A SIMPLE PRODUCT FORMULA FOR CERTAIN KAZHDAN-LUSZTIG R-POLYNOMIALS

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ABSTRACT. We obtain a simple product formula for the Kazhdan-Lusztig R-polynomials, $R_{u,v}(q)$ indexed by permutations "u, v" in the case that v is obtained from u, applying two transpositions (i, j) and (k, l), for $i < k < l < j$. These results are proved combinatorially and include the main result of [7], Sect.5, as special case.

1. INTRODUCTION

The theory of the Kazhdan-Lusztig R-polynomials arises from the Hecke algebra associated to a Coxeter group W (see e.g.[3], Chap.7]) and was introduced by Kazhdan and Lusztig ([4],Sect.2]) with the aim of proving the existence of another family of polynomials, the so-called Kazhdan-Lusztig polynomials. The R-polynomials, as Kazhdan-Lusztig polynomials, are indexed by pairs of elements of W and they are related to the Bruhat order of W. Most of the importance of these polynomials comes from their applications in different contexts, such as topology, algebraic geometry of Schubert varieties and representation theory. Moreover the importance of the R-polynomials stems mainly from the fact that they allow the computation of the Kazhdan-Lusztig polynomials. Although the explicit calculation of the R-polynomials is easier than Kazhdan-Lusztig polynomials calculation, one encounters hard problems to find closed formula for them, even when W is the simmetric group. In recent years purely combinatorial rules to compute the R-polynomials have been found, (see, e.g., [2]). These rules not only make these objects more concrete, but also allow combinatorial reasoning and techniques to be applied to them.

Our aim in this paper is to show that R-polynomials which are indexed by a pair of permutations (u, v), where $v = u(i, j)(k, l)$ and $i < k < l < j$, and $u[k] > u[l]$, factor nicely. This result includes the one proved in [7], which is the case that $l = k + 1$.

The organization of the paper is the following. In the next section we recall some basic definitions, notations, and results, both of an algebraic and combinatorial nature that will be used afterwards. In the third section we define R-polynomials and $\tilde{R}$-polynomials of the simmetric group and recall their properties. In section 4 we exhibit a closed formula for the R-polynomials indexed by the class of permutations described above.

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2. NOTATION AND PRELIMINARIES

In this section we collect some definitions, notation, and results that will be used in the rest of the paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, and $\mathbf{Z}$ be the set of integers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, 3, \dots, a\}$, where $[0] \stackrel{\text{def}}{=} \emptyset$. Given n, $m \in \mathbf{P}$, $n \leq m$, we let $[n, m] = [m] \setminus [n-1]$. We write $S = \{a_1, \dots, a_r\}_<$ to mean that $S = \{a_1, \dots, a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted with $|A|$. Given a set T we will let $S(T)$ be the set of all bijections of T in itself and $S_n \stackrel{\text{def}}{=} S([n])$.

If $\sigma \in S(T)$ and $T = \{t_1, \dots, t_r\}_< \subseteq \mathbf{P}$ then we write $\sigma = \sigma_1 \dots \sigma_n$ to mean that $\sigma(t_i) = \sigma_i$, for $i = 1, \dots, n$. If $\sigma \in S_n$ then we will also write σ on *disjoint cycle form* (see, e.g., [[8],p.17]) and we will not usually write the 1-cycles of σ. For example, if $\sigma = 365492187$ then $\sigma = (1, 3, 5, 9, 7)(2, 6)$.

Given $\sigma, \tau \in S_n$ then $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(1, 3) = (1, 3, 2)$. We refer to [3] for general Coxeter group notation and terminology. Given a Coxeter system (W, S) and $\sigma \in W$ we denote

$$D(\sigma) \stackrel{\text{def}}{=} \{s \in S : \ell(\sigma s) < \ell(\sigma)\};$$

$D(\sigma)$ is called the *descent set of* σ. An element of $D(\sigma)$ is also called *right descent*, this is because we can consider a left-handed property and define the *left-descent set of* σ as

$$D_L(\sigma) \stackrel{\text{def}}{=} \{s \in S : \ell(s\sigma) < \ell(\sigma)\}.$$

We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{wsw^{-1} : s \in S, w \in W\}$, which is called the *reflection set of* W.

We will always assume that W is partially ordered by (strong) *Bruhat order*. We recall (see, e.g., [[3], Sect 5.9]) that this means that if $u, v \in W$, $u \leq v$ iff $\exists t_1, \dots, t_r \in T$, for $r \in \mathbf{N}$ such that:

(i) $v = ut_1t_2 \dots t_r$
(ii) $\ell(ut_1 \dots t_{i+1}) = \ell(ut_1 \dots t_i)$ for $i = 0, \dots, r-1$.

The polynomials $R_{x,w}(q)$ defined by the next theorem are called the *R-polynomials of W*:

Theorem 2.1. *There is a unique family of polynomials* $\{R_{x,w}(q)\}_{x,w \in W} \subseteq \mathbf{Z}[q]$ *such that:*

(i) $R_{x,w}(q) = 0$, *if* $x \not\leq w$;
(ii) $R_{x,w}(q) = 1$, *if* $x = w$;
(iii) $R_{x,w}(q) = \begin{cases} R_{xs,ws}(q), & \text{if } s \in D(x) \\ (q-1)R_{x,ws}(q) + qR_{xs,ws}(q), & \text{if } s \notin D(x) \end{cases}$
if $x < w$ *and* $s \in D(w)$.

See[[3], Sect. 7.5] for a proof.

This theorem gives an inductive procedure to compute the R-polynomials of W because $\ell(ws) < \ell(w)$.

From now on we assume $W = S_n$ and $S = \{s_1, \dots, s_{n-1}\}$, where $s_i \stackrel{\text{def}}{=} (i, i+1)$, for $i \in [n-1]$. For this Coxeter group combinatorial descriptions of Bruhat order, lenght function and descent set, are well known and synthetized in the following results. For $u \in S_n$ and $i \in [n]$, let $\{u^{i,1}, \dots, u^{i,i}\}_< \stackrel{\text{def}}{=} \{u(1), \dots, u(i)\}$.

Theorem 2.2. *Let $u, v \in S_n$. Then $u \leq v$ iff $u^{i,j} \leq v^{i,j}$ for every $1 \leq j \leq i \leq n-1$.*

A proof of this result can be found in [5], Chapter 1.

For example: if $u = 14325$ and $v = 52341$ then $(u^{1,1}, u^{2,1}, u^{2,2}, u^{3,1}, u^{3,2}, u^{3,3}, u^{4,1}, u^{4,2}, u^{4,3}, u^{4,4}) = (1,1,4,1,3,4,1,2,3,4)$ and $(v^{1,1}, v^{2,1}, v^{2,2}, v^{3,1}, v^{3,2}, v^{3,3}, v^{4,1}, v^{4,2}, v^{4,3}, v^{4,4}) = (5,2,5,2,3,5,2,3,4,5)$, so $u < v$.

Proposition 2.3. *Let $w \in S_n$, and $i \in [n-1]$. Then*

(i) $\ell(w) = inv(w) \stackrel{\text{def}}{=} |\{(i,j) \in [n] \times [n] : i, j, w(i) > w(j)\}|$*; the number inv(w) is usually known as inversions of w.*
(ii) $s_i \in D(u)$ *iff* $u(i) > u(i+1)$.

We refer the reader to [5] for a proof. Note that the proposition implicitly provides also a characterization of the left descents because $s_i \in D_L(u)$ iff $s_i \in D(u^{-1})$.

For example, if $u = 13524$ then $inv(u) = |\{(2,4),(3,4),(3,5)\}| = 3$, $D(u) = \{(3,5)\}$ and $D_L(u) = \{(2,3)(4,5)\}$. In the rest of the paper a (right) descent $(i, i+1)$ may be written briefly as i.

Finally we introduce a distance-function on S_n, it will be used on the proof of the main result of this paper: for $u \in S_n$, let $d(u,v) \stackrel{\text{def}}{=} \max\{i \in [n] : u^{-1}(i) \neq v^{-1}(i)\}$, where $\max\{\emptyset\} \stackrel{\text{def}}{=} 0$.

For example, $d(18263574, 28745361) = \max\{1,2,3,4,5,6,7\} = 7$. For the properties of this function we refer the reader to [1].

3. THE R-POLYNOMIALS OF THE SIMMETRIC GROUP

In this section we introduce the family of $\tilde{R}_{u,v}(t)$, which gives a combinatorial interpretation of the R-polynomial of S_n, see [1]. We define these polynomials in the next:

Theorem 3.1. *Let $u, v \in S_n$; then there exists a unique polynomial $\tilde{R}_{u,v}(q) \subseteq \mathbf{N}[q]$ such that*

$$R_{u,v} = q^{(\ell(v)-\ell(u))/2} \tilde{R}_{u,v}(q^{1/2} - q^{-1/2}).$$

This is the fundamental result in [[1], Corollary 3.8]

From Theorem 2.1 and Theorem 3.1 it follows that

Theorem 3.2. *Let $u, v \in S_n$ such that $u \leq v$.Then, for every $s \in D(v)$, we have that*

$$\tilde{R}_{u,v}(t) = \begin{cases} \tilde{R}_{us,vs}(t), & \text{if } s \in D(u) \\ \tilde{R}_{us,vs}(t) + t\tilde{R}_{u,vs}(t), & \text{if } s \notin D(u). \end{cases}$$

We note that one of the advantages of working with the polynomials $\tilde{R}_{u,v}(t)$ is that they have positive coefficients while the R-polynomials have integer coefficients, and thanks to Theorem 3.1 every result on the $\tilde{R}$-polynomials can be traslated into a result on R-polynomials. Moreover Theorem 3.2 gives an inductive procedure to compute $\tilde{R}_{u,v}(t)$ since inv($v(i, i+1)$)=inv(v) $- 1$.

There is one more general fact on the $\tilde{R}_{u,v}(t)$ which we will use:

Proposition 3.3. *Let $u, v \in S_n$; then*

$$\tilde{R}_{u,v}(t) = \tilde{R}_{u^{-1},v^{-1}}(t) = \tilde{R}_{n+1-v(1)\dots n+1-v(n), n+1-u(1)\dots n+1-u(n)}(t) = \tilde{R}_{v(n)\dots v(1), u(n)\dots u(1)}(t).$$

The above result can be proved easily using properties of Hecke algebra and 3.1 [see [3],Proposition 7.6]. The left version of Theorem 3.2 follows easily from 3.3: Let $u, v \in S_n$ such that $u \leq v$. Then, for every $s \in D_L(v)$, we have

$$\tilde{R}_{u,v}(t) = \begin{cases} \tilde{R}_{su,sv}(t), & \text{if } s \in D_L(u) \\ \tilde{R}_{su,sv}(t) + t\tilde{R}_{u,sv}(t), & \text{if } s \notin D_L(u). \end{cases}$$

To conclude this collection of results we give the next:

Lemma 3.4. *Let* $1 < k < l < n$. *Then*

$$\tilde{R}_{(k,l),(1,n)}(t) = (1+t^2)\tilde{R}_{(n-l+1,n-k+1),(1,n-1)}(t),$$

for every $(k,l) \neq (2, n-1)$

This lemma has been proved in [7],lemma 5.3, for the case $l = k+1$. However that proof carries over to give this more general result.

Finally we note that a general closed formula for the R-polynomials does not exist; for example,

$$\tilde{R}_{12345,54321}(t) = t^2(1 + 5t^2 + 10t^4 + 6t^6 + t^8)$$

and

$$\tilde{R}_{123456,654321}(t) = t^3(1 + 9t^2 + 39t^4 + 57t^6 + 36t^8 + 10t^{10} + t^{12}),$$

and these factors are irreducibile over the field of rational numbers.

However, there are several general classes of permutations for which explicit formulas exist. We refer the reader to [2] for a survey of the main results known in this direction.

4. MAIN RESULT

In this section we prove our main result: it includes the formula contained in [7], Theorem 5.4. This result is a product formula for the R-polynomials of a pair of permutations (u, v), where v is obtained by swapping four elements of u, i.e. $v = u(i,j)(k,l)$ for $i < k < l < j$ and $u(l) < u(k)$ with $k + 1 \leq l$. The proof is based on lemma 4.2 which permit to reduce the computation of $\tilde{R}_{u,v}(t)$ to the one of $\tilde{R}_{(k,l),(1,n)}(t)$.

Lemma 4.1. *Let* $u \in S_n$, $u(k) > u(l)$, $1 \leq i < k < l \leq j$ *and suppose that* $v = u(i,j)(k,l)$. *Then* $u \leq v$ *iff* $u(i) < u(l) < u(k) < u(j)$.

The above Lemma follows from Theorem 2.2, we leave its verification to the reader.

Now we prove the fundamental:

Lemma 4.2. *Let* $u \in S_n$, $u(k) > u(l)$, $1 = i < k < l < j = n$ *and suppose that* $v = u(1,n)(k,l)$. *If* $D(u) \cap D(v) = \emptyset$, $u(1) = 1$ *and* $u(n) = n$, *then* $\tilde{R}_{u,v}(t) = \tilde{R}_{(k,l),(1,n)}(t)$.

Proof. We assume $k + 1 < l$, the result being known for $l = k + 1$.

Consider $u = 1u(2)u(3)\dots u(k-1)u(k)u(k+1)\dots u(l-1)u(l)u(l+1)\dots u(n-1)n$, thus $v = nu(2)u(3)\dots u(k-1)u(l)u(k+1)\dots u(l-1)u(k)u(l+1)\dots u(n-1)1n$. By the assumptions we have the following characterization of the sets $D(u)$ and $D(v)$:

- $D(u) = \{k, l-1\}$.
  From $D(u) \cap D(v) = \emptyset$ and $u(k) > u(l)$ it follows that $u(l) < u(k+1)$ (in fact if $u(k+1) < u(l)$, then $k \in D(u) \cap D(v)$), and this together with the fact that $u(k+1) < u(k+2) < \cdots < u(l-1)$ implies that $l-1 \in D(u)$. Moreover from the previous considerations we can easily deduce that $u(k+1) < u(k)$ because $u(l-1)$ must be less than $u(k)$ (the contrary will produce the contradictions that $l-1 \in D(u) \cap D(v)$) hence $k \in D(u)$.
- $\{1, k-1, l, n-1\} \supseteq D(v) \supseteq \{1, n-1\}$

From this characterization, it follows that there are four cases to consider depending on $D(v)$.
We begin by considering the simplest case, $\boldsymbol{D(v) = \{1, n-1\}}$.
This forces that $1 < u(2) < u(3) < \cdots < u(k-1) < u(l) < u(k+1) < u(k+2) < \cdots < u(l-1) < u(k) < u(l+1) < \cdots < u(n-1) < n$ then $u(i) = i$ for every $i \in [k-1] \cup [k-1, l-1] \cup [l+1, n-1]$, $u(l) = k$ and $u(k) = l$. Therefore $u = (k, l)$ and $v = (1, n)$.

Suppose that $\boldsymbol{D(v) = \{1, k-1, n-1\}}$.
Since $k-1 \in D(v)$ then being $u(k-1) > u(l)$, there are two possibilities:

(a) $u(k-1) < u(k+1)$;
(b) $u(k-1) > u(k+1)$.

In every case $u(j) = j$ for every $j \in [l+1, n-1]$ and $u(k) = l$.

We define $p \stackrel{\text{def}}{=} max\{m \in \mathbf{N} : u(m) < u(l)\}$ then
$1 < u(2) < \cdots < u(p-1) < u(p) < u(l) < u(p+1) < \cdots < u(k-1)$. If (a) holds we can easily complete the above inequality chain because it must be $u(k-1) < u(k+1) < \cdots < u(l-1) < u(k) = l$; in other worlds we have completely determined the permutations u and v that satisfy these hypothesis: $u = 1 \ldots p(p+2) \ldots kl(k+1)(k+2) \ldots (l-1)(p+1)(l+1) \ldots (n-1)n$ and $v = n \ldots p(p+2) \ldots k(p+1)(k+1)(k+2) \ldots (l-1)l(l+1) \ldots (n-1)1$.
We see that $p+2$ is on the left of $p+1$ both in u and in v, so $(p+1) \in D_L(v) \cap D_L(u)$. If we apply Theorem 3.2 (left version) to this descent $s = (p+1, p+2)$, we obtain a pair (su, sv) which has the common left descent $(p+2, p+3)$. If we go in this way we obtain a pair (u_1, v_1) which has $(k-1, k)$ as common left descent, more precisely $u_1 = 12 \ldots (k-2)kl(k+1) \ldots (l-1)(k-1)l+1 \ldots n$ and $v_1 = n2 \ldots (k-2)k(k-1)(k+1) \ldots (l-1)l(l+1) \ldots n$; and we have done because $(k-1, k)u_1 = (k, l)$ and $(k-1, k)v_1 = (1, n)$. We illustrate the same situation looking the inverse permutations:

$$\begin{array}{lllll|c|lllllllll}
u^{-}1 = 1\ 2 & & p-1 & l & p+1 & & k-2 & k-1 & k+1 & & l-1 & k & l+1 & & n-1\ n \\
v^{-}1 = n\ 2 & \cdots & p-1 & k & p+1 & \cdots & k-2 & k-1 & k+1 & \cdots & l-1 & l & l+1 & \cdots & n-1\ 1
\end{array}$$

The described application of Theorem 3.2 is equivalent to moving the $\substack{l \\ k}$ column between $\substack{k-1 \\ k-1}$ and $\substack{k+1 \\ k+1}$.

If (b) holds, i.e. $u(k-1) > u(k+1)$, we need to consider two other parameters $t \stackrel{\text{def}}{=} max\{m \in [p+1, k-2] : u(m) < u(k+1)\}$ and $s \stackrel{\text{def}}{=} max\{m \in [k+2, l-1] : u(m) < u(k-1)\}$, thus we have $p < u(l) < u(p+1) < \cdots < u(t) < u(k+1) < u(t+1) < \cdots < u(s) < u(k-1) < u(s+1)$. Since $u(k) = l$, we have that $u(j) = j$, for every $j \in [s+1, l-1]$, $u(k-1) = s$, $u(s) = s-1$, $u(k+2) = k+1$.

On the other side, by definition of p and t, we can conclude that $u(l) = p+1$, $u(p+1) = p+2, \dots, u(t) = t+1, u(k+1) = t+2, u(t+1) = t+3$. This implies that $\{(s-1,s),(t+1,t+2)\} \in D_L(u) \cap D_L(v)$ and by repeated application of Theorem 3.2, we can "eliminate" these left descents to obtain a pair of permutations of the type contained in (a) and so then to the first case.

Obviously to conclude that if $k-1 \in D(v)$ then u and v have a common left descent, it is enough to observe that in both cases, (a) and (b), $(p+1,p+2)$ is a left descent common to u and v; but we decided to be redundant to see better the structure of these permutations at least in one case.

Suppose that $\boldsymbol{D(v) = \{1, l, n-1\}}$.
Under this hypothesis it must be $1 < u(2) < \cdots < u(k-1) < u(l) < u(k+1) < \cdots < u(l-1)$, so that $u(l) = k$, and $u(j) = j$ for every $j \in [2, k-1]$. Being $u(k) > u(l+1)$, we need to define $h \overset{\text{def}}{=} max\{m \in [l+1, n-1] : u(m) < u(k)\}$, then $u(j) = j$ for every $j \in [h+1, n-1]$, $u(k) = h$. There are the next possibilities to consider:

(a) $u(l-1) < u(l+1)$;
(b) $u(l-1) > u(l+1)$.

If (a) holds, then $u(l-1) < u(l+1) < \cdots < u(h) < u(k) < u(h+1)$ thus

$$\begin{array}{lllllllllllllll} u = 1 & 2 & & k-1 & \boxed{h} & k+1 & & l-1 & k & l & & h-1 & h+1 & & n-1 \;\; n \\ v = n & 2 & \cdots & k-1 & k & k+1 & \cdots & l-1 & \boxed{h} & l & \cdots & h-1 & h+1 & \cdots & n-1 \;\; 1 \end{array}.$$

Observe that $(h-1,h) \in D_L(v) \cap D_L(u)$ and we apply theorem 3.2 as illustrated before.

If (b) holds, we define $s \overset{\text{def}}{=} max\{m \in [k+1, l-2] : u(m) < u(l+1)\}$, (note that if this set is empty then $u(l) < u(l+1) < u(k+1)$ that implies $u(l+1) = k+1$) and $t \overset{\text{def}}{=} max\{m \in [l+1, h-1] : u(m) < u(l+1)\}$, hence $k-1 < u(l) < u(k+1) < \cdots < u(s-1) < u(s) < u(l+1) < u(s+1) < \cdots < u(t) < u(l-1) < u(t+1) < \cdots < u(h) < u(k) < u(h+1)$. This means that for $j \in [k] \cup [h+1, n]$, u and v are as in (a), while from $k+1$ to h the two permutations are as follow

$$\begin{array}{lllllllllllllll} \boxed{h} & k+1 & & s-1 & s & s+2 & & t-1 & k & s+1 & & t-2 & t & & h-1 \;\; h+1 \\ k & k+1 & \cdots & s-1 & s & s+2 & \cdots & t-1 & \boxed{h} & s+1 & \cdots & t-2 & t & \cdots & h-1 \;\; h+1 \end{array}.$$

Suppose that $\boldsymbol{D(v) = \{1, k-1, l, n-1\}}$. We know that if $k-1 \in D(v)$ or $l \in D(v)$ then $D_L(u) \cap D_L(v) \neq \emptyset$ so by our assumption we reduce again to the first case.

□

We now prove the main result of this paper.

Theorem 4.3. *Let $u \in S_n$, $u(k) > u(l)$, $1 \leq i < k < l < j \leq n$ and suppose that $v = u(i,j)(k,l)$. Then*

$$\tilde{R}_{u,v}(t) = t^4(1+t^2)^{(inv(v)-inv(u)-4)/2}$$

Proof. We can assume that $i = 1$, $j = n$ and $u(1) = 1$, $u(n) = n$ (this follows from Lemma 4.1 and Proposition 3.3). First we consider the case that $k = 2$ and $l = n-1$, $u = (2, n-1)$ and $v = (1, n)$. In this situation $inv(u) = 2n-7$ while $inv(v) = 2n-3$

so $inv(v) - inv(u) = 4$, and it is easy calcutation that $\tilde{R}_{(2,n-1),(1,n)}(t) = t^4$.

Now we consider $(k,l) \neq (2, n-1)$.
We proceed by induction on $d = d(u,v) = max\{i \in [n] : u^{-1}(i) \neq v^{-1}(i)\}$, and we observe that by definition of u and v we have $d = n$; moreover the case d=4 is trivially true.
It follows from Theorem 3.2 that we can suppose $D(u) \cap D(v) = \emptyset$, so by Lemma 4.2 we have to compute only $\tilde{R}_{(k,l),(1,n)}(t)$ By lemma 3.4 we have that $\tilde{R}_{(k,l),(1,n)}(t) = (1+t^2)\tilde{R}_{(n-l+1,n-k+1),(1,n-1)}$. Observe that $inv((n-l+1, n-k+1)) = 2(l-k)-2 = inv(k,l)-1$ and $inv(1,n-1) = inv(1,n)-1$. We can apply the inductive hypothesis on $\tilde{R}_{(n-l+1,n-k+1),(1,n-1)}$ being $d = n-1$, thus $\tilde{R}_{(k,l),(1,n)}(t) = (1+t^2)t^4(1+t^2)^{(inv(1,n)-1-inv(k,l)-1-4)/2)} = t^4(1+t^2)^{(inv(1,n)-inv(k,l)-4)/2}$, so the thesis. □

Corollary 4.4. *Let* $u \in S_n$, $u(k) > u(l)$, $1 \leq i < k < l < j \leq n$ *and suppose that* $v = u(i,j)(k,l)$. *Then* $R_{u,v}(q) = (q-1)^4(q^2-q+1)^{(inv(v)-inv(u)-4)/2}$

REMARK : Theorem 4.3 can be deduced from lemma 4.2 and from [6], Corollary 4.2, but for the readers convenience we prefered to give a self-contained proof.

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