Terracini loci of curves

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Abstract

We study subsets S of curves X whose double structure does not impose independent conditions to a linear series L, but there are divisors $D \in |L|$ singular at all points of S. These subsets form the Terracini loci of X. We investigate Terracini loci, with a special look towards their non-emptiness, mainly in the case of canonical curves, and in the case of space curves.

1 Introduction

Terracini loci $\mathbb{T}(X, L, x)$ of a projective variety X (over an algebraically closed field of characteristic 0) are subsets of the set $\mathcal{S}(X, x)$ of all reduced finite subsets $S \subset X_{\mathrm{reg}}$ of cardinality x, with the property that the double scheme $2S$ on S does not impose independent conditions to the linear series L, and there are divisors in $|L|$ passing through $2S$. Terracini loci are certainly involved in the study of interpolation properties of the image of X in the map induced by L, but they also assume great importance in the theory of secant varieties to embedded varieties X, for they are connected with points of the abstract secant variety of X in which the differential of the map to the embedded secant variety drops rank. More details on the initial properties of Terracini loci can be found in [1].

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In this paper, we focus the attention to Terracini loci of curves. Even if curves are never defective, so that their secant varieties always have the expected dimension, yet there are special points p in which the Terracini Lemma fails to provide the dimension of the Zariski tangent space to the secant variety. This happens typically when the set of points in X which generates p belongs to some Terracini locus.

We are mainly concerned with the problem of the non-emptiness of a Terracini locus $\mathbb{T}(X, L, x)$. Certainly $\mathbb{T}(X, L, x)$ is empty when X is a rational normal curve in $\mathbb{P}^r$, L is the complete hyperplane linear series and $x \leq (r + 1)/2$, because every subscheme of length $\leq r + 1$ in X is linearly independent. We will see that, for odd r, the emptiness of $\mathbb{T}(X, L, (r + 1)/2)$ characterizes rational normal curves (Proposition 7). This means that there is a large mass of examples of non-empty Terracini loci of curves.

Our analysis considers mainly two cases: canonically embedded curves (Sect. 4) and curves in $\mathbb{P}^3$ (Sect. 5).

In the former case, the series L is the complete canonical linear series K_X, and one can easily expect that the existence of sets in $\mathbb{T}(X, K_X, x)$ strongly depends on the geometry of linear series on the curve. We show indeed that subsets of theta-characteristics on X prove that $\mathbb{T}(X, K_X, x)$ is non-empty when $g/2 \leq x \leq g - 1$. For $x < g/2$ we see (Proposition 4) that the non-emptiness of $\mathbb{T}(X, K_X, x)$ is linked to the gonality of X.

For space curves, we use induction on the degree and genus, with the technique of smoothing reducible curves, to prove that $\mathbb{T}(X, L, x)$ is non-empty for some smooth curves of degree d and genus $g \leq d - 3$, when $6 \leq 2x \leq d$ (Theorem 3). We also prove the existence of smooth special curves with non-empty Terracini loci in some components of the Hilbert scheme of space curves even outside the Brill-Noether range (Theorem 4 and Remark 7).

We would like to thank the anonymous referees for precious observations on the preliminary versions of the paper.

2 Preliminaries

We work over an algebraically closed field of characteristic 0.

2.1 Notation

For any 0-dimensional subscheme S we denote with $\ell(S)$ the length of S. When S is reduced, then $\ell(S)$ is the cardinality of S.

Let X be an integral projective variety. For any point $P \in X_{\text{reg}}$, with homogeneous maximal ideal $m_{P,X}$, we denote with $(2P, X)$ the subscheme of X defined by $m_{P,X}^2$. If $S = \{P_1, \ldots, P_x\} \subset X$ is a finite set of points, then we denote with $(2S, X)$ the non-reduced scheme $2S = \bigcup_{i=1}^{x} (2P_i, X)$.

When it is clear which X we refer to, then we will write simply $2S$ instead of $(2S, X)$.

For any scheme X and for any integer x we denote with $\mathcal{S}(X, x)$ the set of all reduced finite subsets $S \subset X_{\mathrm{reg}}$ of length x. $\mathcal{S}(X, x)$ is an open subset of the symmetric product $X^{(x)}$.

Let X be a projective curve, and let S be a finite subset of X_{reg}. When S identifies a Cartier divisor on X, by abuse, we will continue to denote it with S. We will also denote with $|S|$ the complete linear series associated to S, and we will denote with $h^0(S)$ the dimension of the space of sections of the associated line bundle. We will use the same convention for $2S$.

For any scheme $Z \subset \mathbb{P}^r$ let $\langle Z \rangle \subseteq \mathbb{P}^r$ denote the minimal linear subspace of $\mathbb{P}^r$ containing Z.

2.2 Terracini loci

We recall from [1] the definition of Terracini locus of a projective variety.

Definition 1 Let X be an integral projective variety, L a line bundle on X and $V \subseteq H^0(L)$ a linear subspace. Set $m := \dim X$. Fix $S \in \mathcal{S}(X, x)$. We say that S is in the Terracini locus $\mathbb{T}(X, L, V, x)$ if $V(-(2S, X)) \neq (0)$ and $\dim V(-(2S, X)) > \dim V - x(m + 1)$.

We say that the integer $\delta(S, X, L, V) := \dim V(-(2S, X)) - \dim V + x(m + 1)$ is the (Terracini) defect of S with respect to (L, V).

When $V = H^0(L)$, we will drop V in the notation.

We will consider, throughout the paper, mainly the case where X is a curve, i.e. $m = 1$. In this situation a finite set S of length x lies in the Terracini locus when $\dim V(-(2S, X)) > \max\{\dim V - 2x, 0\}$.

We can extend the definition of Terracini loci to include some non-reduced 0-dimensional subschemes.

Definition 2 Let $C \subset \mathbb{P}^r$ be a smooth and connected non-degenerate curve. For each positive integer x let $C^{(x)}$ denote the symmetric product of x copies of C. The variety $C^{(x)}$ is a connected projective variety of dimension x parametrizing the degree x zero-dimensional schemes of X, and there is a non-empty Zariski open subset $S(C, x) \subset C^{(x)}$ which parametrizes subsets of cardinality x. For each positive integer x let $\tilde{\mathbb{T}}(C, x)$ denote the set of all $Z \in C^{(x)}$ such that $\dim\langle 2Z \rangle \leq 2x - 2$ and $\langle 2Z \rangle \neq \emptyset$. Obviously, $\mathbb{T}(C, 1) = \tilde{\mathbb{T}}(C, 1) = \emptyset$. The inclusion $S(C, x) \subset C^{(x)}$ induces an inclusion $\mathbb{T}(C, x) \subseteq \tilde{\mathbb{T}}(C, x)$. The semicontinuity theorem for cohomology gives that $\tilde{\mathbb{T}}(C, x)$ contains the closure of $\mathbb{T}(C, x)$ in $C^{(x)}$. Sometimes, $\tilde{\mathbb{T}}(C, x)$ is not equal to the closure of $\mathbb{T}(C, x)$ in $C^{(x)}$ (Example 1).

Example 1 Fix an even integer $d \geq 4$, a line $D \subset \mathbb{P}^2$ and $p \in D$. There is a smooth degree d curve $C \subset \mathbb{P}^2$ such that $p \in C$ and $D \cap C = dp$, i.e. p is a total ramification point of C with D as its tangent line. Since C has only finitely many multitangent lines, $\mathbb{T}(C, x)$ is finite and hence it is closed in $\tilde{\mathbb{T}}(C, x)$. Since d is even and $d \geq 4$, $\frac{d}{2} p$ lies in $\tilde{\mathbb{T}}(C, d/2) \backslash \mathbb{T}(C, d/2)$.

3 Generalities

Example 2 Let $X \subset \mathbb{P}^N$ be an integral curve. Here we take $L = \mathcal{O}_X(1)$ and $V =$ the linear series of hyperplanes. For a general $S = \{P_1, \ldots, P_x\} \subset X_{\text{reg}}$ write $2S$ for $(2S, X)$. The elements of $V(-2S)$ correspond to hyperplanes containing the tangent lines to X at the points P_i's. It follows that S lies in the Terracini locus when the span of the x tangent lines at the points of X has dimension smaller than the expected one. Then by definition $\mathbb{T}(X, L, V, 1)$ is always empty.

On the other hand, when the map induced by the linear series V is not birational onto the image the Terracini locus $\mathbb{T}(X, L, V, 1)$ may contain some points.

Example 3 Let X be a smooth hyperelliptic curve of genus $g \geq 2$. We can describe all the Terracini loci with respect to $L = K_X$ and $V = H^0(K_X)$.

Let $B \subset X$ be the set of the Weierstrass points of X. For any integer x let $\mathcal{S}(B, x)$ denote the set of all subsets of B with cardinality x. The set B is the ramification locus of the g_2^1 of X. Since we work in characteristic 0, $\ell(B) = 2g + 2$. Fix a positive integer x and take $S \in S(X, x)$. If $x \geq g$, then $\deg(K_X(-(2S, X))) < 0$ and hence $h^0(K_X(-(2S, X))) = 0$. Thus $\mathbb{T}(X, K_X, x) = \emptyset$ for all $x \geq g$.

Assume that $1 \leq x \leq g - 1$. Let $h : X \to \mathbb{P}^1$ be the morphism associated to the g_2^1 of X. The linear system $|K_X|$ is the minimal sum of $g - 1$ copies of the g_2^1 of X. Hence every base-point free special line bundle on X is the sum of at most $g - 1$ copies of the g_2^1. Thus $\mathbb{T}(X, K_X, 1) = B$ and $\mathbb{T}(X, K_X, g - 1)) = \mathcal{S}(B, g - 1)$. More generally, S belongs to $\mathbb{T}(X, K_X, x)$ if and only if we have that either $S \cap B \neq \emptyset$ or there are $p, q \in S$ such that $p \neq q$ and $h(p) = h(q)$ for these conditions are equivalent to $h^0(K_X(-(2S, X))) > g - 2x$.

It is easy to realize that, for each $x \in \{2, \ldots, g - 2\}$, the elements of $\mathbb{T}(X, K_X, x)$ with (maximal) defect x are the elements of $\mathcal{S}(B, x)$. If $e := \#(S \cap B)$ and S has f distinct sets $\{p_i, q_i\}$, $1 \leq i \leq f$, with $p_i \neq q_i$ and $h(p_i) = h(q_i)$, then $h^0(K_X(-2S)) = g - 2x + e + f$.

For the rest of the section let us go back to the case in which $X \subset \mathbb{P}^N$ is an integral curve, and we take $L = \mathcal{O}_X(1)$ and $V =$ the linear series of hyperplanes.

Example 4 Let us consider what happens when $x = 2$ and X is a plane curve. Thus $\max\{\dim V - 2x, 0\} = 0$. Then $S = \{P, Q\} \in \mathcal{S}(2)$ belongs to the Terracini locus if and only if the tangent lines to X in P and Q coincide. Since in characteristic 0 not every tangent line is bitangent, then $\mathbb{T}(X, L, V, 2)$ is either empty or finite.

If $N > 2$, the tangent lines to two general points of X span a 3-dimensional linear subspace (recall that we work in characteristic 0). The set $S = \{P, Q\}$ lies in the Terracini locus $\mathbb{T}(X, L, V, 2)$ when the tangent lines in P, Q meet at some point P_0, i.e. there exists a plane containing the two tangent lines. In this case, the projection of X from P_0 is a curve with (at least) two cusps.

4 Canonically embedded curves

Let us turn now to the case where X is a smooth curve of genus g and we consider the complete canonical linear series $L = K_X$. We will describe in several cases the locus $\mathbb{T}(X, K_X, x)$.

Since we already treated the case of hyperelliptic curves in Example 3, we assume that $g \geq 3$ and K_X embeds X in $\mathbb{P}^{g-1}$.

We start with a very easy observation, which shows that we need to distinguish two cases, depending if x is smaller than $g/2$ or not.

Proposition 1 *A reduced set S of length x lies in $\mathbb{T}(X, K_X, x)$ if and only if either $2x < g$ and $h^0(2S) > 1$, i.e. the linear series $|2S|$ is not a singleton, or $2x \geq g$ and $h^0(K_X - 2S) > 0$, i.e. $2S$ is special.*

Proof By definition S lies in $\mathbb{T}(X, K_X, x)$ if and only if $h^0(K_X - 2S) > \min\{0, h^0(K_X) - 2x\}$. Since $h^0(K_X) = g$, we distinguish between $2x < g$ and $2x \geq g$. In the latter case S lies in $\mathbb{T}(X, K_X, x)$ if and only if $h^0(K_X - 2S) > 0$. In the former case, since by Riemann–Roch $h^0(2S) = 2x - g + 1 + h^0(K_X - 2S)$, S lies in $\mathbb{T}(X, K_X, x)$ if and only if $h^0(2S) > 1$. □

It follows from the previous proposition that the Terracini locus $\mathbb{T}(X, K_X, x)$ is empty if $x \geq g$, because in this case the degree of $K_X - 2S$ is negative.

Let us consider the extremal case $x = g - 1$.

Example 5 A set S (resp. scheme) of length $g - 1$ belongs to $\mathbb{T}(X, K_X, g - 1)$ (resp. $\tilde{\mathbb{T}}(X, K_X, g - 1)$) if and only if $h^0(K_X - 2S) > 0$ which, for degree reasons, implies that $2S$ is a canonical divisor. Thus subsets $S \in \mathbb{T}(X, K_X, g-1)$ correspond to divisors in some non empty linear series G such that $2G = K_X$, i.e. a *theta-characteristic* of X.

It is well known ([5]) that X has a finite number, exactly 2^{2g}, theta-characteristics. A theta-characteristic is odd or even, depending on the parity of $h^0(G)$. The number of odd theta-characteristics is $2^{g-1}(2^g - 1)$ while there are $2^{g-1}(2^g + 1)$ even theta-characteristics.

Now assume that X is general in the moduli space $\mathcal{M}_g$. In this case, by [5] Corollary 1.11, $h^0(G) \leq 1$ for every theta-characteristic G on X, and for each odd theta-characteristic G on X the divisor D with $\{D\} = |G|$ is reduced. Thus for $X \in \mathcal{M}_g$ general the Terracini locus $\mathbb{T}(X, K_X, g - 1)$ is finite, of cardinality $2^{g-1}(2^g - 1)$.

There are $X \in \mathcal{M}_g$ with theta-characteristics G such that $h^0(G) \geq 2$. For such curves $\mathbb{T}(X, K_X, g - 1)$ is infinite.

On the contrary, a natural question is to ask if there are $X \in \mathcal{M}_g$ such that $\mathbb{T}(X, K_X, g - 1) = \emptyset$, i.e. no reduced divisor is the zero-locus of an effective theta-characteristic.

In the case $g = 3$ this is equivalent to ask if there is a smooth degree 4 plane curve X with 28 flexes of higher order, i.e. 28 lines $L \subset \mathbb{P}^2$ meeting X at a unique point. The total weight of all flexes of a smooth plane quartic is 24, because its Hessian determinant has degree 6. Thus there is no such X for $g = 3$. For any X we have $\#\tilde{\mathbb{T}}(X, K_X, g - 1) \geq 2^{g-1}(2^g - 1)$ and either $\#\tilde{\mathbb{T}}(X, K_X, g - 1) = 2^{g-1}(2^g - 1)$ (case $h^0(G) \leq 1$ for all theta-characteristic G of X) or $\dim \tilde{\mathbb{T}}(X, K_X, g - 1) > 0$.

In any case, taking subsets of a theta-characteristic, we obtain immediately the following

Proposition 2 *Fix an integer x such that $g/2 \leq x \leq g-1$. Then the Terracini locus $\tilde{\mathbb{T}}(X, K_X, x)$ is non empty.*

Of course when X has a positive dimensional theta-characteristic, then also $\mathbb{T}(X, K_X, x)$ is infinite for all x between $g/2$ and $g-1$.

Remark 1 Fix an integer x such that $g/2 \leq x \leq g-2$. One can try to extend elements of $\mathbb{T}(X, K_X, x)$ to elements $\mathbb{T}(X, K_X, x+1)$, with the addition of suitable points. Respect to this, we can observe:

(a) Assume that there exists $S \in \mathbb{T}(X, K_X, x)$ with $\dim\langle 2S\rangle = g-2$, i.e. $H :=$ $\langle 2S \rangle$ is a hyperplane. The point $p \in X\backslash S$ has the property that $S \cup \{p\}$ lies in $\mathbb{T}(X, K_X, x+1)$ if and only if $p \in H$ and H is tangent to X at p. There is $Z \in \tilde{\mathbb{T}}(K_X, x+1)$ containing S if and only if either H is tangent to X at a point of $X\backslash S$ or there is $o \in S$ such that the scheme $H \cap X$ contains o with multiplicity at least 4.
(b) Assume that there exists $S \in \mathbb{T}(X, K_X, x)$ with $\dim\langle 2S\rangle = g-3$. The differential of the rational map $\phi : X\backslash X \cap \langle 2S\rangle \to \mathbb{P}^1$ induced by the linear projection from $\langle 2S\rangle$ shows that there are only finitely many $p \in X\backslash S$ such that $S \cup \{p\} \in \mathbb{T}(X, K_X, x+1)$. There are at least 2 such points p, because ϕ extends to a morphism $\psi : X \to \mathbb{P}^1$ by the smoothness of X and X has at least 2 ramification points, because $g > 0$.
(c) Assume that there exists $S \in \mathbb{T}(X, K_X, x)$ with $\dim\langle 2S\rangle = g-4-2k-\epsilon$, with k non-negative integer and $\epsilon \in \{0, 1\}$. Then $S \cup \{p_1, \ldots, p_{k+1}\}$ lies in $\mathbb{T}(X, K_X, x+k+1)$ for a general $(p_1, \ldots, p_{k+1}) \in X^{k+1}$.

Let us now consider the case $x < g/2$.

Proposition 3 *Assume $x < (g-1)/2$, and assume that $\mathbb{T}(X, K_X, x)$ is non empty. Then*

$$\dim(\mathbb{T}(X, K_X, x+1)) \geq 1 + \dim(\mathbb{T}(X, K_X, x)).$$

Proof Pick $S \in \mathbb{T}(X, K_X, x)$ and p general in X. Since $2x < g$ then by Proposition 1 we have $h^0(2S) \geq 2$. It follows $h^0(2(S \cup \{p\})) \geq h^0(2S) \geq 2$.

When $x+1 < g/2$, the last inequality is sufficient to conclude that $S \cup \{p\}$ sits in $T(X, K_X, x+1)$, by Proposition 1 again, thus the inequality on the dimensions holds.

Assume $x = (g-2)/2$. Since $h^0(2(S \cup \{p\})) \geq 2$, then by Riemann–Roch

$$h^1(2(S \cup \{p\})) \geq g - 1 - 2\left(\frac{g-2}{2} + 1\right) + 2 = 1.$$

Thus $2(S \cup \{p\})$ is special, hence it belongs to $T(X, K_X, x+1)$, by Proposition 1, and we conclude as before. $\square$

Corollary 1 *Fix the minimal integer x such that the Terracini locus $\mathbb{T}(X, K_X, x)$ is non empty. Then for all y with $x \leq y \leq g - 1$ the Terracini locus $\mathbb{T}(X, K_X, y)$ is non-empty.*

We saw in the previous section that $\mathbb{T}(X, K_X, 1)$ is non-empty if and only if X is hyperelliptic. The last 2 propositions of this section link the gonality k of X with the minimal x such that $\mathbb{T}(X, K_X, x) \neq \emptyset$.

Proposition 4 *If $\mathbb{T}(X, K_X, x) \neq \emptyset$, then X has a linear series g_{2x}^1. For the converse, if X has a linear series g_k^1 with $k < g/2$, then $\dim(\mathbb{T}(X, K_X, k)) \geq 1$.*

Proof Recall that we are assuming $g \geq 3$ and X not hyperelliptic. The first assertion is trivial if $x \geq g/2$, for every X satisfies $\mathbb{T}(X, K_X, x) \neq \emptyset$ and every X has a linear series g_{2x}^1. When $x < g/2$, the first assertion follows immediately by Proposition 1.

For the second assertion, consider the degree k map $f : X \to \mathbb{P}^1$ associated to the linear series g_k^1. For any $p \in \mathbb{P}^1$ let $D_p := f^{-1}(p)$ denote the associated degree k divisor. Note that $h^0(2D_p) \geq h^0(\mathcal{O}_{\mathbb{P}^1}(2p)) = 3$. Then, as in Proposition 1, a general D_p belongs to $\tilde{\mathbb{T}}(X, K_X, k)$. Since we work in characteristic 0, then a general D_p is formed by k distinct points, hence it belongs to $\mathbb{T}(X, K_X, k)$. □

Before we can refine the previous proposition, let us see what happens for trigonal curves.

Example 6 Let X be a smooth trigonal curve of genus $g \geq 4$, canonically embedded in $\mathbb{P}^{g-1}$, and let $f : X \to \mathbb{P}^1$ be the degree 3 morphism associated to the g_3^1. By the Castelnuovo's inequality [8], f is unique if $g \geq 5$. Let $\Sigma \subset X$ denote the set of all ramification points of the map and let $\Sigma' \subseteq \Sigma$ denote the set of all $p \in \Sigma$ which belong to fibers of cardinality 2. The points $p \in \Sigma \setminus \Sigma'$ are called the *total ramification points* of f, because the fiber containing p is supported at p.

$\Sigma' = \Sigma$ if X is a general trigonal curve of genus g, but there are trigonal curves in which the equality fails, and also trigonal curves with $\Sigma' = \emptyset$, e.g. the degree 3 Galois coverings of $\mathbb{P}^1$.

Take $p \in \Sigma'$ and consider the point $q_p \neq p$ in the fiber of f through p. We claim that $S = \{p, q_p\}$ belongs to $\mathbb{T}(X, K_X, 2)$, so that $\mathbb{T}(X, K_X, 2)$ is non-empty.

To prove the claim, consider that $h^0(2p + q_p) = 2$. Then $h^0(2p + 2q_p) = h^0(2S) \geq 2$, which proves the claim when $g > 4$, by Proposition 1. For $g = 4$, since $h^0(2S) \geq 2$, the divisor $2S$ is special by Riemann–Roch, and the claim follows again by Proposition 1.

Definition 3 Given a linear series G on X which is a g_d^1, G is *tamely ramified* if there is a non-reduced divisor D in G in which all the points appear with coefficient ≤ 2.

Proposition 5 *Let X be a smooth curve of genus $g \geq 4$, canonically embedded in $\mathbb{P}^{g-1}$. Assume that X has a base point free pencil $R \in \mathrm{Pic}^k(X)$ such that $h^1(R^{\otimes 2}) > 0$, and let $f : X \to \mathbb{P}^1$ be the degree k morphism induced by $|R|$. Then $\tilde{\mathbb{T}}(X, K_X, k-1) \neq \emptyset$. Assume that the f is tamely ramified. Then $\mathbb{T}(X, K_X, k-1) \neq \emptyset$.*

Proof If $k-1 \geq g/2$ the claim follows from Proposition 2. Assume $k-1 < g/2$. Since $g > 0$ and $\mathbb{P}^1$ is algebraically simply connected, there is a divisor $A = m_1 q_1 + \cdots + m_s q_s \in |R|$ with $m_1 \geq 2$. Since $2A$ is a special divisor, $A \in \tilde{\mathbb{T}}(X, K_X, k-1)$. Now assume that f is tamely ramified. Thus all $A \in |R|$ have profiles with multiplicities 1 or 2. Since $g > 0$, $A = 2p_1 + \cdots + 2p_i + p_{i+1} + \cdots + p_j \in |R|$ with $p_1, \ldots, p_j$ distinct points and $i \geq 1$. Note that $i + j = k$, so that $j < k$. Then $S = \{p_1, \ldots, p_j\}$ sits in $\mathbb{T}(X, K_X, j)$. Namely $j \leq k-1 < g/2$ and $h^0(R) \geq 2$ implies $h^0(2S) \geq 2$. Then $\mathbb{T}(X, K_X, j) \neq \emptyset$, and the claim follows from Corollary 1. $\qquad\square$

5 Embedded curves

In this section we consider reduced and locally complete intersection curves X embedded in a projective space $X \subset \mathbb{P}^r$.

Notation 1 We denote with N_X the normal bundle of X.

For brevity, we will denote with $\mathbb{T}(X, x)$ the Terracini locus of X with respect to the (non-necessarily complete) linear series of hyperplane divisors.

For all integers $g \geq 0$, $r \geq 3$ and $d \geq r + g$, we will denote with $H(d, g, r)$ the set of all smooth and non-degenerate curves $X \subset \mathbb{P}^r$ of degree d and genus g such that $h^1(\mathcal{O}_X(1)) = 0$.

Fix $X \in H(d, g, r)$. Since X is smooth, N_X is a quotient of $O_X(1)^{\oplus r}$ by the Euler's sequence. Thus $h^1(N_X) = 0$. Therefore $H(d, g, r)$ is smooth, and

$$\dim H(d, g, r) = (r+1)d + (r-3)(1-g).$$

Since all non-special line bundles of degree d on a curve of genus g have the same number of sections, and all smooth genus g curves are parametrized by an irreducible variety, $H(d, g, r)$ is irreducible. See [4].

An example of J. Harris, generalized by Ein in [4], shows that both claims on the irreducibility and the dimension fail, for large r, if we drop the assumption on the vanishing of $h^1(\mathcal{O}_X(1))$. On the other hand, we will work mainly with curves in $\mathbb{P}^3$, and in this case we can drop the assumption by [3].

It is immediate, by Bezout formula, that $\mathbb{T}(X, x)$ is empty if $x > d/2$. So we analyze the case $2x \leq d$.

Remark 2 Fix x with $2x < r$ and let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible curve such that $\mathbb{T}(X, x) \neq \emptyset$. Then also $\mathbb{T}(X, x+1) \neq \emptyset$.

Indeed if $S \in \mathbb{T}(X, x)$, then the scheme $(2S, X)$ lies in a linear system of hyperplanes of dimension at least $r - 2x + 1 \geq 2$. Thus, for $p \in X$ general, the scheme $(2(S \cup \{p\}), X)$ lies in a linear system of hyperplanes of dimension at least $r - 2x - 1 \geq 0$.

Notation 2 Take $X \subset \mathbb{P}^r$. An *arrow* in $\mathbb{P}^r$ is a non-reduced scheme of length 2. The set of all arrows in $\mathbb{P}^r$ supported at p is closed in the Hilbert scheme, and it has dimension $r - 1$. Thus the set of all arrows in $\mathbb{P}^r$ has dimension $2r - 1$.

Note that an arrow w with support p determines a line r_w through p. If $p \in X$ and X is smooth at p, then r_w is tangent to X exactly when X contains the arrow w.

In the study of Terracini loci (even not on curves) the paper [2] is very useful and we explain it in the following remark used in the proof of Proposition 7.

Remark 3 Fix an integral projective curve X in $\mathbb{P}^r$, and let W be the image of the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_X(1))$. For any zero-dimensional scheme $Z \subset X$ set $W(-Z) := H^0(\mathcal{I}_Z(1)) \cap W$. Fix integers $s > 0$ and $e_i > 0$, $i = 1, \ldots, s$. Let Z be a general subscheme of X_{reg} with s connected components of length $e_1, \ldots, e_s$. Then we claim that the main result of [2] and its proof imply that:

$$\dim W(-Z) = \max\{0, \dim W - e_1 - \cdots - e_s\}.$$

Namely the proof there yields that a general scheme in X (hence obviously curvilinear) which sits in no hypersurfaces of degree d imposes independent conditions to the linear system of hypersurfaces of degree d. Notice that the scheme Z is a Cartier divisor of X, and the integer $\dim W(-Z)$ is the codimension of the linear space $\langle Z \rangle$ in $\mathbb{P}^r$.

For rational curves we can easily show the following.

Proposition 6 *Fix integers $d \geq r \geq 3$ and x with $r \leq 2x \leq d$. Then there is a smooth and non-degenerate rational curve $X \subset \mathbb{P}^r$ such that $\mathbb{T}(X, x) \neq \emptyset$.*

Proof Fix a hyperplane $H \subset \mathbb{P}^r$. Let $Z \subset H$ be a general union of x arrows and let Z' be a general set of $d - 2x$ points. By Perrin [9, Theorem 1.6] there is a smooth and non-degenerate rational curve $X \subset \mathbb{P}^r$ such that $(Z \cup Z') = X \cap H$. Let $S \subset H$ be the reduction of Z. Since X is smooth and $2x \geq r$, then $S \in \mathbb{T}(X, x)$. □

Indeed, we have a characterization of rational normal curves in terms of Terracini loci.

Proposition 7 *Let $r \geq 3$ be an odd integer and $X \subset \mathbb{P}^r$ a smooth, connected and non-degenerate curve. Then X is a rational normal curve if and only if $\mathbb{T}(X, (r+1)/2)) = \emptyset$.*

Proof Set $x := (r+1)/2$.

The "if" part is true, because if X is a rational normal curve each zero-dimensional scheme $Z \subset X$ of degree $r + 1$ is linearly independent.

Now assume $\mathbb{T}(X, x) = \emptyset$. Set $d := \deg(X)$ and fix a general $S \subset X$ of cardinality $x - 1$, say $S = \{p_1, \ldots, p_{x-1}\}$. Let V be the linear span of the double scheme $(2S, X)$. For all positive integers $a_1, \ldots, a_{x-1}$ set $V(a_1, \ldots, a_{x-1}) := \langle a_1 p_1 + \cdots + a_{x-1} p_{x-1} \rangle$. Note that $V = V(2, \ldots, 2)$. Since S is general in X, Remark 3 applied to the curve X gives

$$\dim V(a_1, \ldots, a_{x-1}) = \min \left\{ r, x - 2 + \sum_{i=1}^{x-1} (a_i - 1) \right\}.$$

Thus $\dim V = r - 2$. Since $\dim V = r - 2$ and $\dim V(a_1, \ldots, a_{x-1}) = r - 1$ if there is $j \in \{1, \ldots, x - 1\}$ such that $a_j = 3$ and $a_i = 2$ for all $i \neq j$, the scheme $V \cap X$ contains each p_i with multiplicity 2. First assume $(V \cap X)_{\mathrm{red}} \neq S$ and take $o \in (V \cap X)_{\mathrm{red}} \setminus S$. Since $\dim \langle 2o \cup V \rangle \leq \dim V + 1$, $S \cup \{o\} \in \mathbb{T}(X, x)$, a contradiction. Thus $(V \cap X)_{\mathrm{red}} = S$. Since we proved that each p_i appear with multiplicy 2 in the scheme-theoretic intersection $V \cap X$, we have $V \cap X = (2S, X)$, which has degree $2x - 2 = r - 1$. Let u denote the linear projection from V to $\mathbb{P}^1$. Since X is smooth, $u_{|X \setminus S}$ extends to a morphism $u' : X \to \mathbb{P}^1$, and the degree of u' is $d - r + 1$. The assumption $\mathbb{T}(X, x) = \emptyset$ implies that u' has no ramification point, except possibly at the points of S. Fix $a \in S$, say $a = p_1$. The point p_1 is a ramification point of u' only if $V(4, 2, \ldots, 2)$ is a hyperplane. This is false, because $\dim V(4, 2, \ldots, 2) = r$. Hence u' has no ramification points. This is possible only if $d = r$, hence X is a rational normal curve. $\qquad \square$

Since in the rest of the paper we will often argue by induction, taking the smoothing of nodal, reducible curves, we need some preliminary results on normal bundles of reducible curves.

Remark 4 Let $X \subset \mathbb{P}^r$ be a reduced curve with only locally complete intersection singularities. N_X is a vector bundle of rank $(r - 1)$ on X and

$$\deg(N_X) = (r + 1) \deg(X) + (r - 1)(1 - p_a(X)).$$

There is a map $\phi : T\mathbb{P}^r_{|X} \to N_X$ which is surjective outside $\mathrm{Sing}(X)$. Consider the restriction to X of the Euler's sequence of $T\mathbb{P}^r$:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus(r+1)} \to T\mathbb{P}^r_{|X} \to 0 \tag{1}$$

Now assume $h^1(\mathcal{O}_X(1)) = 0$. Since $\dim X = 1$, then $h^2(\mathcal{O}_X) = 0$. Thus (1) gives $h^1(T\mathbb{P}^r_{|X}) = 0$.

The map ϕ induces the following exact sequence of coherent sheaves on X:

$$0 \to \mathrm{Ker}(\phi) \to T\mathbb{P}^r_{|X} \to \mathrm{Im}(\phi) \to 0 \tag{2}$$

Since $\dim X = 1$, $h^2(\mathrm{Ker}(\phi)) = 0$. Thus (2) gives $h^1(\mathrm{Im}(\phi)) = 0$.

If X is smooth, then $N_X = \mathrm{Im}(\phi)$ and hence $h^1(N_X) = 0$.

Now assume X singular. Consider the exact sequence

$$0 \to \mathrm{Im}(\phi) \to N_X \to N_X/\mathrm{Im}(\phi) \to 0 \tag{3}$$

Since $N_X/\mathrm{Im}(\phi)$ is supported by the finite set $\mathrm{Sing}(X)$, then $h^1(N_X/\mathrm{Im}(\phi)) = 0$. Thus (3) gives $h^1(N_X) = 0$ even if the non-special curve is singular.

If X is smooth and rational, then $h^1(\mathcal{O}_X) = 0$. As above we obtain that $h^1(T\mathbb{P}^r(-1)_{|X}) = 0$ and $h^1(N_X(-1)) = 0$.

In particular we study the case of curves in $\mathbb{P}^3$.

Lemma 1 *Let $C \subset \mathbb{P}^3$ be a reduced curve and let $C' \subset \mathbb{P}^3$ be a smooth conic that meets C at $i \leq 3$ points $p_1, \ldots, p_i \in C_{reg}$ with $C \cup C'$ nodal at each p_i. Let H be the plane spanned by C'. If $i > 1$, assume also that $T_{p_1} C \not\subseteq H$, $T_{p_2} C \not\subseteq H$ and $T_{p_1} C \cap T_{p_2} C = \emptyset$.*

(1) *If $i = 1$ then $N_{C \cup C' | C'}$ is the direct sum of two line bundles, one of degree 3 and one of degree 4.*

(2) *If $i = 2$ then $N_{C \cup C' | C'}$ is the direct sum of two line bundles, both of degree 4.*

(3) *If $i = 3$, then $N_{C \cup C' | C'}$ is the direct sum of two line bundles, one of degree 4 and one of degree 5. In any case $h^1(N_{C \cup C' | C'}(-1)) = 0$.*

Proof Note that $N_{C'} \cong \mathcal{O}_{C'}(2) \oplus \mathcal{O}_{C'}(1)$, i.e. $N_{C'}$ is the direct sum of a degree 4 line bundle and a degree 2 line bundle.

We prove (2) first. Set $Y := T_{p_1} C \cup T_{p_2} C \cup C'$. By Hartshorne and Hirschowitz [7, Corollary 3.2] the bundle $N_{C \cup C' | C'}$ is obtained from $N_{C'}$ by making two positive elementary transformations (in the sense of [7, §2]), which depend uniquely on C', p_1, p_2, and the lines $T_{p_1} C$, $T_{p_2} C$. Thus $N_{C \cup C' | C'} \simeq N_{Y | C'}$. Since Y is connected and $\deg(Y) = 4$, $h^0(\mathcal{O}_Y(2)) < 10$. Thus $h^0(\mathcal{I}_Y(2)) > 0$. Since $T_{p_1} C \cap T_{p_2} C = \emptyset$, Y is contained neither in a reducible quadric, nor in a quadric cone. Thus Y lies in a smooth quadric Q with, say, $T_{p_1} C \in |\mathcal{O}_Q(1, 0)|$. Then also $T_{p_2} C$, which does not meet $T_{p_1} C$, lies in $|\mathcal{O}_Q(1, 0)|$. Obviously $C' \in |\mathcal{O}_Q(1, 1)|$, and we conclude that $Y \in |\mathcal{O}_Q(3, 1)|$. Thus N_Y is an extension of $\mathcal{O}_Y(2)$ by the restriction to Y of the line bundle $\mathcal{O}_Q(3, 1)$. Hence $N_{Y|C'}$ is an extension of two line bundles of degree 4.

(1) By Hartshorne and Hirschowitz [7, Corollary 3.2] the bundle $N_{C \cup C' | C'}$ is obtained from $N_{C'}$ by making one general positive elementary transformation, and as above one general positive elementary transformation of $N_{C \cup C' | C'}$ is a direct sum of a line bundle of degree 4 and a line bundle of degree 3.

(3) Here $N_{C \cup C' | C'}$ is obtained from $N_{C'}$ by making 3 positive elementary transformations, one at p_1, one at p_2, and one at p_3. Hence $N_{C \cup C' | C'}$ is obtained from one positive elementary transformation on the vector bundle E on C' obtained by making positive elementary transformations only at p_1 and p_2. By 1) E is isomorphic to $\mathcal{O}_{C'}(2) \oplus \mathcal{O}_{C'}(2)$. Since any vector bundle on C' splits in a direct sum of line bundles, the claim follows. □

Remark 5 We will use repeatedly a result of J. Kleppe, who proved that the functor of deformations inside $\mathbb{P}^r$ of a (possibly reducible) curve $T \subset \mathbb{P}^r$ with fixed hyperplane section $H \cap T$ has tangent space $H^0(T, N_T(-1))$ and obstruction space $H^1(N_T(-1))$. See e.g. [9, Th. 1.5].

Remark 6 Let C and D be locally complete intersection space curves. Assume $S := C \cap D = C_{reg} \cap D_{reg}$ and that at each $p \in S$ the tangent lines of C and D at p are different. Set $Y := C \cup D$. Call N_C^+ (resp. N_D^+) the rank 2 vector bundle on C (resp. D) obtained from N_C (resp. N_D) making a positive elementary transformation in the direction of $T_p D$ (resp. $T_p C$) at all $p \in S$. We have $\deg(N_C^+) = \deg(N_C) + \#S$ and $\deg(N_D^+) = \deg(N_D) + \#S$. By Hartshorne and Hirschowitz [7, Cor. 3.2] we have $N_C^+ \cong N_{Y|D}$ and $N_D^+ \cong N_{Y|D}$. Thus the Mayer-Vietoris exact sequence of the vector

bundle N_Y gives the following exact sequence

$$0 \to N_{C \cup D} \to N_C^+ \oplus N_D^+ \to N_{Y|S} \to 0. \tag{4}$$

Thus to prove that $h^1(N_Y(-1)) = 0$ it is sufficient to prove that $h^1(N_C^+(-1)) = h^1(N_D^+(-1)) = 0$ and that the restriction map $H^0(D, N_D^+(-1)) \to H^0(N_{Y|S})$ is surjective.

A similar statement holds for the proof that $h^1(N_{C \cup D}) = 0$.

Lemma 2 *Let $H \subset \mathbb{P}^3$ be a plane and let $C' \subset H$ be a smooth conic. Fix 3 distinct points p_1, p_2, p_3 of C' and lines L_1, L_2, L_3 such that $H \cap L_i = \{p_i\}$ for all i. Let E (resp. F) be the vector bundle obtained from $N_{C'}$ making positive elementary transformations at p_1 and p_2 (resp. p_1, p_2 and p_3) with respect to L_1 and L_2 (resp. L_1, L_2 and L_3). Then E is a direct sum of 2 line bundles of degree 4 and F is a direct sum of a line bundle of degree 5 and a line bundle of degree 4.*

Proof Set $X := C' \cup L_1 \cup L_2$ and $Y := X \cup L_3$. Since $H \cap L_i = \{p_i\}$, then L_i is transversal to H. Thus X and Y are nodal at p_1, p_2 and p_3. Note that $E \cong N_{X|C'}$ and $F \cong N_{Y|C'}$.

First assume $L_1 \cap L_2 \neq \emptyset$. Thus X is nodal with 3 nodes and arithmetic genus 1. Call M the plane containing $L_1 \cup L_2$. To prove that $E \cong \mathcal{O}_{C'}(2) \oplus \mathcal{O}_{C'}(2)$ it is sufficient to prove that X is the complete intersection of $M \cup H$ and a quadric. This is true, because $h^0(\mathcal{O}_X(2)) = 8$, $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$ and hence $h^0(\mathcal{I}_X(2)) \geq 2$.

Now assume $L_1 \cap L_2 = \emptyset$. In this case X is contained neither in a reducible quadric nor in a quadric cone. Since $h^0(\mathcal{O}_X(2)) = 9$, X is contained in a smooth quadric, Q. Call $|\mathcal{O}_Q(1, 0)|$ the ruling of Q containing L_1, and hence also containing L_2. Since $C' \in |\mathcal{O}_Q(1, 1)|$, then $X \in |\mathcal{O}_Q(3, 1)|$. Since $N_Q \cong \mathcal{O}_Q(2)$, we have an exact sequence

$$0 \to N_{X,Q} \to N_X \to N_X(2) \to 0 \tag{5}$$

We have $N_{X.Q} \cong \mathcal{O}_X(3, 1)$ so that its restriction to C' has degree 4. Since (5) is an exact sequence of vector bundles, its restriction to C' is an exact sequence of vector bundles on $C' \cong \mathbb{P}^1$ in which the leftmost and the rightmost terms are line bundles of degree 4. Since $C' \cong \mathbb{P}^1$, E is a direct sum of two line bundles of degree 4.

The bundle F is obtained from E making a positive elementary transformation, and all rank 2 vector bundles on $\mathbb{P}^1$ split. Hence F is a direct sum of a line bundle of degree 5 and a line bundle of degree 4. □

Now we are ready to prove some non-emptiness results in $\mathbb{P}^3$. Recall that $H(d, g, 3)$ denotes the set of smooth and non-degenerate curves space curves X of degree d and genus g, such that $h^1(\mathcal{O}_X(1)) = 0$.

Theorem 3 *Fix integers $g \geq 0$, $d \geq g + 3$ and x such that $6 \leq 2x \leq d$. Then there is $X \in H(d, g, 3)$ such that $\mathbb{T}(X, x) \neq \emptyset$ and $h^1(N_X(-1)) = 0$.*

Proof Fix a plane $H \subset \mathbb{P}^3$. We will find $X \in H(d, g, 3)$ such that X is tangent to H at x points of H spanning H. We first dispose of the case $x = \lfloor d/2 \rfloor$ and $d = g + 3$ in steps (a) and (b), leaving the case $x = \lfloor d/2 \rfloor$ and $d > g + 3$ to step (c) and the case $3 \le x < \lfloor d/2 \rfloor$ to step (d).

(a) Assume d even and $x = d/2$.

(a1) Assume $d = 6$. By Proposition 7 $\mathbb{T}(C, 2) \ne \emptyset$ for each smooth curve C of genus 1 and degree 4. Hence by a change of coordinates we find a smooth curve $C \subset \mathbb{P}^3$ of degree 4 and genus 1 which is tangent to H at 2 distinct points, say q_1 and q_2. Since C is the complete intersection of 2 quadric surfaces, the normal bundle of C splits as $N_C \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$. Since C has genus 1, we get $h^1(N_C(-1)) = 0$. Fix a general $q_3 \in H$ and let $M \subset \mathbb{P}^3$ be a general plane containing q_3. Since q_3 and M are general, we may assume that M is transversal to C, $C \cap M \cap H = \emptyset$, and $C \cap M$ spans M. Fix three distinct points $\{p_1, p_2, p_3\} \in C \cap M$ which span M. There is a smooth conic C' containing $\{p_1, p_2, p_3, q_3\}$, tangent in q_3 to the line $H \cap M$ but not containing the fourth point p_4 of $C \cap M$. Indeed, q_3 is general in $H \cap M$, while there are only at most two conics passing through p_1, p_2, p_3, p_4 and tangent to $H \cap M$. Since M is transversal to C, the curve $Y := C \cup C'$ is nodal. By Lemma 1 the vector bundle $N_{C'}^+(-1)$ is the direct sum of a line bundle of degree 2 and a line bundle of degree 3. Since line bundles of degree 2, 3 separate any set of three points in the smooth conic C', the restriction map $H^0(C', N_{C'}^+(-1)) \to H^0(N_{C'}^+(-1)_{|\{p_1, p_2, p_3\}})$ is surjective. Remark 6 gives $h^1(N_Y(-1)) = 0$, so that Y is smoothable by [7, Theorem 4.1]. By semicontinuity, a general member X_0 of a smoothing family of Y satisfies $h^1(N_{X_0}(-1)) = 0$. By construction $\mathbb{T}(Y, 3) \ne \emptyset$. Yet, in order to conclude, we need more: we need to smooth Y in a family of space curves whose elements Y_λ satisfies $\mathbb{T}(Y_\lambda, 3) \ne \emptyset$. In other words, we need:

Claim 1 There are an affine smooth and connected curve Δ, $o \in \Delta$ and a flat family $\{Y_\lambda\}_{\lambda \in \Delta}$ of space curves such that $Y_0 = Y$, the general element of the family is smooth, and $\mathbb{T}(Y_\lambda, 3) \ne \emptyset$ for all $\lambda \in \Delta$.

Proof of Claim 1 Set $Z' := (2q_1, C) \cup (2q_2, C)$, $Z'' := (2q_3, C')$ and $Z := Z' \cup Z''$. Note that $Z \cap C = Z'$ and $Z \cap C' = Z''$. Since q_1, q_2 and q_3 are smooth points of Y, Z is a degree 6 Cartier divisor of Y. Thus $N_Y(-Z)$ is a rank 2 vector bundle on Y with $\deg(N_Y(-Z)) = \deg(N_Y) - 12$. The vector space $H^0(N_Y(-Z))$ is the tangent space to the functor of deformations of Y inside $\mathbb{P}^3$ in families of curves containing Z, while $H^1(N_Y(-Z))$ is an obstruction space of this functor [9, Th. 1.5]. To prove Claim 1 it is sufficient to find a smoothing family $\{Y_\lambda\}_{\lambda \in \Delta}$ of $Y = Y_o$ such that $Z \subset Y_\lambda$ for all $\lambda \in \Delta$. Note that $N_{C'}(-Z) = N_{C'}(-Z'')$ is a direct sum of degree 2 line bundles on $C' \cong \mathbb{P}^1$. Thus $h^1(N_{C'}(-Z - p_i)) = 0$ for all i. In particular (with the terminology of [7]) $h^1(N_C(-Z)^-) = 0$. Since $N_C \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$, and $\deg(Z \cap C) = 4$, $h^1(N_C(-Z - p_1 - p_2 - p_3)) = 0$. Thus $h^1(F) = 0$ for every vector bundle F on C obtained from $N_C(-Z - p_1 - p_2 - p_3)$ making finitely many positive elementary transformations. Consider the Mayer–Vietoris exact sequence

$$0 \to N_Y(-Z) \to N_Y(-Z)_{|C} \oplus N_Y(-Z)_{|C'} \to N_Y(-Z)_{|\{p_1, p_2, p_3\}} \to 0 \quad (6)$$

Since $N_Y(-Z)_{|C}(-p_1 - p_2 - p_3)$ is obtained from $N_C(-Z - p_1 - p_2 - p_3)$ by making positive elementary transformations, $h^1(N_Y(-Z)_{|C}(-p_1 - p_2 - p_3)) = 0$ (here we consider p_1, p_2, p_3 as points of the smooth curve C). The sequence

$$0 \to N_Y(-Z)_{|C}(-p_1 - p_2 - p_3) \to N_Y(-Z)_{|C} \to N_Y(-Z)_{|\{p_1, p_2, p_3\}} \to 0$$

shows that the map $\phi : H^0(N_Y(-Z)_{|C}) \to H^0(N_Y(-Z)_{|\{p_1, p_2, p_3\}})$ is surjective, and $h^1(N_Y(-Z)_{|C}) = 0$. We also know that $N_Y(-Z)_{|C'}$ is a direct sum of 2 line bundles of non-negative degree on the smooth conic C', so that $h^1(N_Y(-Z)_{|C'}) = 0$. Hence the surjectivity of ϕ gives a fortiori, in sequence (6), that $h^1(N_Y(-Z)) = 0$. Thus we may apply the proof of [7, Th.4.1] since the deformation functor of Y which maintains Z fixed is unobstructed. We obtain a family $\{Y_\lambda\}_{\lambda \in \Delta}$ as in the statement, in which the general element contains Z, hence it is tangent to H at three points.   □

(a2) Assume $d \geq 8$ and that the theorem is true for the triples (d', g', x') such that $x = d'/2$, $d' = g' + 3$ and $d' \leq d - 2$. Take a solution C for $(d', g', x') = (d - 2, g - 2, x - 1)$ and use the proof of step (a1) with this C instead of an elliptic curve, as follows. By induction, there is a plane H which contains a set Z' of $x - 1$ arrows in C. Then take a general plane M and three points o_1, o_2, o_3 of $M \cap C$. Fix a conic $C' \subset M$ passing through o_1, o_2, o_3 and tangent to H at a point p. Since the base locus of the family of conics in M passing through o_1, o_2, o_3 and tangent to H is exactly given by o_1, o_2, o_3, we may assume that C' misses any other point of $M \cap C$. Set $Y = C \cup C'$, $Z'' = (2p, C')$, and $Z = Z' \cup Z''$.

Claim 2: $h^1(N_Y(-Z)) = 0$.

Proof of Claim 2 The bundle $N_{C'}^+(-Z'' - o_1 - o_2 - o_3)$ is a direct sum of a line bundle of degree 0 and a line bundle of degree -1. Thus $h^1(C', N_{Y|C'}(-Z - o_1 - o_2 - o_3)) = 0$. As in Claim 1, just working on C' instead of C and with o_1, o_2, o_3 instead of p_1, p_2, p_3, this implies that the restriction map $H^0(N_Y(-Z)_{|C'}) \to H^0(N_Y(-Z)_{|\{o_1, o_2, o_3\}})$ is surjective. Then the analogue of (6) gives $h^1(N_Y(-Z)) = 0$.   □

So, we can continue the induction by taking a general element X in a family $\{Y_\lambda\}_{\lambda \in \Delta}$ giving a smoothing of Y and fixing Z. Notice that at any step X satisfies $h^1(N_X(-Z)) = 0$ by semicontinuity, since $h^1(N_Y(-Z)) = 0$.

(b) Assume d odd and $x = (d - 1)/2$. Since $x \geq 3$, $d \geq 7$. By Proposition 7 each $C \in H(5, 2, 3)$ has $\mathbb{T}(C, 2) \neq \emptyset$. To adapt the proof of step (a) we first find $C \in H(5, 2, 3)$ with $h^1(N_C(-1)) = 0$. Let $Y' \subset \mathbb{P}^3$ be a smooth rational cubic. Take a plane $M \subset \mathbb{P}^3$ transversal to Y' and let $C' \subset M$ be a smooth conic containing $Y' \cap M$. Set $T := Y' \cup C'$. Lemma 1 gives $h^1(N_T(-1)) = 0$. By Hartshorne and Hirschowitz [7, Theorem 4.1] and Lemma 1, T is smoothable in a family preserving the scheme-theoretic intersection $(Y' \cup C') \cap H$ as in step (a1). By semicontinuity, a general element C of the family has $h^1(N_C(-1)) = 0$. Next we proceed by induction as in step (a), adding suitable conics C', constructed as in step (a), to curves C which are solutions for the triple $(d', g', x') = (d - 2, g - 2, x - 1)$, and taking a general smoothing X of $C \cup C'$. To see that we get $h^1(N_X(-1)) = 0$ we apply Lemma 2 to C'.

(c) Assume $d > g + 3$. If $d - g - 3$ is even, start with a curve of genus g and degree $d' = g + 3$, constructed as in step (a2) when d' is even, or constructed as in step (b) if d' is odd. Continue for $(d - g - 3)/2$ steps, by adding to the previously constructed curve C a smooth conic C', tangent to H, with $\#(C' \cap C) = 1$ and $C' \cup C$ nodal. By Lemma 1 we always get $h^1(N_{C \cup C'}(-1)) = 0$. After $(d - g - 3)/2$ steps we get the claimed curve.

Assume that $d - g - 3$ is odd. When $g = 0$, start with a rational quartic, by using Proposition 7. When $g \geq 1$, start with a curve C of genus $g - 1$ and degree $d' = g + 2$, constructed as in step (a2) when d' is even, or constructed as in step (b) if d' is odd. In the first step add to C a smooth conic C', tangent to H, which meets the curve C at two points whose tangent lines t_1, t_2 to C are disjoint and different from the tangent lines to C'. We may take such a conic C' in a plane which does not contain t_1, t_2. Then continue for $(d - g - 4)/2$ steps, by adding to the previously constructed curve C a smooth conic C', tangent to H, which intersects C at a unique point and with $C \cup C'$ nodal. In any case, the assumptions of Lemma 1 hold for $C \cup C'$. Thus by Lemma 1 we always get $h^1(N_{C \cup C'}(-1)) = 0$, and we can continue the induction, by taking a smoothing of $C \cup C'$ which preserves the intersection with H.

(d) Assume $3 \leq x \leq \lfloor d/2 \rfloor$. We start with some curve Y such that $\mathbb{T}(Y, 2) \neq \emptyset$ and $h^1(N_Y(-1)) = 0$. We take Y of genus 1 and degree 4 if d is even or genus 2 and degree 5 if g is odd. Then we continue as in steps (a), (b) and (c) above, except that in $\lfloor d/2 \rfloor - x$ steps we add a smooth conic C' not tangent to H. □

For space curves $X \subset \mathbb{P}^3$ with $h^1(\mathcal{O}_X(1)) \neq 0$ we prove the following result.

Theorem 4 *Fix integers $x \geq 3$, $g \geq 0$ and d such that $g \geq 2x + 2$. Write $g = 2x + 1 + 5s - q$, $0 \leq q \leq 4$, and assume $d \geq 2x + 4 + 3s$. Then there is a smooth and connected curve $X \subset \mathbb{P}^3$ such that $\deg(X) = d$, $p_a(X) = g$, $h^1(N_X) = 0$, $h^1(\mathcal{O}_X(2)) = 0$ and $\mathbb{T}(X, x) \neq \emptyset$.*

Remark 7 In Theorem 4, for each fixed x we find g_0 such that for all $g \geq g_0$ and all $d \geq \frac{3}{4}g + 3$ there is a smooth curve $X \subset \mathbb{P}^3$ of genus g and degree d with $\mathbb{T}(X, x) \neq \emptyset$. The same is true for a slowing increasing function $x(g)$ of g.

Note that for a fixed x and for $g \gg x$ these curves X cover a range of degrees and genera larger that the Brill-Noether range $d \geq \frac{3}{4}g + 3$.

For the proof of Theorem 4, we need a series of preliminary lemmas.

Lemma 5 *Fix an integer $e \in \{1, 2, 3, 4\}$. Let $C \subset \mathbb{P}^3$ be an integral and non-degenerate curve of degree d. If $e = 4$ assume $d \geq 4$. Take a union $Y \subset \mathbb{P}^3$ of finitely many curves such that $C \nsubseteq Y$. Then there is a smooth conic C' such that $\#(C \cap C') = e$, $C \cup C'$ is nodal at each point of $C \cap C'$, and $C' \cap Y = \emptyset$.*

Proof Take a general plane $M \subset \mathbb{P}^3$. The plane M is transversal to C, $Y \cap M$ is finite and $Y \cap C \cap M = \emptyset$. By the trisecant lemma no 3 of the d points of $C \cap M$ are collinear. Fix $S \subseteq C \cap M$ such $\#S = e$. Since no 3 of the points of S are collinear, S is the scheme-theoretic base locus of the $(5 - e)$-dimensional linear space $|\mathcal{I}_S(2)|$. A general element of $|\mathcal{I}_S(2)|$ is smooth. Thus there is a smooth $C' \in |\mathcal{I}_S(2)|$ such that $C' \cap Y = \emptyset$ and $C' \cap C = S$. Since M is transversal to C and $C' \subset M$, then $C \cup C'$ is nodal at each point of S. □

Lemma 6 *Let $C \subset \mathbb{P}^3$ be an integral and non-degenerate curve of degree* 4. *For a general $S \subset C$ of length* 6 *there is a rational normal curve $T_S \subset \mathbb{P}^3$ such that $S = C \cap T_S$ and $T_S \cup C$ is nodal.*

Moreover, if $p_a(C) = 1$ then $T_{S_1} \cap T_{S_2} = \emptyset$ for a general $S_1 \times S_2 \subset C \times C$ such that both S_1 and S_2 have length 6.

Proof Let $\mathcal{U}$ denote the set of all $A \subset \mathbb{P}^3$ such that $\#A = 6$ and A is in linear general position. For every $A \in \mathcal{U}$ there is a unique rational normal curve T_A containing A. Set $\mathcal{U}(C) := \{S \in \mathcal{U} \mid S \subset C\}$. Since C is integral and non-degenerate, $\mathcal{U}(C)$ is an integral quasi-projective variety of dimension 6. The set S is a general element of $\mathcal{U}(C)$. Since $\deg(C) \neq 3$, then $C \neq T_S$. We need to prove that S is equal to $C \cap T_S$ (set-theoretically), that $T \cup C$ is nodal, and the last assertion, concerning a general $S_1 \times S_2 \subset C \times C$, when $p_a(C) = 1$.

If $p_a(C) = 1$, the curve C is the complete intersection of 2 quadrics and (since it has at most one singular point and with embedding dimension 2) it is contained in a smooth quadric Q, say $C \in |\mathcal{O}_Q(2, 2)|$. Fix a general $S \subset C$ of length 6. By the generality of S, since $h^0(\mathcal{O}_Q(1, 2)) = h^0(\mathcal{O}_Q(2, 1)) = 6$, we get $h^0(\mathcal{I}_{S,Q}(2, 1)) = h^0(\mathcal{I}_{S,Q}(1, 2)) = 0$. Thus $T_S \not\subseteq Q$. Bezout theorem gives $S = T_S \cap Q$ as schemes. Thus $T_S \cap C = S$ and $T_S \cup C$ is nodal. If $p_a(C) = 0$ then $h^0(\mathcal{I}_C(2)) = 1$ and the unique quadric surface Q containing C is smooth [6, Ex. V.2.9], with either $C \in |\mathcal{O}_Q(1, 3)|$ or $C \in |\mathcal{O}_Q(3, 1)|$. We conclude as in the case $p_a(C) = 1$.

Now we prove the last claim, for $p_a(C) = 1$. It is sufficient to find $S_1, S_2 \in \mathcal{U}(C)$ such that $T_{S_1} \cap T_{S_2} = \emptyset$. The pencil $|\mathcal{I}_C(2)|$ has only finitely many (i.e. 4) singular elements. Take smooth quadrics $Q_1, Q_2 \in |\mathcal{I}_S(2)|$ such that $Q_1 \neq Q_2$. Note that $C \in |\mathcal{O}_{Q_i}(2, 2)|$, $i = 1, 2$. Take a general $T_i \in |\mathcal{O}_{Q_i}(2, 1)|$. Each T_i is a rational normal curve and $\deg(T_i \cap C) = 6$. Bertini Theorem gives that $S_i := T_i \cap C$ has cardinality 6. Since $S_i \subset T_i$ and T_i is a rational normal curve, S_i is in linear general position. Thus $T_i = T_{S_i}$. Since $T_{S_i} \neq C$, we have $T_{S_1} \not\subseteq Q_2$ and $T_{S_2} \not\subseteq Q_1$. Since $T_i \subset Q_i$, $i = 1, 2$, we get $T_{S_1} \cap Q_2 = T_{S_1} \cap Q_1 \cap Q_2 = T_{S_1} \cap C = S_1$. Since $T_{S_2} \subset Q_2$, then $T_{S_1} \cap T_{S_2} \subseteq T_{S_1} \cap Q_2 = S_1$. Since $S_1 \cap S_2 = \emptyset$ and $T_{S_2} \cap C = S_2$, we get $T_{S_1} \cap T_{S_2} = \emptyset$. □

Lemma 7 *For $q = 0, \ldots, 4$ and $x \geq 2$ the space $H(2x + 4, 2x + 1 - q, 3)$ contains smooth curves X_0 which satisfy the following conditions.*

1. $\mathbb{T}(X_0, x)$ *is non-empty, and there exists a plane H which contains a union of x arrows $Z \subset X_0$ with $h^1(N_{X_0}(-Z)) = 0$.*
2. *For a general choice of s subsets $S_1, \ldots, S_s \subset X_0$ of cardinality* 6, *there are disjoint rational normal curves $T_1, \ldots, T_s$ such that for all i the union $X_0 \cup T_i$ is nodal, and T_i meets X_0 exactly at S_i.*

Proof We make induction on $x \geq 2$.

To deal with the case $x = 2$ (i.e. $d = 8$ and $g = 5 - q$) we start with a smooth elliptic quartic curve C. We know that $H^1(N_C(-1)) = 0$ and $\mathbb{T}(C, 2)$ is non-empty. Since $\mathbb{T}(C, 2) \neq \emptyset$, there is a subset $Z \subset C$ of cardinality 2 such that the tangent lines to C at the points of Z lie in a plane H. By Lemma 6, for a general choice of s subsets $S_1, \ldots, S_s \subset C$ of cardinality 6 we find disjoint rational normal curves $T_1, \ldots, T_s$

such that $C \cup T_i$ is nodal and $T_i \cap C = S_i$ for all i. Take a general plane $H' \subset \mathbb{P}^3$. Thus H' is transversal to C and $Z \cap H' = \emptyset$. Fix $e \in \{0, 1, 2, 3\}$ and $W \subset H' \cap C$ such that $\#W = 4 - e$. Take a general smooth conic $C' \subset H'$ containing W. Since $\#W \leq 4$ and C' is general, $T_i \cap C' = \emptyset$ for all i. Set $Y := C \cup C'$. The normal bundle $N_Y(-Z)_{|C}$ is obtained from $N_C(-Z)$ by making positive elementary transformations, so that $h^1(N_Y(-Z)_{|C}) = 0$. Recall that $1 \leq \#W \leq 4$. For each $o \in W$ the tangent line T_oC of C at o is transversal to H'. If $\#W = 4$ Lemma 2 shows that the restriction map $\rho : H^0(N_Y(-Z)_{|C'}) \to H^0(N_Y(-Z)_{|W})$ surjects, so that the exact sequence

$$0 \to N_Y(-Z) \to N_Y(-Z)_{|C} \oplus N_Y(-Z)_{|C'} \to N_Y(-Z)_{|W} \to 0 \qquad (7)$$

proves that $h^1(N_Y(-Z)) = 0$. If $\#W \leq 3$, then the surjectivity of ρ only use that $N_{C'}$ is a direct sum of 2 line bundles of degree ≥ 2 and that $N_Y(-Z)_{|C'}$ is obtained from $N_{C'}$ making positive elementary transformations. By semicontinuity, we can find a smoothing D of $C \cup C'$ which preserves Z. Thus we get curves D in $H(6, g, 3)$, for $g = \#W \in \{1, 2, 3, 4\}$, such that $\mathbb{T}(D, 2) \neq \emptyset$, $h^1(N_D(-Z)) = 0$, and such that for general subsets $S_1, \ldots, S_s \subset X_0$ of cardinality 6 we have the disjoint rational normal curves $T_1, \ldots, T_s$, as in the statement. Then consider a conic C'' which meets D in one or in two general points. As above, in both cases $Y' = C'' \cup D$ satisfies $h^1(N_{Y'}(-Z)) = 0$, and we can find a general smoothing X_0 of Y' which preserves Z. X_0 has degree 8, and for its genus g we can obtain any number between 1 and 5, because $\#W$ is any integer between 1 and 4. Moreover, $h^1(N_{X_0}(-Z)) = 0$, $\mathbb{T}(X_0, 2) \neq \emptyset$, and by semicontinuity X_0 satisfies condition 2 of the statement. This concludes the case $x = 2$.

Assume we constructed the required curve $X \in H(2(x-1)+4, 2(x-1)+1-q, 3)$. There exists a subscheme $Z \subset X$ formed by $x - 1$ arrows which is contained in a plane H, and moreover $h^1(N_X(-Z)) = 0$. Take a general plane M which is transversal to both H and X, and misses Z. As in the proof of (a1) of Theorem 3, for $p_1, p_2, p_3 \in M \cap X$ there exists a smooth conic C' passing through p_1, p_2, p_3, tangent to H at $p_4 \notin X$, which misses the remaining points of $M \cap X$. Take $Y = X \cup C'$, and define Z_0 as the union of Z and the arrow at p_4 tangent to C'. Notice that Z_0 lies in H. As in the proof of Claim 1, the vanishing of $H^1(N_X(-Z))$ implies the surjectivity of the map $H^0(N_Y(-Z_0)_{|X}) \to H^0(N_Y(-Z_0)_{|\{p_1,p_2,p_3\}})$, and the analogue of sequence (6) shows that $h^1(N_Y(-Z_0)) = 0$. Thus there exists a smoothing $X_0 \in H(2x + 4, 2x + 1 - q, 3)$ of Y which preserves Z_0. The existence of $Z_0 \subset X_0$ implies $\mathbb{T}(X_0, x) \neq \emptyset$. By semicontinuity $h^1(N_{X_0}(-Z_0)) = 0$, and X_0 satisfies condition 2 of the statement. $\qquad \square$

We notice that, in the previous lemma, the condition $h^1(N_{X_0}(-Z)) = 0$ implies $h^1(N_{X_0}) = 0$.

Lemma 8 *Take a reduced curve $C \subset \mathbb{P}^3$ such that $h^1(\mathcal{O}_C(2)) = 0$. Let $T \subset \mathbb{P}^3$ be a smooth rational curve which is not an irreducible component of C. Set $f := \deg(T)$ and $e := \deg(C \cap T)$. If $e \leq 2f + 1$, then $h^1(\mathcal{O}_{C \cup T}(2)) = 0$.*

Proof Consider the Mayer–Vietoris exact sequence

$$0 \to \mathcal{O}_{C \cup T}(2) \to \mathcal{O}_C(2) \oplus \mathcal{O}_T(2) \to \mathcal{O}_{T \cap C}(2) \to 0 \tag{8}$$

By assumption $h^1(\mathcal{O}_C(2)) = 0$. Obviously, $h^1(\mathcal{O}_T(2)) = 0$. Since $e \leq 2f + 1$, $T \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_T(2)) = 2f$, the restriction map $H^0(\mathcal{O}_T(2)) \to H^0(\mathcal{O}_{C \cap T}(2))$ is surjective. Hence (8) gives $h^1(\mathcal{O}_{C \cup T}(2)) = 0$. □

Lemma 9 *Let $C \subset \mathbb{P}^3$ be an integral locally complete intersection curve and $Z \subset C_{\mathrm{reg}}$ a zero-dimensional scheme. Assume $h^1(N_C(-Z)) = 0$. Take $S \subset C_{\mathrm{reg}}$ such that $\#S = 6$, $S \cap Z = \emptyset$, S is in linearly general position and the only rational normal curve T containing S meets C only at S and with $C \cup T$ nodal at each point of S. Set $Y := C \cup T$. Then $h^1(N_Y(-Z)) = 0$.*

Proof By assumption S is the scheme-theoretic intersection of C and T. Thus we have the following Mayer–Vietoris exact sequence

$$0 \to N_Y(-Z) \to N_Y(-Z)_{|C} \oplus N_Y(-Z)_{|T} \to N_Y(-Z)_{|S} \to 0 \tag{9}$$

Since $Z \cap T = \emptyset$, $N_Y(-Z)_{|T} \cong N_{Y|T}$. Since $N_Y(-Z)_{|C}$ is obtained from $N_C(-Z)$ making positive elementary transformations and $h^1(N_C(-Z)) = 0$, $h^1(N_Y(-Z)_{|C}) = 0$. Since N_T is a direct sum of 2 line bundles of degree 5, $\#S = 6$ and $N_Y(-Z)_{|T}$ is obtained from N_T making positive elementary transformations, $h^1(N_Y(-Z)_{|T}) = 0$ and the restriction map $H^0(N_Y(-Z)_{|T}) \to N_Y(-Z)_{|S}$ is surjective. Thus (9) gives $h^1(N_Y(-Z)) = 0$. □

Now we are ready for the proof of Theorem 4.

Proof of Theorem 4 Recall that $g = 2x + 1 + 5s - q$, where $q \in \{0, \ldots, 4\}$. By assumption $d \geq 2x + 4 + 3s$.

We first dispose of the case $d = 2x + 4 + 3s$. The case $s = 0$ is covered by Lemma 7. Consider a curve $X_0 \in H(4x + 2, 2x + 1, 3)$ as in the statement of Lemma 7. Since $\mathbb{T}(X_0, x)$ is non-empty, we can take a set Z of x coplanar arrows in X_0 supported at x points, with $h^1(N_{X_0}(-Z)) = 0$. Moreover, for a choice of s general subsets $S_1, \ldots, S_s \subset X_0$ of cardinality 6 there are disjoint rational normal curves $T_1, \ldots, T_s$ such that for all i the union $X_0 \cup T_i$ is nodal, and T_i meets X_0 exactly at S_i. Define $Y = X_0 \cup T_1 \cdots \cup T_s$. Then $h^1(N_Y(-Z)) = 0$. Arguing by induction on s one finds a smoothing X_s of Y which preserves Z. The curve X_s belongs to $H(2x + 4 + 3s, 1 + 2x + 5s - q, 3)$. The existence of Z provides that $\mathbb{T}(X_s, x) \neq \emptyset$. By semicontinuity we also know that $h^1(N_{X_s}(-Z)) = 0$.

Finally assume $d = (2x + 4 + 3s) + t$, for some $t > 0$. Then we obtain the required curve in $H(d, g, 3)$ by induction on t. We start with $X_{s,0} = X_s$. Then we construct $X_{s,t+1}$ from $X_{s,t}$ by adding a line ℓ which meets $X_{s,t}$ at one general point, so that $X_{s,t} \cup \ell$ is nodal, and taking a smooth deformation which fixes Z. The condition $h^1(\mathcal{O}_X(2)) = 0$ follows by applying several times Lemma 8 applied to smooth rational curves of degree ≤ 3. □

Declarations

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