CT].
[19] E. Karlsson, C. Keller, L. Müller, J. Pulmann, Deformation quantization via categorical factorization homology, arXiv:2410.12516 [math.QA].
[20] C. Kassel, Quantum Groups, Grad. Texts in Math., vol. 155, Springer-Verlag, New York, 1995.
[21] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Math. Soc. Lecture Note Ser., vol. 64, Cambridge University Press, Cambridge-New York, 1982, Reprints in Theory Appl. Categ. 10 (2005) 1–136.
[22] M. Khovanov, A.D. Lauda, A diagrammatic approach to categorification of quantum groups. I, Represent. Theory 13 (2009) 309–347, arXiv:0803.4121 [math.QA].
[23] A. Kraft, J. Schnitzer, An introduction to L_∞-algebras and their homotopy theory for the working mathematician, Rev. Math. Phys. 36 (01) (2024) 2330006, arXiv:2207.01861 [math.QA].
[24] J. Lurie, Derived algebraic geometry X: formal moduli problems, http://www.math.harvard.edu/~lurie/papers/DAG-X.pdf.
[25] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[26] S. Majid, Strict quantum 2-groups, arXiv:1208.6265 [math.QA].
[27] T. Pantev, B. Toën, M. Vaquié, G. Vezzosi, Shifted symplectic structures, Publ. Math. Inst. Hautes Études Sci. 117 (2013) 271–328, arXiv:1111.3209 [math.AG].
[28] J.P. Pridham, Shifted Poisson and symplectic structures on derived N-stacks, J. Topol. 10 (1) (2017) 178–210, arXiv:1504.01940 [math.AG].
[29] J.P. Pridham, An outline of shifted Poisson structures and deformation quantisation in derived differential geometry, arXiv:1804.07622 [math.DG].
[30] J.P. Pridham, A note on étale atlases for Artin stacks and Lie groupoids, Poisson structures and quantisation, J. Geom. Phys. 203 (2024) 105266, arXiv:1905.09255 [math.AG].
[31] E. Riehl, Categorical Homotopy Theory, New Math. Monogr., vol. 24, Cambridge University Press, Cambridge, 2014.
[32] S. Rivière, F. Wagemann, On the string Lie algebra, Algebr. Represent. Theory 18 (4) (2015) 1071–1099, arXiv:1407.6806 [math.QA].
[33] R. Rouquier, 2-Kac-Moody algebras, arXiv:0812.5023 [math.RT].
[34] P. Safronov, Poisson-Lie structures as shifted Poisson structures, Adv. Math. 381 (2021) 107633, arXiv:1706.02623 [math.AG].
[35] A. Schenkel, B. Vicedo, 5d 2-Chern-Simons theory and 3d integrable field theories, Commun. Math. Phys. 405 (12) (2024) 293, arXiv:2405.08083 [hep-th].