PERFECTLY MATCHED LAYERS WITH HIGH RATE DAMPING FOR HYPERBOLIC SYSTEMS

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Abstract

We propose a simple method for constructing non-reflecting boundary conditions via Perfectly Matched Layer approach. The basic idea of the method is to build a layer with high rate damping properties with are provided by adding the stiff relaxation source terms to all equations of the system. No complicated modification of the system to be solved is then required.

1 Introduction

The computational problem of wave propagation in infinite domains arises in acoustics, seismology and electromagnetic waves phenomena. The numerical study is usually reformulated in a bounded artificial computational domain in which infinity is modelled by some properties of the numerical boundary or its neighborhood. The purpose of such a reformulation of the problem is to avoid wave reflection from the numerical boundary.

The problem of constructing numerical boundary conditions has been intensively studied in recent decades. As a result, two main approaches have been developed. The first approach is the so called "Absorbing Boundary Conditions" (ABC) approach and consists of formulating direct boundary conditions for the computational region, which would eliminate wave reflection from the boundary. We refer the reader to some recent papers [8, 9] and bibliography therein.

Another approach is the "Perfectly Matched Layer" (PML) approach in which the computational domain is surrounded by an additional boundary layer. The main idea is to let the waves propagate out of the computational region. Therefore, it is crucial that the boundary layer does not generate waves propagating back to the computational region. Various versions of the PML approach have been proposed in the past, which are based on introducing new artificial variables and differential equations for these variables. See [5, 4, 2] and references therein.

In this paper we propose a simple, reliable and efficient PML method suitable for solving wave propagation problems described by hyperbolic systems, either linear or nonlinear. The basic idea of the method is to formulate a perfectly matched layer as a layer

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with high rate damping properties. These properties are provided by adding the stiff relaxation source terms to all equations of the system. An asymptotic analysis for small value of relaxation time τ allow us to make a conclusion that the amplitude of the wave reflected from the PML is of the second order on τ. Therefore the influence of the angle of incidence and wavelength of the wave incoming to the PML from the computational domain on the reflected wave can be made negligible. The main advantages of the method proposed here are its simplicity and generality. No complicated modification of the system to be solved is required. The structure of the source terms allows one to use practically any advection scheme without significant changes. No stability problem exists either.

A similar idea to use source terms with soft damping rate of acoustic waves in the certain and rather large buffer layer (comparable with computational domain) was proposed in [3]. In the quoted paper the mesh stretching and filtering together with non-reflecting boundary conditions have been used in order to provide outflow disturbance without reflection for aerodynamic sound generation problems. The absorbing layers with relaxation damping have been proposed also for electromagnetic equations [1] and for linearized Euler equations [5]. In these papers absorbing technique requires additional reformulation the governing equations (splitting in the coordinate direction).

The rest of the paper organized as follows. In Section 2 we formulate the idea of the new perfectly matched layer for the general case of nonlinear systems of conservation laws and the method of its numerical solving. In Section 3 we analyze the idea as applied to linear acoustics. In Section 4 we discuss the application of the method to two-dimensional equations and present some numerical examples. Conclusions are drawn in Section 5.

2 Equations for perfectly matched layer

We study the wave propagation phenomena described by a hyperbolic system of conservation laws in an infinite spacial domain. Generally speaking, the system can be nonlinear. In the three dimensional Cartesian coordinate system x_i it can be written as

$$\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}F(Q) + \frac{\partial}{\partial y}G(Q) + \frac{\partial}{\partial z}H(Q) = 0, \tag{1}$$

where Q is the vector of conserved variables, $F(Q)$, $G(Q)$, $H(Q)$, are the flux vectors in the coordinate directions.

To solve the problem numerically we apply the PML strategy, which consists of defining the finite computational domain D in which we intend to obtain a solution, and then constructing a surrounding absorbing boundary layer of a prescribed width. This boundary layer must not affect the basic computational domain, meaning that there are no waves coming back to the basic domain from the boundary layer.

Suppose that the values Q_∞ of the conserved variables at infinity are known: $Q \to Q_\infty$ if $x_i \to \infty$. We formulate the equations which describe the wave propagation in the perfectly matched layer by adding a relaxation source term to (1):

$$\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}F(Q) + \frac{\partial}{\partial y}G(Q) + \frac{\partial}{\partial z}H(Q) = -\frac{1}{\tau}(Q - Q_\infty), \tag{2}$$

where τ is the relaxation time which can be a function of variables Q. We take $\tau = \infty$ inside our computational domain whereas in the PML layer τ is taken to be small.

The idea of adding such relaxation term is to provide rapid (exponential) damping of all variables and waves inside the layer. It is clear intuitively that the decrease of the wave amplitude will be faster if the relaxation time tends to zero. Note that the structure and the type of equations *does not change* making it possible to use the same numerical method as for the original system (1).

We now proceed to describe the numerical procedure to be used in the new PML. Suppose that we have a one-step numerical scheme for solving (1), which gives us a result in the form

$$\frac{Q^{n+1} - Q^n}{\Delta t} = \mathcal{L}(Q^n).$$

Here Δt is the time step. The simplest numerical procedure is the one step implicit approximation of the source term in the system (2). That is we use the following formula:

$$\frac{Q^{n+1} - Q^n}{\Delta t} = \mathcal{L}(Q^n) - \frac{1}{\tau}(Q^{n+1} - Q_\infty). \tag{3}$$

It gives us the result in the form

$$\left(1 + \frac{\Delta t}{\tau}\right) Q^{n+1} = \frac{\Delta t}{\tau} Q_\infty + Q^n + \Delta t \mathcal{L}(Q^n)$$

¿From the latter formula we see that if the relaxation time τ is very small (in particular τ much less than Δt) then the value of Q^{n+1} is very close to its value at infinity Q_∞.

3 Acoustic wave propagation in the PML

In this section we study the method as applied to acoustic waves leaving the computational domain. We suppose that the wave of small amplitude can be obtained as a solution of a linearized isentropic Euler equations which are simply the acoustic equations. Without loss of generality we can assume that the mass density and speed of sound are equal to unity. Then the acoustic equations with the added relaxation source terms read as follows:

$$\begin{aligned} &\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{p}{\tau}, \\ &\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = -\frac{u}{\tau}, \\ &\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = -\frac{v}{\tau}. \end{aligned} \tag{4}$$

Here p, u, v are non-dimensional pressure, and components of velocities in x and y directions, respectively. Inside the computational domain τ is supposed to be infinity $(+\infty)$, hence source terms vanish, whereas in the PML τ is finite and small enough to provide a high rate of decay of waves amplitude.

We shall study the solution to system (4) in the PML $(x, y) \in [0, h] \times (-\infty, +\infty)$ supposing that the computational region is located from the left side of PML ($x \leq 0$).

Following the analysis in [4] we suppose that a harmonic plane wave of the form

$$(p, u, v)^T = (1, \alpha, \beta)^T \exp(i\omega(t - \alpha x - \beta y))$$

propagates from the main computational domain to the PML. Here α, β ($\alpha^2 + \beta^2 = 1$) represent the angle of wave incidence and ω is the normalized frequency of incoming wave.

We seek the solution inside the PML in the following form

$$(p, u, v)^T = (P(x), U(x), V(x))^T \exp(i\omega(t - \beta y)).$$

Substituting this representation of the solution into (4) we obtain a system of ordinary differential equations for $P(x), U(x), V(x)$:

$$\begin{aligned} &\frac{dP}{dx} + \left(i\omega + \frac{1}{\tau}\right) U = 0, \\ &\frac{dU}{dx} + \left(i\omega + \frac{1}{\tau}\right) P - i\omega\beta V = 0, \\ &\left(i\omega + \frac{1}{\tau}\right) V - i\omega\beta P = 0. \end{aligned} \tag{5}$$

For the rest of the section we study the functions $P(x), U(x)$ only. The solution of (5) can be written as a combination of the two exponents:

$$P(x) = P_1 e^{kx} + P_2 e^{-kx}, \quad U(x) = -P_1 \frac{k}{\Omega} e^{kx} + P_2 \frac{k}{\Omega} e^{-kx}, \tag{6}$$

where

$$k = \sqrt{\Omega^2 + \beta^2\omega^2}, \quad \Omega = i\omega + \frac{1}{\tau}, \tag{7}$$

and P_1, P_2 are constants which can be found from the boundary conditions for PML.

Our goal is to prove that the choice of the right boundary condition does not affect the absorbtion properties of the PML provided the relaxation time τ is sufficiently small. As the left boundary condition we take the value of the Riemann invariant which comes to the PML from the computational domain. This means that this Riemann invariant is continuous across the interface between the PML and computational domain. We remark that this requirement is reasonable from the point of view of the theory of hyperbolic equations. Thus we take

$$P(0) + U(0) = 1 + \alpha.$$

Using the representation (6) of the solution we obtain the following relation between P_1 and P_2:

$$P_2 = -\frac{\Omega - k}{\Omega + k} P_1 + (1+\alpha)\frac{\Omega}{\Omega + k}.$$

Now expressions for $P(x)$ and $Q(x)$ can be transformed into

$$P(x) = P_1 \frac{(\Omega + k)e^{kx} - (\Omega - k)e^{-kx}}{\Omega + k} + (1+\alpha)\frac{\Omega}{\Omega + k}e^{-kx}$$
$$U(x) = -P_1 k \frac{(\Omega + k)e^{kx} + (\Omega - k)e^{-kx}}{\Omega(\Omega + k)} + (1+\alpha)\frac{k}{\Omega + k}e^{-kx}. \quad (8)$$

Here only one constant P_1 must be determined with the use of right boundary condition for PML.

We are interested in the invariant $P(x) - U(x)$ and its value at $x = 0$, because this value gives an estimate of the amplitude of waves generated by the PML and propagating into the main computational region. Such a wave propagates into main computational region and must be damped by the PML. This invariant can be easily obtained from (8):

$$P(x) - U(x) = P_1 \frac{(\Omega + k)^2 e^{kx} - (\Omega - k)^2 e^{-kx}}{\Omega + k} + (1+\alpha)\frac{\Omega - k}{\Omega + k}e^{-kx} \quad (9)$$

and its value at $x = 0$ is

$$P(0) - U(0) = P_1 \frac{2\Omega k}{\Omega + k} + (1+\alpha)\frac{\Omega - k}{\Omega + k}. \quad (10)$$

We shall study an asymptotic solution behavior assuming τ sufficiently small. Then the following asymptotic formula can be used:

$$k = \frac{1}{\tau} + i\omega + \frac{\beta^2}{3}\omega^2\tau + O(\tau^2) = \Omega + \frac{\beta^2}{3}\omega^2\tau + O(\tau^2). \quad (11)$$

Now we derive the solution for the case of general right boundary condition for the PML which we take in the form

$$aP(h) + bU(h) = 0,$$

where a, b are an arbitrary constants. Using (8) we obtain the value for P_1:

$$P_1 = -\frac{(1+\alpha)(a\Omega + bk)e^{-kh}}{(a\Omega - bk)(\Omega + k)e^{kh} - (a\Omega + bk)(a\Omega - bk)e^{-kh}}. \quad (12)$$

Hence the solution and the invariant $P(x) - U(x)$ in particular can be obtained by substituting (12) into (8) and (9) accordingly.

Now the study of the asymptotic behavior should be based on the fact that

$$\tau^{-\gamma} e^{-\frac{h}{\tau}} \to 0, \quad \text{if} \quad \tau \to 0,$$

where $\gamma \geq 0$. Using this asymptotic behavior one can prove that

$$P_1 \backsim e^{-\frac{2h}{\tau}} \to 0, \quad \text{if} \quad \tau \to 0.$$

This leads us to the conclusion that the asymptotic behavior of the solution, invariant $P(x) - U(x)$ and its value at $x = 0$ are as follows:

$$P(x) \backsim (1+\alpha)\frac{\Omega}{\Omega + k} e^{-kx},$$

$$U(x) \backsim (1+\alpha)\frac{k}{\Omega + k} e^{-kx},$$

$$P(x) - U(x) \backsim (1+\alpha)\frac{\Omega - k}{\Omega + k} e^{-kx},$$

$$P(0) - U(0) \backsim (1+\alpha)\frac{\Omega - k}{\Omega + k}.$$

In particular we have

$$P(0) - U(0) \simeq -(1+\alpha)\frac{\beta^2\omega^2}{6}\tau^2 + O(\tau^3).$$

So the asymptotic value for the Riemann invariant which generates waves propagating from the PML to the computational domain is of order τ^2 and does not depend on the type of right boundary condition. Moreover the influence of the angle of incidence β and frequency ω of incoming wave can be made negligible. This allows us to choose a very simple numerical algorithm for the computations at the right PML boundary.

4 Numerical example

Here we show some numerical results as applied to the two-dimensional hyperbolic systems. The proposed idea is mostly easily implemented in the framework of one-step Godunov-type methods. For background information see e.g. [11, 7]. The operator $\mathcal{L}$ in (3) takes the following form:

$$\mathcal{L}_{ij} = -\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} - \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y}$$

and the PML update formula is given by

$$Q_{ij}^{n+1} = \frac{\frac{\Delta t}{\tau} Q_\infty}{1 + \frac{\Delta t}{\tau}} + \frac{Q_{ij}^n + \Delta t \mathcal{L}_{ij}(Q^n)}{1 + \frac{\Delta t}{\tau}} \tag{13}$$

Alternatively, one can use advection schemes with Runge-Kutta time marching, e.g. [6]. In this case the relaxation step for the source term is executed after the all Runge-Kutta stages are carried out. Effectively, we could regard this as a time-splitting procedure. The whole time step dt is divided into two sub steps: i) carry out Runge-Kutta method ii) apply the relaxation step as given by (13) but without the spatial operator:

$$Q_{ij}^{n+1} = \frac{\frac{\Delta t}{\tau} Q_\infty + Q_{ij}^{RK}}{1 + \frac{\Delta t}{\tau}} \tag{14}$$

where Q_{ij}^{RK} is the value of the vector of conservative variables, obtained after the Runge-Kutta time stepping.

Below we use the one-step ADER3 scheme, see [12, 10] and references therein. This scheme is uniformly third-order accurate in time and fifth order accurate in space.

We first tested our method on a two-dimensional explosion test problem for the non-linear two-dimensional Euler equation from [11]. In the computations we take the width of the PML layer to be equal to ten cells. The relaxation time is taken to be $\tau = 10^{-2}$ inside the layer when the equations are written in a conventional non-dimensional form. This problem is an analog of the shock-tube problems in one space dimension. The initial condition consists of two regions of constant but different values of gas parameters separated by a cylindrical surface. The solution involves a cylindrical shock wave leaving the computational domain and is thus appropriate for assessing the robustness of the method. Numerical results omitted here show that the proposed PML algorithm works well without any stabilization or filtering used in [4, 5].

Secondly, we apply the method to a standard acoustic test problem [4, 5]. We solve the two-dimensional linearized Euler equations of the form (1) with (again in the non-dimensional form)

$$Q = \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix}, \quad F(Q) = \begin{pmatrix} u_0 & \rho_0 & 0 & 0 \\ 0 & u_0 & 0 & 1/\rho_0 \\ 0 & 0 & u_0 & 0 \\ 0 & \gamma p_0 & 0 & u_0 \end{pmatrix} Q, \quad G(Q) = \begin{pmatrix} v_0 & 0 & \rho_0 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & v_0 & 1/\rho_0 \\ 0 & 0 & \gamma p_0 & v_0 \end{pmatrix} Q$$

in a spatial domain of $[-50, 50] \times [-50, 50]$. Here we take $\gamma = 1.4$, $\rho_0 = p_0 = 1$, $v_0 = 0$, $u_0 = 0.5\sqrt{\gamma}$. The initial conditions include an acoustic pulse centered at (x_a, y_a) and a vorticity and entropy pulses both centred at (x_b, y_b) and are given by

$$\begin{aligned}
\rho &= \exp\left(-(r_a/3)^2 \log 2\right) + 0.1 \exp\left(-(r_b/4)^2 \log 2\right), \\
p &= \gamma \ \exp\left(-(r_a/3)^2 \log 2\right), \\
u &= \sqrt{\gamma}\, 0.05(y - y_b) \exp\left(-(r_b/4)^2 \log 2\right), \\
v &= -\sqrt{\gamma}\, 0.05(x - x_b) \exp\left(-(r_b/4)^2 \log 2\right)
\end{aligned}$$

where $r_{a,b}^2 = (x - x_{a,b})^2 + (y - y_{a,b})^2$, $(x_a, y_a) = (-25, 0)$, $(x_b, y_b) = (25, 0)$. Note that factors proportional to γ appear due to the fact that our choice of non-dimensional variables is different from that of [5, 4]. We use $\Delta x = \Delta y = 1$.

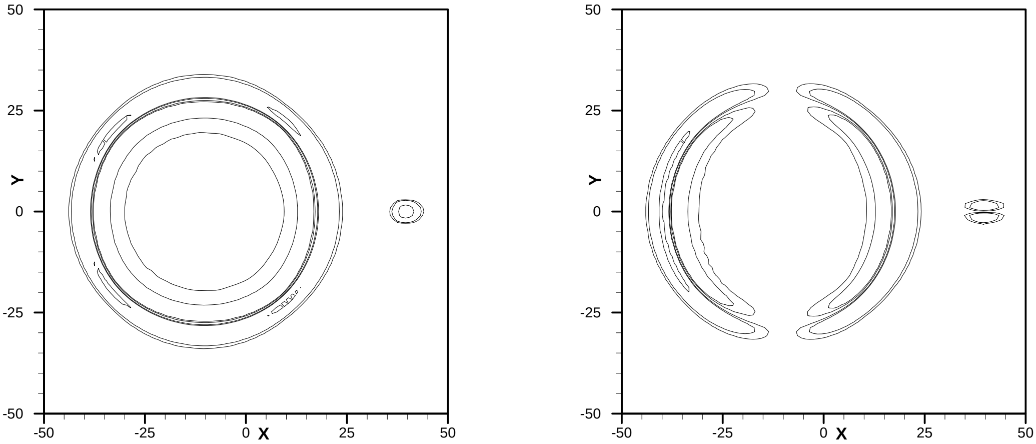

Figure 1: Density (left) and x component of velocity (right) for $t = 25$.

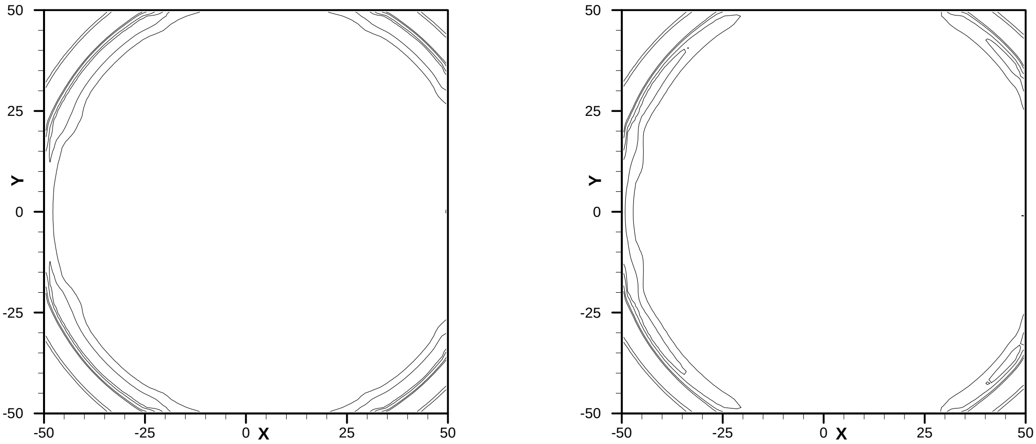

Figure 2: Density (left) and x component of velocity (right) for $t = 50$.

Figs. 1, 2 show the counter lines of density and x component of velocity for output times: $t = 25$ and $t = 50$. Comparing our results with those reported in the literature [5, 4] we conclude that their quality is comparable. No reflections from the boundary take place.

As time elapses, the pressure waves leave the computational domain. For example, Fig. 3 illustrates pressure distribution along the x axis for $t = 60$. Here symbols correspond to our numerical solution whereas the solid line represents the reference solution obtained on a larger domain. As is seen, no spurious waves reflect from the PLM layer back to the domain.

5 Conclusions

We have presented a new variant of the perfectly matched layer approach to construction of non-reflecting boundary conditions. The method is exceedingly simple, robust, does not involve altering of the governing equations and does not need any filters for stability. Numerical results demonstrate that its performance is similar to the other PML schemes,

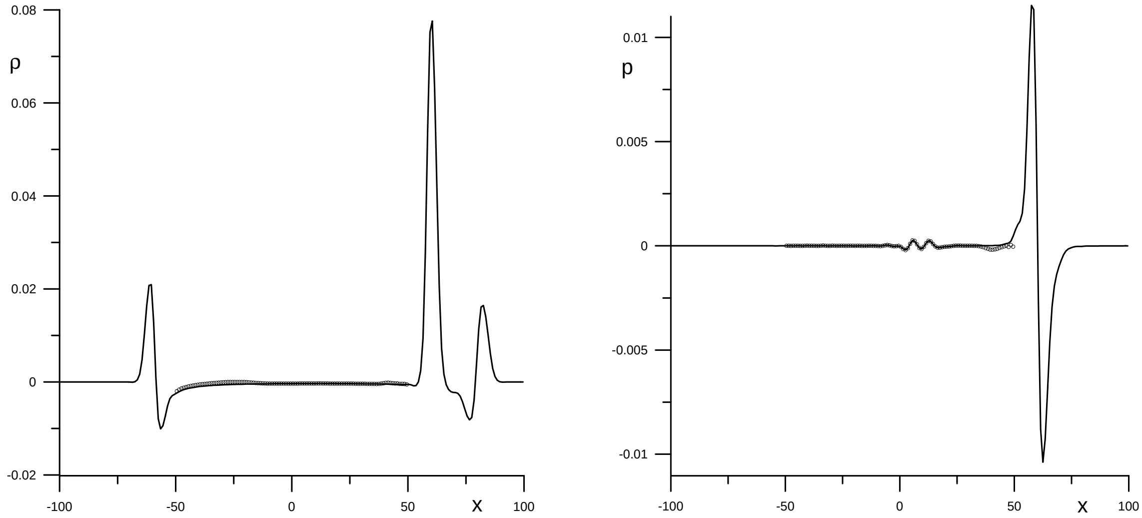

Figure 3: Density (left) and pressure (right) distribution along the x axis for $t = 60$.

presented in the literature.

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