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Non trivial string backgrounds: Tachyons in String Field Theory and Plane-waves in DLCQ Strings

Doctoral Thesis of

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Introduction

In its usual first-quantized formulation, string theory was understood only as a *perturbative* theory, arising from the study of S-matrices and conceived of as a new tool to describe interactions of massless particles including gravitons as well as the infinite tower of massive particles associated with excited string states. The usual perturbative approach to string theory is that of a first-quantized theory whose degrees of freedom are the fields living on the two-dimensional world-sheet described by the string and mapped into the "target" space-time as a set of coordinates X^μ. The world-sheet Polyakov action provides the means to derive any amplitudes of external *on-shell* strings, by calculating correlation functions on the string world-sheet using techniques of two-dimensional conformal field theory.

String Field Theory and the AdS/CFT correspondence, each of which is analyzed in some aspects in this thesis, arise as true *nonperturbative* descriptions of string theory. Though in completely different frameworks - the latter is formulated in the fixed specetime background $\mathrm{AdS}_5 \times S^5$ while the former should be in principle background independent - these two approaches should in some sense fulfill one of the primary theoretical goals of string theory: the formulation of a nonperturbative theory of quantum gravity.

In this thesis, two main approaches to String Field Theory (SFT) will be analyzed, Cubic SFT [1] and Boundary SFT [2, 3, 4, 5]. While in the former the basic idea is to promote to spacetime gauge symmetries those invariances which ensure physical consistency to the world-sheet theory, the latter addresses the question of finding an appropriate gauge invariant Lagrangian on the "space of all two-dimensional field theories".

One of the most interesting problems in string theory, that goes beyond those accessible to the perturbative formulation of the theory, is to understand how the background space-time on which the string propagates arises in a self-consistent way. Due to the existence of a tachyon in the bosonic string theory, the 26-dimensional Minkowski space background about which the string is quantized is unstable. An unstable state should decay to something and the nature of both the decay process and the endpoint of the decay are crucial questions. Some understanding of this process has been achieved for the open bosonic string, thanks to the key idea of Sen [6]. The open bosonic string tachyon reflects the instability of the D-25 brane. This unstable D-brane should decay by condensation of the open string tachyon field, *i.e.* by the tachyon building a nonzero vacuum expectation value so as to minimize its effective potential. A precise claim was formulated by Sen in a set of three conjectures, which we describe here for the case of open bosonic string theory [1] 1) The tachyon effective potential has a local minimum, and the difference in energy between the local maximum and the local minimum of the potential must exactly cancel the tension T_{25} of the space-filling $D25$-brane; 2) There are also intermediate unstable states, solitonic configuration of the tachyon condensate, which are the D-branes of all dimensions between 25 and 0; 3) The locally stable vacuum of the system is the

[1] These conjectures generalize to any unstable system of D-branes in string theory.

closed string vacuum, in which the $D25$-brane is absent and no conventional open string excitations exist [2].

The first and the second conjectures can be proved exactly in the context of Boundary String Field Theory (BSFT) [2]-[5]. It is formulated as a problem in boundary conformal field theory - a boundary interaction with arbitrary operators is added to the free world-sheet theory on the disk. The configuration space of open string field theory is then taken to be the space of all possible boundary operators modulo gauge transformations and field redefinitions. If the boundary perturbation involves terms which correspond to the massive states in the first-quantized open string Hilbert space, the theory may be ill-defined as a two-dimensional field theory - such terms represent non-renormalizable interactions on the world-sheet. Consequently, an ultraviolet cut-off parameter has in general to be introduced [7]. In the special case of tachyon condensation, however, one has only to deal with the relevant perturbation $T(X)$ [3] and the problem of ultraviolet divergences is avoidable. The tachyon field (not the whole string field) acquires the nonvanishing vacuum expectation value, so one can proceed without any approximation scheme [4]. This fact means that one can obtain some *exact* results about the off-shell tachyon physics, as the tachyon potential and the description of the D-branes as tachyonic solitons [10, 11]. If $T(X)$ and the other fields are adjusted so that the sigma model that they define is at an infrared fixed point of the renormalization group (RG) these couplings arise as background fields, being solutions of the classical equations of motion of string theory. Witten and Shatashvili [2, 4] have argued that these equations of motion come from an action which can be derived from the disk partition function Z through a prescription involving the β-functions of the couplings. The usual choice of coordinates in the space of string fields is the one in which the β-functions are exactly linear, and is well suited to studying processes which are far off-shell, such as tachyon condensation. A complete renormalization of the theory in fact makes the β-function *non-linear* in $T(X)$ [41] so that, since the vanishing of the β-function is the field equation for T, these nonlinear terms describe tachyon scattering. One of our goals, which will be directed in the first chapter of this thesis, is to construct non-linear expressions for the β-function which is valid away from the RG fixed point [13]. With this expression we shall derive the Witten-Shatashvili (WS) space-time action S in a very simple form, which is universal as it does not depend on how many couplings are switched on. The knowledge of the non-linear tachyon β-function is very important also for another reason. The solutions of the equation $\beta^T = 0$ give the conformal fixed points, the backgrounds consistent with the string dynamics. It is only with a non-linear β-function that the equations of motion for the WS action can be made identical to the RG fixed point equation $\beta^T = 0$ in the case of slowly varying tachyon profiles. The solitons thus derived, as we will see, are lower dimensional D-branes for which the finite value of S provides a quite accurate prediction

[2] A fundamental step forward our understanding of open SFT has been made recently by Schnabl [8]. An analytic form for the nontrivial vacuum of Cubic SFT was found, in which the first and the third [9] conjectures can be proved analytically.

[3] By classical power counting, the tachyon field has dimension one.

[4] Such as the *level truncation method* in Cubic String Field Theory.

of the D-brane tension.

The tachyon effective action up to the third power in the fields is known exactly also from Cubic SFT, and it seems clear that they should be related through a particular choice of coordinates on the space of string fields (or worldsheet couplings). We shall derive in Chapter 1 such a field-redefinition, showing that it is well suited to confirm the first Sen conjecture.

In the Cubic formulation of string field theory *all* the infinite number of degrees of freedom associated with the states in the string Fock space are encoded in a "string field". The latter can be thought of as a functional of string embeddings $\Psi\,[x^\mu(\sigma), c(\sigma), b(\sigma)]$, where x, c and b are the matter, ghost and antighost fields describing a one-dimensional string of coordinate σ. For concreteness, this string field is expanded in terms of first-quantized modes; for every vertex operator that appears in the first-quantized theory, there corresponds an element in the string field. String Field Theory is defined by giving an action functional $\mathcal{L}(\Psi)$ depending on the string field. Witten's proposal [1] for the bosonic open string consists in a quadratic term - which reproduces the free bosonic string in a BRST formalism - plus a cubic interaction, that from the point of view of conventional field theory seems more exotic, containing exponentials of derivatives. The appearance in the action of an infinite set of spacetime fields forced people working on the subject to adopt a systematic approximation scheme, known as *level truncation* [14], which can be used to solve numerically the theory. It involves dropping all fields over a certain level - which is defined as the sum of the level numbers of the creation operators acting on the oscillator vacuum to create the state associated with the field. Beyond this issue, the unbounded number of derivatives at all orders makes even the classical time-dependence of the string field difficult to analyze. The string field seems to obey a differential equation of infinite order, thus suggesting that an infinite number of boundary conditions are needed.

Despite these difficulties, the cubic action gives a systematic way of constructing perturbative string amplitudes in terms of vertices and propagators; in principle, this approach is easier to generalize than the world-sheet approach, that requires conformal field theory on higher genus Riemann surfaces. Indeed, formal arguments have demonstrated many years ago that in a particular gauge (Feynman-Siegel gauge) Witten's cubic bosonic open string field theory gives rise to a diagrammatic expansion which precisely covers the moduli space of Riemann surfaces of arbitrary genus with at least one boundary and with an arbitrary number of open string punctures on the boundaries [15, 16]. An example of this correspondence is the tree level 4-point Veneziano amplitude, which was first computed through conformal mapping techniques by Giddings [17], that reproduced it correctly in the Cubic SFT formalism. This calculation was generalized by Samuel and Sloan [29, 19] to off-shell momenta - a procedure that allows in general the calculation of any amplitude [20]. Explicit computations of perturbative amplitudes using string field theory, however, have only been done for the tree level 4-point function, as said, and the one-loop 2-point function [21]. These results are defined implicitly by involved relations between elliptic integrals. Although numerical approximations are necessary to get concrete numbers, amplitudes computed with this method are in principle exact and can be used to derive some very accurate results [22].

On the other hand, closed-form expressions for any perturbative amplitude even at higher loop order can be derived, rather than trying to relate string field theory calculations to conformal field theory, by using the oscillator representation [23] of the vertices and propagators in Cubic SFT and using standard squeezed states techniques [24]. The only complications is the appearance, in the final expression for each diagram, of infinite-dimensional matrices whose elements are called Neumann coefficients. While no analytical way is known at present to exactly calculate such expressions, one can evaluate the amplitudes to a high degree of precision truncating the Neumann matrices to finite size [25]. This truncation method, known as level truncation *on oscillators*, is more powerful than the method of level truncation *on fields* used to address the tachyon condensation problem in [14, 26, 27, 28], since rather than having to include a number of fields which grows exponentially in the level, we simply need to evaluate the determinant of some matrices whose size grows linearly in the truncation level.

In the second and third chapters of this thesis we will use all the technologies above described to carry on explicit calculations for the off-shell tachyon amplitudes. We will start by explicitly solving Samuel's elliptic equations for the four-tachyon amplitude, obtaining a new series solution for the off-shell factor in terms of the original coordinates used in [29]. We will perform then a level truncated analysis of the four tachyon amplitude, truncating *on oscillators*, to derive the quartic term in the tachyon effective action up to level $L = 14$. We will finally provide higher order terms in the effective action by using the standard field theory approach that makes use of the level truncation *on fields* [14].

From these numerical results we shall extract off-shell informations both on the non-perturbative stable vacuum and on the tachyon dynamics, improving the numerical approximation for the evaluation of the exact quartic self-coupling c_4 in the tachyon potential and the first few coefficients of a time-dependent solution of Cubic SFT expressed in powers of e^t.

String theory must eventually address cosmological issues and hence it is crucial to understand the role of time dependent solutions of the theory. The *rolling tachyon* is an example of such a solution describing the decay of unstable space-filling D-branes. An important aspect of the open string tachyon which is not yet fully understood, however, is the dynamical process through which the tachyon rolls from the unstable vacuum to the stable vacuum. A review of previous work on this problem is given in [30].

The boundary states approach to the rolling tachyon is the one that initiated the new investigation on time dependent solutions in string theory [31], and is based on the correspondence between conformal field theories and classical solutions of SFT equation of motion. Analogue computations using CFT and RG flow analysis show that the tachyon should monotonically roll towards the true vacuum, but should not arrive at the true vacuum in finite time [31]-[49]. In the decay, the energy density remains constant and the pressure approaches zero from negative values as the tachyon rolls toward its stable minimum.

These conformal field theory methods provide an indirect way of constructing solutions of the classical equations of motion without knowing the effective action. A more direct derivation of the classical solutions can be realized by explicitly constructing the tachyon effective action. Namely one starts from a string field theory in which, in principle, the coupling of the tachyon to the infinite tower of other fields associated with massive open string states could be taken into account. String field theory should then be a natural setting for the study of time dependent rolling tachyon solutions. In the Boundary SFT approach to string field theory [2] a rolling tachyon solution has been found that can be directly associated with a given two dimensional conformal field theory [49, 50, 51]. The strategy adopted is to deform the world-sheet CFT of an unstable D-brane by an exactly marginal operator and interpret the deformed CFT as a time-dependent solution to the classical string equations of motion. Starting from a bare homogeneous tachyon profile $T(t) = e^t$, interpreted as a perturbation at $t = -\infty$ displacing the tachyon infinitesimally from the unstable maximum, one derives the partition function of the world-sheet theory, the world-sheet action and finally the energy-momentum tensor. As expected from the CFT picture, the energy T_{00} associated with this tachyon profile does not depend on time, which is just the statement of conservation of energy. Moreover, one gets vanishing spatial components of the energy-momentum tensor, $T_{ij} \to 0$, as $t \to \infty$, so that the pressure vanishes and the decay product is pressureless matter, as in [31]. This form of *tachyon matter* could have astrophysical consequences and it then seems of utmost importance to confirm its existence.

The direct approach based on the analysis of the classical equations of motion of Cubic SFT [1] is generally believed to be equivalent to the approach based on two dimensional conformal field theory. This equivalence is however less than manifest, the rolling tachyon dynamics being in this framework much more complicated [5].

An analysis of CSFT time-dependent tachyonic solutions has been addressed in refs. [54, 55, 56, 22] by using level truncation methods. It turns out that the tachyon rolls down toward the vacuum, goes far beyond it then turns around and begins to oscillate with ever increasing amplitudes, still the energy being conserved. The pressure starts from negative value at time $t = 0$, forcing the tachyon roll to the vacuum. But instead of decreasing to zero as $t \to \infty$, it oscillates without bound at large times, thus not realizing tachyon matter. In [56], in particular, a systematic level-truncation analysis was carried out for a time-dependent tachyon solution expressed as a power series in e^t. We shall review this procedure in Chapter 3, where we will improve the accuracy of the first coefficients with the use of the exact four-tachyon off-shell amplitude derived in the previous chapter. The qualitative behavior of wild oscillations found in [56] is substantially reproduced, even if the amplitudes at the turnaround points beyond the first are sensibly diminished. An alternative procedure to find a solution of the CSFT equations of motion at the lowest order in the level truncation scheme will be presented [57], which shows many appealing analytic features but confirms a pathological behavior of the tachyon past the origin of time. In [56] an argument was done to reconcile this discrepancy, based on the field redef-

[5] Relevant exceptions, in a framework strictly related to Cubic SFT such as Vacuum SFT, are the time-dependent solutions interpolating between the nonperturbative and the true vacuum analyzed in [52, 53].

inition derived in Chapter 1 that takes the CSFT action to the BSFT action [13]. We will discuss this argument, rising the question whether this field redefinition is a real tool to reconcile the apparent discrepancy with the results of BSFT or the problem is still open.

The final part of this thesis concerns AdS/CFT correspondence, which in its purest form asserts an exact duality between a ten-dimensional type IIB superstring theory on $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in flat four dimensional Minkowski spacetime [58, 59, 60]. Though it has many spectacular successes, it is still a conjecture and it is not yet clear whether it is an exact correspondence, or is only valid in some limits of the two theories. Given its potential importance as a quantitative tool for strongly coupled gauge and string theory, it is important to check it wherever possible.

AdS/CFT is a strong coupling – weak coupling duality. This makes it powerful, as it can be used to compute the strong coupling limit of either theory using the weak coupling limit of the other. On the other hand, it makes it difficult to check since it is not easy to find situations where approximate computations in both theories have an overlapping domain of validity. More recently it has been realized that some large quantum number limits yield domains where accurate computations in both gauge theory and string theory could be done and compared directly with each other. The first and most powerful of these is the BMN limit [61]. It began with the observation [62, 63] that the Penrose limit of the $AdS_5 \times S^5$ metric and 5-form field strength of the string background are the maximally supersymmetric pp-wave metric and a constant self-dual 5-form respectively. Then Metsaev [64] found an exact solution of the non-interacting type IIB Green-Schwarz string in the pp-wave background. Shortly afterward, Berenstein, Maldacena and Nastase (BMN) [61] noted that one could take a similar limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory by considering states with large R-charge. They identified the Yang-Mills operators corresponding to the free string states on the pp-wave background.

The AdS/CFT correspondence predicts that the spectrum of scaling dimensions and charges under R-symmetry of these operators in the 't Hooft planar limit [65] of Yang-Mills theory should match the free string spectrum. The computation of such anomalous dimension is usually non trivial, but the situation has profoundly changed since it has been recently realized [66] that it is possible to construct an effective Hamiltonian describing the matrix of the one-loop anomalous dimension of the composite operators of scalar fields of $\mathcal{N} = 4$ SYM theory. In the planar limit this Hamiltonian is in correspondence with the one of an integrable $SO(6)$ XXX spin chain. Such a relation with an integrable system can then be used, by means of the algebraic Bethe ansatz [67], to compute the anomalous dimension of the gauge theory operators.

The idea that the planar limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and its string theory dual, the IIB superstrings propagating on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, could both be exactly integrable has attracted a good deal of attention [66], [68]-[84]. In particular, the gauge theory results have progressed to the point where integrability has been checked explicitly up to three loop order [73] and there are now proposals for integrable systems in various sectors of the theory which would be equivalent to planar

Yang-Mills theory to all orders in its loop expansion [73, 76, 85, 86].

If string theory on $AdS_5 \times S^5$ is integrable, the theory on simple orbifolds of that space would also be expected to be integrable. In the Yang-Mills dual, orbifolding reduces the amount of supersymmetry and this gives some hope of finding integrability in theories with less supersymmetry [87]-[90]. In the final part of this thesis, we shall consider the issue of integrability of an $\mathcal{N} = 2$ supersymmetric $SU(N)^M$ quiver gauge theory [91] which can be obtained as a particular Z_M-orbifold of $\mathcal{N} = 4$ [92]. This system is also conjectured to be integrable using a twisted version of the Bethe ansatz [93]. Its string theory dual is IIB superstrings on the space $\mathrm{AdS}_5 \times \mathrm{S}^5/Z_M$.

Thus far, explicit solutions of string theory on these backgrounds are not known. Quantitative results are limited to the supergravity limit, or to some large quantum number limits [94, 95, 96, 90]. For example, a Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5/Z_M$, together with a large order limit of the orbifold group, $M \to \infty$ can be taken in such a way that it obtains a plane-wave [64] with a periodically identified null coordinate. The IIB superstring can be solved explicitly in this background. Mukhi, Rangamani and Verlinde (MRV) [92] observed that it is possible to find the Yang-Mills dual of this theory by taking an analog of the BMN limit [97, 98] of the $\mathcal{N} = 2$ quiver gauge theory. In that limit, they found a beautiful matching of the discrete light-cone quantized (DLCQ) free string spectrum and planar conformal dimensions of the appropriate Yang-Mills operators.

In the fourth chapter of this thesis we will present a computation of the leading finite size correction to the MRV limit, focusing on planar Yang-Mills theory and non-interacting strings, both using an effective Hamiltonian technique and a twisted version of a long range spin chain Bethe ansatz [76]. The results in the string dual show a three loop disagreement very similar to the known one in the near BMN limit of N = 4 super Yang-Mills theory [70, 73], which can be solved by adding a "dressing factor" to the twisted Bethe ansatz. These findings are shown to be consistent with integrability of the $\mathcal{N} = 2$ SYM theory [99].

This thesis contains five chapters. In the first one we will give an introduction to the Boundary formulation of SFT, then proceed in the evaluation of the two objects defining the space-time action when a tachyonic boundary perturbation is considered - the partition function on the disk and a non-linear tachyon β-function. We will check the correctness of the latter by solving the fixed point equation perturbatively, thus generating the correct scattering amplitudes of open string theory. We will then calculate the space-time action both in a derivative expansion to any power of the tachyon field and to the third power of the field and to all orders in dereivatives. Its solitonic solutions will be showed to verify in a very accurate way the second Sen conjecture. A field redefinition relating the two actions in the Boundary and Cubic SFT formulations will be also derived.

In the second chapter, we will briefly present Cubic SFT and the ways to implement off-shell scattering amplitude calculations there. We will then derive an analytic series solution of the elliptic equations providing the four-tachyon off-shell amplitude. This will be used to increase the precision in the evaluation of the exact quartic self-coupling coefficient in the tachyon potential and, later in Chapter 3, to improve the accuracy of the coefficients of a time dependent solution.

In the third chapter, we will describe two different approaches for deriving tachyonic time-dependent solutions in Cubic SFT. The first one will be a level truncated analysis of the equation of motion for a solution expressed as a power series in the exponentials e^t. Then the analytic procedure based on the diffusion equation satisfied by a given tachyon profile will be presented. It will be showed that both the solutions present a pathological behavior after the origin of the time, and it will be discussed the field redefinition that would map these problematic trajectories in a “well behaved” rolling tachyon solution of the type existing in the Boundary formulation of SFT.

In the fourth chapter, we will first give an introduction to the tools for studying the integrability of $\mathcal{N} = 4$ SYM theory. Then the structure of the orbifolded $\mathcal{N} = \in$ theory will be presented, together with its string dual type IIB superstrings on $\mathrm{AdS}_5 \times \mathrm{S}^5/Z_M$. The spectrum of both the theories will be then computed and compared, and the three-loop disagreement solved by means of a phase factor added to the sring Bethe ansatz for the string sigma model on the orbifolded background $\mathrm{AdS}_5 \times \mathrm{S}^5/Z_M$.

In Chapter 5 we will summarize our results and discuss open problems and possible applications.

The results described in this thesis have also appeared in four publications during the past three years [13, 57, 22, 99].

Chapter 1

The tachyon effective action in boundary string field theory

1.1 Introduction

Boundary open string field theory [2]-[5] has been useful for finding the classical tachyon potential energy functional and the leading derivative terms in the tachyon effective action [10, 11, 100, 101]. It is formulated as a problem in boundary conformal field theory. One begins with the partition function of open-string theory where the world-sheet is a disk. The strings in the bulk are considered to be on-shell and a boundary interaction with arbitrary operators is added. The configuration space of open string field theory is then taken to be the space of all possible boundary operators modulo gauge transformations and field redefinitions. Renormalization fixed points, which correspond to conformal field theories, are solutions of classical equations of motion and should be viewed as the solutions of classical string field theory. The subject of this chapter is the construction of the space-time tachyon effective action in background independent open string field theory is the subject of the present chapter.

By classical power-counting the tachyon field $T(X)$ has dimension one and is a relevant operator. If $T(X)$ is the only interaction, the field theory is perturbatively super-renormalizable. If $T(X)$ and the other fields are adjusted so that the sigma model that they define is at an infrared fixed point of the renormalization group (RG), these background fields are a solution of the classical equations of motion of string theory. Witten and Shatashvili [2, 4] have argued that these equations of motion come from an action which can be derived from the disk partition function Z by a prescription which we shall make use of below. According to this prescription the effective action for a generic coupling constant g^i (which can be identified with the tachyon, the gauge or any other field that correspond to excitations of the open bosonic string) is related to the renormalized partition function of open string theory on the disk, $Z(g^i)$, through

$$S = \left(1 - \beta^i \frac{\delta}{\delta g^i}\right) Z(g^i) \quad , \tag{1.1}$$

where β^i is the beta-function[1] of the coupling g^i. Note that (1.1) fixes the additive ambiguity in S by requiring that at RG fixed points g^*, in which $\beta^i(g^*) = 0$,

$$S(g^*) = Z(g^*) \quad . \tag{1.2}$$

The derivative of the action S with respect to the coupling constant g^i must be related to the β-function through a metric according to

$$\frac{\partial S}{\partial g^i} = -\beta^j G_{ij}(g) \; . \tag{1.3}$$

G_{ij} should be a non-degenerate metric, otherwise there would be an extra zero which could not be interpreted as a conformal field theory on the world sheet. Eq.(1.3) indicates that the RG flow is actually a gradient flow. The prescription (1.1) provides a definition of the metric G_{ij} in the space of couplings.

The β-functions appearing in (1.1) are in general non-linear functions of the couplings g^i. When the linear parts of the β^i (*i.e* the anomalous dimensions λ_i of the corresponding coupling) satisfy a so called "resonant condition", the non linear parts of the β-function cannot be removed by a coordinate redefinition in the space of couplings [5]. Such resonant condition is nothing but the mass-shell condition so that, near the mass-shell, the β-functions are necessarily non-linear.

However, when the resonant condition does not hold, a possible choice of coordinates on the space of string fields is one in which the β-functions are exactly linear. This choice can always be made locally [11] and is well suited to studying processes which are far off-shell, such as tachyon condensation. These coordinates, however, become singular when the components of the string field (*e.g.* $T(X)$, $A_\mu(X)$ *etc.*) go on-shell. These coordinates can be used to construct, for example, the tachyon effective potential, but become singular when one tries to derive an effective action which reproduces the on-shell amplitudes. In particular, if the Veneziano amplitude needs to emerge from the tachyon effective action it is necessary to consider the whole non-linear β-function in (1.1). A complete renormalization of the theory in fact makes the β-function non-linear in $T(X)$ [41] so that, since the vanishing of the β-function is the field equation for T, these nonlinear terms describe tachyon scattering. One of our goals is to construct non-linear expressions for the β-functions which are valid away from the RG fixed point. With these expressions for the non-linear tachyon β-function we shall construct the Witten-Shatashvili (WS) space-time action (1.1). We shall prove that (1.1) has the following very simple form in the coupling space coordinates in which the tachyon β-function is non-linear

$$S = K \int d^{26}X \left[1 - T_R(X) + \beta^T(X)\right] \quad , \tag{1.4}$$

where T_R is the renormalized tachyon field and K is a constant related to the D25-brane tension. This formula is universal as it does not depend on how many couplings are

[1] In our case the β function is positive for relevant perturbations. In some other papers on the subject, *e.g.*[11], the opposite conventions are used.

switched on. Eq. (1.4) arises from the expression that links the renormalized tachyon field to the partition function that appears in (1.1), namely $Z = K \int d^{26}X(1 - T_R)$. T_R is then a non-linear function of the bare coupling T and in these coordinates the β-function is non-linear. When couplings other than the tachyon are introduced in Z, β^T will depend on them so that S will provide the space-time effective action also for these couplings.

With this prescription we shall compute the non-linear β-function β^T for the tachyon field up to the third order in powers of the field and to any order in derivatives of the field. From this we shall show that the solutions of the RG fixed point equations generate the three and four-point open bosonic string scattering amplitudes involving tachyons. Then, with the same renormalization prescription, we shall compute β^T to the leading orders in derivatives but to any power of the tachyon field and we shall show that S coincides with the one-found in [10, 11, 100]. Obviously, S up to the first three powers of T and expanded to the leading order in powers of derivatives can be obtained from both calculations and the results coincide. In the case of profiles $T_R(k)$ that have support near the on-shell momentum $k^2 \simeq 1$ the equation $\beta^T(k) = 0$ can be derived as the equation of motion of an action. We shall show that this action coincides with the tachyon effective action computed, for the almost on-shell profiles, from the cubic string field theory up to the fourth power of the tachyon field.

The knowledge of the non-linear tachyon β-function is very important also for another reason. The solutions of the equation $\beta^T = 0$ give the conformal fixed points, the backgrounds that are consistent with the string dynamics. In the case of slowly varying tachyon profiles, we shall show that the equations of motion for the WS action can be made identical to the RG fixed point equation $\beta^T = 0$. We shall find solutions of this equation to which correspond a finite value of the WS action. Being solutions of the RG equations, these solitons are lower dimensional D-branes for which the finite value of S provides a quite accurate prediction of the D-brane tension. In this chapter we shall also show that the WS action constructed in terms of a linear β-function [42] is related to the action (1.4) by a field redefinition, and that this field redefinition becomes singular on-shell. This is in agreement with the Poincaré-Dulac theorem [43] used in [5] to prove that when the resonant condition holds, namely near the on-shellness, the β-function has to be non-linear.

The tachyon effective action up to the third power in the fields is known exactly also from the cubic string field theory [1]. This raises the interesting question of how the action S obtained here is related to the cubic SFT. It seems clear that the cubic SFT must correspond to (1.1,1.3) for a particular choice of coordinates on the space of string fields (or worldsheet couplings). The two sets of coordinates are related by a complicated transformation which we shall derive in this chapter. The cubic SFT parametrization of worldsheet RG is regular close to the mass shell. It very well reproduces tachyon scattering [102], to it must correspond a non-linear beta-function. Thus a coordinate transformation that relates the two effective actions needs a non-linear beta function in the definition (1.1). We shall show that this field redefinition exists and that it is non-singular on-shell only when K in (1.4) coincides with the tension of the D25-brane, in agreement with all the conjectures involving tachyon condensation [6, 44, 45].

1.2 Boundary string field theory

In Witten's construction of open boundary string field theory [2] the space of all two dimensional worldsheet field theories on the unit disk, which are conformal in the interior of the disk but have arbitrary boundary interactions, is described by the world-sheet action

$$\mathcal{S} = \mathcal{S}_0 + \int_0^{2\pi} \frac{d\tau}{2\pi} \mathcal{V} \tag{1.5}$$

where $\mathcal{S}_0$ is a free action describing an open plus closed conformal background and $\mathcal{V}$ is a general perturbation defined on the disk boundary. We will discuss the twenty six dimensional bosonic string, for which (1.5) can be expressed in terms of a derivative expansion (or level expansion) of the form

$$\mathcal{V} = T(X) + A_\mu(X)\partial_\tau X^\mu + B_{\mu\nu}(X)\partial_\tau X^\mu \partial_\tau X^\nu + C_\mu(X)\partial_\tau^2 X^\mu + \cdots \tag{1.6}$$

Without the perturbation $\mathcal{V}$ the boundary conditions on X are $\partial_r X^\mu|_{r=1} = 0$, where r is the radial variable on the disk.

$\mathcal{V}$ is a ghost number zero operator and it is useful to introduce a ghost number one operator $\mathcal{O}$ via

$$\mathcal{V} = b_{-1}\mathcal{O}. \tag{1.7}$$

We shall consider the simplest case in which ghosts decouple from matter so that, as in (1.6), $\mathcal{V}$ is constructed out of matter fields alone

$$\mathcal{O} = c\mathcal{V} \ . \tag{1.8}$$

The space-time string field theory action S is defined through its derivative dS which is a two point function computed with the worldsheet action (1.5). More generally one can introduce some basis elements $\mathcal{V}_i$ for operators of ghost number 0 so that the space of boundary perturbations $\mathcal{V}$ can be parametrized as

$$\mathcal{V} = \sum_i g^i \mathcal{V}_i \tag{1.9}$$

where the coefficients g^i are couplings on the world-sheet theory, which are regarded as fields from the space-time point of view, and $\mathcal{O} = \sum_i g^i \mathcal{O}_i$. In this parametrization the space-time action is defined through its derivatives with respect to the couplings and has the form

$$\frac{\partial S}{\partial g^i} = \frac{K}{2} \int_0^{2\pi} \frac{d\tau}{2\pi} \int_0^{2\pi} \frac{d\tau'}{2\pi} \langle \mathcal{O}_i(\tau)\{Q, \mathcal{O}(\tau')\}\rangle_g \ \ , \tag{1.10}$$

where Q is the BRST charge and the correlator is evaluated with the full perturbed worldsheet action $\mathcal{S}$.

If $\mathcal{V}_i$ is a conformal primary field of dimension Δ_i, for $\mathcal{O}$'s of the form (1.8), one has

$$\{Q, c\mathcal{V}_i\} = (1-\Delta_i)c\partial_\tau c\mathcal{V}_i \ , \tag{1.11}$$

so that from (1.10) one gets

$$\frac{\partial S}{\partial g^i} = -(1-\Delta_j) g^j G_{ij}(g) \ , \tag{1.12}$$

where

$$G_{ij} = 2K \int_0^{2\pi} \frac{d\tau}{2\pi} \int_0^{2\pi} \frac{d\tau'}{2\pi} \sin^2(\frac{\tau-\tau'}{2}) \langle \mathcal{V}_i(\tau) \mathcal{V}_j(\tau') \rangle_g \ . \tag{1.13}$$

Eq.(1.12) cannot be true in general, since it does not transform covariantly under reparametrizations of the space of theories, $g^j \to f^j(g^i)$. Indeed, $\partial_i S$ and G_{ij} transform as tensors, (the latter is the metric on the space of worldsheet theories), but g^i does not.

The correct covariant generalization of (1.12) was given in [4, 5]. The worldsheet RG defines a natural vector field on the space of theories: the β-function $\beta^i(g)$, which transforms as a covariant vector under reparametrizations of g^i. The covariant form of (1.12) is thus (1.3). If we assume that total derivatives inside the correlation function decouple and that there are no contact terms, it turns out that the β-function in (1.1) is the linear β-function. This implies that the equations of motion derived from the action (1.1) are just linear. However, as shown by Shatashvili [4, 5], contact terms show up in the computation on the world-sheet and cannot be ignored. The point is that the operator Q, which is constructed out of the BRST operator in the bulk and should be independent on the couplings because the perturbation is on the boundary, actually depends on the couplings when the contour integral approaches the boundary of the disk. A way to fix the structure of the contact terms is to consider that, since dS is a one-form, the derivative of dS should be zero independently of the choice of the contact terms that one makes in the computation. This leads to the following formula for the vector field in equation (1.1)

$$\beta^i = (1-\Delta_i) g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \cdots \tag{1.14}$$

This is an expression for the β-function with all the non-linear terms. According to the Poincaré-Dulac Theorem about vector fields (whose relevance to the β-function related issues was stressed many times by Zamolodchikov [43]) every vector field can be linearized by an appropriate redefinition of the coordinates up to the resonant term. In the second order of equation (1.14) the resonance condition is given by

$$\Delta_j + \Delta_k - \Delta_i = 1 \ . \tag{1.15}$$

The resonance condition means that the β-function cannot be linearized by a coordinate transformation and that all the non-linear terms cannot be removed from the β-function equation (1.14). When g^i is the tachyon field $T(k)$, the resonant condition (1.15) corresponds to the mass-shell conditions for three tachyons. We shall prove in what follows that the WS action S up to the third order in the tachyon fields, constructed in terms of the linear β-function [42], is related to the S made of a non-linear β-function by a field redefinition, but that this field redefinition becomes singular on-shell.

1.3 Integration over the bulk variables

Let us now restrict ourselves to the specific example of open strings propagating in a tachyon background. The partition function reads

$$Z = \int [dX^\mu(\sigma,\tau)] \exp\left(-S[X]\right) \ , \tag{1.16}$$

where the action is

$$S[X] = \int d\sigma d\tau \frac{1}{4\pi} \partial_a X(\sigma,\tau) \cdot \partial_a X(\sigma,\tau) + \int_0^{2\pi} \frac{d\tau}{2\pi} T(X(\tau)) \ . \tag{1.17}$$

Here, the first term in (1.17) is the bulk action and is integrated over the volume of the unit disk. The second term in (1.17) is integrated on the circle which is the boundary of the unit disk and describes the interactions. The scalar fields X^μ have D components with $\mu = 1, ..., D$ and we shall assume $D = 26$ in what follows for a critical string. We are working in a system of units where $\alpha' = 1$.

We begin with the observation that the bulk excitations can be integrated out of (1.16) to get an effective non-local field theory which lives on the boundary [46]. To do this we write the field in the bulk as [101]

$$X = X_{\text{cl}} + X_{\text{qu}} \ ,$$

where

$$\partial^2 X_{\text{cl}} = 0$$

and X_{cl} approaches the fixed (for now) boundary value of X,

$$X_{\text{cl}} \to X_{\text{bdry}} \text{ and } X_{\text{qu}} \to 0 \ .$$

Then, in the bulk, the functional measure is $dX = dX_{\text{qu}}$ and

$$S = \int \frac{d^2\sigma}{4\pi} \partial X_{\text{qu}} \cdot \partial X_{\text{qu}} + \int \frac{d\tau}{2\pi} \left\{ \frac{1}{2} X^\mu |i\partial_\tau| X^\mu + T(X) \right\} \ , \tag{1.18}$$

where we omitted the cl index in the last integral. Then, the integration of X_{qu} produces a multiplicative constant in the partition function - the partition function of the Dirichlet string, which we shall denote K. The kinetic term in the boundary action is non-local. The absolute value of the derivative operator is defined by the Fourier transform,

$$|i\partial_\tau| \delta(\tau - \tau') = \sum_n \frac{|n|}{2\pi} e^{in(\tau-\tau')} \ .$$

The partition function of the boundary theory is then

$$Z(J) = K \int [dX_\mu] e^{-\int_0^{2\pi} \frac{d\tau}{2\pi} \left(\frac{1}{2} X^\mu |i\partial| X^\mu + T(X) - J\cdot X \right)} \ , \tag{1.19}$$

where we have added a source $J^\mu(\tau)$ so that the path integral can be used as a generating functional for correlators of the fields X^μ restricted to the boundary. In particular, this source will allow us to compute the correlation functions of vertex operators of open string degrees of freedom. The remaining path integral over the boundary $X^\mu(\tau)$ defines a one-dimensional field theory with non-local kinetic term. If the tachyon field were absent ($T=0$), the further integration over $X^\mu(\tau)$ would give a factor which converts the Dirichlet string partition function to the Neumann string partition function.

1.4 Partition function on the disk and the renormalized tachyon field

When only the tachyon field is considered as a boundary perturbation, the Witten-Shatashvili action is given by

$$S = \left(1 - \int \beta^T \frac{\delta}{\delta T}\right) Z \ , \tag{1.20}$$

where Z is the partition function of the boundary theory on the disk and β^T is the tachyon β-function. It is useful to introduce a constant source term k for the zero mode of the X field, the integral over the zero mode variable will just provide the energy-momentum conservation δ-function. The partition function (1.19) in the presence of this constant source reads

$$Z(k) = K \int [dX_\mu] e^{-\int_0^{2\pi} \frac{d\tau}{2\pi}\left(\frac{1}{2}X^\mu|i\partial_\tau|X^\mu + T(X) - ik\cdot\hat{X}\right)} \ , \tag{1.21}$$

where $\hat{X}$ is the zero mode which is defined by

$$\hat{X}^\mu = \int_0^{2\pi} \frac{d\tau}{2\pi} X^\mu(\tau) \ . \tag{1.22}$$

In this section we shall expand the exponential in eq.(1.21) in powers of $T(X)$. The first non-trivial term is

$$Z^{(1)}(k) = -K \int [dX_\mu] \int dk_1 \int_0^{2\pi} \frac{d\tau_1}{2\pi} T(k_1) e^{-\int_0^{2\pi} \frac{d\tau}{2\pi}\left(\frac{1}{2}X^\mu|i\partial_\tau|X^\mu\right) - ik\hat{X} + ik_1 X(\tau_1)} \ . \tag{1.23}$$

The functional integral over the non-zero modes of $X(\tau)$ gives

$$Z^{(1)}(k) = -K \int d\hat{X}_\mu \int dk_1 T(k_1) e^{-\frac{k_1^2}{2}G(0) + i(k_1-k)\hat{X}} \ , \tag{1.24}$$

where $G(\tau)$ is the Green function of the operator $|i\partial_\tau|$

$$G(\tau_1 - \tau_2) = 2\sum_{n=1}^{\infty} e^{\epsilon n} \frac{\cos n\,(\tau_1-\tau_2)}{n} = -\log\left[1 - 2e^{-\epsilon}\cos(\tau_1-\tau_2) + e^{-2\epsilon}\right] \tag{1.25}$$

and ϵ is an ultraviolet cut-off. In all the calculations we shall use the following prescription for $G(\tau)$

$$G(\tau) = \begin{cases} -\log\left[c\sin^2\left(\frac{\tau}{2}\right)\right] & \tau \neq 0 \\ -2\log\epsilon & \tau = 0 \end{cases} \tag{1.26}$$

The coefficient c reflects the ambiguity involved in subtracting the divergent terms. Its value is scheme dependent and should be fixed by some renormalization prescription. We choose the value $c = 4$ for later convenience. This arbitrariness was discussed in [100, 101]. The integrals over the zero-modes in eq.(1.24) give a 26-dimensional δ-function so that

$$-Z^{(1)}(k) = KT(k)\epsilon^{k^2-1} \tag{1.27}$$

and we can identify

$$T_R(k) \equiv T(k)\epsilon^{k^2-1} = -\frac{Z^{(1)}(k)}{K} \ . \tag{1.28}$$

This equation provides the renormalized coupling T_R in terms of the bare coupling T to the lowest order in perturbation theory. $1-k^2$ is the anomalous dimension of the tachyon field. The second order term in T is given by

$$Z^{(2)}(k) = K\int_0^{2\pi}\frac{d\tau_1}{4\pi}\frac{d\tau_2}{2\pi}\int dk_1dk_2T(k_1)T(k_2)\left\langle e^{ik_1X(\tau_1)}e^{ik_2X(\tau_2)}e^{-ik\hat{X}}\right\rangle \ . \tag{1.29}$$

Again in (1.29) the integral over the zero modes $\hat{X}^\mu$ gives just a 26-dimensional δ-function, $\delta(k-k_1-k_2)$, and we can perform the integral over the non-zero modes of $X(\tau)$ to get

$$\begin{aligned} Z^{(2)}(k) = \quad & K\int_0^{2\pi}\frac{d\tau_1}{4\pi}\frac{d\tau_2}{2\pi}\int dk_1dk_2(2\pi)^D\delta(k-k_1-k_2)\,T(k_1)T(k_2) \\ & \exp\left[-\frac{1}{2}\left(k_1^2+k_2^2\right)G(0) - k_1k_2G(\tau_1-\tau_2)\right] \ . \end{aligned} \tag{1.30}$$

The integral in (1.30) becomes

$$\begin{aligned} Z^{(2)}(k) = \quad & K\int dk_1dk_2(2\pi)^D\delta(k-k_1-k_2)\,\epsilon^{k_1^2+k_2^2-2}T(k_1)T(k_2) \\ & \int_0^{2\pi}\frac{d\tau_1}{4\pi}\frac{d\tau_2}{2\pi}\left[4\sin^2\left(\frac{\tau_1-\tau_2}{2}\right)\right]^{k_1k_2} \ . \end{aligned} \tag{1.31}$$

The integral over the relative variable $x = (\tau_1-\tau_2)/2$ does not need regularization, it converges when $1+2k_1k_2 > 0$, providing the result

$$Z^{(2)}(k) = \frac{K}{2}\int dk_1dk_2(2\pi)^D\delta(k-k_1-k_2)\,\epsilon^{k_1^2+k_2^2-2}T(k_1)T(k_2)\frac{\Gamma(1+2k_1k_2)}{\Gamma^2(1+k_1k_2)} \ . \tag{1.32}$$

The integrand in (1.32) can be analytically continued also to the region where $1+2k_1k_2 < 0$, so that the integral can be performed.

To the second order in perturbation theory the renormalized coupling in terms of the bare coupling reads

$$
\begin{aligned}
T_R(k) &= -\frac{Z^{(1)}(k)+Z^{(2)}(k)}{K} \\
&= \epsilon^{k^2-1}\left[T(k)-\frac{1}{2}\int dk_1dk_2(2\pi)^D\delta\left(k-k_1-k_2\right)T(k_1)T(k_2)\epsilon^{-(1+2k_1k_2)}\frac{\Gamma\left(1+2k_1k_2\right)}{\Gamma^2\left(1+k_1k_2\right)}\right] \ .
\end{aligned}
\tag{1.33}
$$

The third order contribution to the partition function is given by

$$
Z^{(3)}(k) = -\frac{K}{3!}\int dk_1dk_2dk_3(2\pi)^D\delta\left(k-\sum_{i=1}^{3}k_i\right)\epsilon^{\sum_{i=1}^{3}k_i^2-3}T(k_1)T(k_2)T(k_3)I(k_1,k_2,k_3) \ , \tag{1.34}
$$

where $I(k_1,k_2,k_3)$ is the integral

$$
\begin{aligned}
I(k_1,k_2,k_3) = \quad & \frac{2^{2k_1k_2+2k_2k_3+2k_1k_3}}{(2\pi)^3}\int_0^{2\pi}d\tau_1d\tau_2d\tau_3\ \left[\sin^2\left(\frac{\tau_1-\tau_2}{2}\right)\right]^{k_1k_2} \\
& \left[\sin^2\left(\frac{\tau_2-\tau_3}{2}\right)\right]^{k_2k_3}\left[\sin^2\left(\frac{\tau_1-\tau_3}{2}\right)\right]^{k_1k_3} \ .
\end{aligned}
\tag{1.35}
$$

The complete computation of $I(k_1,k_2,k_3)$ will be given in Appendix A. The result is given by the completely symmetric formula

$$
I(a_1,a_2,a_3) = \frac{\Gamma(1+a_1+a_2+a_3)\Gamma(1+2a_1)\Gamma(1+2a_2)\Gamma(1+2a_3)}{\Gamma(1+a_1)\Gamma(1+a_2)\Gamma(1+a_3)\Gamma(1+a_1+a_2)\Gamma(1+a_2+a_3)\Gamma(1+a_1+a_3)} \ , \tag{1.36}
$$

where we have set $a_1 = k_1k_2$, $a_2 = k_2k_3$ and $a_3 = k_1k_3$. The integral (1.35) converges when $1+a_1+a_2+a_3>0$, but its result (1.36) can be analytically continued also outside this convergence region. The result (1.36) is in agreement with the one obtained, with a different procedure, in [42] but does not coincide with the one provided in the appendix of ref. [11]. Up to the third order in powers of T and to all orders in k_i the relation between the bare and the renormalized couplings reads

$$
\begin{aligned}
T_R(k) &= -\frac{Z^{(1)}(k)+Z^{(2)}(k)+Z^{(3)}(k)}{K} \\
&= \epsilon^{k^2-1}\left[T(k)-\frac{1}{2}\int dk_1dk_2(2\pi)^D\delta\left(k-\sum_{i=1}^{3}k_i\right)T(k_1)T(k_2)\epsilon^{-(1+2k_1k_2)}\frac{\Gamma\left(1+2k_1k_2\right)}{\Gamma^2\left(1+k_1k_2\right)}\right. \\
&\left.+\int dk_1dk_2dk_3\frac{(2\pi)^D}{3!}\delta\left(k-\sum_{i=1}^{3}k_i\right)T(k_1)T(k_2)T(k_3)\epsilon^{-2(1+\sum_{i<j}k_ik_j)}I(k_1,k_2,k_3)\right] \ .
\end{aligned}
\tag{1.37}
$$

In section 1.6 we shall use this expression to construct the non-linear β-function.

The renormalized tachyon field can be constructed to all powers of the bare tachyon field in the case in which the tachyon profile appearing in (1.21) is a slowly varying function of X^μ. In this case one can consider an expansion of (1.21) in powers of derivatives of T. To this purpose consider the n-th term in the expansion of (1.21) in powers of $T(X(\tau))$, $Z^{(n)}(k)$. Taking the Fourier transform of the tachyon field and performing all the contractions of the $X(\tau_i)$ fields, for $Z^{(n)}(k)$ we get

$$\begin{aligned} Z^{(n)}(k) = K\frac{(-1)^n}{n!}\epsilon^{-n}\int\prod_{i=1}^{n}dk_iT(k_i)\int_0^{2\pi}\prod_{i=1}^{n}\left(\frac{d\tau_i}{2\pi}\right) \\ e^{-\sum_{i=1}^{n}\frac{k_i^2}{2}G(0)-\sum_{i<j}^{n}k_ik_jG(\tau_i-\tau_j)}\delta\left(k-\sum_{i=1}^{n}k_i\right) \ . \end{aligned} \tag{1.38}$$

Note that with our regularization prescription the dependence on the cut-off in (1.38) comes only from the zero distance propagator $G(0)$ and from the explicit scale dependence of the tachyon field. If the tachyon profile is a slowly varying function of X^μ we can expand inside the integrand of (1.38) in powers of the momenta k_i. The leading and next to leading terms in this expansion read

$$\begin{aligned} Z^{(n)}(k) = \quad & K\frac{(-1)^n}{n!}\prod_{i=1}^{n}\int dk_i\delta\left(k-\sum_{i=1}^{n}k_i\right)\epsilon^{-n}\prod_{i=1}^{n}T(k_i) \\ & \left(1+\sum_{i=1}^{n}k_i^2\log\epsilon+\sum_{i<j}^{n}k_ik_j\log\frac{c}{4}\right) \ , \end{aligned} \tag{1.39}$$

where the last term comes from the integral over a couple of τ variables of the propagator $G(\tau_i-\tau_j)$, the other integrations over τ_k $k\neq i,j$ being trivial. Here we have kept explicit the ambiguity c appearing in the propagator (1.26) to show that the result greatly simplifies with the choice $c=4$. Unless otherwise stated, we shall adopt this choice throughout this chapter. As before, the renormalized tachyon field $T_R(k)$ can be obtained from (1.39) by summing over n from 1 to ∞, changing sign and dividing by K. Taking the Fourier transform of $T_R(k)$ with $c=4$, to all orders in the bare tachyon field and to the leading order in derivatives, we get the renormalized tachyon field $T_R(X)$

$$T_R(X) = 1-\exp\left\{-\frac{1}{\epsilon}\left[T(X)-\triangle T(X)\log\epsilon\right]\right\} \tag{1.40}$$

where $\triangle$ is the Laplacian. Again in section 1.6 we shall use this expression to compute the non-linear tachyon β-function.

From eqs.(1.28,1.33,1.37,1.40) it is clear that the general relation between the renormalized tachyon field $T_R(X)$ and the partition function $Z\equiv Z(k=0)$ is simply

$$Z = K\int d^{26}X\left[1-T_R(X)\right] \ . \tag{1.41}$$

This expression is true also when other couplings are present. T_R in this case would be a non linear function also of the other bare couplings but its relation with the partition function of the theory would always be given by (1.41). We shall prove eq.(1.41) in the next section.

1.5 Background-field method

The partition function of the boundary theory on the disk in general is given by

$$Z = K \int [dX_\mu] e^{-\left(S_0[X] + \int_0^{2\pi} \frac{d\tau}{2\pi} \mathcal{V}[X(\tau)]\right)} , \tag{1.42}$$

where $S_0 = \int d\tau X^\mu |i\partial_\tau| X^\mu$ and $\mathcal{V}[X(\tau)]$ is given in (1.6). Our goal is to determine the relationship between the renormalized and the bare couplings of the one-dimensional field theory. To this purpose we shall make use of the background field method [41]. We expand the fields X^μ around a classical background X_0^μ which satisfies the equations of motion and which varies slowly compared to the cut-off scale,

$$X^\mu = X_0^\mu + Y^\mu .$$

The effective action is $S_{\text{eff}}[X_0] = -\log Z[X_0]$ and the aim of the renormalization process is to rewrite the local terms of $S_{\text{eff}}[X_0]$ in terms of renormalized couplings in such a way that $S_{\text{eff}}[X_0]$ has the same form of the original action

$$S_{\text{eff}}[X_0]\Big|_{\text{local}} = S_0[X_0] + \int_0^{2\pi} \frac{d\tau}{2\pi} \mathcal{V}_R[X_0(\tau)] . \tag{1.43}$$

$Z[X_0]$ can be conveniently calculated in powers of the boundary interaction $\mathcal{V}$. The first order for example reads, up to the multiplicative constant K,

$$\begin{aligned} -\int_0^{2\pi} \frac{d\tau}{2\pi} \int dk e^{ikX_0} \quad & \langle [T(k) + A_\mu(k)\partial_\tau(X_0^\mu + Y^\mu) + B_{\mu\nu}(k)\partial_\tau(X_0^\mu + Y^\mu)\partial_\tau(X_0^\nu + Y^\nu) \\ & + C_\mu(k)\partial_\tau^2(X_0^\mu + Y^\mu) + \cdots]\, e^{ikY} \rangle . \end{aligned} \tag{1.44}$$

The renormalized couplings $T_R(k)$ will be given by the opposite of the coefficient of the term in (1.44) that does not contain X_0 derivatives. Analogously, the renormalized $A_\mu^R(k)$ will be determined by the coefficient of $\partial_\tau X_0^\mu$, $B_{\mu\nu}^R(k)$ by the coefficient of $\partial_\tau X_0^\mu \partial_\tau X_0^\nu$ and so on. The second order term in the expansion of $Z[X_0]$ is

$$\begin{aligned} & \int_0^{2\pi} \frac{d\tau_1}{2\pi} \int_0^{2\pi} \frac{d\tau_2}{4\pi} \int dk_1 dk_2 e^{ik_1 X_0(\tau_1) + ik_2 X_0(\tau_2)} \langle e^{ik_1 Y(\tau_1) + ik_2 Y(\tau_2)} \\ & [T(k_1) + A_\mu(k_1)\partial_{\tau_1}(X_0^\mu + Y^\mu) + \dots] [T(k_2) + A_\nu(k_2)\partial_{\tau_2}(X_0^\nu + Y^\nu) + \dots] \rangle \end{aligned} \tag{1.45}$$

An expansion of the background field X_0 in powers of its derivatives is required to determine the coefficients of 1, $\partial_\tau X_0^\mu$, $\partial_\tau X_0^\mu \partial_\tau X_0^\nu$, ...,

$$X_0(\tau_2) = X_0(\tau_1) + (\tau_2 - \tau_1)\partial_{\tau_1} X_0(\tau_1) + \dots . \tag{1.46}$$

If we are interested in renormalization of couplings of the form $\exp[ikX_0]$, namely in the renormalized tachyon field $T_R(k)$, we can disregard the terms in (1.46,1.45) involving derivatives acting on X_0. For example, at the second order, the only non-vanishing terms in T and A_μ contributing to T_R are

$$T_R(k) = - \int dk_1 \int dk_2 \delta(k-k_1-k_2) \int_0^{2\pi} \frac{d\tau_2}{4\pi} \langle e^{ik_1 Y(\tau_1)+ik_2 Y(\tau_2)} \left[T(k_1)T(k_2) + A_\mu(k_1)A_\nu(k_2)\partial_{\tau_1}Y^\mu \partial_{\tau_2}Y^\nu + \ldots\right]\rangle \ , \tag{1.47}$$

where the correlator does not depend on τ_1 since the propagator (1.26) of $X(\tau)$ and its derivatives are periodic functions on the unit circle. It is not difficult to see that $T_R(k)$ in (1.47) coincides with the opposite of the second order term in the expansion of the partition function

$$Z(k) = \int [dY_\mu] e^{-\left(S_0[Y] + \int_0^{2\pi} \frac{d\tau}{2\pi} \mathcal{V}[Y(\tau)]\right) - ik\hat{Y}} \tag{1.48}$$

in powers of the couplings. Here k is a constant source for the zero mode of the Y^μ field, $\hat{Y}^\mu$ (1.22). Such a constant source will just provide the δ-function in (1.47) that imposes the energy-momentum conservation. This will be true at any order in the expansion in powers of the coupling fields. Therefore, to all orders in whatever coupling, the expression for the renormalized tachyon field $T_R(X)$ is related to the partition function $Z = Z(k=0)$ precisely by (1.41), which is the relation that we wanted to prove. Note that T_R depends not only on the bare tachyon field but also on the other coupling fields (in particular T_R will exists also if one starts from a boundary interaction that does not contain the bare tachyon). As a consequence, the tachyon β function will contain for example also the gauge field [47], and this is as it should be, since the solution of the equation $\beta^T = 0$ will then describe the scattering of a tachyon by other excitations (*e.g.* from (1.47) by two vector fields).

1.6 β-function

In this section we shall perform a calculation of the non-linear tachyon β-function. The resulting expression will then be used to derive the Witten-Shatashvili action (1.4,1.20). Following [41], the most general RG equations for a set of couplings g^i can be written as

$$\beta^i \equiv \frac{dg^i}{dt} = \lambda_i g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \cdots \quad , \tag{1.49}$$

where the scale t is $t = -\log\epsilon$, λ_i are the anomalous dimensions corresponding to the couplings g^i and there is no summation in the first term on the right-hand side. This equation has the solution

$$g^i(t) = e^{\lambda_i t} g^i(0) + \left[e^{(\lambda_j+\lambda_k)t} - e^{\lambda_i t}\right] \frac{\alpha^i_{jk}}{\lambda_j + \lambda_k - \lambda_i} g^j(0) g^k(0) + b^i_{jkl}(t) g^j(0) g^k(0) g^l(0) + \cdots \quad , \tag{1.50}$$

where $g^i(0)$ are the bare couplings and

$$
\begin{aligned}
b^i_{jkl}(t)g^j(0)g^k(0)g^l(0) = &\left[\left(\frac{2\alpha^i_{jm}\alpha^m_{kl}}{\lambda_j+\lambda_m-\lambda_i}-\gamma^i_{jkl}\right)\frac{e^{\lambda_i t}}{\lambda_j+\lambda_k+\lambda_l-\lambda_i}\right.\\
&+\left(\frac{2\alpha^i_{jm}\alpha^m_{kl}}{\lambda_k+\lambda_l-\lambda_m}+\gamma^i_{jkl}\right)\frac{e^{(\lambda_j+\lambda_k+\lambda_l)t}}{\lambda_j+\lambda_k+\lambda_l-\lambda_i}\\
&\left.-\frac{2\alpha^i_{jm}\alpha^m_{kl}}{(\lambda_j+\lambda_m-\lambda_i)(\lambda_k+\lambda_l-\lambda_m)}e^{(\lambda_j+\lambda_k)t}\right]g^j(0)g^k(0)g^l(0)\ . \qquad (1.51)
\end{aligned}
$$

Let us now consider the case of interest for our purpose: open strings propagating in a tachyon background. In this case the coupling g^i is the tachyon field $T(k)$. Then $\lambda_i = 1-k^2$ and $\lambda_j = 1-k_j^2$. Comparing the general solution (1.50) with eq.(1.37) derived in the previous section, we will be able to identify the renormalized tachyon field in terms of the bare field up to the third order in powers of the field and to all orders in its derivatives. In the second order term of (1.37) the coefficient proportional to $e^{\lambda_i t} = \epsilon^{1-k^2}$ appearing in (1.49) is absent. This is due to the fact that the convergence condition for the integral (1.31), $1+2k_1k_2>0$, implies that $\lambda_j+\lambda_k>\lambda_i$ so that in the limit $t\to\infty$ the dominant contribution comes from $e^{(\lambda_j+\lambda_k)t}$. From similar arguments, the first and the second terms of the right-hand side of (1.51) are negligible compared to the second term, due to the convergence conditions for the integral $I(k_1,k_2,k_3)$ computed in the previous section. This is a general feature of our renormalization procedure. At the n-th order in the bare coupling in the expansion (1.50), the renormalized coupling will contain only the term of the form

$$
e^{t\sum_{k=1}^n \lambda_k}\ . \qquad (1.52)
$$

This is due to the fact that the integrals over the τ's do not need an explicit regulator, rather they can be evaluated in a specific region of the k_i variables and then analytically continued. Therefore the only dependence on the cut-off does not come from such integrals but from the propagators (1.26) evaluated at zero distance.

Comparing our result for the renormalized tachyon field (1.37) with the general expressions (1.50,1.51), for the coefficients in the expansion (1.49) we find

$$
\begin{aligned}
\alpha^i_{jk} &= -\frac{1}{2}\frac{\Gamma(2+2k_jk_k)}{\Gamma^2(1+k_jk_k)}\delta(k-k_j-k_k)\\
\gamma^i_{jkl} &= \frac{1}{3!}\int dk_jdk_kdk_l\delta(k-k_j-k_k-k_l)\left[2(1+k_jk_k+k_jk_l+k_kk_l)I(k_j,k_k,k_l)\right.\\
&\left.-\left(\frac{\Gamma(2+2k_jk_k+2k_jk_l)\Gamma(1+2k_kk_l)}{\Gamma^2(1+k_jk_k+k_jk_l)\Gamma^2(1+k_kk_l)}+\text{cycl.}\right)\right]\ , \qquad (1.53)
\end{aligned}
$$

where $I(k_j,k_k,k_l)$ is given in equation (1.36). The perturbative expression for the β-function up to the third order in the tachyon field obtained using this procedure therefore is

$$
\beta^T(k) = (1-k^2)T_R(k) - \frac{1}{2}\int dk_1dk_2(2\pi)^D\delta(k-k_1-k_2)T_R(k_1)T_R(k_2)\frac{\Gamma(2+2k_1k_2)}{\Gamma^2(1+k_1k_2)}
$$

$$+\frac{1}{3!}\int dk_1dk_2dk_3(2\pi)^D\delta(k-k_1-k_2-k_3)T_R(k_1)T_R(k_2)T_R(k_3)$$
$$\left[2(1+k_1k_2+k_1k_3+k_2k_3)I(k_1,k_2,k_3)-\left(\frac{\Gamma(2+2k_1k_2+2k_1k_3)\Gamma(1+2k_2k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)\Gamma^2(1+k_2k_3)}+\text{cycl.}\right)\right]. \tag{1.54}$$

We have thus succeeded in deriving a β-function for tachyon backgrounds which do not satisfy the linearized on-shell condition. Exactly the same result can be obtained by taking the derivative of (1.37) (or of the opposite of $Z(k)$) with respect to the logarithm of the cut-off $-\log\epsilon$. The result obtained in this way must then be expressed in terms of the renormalized field by inverting (1.37) and it coincides with (1.54).

It is interesting to note that all the known conformal tachyon profiles, like e^{iX^0} or $\cos X^i$ where i is a space index, are solutions of the equation $\beta^T(X)=0$, where $\beta^T(X)$ is the Fourier transform of (1.54). These solutions and perturbations around them have been used to construct tachyon effective actions around the on-shellness [48, 36, 38, 37, 40] and to study the problem of the rolling tachyon [31, 32, 33, 34, 35, 49, 39].

That the non-linear β-function (1.54) is the correct one can be shown by solving the $\beta^T(k)=0$ equation perturbatively. The solution of this equation will generate the correct scattering amplitudes of open string theory [41]. This in turn will show the validity of the general formula (1.41). To the lowest order the equation is $(1-k^2)T_0(k)=0$, so that the solution $T_0(k)$ satisfies the linearized on-shell condition. By writing $T(k)=T_0(k)+T_1(k)$ and substituting into the equation $\beta_T(k)=0$, to the next order we find

$$T_1(k)=\frac{1}{2}\int dk_1dk_2(2\pi)^D\delta(k-k_1-k_2)T_0(k_1)T_0(k_2)\frac{\Gamma(k^2)}{(1-k^2)\,\Gamma^2\,(k^2/2)}\ . \tag{1.55}$$

The presence of the couplings T_0 in (1.55) sets two of the three k_i on-shell. To pick out the propagator pole corresponding to the third k we set it on-shell too. The scattering amplitude for three on-shell tachyons is given by the residue of the pole and is $1/2\pi$ with our normalization.

The calculation at the next order proceed in a similar fashion. One sets $T(k)=T_0(k)+T_1(k)+T_2(k)$ and finds

$$T_2(k)=-\frac{(2\pi)^D}{3!(1-k^2)}\int dk_1dk_2dk_3\delta(k-k_1-k_2-k_3)T_0(k_1)T_0(k_2)T_0(k_3)I(k_1,k_2,k_3)$$
$$\left\{2\left(1+\sum_{i<j}k_ik_j\right)I(k_1,k_2,k_3)-\left[\frac{\Gamma(2+2k_1k_2+2k_1k_3)\Gamma(1+2k_2k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)\Gamma^2(1+k_2k_3)}+\text{cycl.}\right]\right.$$
$$\left.-\left[\frac{\Gamma(2+2k_1k_2+2k_1k_3)\Gamma(2+2k_2k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)\Gamma^2(1+k_2k_3)[1-(k_2+k_3)^2]}+\text{cycl.}\right]\right\}\ . \tag{1.56}$$

When all the tachyons are on-shell, the last two terms on eq. (1.56) cancel and, as it should be for consistency, the residue of the pole in k is the scattering amplitude of four on-shell tachyons. It is given by

$$\frac{\Gamma(1+2k_1k_2)\Gamma(1+2k_2k_3)\Gamma(1+2k_1k_3)}{\Gamma(1+k_1k_2)\Gamma(1+k_2k_3)\Gamma(1+k_1k_3)\Gamma(1+k_1k_2+k_2k_3)\Gamma(1+k_2k_3+k_1k_3)\Gamma(1+k_1k_2+k_1k_3)}, \tag{1.57}$$

where the on-shell condition is $1+k_1k_2+k_2k_3+k_1k_3=0$. By means of the on-shell condition, from the above expression, we recover, up to a normalization constant, the Veneziano amplitude, the scattering amplitude of four on-shell tachyons. Eq.(1.57) in fact becomes

$$\begin{aligned}&\frac{1}{\pi^3}\Gamma(1+2k_1k_2)\Gamma(1+2k_2k_3)\Gamma(1+2k_1k_3)\sin(\pi k_1)\sin(\pi k_2)\sin(\pi k_3)\\ &=\frac{1}{(2\pi)^2}\left[B\left(1+2k_1k_2,1+2k_2k_3\right)+\text{cycl.}\right],\end{aligned} \tag{1.58}$$

where $B(x,y)$ is the Euler beta function. The expression between square brackets is just the Veneziano amplitude. The ambiguity c appearing in the propagator (1.26) could be kept undetermined throughout the calculations of the scattering amplitudes. It is not difficult to see that this would just consistently change the normalization of the on-shell amplitudes.

For tachyon profiles $T_R(k)$ supported over near on-shell momentum $k^2\simeq 1$, the equation of motion $\beta^T=0$ with β^T given in (1.54) becomes

$$\begin{aligned}\beta^T(k) &= (1-k^2)T_R(k)-\frac{(2\pi)^D}{2\pi}\int dk_1dk_2\delta(k-k_1-k_2)T_R(k_1)T_R(k_2)\\ &+ \frac{(2\pi)^D}{3!(2\pi)^2}\int dk_1dk_2dk_3\delta(k-k_1-k_2-k_3)T_R(k_1)T_R(k_2)T_R(k_3)\\ &\quad \{[B\left(1+2k_1k_2,1+2k_2k_3\right)+\text{cycl.}]+2\pi\tan(\pi k_1k_2)\tan(\pi k_1k_3)\tan(\pi k_2k_3)\}=0\,.\end{aligned} \tag{1.59}$$

The coefficients of the quadratic and cubic terms in (1.59) are symmetric with respect to all the k_i and k when these are on the mass-shell. Thus (1.59) can be derived as the equation of motion of an effective action. Such effective action for near on-shell tachyons up to the fourth order in powers of the tachyon fields can be derived from the results of the cubic string field theory. In [25] it was shown that the cubic SFT reproduces the Veneziano amplitude with great accuracy already at level $L=50$. The tachyon effective action arising from the cubic string field theory for near on shell tachyon profiles $\Phi(k)$ therefore reads

$$\begin{aligned}S_C &= 2\pi^2T_{25}(2\pi)^D\left\{-\frac{1}{2}\int dk\Phi(k)\Phi(-k)\left(1-k^2\right)+\frac{1}{3}\int\prod_{i=1}^{3}dk_i\Phi(k_i)\delta\left(\sum_{i=1}^{3}k_i\right)\right.\\ &+\left.\frac{1}{4!}\int\prod_{i=1}^{4}dk_i\Phi(k_i)\delta\left(\sum_{i=1}^{4}k_i\right)\left[B\left(1+2k_1k_2,1+2k_2k_3\right)+\text{cycl.}\right]\right\},\end{aligned} \tag{1.60}$$

where the tachyon momenta in the fourth order term satisfy

$$k_1 = (0,1,0,0,\dots,0) \qquad\qquad k_2 = (0,\sin\theta,\cos\theta,0,\dots,0) \tag{1.61}$$

$$k_3 = (0,-1,0,0,\dots,0) \qquad\qquad k_4 = (0,-\sin\theta,-\cos\theta,0,\dots,0)\ . \tag{1.62}$$

Since the Veneziano amplitude is completely symmetric in the four momenta k_i, it is not difficult to see that the equation of motion deriving from (1.60) becomes precisely (1.59) once the simple field rescaling $T = 2\pi\Phi$ is performed. Thus the cubic string field theory for almost on-shell tachyons reproduces the non-linear $\beta^T = 0$ equation of motion.

In section 1.4 we also derived the renormalized tachyon field for the case of a slowly varying tachyon profile, to all orders in the bare field and to the leading order in derivatives, eq.(1.40). From this we can easily compute the corresponding β function. The task in this case is much simpler, as we just need to take the derivative of (1.40) with respect to $-\log\epsilon$

$$\beta(X) = \frac{\partial T_R(X)}{\partial(-\log\epsilon)} = \frac{1}{\epsilon}\exp\left(-\frac{T(X)}{\epsilon}\right)\left\{T(X) + \triangle T(X)\left[1 - \left(1 - \frac{T(X)}{\epsilon}\right)\log\epsilon\right]\right\}\ . \tag{1.63}$$

Then we have to invert the relation (1.40) between T_R and T. To the leading order in derivatives one has

$$T(X) = -\epsilon\left\{\left[1 + (\log\epsilon)\triangle\right]\log(1 - T_R(X))\right\}\ , \tag{1.64}$$

from which it is clear that the admissible range for T_R is $-\infty \le T_R \le 1$. Plugging (1.64) into (1.63) we get the non-linear tachyon β-function to all powers of the renormalized tachyon and to the leading order in its derivatives

$$\beta^T(X) = (1 - T_R(X))\left[-\log(1 - T_R(X)) - \triangle\log(1 - T_R(X))\right]\ . \tag{1.65}$$

$\beta^T(X) = 0$ is the tachyon equation of motion for a slowly varying tachyon profile.

Since in our calculations of the non-linear β-function we have always used the same coordinates in the space of string fields, the two results (1.65) and (1.54) should coincide when expanded up to the third power of the field and to the leading order in derivatives, respectively. This is indeed the case and the result in both cases reads

$$\beta^T(X) = \triangle T_R + \partial_\mu T_R\partial_\mu T_R + T_R\partial_\mu T_R\partial_\mu T_R\ . \tag{1.66}$$

It is interesting to compute the β-function also in the case in which the ambiguity constant c appearing in (1.26) is kept undetermined. $T_R(k)$ can be easily obtained as before from (1.39) without fixing $c = 4$. By taking the Fourier transform and by differentiating with respect to $-\log\epsilon$, the β-function expressed in terms of the renormalized tachyon field $T_R(X)$ turns out to be

$$\beta^T(X) = (1 - T_R)\left[-\log(1 - T_R) + \frac{\triangle T_R}{1 - T_R} + \left(1 + \frac{1}{2}\log\frac{c}{4}\right)\frac{\partial_\mu T_R\partial_\mu T_R}{(1 - T_R)^2}\right]\ . \tag{1.67}$$

In the next section we shall use also this form of the β-function to construct the Witten-Shatashvili action.

1.7 Witten-Shatashvili action

In this section we shall compute the Witten-Shatashvili action. From the simple expression that relates the partition function to the renormalized tachyon (1.41) it is easy to deduce a simple and universal form for the WS action of the open bosonic string theory

$$S = \left(1 - \int \beta^T \frac{\delta}{\delta T_R}\right) Z[T_R] = K \int d^D X \left[1 - T_R(X) + \beta^T(X)\right] \ . \tag{1.68}$$

This can now be computed in both the cases analyzed in the previous sections. We shall show that the expressions for S that we will obtain are consistent both with the known results on the tachyon potential [11] and with the expected on-shell behavior. Thus a choice of coordinates in the space of couplings in which the tachyon β-function is non-linear allows one to find not only a simple general formula for the WS action, but provides also a space-time tachyon effective action that describes tachyon physics from the far-off shell to the near on-shell regions.

Let us start with the evaluation of (1.68) up to the third order in the expansion of the tachyon field using the non-linear β-function (1.54). A similar computation was done in [11, 42] by means of the linear β-function, $\beta(k) = (1-k^2)\,T(k)$. We shall later compare the two results. From the renormalized field (1.37) and the β-function (1.54) we arrive at the following expression for the Witten action

$$\begin{aligned} S = K \Bigg\{ & 1 - \frac{1}{2}\int dk (2\pi)^D T_R(k) T_R(-k) \frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)} \\ & + \frac{1}{3!}\int dk_1 dk_2 dk_3 (2\pi)^D T_R(k_1) T_R(k_2) T_R(k_3) \delta(k_1+k_2+k_3) \\ & \left[2\left(1 + \sum_{i<j} k_i k_j\right) I(k_1,k_2,k_3) - \left(\frac{\Gamma(1+2k_2k_3)\Gamma(2+2k_1k_2+2k_1k_3)}{\Gamma^2(1+k_2k_3)\Gamma^2(1+k_1k_2+k_1k_3)} + \text{cycl.} \right) \right] \Bigg\} . \end{aligned} \tag{1.69}$$

The propagator coming from the quadratic term in (1.69) exhibits the required pole at $k^2 = 1$. There are however also an infinite number of other zeroes and poles. We shall show that these are due to the metric in the coupling space appearing in (1.3). The equations of motion derived from the action (1.69) are

$$\begin{aligned} \frac{\delta S}{\delta T_R(-k)} = & -K \frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)} (2\pi)^D T(k) \\ & + \frac{K}{2} \int dk_1 dk' (2\pi)^D \delta\left(k_1 + k' - k\right) T_R(k_1) T_R(k') \cdot \\ & \cdot \Bigg\{ 2(1 - k_1 k + k_1 k' - kk') I(-k, k_1, k') - \frac{\Gamma(1-2kk_1)\Gamma(2-2kk'+2k_1k')}{\Gamma^2(1-kk_1)\Gamma^2(1-kk'+k'k_1)} \\ & - \frac{\Gamma(1-2kk')\Gamma(2-2kk_1+2k'k_1)}{\Gamma^2(1-kk')\Gamma^2(1-kk_1+k'k_1)} - \frac{\Gamma(1+2k'k_1)\Gamma(2-2kk'-2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1-kk'-kk_1)} \Bigg\} . \end{aligned} \tag{1.70}$$

As we did for the equation $\beta^T = 0$ in the previous section, by solving these equations perturbatively it is possible to recover the scattering amplitudes for three on-shell tachyons.

To the lowest order the equation is

$$\frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)}T_0(k)=0\ . \tag{1.71}$$

At variance with the lowest order solution of $\beta^T=0$, there are infinite possible solutions of (1.71). We choose the solution for which the tachyon field $T_0(k)$ is on the mass-shell, which corresponds to a consistent string theory background. This choice is also a solution of $\beta^T=0$ to the lowest order. As we shall show, the other possible zeroes of (1.71) could be interpreted as zeroes of the metric in the space of couplings through eq.(1.3). With such a choice of $T_0(k)$, to the next order we recover the scattering amplitudes for three on-shell tachyons. By writing $T(k)=T_0(k)+T_1(k)$ and substituting it into (3.11) we find

$$\begin{aligned}T_1(k)=&\frac{\Gamma^2(1-k^2)}{2\Gamma(2-2k^2)}\int dk_1dk'(2\pi)^D\delta(k-k_1-k')T_0(k_1)T_0(k')\\ &\Bigg\{2(1-k_1k+k_1k'-kk')I(-k,k_1,k')-\frac{\Gamma(1-2kk_1)\Gamma(2-2kk'+2k_1k')}{\Gamma^2(1-kk_1)\Gamma^2(1-kk'+k'k_1)}\\ &-\frac{\Gamma(1-2kk')\Gamma(2-2kk_1+2k'k_1)}{\Gamma^2(1-kk')\Gamma^2(1-kk_1+k'k_1)}-\frac{\Gamma(1+2k'k_1)\Gamma(2-2kk'-2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1-kk'-kk_1)}\Bigg\}.\end{aligned}\tag{1.72}$$

Since the two couplings T_0 satisfy the on-shell condition, k_1 and k' are on-shell. To pick out the propagator pole corresponding to the third k we set it on-shell too. The scattering amplitude for three on-shell tachyons is given again by the residue of the pole and with our normalization is $(2\pi)^{-1}$, in precise agreement with the result obtained in the previous section.

The equations (3.11) must be related to the equation $\beta^T=0$ through a metric $G_{T(k)T(k')}$ as in (1.3), which in this case becomes

$$\frac{\delta S}{\delta T_R(k)}=-\int dk' G_{T(k)T(k')}\beta^{T(k')}\ . \tag{1.73}$$

The Witten-Shatashvili formulation of string field theory provides a prescription for the metric $G_{T(k)T(k')}$ which can then be computed explicitly. To the first two orders in powers of T_R, it is given by

$$\begin{aligned}G_{T(k)T(k')}\quad&=K\frac{(2\pi)^D\Gamma(2-2k^2)}{(1-k^2)\Gamma^2(1-k^2)}\delta\left(k+k'\right)-\frac{K}{2}\int dk_1(2\pi)^D\delta\left(k+k'+k_1\right)\frac{T_R(k_1)}{1-k'^2}\cdot\\ &\cdot\Bigg\{2(1+k_1k+k_1k'+kk')I(k_1,k,k')-\frac{\Gamma(1+2kk_1)\Gamma(2+2kk'+2k_1k')}{\Gamma^2(1+kk_1)\Gamma^2(1+kk'+k'k_1)}\\ &-\frac{\Gamma(1+2kk')\Gamma(2+2kk_1+2k'k_1)}{\Gamma^2(1+kk')\Gamma^2(1+kk_1+k'k_1)}-\frac{\Gamma(1+2k'k_1)\Gamma(2+2kk'+2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1+kk'+kk_1)}\\ &-\frac{\Gamma(2+2k'k_1)\Gamma(2+2kk'+2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1+kk'+kk_1)(1+kk'+kk_1)}\Bigg\}\ .\end{aligned}\tag{1.74}$$

The first term in this metric coincides with (1.13) for a conformal primary given by the tachyon vertex. From (1.74) it is possible to see that the infinite number of zeroes and poles that the second order term in eq.(1.69) exhibits at $k^2 = 1 + n$ and $k^2 = 3/2 + n$, respectively, is in fact due to the metric. This is true except for the zero corresponding to the tachyon mass-shell $k^2 = 1$. In fact the metric (1.74) is regular for $k^2 = 1$. This indicates that the kinetic term in eq.(1.69) exhibits the required zero at the tachyon mass-shell and the metric (1.74) can be made responsible for the other extra zeroes and poles. If these zeroes and poles are just an artifact of the expansion in powers of T, it is an open question. It would be interesting to consider for example an expansion around $k^2 = 1+n$ to all orders in T and check if in this case one would still find that the kinetic term exhibits a zero at $k^2 = 1 + n$.

Let us turn now to the cubic term in eq.(1.69). If one or two tachyons are on-shell, then the cubic term vanishes. This means that any exchange diagram involving the cubic term vanishes [42]. When all the three tachyons are on-shell, the scattering amplitude for three on-shell tachyons should arise directly as the coefficient of the cubic term. However, the cubic term in (1.69) is ill-defined on shell. Nonetheless, with the most obvious regularization (*i.e.* by going on-shell symmetrically by giving to the three tachyons an identical small mass m, $k_i^2 = 1+m^2$ and then by taking the $m \to 0$ limit) one gets a finite result for the scattering amplitude [42]. Recalling the first of eqs.(1.53) we conclude that this scattering amplitude is $(2\pi)^{-1}$ with our normalization. Also the cubic term in (1.69) has a sequence of poles at finite distances from the tachyon mass-shell. This is related to the fact that the set of couplings that we have taken into account is not complete. If we get far enough from the tachyon mass-shell, we run into the poles due to all the other string states which have not been subtracted.

In the next section we shall compare (1.69) with the corresponding action derived from the cubic string field theory. Here we would like to show that, by means of a field redefinition, (1.69) can be rewritten in the form of the WS action obtained from a linear β-function [42], but that this field redefinition becomes singular on-shell. The partition function up to the third order in the bare tachyon field is again given by

$$\begin{aligned} Z(k) &= K\delta(k) - K\epsilon^{k^2-1}\Big[T(k) \\ &-\frac{1}{2}\int dk_1 dk_2 (2\pi)^D \delta\left(k-k_1-k_2\right)\epsilon^{-(1+2k_1k_2)}T(k_1)T(k_2)\frac{\Gamma\left(1+2k_1k_2\right)}{\Gamma^2\left(1+k_1k_2\right)} \\ &+\frac{1}{3!}\int dk_1 dk_2 dk_3 (2\pi)^D \delta\left(k-\sum_{i=1}^{3}k_i\right)\epsilon^{-2(1+\sum_{i<j}k_ik_j)}T(k_1)T(k_2)T(k_3)I(k_1,k_2,k_3)\Big] \ , \end{aligned} \tag{1.75}$$

where we have used (1.37). If instead of following the general procedure of ref. [41] one renormalizes the theory simply by normal ordering, the β-function turns out to be linear. Thus the renormalized field to all orders in the bare field would just be

$$\phi_R(k) = T(k)\epsilon^{k^2-1} \ , \tag{1.76}$$

so that $\beta(k) = (1-k^2)\,\phi_R(k)$. The WS action with a linear β-function up to the third order in the tachyon field then reads

$$S_L = K\left\{1 - \frac{1}{2}\int dk (2\pi)^D \phi_R(k)\phi_R(-k)\frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)}\right.$$

$$+\frac{1}{3!}\int dk_1dk_2dk_3(2\pi)^D\phi_R(k_1)\phi_R(k_2)\phi_R(k_3)\delta\left(\sum_{i=1}^{3}k_i\right)2\left(1+\sum_{i<j=2}^{3}k_ik_j\right)I(k_1,k_2,k_3)\Bigg\} \tag{1.77}$$

in agreement with what found in [42]. If we assume that the fields ϕ_R and T_R are related as follows

$$\phi_R(k) = T_R(k) + \int dk_1 f(k,k_1)T_R(k_1)T_R(k-k_1) + \dots\ , \tag{1.78}$$

by comparing the cubic terms in (1.69) and (1.77) one finds

$$\begin{aligned}&\left[f(k_2+k_3,k_2)\frac{\Gamma(2+2k_1k_2+2k_1k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)}+\text{cycl.}\right]\\ =\ &\frac{1}{2}\left[\frac{\Gamma(1+2k_2k_3)}{\Gamma^2(1+k_2k_3)}\frac{\Gamma(2+2k_1k_2+2k_1k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)}+\text{cycl.}\right]\ ,\end{aligned} \tag{1.79}$$

so that the solution for f is $f(k_1+k_2,k_1)=\Gamma(1+2k_1k_2)/(2\Gamma^2(1+k_1k_2))$ and the field redefinition becomes

$$\phi_R(k) = T_R(k) + \int dk_1dk_2\frac{\Gamma(1+2k_1k_2)}{2\Gamma^2(1+k_1k_2)}T_R(k_1)T_R(k_2)\delta(k-k_1-k_2)\ . \tag{1.80}$$

It is not difficult to see that if we evaluate this relation when the three tachyon fields are on-shell it becomes singular since $f(k,k_1)$ has a pole. This is in agreement with the Poincaré-Dulac theorem [43] used in [5] to prove that when the resonant condition (1.15) holds, namely near the on-shellness, the β-function has to be non-linear. We showed in fact that the field redefinition that gives from S the WS action constructed in terms of a linear β-function, S_L, becomes singular on-shell.

Let us now turn to the WS action computed in an expansion to the leading order in derivatives and to all orders in the powers of the tachyon fields. If we keep the renormalization ambiguity c undetermined, the β-function is given in (1.67). Using (1.4), S then reads

$$S = K\int dX\,(1-T_R)\left[1-\log(1-T_R)+\left(1+\frac{1}{2}\log\frac{c}{4}\right)\frac{\partial_\mu T_R\partial_\mu T_R}{(1-T_R)^2}\right]\ , \tag{1.81}$$

where $-\infty \le T_R \le 1$. With the field redefinition

$$1-T_R = e^{-\tilde{T}} \tag{1.82}$$

S becomes

$$S = K\int dX e^{-\tilde{T}}\left[\left(1+\frac{1}{2}\log\frac{c}{4}\right)\partial_\mu\tilde{T}\partial_\mu\tilde{T}+1+\tilde{T}\right]\ , \tag{1.83}$$

which for $c=4$ coincides with the space-time tachyon action found in [10, 11]. In particular we shall show in the next section that K coincides with the tension of the D25-brane, $K=T_{25}$, in agreement with the results of ref. [11]. It is not difficult to show that (1.83) can be rewritten, by means of a field redefinition, in the form found in [100] where the renormalization ambiguity was also discussed.

Note that (1.82) is the coordinate transformation in the coupling space that leads form the non-linear β-function (1.65) to the linear beta function $\beta^T = (1 + \triangle)T$. The β-function in fact is a covariant vector in the coupling space and as such it transforms.

We have left the ambiguity c in (1.83) undetermined because we want to show that it is possible to fix c in such a way that the equation of motion deriving from (1.83) coincides with the equation $\beta^T = 0$ with β^T given in (1.67). In fact, in terms of the coordinates (1.82), this equation reads

$$\beta^{\tilde{T}} = \tilde{T} + \triangle\tilde{T} + \frac{1}{2}\log\frac{c}{4}\partial_\mu\tilde{T}\partial_\mu\tilde{T} = 0 \ . \tag{1.84}$$

where we have kept into account that $\beta^{\tilde{T}}$ transforms like a covariant vector in the space of worldsheet theories. Choosing $\log(c/4) = -1$, eq.(1.84) becomes the equation of motion of the action (1.83). This is important because if we find finite action solutions of the equation (1.84), these would be at the same time solutions of the renormalization group equations *and* solitons of the tachyon effective action (1.83). These could then be interpreted as lower dimensional branes. Being solutions of the renormalization group equations they are interpreted as background consistent with the string dynamics, being solitons they must describe branes. The finite action solutions of eq.(1.84) are easy to find

$$\tilde{T}(X) = -n + \frac{1}{2}\sum_{i=1}^{n}(X^i)^2 \ . \tag{1.85}$$

These codimension n solitons can be interpreted as D$(25-n)$-branes. $26-n$ are in fact the number of coordinates on which the profile $\tilde{T}(X)$ does not depend. Substituting the solution (1.85) into the action (1.83) with $\log(c/4) = -1$ we get

$$S = T_{25}(e\sqrt{2\pi})^n V_{26-n} \ . \tag{1.86}$$

Comparing this with the expected result $T_{25-n}V_{26-n}$ we derive the following ratio between the brane tensions

$$R_n = \frac{T_{25-n}}{T_{25}} = (\frac{e}{\sqrt{2\pi}}2\pi)^n \ . \tag{1.87}$$

With our notation, $\alpha' = 1$, the exact tension ratio should be $R_n = (2\pi)^n$. Thus R_n differs from the one given in (1.87) by a factor $e/\sqrt{2\pi} = 1.084$. It is remarkable that a small derivatives expansion of the WS action truncated just to the second order provides a result with the 93% of accuracy. In particular the result (1.87) is much closer to the exact tension ratio then the one found in [11] with analogous procedure. The solutions of the equations of motion of the WS action considered in [11] were not in fact solutions of the equation $\beta^T = 0$, so that they could not be interpreted as consistent string backgrounds (this was already noticed by the authors of [11] and for this reason the exact tension ratio was obtained with a different procedure). The equations of motion deriving from the WS action are in fact related to the β-function through (1.3) where the metric should in principle be non-degenerate. However, if the metric is computed in some approximation, it could be singular and present solutions that introduce physics beyond that contained in the β-functions. The action (1.83) with $\log(c/4) = -1$ gives an equation of the form (1.3) with the non-degenerate metric $e^{-\tilde{T}}$. The solution of this equation can be at the same time a soliton and a conformal RG fixed point.

In conclusion the general formula (1.4) reproduces all the expected results on tachyon effective actions both in the far off-shell and in the near on-shell regions.

1.8 Cubic vs. Witten-Shatashvili tachyon effective actions

In this section we shall compare the result (1.69), which gives the WS action up to the third order in the powers of the renormalized tachyon field $T_R(k)$, with the tachyon effective action S_C computed with the cubic open string field theory of [1]. Like (1.69), S_C is known exactly up to the third power in the tachyon field. A similar comparison was already done in [11] where, however, the WS action constructed in terms of the linear tachyon β-function $\beta^T(k) = (1-k^2)T_R(k)$ was used. With such a choice of coordinates in the space of string fields, the relation between the tachyon fields of the cubic and the WS string field theory becomes singular on-shell [11]. The cubic string field theory parametrization of worldsheet RG is regular on-shell and it very well reproduces the tachyon scattering amplitudes [102], thus indicating that to it should correspond a non-linear β-function. We shall show that, comparing the result (1.69) for the WS action with the corresponding cubic string field theory action, a field redefinition between the tachyon fields in the two formulations can be found which is non-singular on-shell. In particular, by requiring the regularity of the coordinate transformation that links the cubic tachyon effective action to the WS action (1.69) we find that the overall normalization constant K in the WS action (1.69) is precisely the tension of the D25-brane. This is in agreement with all the conjectures involving tachyon condensation and with the result $K = T_{25}$ derived from the tachyon potential.

For a tachyon field $\phi(k)$, the cubic string field theory action can be written as [1, 103]

$$\begin{aligned} S_C &= 2\pi^2 T_{25}\left[-\frac{1}{2}\int dk(2\pi)^D\phi(k)\phi(-k)\left(1-k^2\right)\right. \\ &\left. +\frac{1}{3}\int dk_1dk_2dk_3(2\pi)^D\delta(k_1+k_2+k_3)\phi(k_1)\phi(k_2)\phi(k_3)\left(\frac{3\sqrt{3}}{4}\right)^{3-k_1^2-k_2^2-k_3^2}\right]. \end{aligned} \tag{1.88}$$

The normalization factor $2\pi^2T_{25}$ was derived in [44] and will be important for our analysis. Let us assume that the relation between the fields $\Phi(k)$ and $T_R(k)$ of the two theories is of the form

$$\phi(k) = f_1(k)T_R(k) + \int dk_1dk_2f_2(k,k_1)T_R(k_1)T_R(k_2)\delta(k-k_1-k_2) + \cdots , \tag{1.89}$$

where $f_1(k) = f_1(-k)$ from the reality of the tachyon field. The cubic string field theory action (1.88) becomes

$$\begin{aligned} S_C &= 2\pi^2 T_{25} \left\{ -\frac{1}{2} \int dk (2\pi)^D T_R(k) T_R(-k) \left(1-k^2\right) \left(f_1(k)\right)^2 \right. \\ &- \int dk_1 dk_2 dk_3 (2\pi)^D \delta(k_1+k_2+k_3) T_R(k_1) T_R(k_2) T_R(k_3) \\ &\left. \left[(1+k_1k_2+k_1k_3)\, f_1(k_1) f_2(k_2+k_3,k_3) - \frac{1}{3} f_1(k_1) f_1(k_2) f_1(k_3) \left(\frac{3\sqrt{3}}{4}\right)^{3-\sum_i k_i^2} \right] \right\} . \end{aligned} \tag{1.90}$$

By comparing the second order term of eq.(1.88) with the corresponding term of eq.(1.69) we find

$$\left(f_1(k)\right)^2 = \frac{K}{2\pi^2 T_{25}} \frac{\Gamma(2-2k^2)}{(1-k^2)\Gamma^2(1-k^2)} \ . \tag{1.91}$$

When the tachyon field is on the mass-shell, $f_1(k)$ is regular and takes the value

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{K}{T_{25}}} \ . \tag{1.92}$$

From the comparison of the cubic terms in (1.90) and in (1.69) we get

$$\begin{aligned} \left[1-(k_2+k_3)^2\right] f_2(k_2+k_3,k_3) &= \frac{f_1(k_2)f_1(k_3)}{3} \left(\frac{3\sqrt{3}}{4}\right)^{3-\sum_i k_i^2} - \frac{K}{3! 2\pi^2 T_{25} f_1(k_2+k_3)} \\ &\left[2(1-k_2k_3-k_2^2-k_3^2) I(-k_2-k_3,k_2,k_3) - \left(\frac{\Gamma(1+2k_2k_3)\Gamma(2-2(k_2+k_3)^2)}{\Gamma^2(1+k_2k_3)\Gamma^2(1-(k_2+k_3)^2)} + \text{cycl.} \right) \right] . \end{aligned} \tag{1.93}$$

We can fix the value of the normalization constant K by requiring the regularity of the function $f_2(k_2+k_3,k_3)$ when the three tachyons are on-shell. The on-shell condition is

$$2(k_1k_2+k_2k_3+k_1k_3)+3=0 \ ,$$

and the factor between square brackets in (3.57) is, on-shell, $(2\pi)^{-1}$. Consequently, requiring the regularity of the function f_2 when the three tachyons are on-shell, eq. (3.57) simply becomes

$$K = T_{25} \ . \tag{1.94}$$

When (1.94) is satisfied, the field redefinition that links the boundary and the cubic string field theory tachyon effective actions is regular on-shell. This result shows that the tachyon dynamics described by the WS string field theory reproduces the conjectured relations involving tachyon condensation. With analogous procedure, it is not difficult to show that the coordinate transformation between the WS tachyon effective action constructed in terms of the linear β-function (1.77), with $K = T_{25}$, and (1.88) is, as expected, singular on-shell.

Chapter 2

A solution to the 4-tachyon off-shell amplitude in cubic string field theory

2.1 Introduction

2.1.1 Basics of CSFT formalism

The action proposed in [1] for the open bosonic string field theory is

$$S = -\frac{1}{2}\int \Psi \star Q\Psi - \frac{g}{3}\int \Psi \star \Psi \star \Psi \qquad , \tag{2.1}$$

where g is interpreted as the (open) string coupling constant. The field Ψ is a string field taking values in a graded algebra $\mathcal{A}$. Associated with the algebra $\mathcal{A}$ there are

- A star product
$$\star : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$
under which the degree G is additive ($G_{\Psi\star\Phi} = G_\Psi + G_\Phi$).

- A kinetic operator
$$Q : \mathcal{A} \to \mathcal{A}$$
of degree $G = 1$ ($G_{Q\Psi} = 1 + G_\Psi$).

- An integration operator
$$\int : \mathcal{A} \to C$$
that vanishes for all Ψ with degree $G_\Psi \neq 3$. The action (2.1) is then nonvanishing only for a string field Ψ of degree 1.

The elements Q, $\star$, $\int$ defining the string fieldtheory are assumed to satisfy the following properties

(a) Q is nilpotent: $Q^2\Psi = 0, \quad \forall \Psi \in \mathcal{A}$.

(b) $\int Q\Psi = 0, \quad \forall\Psi \in \mathcal{A}$.

(c) Q is the derivative operator of the star product:
$Q(\Psi \star \Phi) = (Q\Psi) \star \Phi + (-1)^{G_\Psi}\Psi \star (Q\Phi), \;\; \forall\Psi, \Phi \in \mathcal{A}$.

(d) The star product inside the integral is non commutative:
$\int \Psi \star \Phi = (-1)^{G_\Psi G_\Phi} \int \Phi \star \Psi, \quad \forall\Psi, \Phi \in \mathcal{A}$.

(e) The star product is associative:
$(\Phi \star \Psi) \star \Xi = \Phi \star (\Psi \star \Xi), \quad \forall\Phi, \Psi, \Xi \in \mathcal{A}$.

This suggest that each element or operation in (2.1) has its counterpart in the theory of differential forms on a manifold. The string field Ψ is like a matrix-valued one-form, the operator Q is the analog of the exterior derivative, the star product $\star$ and the string integral $\int$ correspond respectively to the wedge product of forms (the only difference is that $\Psi \star \Phi$ and $\Phi \star \Psi$ have no simple relation in $\mathcal{A}$) and to the integral of forms over a manifold times the trace in matrix space.

When the axioms above are satisfied, the action (2.1) is invariant under the gauge transformations

$$\delta\Psi = Q\Lambda + \Psi \star \Lambda - \Lambda \star \Psi \qquad , \tag{2.2}$$

for any gauge parameter $\Lambda \in \mathcal{A}$ with degree $G = 0$.

When g is taken to vanish, the equation of motion derived from (2.1) and the gauge transformation (2.2) simply become $Q\Psi = 0$ and $\delta\Psi = Q\Lambda$. Thus, if the kinetic operator Q is chosen to be the BRST operator Q_B and the degree G of the $\mathcal{A}$ algebra is identified with the ghost number, these equations coincide respectively with the physical condition for the first quantized string and with the cohomology of $Q = Q_B$. When $g = 0$ this string field theory gives precisely the structure needed to describe the free bosonic string. The extra structure appearing in (2.1) is motivated by finding a simple interacting extension of the free theory consistent with the perturbative expansion of open bosonic string theory.

All the needed axioms are then satisfied when $\mathcal{A}$ is taken to be the Hilbert space of string fields

$$\mathcal{A} = \{\Psi\,[x(\sigma); c(\sigma); b(\sigma)]\}$$

which can be described as a functional of the matter, ghost ad antighost fields describing an open string in 26 dimensions with $0 \le \sigma \le \pi$. The string field Ψ can be written as an expansion over open string Fock space states of the first quantized open string theory [14]

$$\begin{aligned} |\Psi\rangle \;\; = \;\; & \big[\phi(x) + A_\mu(x)\alpha^\mu_{-1} + i\alpha(x)b_{-1}c_0 + iB_\mu(x)\alpha^\mu_{-2} + B_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\nu_{-1} \\ & +\beta_0(x)b_{-2}c_0 + \beta_1(x)b_{-1}c_{-1} + ik_\mu(x)\alpha^\mu_{-1}b_{-1}c_0 + \cdots\big]\,|0\rangle, \end{aligned} \tag{2.3}$$

with coefficients given by an infinite family of space-time fields. Here, x is the center of mass coordinate, ϕ is the tachyon, A_μ is the massless vector, $B_{\mu\nu}$ is a symmetric two tensor at the first massive level, etc., these descriptions applying in the canonical vacuum. The fields α, β_0, β_1 etc. are auxiliary fields. The first-quantized string vacuum is $|0\rangle = c_1|\Omega\rangle$, where $\Omega\rangle$ is the SL$(2, \mathbf{R})$ invariant vacuum. The states created by applying $\alpha^\mu_{-n}, n > 0$,

$c_{-n}, n \geq 0$, $b_{-n}, n > 0$, together with $|0\rangle$ itself, are solutions to the first-quantized theory in a harmonic-oscillator representation. All fields are real; the factors of i are needed to ensure the reality condition for the string field Ψ. The level of a state is defined as its mass squared in units of 1/a' as measured above the tachyon-mass squared. The tachyon 0 is at level 0, A_μ is at level 1, etc. and the level of a field is defined to be the level of the state associated with it. Now that we have defined the level number for the expansion of the string field, level of each term in the action is also defined to be the sum of the levels of the fields involved. For example, if states $|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle$ have level n_1, n_2, n_3 respectively, we assign level $n_1 + n_2 + n_3$ to the interaction term $\Psi_1 \star \Psi_2 \star \Phi_3$. Then *truncation to level* L means to keep only those terms with level equal to or less then L. The *level* (L, I) *truncation* of the full string field theory involves dropping all fields at level $N > L$, and disregarding any interaction term between three fields whose levels add up to a number that is greater then I.

Gauge fixing is necessary to render the functional integral well defined. We choose the Feynman Siegel gauge

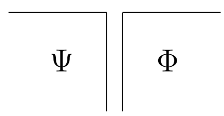
$$b_0 \Psi = 0 \quad . \tag{2.4}$$

The interaction in Eq.2.1 is characterized as follows. The star product $\star$ acts on string functional Ψ, Φ by gluing the right half of the string represented by Ψ to the left half of the one represented by Φ, and the interaction still glues the remaining half of the strings.

This gluing is realized by means of delta functions, and factorizes into separate matter and ghost parts. Defining the embeddings of the strings $\Psi, \Phi, \Psi \star \Phi$ respectively as $x_\Psi(\sigma)$, $x_\Phi(\sigma)$ and $x_{\Psi\star\Phi}(\sigma)$ with $(0 \leq \sigma \geq \pi)$, then the matter part of the product $\star$ reads

$$[\Psi \star \Phi]\left(x_{\Psi\star\Phi}(\sigma)\right) = \int \mathcal{D}x_\Psi(\sigma)\mathcal{D}x_\Phi(\sigma)\Psi\left(\mathcal{D}x_\Psi(\sigma)\right)\Phi\left(\mathcal{D}x_\Phi(\sigma)\right)$$
$$\prod\nolimits_{0\leq\sigma\leq\frac{\pi}{2}} \delta\left(x_\Phi(\sigma) - x_\Psi(\pi-\sigma)\right)\delta\left(x_\Psi(\sigma) - x_{\Psi\star\Phi}(\sigma)\right)\delta\left(x_{\Psi\star\Phi}(\pi-\sigma) - x_\Phi(\pi-\sigma)\right) \tag{2.5}$$

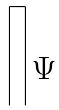
While the first delta function glues the left half of Ψ with the right half of Φ, the second delta function gives the condition that the right half of Ψ coincides with the right half of $\Psi \star \Phi$ and the third one imposes that the left half of Ψ coincides with the left half of $\Psi \star \Phi$.



Similarly, the matter part of the integral over a string field is given by

$$\int \Psi = \int \mathcal{D}x_\Psi(\sigma) \quad \Psi\left(x_\Psi(\sigma)\right) \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta\left(x_\Psi(\sigma) - x_\Psi(\pi-\sigma)\right) \tag{2.6}$$

This corresponds to gluing the left and right halves of the string together with a delta function interaction.



The ghost part of all the operations is analogously defined.

The rather formal derivation of (2.1) is not suitable for concrete calculations, since the star product $\star$ and the integration operator $\int$ have been defined only geometrically as the gluing procedure. There is a more convenient form for the CSFT action

$$S = -\frac{1}{2}\langle\Psi, Q\Psi\rangle - \frac{g}{3}\langle\Psi, \Psi\star\Psi\rangle \tag{2.7}$$

where $\langle\cdot,\cdot\rangle$ is a bilinear inner product related to the $\star$ product and the integration through

$$\langle\Psi,\Phi\rangle = \int \Psi\star\Phi, \qquad \langle\Psi,\Phi\star\Xi\rangle = \int \Psi\star\Phi\star\Xi. \tag{2.8}$$

The form (2.7) allows to relate the geometrical construction seen above to more familiar formulations of CSFT, following from a conformal field theory (CFT) approach [104] and from an operator formulation [23] of the $\star$ product, that we briefly resume in what follows.

Before showing how explicit calculations can be performed, it's worth to mention an additional algebraic structure of open string theory. It arises from the twist operation, which reverses the parametrization of a string, and is summarized by the existence of an operator Ω that satisfies the properties

$$\begin{aligned} &\Omega(QA) = Q(\Omega A)\\ &\langle\,\Omega A\,,\,\Omega B\,\rangle = \langle\,A\,,\,B\,\rangle\\ &\Omega\,(A * B) = (-)^{AB+1}\,\Omega(B) * \Omega(A)\,. \end{aligned} \tag{2.9}$$

Up to a sign, twisting the star product of string fields amounts to multiplying the twisted states in opposite order, as follows from the basic multiplication rule, for which the second half of the first string must be glued to the first half of the second one. Applying this to the string field gives [1]

$$\Omega(\Phi * \Phi) = +\,(\Omega\Phi) * (\Omega\Phi) \tag{2.10}$$

From the above formulas it turns out that the string field action (2.7) is twist invariant. From an heuristic point of view, this follows from the fact that the two- and the three-string vertices are invariant under reflection. The twist eigenvalue of a state is given as $(-1)^N$, where N is the number eigenvalue of the state, defined with $N = 0$ for the zero momentum tachyon. In terms of level, string states at odd level are twist odd and states at even levels are twist even.

[1]This relation follows from the odd Grassmanality of the string field, for which cfr. [105].

2.1.2 The CFT approach

The CFT language represents the complete gluing of the constituents Ψ fields in the 3-string vertex through the conformal mapping of three upper half-disk, each of which represents the propagation of one of the three open strings, to the interior of a unit disk in a conformal plane. The three world-sheets are pictured in Figure 2.1. In the infinite past ($\tau \to \infty$, $z = e^{\tau - i\sigma} = 0$, P_r in the figure) - in the CFT language, a vertex operator was inserted in $z_r = 0$ - each string appeared, then propagated radially to reach the Q interaction point $|z_r| = 1$.

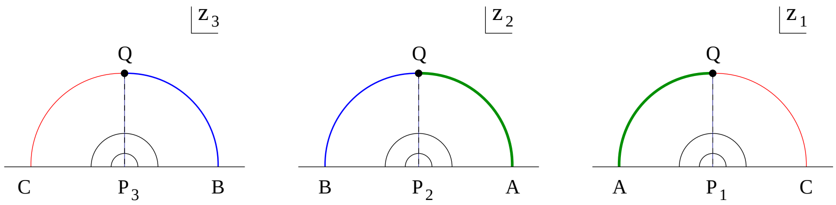


Figure 2.1: 3 half–disks representing three strings that propagate from infinite past $\mathrm{P_i}$ to the interaction point Q.

The interaction that defines the vertex is built by gluing the three half-disks to form a single 'three-punctured' disk in the conformal plane of global coordinate w. The transformation needed should map the common interaction point Q to the center $w = 0$ of the unit disk, and map the open string boundaries - the line segments $-1 \le \mathrm{Re(z_r)} \le 1$ in Figure 2.1 - to the boundary of the unit disk. The explicit maps that send the three half-disk to three wedges in the w plane of Figure 2.2, with punctures at $e^{\frac{2\pi i}{3}}$, 1 and $e^{-\frac{2\pi i}{3}}$ are

$$\begin{aligned} f_1(\xi_1) &= e^{\frac{2\pi i}{3}} \left(\frac{1 + i\xi_1}{1 - i\xi_1} \right)^{\frac{2}{3}} , \\ f_2(\xi_2) &= \left(\frac{1 + i\xi_2}{1 - i\xi_2} \right)^{\frac{2}{3}} , \\ f(\xi_3) &= e^{-\frac{2\pi i}{3}} \left(\frac{1 + i\xi_3}{1 - i\xi_3} \right)^{\frac{2}{3}} , \end{aligned} \tag{2.11}$$

This specification of the functions $f_r(z_r)$ gives the definition of the cubic vertex

$$\int \Psi \star \Psi \star \Psi = \langle f_1 \circ \Psi(0) f_2 \circ \Psi(0) f_3 \circ \Psi(0) \rangle_D \tag{2.12}$$

where $\langle \cdots \rangle_D$ is the correlator over the unit disk and the conformal transformation of $\Psi(0)$ - *i.e.*, in the CFT language, the expression for the vertex operator $\mathcal{O}(z_r = 0)$ associated to the string state Ψ in terms of $f_r(z_r = 0)$ - is defined as

$$f_r \circ \Psi(0) = \left(f_r'(0)\right)^h \Psi\left(f_r(0)\right)$$

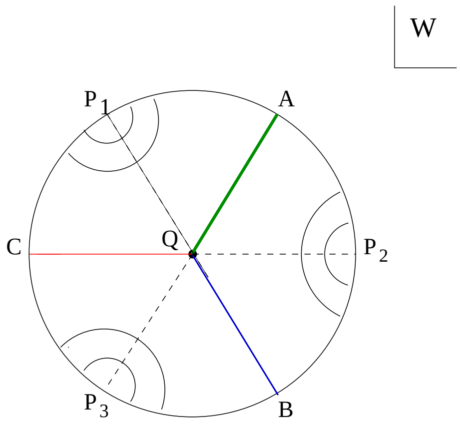


Figure 2.2: The cubic vertex as a 3–punctured unit disk.

for a primary field of conformal weight h.

The generalization of (2.11) to the gluing of n strings is written as

$$f_r(z_r) = e^{\frac{2\pi i}{n}(r-1)} \left(\frac{1+iz_r}{1-iz_r} \right)^{\frac{2}{n}} , \qquad 1 \leq r \leq n \quad . \tag{2.13}$$

Each half-disk is mapped by f_r to a $2\pi/n$ wedge, and these n wedges are collected and rotated in such a way to make a unit disk. The formula (2.13) applies to the kinetic term of the CSFT action, for which the maps result

$$g_1(z_1) = \left(\frac{1+iz_1}{1-iz_1} \right) , \qquad\qquad g_2(z_2) = -\left(\frac{1+iz_1}{1-iz_1} \right) \tag{2.14}$$

thus defining the quadratic term in (2.1) as

$$\int \Psi \star Q\Psi = \langle f_1 \circ \Psi(0) f_2 \circ Q\Psi(0) \rangle_D \quad . \tag{2.15}$$

2.1.3 The operator formulation

In the operator formulation opened up in [23], the action (2.1) is written as

$$S = -\frac{1}{2} \langle V_{12}^{(2)} | \Psi \rangle_1 | Q\Psi \rangle_2 - \frac{g}{3} \langle V^{(3)} |_{123} | \Psi \rangle_1 | \Psi \rangle_2 | \Psi \rangle_3 \quad . \tag{2.16}$$

in terms of the 2-string and the 3-string verteces

$$\langle V_{12}^{(2)} | \in \mathcal{H}_1^* \otimes \mathcal{H}_2^* \qquad\qquad \langle V_{123}^{(3)} | \in \mathcal{H}_1^* \otimes \mathcal{H}_2^* \otimes \mathcal{H}_3^* \quad . \tag{2.17}$$

In (2.17), the subscripts distinguish between different copies of the string Fock space $\mathcal{H}$ referred to different strings. Here $bpz : \mathcal{H} \to \mathcal{H}^*$ is BPZ conjugation. The BPZ of a primary field $\phi(z)$ of conformal dimension d creating a state in the infinite past ($\tau \to -\infty$, $z = e^{\tau - i\sigma} \to 0$) by operator state correspondence, is defined as

$$\langle bpz(\phi)| = \langle 0| \lim_{z\to 0} \phi\left(-\frac{1}{z}\right) \tag{2.18}$$

The action of BPZ on the modes of the primary field

$$\phi_n = \oint \frac{dz}{2\pi i} z^{n+d-1}\phi(z)\,. \tag{2.19}$$

reads

$$bpz(\phi_n) = (-1)^{n+d}\phi_{-n}\,. \tag{2.20}$$

This equation defines BPZ conjugation when we supplement it with the rule

$$bpz\Big(\phi_{n_1}\cdots\phi_{n_p}|0\rangle\Big) = \langle 0|bpz(\phi_{n_1})\cdots bpz(\phi_{n_p})\,. \tag{2.21}$$

The form of the vertices (2.17) is determined by writing the gluing conditions (2.5,2.6) in terms of the oscillators of the first quantized theory. One starts from associating the string functional $\Psi\left[x(\sigma)\right]$ with a function $\Psi\left(\{x_n\}\right)$ of the infinite family of string mode amplitudes. In this way the overlap integral combining (2.5,2.6) can be written in term of modes

$$\int \Psi \star \Phi = \int \prod_{n=0}^{\infty} dx_n dy_n\ \delta\left(x_n - (-1)^n y_n\right)\Psi\left(\{x_n\}\right)\Phi\left(\{y_n\}\right) \tag{2.22}$$

Using in the above formula a standard result for the delta-function as a squeezed state representation in terms of the two-harmonic oscillator Fock space

$$\delta(x \pm y) \to \exp\left(\pm a^\dagger_{(x)} a^\dagger_{(y)}\right)\left(|0\rangle_x \otimes |0\rangle_y\right)\,. \tag{2.23}$$

we end with the following squeezed state representation for the two-string vertex

$$\langle V_{12}^{(2)}|_{matter} = \left(\langle 0| \otimes \langle 0|\right) e^{\sum_{n,m=0}^{\infty} -a_n^{(1)} C_{nm} a_m^{(2)}}$$

where $C_{mn} = (-1)^n\delta_{nm}$ is the BPZ conjugation matrix, an infinite-size matrix connecting the oscillator modes of the two single-string Fock spaces. This expression can be translated in the momentum space, including the ghost sector one ends with

$$\langle V_{12}^{(2)}| = \int d^{26}p\ \langle p|^{(1)} \otimes \langle p|^{(2)} \left(c_0^{(1)} + c_0^{(2)}\right) e^{-\sum_{n=1}^{\infty}(-1)^n\left[a_n^{(1)}\cdot a_n^{(2)} + c_n^{(1)} b_n^{(2)} + c_n^{(2)} b_n^{(1)}\right]} \tag{2.24}$$

A direct calculation from the CFT approach, computing the two-point function of an arbitrary pair of states on the disk, gives the same result for $V^{(2)}$.

From (2.24) we can derive the quadratic term of the CSFT action (2.16). The gauge condition (2.4) sets to zero terms in the expansion of the string field Ψ (2.3) containing

c_0. In this gauge, the kinetic piece S_2 of the action in terms of particle fields to level two is thus

$$\begin{aligned} S_2 &= -\frac{1}{2}\langle V_{12}^{(2)}|\Psi\rangle_1|Q\Psi\rangle_2 \\ &= \frac{1}{2}\int d^{26}x \, \big[\phi^2 - B_{\mu\nu}B^{\mu\nu} + \beta_1^2 + \cdots \\ &\qquad -\partial_\mu\phi\,\partial^\mu\phi - \partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\mu B_{\nu\lambda}\,\partial^\mu B^{\nu\lambda} + \partial_\mu\beta_1\,\partial^\mu\beta_1 + \cdots\big] \end{aligned} \tag{2.25}$$

The procedure for obtaining the 3-string vertex in the pure operator formalism [23] is more involved and we will not give it in details. A possible way to proceed is to start by making an ansatz for $\langle V_{123}^{(3)}|$

$$\begin{aligned} \langle V_{123}^{(3)}| = & \int d^{26}p_1 d^{26}p_2 d^{26}p_3\, \delta(p_1+p_2+p_3)\langle p_1|^{(1)}c_0^{(1)} \otimes \langle p_2|^{(2)}c_0^{(2)} \otimes \langle p_3|^{(3)}c_0^{(3)}\cdot \\ & \cdot e^{-\frac{1}{2}\sum_{r,s=1}^{3}\left[\alpha_n^{(r)\mu}N_{nm}^{rs}\eta_{\mu\nu}\alpha_m^{(s)\nu} + c_n^{(r)}X_{nm}^{rs}b_m^{(s)}\right]} \end{aligned} \tag{2.26}$$

and then calculate the quantities N_{mn}^{rs} and X_{mn}^{rs}, called *Neumann coefficients*, in terms of conformal transformations. This is always possible, since due to (2.7,2.16) the vertex $\langle V_{123}^{(3)}|$ is such that

$$\langle \Psi, \Phi, \Xi\rangle \equiv \langle V_{123}^{(3)}|\Psi\rangle_{(1)}|\Phi\rangle_{(2)}|\Xi\rangle_{(3)}\,. \tag{2.27}$$

and in (2.12) it was provided a definition of the left hand side of this equation. It turns out that the coefficients N_{nm}^{rs}, X_{nm}^{rs} are given in terms of the 6-string Neumann functions $\bar{N}_{nm}^{rs}$, $1 \le r, s \le 6$ through

$$\begin{aligned} N_{nm}^{rs} &= \frac{1}{2}(\bar{N}_{nm}^{rs} + \bar{N}_{nm}^{r(s+3)} + \bar{N}_{nm}^{(r+3)s} + \bar{N}_{nm}^{(r+3)(s+3)}) \\ X_{nm}^{rs} &= -m(\bar{N}_{nm}^{rs} - \bar{N}_{nm}^{r(s+3)}), \qquad s = r, r+2 \\ X_{nm}^{rs} &= m(\bar{N}_{nm}^{rs} - \bar{N}_{nm}^{r(s+3)}), \qquad s = r+1 \end{aligned} \tag{2.28}$$

For $s < r$ the values of X_{nm}^{rs} are fixed by using (2.28) and the cyclic symmetry of the coefficients under $r \to (r \bmod 3)+1$, $s \to (s \bmod 3)+1$. When all momenta are zero the Neumann functions with $n, m > 0$, are given, through explicit CFT calculations, by

$$\bar{N}_{nm}^{rs} = \frac{1}{nm}\oint_{z^r}\frac{dz}{2\pi i}\oint_{z^s}\frac{dw}{2\pi i}\frac{1}{(z-w)^2}(-1)^{n(r-1)+m(s-1)}(f(z))^{(-1)^r n}(f(w))^{(-1)^s m} \tag{2.29}$$

with

$$f(z) = \frac{z(z^2-3)}{3z^2-1} \tag{2.30}$$

and

$$z^1, \ldots, z^6 = \sqrt{3}, 1/\sqrt{3}, 0, -1/\sqrt{3}, -\sqrt{3}, \infty. \tag{2.31}$$

An equivalent formula for $\langle V_{123}^{(3)}|$ that will be useful in what follows is

$$\langle V_{123}^{(3)}| = \int d^{26}p_1 d^{26}p_2 d^{26}p_3\, \delta(p_1+p_2+p_3)\langle p_1|^{(1)}c_0^{(1)} \otimes \langle p_2|^{(2)}c_0^{(2)} \otimes \langle p_3|^{(3)}c_0^{(3)}\cdot$$

$$\cdot e^{-\frac{1}{2}\sum_{r,s=1}^{3}\left[a_m^{(r)}V_{mn}^{rs}a_n^{(s)}+2a_m^{(r)}V_{m0}^{rs}p^{(s)}+p^{(r)}V_{00}^{rs}p^{(s)}-2c_m^{(r)}X_{mn}^{rs}b_n^{(s)}\right]} \tag{2.32}$$

where the Neumann coefficients V_{mn}^{rs} are related to the 6-string Neumann functions (2.29) as described in Appendix B.

From (2.26) or (2.32) we can derive the cubic interaction term S_3 in the string field theory action (2.16) for an arbitrary set of 3 fields. In the Feynman-Siegel gauge and for the fields appearing in (2.3), the interactions at zero momentum are

$$\begin{aligned} S_3 &= -\frac{g}{3}\langle V^{(3)}|_{123}|\Psi\rangle_1|\Psi\rangle_2|\Psi\rangle_3 \\ &= \kappa g\left(-\phi^3+\frac{5}{3^2\sqrt{2}}B_\mu^{\ \mu}\phi^2+\frac{11}{3^2}\beta_1\phi^2-\frac{2^4}{3^2}\phi A_\mu A^\mu+\frac{5\cdot 2^3\sqrt{2}}{3^5}B_\mu^{\ \mu}A_\nu A^\nu\right. \\ &\qquad \left.-\frac{2^8\sqrt{2}}{3^5}B^{\mu\nu}A_\mu A_\nu+\frac{11\cdot 2^4}{3^5}\beta A_\mu A^\mu+\cdots\right) \end{aligned}$$

where

$$\kappa=\frac{3^{7/2}}{2^7}.$$

2.1.4 Off-shell amplitudes

Following the classification of ref.[106], there are four possible approaches for computing off-shell amplitudes that we briefly describe here since three of them will be used later in this chapter.

a) Field theory approach

The string field contains an infinite number of component fields, whose number grows exponentially with the mass level L. In this approach one can approximate the calculations by truncating the string field up to some fixed level L [107], for this reason it is called “level truncation on fields”. For example one can construct the CSFT lagrangian by means of a truncated string field up to some level L and then compute the cubic terms for each of the field components at the desired level. From this classical action one can then derive the tree level effective action for some field component (*e.g.* the tachyon) by integrating out all the other ones through the solution of their equations of motion.

We shall use this procedure in Sections 2.4-3.1 to derive the perturbative tachyon effective action.

b) Conformal mapping

With this method Giddings [17] reproduced the on-shell Veneziano amplitude directly from Witten’s CSFT. He gave an explicit conformal map that takes the Riemann surfaces defined by the Witten diagrams to the standard disc with four tachyon vertex operators on the boundary. Following Giddings’ procedure and with some additional analysis -related to the oscillator method in c)- Samuel [29] and Sloan [19] computed the off-shell Veneziano amplitude. This procedure allows in principle the calculation of any amplitude [20]. Amplitudes computed using this method are exact, although numerical approximations are necessary to get concrete numbers for them. We shall solve Samuel's equations to derive, from the 4-tachyon off shell amplitude, some very accurate numerical approximations of the quartic coupling of the tachyon potential and of the coefficients for a time dependent solution of CSFT.

c) Oscillator method

Perturbative amplitudes can be directly evaluated using the oscillator representation of the vertices and propagators in CSFT. The vertex and the propagator can be written completely in terms of squeezed states [24], *i.e.* in terms of exponentials of quadratic forms in the oscillators creating and annihilating operators. In this way the complete set of amplitudes associated with a Feynmann diagram results in an integral over the internal momenta that can be evaluated using standard squeezed techniques. Any perturbative amplitude is then given in a closed-form expression containing infinite-dimensional Neumann matrices. While no analytical way is known at present to exactly calculate such expressions, one can evaluate the amplitudes to a high degree of precision truncating the Neumann matrices to finite size [25]. This means truncate the levels of oscillators in the string states which are considered, this is the reason for which this method is known as "level truncation on oscillators". Rather then having to include a number of fields which grows exponentially in the level, with this procedure one simply needs to evaluate quantities, as the determinant of the Neumann matrices, whose size grows linearly in the truncation level. A specific example of this method is given in Appendix B and will be used in Section 3.1.

d) Moyal string field theory

In this alternative formulation of SFT the string joining star product is identified with the Moyal product. Calculations performed using this method reproduce directly the expressions for the off-shell amplitudes as for example the 3-point and 4-point tachyon amplitudes [108]. Some numerical results [109] achieved with this procedure are comparable to those obtained using the methods (a)-(c).

In this chapter we mainly focus on the four tachyon amplitude which we evaluate by solving explicitly Samuel's elliptic equations for the off-shell factor (method (b)). We have obtained a new series solution for the off-shell factor introduced by Samuel [29], which, at variance with the one found in [108], provides the off-shell factor in terms of the original coordinates used in [29].

As a test for the solution we shall first improve the numerical approximation for the evaluation of the exact quartic self-coupling c_4 in the tachyon potential. This was computed for the first time in [107] and repeated to a higher degree of precision in [25]. Our results provides c_4 with a precision that goes up to the ninth significative digit and is in complete agreement with the extrapolations of ref. [110]. The solutions of the 4-tachyon off-shell amplitude that we have found therefore is a very useful tool for providing precise tests of CSFT. The agreement with previous work on the subject, both on the quartic tachyon coupling and on the CSFT rolling tachyon, is an excellent test for the accuracy of our off-shell solution. A second application will be descripted in the next chapter, where we shall improve the CSFT time-dependent solution given in [56] as a sum in powers of e^t.

The chapter is organized as follows. In Section 2.2 we review the derivation of the off-shell four tachyon amplitude following ref. [29]. Explicit formulas for the Neumann coefficients involved in the oscillator formalism are reported in Appendix A. A brief review of the level truncation method is also given and a specific example is provided in Appendix B. In Section 2.3 we develop the tools needed to perform the computations of Section 2.4. A solution to the elliptic equations defining the off-shell amplitude is derived, obtaining a useful expansion of $\kappa(x)$ in powers of the Koba-Nielsen variable x. This analysis improves the accuracy in the evaluation of the quartic coupling of the tachyon potential, which is performed in Section 2.4. In the next chapter, we will use the exact four-point amplitude to study the first few coefficients of the rolling tachyon solution expressed as a sum of exponentials e^{nt}, and we will compare the corresponding solution with the ones obtained in the level truncation scheme and in the field theory approach.

Our calculations were performed using the symbolic manipulation program *Mathematica*.

2.2 The 4-tachyon off-shell amplitude

The first step in computing the off-shell four tachyon amplitude in CSFT is to relate it to a world-sheet process. Four-point amplitudes involve one propagator and two vertices. After gauge fixing, we use the Feynman-Siegel gauge (2.4), the propagator becomes b_0/L_0 where L_0 is the Virasoro generator for the intermediate state including ghosts

$$L_0 = p \cdot p - 1 + \sum_{n=1}^{\infty} \left(\alpha_{-n} \cdot \alpha_n + n b_{-n} c_n + n c_{-n} b_n \right) \ . \tag{2.33}$$

Writing the propagator

$$\frac{b_0}{L_0} = b_0 \int_0^\infty dT e^{-TL_0} \ ,$$

the Schwinger parameter T can be interpreted in the world-sheet language as a modular parameter describing the lenght of a propagating strip, inserted into the amplitude.

2.2.1 Conformal mapping: on-shell amplitude

A closed analytical expression for the off-shell four tachyon amplitude in CSFT [1] was derived in [29] by following Gidding's analysis of the on-shell Veneziano amplitude [17]. Giddings gave an explicit conformal map that takes the Riemann surfaces defined by the world-sheet diagrams for the four-tachyon scattering amplitude to the standard disc with four tachyon vertex operators on the boundary. This conformal map is defined in terms of four parameters $\alpha,\ \beta,\ \gamma,\ \delta$. The four parameters are not independent variables. They satisfy the relations

$$\alpha\beta = 1 \qquad \gamma\delta = 1 \tag{2.34}$$

and

$$\frac{1}{2} = \Lambda_0(\theta_1, k) - \Lambda_0(\theta_2, k) , \tag{2.35}$$

where $\Lambda_0(\theta, k)$ is defined by

$$\Lambda_0(\theta, k) = \frac{2}{\pi} \left(E(k)F(\theta, k') + K(k)E(\theta, k') - K(k)F(\theta, k') \right) \quad . \tag{2.36}$$

In (2.36) $K(k)$ and $E(k)$ are complete elliptic functions of the first and second kinds, $F(\theta, k)$ is the incomplete elliptic integral of the first kind (we follow the notation of ref.[111]). The parameters θ_1, θ_2, k and k' satisfy

$$k^2 = \frac{\gamma^2}{\delta^2} \qquad\qquad k'^2 = 1 - k^2 \tag{2.37}$$

$$\sin^2\theta_1 = \frac{\beta^2}{\beta^2 + \gamma^2} \qquad\qquad \sin^2\theta_2 = \frac{\alpha^2}{\alpha^2 + \gamma^2} \, . \tag{2.38}$$

By convention $\beta > \alpha$ and $\delta > \gamma$. Because of (2.34) and (2.35) only one variable is independent. By convention this is taken to be α, that is related to T, the lenght of the intermediate strip, by

$$\frac{T}{2} = K(k') \left[Z(\theta_2, k') - Z(\theta_1, k') \right] \tag{2.39}$$

where $Z(\theta, k)$ is defined through the ordinary elliptic functions

$$Z(\theta, k) = K(k)E(\theta, k) - E(k)F(\theta, k) \, . \tag{2.40}$$

The parameter α is finally related to the Koba-Nielsen variable x - in terms of which the standard formula for the Veneziano amplitude is written - through

$$x = \left(\frac{(1-\alpha^2)}{(1+\alpha^2)} \right)^2 , \qquad\qquad \alpha = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \, . \tag{2.41}$$

Using this conformal map Giddings managed to derive the Veneziano amplitude from CSFT. Because of the cubic vertex, in CSFT there are six relevant Feynman diagrams for four particles processes (fig.2.3). The contribution from the graph (a) in fig.2.3, the s-channel amplitude, was computed in [17] to be

$$A_s(p_1, p_2, p_3, p_4) = \int_{\alpha_0}^{0} d\alpha \, 2A_G \, \frac{dT}{d\alpha} \, (\beta - \alpha)^{2(p_1\cdot p_2 + p_3 \cdot p_4)} (\beta + \alpha)^{2(p_1 \cdot p_3 + p_2 \cdot p_4)} (2\alpha)^{2(p_2 \cdot p_3)} (2\beta)^{2(p_1 \cdot p_4)} \tag{2.42}$$

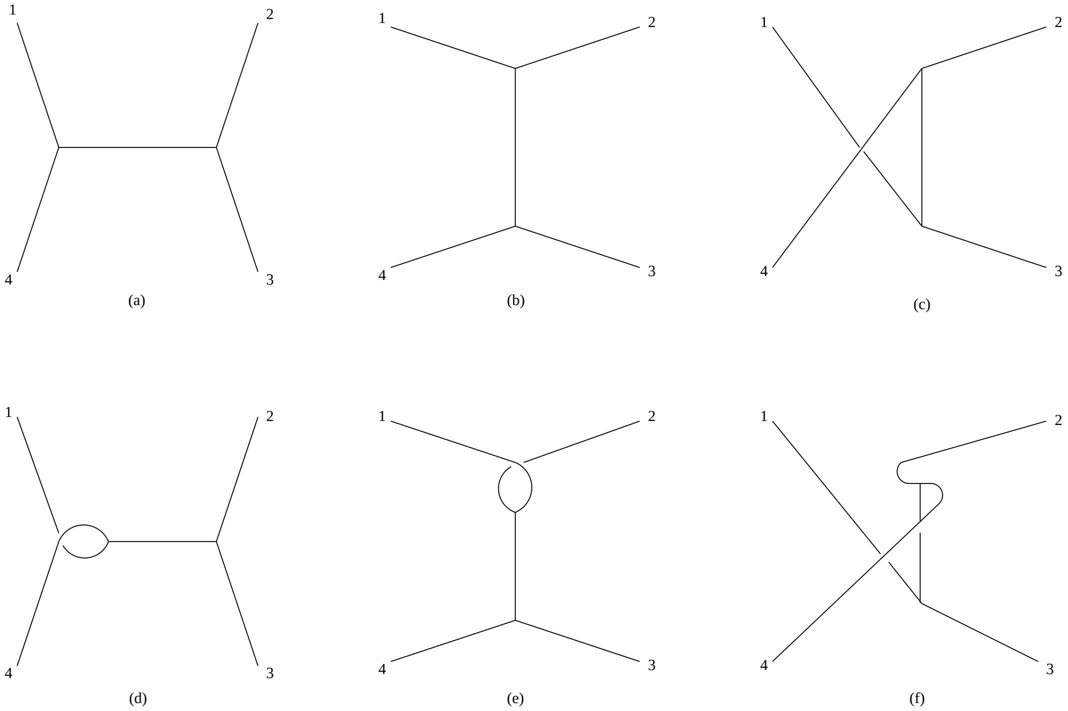


Figure 2.3: The relevant Feynman diagrams for the four particles scattering.

where the integration limits $\alpha_0 = \sqrt{2} - 1$ and $\alpha = 0$ correspond to $T = 0$ and $T = \infty$ respectively , $2A_G$ is the ghost contribution and is given by

$$2A_G = 8\frac{1}{2\pi}\sqrt{\alpha^2 + \gamma^2}\sqrt{\beta^2 + \gamma^2}(\beta^2 - \alpha^2)K(\gamma^2) \tag{2.43}$$

and the Jacobian factor almost cancels the ghost factor

$$\frac{dT}{d\alpha} = -\frac{4(\beta^2 - \alpha^2)}{\alpha A_G} \quad . \tag{2.44}$$

2.2.2 Oscillator method: off-shell amplitude

Samuel derived a perturbative off-shell string amplitude [29] directly from string field theory by requiring that it reproduces Gidding's result (2.42) when the momenta are set on-shell. We now briefly review Samuel results.

Let

$$\frac{g}{2}\langle V^{(3)}_{41I}|\langle V^{(3)}_{23J}|b_0 e^{-TL_0}|V^{(2)}_{IJ}\rangle = \langle V^{(4)}_{1234}| \tag{2.45}$$

denote the vertex function associated with the graph (a) in fig.2.3, where the subscripts $1, 2, 3, 4, I$ and J indicate Fock-space labels. The full contribution to the diagram is

$$\int_0^\infty dT \langle V^{(4)}_{1234}||\Psi^{(4)}_4\rangle|\Psi^{(3)}_3\rangle|\Psi^{(2)}_2\rangle|\Psi^{(1)}_1\rangle \tag{2.46}$$

where $|\Psi_r^{(r)}\rangle$ is the Fock-space representation of the external states. The explicit oscillator representations of $\langle V^{(2)}|$ and $\langle V^{(3)}|$ (see Section 2.1.3)

$$\langle V_{12}^{(2)}| = \int d^{26}p \; \langle p|^{(1)} \otimes \langle -p|^{(2)} \left(c_0^{(1)} + c_0^{(2)}\right) e^{-\sum_{n=1}^{\infty}(-1)^n\left[a_n^{(1)}\cdot a_n^{(2)} + c_n^{(1)}b_n^{(2)} + c_n^{(2)}b_n^{(1)}\right]} \tag{2.47}$$

$$\begin{aligned}\langle V_{123}^{(3)}| = \quad & \int d^{26}p_1 d^{26}p_2 d^{26}p_3 \; \delta(p_1+p_2+p_3)\langle p_1|^{(1)}c_0^{(1)} \otimes \langle p_2|^{(2)}c_0^{(2)} \otimes \langle p_3|^{(3)}c_0^{(3)} \cdot \\ & \cdot e^{-\frac{1}{2}\sum_{r,s=1}^{3}\left[a_m^{(r)}V_{mn}^{rs}a_n^{(s)} + 2a_m^{(r)}V_{m0}^{rs}p^{(s)} + p^{(r)}V_{00}^{rs}p^{(s)} - 2c_m^{(r)}X_{mn}^{rs}b_n^{(s)}\right]}\end{aligned} \tag{2.48}$$

show that all the terms in (2.46) are given in terms of exponentials of quadratic expressions in the oscillators. Using standard squeezed state techniques [24], closed-form expressions can be given for any perturbative amplitude. In the case of the four tachyon amplitude corresponding to the first diagram of fig.2.3, this procedure gives[2]

$$A_4(p_1,p_2,p_3,p_4) = \frac{\lambda_c^2 g^2}{2}\,\delta(\textstyle\sum_i p_i)\int_0^\infty dT\, e^T \det\left(\frac{1-(\tilde{X}^{11})^2}{1-(\tilde{V}^{11})^2}\right) e^{-\frac{1}{2}p_i Q^{ij} p_j} \tag{2.49}$$

where λ_c is a constant related to the Neumann coefficient for the three tachyon vertex, $\lambda_c = e^{3V_{00}^{11}} = \frac{3^{9/2}}{2^6}$. In this formula $\tilde{V}^{11}$ and $\tilde{X}^{11}$ are defined by

$$\tilde{V}_{mn}^{11} = e^{-\frac{(m+n)}{2}T}V_{mn}^{11} \qquad \tilde{X}_{mn}^{11} = e^{-\frac{(m+n)}{2}T}X_{mn}^{11} \tag{2.50}$$

where V^{rs} and X^{rs} are infinite-dimensional matrices

$$V^{rs} = \begin{pmatrix} V_{11}^{rs} & V_{12}^{rs} & \dots & V_{mn}^{rs} & \dots \\ V_{21}^{rs} & V_{22}^{rs} & \dots & V_{m+1,n}^{rs} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad X^{rs} = \begin{pmatrix} X_{11}^{rs} & X_{12}^{rs} & \dots & X_{mn}^{rs} & \dots \\ X_{21}^{rs} & X_{22}^{rs} & \dots & X_{m+1,n}^{rs} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{2.51}$$

whose elements are matter and ghost Neumann coefficients of the cubic string field theory vertex, for which exact expressions are given in the Appendix B. Q^{ij} are defined as

$$\begin{aligned} Q^{ij} = \quad & V_{0m}^{iI}\left(\frac{1}{1-(\tilde{V}^{11})^2}\right)_{mn}\tilde{V}_{np}^{11}V_{p0}^{Ij} + V_{00}^{11} - T(2-\delta_{ij}) \qquad i,j=1,2 \text{ or } i,j=3,4 \\ Q^{ij} = \quad & -V_{0m}^{iI}\left(\frac{1}{1-(\tilde{V}^{11})^2}\right)_{mn} C\,\tilde{V}_{np}^{11}V_{p0}^{Ij} \qquad i=1,2 \text{ and } j=3,4 \text{ or } i=3,4 \text{ and } j=1,2 \end{aligned} \tag{2.52}$$

where $m,n,p \geq 1$, $C = \delta_{mn}(-1)^n$ and the sum over I denotes a sum over the intermediate states.

The two expressions (2.42) and (2.49) should both represent the contribution to the four tachyon amplitude coming from the diagram (a) in fig.2.3 when the momenta are

[2]We follow the notation of refs.[25, 112].

on-shell. To relate them in the proper way, a general procedure was developed in [29] for computing the functions Q^{ij} appearing in (2.49) from the Giddings map, giving

$$
\begin{aligned}
Q^{11} &= Q^{44} = \ln\alpha - \ln\kappa, & Q^{22} &= Q^{33} = -\ln\alpha - \ln\kappa \\
Q^{12} &= Q^{21} = -\ln|\alpha-\beta|, & Q^{13} &= Q^{31} = -\ln(\alpha+\beta) \\
Q^{14} &= Q^{41} = -\ln(2\beta), & Q^{23} &= Q^{32} = -\ln(2\alpha) \\
Q^{24} &= Q^{42} = -\ln(\alpha+\beta), & Q^{34} &= Q^{43} = -\ln|\alpha-\beta|
\end{aligned}
\tag{2.53}
$$

where κ is given as an integral

$$
\ln(\kappa) = -2\alpha\frac{(\beta^2-\alpha^2)}{\sqrt{(\alpha^2+\gamma^2)(\alpha^2+\delta^2)}}\int_1^\infty dw\,\ln(w-1)\frac{d}{dw}\left(\frac{\sqrt{(w^2+\alpha^2\gamma^2)(w^2+\alpha^2\delta^2)}}{(w+1)(\beta^2w^2-\alpha^2)}\right) \quad . \tag{2.54}
$$

As already noticed, although $\alpha,\beta,\gamma,\delta$ all appear in the above equation, there is only one independent variable, so that the function κ in (2.54) is actually a function of α. The substitution of (2.53) in (2.49) leads to the following formula

$$
\begin{aligned}
A_4(p_1,p_2,p_3,p_4) &= \lambda_c^2\frac{g^2}{2}\int_{\alpha_0}^{0} d\alpha\frac{dT}{d\alpha}e^T\det\left(\frac{1-(\tilde{X}^{11})^2}{1-(\tilde{V}^{11})^2}\right)[\kappa(\alpha)]^{\sum_{i=1}^4 p_i^2}(\alpha)^{-(p_1^2+p_4^2)+p_2^2+p_3^2} \\
&\quad |\alpha-\beta|^{2(p_1\cdot p_2+p_3\cdot 4)}(\beta+\alpha)^{2(p_1\cdot p_3+p_2\cdot 4)}(2\alpha)^{2(p_2\cdot p_3)}(2\beta)^{2(p_1\cdot p_4)}
\end{aligned}
\tag{2.55}
$$

Comparing the two expressions (2.42) and (2.55) on shell ($p_i^2=1$), one can see that the momentum dependence matches and for the momentum independent part the following identity holds

$$
\lambda_c^2\left(\frac{dT}{d\alpha}\right)e^T\det\left(\frac{1-(\tilde{X}^{11})^2}{1-(\tilde{V}^{11})^2}\right) = 2A_g\frac{dT}{d\alpha}\frac{1}{[\kappa(\alpha)]^4} \quad . \tag{2.56}
$$

By trading the variable α for the Koba-Nielsen variable x through (2.41) in (2.55), the contribution from the first graph in fig.2.3 becomes

$$
A_4(p_1,p_2,p_3,p_4) = \frac{g^2}{2}\int_{\frac{1}{2}}^{1} dx\; x^{p_1\cdot p_2+p_3\cdot p_4}(1-x)^{(p_1+p_4)^2-2}\left(\frac{\kappa(x)}{2}\right)^{\sum_{i=1}^4 p_i^2-4} \tag{2.57}
$$

The remaining diagrams (b),(c),(d),(e),(f) of fig.2.3 can be obtained from the first one by a suitable permutation of the string labels, *i.e.* by permuting the momenta in (2.57), and the total four-point tachyon amplitude is the sum of these six contributions. Notice that the Veneziano amplitude is exactly reproduced when $p_i^2=1$ in (2.57) and the additional factor containing $\kappa(x)$ goes to 1.

2.2.3 Level truncation

The infinite-dimensional matrices (2.51) appearing in the final expression for a given diagram are expressed in terms of the Neumann coefficients of Witten's vertex. The level truncation method we use in this chapter consists in the truncation on the level of oscillators associated with the Neumann coefficients. This procedure is somewhat different from the original method of level truncation [107] (method a) section 2.1.4), in which one calculates the SFT action by only including in the string field expansion contributions up to a fixed total oscillator level. While the latter approach involves computations with a number of fields that grows exponentially in the level, in the former one has to calculate the determinant of some matrices whose size grows linearly in the truncation level.

Let us explicitly remind the procedure [112] in the case of a tree diagram with four external fields as (2.49), in which there is a single internal propagator with Schwinger parameter T. One starts with a suitable change of coordinates in (2.49)

$$\sigma = e^{-T} \tag{2.58}$$

then expands in powers of σ, so getting an expression of the form

$$\int_0^1 \frac{d\sigma}{\sigma^2}\sigma^{p^2}\sum_{n=0}^{\infty} c_n(p_i)\sigma^n = \sum_{n=0}^{\infty}\frac{c_n(p^i)}{p^2+n-1} \quad , \tag{2.59}$$

where $p = p_1 + p_2 = p_3 + p_4$ represents the momentum of the intermediate state. The poles $p^2 = 1-n$ in (2.59) clearly correspond to the contributions of intermediate particles as the tachyon ($n = 0$), the gauge field ($n = 1$) and all the other open string massive fields. Truncate all the matrices to size $L \times L$ means to truncate the sum in (2.59) to $n = L$, thus imposing a limit on the mass of the intermediate states.

The analysis can be simplified by noting that, as seen in Section 2.1.1, in CSFT the combined level of fields coupled by a cubic interaction must be even. For example, there is no vertex coupling two tachyons (level zero) with the gauge boson (level 1). It follows that there are no tree level Feynman diagrams with all external tachyons and internal fields of odd level. Thus, in calculating the tachyonic effective action we may set odd level fields to 0, *i.e.* only even powers of σ in the expansion (2.59), need to be considered. An explicit example of the procedure above explained is given in Appendix C, where the four tachyon amplitude at level $L = 2$ is derived in the time-dependent case.

2.3 Solution for the function $\kappa(x)$

As shown in the previous section the off-shell 4-point string amplitudes are completely determined once the function $\kappa(x)$ defined by (2.54) is known. To determine the function $\kappa(\alpha, \gamma)$ we have first to solve eq.(2.35) for one of the two variables in terms of the other, so that the function κ will be a function of only one of the two α or γ. Since the four point amplitude is written in terms of an integral over x, which is easily related to α through (2.41), it would be more natural to solve for γ as a function of α then the opposite. The solution can be found numerically and for γ as a function of x is given by the solid line in fig.2.4. γ goes from 0 to 1 while x goes from 1 to 1/2 and α goes from 0 to $\sqrt{2}-1$. To check for the accuracy of the solution, we have found two different expansions: 1) A power series in α which gives γ in a neighbor of 0 and can be inverted so as to give α as a function of γ around 0. 2) An expansion of α around $\sqrt{2}-1$ as an expansion in $1-\gamma$, this series cannot be inverted due to the presence of terms of the type $(1-\gamma)^m \log(1-\gamma)^n$. We have found a general procedure to obtain as many terms as necessary in both expansions and the function $\alpha(\gamma)$ can be determined in the whole range $0 \le \gamma \le 1$. As we shall show in fact the two series for $\alpha(\gamma)$ overlap in an extended interval that goes from $\gamma \sim 0.6$ to $\gamma \sim 0.7$.

2.3.1 γ and α around 0

By using the integral representations of the elliptic functions [111] it is possible to write the equation (2.35) in a useful form

$$E(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}} - (1-\gamma^4)K(\gamma^2)\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}(\sqrt{1+t^2})^3} = \frac{\pi}{4} \quad (2.60)$$

To expand (2.60) for small γ and α we have to divide the integration region into three intervals in such a way that the square roots in the denominators of (2.60) can be consistently expanded and the integrals in t performed. For example consider the integral in the first term of (2.60), it can be rewritten as

$$\begin{aligned} &\int_{\alpha\gamma}^{\gamma/\alpha} dt \frac{1}{\sqrt{t^2+\gamma^4}\sqrt{1+t^2}} = \\ &\int_{\alpha\gamma}^{\gamma^2} dt \frac{1}{\gamma^2\sqrt{1+\frac{t^2}{\gamma^4}}\sqrt{1+t^2}} + \int_{\gamma^2}^{1} dt \frac{1}{t\sqrt{1+\frac{\gamma^4}{t^2}}\sqrt{1+t^2}} + \int_{1}^{\frac{\gamma}{\alpha}} dt \frac{1}{t^2\sqrt{1+\frac{\gamma^4}{t^2}}\sqrt{1+\frac{1}{t^2}}} \end{aligned} \quad (2.61)$$

In each integral of the rhs the integration domain is contained in the convergence radius of the Taylor expansions of the square roots containing γ, so that they can be safely expanded and the integrals in t performed.

With this procedure one gets the following equation equivalent to (2.60)

$$\begin{aligned} &E(\gamma^2)\sum_{n,k=0}^{\infty} \frac{\Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}-k)n!k!} \\ &\left\{\frac{2}{2n+2k+1}\left[\gamma^{4k} - \left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right] + (1-\delta_{kn})\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n} - \delta_{kn}\gamma^{4n}\ln\gamma^2\right\} \\ &-(1-\gamma^4)K(\gamma^2)\sum_{n,k=0}^{\infty} \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}-n)\Gamma(-\frac{1}{2}-k)n!k!} \\ &\left\{\frac{1}{2n+2k+1}\left[\gamma^{4k} - \left(\frac{\alpha}{\gamma}\right)^{2n+1}(\alpha\gamma)^{2k}\right] + (1-\delta_{kn})\frac{\gamma^{4n}-\gamma^{4k}}{2k-2n} - \delta_{kn}\gamma^{4n}\ln\gamma^2 \right. \\ &\left. +\frac{1}{2n+2k+3}\left[\gamma^{4n} - \left(\frac{\alpha}{\gamma}\right)^{2k+3}(\alpha\gamma)^{2n}\right]\right\} = \frac{\pi}{4} \end{aligned} \quad (2.62)$$

The series containing $\ln\gamma^2$ can be resummed, the first gives $\frac{2}{\pi}K(\gamma^2)$ the second $\frac{2}{\pi(1-\gamma^4)}E(\gamma^2)$. Hence these terms cancel and $\ln\gamma^2$ actually disappears from the equation. As a consequence one can write γ as a power series in α whose coefficients are determined requiring that eq.(2.62) is satisfied. γ turns out to contain only the powers α^{4n+1}, $n \in \mathbb{N}$. We have determined the first 12 terms of this series to get a very good approximation for γ in an extended neighbor of zero (in which sense it is an extended neighbor will be clarified later)

$$\gamma = \sqrt{3}\alpha\left(1 + 5\alpha^4 + \frac{1041}{16}\alpha^8 + \frac{38719}{32}\alpha^{12} + \frac{109062913}{4096}\alpha^{16} + \frac{5278728465}{8192}\alpha^{20} + \frac{2172202186251}{131072}\alpha^{24} + \frac{116561474500179}{262144}\alpha^{28} + \frac{3303689940814193505}{268435456}\alpha^{32} + \frac{187301165958864015157}{536870912}\alpha^{36} + \frac{86571446884950765378149}{8589934592}\alpha^{40} + \frac{5078927050639748451791733}{17179869184}\alpha^{44} + O(\alpha^{48})\right) \tag{2.63}$$

Any higher order in (2.63) can be in principle computed from (2.62). Using (2.41) we can plot γ as a function of x and compare it to the graph obtained from the numerical solution of eq.(2.60). As it is clear from fig.2.4 $\gamma(x)$ has in $x = 1/2$ a vertical tangent, thus showing the presence of a branch point which cannot be gotten from a power series of the form (2.63). Nevertheless (2.63) gives a very good approximation for $\gamma(x)$ except in a small neighbor of $x = 1/2$. In particular the agreement between the values of γ obtained from the series (2.63) and the numerical values is on the 15-th significative digit for $0.8 \le x \le 1$, where the series (2.63) is expected to give exact results, thus providing a precision test for the accuracy of the numerical solution. Moreover, the expansion (2.63) can be iteratively inverted to give a series for α as a function of γ

$$\alpha = \frac{\gamma}{\sqrt{3}}\left(1 - \frac{5}{9}\gamma^4 + \frac{959}{1296}\gamma^8 - \frac{10993}{7776}\gamma^{12} + \frac{83359631}{26873856}\gamma^{16} - \frac{3579242677}{483729408}\gamma^{20} + \frac{1297273056905}{69657034752}\gamma^{24} - \frac{6783253984031}{139314069504}\gamma^{28} + \frac{168109910408625655}{1283918464548864}\gamma^{32} - \frac{24949101849547687507}{69331597085638656}\gamma^{36} + \frac{10046339553062261150885}{9983749980331966464}\gamma^{40} - \frac{512861712698825472832315}{179707499645975396352}\gamma^{44} + O(\gamma^{48})\right) \tag{2.64}$$

By plugging the expansion (2.63) in (2.54) and using (2.41), the corresponding expansion for $\kappa(\alpha)$ can be found by means of numerical integration

$$\kappa(\alpha) = \frac{8}{3\sqrt{3}}\exp\left[-2.5\,\alpha^4 - 7.1562\,\alpha^8 - 75.927\,\alpha^{12} - 1238.7\,\alpha^{16} - 24301\,\alpha^{20} - 531290\,\alpha^{24} - 1.2489\cdot 10^7\,\alpha^{28} - 3.0923\cdot 10^8\,\alpha^{32} - 7.9627\cdot 10^9\,\alpha^{36} - 2.1140\cdot 10^{11}\,\alpha^{40} - 5.7517\cdot 10^{12}\,\alpha^{44}\right] + O(\alpha^{48})\,. \tag{2.65}$$

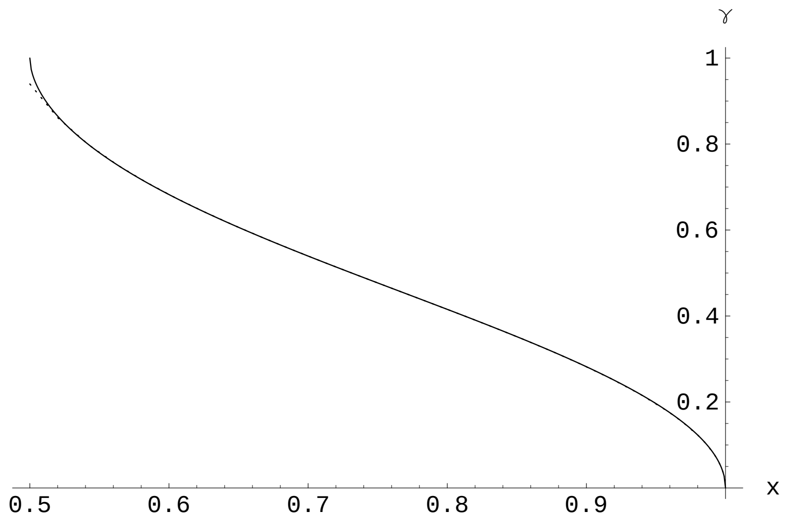


Figure 2.4: Plots of $\gamma(x)$: the solid line is the numerical solution of the elliptic equation, the dashed line is the power series.

2.3.2 γ around 1 and α around $\sqrt{2}-1$

Around $x = 1/2$, *i.e* $\alpha = \sqrt{2}-1$ and $\gamma = 1$, it is possible to obtain only x (or α) as a function of γ and not the opposite. Such an expansion can be obtained by first expanding eq.(2.60) around $\gamma = 1$ and then looking for an expansion of α in terms of powers of $1-\gamma$ and $\ln(1-\gamma)$

$$\alpha = \sqrt{2}-1+a_1(1-\gamma)+a_2(1-\gamma)^2+\cdots+b_1(1-\gamma)\ln(1-\gamma)+ \\ b_2(1-\gamma)^2\ln(1-\gamma)+\cdots+c_1(1-\gamma)(\ln(1-\gamma))^2+c_2(1-\gamma)^2(\ln(1-\gamma))^2+\ldots \tag{2.66}$$

The coefficients in (2.66) are determined by requiring that (2.60) is satisfied. We provide here directly the expansion of x as a function of $1-\gamma$ up to the ninth order

$$x = \frac{1}{2}+\frac{1}{8}(1-\gamma)^2\left[1-2\log\left(\tfrac{1-\gamma}{4}\right)\right]-\frac{1}{4}(1-\gamma)^3\log\left(\tfrac{1-\gamma}{4}\right)- \\ \frac{1}{16}(1-\gamma)^4\left[1+3\log\left(\tfrac{1-\gamma}{4}\right)\right]-\frac{1}{96}(1-\gamma)^5\left[7+12\log\left(\tfrac{1-\gamma}{4}\right)\right]+ \\ \frac{1}{1536}(1-\gamma)^6\left[-97-108\log\left(\tfrac{1-\gamma}{4}\right)-24\log^2\left(\tfrac{1-\gamma}{4}\right)+64\log^3\left(\tfrac{1-\gamma}{4}\right)\right]- \\ \frac{1}{2560}(1-\gamma)^7\left[119+100\log\left(\tfrac{1-\gamma}{4}\right)-40\log^2\left(\tfrac{1-\gamma}{4}\right)-320\log^3\left(\tfrac{1-\gamma}{4}\right)\right]+ \\ \frac{1}{10240}(1-\gamma)^8\left[-321-60\log\left(\tfrac{1-\gamma}{4}\right)+1240\log^2\left(\tfrac{1-\gamma}{4}\right)+2240\log^3\left(\tfrac{1-\gamma}{4}\right)\right]+ \\ \frac{1}{107520}(1-\gamma)^9\left[-1871+5740\log\left(\tfrac{1-\gamma}{4}\right)+29120\log^2\left(\tfrac{1-\gamma}{4}\right)+31360\log^3\left(\tfrac{1-\gamma}{4}\right)\right]+\ldots \tag{2.67}$$

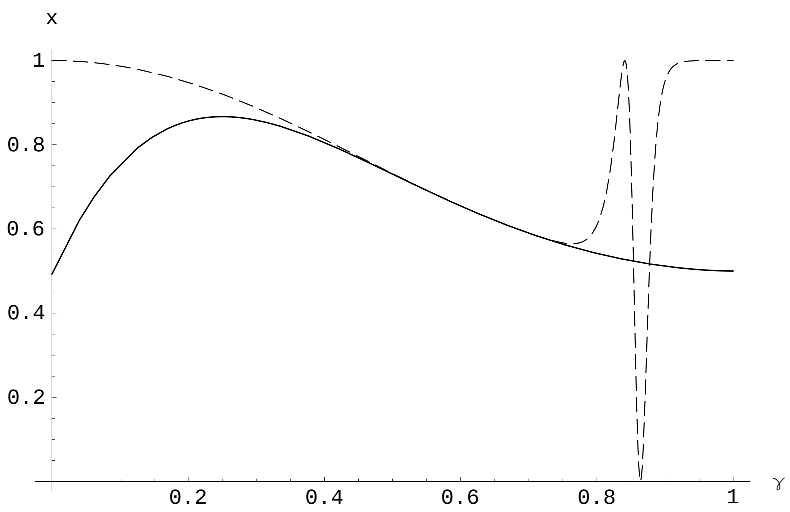


Figure 2.5: Plots of $x(\gamma)$: the dashed line gives the expansion of $x(\gamma)$ which holds in a neighbor of $\gamma = 1$, the solid line gives the expansion of $x(\gamma)$ around $\gamma = 0$.

From (2.64) one can easily get x as a function of γ in the region $x \sim 1$ ($\gamma \sim 0$) so that $x(\gamma)$ can be obtained for the whole range $1/2 \leq x \leq 1$. The two expansions in fact overlap in a long range for $0.3 \leq \gamma \leq 0.7$ as it is shown in fig.2.5. They have an excellent agreement up to the 13-th significative digit for $0.6 \leq \gamma \leq 0.7$.

2.4 Coefficient of the Quartic Tachyon Potential

As first check for the solution derived in Section 2.3 we consider the effective quartic term in the tachyon field ϕ^4 that arises from integrating out all the massive scalar fields in the theory. For each massive scalar field ψ in (2.33) quadratic term and coupling to ϕ^2 are given by

$$S_\psi = \frac{a}{2}\psi^2 + \kappa g c \psi \phi^2 \tag{2.68}$$

there is a term in the effective potential for ϕ of the form

$$S_{\phi^4} = -\kappa^2 g^2 \frac{c^2}{2a}\phi^4.$$

The static tachyon potential has thus the general form[3]

$$V_T = \frac{1}{2}\phi^2 - g\,k\;\phi^3 + g^2 k^2\, c_4 \phi^4 + \ldots \tag{2.69}$$

where g is the string coupling constant, $k = \frac{3^{7/2}}{2^7}$ and c_4 is the object of our analysis.

[3]We follow the notation of refs.[27, 25].

The four point tachyon potential is obtained from the off-shell four tachyon amplitude by setting to zero the external momenta and by explicitly subtracting out the term with the tachyon on the internal line. The amplitude is the sum of the six Feynman diagrams shown in fig.2.3, the first of which gives the contribution (2.57) that can be usefully rewritten in terms of the Mandelstam variables

$$A_4(s,t,u) = \frac{g^2\lambda_c^2}{2}\int_{\frac{1}{2}}^{1} dx\, x^{\frac{t-s-u}{2}}(1-x)^{s-2}\left(\frac{\kappa(x)}{2}\right)^{t+s+u-4} \quad . \tag{2.70}$$

To get explicitly the first diagram contribution to the amplitude one can set $t = u = 0$ in (2.70), A_4 can then be defined through an analitical continuation of (2.70) to the region $s \leq 1$. This can be achieved by adding and subtracting the pole in $x = 1$ in the integrand of (2.70)

$$\begin{aligned}\int_{\frac{1}{2}}^{1} dx\, x^{-\frac{s}{2}}(1-x)^{s-2}\left(\frac{\kappa(x)}{2}\right)^{s-4} &= \int_{\frac{1}{2}}^{1} dx\, x^{-\frac{s}{2}}(1-x)^{s-2}\left[\left(\frac{\kappa(x)}{2}\right)^{s-4} - \left(\frac{\kappa(1)}{2}\right)^{s-4}\right] \\ &+ \left(\frac{\kappa(1)}{2}\right)^{s-4}\int_{\frac{1}{2}}^{1} dx\, x^{-\frac{s}{2}}(1-x)^{s-2} \quad .\end{aligned} \tag{2.71}$$

where the first integral is now well defined in $s = 0$. When $Re[s] > 1$ the last integral in (2.71) gives

$$\frac{2^{s-2}}{\sqrt{\pi}}\Gamma(1-\frac{s}{2})\Gamma(\frac{s}{2}-\frac{1}{2}) + \frac{2^{2-\frac{s}{2}}}{s-2}\,{}_2F_1\left(1, 2-s; 2-\frac{s}{2}; -1\right)$$

that has a well defined limit for $s \to 0$, so that the four point tachyon potential can be written

$$A_4(0,0,0) = \frac{g^2\lambda_c^2}{2}\left[\int_{\frac{1}{2}}^{1} dx\left(\left(\frac{2}{\kappa(x)}\right)^4 - \left(\frac{2}{\kappa(1)}\right)^4\right)(1-x)^{-2} - \frac{3}{2}\left(\frac{2}{\kappa(1)}\right)^4\right] \tag{2.72}$$

As already pointed out, the function $\kappa(x)$ in (3.42) can be evaluated numerically in the whole interval $\frac{1}{2} < x < 1$, by using the numerical solution of eq.(2.60) graphed by the solid line in fig.2.4. The integrand in (3.42) is regular at $x = 1$, as can be easily checked by studying the behavior of (2.65) in a neighbor of $\alpha = 0$. However, problems are expected in the numerical evaluation of the integral in a neighbor of $x = 1$ due to the product of a pole times a zero. To circumvent possible computational problems we divided the interval $\frac{1}{2} < x < 1$ into two parts . For $x \in [\frac{1}{2}, 0.95]$ we used numerical evaluation of $\kappa(x)$, by plugging the numerical solutions of (2.60) in (2.54). For $x \in [0.95, 1]$ we used the analitical expression obtained substituting (2.41) in (2.65). By summing the two contributions we have found the value $A4(0,0,0) = -\frac{g^2}{2}2.94497480(2)$. To get the the quartic term of the tachyon effective potential we have to subtract [107] from (3.42) the contribution from the internal tachyon line

$$A4_t(s,t,u) = \frac{g^2}{2}\lambda_c^{2-s-\frac{t+u}{3}}\frac{1}{s-1} \tag{2.73}$$

evaluated at $s = t = u = 0$. Each graph in fig.2.3 contributes equally, so that for the quartic tachyon coupling one eventually gets

$$g^2 k^2 c_4 = \frac{6}{4!} \left[A4(0,0,0) - A4_t(0,0,0) \right] = \frac{6}{4!} \frac{g^2}{2} (-2.94497480(2) + \frac{3^9}{2^{12}}) = \frac{g^2}{4!} \, 5.5813353(1) \tag{2.74}$$

where the factor 1/4! is required to recover the units of [27, 25]. The numerical evaluation of the coefficient c_4 from the exact four tachyon amplitude was given in [29] to an accuracy of 1%, $c_4 \approx 1.75(2)$, and in [25] to an accuracy of 0.1%, $c_4 \approx 1.742(1)$. We have repeated this calculation to an higher degree of precision, and the result (2.74) gives

$$c_4 \approx 1.74220008(3) \; . \tag{2.75}$$

This coefficient was calculated using the level truncation scheme up to level $L = 20$ in [27], and improved up to level $L = 28$ in [110], thus obtaining $c_{4,L=28} \simeq -1.70028$, with a discrepancy of 2.4% with respect to (2.75). In the same paper, a procedure to extrapolate the known level truncated results and predict the asymptotic $L \to \infty$ value for c_4 was described, giving an extimated value $c_{4,L\to\infty} = 1.7422006(9)$ that agrees within the 10^{-7} of accuracy with our exact result (2.75).

Chapter 3

Rolling tachyon solutions in cubic string field theory

In this chapter, we will describe two different approaches for deriving tachyonic time-dependent solutions in Cubic SFT. The first one will be a level truncated analysis of the equation of motion for a solution expressed as a power series in the exponentials e^t. Then, an analytic procedure based on the diffusion equation satisfied by a given tachyon profile will be presented.

An analysis of Cubic SFT time-dependent tachyonic solutions has been addressed in refs. [54, 55, 56] by using level truncation methods. It turns out that the tachyon rolls down toward the vacuum, goes far beyond it then turns around and begins to oscillate with ever increasing amplitudes. In [56], in particular, a systematic level-truncation analysis was carried out for a trajectory $\phi(t)$ expressed as a power series in e^t. Increasing the level of truncation in CSFT leads to a well defined trajectory up to some upper bound $t = t_b$. For the first turnaround points, the leading terms in the CSFT solution are those with small powers of e^t. Consequently, the very accurate value of the 4-tachyon amplitude that we have derived in the previous chapter would improve the solution of ref. [56], at least up to the first extrema of the trajectory. In Section 3.1 we review the procedure followed in [56] and improve the accuracy of the first coefficients of the pure exponential tachyon solution. The trajectories $\phi(t)$, obtained by computing the ϕ^4 term in the effective action exactly and the terms up to ϕ^7 in an $L = 2$ approximation, show that indeed the position of the first turnaround point does not change significantly with the improvement in the ϕ^4 term. This suggests the possibility that this value could have the physical meaning of an inversion point. The second turnaround point instead changes position and amplitude compared to the one found in [56]. However, the inclusion of higher order terms in the lagrangian - calculated through the standard field theory procedure that uses level truncation *on fields* - does not produce significative changes, so that the trajectory seems again to stabilize. This substantially confirms that for $t > 0$ the tachyon does not roll towards the stable non-perturbative minimum of the potential - and does not represent tachyon matter, as can be checked by an analysis of the pressure. The qualitative behavior of wild oscillations is reproduced, even if the amplitudes at the turnaround points beyond the first are sensibly diminished.

An alternative analytic procedure to find a solution of the CSFT equations of motion at the lowest order $(0,0)$ in the level truncation scheme was used in [57]. It is based on the following steps: i) Treat the CSFT coupling constant λ as an independent variable in the equation of motion; ii) Find an exact solution for $\lambda = 1$ and interpret it as an initial condition for an evolution equation with respect to λ; iii) Evolve the equation of motion to find its solution, valid for $\lambda < 1$ and iv) Try to analytically continue this solution to the physically meaningful region $\lambda = 3^{(9/2)}/2^6$. We will review this procedure in Section 3.2.

In [56] it was also shown that the non-local field redefinition derived in Section 1.8, which takes the CSFT action to the BSFT action [13], also maps the first two coefficients of the wildly oscillating CSFT solution to the well-behaved BSFT exponential solution. We will review and discuss this approach in Section 3.3, where we will rise the question whether this field redefinition is a real tool to reconcile the apparent discrepancy with the results of BSFT or the problem is still open.

3.1 The level truncated analysis

The full CSFT action (2.1) contains an infinite number of fields, coupled through cubic terms which contain exponentials of derivatives. Being the action nonlocal, an initial value problem is highly nontrivial

Nevertheless, a solution valid for all times can be systematically developed by assuming that as $t \to -\infty$ it approaches the perturbative vacuum at $\phi = 0$. In this limit the equation of motion is the free equation for the tachyon field $\ddot{\phi}(t) = \phi(t)$, with solution $\phi(t) = ce^t$. For $t \ll 0$, one can perform a perturbative expansion in the small parameter e^t. A level truncated analysis of the tachyon dynamics was carried out in [56] for a perturbative solution given as a sum of exponentials of the form [1]

$$\phi(t) = \sum_{n>0} a_n e^{nt} \quad . \tag{3.1}$$

The infinite number of fields of CSFT represents an additional complication. One can, however, systematically integrate out any finite set of fields to arrive at an effective action for the tachyon field. This can be done by using two of the approaches described in Section 2.1.4, *i.e* “level truncation on the fields” or “level truncation on oscillators”. Integrating out all the massive fields at tree level results in a tachyon effective action that can be written in terms of the (temporal) Fourier modes $\phi(k)$ of $\phi(t)$ as

$$S[\phi] = \sum_n \frac{g^{n-2}}{n!} \int \prod_{i=1}^{n} (2\pi dk_i) \delta(\sum_i k_i) \phi(k_1) \dots \phi(k_n) A_n(k_1, \dots, k_n) \tag{3.2}$$

[1] A related approach was taken in [54, 55], where an expansion in cosh$t$ was proposed. In most previous work on this problem, solutions have been constructed using Wick rotation of periodic solutions; in this case one works directly with the real solution which is a sum of exponentials.

where the functions $A_n(k_1, \ldots, k_n)$ determine the derivative structure of the terms at order $g^{n-2}\phi^n$.

The coefficients in (3.1) can be determined by perturbatively solving the CSFT equation of motion. Defining $\phi_n = a_n\, e^{nt}$, these will have the general form

$$(\partial_t^2 - 1)\phi_n = \sum_p \sum_{m_1+m_2+\ldots m_p=n} F_p(\phi_{m_1}, \phi_{m_2}, \ldots, \phi_{m_p}), \tag{3.3}$$

where the specific form of the F_p follows by differentiating (CSFT) with respect to $\phi(t)$. The functions A_n appearing in (3.2), and thus the corresponding F_{n-1}'s, can in principle be computed for arbitrary n at any finite level of truncation. An alternative approach - more efficient for computing F_n at small n but large truncation level - is the level truncation method reviewed in Section 2.2.3 and in Appendix B. In general, independently of the method used to compute it, F_n results in a complicated momentum-dependent function of its arguments.

Equations (3.3) can be solved for $a_{n>1}$ iteratively in n. Having solved the equations for $a_2, \ldots, a_{n-1}$ we can plug them in via (3.1) on the right hand side of (3.3) to determine a_n. Since in the action (3.2) the coefficients A_2 and A_3 are exactly known,

$$A_2(k_1, k_2) = 1 - k_1 k_2\ , \qquad A_3(k_1, k_2, k_3) = -2\left(\frac{3\sqrt{3}}{4}\right)^{3+k_1^2+k_2^2+k_3^2}, \tag{3.4}$$

the first two coefficients in (3.1) are exact and can be normalized as $a_1 = 1$, $a_2 = -64/(243\sqrt{3})$.

In [56] an $L = 2$ approximation was explicitly provided for the coefficients $a_3 \ldots a_6$ in the sum (3.1)

$$\phi(t) \cong e^t - \frac{64}{243\sqrt{3}}e^{2t} + 0.002187\, e^{3t} - 3.9258\, 10^{-6}\, e^{4t} + 4.9407\, 10^{-10}\, e^{5t} - 6.3227\, 10^{-12} e^{6t} \tag{3.5}$$

Plotting the result in Fig.3.1 it can be observed that for small enough t the term e^t dominates and the solution decays as e^t at $-\infty$. Then, as t grows, the second term in (3.5) becomes important. The solution turns around and $\phi(t)$ becomes negative, with the major contribution coming from e^{2t}. Then the next mode, e^{3t} becomes dominating and so on. The solution $\phi(t)$ around the first two turnaround points is shown in Figure 3.1. Note that the trajectory passes through the minimum of the static potential, which is at $\phi \sim 0.545$ [26, 54], well before the first turnaround point.

For negative t, Eq.(3.5) describes the rolling of the tachyon off the unstable maximum along the potential. The physical interpretation for positive t is more problematic. Before exponentially exploding $\phi(t)$ presents an oscillatory behavior with increasing amplitudes that makes the rolling tachyon dynamics in the framework of CSFT for positive t difficult to interpret. In ref. [56], however, it was shown that the trajectory $\phi(t)$ is well-defined. Increasing both the level of truncation and the number of terms retained in the power series (3.1) changes the position of the first two turnaround points only slightly, leading

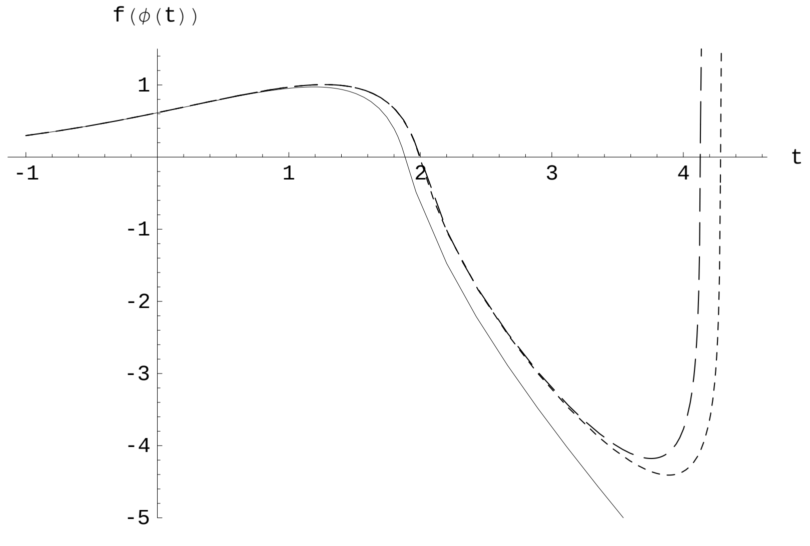


Figure 3.1: The solution $\phi(t)$ of [56] up to the first two turnaround points, including fields up to level $L = 2$. The solid line graphs the approximation $\phi(t) = e^t + c_2 e^{2t}$. The long dashed line graphs $\phi(t) = e^t + c_2 e^{2t} + c_3 e^{3t}$. The approximate solutions computed up to e^{4t}, e^{5t} and e^{6t} are very close in this range of t and are all represented in the short dashed line. One can see that after going through the first turnaround point with coordinates $(t, \phi(t)) \sim (1.27, 1.8)$ the solution decreases, reaching the second turnaround at around $(t, \phi(t)) \sim (3.9, \text{-}81)$. The function $f(\phi(t)) = \mathrm{sign}(\phi(t)) \log(1 + |\phi(t)|)$ is graphed to show both turnaround points clearly on the same scale.

to a convergent value of $\phi(t)$ for any fixed t as far as the second turnaround point. The trajectories $\phi(t)$, obtained by computing the ϕ^4 term in the effective action up to $L = 16$, show that indeed the position of the first two turnaround points seems to stabilize [56]. The expansion (3.5) for $t > 0$ would be justified at least up to those points. For $t > 0$, the tachyon does not roll towards the stable non-perturbative minimum of the potential.

We shall now study how this solution is modified by using the exact value of the 4-tachyon term in the effective action for homogeneous time dependent profiles. The exact value of the coefficient a_3 can be obtained by computing integrals of the type (2.57), that in the time-dependent case read

$$A_4(p_1, p_2, p_3, p_4) = \frac{g^2}{2} \int_{\frac{1}{2}}^{1} dx \; x^{-p_1 \cdot p_2 - p_3 \cdot p_4} (1-x)^{-(p_1+p_4)^2 - 2} \left(\frac{\kappa(x)}{2} \right)^{-\sum_{i=1}^{4} p_i^2 - 4} \tag{3.6}$$

To get the equations of motion the function A_4 in (3.2) has to be evaluated for imaginary integer values of the field modes so that (3.6) is regular and does not need any analytical continuation. In the evaluation of a_3, the relevant integral (3.6) over the Kobe-Nielsen variable is

$$A_4(-i, -i, -i, 3i) = \frac{g^2}{2} \int_{\frac{1}{2}}^{1} dx \; x^{-2} (1-x)^2 \left(\frac{\kappa(x)}{2} \right)^{8} \tag{3.7}$$

Summing all the diagrams in fig.2.3 and subtracting the corresponding contributions coming from the internal tachyon line, $A_{4t} = 2^{29}/3^{22}$, we get $a_3 = 0.00241475435(3)$. This value, which is exact, can be compared with the corresponding ones obtained through the level truncation approximation. The first column of Table 3.1 shows the sequence of

Level	a_3	a_4	a_5	a_6
2	0.002187797562	$-3.7830611\,10^{-6}$	$4.1448524\,10^{-9}$	$-4.7728992\,10^{-13}$
4	0.002245884478	$-4.3957017\,10^{-6}$	$4.6338501\,10^{-9}$	$-5.4000742\,10^{-13}$
6	0.002281097505	$-4.5437634\,10^{-6}$	$4.7480437\,10^{-9}$	$-6.2618454\,10^{-13}$
8	0.002304369408	$-4.6509193\,10^{-6}$	$4.8933743\,10^{-9}$	$-6.7366480\,10^{-13}$
10	0.002320816678	$-4.7282645\,10^{-6}$	$4.9938778\,10^{-9}$	$-6.9213556\,10^{-13}$
12	0.002333033369	$-4.7867688\,10^{-6}$	$5.0729134\,10^{-9}$	$-7.0850857\,10^{-13}$
14	0.002340032469	$-4.8250629\,10^{-6}$	$5.1236425\,10^{-9}$	$-7.2267875\,10^{-13}$
16	0.002342489534	$-4.8443632\,10^{-6}$	$5.1338898\,10^{-9}$	$-7.3568697\,10^{-13}$
ExactA_4	0.00241475435(3)	$-5.205903(1)\,10^{-6}$	$5.692641(2)\,10^{-9}$	$-8.338132(4)\,10^{-13}$

Table 3.1: First few coefficients a_n of the time-dependent solution $\sum_n a_n e^{nt}$ at various levels of truncation, when only the contribution from the quartic term in the effective action is considered in the EOM. In the last row the exact four tachyon amplitude is used for the calculations.

the first approximate values of the $a_3(L)$ coefficients up to $L = 14$. The level sequence is perfectly consistent with the exact value given in the last row (first column), which should then be considered as the limit $a_3(L \to \infty)$.

The amplitude (2.57) can be used to improve the accuracy of the remaining coefficients a_n, $n \geq 4$. The exact evaluation of a_4 would require the knowledge of $A_5(p_1, \ldots, p_5)$, for which an expression analog to (2.57) is not known. However, when solving the CSFT equation of motion, one can easily see that the dominant contribution to a_4 comes from the lower order amplitudes $A_2(p_1, p_2)$, $A_3(p_1, p_2, p_3)$, $A_4(p_1, p_2, p_3, p_4)$. Therefore, for a precise evaluation of a_4 seems more relevant to know these lower order amplitudes exactly, rather than $A_5(p_1, \ldots, p_5)$ approximate in levels. The remaining columns in Table 3.1 give the behavior of the coefficients a_4, a_5, a_6 for increasing levels of truncation, when only the contribution from the quartic term in the effective action is considered in the equations of motion. The last row gives the corresponding value obtained from the exact amplitude (2.57) (*i.e.* limit $L \to \infty$). As can be seen from Table 3.1, for any fixed L, $|a_n(L)| < |a_n(L \to \infty)|$. Notice that the same property holds also in the calculation of the coefficient of the quartic tachyon potential derived in Section 2.4. Indeed, up to $L = 28$ [110], $|c_{4,L}| < |c_4|$. Moreover, for any fixed n, the sequence $(a_n(L+2) - a_n(L))$ goes like $C_n a_n(L)/L$, C_n being a constant, confirming the $1/L$ behavior of the leading correction [102, 110]. The results given in the last row of Table 3.1 provide the first few coefficients of the trial solution (3.1).

We can now include in the computation of a_4, a_5, a_6 the $L = 2$ truncated expressions for $A_5(p_1, \ldots, p_5)$, $A_6(p_1, \ldots, p_6)$, $A_7(p_1, \ldots, p_7)$. The numerical results are listed in Table 3.2. The $L = 2$ truncated $A_7(p_1, \ldots, p_7)$, however, gives a contribution to a_6 which is not reliable, since increasing the order of the effective action higher level field components become more and more important. The inclusion of the a_6 coefficient, in any case, does not change the behavior of the solution around the first two turnaround points. This is the region where we shall mainly focus, only here the solution with the first few coefficients is reliable.

Effective action	a_3	a_4	a_5
A_4^{exact}	0.00241475435(3)	$-5.205903(1)\,10^{-6}$	$5.692641(2)\,10^{-9}$
A_4^{exact}, $A_5^{L=2}$	0.00241475435(3)	$-5.348643(1)10^{-6}$	$3.231846(1)10^{-9}$
A_4^{exact}, $A_5^{L=2}$, $A_6^{L=2}$	0.00241475435(3)	$-5.348643(1)10^{-6}$	$2.0650063(5)10^{-9}$

Table 3.2: First few coefficients a_n of the time-dependent solution $\sum_n a_n e^{nt}$. The first column indicates which terms of the effective action are considered in the EOM.

In fig.3.2 we show how the solution changes at the second turnaround point by introducing higher order terms of the effective action. The higher group of trajectories is obtained by using the exact value for the four-tachyon effective action and adding to it the level $L = 2$ five and six tachyon effective action, the lower group by using only $L = 2$ terms (the solid line in this group represent the solution of ref. [56] up to the e^{5t} power). As it is manifest from the figure the use of an exact A_4 leads to a decreasing of the amplitude of the oscillations by at least the 20%. This is however not enough to change the qualitative behavior of the solutions which maintains huge oscillations and does not provide a physically meaningful picture. The best approximation we get is given by the solution obtained using the exact A_4 and the level 2 A_5, A_6. It reads

$$\phi(t) \cong e^t - \frac{64}{243\sqrt{3}} e^{2t} + 0.00241475\, e^{3t} - 5.348643 \quad 10^{-6}\, e^{4t} + 2.0650063 \quad 10^{-9}\, e^{5t} \tag{3.8}$$

and is plotted in fig.3.3 against the solution (3.5) of ref. [56] up to the coefficient of e^{5t}.

For $t < 0$ all the solutions overlap up to the 6-th significative digit. For positive t, all the solutions present the expected oscillatory behavior with ever-growing amplitudes and have constant energy, as has been verified [56] by a perturbative calculation including arbitrary derivative terms in the infinite series defining the energy $T_{00}(t)$, along the lines of [54]. The pressure shows the properties found in [114], where an analog approximate solution was considered using the coshnt basis, starting from negative value at time $t = 0$ to force the tachyon roll to the vacuum. But instead of decreasing to zero as $t \to \infty$, it oscillates without bound at large times. So, this solution does not seem to represent tachyon matter. In CSFT where the action contains infinite derivatives the kinetic energy can be negative and thus the tachyon can move to higher and higher heights on the tachyon potential while conserving the total energy [54]. Whatever solution one chooses, the position of the first extremum seems to be fixed at $t_1 \sim 1.27$ with amplitude $\phi(t_1) \sim 1.74$. In addition, such a position is compatible within the 1% also with [54], where an analog approximate solution was considered using the coshnt basis. This suggests the idea that the first maximum could have a physical meaning. Actually, since the solution describes

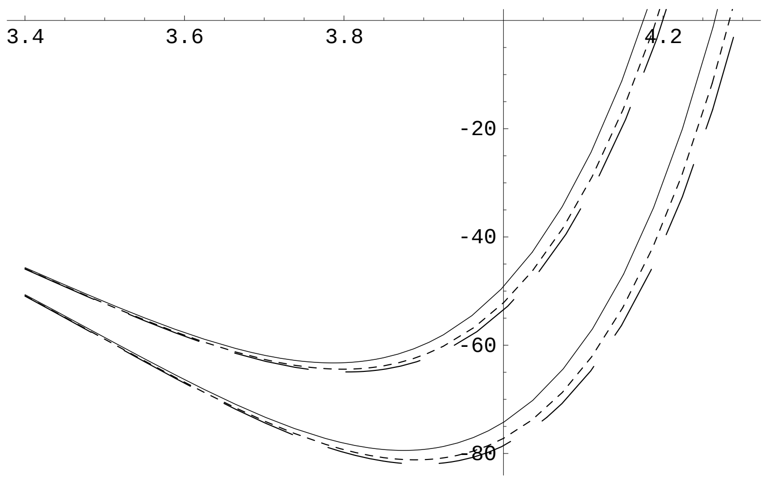


Figure 3.2: Solution at the second turnaround point. The higher group of trajectories is obtained by using the exact value for the four-tachyon effective action (solid line) and adding to it the level $L = 2$ five (long dashed line) and six (dashed line) tachyon effective action, the lower group by using only $L = 2$ terms. The solid line in the lower group represents the solution of ref. [56] up to the e^{5t} power.

the motion of the tachyon rolling off its unstable maximum at $\phi = 0$, the naive energy conservation would confine the motion between $0 \leq \phi(t) \leq \phi_M$, where ϕ_M denotes the maximum value attained by ϕ *i.e.* is the naive inversion point defined by the condition $V_{eff}[0] = V_{eff}[\phi_M]$ on the effective tachyon potential V_{eff}. A natural interpretation for the first maximum is therefore $\phi(t_1) \sim \phi_M$. Numerically, the value $\phi_M \sim 1.7$ is in fact in a qualitative agreement with the available data on the effective tachyon potential [28].

The other extrema, instead, do not have any clear physical meaning. These oscillations undergo wild ever-growing amplitudes, which, however, depend quite significantly on the solution chosen. In passing from (3.5) to (3.8), both positions of the second turnaround points and their amplitudes change. For instance, as shown in fig.3.3, the amplitude of the second turnaround point is lowered by a 20% factor, the third one by an order of magnitude.

In conclusion, it seems that up to the first turnaround point all the solutions (3.5), (3.8), (3.30), practically coincide. After the first turnaround point, the wild oscillations with increasing amplitudes found in refs.[54, 56] are confirmed. Although the qualitative behavior is reproduced, the oscillations in (3.8) are sensibly reduced when compared to those in ref.[56]. Up to the second turnaround point, where low powers of e^t dominate, (3.8) provides a more accurate estimate for the trajectory of the rolling tachyon in CSFT.

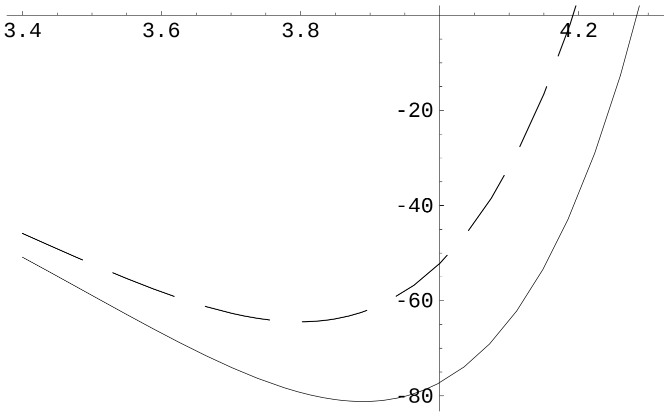


Figure 3.3: Second turnaround point for the solution (solid line) given in ref. [56] and the solution (dashed line) obtained using the exact A_4 and the level 2 A_5, A_6.

3.2 An analytical approach

In [57] an alternative analytic procedure was used to find a solution of the cubic string field theory equations of motion at the lowest order, $(0, 0)$, in the level truncation scheme [54, 55].

At this order one considers only the tachyon field, disregarding any interaction term between the tachyon and other massive level L fields, and the cubic string field theory action reads

$$S = \frac{1}{g_o^2} \int d^{26}x \left(\frac{1}{2}\, \phi(x)\, (\Box + 1)\, \phi(x) - \frac{1}{3} \lambda_c \left(\lambda_c^{(1/3)\Box} \phi(x) \right)^3 \right), \tag{3.9}$$

where we remind that the coupling λ_c has the value

$$\lambda_c = 3^{9/2}/2^6 = 2.19213\ . \tag{3.10}$$

Considering spatially homogeneous profiles of the form $\phi(t)$, where t is time, the equation of motion derived from (3.9) is

$$(\partial_t^2 - 1)\phi(t) + \lambda_c^{1-\partial_t^2/3} \left(\lambda_c^{-\partial_t^2/3} \phi(t) \right)^2 = 0. \tag{3.11}$$

The procedure followed in [57] is based on the idea that Eq.(3.11) can be generalized to become a non-linear differential equation with an arbitrary parameter λ which substitutes the fixed value (3.10)

$$(\partial_t^2 - 1)\phi(t) + \lambda^{1-\partial_t^2/3} \left(\lambda^{-\partial_t^2/3} \phi(t) \right)^2 = 0. \tag{3.12}$$

Then λ can be treated as an *evolution* parameter. Fixing the initial value $\lambda = 1$ one can easily find an exact solution to (3.12) and then study how this solution evolves to different values of λ keeping its property of being a solution of (3.12). We shall find that the equation governing the evolution in λ is extremely simple and we shall look for a solution of (3.12) for generic λ, setting eventually $\lambda = \lambda_c$ as in (3.10).

3.2.1 Diffusion equation

When $\lambda = 1$, Eq.(3.11) admits a particularly simple exact solution, the following bounce

$$\phi(\log\lambda = 0,\, t) = \frac{3}{2\cosh^2(t/2)} = 6\int_0^\infty \frac{\sigma\cos(\sigma t)}{\sinh(\pi\sigma)}\,d\sigma \quad . \tag{3.13}$$

The boundary conditions of (3.13) are such that $\partial\phi(0,t)/\partial t = 0$ at $t = \pm\infty$.

Now we shall interpret the solution (3.13) as the "initial" condition of an "evolution" equation with respect to the "time" $\log\lambda$. The evolution is driven by the action of infinite derivative operators of the type

$$q^{\partial^2} = e^{\log q\,\partial^2} \equiv \sum_{n=0}^\infty \frac{(\log q)^n}{n!}\partial^{2n} \ , \tag{3.14}$$

which act on the function $\phi(t)$ in (3.12) when $\lambda \neq 1$, that play a crucial role in string field theories and related models.

A particularly convenient redefinition of the tachyon field that leaves invariant the initial condition (3.13) is

$$\Phi(\log\lambda,\, t) = \lambda^{5/3+\partial_t^2/3}\phi(\log\lambda,\, t) \ . \tag{3.15}$$

With this field redefinition Eq.(3.11) transforms into the following

$$(\partial_t^2 - 1)\Phi(\log\lambda, t) + \lambda^{-2/3}\left(\lambda^{-2\partial_t^2/3}\Phi(\log\lambda, t)\right)^2 = 0. \tag{3.16}$$

Since the operator $\lambda^{-2\partial_t^2/3}$ is defined as a power series of $\log\lambda$ through Eq.(3.14), it is natural to look for solutions of Eq.(3.16) of the form

$$\Phi(\log\lambda, t) = \sum_{n=0}^\infty \frac{(\log\lambda)^n}{n!}\phi_n(t) \tag{3.17}$$

It is not difficult to check that at any desired order n in (3.17) the functions $\phi_n(t)$ can *always* be written as finite sums of the form

$$\phi_n(t) = \sum_{k=0}^n \frac{a_k^{(n)}}{\cosh^{2k+2}(t/2)} \ , \tag{3.18}$$

and the differential equation for the tachyon field becomes an algebraic equation for the unknown coefficients $a_k^{(n)}$. Thus, an exact solution of (3.16) can always be obtained as a series representation. However, it is interesting to look for solutions that, although approximate, sum the *whole* series (3.17) rather than to find the exact coefficients $a_k^{(n)}$ at any fixed truncation n of the sum. In fact, it is easy to show that any truncation of the sum (3.17) leads to solutions with wild oscillatory behavior with increasing amplitudes. One may wonder if the resummation of the whole series can smooth such oscillations.

A more convenient representation of $\phi_n(t)$ alternative to (3.18) is given by

$$\phi_n(t) = 6\int_0^\infty \frac{\sigma\cos(\sigma t)}{\sinh(\pi\sigma)} P_n(\sigma)\, d\sigma \ , \tag{3.19}$$

$P_n(\sigma)$ being a polynomial of even powers of σ of degree $2n$. This representation is particularly useful since it provides the $\phi_n(t)$ in terms of eigenfunction of the operator ∂_t^2. The field redefinition (3.15) was chosen in such a way that the form of the coefficients (3.19) becomes particularly simple. This allows an approximate (although very accurate) resummation of the whole series (3.17). With this choice, in fact, the polynomials $P_n(\sigma)$ simply become

$$P_n(\sigma) \simeq \sigma^{2n} \tag{3.20}$$

leading to the following approximate solution of Eq.(3.16)

$$\Phi(\log\lambda,\, t) = 6\int_0^\infty \frac{\sigma\cos(\sigma t)}{\sinh(\pi\sigma)} e^{\log\lambda\,\sigma^2}\, d\sigma \ = 6\lambda^{-\partial_t^2}\int_0^\infty \frac{\sigma\cos(\sigma t)}{\sinh(\pi\sigma)}\, d\sigma, \quad \lambda < 1. \tag{3.21}$$

Note that all the λ-dependence in (3.21) is encoded in the operator $\lambda^{-\partial_t^2}$ acting on the solution of Eq.(3.16) with $\lambda = 1$. In fact $\Phi(\log\lambda = 0, t) \equiv \phi(\log\lambda = 0, t)$ and $\lambda^{-\partial_t^2}$ plays the role of the "evolution" operator (with respect to the "time" $\log\lambda$) acting on the initial condition $\Phi(\log\lambda = 0, t)$,

$$\Phi(\log\lambda, t) = \lambda^{-\partial_t^2}\,\Phi(\log\lambda = 0, t)\ . \tag{3.22}$$

Clearly, the representation (3.21) of the solution $\Phi(\log\lambda, t)$ is valid only for $\lambda \in (0, 1]$. In our case the physically relevant value of λ is the one given in (3.10), which is greater than one. Consequently, we need an analytical continuation of the representation (3.21) to positive values of $\log\lambda$.

Eq.(3.21) shows that the evolution of the tachyon field with respect to the parameter $\log\lambda$ is simply driven by the diffusion equation with negative unitary coefficient. In fact (3.21) satisfies the diffusion equation

$$\frac{\partial\Phi(\log\lambda, t)}{\partial\log\lambda} = -\frac{\partial^2\Phi(\log\lambda, t)}{\partial(t)^2} \tag{3.23}$$

with respect to the "time" variable $\log\lambda$ and the "space" variable t, with "initial" and "boundary" conditions $\Phi(0, t) = 3/[2\cosh^2(t/2)]$, $\Phi(\log\lambda, \pm\infty) = 0$.

Therefore, the action of the operator $q^{\partial_t^2}$ on $\Phi(\log\lambda,\, t)$ can be simply represented as a translation of $\log\lambda$

$$q^{\partial_t^2}\Phi(\log\lambda,\, t) = e^{\log q\,\partial_t^2}\Phi(\log\lambda,\, t) = e^{-\log q\frac{\partial}{\partial\log\lambda}}\Phi(\log\lambda,\, t) = \Phi(\log\lambda - \log q,\, t)\ . \tag{3.24}$$

This remarkable property can only be used thanks to the fact that we have treated the quantity λ as a generic variable. In particular we will make use of the following operator

$$\lambda^{a\partial_t^2}\Phi(\log\lambda,\, t) \ = \ \sum_{n=0}^\infty a^n \frac{(\log\lambda)^n}{n!}\frac{\partial^{2n}}{\partial t^{2n}} T(\log\lambda,\, t)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(-a)^n \frac{(\log\lambda)^n}{n!}\frac{\partial^n}{\partial(\log\lambda)^n}\Phi(\log\lambda,\, t) \\
&= \Phi((1-a)\log\lambda,\, t)\ . \qquad (3.25)
\end{aligned}
$$

where in the second equality we have used the diffusion equation (3.23).

3.2.2 Analytical continuation

The problem of the analytical continuation of the representation (3.21) to positive values of $\log\lambda$ can be faced as follows. Setting $\sigma = -is$ in Eq.(3.21), we rewrite Φ as

$$
\Phi(\log\lambda, t) = \frac{3}{i}\lambda^{-\partial_t^2}\int_{-i\infty}^{+i\infty}\frac{se^{st}}{\sin(\pi s)}\, ds\ . \qquad (3.26)
$$

In Eq.(3.26) the integral can be closed with semi-circles at infinity to the right or to the left depending on the sign of t. Let us choose for instance $t < 0$. Then (3.26) reads

$$
\Phi(\log\lambda, t<0) = -6\lambda^{-\partial_t^2}\sum_{n=1}^{\infty}(-1)^n n e^{nt}\ . \qquad (3.27)
$$

By replacing the operator $\lambda^{-\partial_t^2}$ in Eq.(3.27) with its eigenvalue λ^{-n^2} inside the series, one gets

$$
-6\sum_{n=1}^{\infty}(-1)^n\lambda^{-n^2} n e^{nt}\ , \qquad \lambda > 1\ , \qquad (3.28)
$$

The solutions in cubic string field theory (CSFT) analyzed in Section 3.1 in the level truncation scheme have precisely the form (3.28). In order to make a proper comparison between these tachyon profiles, we remind that the tachyon field $\phi(t)$ appearing in the original form of the level truncated CSFT (3.9) is obtained by the field redefinition (3.15) applied to (3.28) with $\lambda = \lambda_c$. Using (4.22), one has $\phi(t) = \lambda_c^{-5/3}\Phi(\frac{4}{3}\log\lambda_c,\, t)$, namely

$$
\phi(t) = -6\lambda_c^{-5/3}\sum_{n=1}^{\infty}(-1)^n\, n\lambda_c^{-\frac{4}{3}n^2}e^{nt}. \qquad (3.29)
$$

By means of the time translation $t \to t + \log\frac{\lambda^3}{6}$ the previous series becomes

$$
\phi(t) = -6\lambda_c^{-\frac{5}{3}}\sum_{n=1}^{\infty}\left(-\tfrac{1}{6}\right)^n n\lambda_c^{-\frac{4}{3}n^2+3n}e^{nt}\quad . \qquad (3.30)
$$

which reproduces exactly the first two coefficients of the solutions (3.5,3.8). The differences arising for the coefficients of the higher powers in e^t are due to the fact that (3.30) represents a level $(0,0)$ solution, while those analyzed in Section 3.1 present corrections due to the use of effective actions with higher levels of truncation in the fields. Again, the tachyon profile (3.29) presents an oscillatory behavior with ever-growing amplitudes and a position of the first minimum that seems to be fixed at $t \sim 1.27$. It can be easily seen that, up to the first turnaround point, Eq. (3.29) practically coincides with all the solutions written in Section 3.1. The alternative procedure here described would then confirm the CSFT picture of the rolling tachyon dynamics in which the tachyon does not converge monotonically to the true vacuum.

Nevertheless, some further considerations can be done, by strictly obeying the prescription given in (3.22). Despite of the fact that the series (3.28) has an infinite convergence radius for $\lambda > 1$, we should not consider it for positive values of t. In order to be a solution satisfying (3.22) for $t > 0$, it should obey the same Eq. (3.22) with an inverse exponential operator on the left hand side. This means that, acting with $\lambda^{-\partial_t^2}$ on (3.28), we should recover the initial condition $\Phi(\log\lambda, t > 0)$ for positive values of t, which is a series of exponentials of the type e^{-nt}. This is not true, we are thus lead to look for an alternative representation for the tachyon profile, by a different treatment of the infinite derivative operator in front of the series in (3.27).

We make use of a Mellin-Barnes representation for the operator $\lambda^{-\partial_t^2}$,

$$\lambda^{-\partial_t^2} = \sum_{n=0}^{\infty} \frac{(-\log\lambda)^n}{n!}\,\partial_0^{2n} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\,\Gamma(-s)(\log\lambda)^s \partial_t^{2s}\ , \quad \mathrm{Re}\gamma < 0\ . \tag{3.31}$$

Acting with (3.31) in (3.27), we find

$$\begin{aligned}
\Phi(\log\lambda, t < 0) \quad &= -\frac{3}{\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\Gamma(-s)(\log\lambda)^s \sum_{n=1}^{\infty}(-1)^n n^{2s+1} e^{nt} \\
&= \frac{3e^t}{\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\Gamma(-s)(\log\lambda)^s \mathcal{F}(-e^t, -2s-1, 1) \\
&= \frac{12 e^t}{\sqrt{\pi} i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds \frac{(4\log\lambda)^s}{\Gamma(-s-1/2)}\int_0^{\infty}\frac{dy}{y^2}\frac{y^{-2s}}{e^t + e^y}\ ,
\end{aligned} \tag{3.32}$$

where $\mathcal{F}$ is the Lerch Transcendent defined as

$$\begin{aligned}
\mathcal{F}(z, s, v) \quad &= \sum_{n=0}^{\infty}(v+n)^{-s} z^n\ , \qquad |z| < 1\ , \quad v \neq 0, -1, -2, \ldots \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} dy\ \ \frac{y^{s-1} e^{-(v-1)y}}{e^y - z}\ ,
\end{aligned} \tag{3.33}$$

and the last equation in (3.32) follows from the integral representation of $\mathcal{F}$ given in (3.33). The gamma function in (3.32) can be rewritten by using the formula

$$\frac{1}{\Gamma(-s-1/2)} = \frac{1}{2\pi i}\int_C dz\ \ e^z z^{s+1/2} \tag{3.34}$$

where C is the path drawn in Fig.3.4.

Thus, the integral over s in (3.32) can be explicitely performed,

$$\int_{\gamma-i\infty}^{\gamma+i\infty} ds \left(\frac{4z\log\lambda}{y^2}\right)^s = i\pi\ \delta\left[\log y - \log\left(2\sqrt{z\,\log\lambda}\right)\right]\ . \tag{3.35}$$

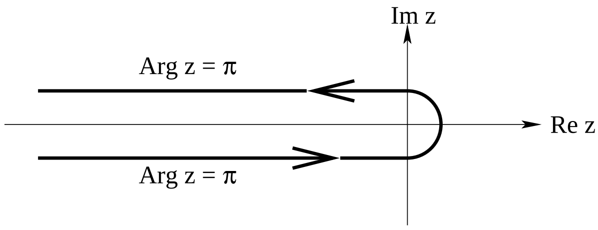


Figure 3.4: Contour C

In turn, integration of the δ-function leads to the expression

$$\Phi(\log\lambda, t<0) = \frac{3}{i\sqrt{\pi\log\lambda}}\int_C dz \;\; \frac{e^z}{1+e^{2\sqrt{z\,\log\lambda}-t}} \quad . \tag{3.36}$$

It is easily realized that the contribution to the integral (3.36) given by the semicircle around the origin vanishes. The lower and upper branches of the path C are parametrized, according to the notation of Fig.3.4, as

$$\begin{aligned} z &= e^{-i\pi}y - i\epsilon\ , \qquad y\in(\infty,0)\ , \\ z &= e^{i\pi}y + i\epsilon\ , \qquad y\in(0,\infty)\ , \end{aligned} \tag{3.37}$$

respectively.

Then, by changing variable $\sigma = \sqrt{t}$ and sending $\epsilon \to 0$, the integral (3.36) can be rewritten as

$$\Phi(\log\lambda, t<0) = \frac{6}{\sqrt{\pi\log\lambda}}\int_0^\infty d\sigma e^{-\sigma^2}\frac{\sigma\sin(2\sigma\sqrt{\log\lambda})}{\cosh(t)+\cos(2\sigma\sqrt{\log\lambda})}\ , \quad \lambda>1\ . \tag{3.38}$$

If we consider the case $t>0$ in (3.26), we obtain for $\Phi(\log\lambda, t>0)$ an expression similar to (3.38) with $t-\epsilon$ replaced by $t+\epsilon$. Therefore Eq. (3.38) can be conveniently written as

$$\Phi(\log\lambda, t) = \frac{6}{\sqrt{\pi\log\lambda}}\int_0^\infty d\sigma e^{-\sigma^2}\frac{\sigma\sin(2\sigma\sqrt{\log\lambda})}{e^\epsilon\cosh(t)+\cos(2\sigma\sqrt{\log\lambda})}\ , \quad \lambda>1\ . \tag{3.39}$$

Clearly, the regulator ϵ has to be set to zero, thus leading to a representation that is valid for all the values of t except the origin, and describes a tachyon that rolls from the top of the potential at early times, passes the minimum and reaches at $t=0$ a point at which the integrand in (3.39) shows an infinite number of poles.

Eq.3.39 presents a cusp at the origin [2], which is obviously a problem both on mathematical and on physical grounds. Mathematically, a cusp prevents the solution even being C^1 - in order to belong to the definition domain of the operator $\lambda^{-\partial_t^2}$ it should be C^∞. Physically, it leads to apparent inconsistencies in the particle interpretation of the solution.

[2]It can be seen that setting ϵ to zero in (3.39) corresponds to consider Eq.3.30 with $e^{n|t|}$, for which a cusp in $t=0$ is evident.

The breakdown of the solution (3.39) at the point $t = 0$ is actually consistent within the picture of a diffusion driven by Eq.3.23, which has a negative unitary coefficient. The natural evolution - with respect to the "time" variable - of the initial solution (3.13) at value $\log \lambda = 1$ is therefore toward negative values of $\log \lambda$, *i.e.* values of λ between 0 and 1. An analytical continuation to positive values of $\log \lambda$ would then be the analogue of a propagation *backward* in time, and just as in the standard heat equation a non analytical behavior is expected [3].

It seems then confirmed that a pathological behavior arises in the dynamical process of a rolling tachyon in CSFT, whatever is the procedure chosen. The field starts from the maximum of the potential at $t \to -\infty$, rolls down to the minimum and past it up to $t = 0$. The origin of the time is a breaking point, after which the tachyon experiences wild oscillations, or becomes singular.

This pathology, when compared with the fairly transparent dynamics of the BSFT rolling tachyon [49] in which the tachyon monotonically rolls towards the true vacuum, rises an obvious puzzle. Which picture is correct? Does the tachyon converge monotonically to the true vacuum or does it present weird (ever-growing oscillations), or singular (cusp) behavior? Is there a problem with the BSFT approach or does the CSFT analysis break down for some reason such as a branch point singularity at a finite value of t? In [56] an argument was done to reconcile this discrepancy, based on the field redefinition we derived in [13] that takes the CSFT action to the BSFT action. We will discuss this argument in Section 3.3.

Before doing that we will present an interesting result related to the tachyon profile (3.39), an explicit calculation of the energy momentum tensor associated. In order to do that, we need a prescription that defines the derivatives of (3.39) in any point of the real axis, except the origin. This can be achieved integrating by parts Eq.(3.39) keeping $\epsilon \neq 0$. After integration by parts, the singularities of the denominator that would appear at $t = 0$ in the $\epsilon \to 0$ limit become logarithmic (integrable) singularities. Then the regulator ϵ can be removed, obtaining

$$\Phi(\log \lambda, t) = \frac{3}{\sqrt{\pi} \log \lambda} \int_0^\infty \frac{d}{d\sigma} \left(\sigma e^{-\sigma^2} \right) \log[\cosh t + \cos(2\sqrt{\log \lambda}\sigma)] d\sigma \ . \tag{3.40}$$

Iterating the procedure, any even derivative of T can be written in a manifestly regular

[3]The fundamental solution $K(x,t)$ of the one-dimensional heat equation

$$\frac{\partial}{\partial t} K(x,t) = \frac{\partial^2}{\partial x^2} K(x,t),$$

where t and x are standard time and space variables, is the heat kernel $K(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}$. This gaussian distribution becomes very broad when $t \to +\infty$, corresponding to the experience that a diffusing substance tends to a widespread, nearly uniform, density. As t tends to 0, $K(x,t)$ approaches the δ-function, thus preventing a propagation backward in time.

way. For example, the formula for the even derivatives of T reads

$$\frac{d^{2n}\Phi(\log\lambda, t)}{d(t)^{2n}} = \frac{3(-1)^n}{2^{2n}\sqrt{\pi}(\log\lambda)^{n+1}} \int_0^\infty \frac{d^{2n+1}}{d\sigma^{2n+1}} \left(\sigma e^{-\sigma^2}\right) \log[\cosh t + \cos(2\sqrt{\log\lambda}\sigma)] d\sigma \; . \tag{3.41}$$

Note that, since ϵ can be eventually removed, (3.40) and (3.41) work as a prescription to define the integral (3.40) with all its derivatives.

Moreover, Eq.(3.40) still satisfies the diffusion equation (3.23).

3.2.3 Energy-momentum tensor

Due to the presence of infinitely many derivatives in the CSFT action, a conserved energy-momentum tensor is expected to be defined through a generalized Noether procedure. As observed in [54], however, this would lead to total derivatives ambiguities in identifying the pressure from the Noether construction of $(-T^i_i)$. The energy-momentum tensor can be therefore calculated by doing a metric variation in the relativistic covariantization of the CSFT action. This can be achieved by first including a metric tensor $g_{\mu\nu}$ in the action (3.9), varying the action S with respect to $g_{\mu\nu}$ and setting afterwards the metric to be flat, $g_{\mu\nu} = \eta_{\mu\nu}$.

It is also possible to add a constant term $-\alpha$ to the action (3.9), it does not contribute to the equation of motion for the tachyon field but it does determine its dynamical behavior. In this way the tachyon potential reads

$$V[\,t\,] = -\frac{1}{2}\,t^2 + \frac{\lambda_c}{3}\,t^3 + \alpha \;\; . \tag{3.42}$$

Since α just gives the height of the maximum of the potential, a natural choice for it would be the one that sets to zero the minimum of the potential. In this case, when coupled to gravity, the potential would not produce a cosmological constant term when the tachyon is at the minimum. At the $(0,0)$ level truncation we are considering, such a constant is $1/(6\lambda_c^2)$, which is the 68% of the brane tension. When all the higher level fields are taken into account, the depth of the "effective" potential increases and the constant that sets to zero the minimum of the potential should reproduce the D-brane tension, which in our units is $1/(2\pi^2)$.

Thus, we consider the action

$$S = \frac{1}{g_o^2} \int d^{26}x\sqrt{-g} \left(\frac{1}{2}\,\phi^2 \, - \frac{1}{2}\, g^{\mu\nu}\delta_\mu\phi\,\delta_\nu\phi - \frac{1}{3}\lambda_c\,\tilde{\phi}^3 - \alpha \right) \quad , \tag{3.43}$$

where $\tilde{\phi} = \lambda_c^{\frac{1}{3}\Box}\phi$. The stress tensor reads

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} \;\; . \tag{3.44}$$

In varying (3.43) with respect to the metric tensor, one has to consider the covariant form of the D'Alembertian operator

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu \;\; . \tag{3.45}$$

The variation of the operator $\lambda_c^{\frac{1}{3}\Box}$ with respect to the metric can be performed by using the following identity

$$\frac{\delta \lambda_c^{\frac{1}{3}\Box}}{\delta g^{\alpha\beta}} = \frac{1}{3} \log\lambda_c \int_0^1 ds\, \lambda_c^{\frac{1}{3}s\Box} \, \frac{\delta\Box}{\delta g^{\alpha\beta}} \, \lambda_c^{\frac{1}{3}(1-s)\Box} \, . \tag{3.46}$$

An alternative way to get the variation of the infinitely many derivatives operator $\lambda_c^{\frac{1}{3}\Box}$ would be through a power series [54, 114] representation of the type (3.14).

However, the property (4.22) of our solution is particularly well suited to deal with operators of the type $\lambda_c^{\frac{1}{3}s\Box}$. In fact, their action on $\Phi(\log\lambda_c, t)$ consists in a trivial translation $\log\lambda_c \to (1+\frac{1}{3}s)\log\lambda_c$. This will permit to write the energy momentum tensor in a simple and closed form, as a bilinear in the fields $\Phi(\log\lambda_c, t)$ containing only *finite* derivatives. The substitution of infinite derivative operators acting on the field with the field itself, but with the parameter λ_c traslated, allows to write the energy momentum tensor in a form analogous to that of an ordinary (finite derivatives) field theory.

Taking the equation of motion (3.11) and Eqs.(3.16),(4.22),(3.43)-(3.46) into account, after some integrations by parts we get the following expression for the energy-momentum tensor

$$\begin{aligned} T_{\alpha\beta} \;=\; & \lambda_c^{-10/3} \Bigg\{ \delta_{\alpha 0}\, \delta_{\beta 0} \left(\partial_t \Phi(\tfrac{4}{3}\log\lambda_c, t)\right)^2 + g_{\alpha\beta} \Bigg[\frac{1}{2} \left(\partial_t \Phi(\tfrac{4}{3}\log\lambda_c, t)\right)^2 \\ & + \frac{1}{2} \left(\Phi(\tfrac{4}{3}\log\lambda_c, t)\right)^2 - \frac{1}{3}\, \Phi(\tfrac{5}{3}\log\lambda_c, t)\, (1-\partial_t^2)\, \Phi(\log\lambda_c, t) - \alpha\lambda_c^{10/3} \Bigg] \\ & - \frac{1}{3}\log\lambda_c \int_0^1 ds \Big[g_{\alpha\beta}\, (1-\partial_t^2)\, \Phi(\tfrac{4-s}{3}\log\lambda_c, t)\, \partial_t^2 \Phi(\tfrac{4+s}{3}\log\lambda_c, t) \\ & + g_{\alpha\beta} (1-\partial_t^2)\, \partial_t \Phi(\tfrac{4-s}{3}\log\lambda_c, t)\, \partial_t T(\tfrac{4+s}{3}\log\lambda_c, t) \Big] \\ & + 2\, \delta_{\alpha 0} \delta_{\beta 0}\, (1-\partial_t^2)\, \partial_t \Phi(\tfrac{4-s}{3}\log\lambda_c, t)\, \partial_t \Phi(\tfrac{4+s}{3}\log\lambda_c, t) \Bigg\} \quad . \end{aligned} \tag{3.47}$$

From (3.47) the explicit form of the energy density $\mathcal{E}(t) = T_{00}$ and the pressure $p(t) = T_{11}$ can be obtained

$$\begin{aligned} \mathcal{E}(t) \;=\; & \lambda_c^{-10/3} \Bigg\{ \frac{1}{2} \left(\partial_t \Phi(\tfrac{4}{3}\log\lambda_c, t)\right)^2 - \frac{1}{2} \left(\Phi(\tfrac{4}{3}\log\lambda_c, t)\right)^2 \\ & + \frac{1}{3} \Phi(\tfrac{5}{3}\log\lambda_c, t)(1-\partial_t^2)\Phi(\log\lambda_c, t) + \alpha\lambda_c^{10/3} \\ & - \frac{1}{3}\log\lambda_c \int_0^1 ds \Big[(1-\partial_t^2)\, \Phi(\tfrac{4-s}{3}\log\lambda_c, t)\, \partial_t^2 \Phi(\tfrac{4+s}{3}\log\lambda_c, t) \\ & - (1-\partial_t^2)\, \partial_t \Phi(\tfrac{4-s}{3}\log\lambda_c, t)\, \partial_t \Phi(\tfrac{4+s}{3}\log\lambda_c, t) \Big] \Bigg\} \quad , \end{aligned} \tag{3.48}$$

$$p(t) \;=\; \lambda_c^{-10/3} \Bigg\{ \frac{1}{2} \left(\partial_t \Phi(\tfrac{4}{3}\log\lambda_c, t)\right)^2 + \frac{1}{2} \left(\Phi(\tfrac{4}{3}\log\lambda_c, t)\right)^2$$

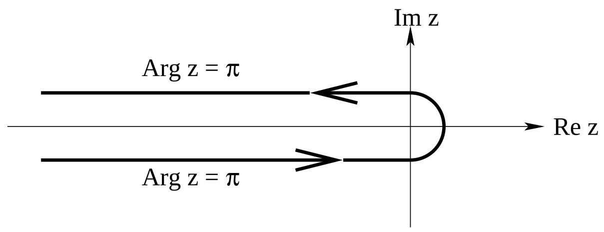


Figure 3.5: Contour C

$$
\begin{aligned}
&-\frac{1}{3}\Phi(\tfrac{5}{3}\log\lambda_c,t)(1-\partial_t^2)\Phi(\log\lambda_c,t)-\alpha\lambda_c^{10/3}\\
&-\frac{1}{3}\log\lambda_c\int_0^1 ds\,\Big[\,(1-\partial_t^2)\,\Phi(\tfrac{4-s}{3}\log\lambda_c,t)\,\partial_t^2\Phi(\tfrac{4+s}{3}\log\lambda_c,t)\\
&+(1-\partial_t^2)\,\partial_t\Phi(\tfrac{4-s}{3}\log\lambda_c,t)\,\partial_t\Phi(\tfrac{4+s}{3}\log\lambda_c,t)\Big]\Big\} \quad . \qquad (3.49)
\end{aligned}
$$

Even if from (3.48) the energy density seems to depend strongly on time, its plot - considered up to the breaking point $t = 0$ - shows that $\mathcal{E}(t)$ is actually a constant, always identical to the chosen height of the maximum of the potential, $\mathcal{E} = \alpha$. The pressure starts from negative ($p = -\alpha$), thus forcing the tachyon to roll towards the minimum, becoming positive between this and the breaking point.

3.3 Are CSFT and BSFT results reconcilable?

The field redefinition derived in [13] and described in Section 1.8 has been used in [56] to map the well-behaved BSFT pure exponential rolling tachyon solution [49] $T_{rolling} = e^t$ into the oscillatory CSFT solution (3.5) up to the second power in e^t. We now review this argument, reminding the results of [13].

In parallel with the CSFT action (3.2)

$$S_{CSFT}[\phi] = \sum_n \frac{g^{n-2}}{n!}\int\prod_{i=1}^{n}(2\pi dk_i)\delta(\sum_i k_i)A_n(k_1,\ldots,k_n)\ \phi(k_1)\ldots\phi(k_n) \quad , \qquad (3.50)$$

with A_2 and A_3 given in (3.4), one considers the BSFT tachyon effective action derived up to the cubic order in Section 1.7

$$S_{BSFT}[T] = \sum_n \frac{g^{n-2}}{n!}\int\prod_{i=1}^{n}(2\pi dk_i)\delta(\sum_i k_i)\ B_n(k_1,\ldots,k_n)\ T(k_1)\cdots T(k_n) \quad . \qquad (3.51)$$

In (3.51) the BSFT tachyon T is the renormalized tachyon with the renormalization scheme [13] described in Section 1.4 and the coefficients $B_n(k_1,\ldots,k_n)$ are exactly known up to the third order [13]. The quadratic term is

$$B_2(k_1,k_2) = -\frac{\Gamma(2-2k_1k_2)}{\Gamma^2(1-k_1k_2)} \qquad (3.52)$$

Defining $a_1 = -w_2 w_3$, $a_2 = -w_1 w_3$, $a_3 = -w_2 w_3$ the cubic cofficient in (3.51) can be written as

$$B_3(w_1, w_2, w_3) = 2(1 + a_1 + a_2 + a_3) I(w_1, w_2, w_3) + J(w_1, w_2, w_3) \tag{3.53}$$

where $I(a_1, a_2, a_3)$ and $J(a_1, a_2, a_3)$ are

$$\begin{aligned} I(a_1,a_2,a_3) &= \frac{\Gamma(1+a_1+a_2+a_3)\Gamma(1+2a_1)\Gamma(1+2a_2)\Gamma(1+2a_3)}{\Gamma(1+a_1)\Gamma(1+a_2)\Gamma(1+a_3)\Gamma(1+a_1+a_2)\Gamma(1+a_1+a_3)\Gamma(1+a_2+a_3)}, \\ J(a_1,a_2,a_3) &= -\frac{\Gamma(1+2a_1)\Gamma(2+2a_2+2a_3)}{\Gamma^2(1+a_1)\Gamma^2(1+a_2+a_3)} + \text{cyclic.} \end{aligned} \tag{3.54}$$

The general field redefinition that relates (3.50) and (3.51) reads

$$\begin{aligned} \varphi(k_1) &= \int dk_2 \delta(k_1 - k_2) f_1(k_1, k_2) T(k_2) \\ &+ \int dk_2 dk_3 \, \delta(k_1 - k_2 - k_3) \; f_2(k_1, k_2, k_3) T(k_2) T(k_3) + \cdots \end{aligned} \tag{3.55}$$

where f_2 is symmetric under $k_2 \longleftrightarrow k_3$. The requirement that this field redefinition maps the CSFT action to the BSFT action imposes conditions on the functions f_i. From the matching of the quadratic and cubic terms it follows for f_1 the expression

$$(f_1(k,k))^2 = (f_1(k))^2 = \frac{B_2(k,-k)}{A_2(k,-k)} \,. \tag{3.56}$$

and for f_2 the relation

$$f_2(k_1,k_2,k_3) = \frac{1}{2A_2(k_1,-k_1)} \left[\frac{B_3(-k_1,k_2,k_3)}{f_1(k_1)} - A_3(-k_1,k_2,k_3) f_1(k_2) f_1(k_3) \right] \tag{3.57}$$

In order to check that this field redefinition maps the rolling tachyon solution of BSFT into the CSFT perturbative solution in powers of e^{nt} one has to plug the pure exponential solution $T_{rolling} = e^t$ into the field redefinition (3.55) and compute the numerical values. As we are considering powers of e^t, we are restricting attention to fields expressed in (imaginary) integers modes $k = i\ n$. The CSFT solution to which the BSFT $T_{rolling}$ is mapped thus reads

$$\phi(t) = f_1(-i)\ e^t + f_2(-2i, -i, -i)\ e^{2t} + \cdots \tag{3.58}$$

It is easy to see, from (3.4,3.52,3.56), that $f_1(-i)$ is a constant. It actually can *always* be set equal to the first coefficient a_1 of a solution expressed as a power series of the type (3.1) by means of a time translation. As for the coefficient $f_2(-2i, -i, -i)$, it has to be evaluated when in (3.55) $T(k_1)$ and $T(k_2)$ are on mass-shell ($k_2^2 = k_3^2 = -1$) while $\varphi(k_1)$ is not ($k_1^2 = -4$. As already noticed in [13], in this case the BSFT cubic coefficient

$B_3(k_1, k_2, k_3)$ in (3.57) has a zero stronger then the one exhibited by $f_1(k_1^2 = -4)$, so that it can be written (reminding that $a_1 = f_1(-i)$)

$$f_2(-2i, -i, -i) = -\frac{1}{2A_2(2i, -2i)} A_3(2i, -i, -i) a_1^2 \tag{3.59}$$

It is not difficult to check that the right-hand side of (3.59) is exactly the formula defining the second coefficient a_2 of the tachyon profile (3.1) given by a sum of exponentials, as results by solving iteratively the equation of motion as described in Section (3.1). This might suggest that the wild-oscillating solutions analyzed in Section 3.1 correspond to the well-behaved $T_{rolling}$ of BSFT [56]. However, it seems to us that a relevant observation is the following. The quadratic and cubic coefficients (3.52) and (3.53) of the BSFT tachyon effective action are such that, when the pure exponential tachyon profile $T_{rolling}$ is plugged in to the BSFT equation of motion [4]

$$T(-k) B_2(k, -k) + \frac{1}{2} \int dk_1 \; B_3(k, -k - k_1, k_1) \; T(k_1) \; T(-k - k_1) = 0 \tag{3.60}$$

each term in (3.60) identically vanish. This amounts in a lack of information on the BSFT side of the field redefinition (3.55) from which follows the formula (3.59). This formula states that $f_2(-2i, -i, -i) \equiv a_2$ because the features of B_2 and B_3 are such that we are left, at this order, only with the CSFT information. On the other end, the first coefficient a_1 can be *always* be recovered through a translation in time. Thus, some more informations should come at least from the analysis of a further coefficient, that would require the knowledge of the quartic coefficient B_4 of the BSFT tachyon effective action - not a trivial task, dealing with integrals more involved that the ones considered in Appendix A. From this one could derive a f_3 function in the field-redefinition (3.55). In any case, we have checked that, in order to reproduce from such a f_3 *exactly* the third coefficient a_3 in (3.1) as it comes by solving iteratively the CSFT equation of motion, the quartic coefficient B_4 of the BSFT tachyon effective action should, again, identically vanish - which is certainly not obvious.

We believe therefore that the problem of reconciling the BSFT and CSFT results about rolling tachyon solutions in string field theory is still far from being completely understood.

[4] The equation of motion are written - in the momentum space - up to the order for which we know the coefficients, *i.e.* the third.

Chapter 4

DLCQ strings on a pp-wave and the integrability of $\mathcal{N} = 2$ SYM

4.1 Introduction

The AdS/CFT duality [58] claims the equivalence of $\mathcal{N} = 4$ extended supersymmetric gauge theory and IIB string theory on the $AdS_5 \times S^5$ background. Unfortunately, quantitative tests of this conjecture are usually prevented by its strong/weak nature.

A first step to face these difficulties was the realization that the Green–Schwarz superstring, evaluated in the light-cone gauge, becomes a free worldsheet theory if one replace the $AdS_5 \times S^5$ background by a *Penrose limit* describing the near neighborhood of an equatorial lightlike geodesic on the S^5 subspace [115, 64]. The energy spectrum of this free theory is simply that of a string moving around the equator of the S^5 and boosted to large angular momentum J. By the AdS/CFT correspondence, these string energies should match the dimensions of operators with large R-charge ($R \sim J$) in strongly coupled four-dimensional $\mathcal{N} = 4$ super Yang–Mills theory. In [61], Berenstein Maldacena and Nastase (BMN) identified the subspace of gauge theory operators corresponding to specific free string excited states (i.e. states with different numbers and types of string oscillator modes applied to the string ground state) and showed that perturbative calculations of the dimensions of these operators are reliable in the large R-charge limit, giving evidence that they agree with the string theory predictions. This resulted in a formidable prediction for the all loop scaling dimensions of the dual gauge theory operators in the corresponding limit, the formula

$$\Delta_n = J + 2\sqrt{1 + \frac{\lambda\, n^2}{J^2}} \tag{4.1}$$

for the simplest two string oscillator mode excitation. The key point was the emergence of the effective gauge theory loop counting parameter $\lambda/J^2 = \lambda'$.

A second fundamental step of our understanding of the AdS/CFT correspondence benefited greatly from the discovery of *integrable* structures both on the gauge theory and string theory side, [66], [68]-[84], which allowed to firmly reproduce the scaling dimensions (4.1) up to the three loop order in gauge theory [68, 72, 116].

A natural question to ask is to what extent we can deform the model and "enlarge" the correspondence while preserving full integrability. In this chapter we propose two ways in this direction. In the first place, if string theory on $AdS_5 \times S^5$ is integrable, the theory on simple *orbifolds* of that space would also be expected to be integrable. In the Yang-Mills dual, orbifolding reduces the amount of supersymmetry and this gives some hope of finding integrability in theories with less supersymmetry [87, 88, 89, 90]. A second way to extend the duality is the realization that this equivalence should not be restricted to the Penrose limit of the geometry (or the large R-charge limit of operator dimensions), i.e. one should go *beyond* the pp-wave [75]. Corrections to the string spectrum should subsequently be compared with an expansion in inverse powers of R-charge of the dimensions of BMN-type gauge theory operators.

In this chapter, we shall consider a particular Z_M-orbifold of $\mathcal{N} = 4$, an $\mathcal{N} = 2$ supersymmetric $SU(N)^M$ quiver gauge theory [91] which can be obtained as a particular Z_M-orbifold of $\mathcal{N} = 4$ [92]. This system is also conjectured to be integrable using a twisted version of the Bethe ansatz [93]. Its string theory dual is IIB superstrings on the space $\mathrm{AdS}_5\times\mathrm{S}^5/Z_M$, for which a Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5/Z_M$ (together with a large order limit of the orbifold group, $M \to \infty$) can be taken in such a way that it obtains a plane-wave with a periodically identified null coordinate. The IIB superstring can be solved explicitly in this background. We will present a computation of the leading finite size corrections to the string and the gauge spectrum in an expansion about this limit. We will focus on the Bethe ansatz techniques for solving integrable spin chains which arise in the gauge theory, that when compared to the string results will in turn confirm the integrability of $\mathcal{N} = 2$ quiver gauge theory within the same framework of $\mathcal{N} = 4$. In what follows we will only sketch some topics needed in the next sections [1].

4.2 Review of the tools

The BMN [61] proposal identifies the energy eigenstates of the $AdS_5 \times S^5$ string with (suitable) composite gauge theory trace operators of the form $\mathcal{O}_\alpha = \mathrm{Tr}(\Phi_{i_1}\, \Phi_{i_2} \ldots \Phi_{i_n})$. Here $(\Phi_i)_{ab}$ are the elementary fields of $\mathcal{N} = 4$ SYM (and their covariant derivatives) in the adjoint representation of $SU(N)$, i.e. $N \times N$ hermitian matrices, evaluated at the same point. The energy eigenvalue E of a string state is conjectured to be equal to the scaling dimension Δ of the dual gauge theory operator.

The standard way to find the scaling dimensions $\Delta_{\mathcal{O}_\alpha}$ of a set of conformal fields $\mathcal{O}_\alpha$ is to consider the two-point functions

$$\langle \mathcal{O}_\alpha(x)\, \mathcal{O}_\beta(0)\rangle = \frac{\delta_{\alpha\beta}}{|x|^{2\Delta_\alpha}}.$$

Classically, these scaling dimensions are simply the sum of the individual dimensions of

[1] A nice review of these topics is [117].

the constituent fields obtained by standard power counting. In quantum theory the scaling dimensions receive anomalous corrections, organized in a double expansion in $\lambda = g_{\rm YM}^2 N$ (loops) and $1/N^2$ (genera)

$$\Delta = \Delta_0 + \sum_{l=1}^{\infty} \lambda^l \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \Delta_{l,g} \, . \tag{4.2}$$

For weak coupling $g_{\rm YM}$, we can compute corrections to the naive dimension by perturbation theory. In practice, extracting these corrections from eq.(4.2) can be problematic. Startin with a set of operators $\mathcal{O}_\alpha$ with the same engineering (tree-level) dimension one generically encounters the phenomenon of mixing: the two-point function is not diagonal in α, β, and one rather has a matrix $\langle \mathcal{O}_\alpha(x) \mathcal{O}_\beta(0) \rangle$ since a generic field does not have a definite scaling dimension. It therefore seems that one has to diagonalize the two-point functions order by order in perturbation theory.

This task is facilitated through the use of the gauge theory dilatation operator acting on states at the origin of space-time (in a radial quantization scheme) as was already done, for one-loop, in [66] (on the planar level) and in [118] (in the BMN limit). Its eigensystem consists of the eigenvalues Δ_α and the eigenstates $\hat{\mathcal{O}}_\alpha$. One thus has

$$D \, \hat{\mathcal{O}}_\alpha = \Delta_\alpha \, \hat{\mathcal{O}}_\alpha .$$

In $\mathcal{N} = 4$ there is a sector of local operators in which the dilatation operator takes a particularly simple form: the $SU(2)$ sector. In this sector, one restricts attention to four of the 6 scalars of $\mathcal{N} = 4$ in the complex combinations $Z = \Phi^1 + i\Phi^2$ and $\Phi = \Phi^3 + i\Phi^4$ and the composite operators [2]

$$\mathrm{Tr}\,(\Phi Z Z Z \Phi \Phi Z \Phi Z Z Z ...) \tag{4.3}$$

with a total number of J Z-fields and $\mathcal{M}$ Φ fields. The full one-loop contribution to the dilatation operator was first worked out, as an "effective vertex", in [98]. The dilatation operator up to one loop reads

$$D_0 = \mathrm{Tr}(Z\bar{Z} + \Phi\bar{\Phi}), \qquad D_{1_{loop}} = -\frac{g_{\rm YM}^2}{8\pi^2} \, \mathrm{Tr}\,[Z, \Phi]\,[\bar{Z}, \bar{\Phi}] \, . \tag{4.4}$$

Note that the tree-level piece D_0 simply measures the length of the incident operator (or spin chain) $L = J + \mathcal{M}$, as the engineering dimension of scalars is one. We shall be exclusively interested in the planar contribution to D, as this sector of the gauge theory corresponds to the "free" $AdS_5 \times S^5$ string. For this it is important to realize that the

[2] In the BMN proposal [61], single string states map to certain single trace operators in gauge theory, made up by taking products of the scalar fields at the same point. In particular, the single string vacuum state is identified with the chiral primary BPS operator

$$\mathrm{Tr}\,(ZZZ....ZZ) = \mathrm{Tr}\,\left(\mathrm{Z}^{\mathrm{J}}\right) \quad .$$

Excitations of the vacuum are obtained by inserting some Φ fields (called *impurities*) in the trace.

planar piece of $D_{1\ loop}$ only acts on two neighboring fields in the chain of Z and Φ fields. By explicitly calculating the action of D on (4.3) [3] it is easy to see that

$$D_{1\ loop}^{\text{planar}} = \frac{\lambda}{8\pi^2} \sum_{i=1}^{L} (\mathbf{1} - P_{i,i+1}) \tag{4.5}$$

where $P_{i,j}$ permutes the fields (or spins) at sites i and j and the periodicity $P_{L,L+1} = P_{1,L}$ of the trace is taken into account. Remarkably, as noticed by Minahan and Zarembo [66], this operator coincides with the Heisenberg $\text{XXX}_{1/2}$ quantum spin chain Hamiltonian. Written in terms of the Pauli matrices $\vec{\sigma}_i$ acting on the spin at site i one finds

$$D_{1\ loop}^{\text{planar}} = \frac{\lambda}{8\pi^2}\mathcal{H}_{\text{XXX}_{1/2}} = \frac{\lambda}{16\pi^2} \sum_{i=1}^{L} (\mathbf{1} - \vec{\sigma}_i \cdot \vec{\sigma}_{i+1})\,. \tag{4.6}$$

Actually, since the Z and the Φ fields are related by an $SU(2)$ R-symmetry subgroup of $SO(6)$, they can be thought as a spin up or spin down configuration, $Z \equiv |\uparrow\rangle$ and $\Phi \equiv |\downarrow\rangle$. The local operators (4.3) of the field theory then can be viewed as quantum mechanical states of a one-dimensional lattice of L sites, an $SU(2)$ spin chain

$$\text{Tr}\,(\Phi ZZZ\Phi\Phi Z\Phi ZZZ...) \quad \Leftrightarrow \quad |\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\ ...\rangle_{\text{cyclic}}\,.$$

In this picture, the vacuum BPS state is identified with the ferromagnetic vacuum state $\text{Tr}\,(ZZZ...Z) \Leftrightarrow |\uparrow\uparrow\uparrow\ ...\uparrow\rangle_{\text{cyclic}}$. The excitations (*impurities*) on this ground states are given by spin flips or *magnons*.

The Heisenberg hamiltonian (4.5) is the prototype of an integrable spin-chain. The integrability of the spin-chain of lenght L amounts to the existence of $L-1$ higher charges Q_k which commute with the Hamiltonian (alias dilatation operator) and among themselves, i.e. $[Q_k, Q_l] = 0$. Explicitly the first two charges of the Heisenberg chain are given by

$$Q_2 := \mathcal{H}_{\text{XXX}_{1/2}} \qquad Q_3 = \sum_{i=1}^{L} (\vec{\sigma}_i \times \vec{\sigma}_{i+1}) \cdot \vec{\sigma}_{i+2}\,. \tag{4.7}$$

The explicit form of all the higher Q_k may be found in [119]. Note that Q_k will involve up to k neighboring spin interactions.

The algebraic Bethe ansatz

The integrability of the Heisenberg spin chain allows its diagonalization by means of the algebraic Bethe ansatz [67], that we briefly review here. The Bethe ansatz determines the energy eigenvalues of a quantum integrable spin chain, giving a set of algebraic equations whose solution directly leads to the energies as well as the eigenvalues of the higher charges.

[3] The fission rule $\text{Tr}A\bar{\Phi}_m B\Phi_n = \delta_{mn}\text{Tr}A\text{Tr}B$ and the fusion rule $\text{Tr}A\bar{\Phi}_m\text{Tr}\Phi_n B = \delta_{mn}\text{Tr}AB$ are useful to see that.

It makes use of the Bethe equations for $\mathcal{M}$ magnons on a chain of length L:

$$e^{ip_jL} = \prod_{l=1\ ;\ l\neq j}^{\mathcal{M}} \frac{\varphi_j - \varphi_l + i}{\varphi_j - \varphi_l - i} = \prod_{l=1\ ;\ l\neq j}^{\mathcal{M}} S(p_j, p_l) \qquad l = 1, \ldots, \mathcal{M} \tag{4.8}$$

The left hand side is a free plane wave phase factor for the j-th magnon, with momentum p_j, going around the chain. The first formula on the right hand side is "almost" one, except for a sequence of *pairwise, elastic* interactions with the $\mathcal{M} - 1$ other magnons, leading to a small phase shift. Without this phase shift, the equation simply leads to the standard momentum quantization condition for a free particle on a circle. The details of the exchange interactions are encoded into the functions $\varphi_j = \varphi(p_j)$, called *rapidities*, and change from model to model. The two-body nature of the scattering is, however, a universal feature leading to integrability. It allows the reduction of an $\mathcal{M}$-body problem to a sequence of two-body problems, as the factorization to 2-body S-matrices show in the second formula on the right hand side [4].

The energy and higher charge eigenvalues are then given by the linear sum of contributions from the individual magnons

$$Q_r = \sum_{k=1}^{\mathcal{M}} q_r(p_k), \qquad H = Q_2.$$

This additive feature is due to the almost complete independence of the individual excitations. However, the details of the contribution of an individual excitation to the r-th charge, $q_r(p_k)$, depend once more on the precise integrable model. For example, the $\text{XXX}_{1/2}$ Bethe ansatz is obtained by setting

$$\varphi(p) = \frac{1}{2}\cot\frac{p}{2}, \qquad q_r(p) = \frac{2^r}{r-1}\sin\left(\frac{(r-1)}{2}p\right)\sin^{r-1}\frac{p}{2}.$$

In order to reinstate the cyclicity of the trace condition one needs to further impose the constraint of a total vanishing momentum

$$\sum_{i=1}^{\mathcal{M}} p_i = 0 \tag{4.9}$$

As an example let us diagonalize the two magnon problem exactly. Due to (4.9) we have $p := p_1 = -p_2$ and the Bethe equations (4.8) reduce to the single equation

$$e^{ipL} = \frac{\cot\frac{p}{2} + i}{\cot\frac{p}{2} - i} = e^{ip} \quad \Rightarrow \quad e^{ip(L-1)} = 1 \quad \Rightarrow \quad p = \frac{2\pi\, n}{L-1}\,. \tag{4.10}$$

[4]We are discussing here an *internal* S-Matrix describing the scattering of elementary excitations on a lattice hidden inside the trace of of gauge invariant composite local operators. It should certainly not be confused with the *external* S-matrix of $\mathcal{N} = 4$ which refers to multi-gluon amplitudes in four-dimensional space time. Recently dramatic progress was also achieved in this direction, see [120], and references therein.

The energy eigenvalue then reads

$$E = Q_2 = q_1 + q_2 = 8\sin^2\left(\frac{\pi\, n}{L-1}\right) . \tag{4.11}$$

Reinserting the prefactor of $\frac{\lambda}{8\pi^2}$ present in (4.6) and writing $L = J + 2$, one gets the one-loop scaling dimension $\Delta_{1\ loop} = \frac{\lambda}{\pi^2}\sin^2(\frac{\pi\, n}{J+1})$ of the two-magnon operators [121, 66]. In the BMN limit $N, J \to \infty$ with λ/J^2 fixed, the scaling dimension takes the famous value $\Delta_{1\ loop} = n^2\,\lambda/J^2$, corresponding to the first term in the expansion of (4.1), *i.e.* in the the level-two energy spectrum of the plane-wave superstring $E_{\text{light-cone}} = \sqrt{1 + n^2\,\lambda/J^2}$ [61].

Hence from the viewpoint of the spin chain the plane-wave limit corresponds to a chain of diverging length $L >> 1$ carrying a finite number of magnons $\mathcal{M}$, which are nothing but the gauge duals of the oscillator excitations of the plane-wave superstring.

Higher loops

Higher loop contributions to the planar dilatation operator in the $SU(2)$ subsector are by now firmly established to the two-loop [68] and three-loop level [72, 116]. In a $s = 1/2$ quantum spin chain language the dilatation operator can be expressed in terms of the spin chain Hamiltonian H as

$$D(g) = L + g^2 H \qquad\qquad Q_2 = H \quad ,$$

where $g^2 = \frac{\lambda}{8\pi^2}$, $H = \sum_{i=1}^{L}(h_2 + g^2 h_4 + g^2 h_6 + \ldots)$ and

$$\begin{aligned}
h_2 &= \frac{1}{2}(1 - \vec{\sigma_i}\vec{\sigma_{i+1}})\,, \\
h_4 &= -(1 - \vec{\sigma_i}\vec{\sigma_{i+1}}) + \frac{1}{4}(1 - \vec{\sigma_i}\vec{\sigma_{i+2}})\,, \\
h_6 &= \frac{15}{4}(1 - \vec{\sigma}_i\vec{\sigma}_{i+1}) - \frac{3}{2}(1 - \vec{\sigma}_i\vec{\sigma}_{i+2}) + \frac{1}{4}(1 - \vec{\sigma}_i\vec{\sigma}_{i+3}) \\
&\quad -\frac{1}{8}(1 - \vec{\sigma}_i\vec{\sigma}_{i+3})(1 - \vec{\sigma}_{i+1}\vec{\sigma}_{i+2}) \\
&\quad +\frac{1}{8}(1 - \vec{\sigma}_i\vec{\sigma}_{i+2})(1 - \vec{\sigma}_{i+1}\vec{\sigma}_{i+3})\,.
\end{aligned} \tag{4.12}$$

In general the k-loop contribution to the dilatation operator involves interactions of $k+1$ neighboring spins, i.e. the full dilatation operator will correspond to a long-range interacting spin-chain hamiltonian. Integrability remains stable up to three loop order [5] and acts in a perturbative sense, the conserved charges of the Heisenberg $\text{XXX}_{1/2}$ chain receive higher order corrections in λ [6]. An additional key property of these higher-loop corrections is that they obey the BMN scaling (4.1) [68, 73], *i.e.* the emergence of the effective

[5] It was recently shown [81] the relation between (4.12) and the strong coupling limit of the Hamiltonian of the one-dimensional Hubbard model at half filling, a well studied integrable model of condensed matter theory.

[6] However, opposed to the situation for the Heisenberg chain [67], there does not yet exist an algebraic construction of the gauge theory charges at higher loops. Nevertheless the first few Q_k have been constructed manually to higher loop-orders [78].

loop-counting parameter $\lambda' := \lambda/J^2$ in the $J \to \infty$ limit leading to the scaling dimensions $\Delta \sim \sqrt{1+\lambda' n^2}$ for two magnon states in quantitative agreement with plane-wave superstrings.

Motivated by these findings Beisert, Dippel and Staudacher [76] turned the logic around and simply *assumed* integrability, BMN scaling and a Feynman diagrammatic origin of the k-loop $SU(2)$ dilatation operator.

Upon these assumptions, their proposal gives an elegant presentation of the problem of computing operator dimensions to all orders in the coupling constant. We emphasize at this point, that we shall only use this proposal up to three loop order, where its equivalence to renormalized Yang-Mills perturbation theory has been firmly established. In fact, we shall mainly be interested in a twisted generalization of it, which is conjectured to describe a Z_M-orbifold of $\mathcal{N}=4$ super-Yang-Mills theory.

In the proposal, the problem for computing eigenvalues of the dilatation operator is summarized in three equations. First, it makes use of the Bethe equations (4.8) for $\mathcal{M}$ magnons on a chain of length L, where the momenta are constrained by the "level-matching condition" (4.9). Then, there is the BDS "all-loop ansatz", which manifestly obeys BMN scaling - that are the remaining two equations. One relates momenta and rapidities, which depends on the 't Hooft coupling λ,

$$\varphi(p_j) = \frac{1}{2}\cot\frac{p_j}{2}\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}}\ . \tag{4.13}$$

The other gives the spectrum of dimensions as a function of the momenta,

$$\Delta = L - \mathcal{M} + \sum_{j=1}^{\mathcal{M}}\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}} \tag{4.14}$$

The program of computing operator dimensions is implemented as seen in the resolution of the Heisenberg chain. Eqs. (4.8) and (4.13) should first be solved to find p_i. The solutions must depend on λ and can in principle be found at least order-by-order in an expansion in λ. Then, the solutions must be inserted into Eq. (4.14) to find the operator dimensions. The statement is that this procedure should yield the dimensions of this class of operators in $\mathcal{N}=4$ super-Yang-Mills theory. Explicit computations and comparison with diagrammatic perturbation theory have shown that this procedure agrees with renormalized Yang-Mills perturbation theory to at least third order, and is conjectured to do so for higher orders. There is a number of quite non-trivial checks of this fact which are outlined in Ref. [76].

4.3 $\mathcal{N}=2$ quiver gauge theory as orbifolded $\mathcal{N}=4$

Before we go on to discuss integrability of the $\mathcal{N}=2$ theory, we pause to review some facts about the structure of the theory and the procedure for computing operator dimensions there.

The $\mathcal{N}=2$ quiver gauge theory with gauge group $SU(N)^M$ is obtained from $\mathcal{N}=4$ with gauge group $SU(MN)$ by a well-known projection. Details of this construction can be found in the literature [91, 87, 122]. The conventions and notation that we use are those of Refs. [92],[90] and some more details can be found there.

We begin with $\mathcal{N}=4$ with a $U(MN)$ gauge group. The orbifold group will be the cyclic group Z_M whose generator γ acts on the six scalar fields of $\mathcal{N}=4$ theory as

$$\gamma:\left(\frac{\phi^1+i\phi^2}{\sqrt{2}},\frac{\phi^3+i\phi^4}{\sqrt{2}},\frac{\phi^5+i\phi^6}{\sqrt{2}}\right)=\left(\omega\frac{\phi^1+i\phi^2}{\sqrt{2}},\omega^{-1}\frac{\phi^3+i\phi^4}{\sqrt{2}},\frac{\phi^5+i\phi^6}{\sqrt{2}}\right),\quad \omega=e^{\frac{2\pi i}{M}} \tag{4.15}$$

The procedure for obtaining the quiver gauge theory from $\mathcal{N}=4$ begins by embedding the orbifold group Z_M, which is a subgroup of the R-symmetry group, into the gauge group. We will assume that Z_M is in the $\mathfrak{su}(2)$ subgroup of the $\mathfrak{su}(4)$ R-symmetry so that orbifolding preserves $\mathcal{N}=2$ supersymmetry. If γ is an element of Z_M, $R(\gamma)$ is the corresponding element of the R-symmetry group and $U(\gamma)$ is a $U(MN)\times U(MN)$ matrix containing N copies of the regular representation of Z_M, we consider that subset of the $\mathcal{N}=4$ fields which obey the constraint

$$X=U(\gamma)\,[R(\gamma)\circ X]\,U^{\dagger}(\gamma) \tag{4.16}$$

This is accomplished by setting to zero all of those components which do not obey this condition. In the present case, choosing $U(\gamma)$ having the $N\times N$ blocks

$$U(\gamma)=\begin{pmatrix} \bar{1} & 0 & 0 & 0 & ... \\ 0 & \omega & 0 & 0 & ... \\ 0 & 0 & \omega^2 & 0 & ... \\ . & . & . & . & ... \\ 0 & 0 & 0 & ... & \omega^{M-1}\end{pmatrix}$$

and the action

$$R(\gamma)Z=\omega Z \quad , \quad R(\gamma)\Phi=\Phi,$$

then some components of the $MN\times MN$ matrix fields are set to zero. The resulting $\mathcal{N}=2$ theory has residual R-symmetry $U(1)\times SU(2)$ and gauge group

$$U(N)^{(1)}\times U(N)^{(2)}\times\cdots U(N)^{(M)}. \tag{4.17}$$

$U(N)^{(M+1)}$ is identified with $U(N)^{(1)}$.

The resulting field content is as follows:

- M vector multiplets
$$(A_{\mu I},\Phi_I,\psi_I,\psi_{\Phi I})\quad,\quad I=1,...,M. \tag{4.18}$$
Φ_I is a complex scalar field and the Weyl fermion $\psi_{\Phi I}$ is its superpartner. A^{μ}_I is the gauge field and ψ_I is the gaugino. All of these fields transform in the adjoint representation of $U(N)^{(I)}$.

- M bi-fundamental hypermultiplets which, in $\mathcal{N}=1$ notation, are

$$(A_I, B_I, \chi_{AI}, \chi_{BI}) \tag{4.19}$$

The complex scalar field A_I and its super-partner ψ_{AI} transform in the $(N_I, \bar{N}_{I+1})$ representation of $U(N)^{(I)} \times U(N)^{(I+1)}$. The pair B_I and χ_{BI} transform in the complex conjugate representation $(\bar{N}_I, N_{I+1})$.

All fields are $MN \times MN$ matrices. With the notation

$$\mathbf{A} = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) \quad , \quad \mathbf{B} = \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4) \quad , \quad \mathbf{\Phi} = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6) \tag{4.20}$$

the elements of the bosonic fields - which are of interest to us - which survive the projection (4.16) are

$$\mathbf{\Phi} \equiv \begin{pmatrix} \Phi_1 & 0 & \cdots & 0 \\ 0 & \Phi_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_M \end{pmatrix} \quad , \quad A_\mu \equiv \begin{pmatrix} A_{\mu 1} & 0 & \cdots & 0 \\ 0 & A_{\mu 2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\mu M} \end{pmatrix} \tag{4.21}$$

$$\mathbf{A} \equiv \begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{M-1} \\ A_M & 0 & 0 & \cdots & 0 \end{pmatrix} \quad , \quad \mathbf{B} \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 & B_M \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{M-1} & 0 \end{pmatrix} \tag{4.22}$$

Each non-vanishing entry of the above matrices is an $N \times N$ matrix and corresponds to a bosonic field of the $\mathcal{N}=2$ theory. Analogous expressions hold for the fermionic superpartners [90]. It is convenient to think of the blocks as being labelled periodically, $A_{M+1} = A_1$, etc. The gauge group is $[U(N)]^M$ with elements labelled by U_I, $I = 1, ..., M$ and each field transforms as

$$A_I \to U_I A_I U_{I+1}^\dagger \quad , \quad \bar{A}_I \to U_{I+1} A_I U_I^\dagger \tag{4.23}$$

$$\Phi_I \to U_I \Phi_I U_I^\dagger \quad , \quad \bar{\Phi}_I \to U_I \bar{\Phi}_I U_I^\dagger \tag{4.24}$$

States of the $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ super-Yang-Mills were words made from Z and Φ,

$$\mathrm{Tr}(ZZ\Phi Z\Phi ZZZZ\Phi ZZZ...)$$

Since the remaining gauge transformations (4.23) and (4.24) now commute with $U(\gamma)$, there are additional gauge invariant twisted operators

$$\mathrm{Tr}\ \left[U(\gamma)^\ell ZZ\Phi Z\Phi ZZZZ\Phi ZZZ...\right] \quad , \quad \ell = 0, 1, ..., M-1 \tag{4.25}$$

These are translated into words with (A_I, Φ_I) by substituting (4.21) and (4.22). We will see that the twist in (4.25) can be identified with the string wrapping number.

IIB String on $AdS_5 \times S^5/\mathbf{Z}_M$

The $\mathcal{N} = 2$ theory is the holographic dual of IIB string theory with background the orbifold $AdS_5 \times S^5/\mathbf{Z}_M$ and with MN units of Ramond-Ramond 5-form flux through the 5-sphere. Since the 5-sphere contains M copies of a fundamental domain that are identified by the orbifold group, there are N units of flux per fundamental domain. The action of the orbifold group is obtained by embedding the 5-sphere in $\mathbf{R}^6 \sim \mathbf{C}^3$ so that

$$\sum_{i=1}^{3} |z_i|^2 = R^2$$

where $(z_1, z_2, z_3) \in \mathbf{C}^3$ and then identifying points as prescribed in (4.15):

$$\big(z_1, z_2, z_3\big) \sim \big(\omega z_1, \omega^{-1} z_2, z_3\big) \quad . \quad \omega = e^{2\pi i/M} \tag{4.26}$$

The radii of AdS_5 and S^5 are equal and are given by

$$R^2 = \sqrt{4\pi g_s \alpha'^2 NM} \ , \tag{4.27}$$

where g_s is the type IIB string coupling. Furthermore, the Yang-Mills theory coupling constant of the parent $\mathcal{N} = 4$ theory is identified with the coupling constant of the parent superstring theory on $AdS_5 \times S^5$,

$$4\pi g_s = g_{YM}^2 \tag{4.28}$$

Double Scaling limit

We shall consider the double scaling limit of both the gauge theory and its string theory dual. The double scaling limit of the string theory is the Penrose limit which obtains the pp-wave background. The radii of AdS_5 and S^5, given by R in (4.27), are put to infinity by scaling both N and M to infinity while keeping g_s small but finite. The parameter which will become the null compactification radius, $R^- = \frac{R^2}{2M}$, is also held fixed in the limit by keeping the ratio $\frac{N}{M}$ fixed.

The metric of $AdS_5 \times S^5/\mathbf{Z}_M$ can be written as:

$$\begin{aligned} ds^2 \;&=\; R^2 \bigg[-\cosh^2\!\rho\, dt^2 + d\rho^2 + \sinh^2\!\rho\, d\Omega_3^2 + \\ &\qquad d\alpha^2 + \sin^2\alpha\, d\theta^2 + \cos^2\alpha\, \left(d\gamma^2 + \cos^2\gamma\, d\chi^2 + \sin^2\gamma\, d\phi^2\right) \bigg] \ . \end{aligned} \tag{4.29}$$

The angles of S^5 are related to the complex coordinates of $\mathbf{C}^3/\mathbf{Z}_M$ by

$$z_1 = R\cos\alpha\cos\gamma\, e^{i\chi}, \quad z_2 = R\cos\alpha\sin\gamma\, e^{i\phi}\,, \quad z_3 = R\sin\alpha e^{i\theta} \tag{4.30}$$

In terms of the angles of S^5 the orbifold described by the action (4.26) is obtained by the identifications

$$\chi \sim \chi + \frac{2\pi}{M}, \qquad \phi \sim \phi - \frac{2\pi}{M} \ . \tag{4.31}$$

To take the Penrose limit it is useful to introduce the coordinates

$$r = \rho R, \quad w = \alpha R, \quad y = \gamma R \ . \tag{4.32}$$

and the light-cone coordinates

$$x^+ = \tfrac{1}{2}\left(t + \chi\right), \quad x^- = \frac{R^2}{2}\left(t - \chi\right) \ . \tag{4.33}$$

After taking the $R \to \infty$ limit and renaming some coordinates, the metric becomes [94]

$$ds^2 = -4dx^+dx^- - \sum_{i=1}^{8}(x^i)^2\, dx^{+2} + \sum_{i=1}^{8} dx^{i2} \ , \tag{4.34}$$

In the geometry (4.34) there is also a Ramond-Ramond flux

$$F_{+1234} = F_{+5678} = const \ . \tag{4.35}$$

So far, with the rescaling (4.32) and (4.33) the only limit that we have taken to obtain (4.34) is that of large R. The orbifold identification (4.31) implies that the light-cone coordinates have the periodicity

$$\begin{aligned} x^+ &\sim x^+ + \frac{\pi}{M} \\ x^- &\sim x^- + \frac{\pi R^2}{M} \ , \end{aligned} \tag{4.36}$$

In the double scaling limit, as R is taken large, M is also taken large so that $R^- = \frac{R^2}{2M}$ is held fixed. In the limit

$$\left(x^+, x^-\right) \sim \left(x^+, x^- + 2\pi R^-\right) \tag{4.37}$$

The periodic direction becomes null. As a consequence the corresponding light-cone momentum $2p^+$ is quantized in units of $\frac{1}{R^-}$.

The conclusion is that the Penrose limit of $AdS_5 \times S^5/\mathbf{Z}_M$ with $M \to \infty$ in this particular way leads to a Discrete Light-Cone Quantization (DLCQ) of the string on a pp-wave background, in which the null coordinate x^- is periodic. Note that the orbifold of the 5-sphere preserves half of the supersymmetries of the original $AdS_5 \times S^5$ solution of string theory. Nonetheless, in the Penrose limit, we recover the maximally supersymmetric plane-wave background.

Discrete light-cone quantization of the string on the pp-wave background is a slight generalization of ref.[64]. One component of the light-cone momentum is quantized as

$$2p^+ = \frac{k}{R^-} \quad , \quad k = 1, 2, 3, ... \tag{4.38}$$

The other component is the light-cone-gauge Hamiltonian,

$$\begin{aligned} 2p^- &= \sum_{n=-\infty}^{\infty}\left(\sum_{i=1}^{8} a_n^{i\dagger}a_n^i + \sum_{\alpha=1}^{8} b_n^{\alpha\dagger}b_n^\alpha\right)\sqrt{1 + \frac{4n^2(R^-)^2}{k^2\alpha'^2}} \\ &= \sum_{n=-\infty}^{\infty}\left(\sum_{i=1}^{8} a_n^{i\dagger}a_n^i + \sum_{\alpha=1}^{8} b_n^{\alpha\dagger}b_n^\alpha\right)\sqrt{1 + \frac{4\pi g_s N}{M}\frac{n^2}{k^2}} \end{aligned} \tag{4.39}$$

where $a^i_n, a^{i\dagger}_n$ and $b^\alpha_n, b^{\alpha\dagger}_n$ are the annihilation and creation operators for the discrete bosonic and fermionic transverse oscillations of the string, respectively. They obey the (anti-) commutation relation

$$\left[a^i_{n_1}, a^{j\dagger}_{n_j}\right] = \delta^{ij}\delta_{n_i n_j} \quad , \quad \left\{b^\alpha_{n_1}, b^{\beta\dagger}_{n_j}\right\} = \delta^{\alpha\beta}\delta_{n_i n_j} \tag{4.40}$$

In the last line of eqn.(4.39) we have written the compactification radius in terms of string background parameters.

There are also wrapped states. If the total number of times that the closed string wraps the compact null direction is ℓ, the level-matching condition is

$$k\ell = \sum_{n=-\infty}^{\infty} n \left(\sum_{i=1}^{8} a^{i\dagger}_n a^i_n + \sum_{\alpha=1}^{8} b^{\alpha\dagger}_n b^\alpha_n \right) \quad , \tag{4.41}$$

States of the string are characterized by their discrete light-cone momentum k and their wrapping number ℓ. The lowest energy state in a given sector is the string sigma model vacuum, $|k, \ell\rangle$ which obeys

$$a^i_n \, |k, \ell\rangle = 0 = b^\alpha_n \, |k, \ell\rangle \quad , \quad \forall n, i, \alpha$$

Other string states are built from the vacuum by acting with transverse oscillators,

$$\prod_{j=1}^{L} a^{i_j\dagger}_{n_j} \prod_{j'=1}^{L'} b^{\alpha_{j'}\dagger}_{n_{j'}} \mid k, \ell\rangle \tag{4.42}$$

The level matching condition reads

$$\sum_{j=1}^{L} n_j + \sum_{j'=1}^{L'} n_{j'} = k\,\ell. \tag{4.43}$$

Matching charges

There are three important quantum numbers that can be matched between the string theory and its gauge theory dual. One is the energy in string theory, which is the quantum operator generating a flow along the Killing vector field $i\partial_t$ of the background. It corresponds to the conformal dimension, Δ, of operators in the gauge theory.

The others are U(1) charges. Two are particularly important to us. One is J' which generates a U(1) which is in the SU(2) subgroup of the R-symmetry

$$A \to e^{i\xi} A \quad , \quad B \to e^{i\xi} B \quad , \quad 0 \le \xi < 2\pi$$

J' which has integer eigenvalues. In the orbifold geometry, it corresponds to the Killing vector $J' = -\frac{i}{2}\left(\partial_\chi + \partial_\phi\right)$. There is an additional $U(1)$ which is not part of the R-symmetry

$$A \to e^{i\zeta} A \quad , \quad B \to e^{-i\zeta} B \quad , \quad 0 \le \zeta < 2\pi/M$$

	Δ	MJ	J'	$2p^-$
A_I	1	$\frac{1}{2}$	$\frac{1}{2}$	0
B_I	1	$-\frac{1}{2}$	$\frac{1}{2}$	1
Φ_I	1	0	0	1

Table 1: Dimensions and charges for bosonic fields

	Δ	MJ	J'	$2p^-$
$\overline{A}_I$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	2
$\overline{B}_I$	1	$\frac{1}{2}$	$-\frac{1}{2}$	1
$\overline{\Phi}_I$	1	0	0	1

Table 2: Dimensions and charges for complex conjugate fields

The domain of the angle ζ is reduced from 2π to $2\pi/M$ by the orbifold identification. This $U(1)$ is generated by J whose eigenvalues are integer multiples of M. In order to normalize it more conveniently, we rename it MJ where J has integer eigenvalues. On the orbifold geometry, it corresponds to the Killing vector $J = -\frac{i}{2M}\left(\partial_\chi - \partial_\phi\right)$.

In summary, charges and Killing vectors are related by

$$\Delta = i\partial_t \quad , \quad J = -\frac{i}{2M}\left(\partial_\chi - \partial_\phi\right) \quad , \quad J' = -\frac{i}{2}\left(\partial_\chi + \partial_\phi\right)$$

We can then recall the combinations of χ, ϕ and t which were used to form the light-cone coordinates x^+ and x^- of the pp-wave geometry to deduce the light-cone momenta

$$\begin{aligned} 2p^- &= i(\partial_t + \partial_\chi) &= \Delta - MJ - J' \\ 2p^+ &= i\frac{(\partial_t - \partial_\chi)}{R^2} &= \frac{\Delta + MJ + J'}{2MR^-} \ . \end{aligned} \tag{4.44}$$

These are the light-cone momenta of string states. We will focus on those states of the gauge theory where these quantum numbers remain finite in the double scaling limit. It will be easy to see that $2p^+$ will turn out to be quantized appropriately in units of integers$/2R^-$ and the values of $2p^-$ which we find in the gauge theory will be compared to the spectrum of the string light-cone Hamiltonian.

The BPS condition $\Delta \geq |MJ + J'|$ implies that keeping $2p^+$ and $2p^-$ finite as $R, M \to \infty$ will clearly only be possible when both Δ and $MJ + J'$ diverge with their difference, $\Delta - (MJ + J')$, remaining finite.

The charges of gauge theory operators are obtained as follows. By convention, the U(1) transformation is generated by $e^{4\pi i J}$ so the A_I and B_I fields that make up the hypermultiplets have fractional charge under J, $\frac{1}{2M}$ and $-\frac{1}{2M}$ respectively. The operator J' generates a $U(1)$ symmetry contained in the $SU(2)_R$ factor of the R-symmetry. Under this $U(1) \subset SU(2)_R$, the fields Φ_I are neutral. On the other hand, the scalars A_I, B_I in the hypermultiplets both have charge $\frac{1}{2}$ under J'. Complex conjugation and supersymmetry give the remaining charge assignments, for the fermions and all the conjugate fields. The dimension and charge assignments, along with the $2p^-$ values, are summarized in Tables 1 and 2 just in the bosonic case, the one of interest to us. In Table 1, A_I, B_I refer to the scalar components of the $\mathcal{N}=1$ chiral superfields that form the $\mathcal{N}=2$ hypermultiplets, Φ_I are the complex scalars in the vector multiplet. Table 2 lists the complex conjugate fields.

The holographic dictionary

In order to identify states in the $\mathcal{N}=2$ gauge theory with finite values of light-cone momenta, as given in (4.44), we first find the appropriate quantum numbers of the field operators. These are tabulated in Table 1 and Table 2. We see that only the fields A_I carry vanishing $2p^-$. By matching quantum numbers, one starts by identifying the operator corresponding to the vacuum state of the string sigma model. We see that the string state $|k,0>$ - which was $\mathrm{Tr}Z^J$ in the parent $\mathcal{N}=4$ theory - corresponds to the gauge invariant composite operator

$$|k,0\rangle \quad \leftrightarrow \quad \mathrm{TR}\left((A_1(x)A_2(x)\dots A_M(x))^k\right)$$

We have indicated the x-dependence of the composite operator. In the following, where from the context it is obvious, we will omit it. Because $A_I(x)$ transforms in the bi-fundamental representation of the gauge group, we are required to form the chains $A_1\dots A_M$ to obtain a gauge invariant operator. This chain can be repeated k times. The conformal dimension of this composite operator is protected by supersymmetry. This protection is inherited from the parent $\mathcal{N}=4$ theory. Thus, its exact conformal dimension is $\Delta=km$ and its exact spectrum is therefore $p^-=0$.

The next step are the first excited states. There are eight states which are created by one bosonic oscillator and eight which are created by a fermionic oscillator. These all add one unit to the Hamiltonian $2p^-$. In Yang-Mills theory, they are gotten by inserting an impurity, for example Φ_I, into the $A_1...A_M$ chains. It should be noticed that there are more possible states with these insertions than occurred in the parent $\mathcal{N}=4$ theory. There, the cyclic property of the trace implies that there is only one possible one-impurity state, $\mathrm{Tr}\Phi \mathrm{Z}^\mathrm{J}$. For the analogous operator in $\mathcal{N}=2$, there are M inequivalent one-impurity states

$$\mathrm{Tr}\left[A_1...A_{I-1}\Phi_I A_I...A_M(A_1...A_M)^{k-1}\right] \quad , \quad I=1,...,M \tag{4.45}$$

that will have $2p^-=1$+corrections. A naive Fourier transform of the 1-impurity state, assuming that the are kM positions that the impurity could take up is

$$\sum_{I=1}^{kM} e^{i\frac{2\pi}{kM}nI}\mathrm{Tr}\left[A_1...A_{I-1}\Phi_I A_I...A_M(A_1...A_M)^{k-1}\right] \quad , \quad n=0,1,...,kM-1$$

The degree of freedom in the dual string theory corresponding to the wave-number n in this Fourier transform is the world-sheet momentum. The level matching condition comes from realizing that the actual periodicity of the operator above is $I\to I+M$, rather than $I\to I+kM$. This requires that $n=k\,\ell$, where ℓ is an integer, which is the level-matching condition. The integer ℓ is dual to the wrapping number of the string around the periodic null direction, and can be identified with the twist appearing in the novel twisted operators of the type (4.25), once they are translated in terms of Z and Φ fields.

If the orbifold symmetry group is not spontaneously broken, ℓ is a good quantum number of the states of the theory and operators with different values of ℓ do not mix with each other. In addition it is known that [122], in the planar limit, the correlation functions of un-twisted operators of the $\mathcal{N}=2$ theory (those with vanishing wrapping number, ℓ=0) are identical to those of their parent operators in $\mathcal{N}=4$ super-Yang-Mills theory once one makes the replacement $\lambda \to \lambda/M$. This means that, for the untwisted operators, the dimension should be identical to that in $\mathcal{N}=4$ super-Yang-Mills theory. This will give a consistency check for some of our computations in the following.

In this Chapter we will be interested in two-impurity operators of the form

$$\mathcal{O}_{IJ} = \mathrm{Tr}\left(A_1...A_{I-1}\Phi_I A_I...A_M(A_1....A_M)^p A_1...A_{J-1}\Phi_J A_J...A_M(A_1....A_M)^{k-p-2}\right) \quad (4.46)$$

where we take I and J as running from 1 to kM. Distinct operators are enumerated by taking $I \leq J$. The number of scalar fields in this operator is $kM+2$. The cyclic property of the trace implies the conditions

$$\mathcal{O}_{I,kM+1} = O_{1I} \quad (4.47)$$

and

$$\mathcal{O}_{I+M,J+M} = \mathcal{O}_{I,J} \quad (4.48)$$

which will be important to us.

The dilatation operator

Just as in $\mathcal{N}=4$ supersymmetric Yang-Mills theory [118, 68], the computation of dimensions of the operators of interest to us can be elegantly summarized by the action of an effective Hamiltonian. This technique was invented in Ref. [118]. The $\mathcal{N}=4$ dilatation operator is known explicitly in terms of its action on fields up to two loop order, and implicitly to three loop order [68, 123, 116]. That part which is known explicitly can be projected, using the orbifold projection, to obtain a dilatation operator for the $\mathcal{N}=2$ theory. Here, we shall be interested in computing dimensions of operators in the scalar $\mathfrak{su}(2)$ sector, so we only retain the parts of the operator which will contribute there. They can be obtained by simply substituting the matrices in Eqs. (4.21) and (4.22) into the analogous terms of the $\mathcal{N}=4$ operator. The result is

$$D = D_{tree} + D_{1\ loop} + D_{2\ loops} \quad (4.49)$$

where

$$D_{tree} = \sum_{L=1}^{M} \mathrm{Tr}\left(A_L\bar{A}_L + \Phi_L\bar{\Phi}_L\right) \quad (4.50)$$

$$D_{1\ loop} = -\frac{g_{YM}^2 M}{8\pi^2} \sum_{L=1}^{M} \mathrm{Tr}(A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L - \Phi_L A_L \bar{A}_L \bar{\Phi}_L + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L) \tag{4.51}$$

$$\begin{aligned}
D_{2\ loops} \;=\; & \frac{g_{YM}^4 N M^2}{64\pi^4} \sum_{L=1}^{M} \mathrm{Tr}(A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L - \Phi_L A_L \bar{A}_L \bar{\Phi}_L + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L) \\
& + \frac{g_{YM}^4 M^2}{128\pi^4} \sum_{L=1}^{M} \mathrm{Tr}(\Phi_L A_L \bar{A}_L A_L \bar{\Phi}_{L+1} \bar{A}_L - A_L \Phi_{L+1} \bar{A}_L A_L \bar{\Phi}_{L+1} \bar{A}_L \\
& \qquad + A_L \Phi_{L+1} A_{L+1} \bar{\Phi}_{L+2} \bar{A}_{L+1} \bar{A}_L - \Phi_L A_L A_{L+1} \bar{\Phi}_{L+2} \bar{A}_{L+1} \bar{A}_L \\
& \qquad + A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \bar{A}_{L-1} A_{L-1} - \Phi_L A_L \bar{A}_L \bar{\Phi}_L \bar{A}_{L-1} A_{L-1} \\
& \qquad + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L A_L \bar{A}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L A_L \bar{A}_L \\
& \qquad + A_L \Phi_{L+1} \bar{A}_L A_L \bar{A}_L \bar{\Phi}_L - \Phi_L A_L \bar{A}_L A_L \bar{A}_L \bar{\Phi}_L \\
& \qquad + \Phi_L A_L A_{L+1} \bar{A}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L - A_L \Phi_{L+1} A_{L+1} \bar{A}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \\
& \qquad + \Phi_L A_L \bar{A}_L \bar{A}_{L-1} \bar{\Phi}_{L-1} A_{L-1} - A_L \Phi_{L+1} \bar{A}_L \bar{A}_{L-1} \bar{\Phi}_{L-1} A_{L-1} \\
& \qquad + A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L A_L \bar{A}_L - \Phi_L A_L \bar{A}_L \bar{\Phi}_L A_L \bar{A}_L) \\
& + \frac{g_{YM}^4 M^2}{128\pi^4} \sum_{L=1}^{M} \mathrm{Tr}(\Phi_L A_L \bar{\Phi}_{L+1} \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \\
& \qquad + A_L \Phi_{L+1} \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L - \Phi_L A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L \\
& \qquad + A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \Phi_L - \Phi_L A_L \bar{\Phi}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \Phi_L \\
& \qquad + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L \Phi_L \bar{\Phi}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \Phi_L \bar{\Phi}_L \\
& \qquad + A_L \Phi_{L+1} \bar{\Phi}_{L+1} \Phi_{L+1} \bar{A}_L \bar{\Phi}_L - \Phi_L A_L \bar{\Phi}_{L+1} \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \\
& \qquad + \Phi_L A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \bar{\Phi}_L - A_L \Phi_{L+1} \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \bar{\Phi}_L \\
& \qquad + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L \Phi_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L \Phi_L \\
& \qquad + A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \Phi_L \bar{\Phi}_L - \Phi_L A_L \bar{A}_L \bar{\Phi}_L \Phi_L \bar{\Phi}_L)
\end{aligned} \tag{4.52}$$

The number of loops which contribute to each order is exhibited in the power of the Yang-Mills coupling constant g_{YM}^2 which precedes each term. Later we will use either the parent $\mathcal{N} = 4$ 't Hooft coupling,

$$\lambda \equiv g_{YM}^2 N M$$

which is important for the planar limit, or the modified 't Hooft coupling

$$\lambda' \equiv \frac{g_{YM}^2 N}{M} = \frac{\lambda}{M^2}$$

which is held constant in the MRV limit. In the latter limit, N and M are both put to infinity so that λ' and the effective string coupling,

$$g_2 \equiv \frac{M}{N}$$

are held constant. The effective string coupling controls the appearance of non-planar diagrams and, to get the planar limit, which we will be interested in, it must also be put to zero. Inspection of the 1-loop and 2-loop dilatation operators shows that, in order for this MRV limit to make sense, their action should be suppressed by some powers of $\frac{1}{M}$ further to those exhibited in Eqs. (4.51) and (4.52). We shall see that this is indeed the case.

The action of the operators in Eqs. (4.50), (4.51) and (4.52) on a composite of the form (4.46) is implemented with the following procedure.

We note that each term in the dilatation operators contains a few $\bar{A}_I$'s and $\bar{\Phi}_I$'s. We take a term in D, and we Wick-contract the $\bar{A}_I$'s and $\bar{\Phi}_I$'s which appear in that term with each occurrence of A_I and Φ_I in the trace (4.46) according to the rules

$$\left\langle \left[\bar{A}_I\right]_{ab} \left[A_J\right]_{cd} \right\rangle_0 = \delta_{IJ}\delta_{ad}\delta_{bc} \quad , \quad \left\langle \left[\bar{\Phi}_I\right]_{ab} \left[\Phi_J\right]_{cd} \right\rangle_0 = \delta_{IJ}\delta_{ad}\delta_{bc}$$

Here we are treating the fields as if they are simply matrices in a Gaussian matrix model, ignoring their space-time dependence and simply substituting them with other fields according to the rules of performing the contractions. The space-time dependence, that of course must be taken into account in order to compute dimensions in renormalized perturbation theory, has already been taken care of in formulating the effective Hamiltonian.

In doing these contractions with the first term in (4.49), the tree-level operator, we find the tree level contribution to the conformal dimension. The procedure merely counts the number of scalar fields, giving $kM + 2$ in the case of (4.46).

When we Wick-contract with the 1-loop and 2-loop terms, (4.51) and (4.52), once all possible contractions are done, we find a superposition of operators where the total number of fields in each operator is the same and the number of impurities in each operator is still two, but the positions of the impurities have been shifted.

All of the operators in the superposition have the same tree-level dimensions. It means that, at the outset, we could have began with linear combinations of them. We could then have chosen the coefficients in the linear combinations in such a way as to diagonalize the action of the dilatation operator. Upon doing this, we would find the eigenvalues, i.e. the dimensions, and the linear combinations that we find would be the scaling operators themselves.

Once the Wick contractions are explicitly performed, the action of the one loop dilatation operator on the operators (4.46) is given by two equations, depending on whether the impurities lie next to each other or not

$$D_{1\ loop} \circ O_{IJ} = \frac{\lambda' M^2}{8\pi^2}\big(- O_{I+1,J} - O_{I-1,J} + 4O_{IJ} - O_{I,J+1} - O_{I,J-1}\big), \quad I < J \tag{4.53}$$

$$D_{1\ loop} \circ O_{II} = \frac{\lambda' M^2}{8\pi^2}\big(- O_{I-1,I} - O_{I,I+1} + 2O_{II}\big) \tag{4.54}$$

At two loops, the action of the dilation operator results in three equations,

$$\begin{aligned} D_{2\ loops} \circ O_{IJ} \quad &= \quad \frac{\lambda'^2 M^4}{128\pi^4}\big(- O_{I-2,J} - O_{I+2,J} + 4O_{I-1,J} + 4O_{I+1,J} \\ &- \quad O_{I,J-2} - O_{I,J+2} + 4O_{I,J-1} + 4O_{I,J+1} - 12O_{IJ}\big) \end{aligned} \tag{4.55}$$

for $J - I \geq 2$ and

$$\begin{aligned} D_{2\ loops} \circ O_{II} &= \frac{\lambda'^2 M^4}{128\pi^4}\big(- O_{I-2,I} + 4O_{I-1,I} - O_{I-1,I-1} \\ &- 4O_{I,I} + 4O_{I,I+1} - O_{I+1,I+1} - O_{I,I+2}\big) \end{aligned} \tag{4.56}$$

$$\begin{aligned} D_{2\ loops} \circ O_{I,I+1} &= \frac{\lambda'^2 M^4}{128\pi^4}\big(- O_{I,I+3} + 4O_{I+1,I+1} + 4O_{I,I+2} - 14O_{I,I+1} \\ &+ 4O_{I,I} + 4O_{I-1,I+1} - O_{I-2,I+1}\big) \end{aligned} \tag{4.57}$$

where the second and the third formulae represent, respectively, the nearest ($I = J$) and the next-to-nearest ($J = I + 1$) neighbor cases. We see explicitly that the dilatation operator is acting like a lattice differential operator on the matrix chains. The result is an effective spin-chain Hamiltonian. The problem of finding the eigenvalues of this Hamiltonian is integrable and can be attacked using the twisted Bethe ansatz, which we summarize in the next subsection.

4.3.1 Twisted Bethe ansatz for the orbifold

The conjecture [93] is that the spectrum of operator dimensions in the $\mathfrak{su}(2)$ sector of the $\mathcal{N} = 2$ quiver theory which is a Z_M orbifold of $\mathcal{N} = 4$ is found by including a simple twist in the Bethe equation (4.8). The other equations, (4.13) and (4.14) are applied unchanged.

For example, for two magnons, the twisted Bethe equations are

$$e^{ip_1(kM+2)} = \omega^\ell \ \frac{\varphi_1 - \varphi_2 + i}{\varphi_1 - \varphi_2 - i} \quad , \quad e^{ip_2(kM+2)} = \omega^\ell \ \frac{\varphi_2 - \varphi_1 + i}{\varphi_2 - \varphi_1 - i} \tag{4.58}$$

Here, as in (4.8), $L = kM+2$ is the length of the chain. The twist is the M'th root of unity factor ω^ℓ in front the right-hand-sides of (4.58). $\omega = e^{\frac{2\pi}{M}i}$ and the integer ℓ is the charge of the state under the U(1) symmetry which is used in the orbifold projection. In the dual string theory, it coincides with the wrapping number of the string world-sheet on the compact null direction. The periodicity of the chain implies now that the "level-matching condition" (4.9) is replaced by

$$\sum_{i=1}^{\mathcal{M}} p_i = \frac{2\pi}{M} \cdot \ell \quad , \quad \ell = \text{ integer} \tag{4.59}$$

Because of (4.59), the twist ω is related to the total world-sheet momentum through $e^{i(p_1+p_2)} = \omega^\ell$. As in the $\mathcal{N} = 4$ theory, the momenta and rapidities are still related by

$$\varphi_1 = \frac{1}{2}\cot\frac{p_1}{2}\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_1}{2}} \quad , \quad \varphi_2 = \frac{1}{2}\cot\frac{p_2}{2}\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_2}{2}} \ . \tag{4.60}$$

and the spectrum is

$$\Delta = kM + \sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_1}{2}} + \sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_2}{2}} \tag{4.61}$$

Multiplying the two equations in (4.58) gives the condition on the total momentum

$$e^{i(p_1+p_2)kM} = 1 \quad \rightarrow \quad p_1+p_2 = \frac{2\pi}{kM}s \quad , \quad s = \text{integer} \tag{4.62}$$

and (4.59) implies

$$s \;=\; k\cdot\ \text{integer} \tag{4.63}$$

It is clear from the form of the equations (4.58) and (4.60) that the momenta, which are their solutions, generally depend on λ and the parameter kM. It is also clear that the momenta which solve them must be small when M is large, $p_i \propto \frac{1}{kM}$. This is also needed for consistency of the MRV limit where $M\to\infty$ and $\lambda\to\infty$ in such a way that $\lambda' = \frac{\lambda}{M^2}$ remains finite. Equation (4.60) also implies that φ_1 and φ_2 are both of order M in that limit. Later in this Chapter, we shall consider the leading corrections to this limit in an expansion in $1/M$. In the remainder of this subsection, for a warmup exercise, we will seek the solutions for p_i in the MRV limit, where $M\to\infty$. In this limit, we hold $\lambda' = \frac{\lambda}{M^2}$ finite.

Even in this limit, we shall not be able to solve equations (4.58) and (4.60) for arbitrary values of λ'. We will be limited to considering a Taylor expansion of Eq. (4.60) in λ' and then seeking momenta which are also expressed as expansions in λ'. We begin with the leading order where we simply set λ' to zero in Eq. (4.60). [7] Then, it is easy to see that the momenta must be given by

$$p_1 = \frac{2\pi}{kM}n_1 + \mathcal{O}\left(\frac{1}{M^2}\right) \quad , \quad p_2 = \frac{2\pi}{kM}n_2 + \mathcal{O}\left(\frac{1}{M^2}\right) \tag{4.64}$$

where n_1 and n_2 are integers. Level matching gives the further condition

$$n_1+n_2 = k\cdot\ell$$

where ℓ is an integer. Then Eq. (4.61) implies

$$\Delta = kM + \sqrt{1+\lambda'\frac{n_1^2}{k^2}} + \sqrt{1+\lambda'\frac{n_2^2}{k^2}} \tag{4.65}$$

which agrees beautifully with the spectrum of DLCQ free strings on the plane-wave background.

[7] We do this by setting λ to zero, but we must be careful to see, a posteriori, that indeed $p_i \sim \mathcal{O}\left(\frac{1}{M}\right)$, so that setting $\lambda=0$ is equivelent to setting $\lambda'=0$. We shall see this shortly, in Eq. (4.64).

4.3.2 Coordinate Bethe ansatz

There is another, equivalent procedure which is sometimes convenient, called the coordinate Bethe ansatz. Since we will make use of it later, we shall review it here for the special case of a two-impurity operator.

Consider the dilatation operator in the form of the difference operators (4.53)-(4.57) which we derived using the effective Hamiltonian. Finding the spectrum of the dilatation operator entails finding the eigenstates and eigenvalues of the combination of difference operators (4.53)-(4.57), operating on the space of two-impurity operators. Here, for illustration, we will review the argument that, to order λ', this is equivalent to the task of solving the twisted Bethe ansatz which was set out in the previous sub-section. Later on in this Chapter, we will show that this also holds to order λ'^2 (and then we will assume that it holds to order λ'^3).

To begin, we take the linear super-position of two-impurity operators

$$\mathcal{O} \equiv \sum_{1\leq I\leq J\leq kM} \Psi_{IJ} O_{IJ} \tag{4.66}$$

Our task is to find the coefficients Ψ_{IJ} in this series so that this operator is an eigenstate of the dilation operator. If we impose the same periodicity conditions on Ψ_{IJ} as the operators O_{IJ} obey in (4.47), the action of the dilatation operator as difference operators in (4.53)-(4.57) is self-adjoint[8] and we can recast the problem of diagonalizing dilatations as the problem of finding eigenvalues for the action of the difference operators acting on the wave-functions Ψ_{IJ}.

The coordinate Bethe ansatz was used in refs. [124] and [90] to find the spectrum of the one-loop operator in the large M limit. To introduce the technique, we shall review the essential parts of the argument here. At one-loop order, the eigenvalue equation is

$$E^{(1)}\Psi_{IJ} = g^2\left(-\Psi_{I+1,J} - \Psi_{I-1,J} + 4\Psi_{IJ} - \Psi_{I,J+1} - \Psi_{I,J-1}\right) \qquad I<J \tag{4.67}$$
$$E^{(1)}\Psi_{IJ} = g^2\left(-\Psi_{I-1,I} - \Psi_{I,I+1} + 2\Psi_{II}\right) \qquad I=J \tag{4.68}$$

where $g^2 = g_{YM}^2 NM/(8\pi^2)$. To look for a solution, we make the plane-wave ansatz

$$\Psi_{IJ} = \mu_1^I \mu_2^J + S_0(\mu_2,\mu_1)\mu_2^I \mu_1^J \tag{4.69}$$

where $\mu_1 = e^{ip_1}$ and $\mu_2 = e^{ip_2}$. Then, Eq. (4.67) yields the eigenvalue,

$$E^{(1)} = \frac{\lambda' M^2}{2\pi^2}\left(\sin^2\frac{p_1}{2} + \sin^2\frac{p_2}{2}\right) \tag{4.70}$$

which is the expansion to first order in λ' of the square roots in (4.61). The problem of finding the allowed values of (p_1, p_2) remains.

Then, (4.68) yields the equation

[8] We note that the detailed form of the contact terms in the difference operators are essential in demonstrating the self-adjoint property.

$$S_0(\mu_2,\mu_1) = -\frac{\mu_1}{\mu_2}\frac{\mu_1\mu_2 - 2\mu_2 + 1}{\mu_1\mu_2 - 2\mu_1 + 1} \tag{4.71}$$

where it should be noticed that $S_0(\mu_1,\mu_2)^{-1} = S_0(\mu_2,\mu_1)$.

The boundary condition $\Psi_{I,kM+1} = \Psi_{1,I}$ gives

$$\mu_2^{kM} = S_0(\mu_2,\mu_1) \quad , \quad \mu_1^{kM} = S_0(\mu_2,\mu_1)^{-1} \tag{4.72}$$

Eqs. (4.72) together with (4.71) are identical to the twisted Bethe equations (4.58), together with (4.60) with λ' set to zero. The level-matching condition is obtained by noticing that

$$\Psi_{I+M,J+M} = \Psi_{IJ} \tag{4.73}$$

implies

$$(\mu_1\mu_2)^M = 1 \tag{4.74}$$

4.3.3 Outline

In the remainder of this Chapter, we shall compute the finite size corrections to the spectrum of dimensions of the two-impurity operators in the $\mathfrak{su}(2)$ bosonic sector that we have been discussing so far. We will use the twisted Bethe ansatz, summarized in Eqs. (4.58)-(4.61) and will compute to three-loop order. We also will check explicitly that the coordinate Bethe ansatz technique which used the difference operator form of the dilatation operator exhibited in Eqs. (4.53)-(4.57) indeed produces the same result to two loop order.

Then, we will adopt the string theory computation which was originally used in Ref. [70] for the near pp-wave limit of $AdS_5 \times S^5$ to the present case of the near DLCQ pp-wave limit of $AdS_5 \times S^5/Z_M$. This is the string theory dual of the "near"-MRV limit of the $\mathcal{N} = 2$ theory. We compute the spectrum of the string in this case, expanded to order $1/M$. On the string side, the expression that is obtained is exact to all orders in λ'. When expanded to third order, we find beautiful agreement with the $\mathcal{N} = 2$ gauge theory prediction up to second order in λ', i.e. two loops, and disagreement at third, or three loop order.

This disagreement is similar to the one which is found in the $\mathcal{N} = 4$ theory in Ref. [73, 76]. In fact, in the de-compactified limit, $k \to \infty, R_- \to \infty$ with $p^+ = k/R_-$ fixed, it approaches that result.

In addition, we show that, like in the case of $\mathcal{N} = 4$ super-Yang-Mills theory, the discrepancy can be taken into account by a dressing factor [79].

4.4 Finite size corrections at one loop

In order to calculate the first finite size corrections to Eq.(4.64) we make the following general ansatz for the magnon momenta

$$
\begin{aligned}
p_1 &= \frac{2n_1\pi}{kM} + \frac{A\pi}{M^2} \\
p_2 &= \frac{2n_2\pi}{kM} - \frac{A\pi}{M^2}
\end{aligned}
\tag{4.75}
$$

Recall that we solve at one loop order by simply setting $\lambda' \to 0$ in the equation for the rapidity (4.60), so that it is given by

$$
\varphi_j = \frac{1}{2}\cot\frac{p_j}{2}. \tag{4.76}
$$

By requiring that the Bethe equations (4.58) are satisfied by (4.75) at both leading and next to leading order in $\frac{1}{M}$ one gets the following value for A

$$
A = \frac{2\,(n_1^2 + n_2^2)}{k^2(n_2 - n_1)} \tag{4.77}
$$

We can then insert this solution in the expression (4.70) for the anomalous dimension in terms of p_i and expand in a $\frac{1}{M}$ series. The first finite size correction to the planar anomalous dimension reads

$$
\Delta_{1\ loop} = \frac{\lambda'}{2}\left[\frac{n_1^2+n_2^2}{k^2} - \left(\frac{2}{kM}\right)\frac{(n_1^2+n_2^2)}{k^2} + O\left(\frac{1}{M^2}\right)\right] \tag{4.78}
$$

As a first consistency check, it is easy to verify that when the $\mathcal{N}=4$ level-matching condition $n_2 = -n_1$ is imposed – this gives the result for the unwrapped, $\ell = 0$ state – recalling that $J = kM$ and the appropriate re-definition of λ', the $\mathcal{N}=4$ result [73, 76] is recovered.

The zeroth order term in (4.78) equals the one-loop free string spectrum in the plane-wave limit and the first finite size correction, $\frac{1}{M}$ order, will be compared with the corresponding $1/R^2$ correction on the string side of the duality.

4.5 Two loops

To find the correction to the dimension at two loops, we must expand (4.60) to linear order in λ' and then use it in (4.58) to find the momenta, also to linear order in λ'. The resulting twisted Bethe equation reads

$$
e^{ip_2(kM+2)} = e^{i(p_1+p_2)}\frac{\frac{1}{2}\cot\frac{p_2}{2} + \frac{\lambda}{8\pi^2}\sin p_2 - \frac{1}{2}\cot\frac{p_1}{2} + \frac{\lambda}{8\pi^2}\sin p_1 + i}{\frac{1}{2}\cot\frac{p_2}{2} + \frac{\lambda}{8\pi^2}\sin p_2 - \frac{1}{2}\cot\frac{p_1}{2} + \frac{\lambda}{8\pi^2}\sin p_1 - i} \tag{4.79}
$$

The simultaneous expansion of the momenta in λ' and $\frac{1}{M}$ will have the form

$$p_1 = \frac{2n_1\pi}{kM} + \frac{A\pi}{M^2} + \lambda'\frac{B\pi}{M^2} + ... \quad , \quad p_2 = \frac{2n_2\pi}{kM} - \frac{A\pi}{M^2} - \lambda'\frac{B\pi}{M^2} + ... \tag{4.80}$$

where A, given in Eq. (4.77), was calculated in the previous section. We could also have included a contribute of order λ'/M to the momenta, but Eq.(4.79), expanded as a power series in λ' and $1/M$, would force it to be zero.

The corrections, indicated by three dots are at least of order $\frac{1}{M^3}$ or $\frac{\lambda'^2}{M^2}$. (In the next Section, we will compute the $\frac{\lambda'^2}{M^2}$ correction.)

B can be fixed by requiring that the Bethe equation (4.79) is satisfied at the first order in the λ' expansion

$$B = \frac{2\, n_1^2 n_2^2}{k^4(n_2 - n_1)} \tag{4.81}$$

To calculate the $O(\lambda'^2)$ contribution to the planar anomalous dimension, one plugs the solution of the Bethe equation into the eigenvalue formula (4.61). Performing a double series expansion, in λ' and $\frac{1}{M}$, we obtain the following expression for the two loops planar anomalous dimension, up to the first finite size correction

$$\Delta_{2\ loops} = \frac{\lambda'^2}{8}\left[-\frac{n_1^4+n_2^4}{k^4} + \left(\frac{4}{kM}\right)\frac{n_1^4+n_1^3 n_2 + n_1 n_2^3 + n_2^4}{k^4} + O\left(\frac{1}{M^2}\right)\right]. \tag{4.82}$$

As a consistency check, we take the case where $\ell = (n_1+n_2)/k = 0$ We see that (4.82) agrees with the $\mathcal{N}=4$ solution [73, 76] in that case.

4.6 Two loops revisited: the perturbative asymptotic Bethe ansatz

In order to diagonalize the two-loop corrected dilatation operator (4.49) the ansatz for the wave-function (4.69) has to be adjusted in a perturbative sense in order to take into account long range interactions. When interactions are included at the next order, the wave-functions are no longer plane waves. The technique which is used, termed as perturbative asymptotic Bethe ansatz (PABA) [125, 79], begins with

$$\Psi_{IJ} = \mu_1^I \mu_2^J\ f(J-I+1,\mu_1,\mu_2) + \mu_2^I\mu_1^J\ f(kM-J+I+1,\mu_1,\mu_2)\ S(\mu_2,\mu_1) \tag{4.83}$$

where the S-matrix and the function f have the perturbative expansions

$$\begin{aligned} S(\mu_2,\mu_1) &= S_0(\mu_2,\mu_1) + \sum_{n=1}^{\infty}(g^2)^n\ S_n(\mu_2,\mu_1) \\ f(J-I+1,\mu_1,\mu_2) &= 1 + \sum_{n=0}^{\infty}(g^2)^{n+|J-I+1|} f_n(\mu_1,\mu_2) \end{aligned} \tag{4.84}$$

where $g^2 = g_{\rm YM}^2 MN/(8\pi^2) = \lambda' M^2/(8\pi^2)$. The number of powers of the coupling in the second of Eqs.(4.84) clearly indicates the interaction range on the lattice.

Note that, once it is determined at the leading order, the wave-function at the next order should be uniquely determined by quantum mechanical perturbation theory. Here, we are postulating that the result of determining it can be put in the form of Eq. (4.83). We will justify this postulate by showing that (4.82) does satisfy the equation to the required order and that the process of finding the solution is encoded in the twisted Bethe ansatz.

To derive the two loop Bethe equations it is sufficient to keep only the following terms in the ansatz (4.83)

$$\begin{aligned}\Psi_{IJ} &= \mu_1^I \mu_2^J \left[1 + g^{2|J-I+1|} f_0(\mu_1,\mu_2)\right] \\ &+ \mu_2^I \mu_1^J \left[S_0(\mu_2,\mu_1) + g^2 S_1(\mu_2,\mu_1)\right]\left[1 + g^{2|kM+1-J+I|} f_0(\mu_1,\mu_2)\right] \end{aligned} \tag{4.85}$$

The boundary conditions $\Psi_{I,kM+1} = \Psi_{1,I}$ on (4.85) imply the Bethe equations

$$\begin{aligned}\mu_2^{kM} &= S_0(\mu_2,\mu_1) + g^2 S_1(\mu_2,\mu_1) \\ \mu_1^{kM} &= [S_0(\mu_2,\mu_1) + g^2 S_1(\mu_2,\mu_1)]^{-1}\end{aligned} \tag{4.86}$$

The Schrödinger equation is obtained, as in Section 4.3.2, by acting on the wave-function Ψ_{IJ} with the dilatation operator as difference operators according to (4.53)-(4.57). In doing so, the two-loop contributions coming from the action of the 1-loop dilatation operator on the order λ' part of the wave-function have to be kept into account. Note that, since $\mu_i = e^{ip_i}$ and in general the p_i's depend on λ', the wave function has an implicit dependence on λ' through its dependence on μ_i.

The difference equation for $J - I \geq 2$ reads

$$\begin{aligned}&(D_{1\ loop} + D_{2\ loop}) \circ \Psi_{IJ} = \\ &g^2\left(-\Psi_{I+1,J} - \Psi_{I-1,J} + 4\Psi_{IJ} - \Psi_{I,J+1} - \Psi_{I,J-1}\right) \\ &\frac{g^4}{2}\left(-\Psi_{I-2,J} - \Psi_{I+2,J} + 4\Psi_{I-1,J} + 4\Psi_{I+1,J}\right. \\ &\left.-\Psi_{I,J-2} - \Psi_{I,J+2} + 4\Psi_{I,J-1} + 4\Psi_{I,J+1} - 12\Psi_{IJ}\right) \qquad J - I \geq 2\end{aligned} \tag{4.87}$$

Using the ansatz (4.85) and keeping only terms up to order g^4 we see that, when $J-I \geq 2$ the dilatation operator acting on the wave-function returns its form times an eigenvalue,

$$(D_{1\ loop} + D_{2\ loop}) \circ \Psi_{IJ} = \left[4g^2\left(\sin^2\frac{p_1}{2} + \sin^2\frac{p_2}{2}\right) - \frac{g^4}{8}\left(\sin^4\frac{p_1}{2} + \sin^4\frac{p_2}{2}\right)\right]\Psi_{IJ} \tag{4.88}$$

In order for (4.85) to be a eigenstate of the dilatation operator up to two loops, this must also be so for the contact terms in the dilatation operator. For this, the following equations must hold:

$$(D_{1\ loop} + D_{2\ loop}) \circ \Psi_{II} =$$

$$
\begin{aligned}
&g^2\left(-\Psi_{I-1,I}-\Psi_{I,I+1}+2\Psi_{I,I}\right)\\
&+\frac{g^4}{2}\left(-\Psi_{I-2,I}+4\Psi_{I-1,I}-\Psi_{I-1,I-1}-4\Psi_{I,I}+4\Psi_{I,I+1}-\Psi_{I+1,I+1}-\Psi_{I,I+2}\right)\\
&\equiv\left[4g^2\left(\sin^2\frac{p_1}{2}+\sin^2\frac{p_2}{2}\right)-\frac{g^4}{8}\left(\sin^4\frac{p_1}{2}+\sin^4\frac{p_2}{2}\right)\right]\Psi_{II}
\end{aligned}
\tag{4.89}
$$

$$
\begin{aligned}
&(D_{1\ loop}+D_{2\ loop})\circ\Psi_{I,I+1}=\\
&g^2\left(-\Psi_{I+1,I+1}-\Psi_{I-1,I+1}+4\Psi_{I,I+1}-\Psi_{I,I+2}-\Psi_{I,I}\right)\\
&+\frac{g^4}{2}\left(-\Psi_{I,I+3}+4\Psi_{I+1,I+1}+4\Psi_{I,I+2}-14\Psi_{I,I+1}+4\Psi_{I,I}+4\Psi_{I-1,I+1}-\Psi_{I-2,I+1}\right)\\
&\equiv\left[4g^2\left(\sin^2\frac{p_1}{2}+\sin^2\frac{p_2}{2}\right)-\frac{g^4}{8}\left(\sin^4\frac{p_1}{2}+\sin^4\frac{p_2}{2}\right)\right]\Psi_{I,I+1}
\end{aligned}
\tag{4.90}
$$

We regard these equations as determining p_i.

Using (4.85) and (4.71) in (4.90) the function $f_0(\mu_1,\mu_2)$ is uniquely derived as

$$
f_0(\mu_1,\mu_2)=-\frac{(\mu_1-1)(\mu_2-1)(\mu_1-\mu_2)}{\mu_2(1+\mu_1(\mu_2-2))}
\tag{4.91}
$$

Plugging (4.91) in (4.89) one can fix also the function $S_1(\mu_1,\mu_2)$ as

$$
S_1(\mu_2,\mu_1)=-\frac{(\mu_1-1)^2(\mu_2-1)^2(\mu_1-\mu_2)(1+\mu_1\mu_2)}{\mu_2^2(1+\mu_1(\mu_2-2))^2}
\tag{4.92}
$$

Using (4.71) and (4.92) the Bethe equation (4.86) becomes

$$
e^{ip_2(kM+2)}=e^{i(p_1+p_2)}\left[\frac{\frac{1}{2}\cot\frac{p_2}{2}-\frac{1}{2}\cot\frac{p_1}{2}+i}{\frac{1}{2}\cot\frac{p_2}{2}-\frac{1}{2}\cot\frac{p_1}{2}-i}-\frac{\lambda}{4\pi^2}\frac{\sin p_1-\sin p_2}{\left(\frac{1}{2}\cot\frac{p_2}{2}-\frac{1}{2}\cot\frac{p_1}{2}-i\right)^2}\right]
\tag{4.93}
$$

This is equivalent to Eq. (4.79) expanded to the first order in λ. We have thus demonstrated that the PABA in Eq. (4.83) solves the eigenvalue equations for the dilatation operator in the form (4.53)-(4.57) and that the process of finding these solutions is equivalent to solving the twisted Bethe equations for the $\mathcal{N}=2$ theory up to two loops.

4.7 Three loops

The three loop operator dimensions cannot be gotten by direct computation in Yang-Mills perturbation theory, or equivalently, by the perturbative asymptotic Bethe ansatz approach that we used for two loops in the previous Section. The reason is that, so far, no explicit expression for the dilatation operator in terms of fields and their derivatives is available at three loop order. Our approach to computing at three loops will therefore be to assume that the twisted Bethe ansatz, summarized in Eqs. (4.58)-(4.61), correctly describes the spectrum and to derive the three-loop correction to operator dimensions from it.

For this purpose we have to keep $O(\lambda^2)$ terms in Eq.(4.58) so that the twisted Bethe equation now reads

$$e^{ip_2(kM+2)} = e^{i(p_1+p_2)}\frac{\frac{1}{2}\cot\frac{p_2}{2}+\frac{\lambda}{8\pi^2}\sin p_2+\frac{\lambda^2}{64\pi^4}\sin p_2(\cos p_2-1)-\frac{1}{2}\cot\frac{p_1}{2}-\frac{\lambda}{8\pi^2}\sin p_1-\frac{\lambda^2}{64\pi^4}\sin p_1(\cos p_1-1)+i}{\frac{1}{2}\cot\frac{p_2}{2}+\frac{\lambda}{8\pi^2}\sin p_2+\frac{\lambda^2}{64\pi^4}\sin p_2(\cos p_2-1)-\frac{1}{2}\cot\frac{p_1}{2}-\frac{\lambda}{8\pi^2}\sin p_1-\frac{\lambda^2}{64\pi^4}\sin p_1(\cos p_1-1)-i} \tag{4.94}$$

We look for a solution of this equation by means of momenta of the following form

$$\begin{aligned} p_1 &= \frac{2n_1\pi}{kM}+\frac{A\pi}{M^2}+\lambda'\frac{B\pi}{M^2}+\lambda'^2\frac{C\pi}{M^2} \\ p_2 &= \frac{2n_2\pi}{kM}-\frac{A\pi}{M^2}-\lambda'\frac{B\pi}{M^2}-\lambda'^2\frac{C\pi}{M^2}, \end{aligned} \tag{4.95}$$

where A and B have been computed at lower loops, Eqs. (4.77) and (4.81). Recall that $\lambda' = \frac{\lambda}{M^2}$. Requiring that the Bethe equations are satisfied at order λ'^2 we fix C as

$$C = \frac{n_1^2 n_2^2\,(n_1^2-n_1n_2+n_2^2)}{2k^6(n_2-n_1)} \tag{4.96}$$

The eigenvalue formula eq.(4.61) expanded up to three loops gives

$$\begin{aligned} \Delta = \;& kM+2+\frac{\lambda' M^2}{2\pi^2}\left(\sin^2\frac{p_1}{2}+\sin^2\frac{p_2}{2}\right)-\frac{\lambda'^2M^4}{8\pi^4}\left(\sin^4\frac{p_1}{2}+\sin^4\frac{p_1}{2}\right) \\ &+\frac{\lambda'^3M^6}{16\pi^6}\left(\sin^6\frac{p_1}{2}+\sin^6\frac{p_2}{2}\right)+O(\lambda'^4) \end{aligned} \tag{4.97}$$

Taking into account the λ' dependence of the momenta given in (4.95) and expanding in λ' and $\frac{1}{M}$, we obtain the planar three loop result up to the first finite size correction

$$\Delta_{3\ loops} = \frac{\lambda'^3}{16}\left[\frac{n_1^6+n_2^6}{k^6}-\left(\frac{2}{kM}\right)\frac{3n_1^6+3n_1^5n_2+4n_1^3n_2^3+3n_1n_2^5+3n_2^6}{k^6}+O\left(\frac{1}{M^2}\right)\right]. \tag{4.98}$$

This result has to be compared with the $1/R^2$ corrections to the pp-wave energy spectrum of the corresponding string states.

As a consistency check, we see that when we set the wrapping number to zero to get the $\mathcal{N}=4$ state, i.e. put $n_2=-n_1$, it provides the $\mathcal{N}=4$ result, in beautiful agreement with the one quoted in Refs. [73], [76].

4.8 On the string side of the duality

In Sections 4.4-4.7, we discussed the expansion to leading order in $\frac{1}{M}$ about the MRV limit of the $\mathcal{N}=2$ quiver gauge theory. In the string dual to the quiver gauge theory - the IIB superstring on the $AdS_5\times S^5/\mathbf{Z}_M$ background we have explored in Section 4.3 - this corresponds to an expansion in the ratio $\frac{1}{M}=\frac{2R_-}{R^2}$ about the pp-wave space-time.

Corrections of this kind have already been analyzed in some detail for the case of $\mathcal{N}=4$ super Yang-Mills theory – string on $AdS_5 \times S^5$ duality in Ref. [70]. They considered the leading correction to the BMN limit, which was an expansion in the inverse R-charge $\frac{1}{J}$ of Yang-Mills theory or $\frac{\alpha'}{R^2}$ in string theory. In this section, we will generalize their computation to the case of the DLCQ string on the pp-wave background. We will compare the result with our computations of $1/M$-corrections in the quiver gauge theory.

The exact spectrum of states of the string theory on the pp-wave background, as well as the DLCQ of the pp-wave background are well-known. Our goal is to find corrections to the energies of these states to order $\frac{2R_-}{R^2}$. The technique to be used is to first find the correction to the string sigma model which arises from an expansion of the space-time metric and other background fields about the pp-wave. This yields an interaction Hamiltonian. The strategy is then to compute corrections to the energy spectrum by evaluating matrix elements of this interaction Hamiltonian in the pp-wave string theory states. The coefficient of the interaction Hamiltonian contains the factor $\frac{2R_-}{R^2}$.

In the case of $AdS_5 \times S^5$ background, the terms in the interaction Hamiltonian which contain two bosonic creation and two bosonic annihilation operators are expressed in terms of the string oscillators as [70]

$$\begin{aligned} H_{BB} \;=\; & -\frac{1}{32p^+R^2}\sum \frac{\delta(n+m+l+p)}{\xi}\times \\ & \Bigg\{2\Big[\xi^2-(1-k_lk_pk_nk_m)+\omega_n\omega_mk_lk_p+\omega_l\omega_pk_nk_m+2\omega_n\omega_lk_mk_p \\ & +2\omega_m\omega_pk_nk_l\Big]a^{\dagger A}_{-n}a^{\dagger A}_{-m}a^B_l a^B_p+4\Big[\xi^2-(1-k_lk_pk_nk_m)-2\omega_n\omega_mk_lk_p+\omega_l\omega_mk_nk_p \\ & -\omega_n\omega_lk_mk_p-\omega_m\omega_pk_nk_l+\omega_n\omega_pk_mk_l\Big]a^{\dagger A}_{-n}a^{\dagger B}_{-l}a^A_m a^B_p+4\Big[8k_lk_pa^{\dagger i}_{-n}a^{\dagger j}_{-l}a^i_m a^j_p \\ & +2(k_lk_p+k_nk_m)a^{\dagger i}_{-n}a^{\dagger i}_{-m}a^j_l a^j_p+(\omega_l\omega_p+k_lk_p-\omega_n\omega_m-k_nk_m)a^{\dagger i}_{-n}a^{\dagger i}_{-m}a^{j'}_l a^{j'}_p \\ & -4(\omega_l\omega_p-k_lk_p)a^{\dagger i}_{-n}a^{\dagger j'}_{-l}a^i_m a^{j'}_p-(i,j\rightleftharpoons i',j')\Big]\Bigg\}, \end{aligned} \tag{4.99}$$

where p^+ is the space-time momentum conjugate to the light-cone coordinate x^-, $\xi \equiv \sqrt{\omega_n\omega_m\omega_l\omega_p}$, $\omega_n=\sqrt{1+k_n^2}$ and $k_n^2=\frac{n^2}{\alpha'^2p^{+2}}=\lambda' n^2$, with $\lambda'=g^2_{YM}N/J^2$. The indices l,m,n,p run from $-\infty$ to $+\infty$. The presence of the R-R flux breaks the transverse $SO(8)$ symmetry of the metric to $SO(4)\times SO(4)$. Consequently the notation distinguishes sums over indices of the transverse coordinates in the first $SO(4)$ $(i,j,..)$, the second $SO(4)$ $(i',j',..)$ and over the full $SO(8)$ $(A,B,..)$. The operators in (4.99) are in a normal-ordered form. Since H_{BB} was derived as a classical object, the correct ordering on the operators is not defined and the ambiguity thus arising can be kept into account by introducing a normal ordering function $N_{BB}(k_n^2)$. Such normal-ordering function can however be set to zero following the prescription of Ref.[70].

The DLCQ version of (4.99) can be obtained by taking into account that the light-cone momentum p^+ along the compactified light-cone direction ($x^-\sim x^-+2\pi R^-$) is quantized as $p^+ = k/(2R_-)$. R_- is related to R through $R_- = R^2/(2M)$ so that $p^+ = kM/R^2$ and $R^2 = \sqrt{4\pi g_s \alpha'^2 NM}$. The Yang-Mills theory coupling constant is then identified with the superstring coupling constant g_s in the usual way $4\pi g_s = g^2_{\rm YM}$ and the double scaling limit is realized by sending both N and M to infinity and keeping the ratio N/M fixed, so that $R_- = \frac{\alpha'}{2}\sqrt{g^2_{YM}\frac{N}{M}} = \frac{\alpha'}{2}\sqrt{\lambda'}$ is also held fixed. As noticed in the introduction, the definition of λ' is in this case related to the YM coupling constant through an analogue of the usual definition $\frac{1}{(\alpha' p^+)^2} = \frac{g^2_{YM} NM}{(kM)^2} \equiv \frac{\lambda'}{k^2}$. This gives for the frequencies ω_n in (4.99) the formula $\omega_n = \sqrt{1+\lambda'\frac{n^2}{k^2}}$.

In the case of the $\mathcal{N}=2$ operator (4.46), the dual string state is the symmetric traceless two-impurity state created by the action of the following combination of bosonic creation operators on the string vacuum[9]

$$|[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}] >= \left[a^{\dagger a}_{n_1} a^{\dagger b}_{n_2} + a^{\dagger b}_{n_1} a^{\dagger a}_{n_2} - \frac{1}{2}\delta^{ab} a^{\dagger g}_{n_1} a^{\dagger g}_{n_2}\right]|0\rangle \tag{4.100}$$

where $n_1 + n_2 = k\,\ell$.

The general matrix elements of the DLCQ version $H^{Z_M}_{BB}$ of (4.99) between space-time bosons built out of bosonic string oscillators have the following explicit form

$$\begin{aligned}
&\langle 0|\, a^A_{-n_2} a^B_{-n_1}\, H^{Z_M}_{BB}\, a^{\dagger C}_{n_1} a^{\dagger D}_{n_2}\,|0\rangle = -\frac{1}{2R^2 p^+}\frac{1}{\sqrt{1+\lambda'\frac{n_1^2}{k^2}}\sqrt{1+\lambda'\frac{n_2^2}{k^2}}}\\
&\Bigg\{\delta^{AB}\delta^{CD}\lambda'\left[\frac{n_1^2}{k^2}+\frac{n_2^2}{k^2}+2\lambda'\frac{n_1^2 n_2^2}{k^4}+2\frac{n_1 n_2}{k^2}\sqrt{1+\lambda'\frac{n_1^2}{k^2}}\sqrt{1+\lambda'\frac{n_2^2}{k^2}}\right]\\
&+\delta^{AC}\delta^{BD}\lambda'\left[\frac{n_1^2}{k^2}+\frac{n_2^2}{k^2}+2\lambda'\frac{n_1^2 n_2^2}{k^4}-2\frac{n_1 n_2}{k^2}\sqrt{1+\lambda'\frac{n_1^2}{k^2}}\sqrt{1+\lambda'\frac{n_2^2}{k^2}}\right]\\
&+\lambda'\left[2\frac{n_1 n_2}{k^2}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}\right)+\frac{(n_1^2+n_2^2)}{k^2}\delta^{ad}\delta^{bc}\right]\\
&-\lambda'\left[2\frac{n_1 n_2}{k^2}\left(\delta^{a'b'}\delta^{c'd'}+\delta^{a'c'}\delta^{b'd'}\right)+\frac{(n_1^2+n_2^2)}{k^2}\delta^{a'd'}\delta^{b'c'}\right]\Bigg\}
\end{aligned} \tag{4.101}$$

where lower-case $SO(4)$ indices $a,b,c,d \in 1,\ldots,4$ mean that the corresponding $SO(8)$ labels A,B,C,D all lie in the first $SO(4)$, while the indices $a',b',c',d' \in 5,\ldots,8$ mean that the $SO(8)$ labels lie in the second $SO(4)$ ($A,B,C,D \in 5,\ldots,8$).

Eq. (4.101) can be used to evaluate the first order correction to the energy of the state (4.100), namely the matrix element $< [\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}]|H^{Z_M}_{BB}|[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}] >$. Summing all the contributes and dividing the result by the norm of the state

$$< [\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}]|[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}] >= 2(1+\frac{1}{2}\delta^{ab})$$

[9]We use the notation of Ref. [70], where the representations of $SO(4)\times SO(4)$ are classified using an $SU(2)$ notation as $SO(4)\approx SU(2)\times SU(2)$.

one gets the desired first curvature correction to the spectrum of the states (4.100). The final result for the energy levels for a two impurity state with discrete light-cone momentum k, exact to all orders in λ', is

$$
\begin{aligned}
E(n_1,n_2) &= \sqrt{1+\lambda'\left(\frac{n_1}{k}\right)^2} + \sqrt{1+\lambda'\left(\frac{n_2}{k}\right)^2} \\
&- \frac{\lambda'}{kM}\left[\frac{\frac{n_1^2}{k^2}+\frac{n_2^2}{k^2}+\lambda'\frac{n_1^2 n_2^2}{k^4}+\frac{n_1 n_2}{k^2}-\frac{n_1 n_2}{k^2}\sqrt{1+\lambda'\left(\frac{n_1}{k}\right)^2}\sqrt{1+\lambda'\left(\frac{n_2}{k}\right)^2}}{\sqrt{1+\lambda'\left(\frac{n_1}{k}\right)^2}\sqrt{1+\lambda'\left(\frac{n_2}{k}\right)^2}}\right] + O\left(\frac{1}{M^2}\right)
\end{aligned}
\tag{4.102}
$$

where the small parameter governing the strength of the perturbation has been converted from $1/(R^2p^+)$ to $1/(kM)$ in order to make the comparison with the finite size corrections of the gauge theory results more clear. Notice that for $n_1 = -n_2$ (4.102) gives back the $\mathcal{N}=4$ result of Ref.[70], as it should.

A λ' expansion of (4.102) up to $O(\lambda'^2)$ shows perfect agreement with the gauge theory calculations at one and two loops, Eqs.(4.78) and (4.82). As for the parent $\mathcal{N}=4$ theory [73, 76], the disagreement between the two sides of the duality is manifest at three loops, where the finite size correction to the string energy

$$
\begin{aligned}
E_{3\ loops} = \frac{\lambda'^3}{16} \quad & \left[\frac{n_1^6+n_2^6}{k^6} - \left(\frac{2}{kM}\right)\frac{3n_1^6+3n_1^5n_2+n_1^4n_2^2+2n_1^3n_2^3+n_1^2n_2^4+3n_1n_2^5+3n_2^6}{k^6}\right. \\
& \left. + O\left(\frac{1}{M^2}\right)\right]
\end{aligned}
\tag{4.103}
$$

does not match its gauge dual result (4.98).

4.9 The S-matrix dressing factor

Integrable structures have been found also in the $AdS_5 \times S^5$ string sigma model: from a classical point of view integral Bethe equations were derived in the thermodynamic limit [74], while quantum corrections are believed to yield discrete equations describing a finite number of excitations.

The agreement between the anomalous dimensions of the $\mathcal{N}=4$ gauge theory operators in the near-BMN limit and the string energies in the near-plane wave limit up to two gauge theory loops suggests that, if we wish to describe the string excitations by the language of a spin chain, the string dynamics should be given by the BDS chain.

The three loop disagreement can actually be encoded by "dressing" the gauge theory S-matrix (*i.e.* the r.h.s. of the Bethe equations for the BDS chain) by a multiplicative factor. From these equations one derives a solution for the momenta of the string excitations which plugged in the BDS dispersion relation (4.61) reproduce the near-plane wave string energies, both in the thermodynamic limit and in the few impurity case [126, 77].

The near-plane wave string energies can therefore be computed in the $AdS_5 \times S^5$ IIB superstring theory by the following Bethe equations:

$$e^{ip_jL} = \prod_{l=1\ ;\ l\neq j}^{\mathcal{M}} S_{string}(p_j, p_l), \tag{4.104}$$

with $L = J + \mathcal{M}$ and

$$S_{string}(p_j, p_l) = \frac{\varphi_j - \varphi_l + i}{\varphi_j - \varphi_l - i} \exp\left\{ 2i \sum_{r=0}^{\infty} \left(\frac{\lambda}{16\pi^2}\right)^{r+2} [q_{r+2}(p_j)q_{r+3}(p_l) - q_{r+2}(p_l)q_{r+3}(p_j)] \right\} \tag{4.105}$$

where the BDS rapidities are defined in (4.13) and the exponential term is the so called dressing factor, expressed as a function of the BDS conserved charges

$$q_r(p_j) = \frac{2\sin\left(\frac{r-1}{2}p_j\right)}{r-1} \left(\frac{\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}} - 1}{\frac{\lambda}{4\pi^2}\sin\frac{p_j}{2}} \right)^{r-1} \tag{4.106}$$

In particular, the second charge $q_2(p_j)$ is the energy of a single excitation and the energy of a string state with $\mathcal{M}$ excitations is given by

$$E = \frac{\lambda}{8\pi^2} \sum_{j=1}^{\mathcal{M}} q_2(p_j) \tag{4.107}$$

We will now discuss the two magnon case in the orbifolded theory and show that the same dressing factor allows one to compute the DLCQ string energies by means of a Bethe ansatz. The two magnon scattering however is not as trivial as in the parent theory, since the excitations are not forced by the level matching condition to carry opposite momenta.

It is not difficult to check that the string spectrum (4.102) coincides with (4.107) up to $O(\lambda'^3)$ with $\mathcal{M} = 2$ if the magnon momenta have the form

$$\begin{aligned} p_1 &= \frac{2n_1\pi}{kM} + \frac{A\pi}{M^2} + \lambda'\frac{B\pi}{M^2} + \lambda'^2\frac{C'\pi}{M^2} \\ p_2 &= \frac{2n_2\pi}{kM} - \frac{A\pi}{M^2} - \lambda'\frac{B\pi}{M^2} - \lambda'^2\frac{C'\pi}{M^2}, \end{aligned} \tag{4.108}$$

with the same A and B found in the gauge theory, Eqs. (4.77) (4.81), and C' given by

$$C' = \frac{n_1^2 n_2^2\,(n_1^2 + n_2^2)}{4k^6(n_1 - n_2)} \tag{4.109}$$

We conjecture that the string S-matrix for the $\mathrm{AdS}_5 \times \mathrm{S}^5/Z_M$ IIB superstring is given by (4.105) with the addition of a twist factor which coincides with the one used in the gauge theory

$$S^{orb.}_{string}(p_j, p_l) = \omega^l \frac{\varphi_j - \varphi_l + i}{\varphi_j - \varphi_l - i}$$

$$\exp\left(2i\sum_{r=0}^{\infty}\left(\frac{\lambda}{16\pi^2}\right)^{r+2}[q_{r+2}(p_j)q_{r+3}(p_l)-q_{r+2}(p_l)q_{r+3}(p_j)]\right) \tag{4.110}$$

with $\omega^l = e^{i(p_1+p_2)}$ for the two magnon case. It is easy to see that the Bethe equations

$$e^{ip_2(kM+2)} = S^{orb.}_{string}(p_2, p_1), \tag{4.111}$$

are in fact satisfied if p_1 and p_2 are exactly (4.108), with the constants A, B and C given in (4.77), (4.81) and (4.109).

Thus we have proved that the dressing factor for the orbifolded theory equals that of the parent theory and therefore, as for the gauge theory, the spectrum can be obtained by just twisting the parent Bethe equations: the three loop disagreement is inherited and does not depend on the orbifold projection.

4.10 Summary

In this Chapter, we have computed the first finite size correction to the anomalous dimension of two-impurity states about the double scaling limit of the $\mathcal{N}=2$ quiver gauge theory and the analogous quantity in the IIB superstring propagating on the plane-wave background with a periodically identified null coordinate.

In the gauge theory the anomalous dimensions are computed by two independent techniques that agree with each other. We have solved, up to three loops and the first finite size correction, the twisted Bethe equations conjectured in Ref. [93] for the orbifolded theory. Then we have provided an ansatz for the eigenstate of the dilatation operator that up to two loops gives the same spectrum derived with the other procedure. The eigenvalue equation for this wave function reduces to the twisted Bethe equation.

On the string theory side the computation is done by evaluating the first curvature correction to the pp-wave DLCQ spectrum of a bosonic two excitation state.

We have found that the gauge theory and the string theory results agree up to two loop order, but there is a disagreement at three loops. This disagreement is similar to, and a slight generalization of the one which is known to exist at three loop order in the analogous computation in $\mathcal{N}=4$ super Yang-Mills theory expanded about the BMN limit [73, 76].

In Summary, the results of this Chapter are

$$\begin{aligned}
\Delta_{YM} &= kM+2+\frac{\lambda'}{2}\left[\frac{n_1^2+n_2^2}{k^2}\right]-\frac{\lambda'^2}{8}\left[\frac{n_1^4+n_2^4}{k^4}\right]+\frac{\lambda'^3}{16}\left[\frac{n_1^6+n_2^6}{k^6}\right]+...\\
&+ \frac{\lambda'}{kM}\left[-\frac{(n_1^2+n_2^2)}{k^2}+\frac{\lambda'}{2}\frac{n_1^4+n_1^3n_2+n_1n_2^3+n_2^4}{k^4}\right.\\
&- \left.\frac{\lambda'^2}{8}\frac{3n_1^6+3n_1^5n_2+4n_1^3n_2^3+3n_1n_2^5+3n_2^6}{k^6}+...\right]
\end{aligned} \tag{4.112}$$

$$
\begin{aligned}
\Delta_{string} &= kM + 2 + \frac{\lambda'}{2}\left[\frac{n_1^2+n_2^2}{k^2}\right] - \frac{\lambda'^2}{8}\left[\frac{n_1^4+n_2^4}{k^4}\right] + \frac{\lambda'^3}{16}\left[\frac{n_1^6+n_2^6}{k^6}\right] + \ldots \\
&+ \frac{\lambda'}{kM}\left[-\frac{(n_1^2+n_2^2)}{k^2} + \frac{\lambda'}{2}\frac{n_1^4+n_1^3n_2+n_1n_2^3+n_2^4}{k^4}\right. \\
&- \left.\frac{\lambda'^2}{8}\frac{3n_1^6+3n_1^5n_2+n_1^4n_2^2+2n_1^3n_2^3+n_1^2n_2^4+3n_1n_2^5+3n_2^6}{k^6} + \ldots\right] \qquad (4.113)
\end{aligned}
$$

The first two lines of each of the above expressions are identical and they differ in the third line.

We have finally shown that the DLCQ string spectrum is obtained by twisting the string Bethe ansatz proposed in Ref. [77]. The three loop disagreement is encoded in a "dressing factor" added to the gauge theory S-matrix, which coincides with the one of the $\mathcal{N}=4$ theory.

Our computations are consistent with integrability of $\mathcal{N}=2$ quiver gauge theory in the MRV limit and its string theory dual, DLCQ type IIB superstring theory on a plane wave background with a compactified null direction.

Chapter 5

Discussion

In this thesis we have described the work presented in the papers [13, 57, 22, 99]. In the most part of the thesis we investigated aspects of tachyon dynamics in String Field Theory (SFT), while the final chapter concerns the issue of integrability of $\mathcal{N} = 2$ SYM on a plane-wave background in the framework of AdS/CFT correspondence.

In [13], we have derived some exact results for the non-linear tachyon β-function of the open bosonic string theory. We have shown its relevance in the construction of the Witten-Shatashvili (WS) bosonic string field theory in its Boundary formulation [2]. When a non-linear renormalization of the tachyon field is considered [41], the WS action in fact results in a simple formula in terms of the disk partition function and β_T. In the case in which the tachyon profile is a slowly varying function of the embedding coordinates of the string this formula can be used to derive the exact tachyon potential and the first derivative terms of the effective action. The action holds also when the tachyon coupling $T(k)$ is small and has support near the mass-shell. In fact, we have showed that the WS action constructed in terms of a linear β-function [42] is related to the one derived in [13] by a field redefinition that becomes singular on-shell. This is in agreement with the Poincaré-Dulac theorem [43], used to prove [5] that near the on-shellness the β-function *has* to be non-linear. The perturbative solutions of the equation $\beta^T = 0$ have been thus shown to provide the expected scattering amplitudes of on-shell tachyons.

The explicit form of the WS action constructed from the tachyon non-linear β-function is in precise agreement with all the conjectures involving tachyon condensation. In particular its normalization can be fixed either by studying the exact tachyon potential or by finding the field redefinition that maps the WS action into the effective tachyon action coming from the Cubic formulation of SFT. This field redefinition is non-singular on-shell only if the normalization constant coincides with the tension of the D25-brane.

The knowledge of the non-linear tachyon β-function is very important also for another reason. The solutions of the equation $\beta^T = 0$ give the conformal fixed points, the backgrounds that are consistent with the string dynamics. In the case of slowly varying tachyon profiles, we showed that the equations of motion for the WS action can be made identical to the RG fixed point equation $\beta^T = 0$. This can be done for a particular choice of the renormalization prescription ambiguity. We have found soliton solutions of this equation to which correspond a finite value of the WS action. Being solutions of the RG equations these solitons are lower dimensional D-branes for which the finite value of S provides a very accurate estimate of the D-brane tension.

When other excited string modes are present, say modes of the vector field A_μ, the form of the WS action is still given by (1.4) where the renormalized tachyon field in this case depends also on the other string couplings [127]. Whether our approach would help in treating also non-renormalizable interactions it is not yet clear.

In [22] we explored various technologies useful for deriving off-shell tachyon amplitudes. The series solution for the off-shell factor appearing in the four-tachyon amplitude [29] results as a very useful tool for providing precise tests of CSFT. It has improved the numerical approximation for the evaluation of the exact quartic self-coupling c_4 in the tachyon potential and signaled a slightly different behavior in the wild oscillations presented by the "rolling" tachyon solution analyzed in [56]. Both the level truncation *on fields* and *on oscillators* were used to derive higher order terms in the tachyon effective action and so higher order coefficients in the time-dependent solution. The sequences thus found for each coefficient of the solution - corresponding to growing truncation levels used in the evaluation of the amplitudes - well converge toward the value found through the exact amplitude, and exhibit a $1/L$ behavior.

The time-dependent tachyon solutions analyzed in [56, 22], a power series in e^t, exhibits a behavior that is drastically different from the one showed in the Boundary formulation of SFT [49] (as well in frameworks such CFT, boundary state approach, RG flow analysis, etc.), in which the tachyon monotonically rolls from the perturbative unstable vacuum towards the true vacuum. The Cubic SFT tachyon starts from the maximum of the potential at $t = -\infty$, rolls down past the minimum and undergoes evergrowing oscillations, still the energy being conserved. The pressure oscillates similarly, thus not representing tachyon matter. In [56] an argument was done to reconcile this discrepancy, based on the field redefinition we derived in [13], that would map the CSFT oscillating solution to the BSFT "well-behaved" one. As we explained at the end of Chapter 3, it seems to us that some more informations about this mapping should come at least from the knowledge of a further order - the third - of this field redefinition. This would be an important step forward a meaningful interpretation of the puzzling behavior of the Cubic SFT solution.

Another point of view about this discrepancy is related to an important general issue of Cubic open string field theory, the gauge fixing. To perform explicit calculations in string field theory, the gauge symmetry of the cubic SFT action must be fixed. Almost all the studies in Cubic SFT - included the results quoted above - are conducted in Feynman–Siegel gauge. This gauge, however, is known to show a pathological behavior in the effective tachyon potential [28]. Namely, branch points appear in both side of larger and smaller field values of the tachyon so that one could not go beyond this small region [1]. These branch points arise because the trajectory in field space associ-

[1] It is worth mentioning again here that in the Boundary SFT approach the tachyon potential can be computed exactly [10, 11]. In this formulation, there is no branch point in the effective potential. On the other hand, in that case the nontrivial vacuum arises only as the tachyon field goes to infinity, so it is harder to study the physics of the stable vacuum from this point of view.

ated with this potential encounters the boundary of the region of Feynman-Siegel gauge validity [128].

As already noticed in [56], it is interesting to compare the behavior of the perturbative expansion of the CSFT time-dependent solution in powers of e^t with the perturbative expansion of the effective tachyon potential $V(\varphi)$. As found in [28], a branch point associated with the brakdown of Feynman-Siegel gauge is encountered at positive φ just *past* the minimum. As a consequence, the power series expansion for $V(\varphi)$ fails to converge in that region. The branch points in the tachyon potential prevent us going beyond a small region of tachyon field value. Therefore an alternative useful gauge choice has been desirable for convincing the known results or extending the analysis. Might the wild oscillations of the tachyon trajectory $\varphi(t)$ indicate a similar breakdown of the perturbative expansion? It is interesting to note that the analytic solution for the CSFT nontrivial tachyon vacuum by Schnabl [8] uses a *different* gauge choice then the Feynman–Siegel one. It will be interesting, and not surprising, to see if this gauge has better features with regard to the wild oscillating behavior of the CSFT solution analyzed in [56, 22].

In [57], a time-dependent solution was found at the lowest order, $(0, 0)$, in the level truncation scheme. There, an alternative analytic procedure was used, based on the treatment of the coupling constant as an independent variable. The solution satisfies a "diffusion" equation, a remarkable property thanks to which the action of the infinite derivative operators of Cubic SFT on it can be simply represented as a trivial translation in this alternative variable. The energy-momentum tensor can thus be written in a simple and closed form, as a bilinear in the tachyon fields containing only finite derivatives. The dynamical picture is, however, again problematic. A pathological behavior arises also in this case, a singularity appears at the origin of time - again after the tachyon have reached the minimum of its potential.

About possible cosmological implications, it should be noticed that in boundary string field theory and in most of the models used to study tachyon driven cosmology, the stable minimum of the potential is taken at infinite values of the tachyon field [10, 11, 13, 32, 51, 129]. The tachyon thus cannot roll beyond its minimum. One of the main objections to the rolling tachyon as a mechanism for inflation is that reheating and creation of matter in models where the minimum of the potential is at $T \to \infty$ is problematic because the tachyon field in such theory does not oscillate [130, 131]. In cubic string field theory the minimum of the potential is at finite values of the tachyon field. Therefore, it would be interesting to see if the coupling of the free theory to a Friedman-Robertson-Walker metric [131], and the consequent inclusion of a Hubble friction term, might lead to damped oscillations around the stable minimum of the potential well. This would provide an alternative to the Born-Infeld type effective action that has been so extensively used in the study of tachyon cosmology [131, 130, 132, 133, 134]. Cubic string field theory might then open interesting perspectives in tachyon cosmology [135].

In the paper [99], we have approached a very active area of research of the last few years, whose main idea is applying the methods of integrable systems to leading problems in gauge and string theory, and in a special effort to the AdS/CFT correspondence.

In particular, we have computed the planar finite size corrections to the spectrum of the dilatation operator acting on two-impurity states of a certain limit of conformal $\mathcal{N}=2$ quiver gauge field theory which is a Z_M-orbifold of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. We matched the result to the string dual, IIB superstrings propagating on a pp-wave background with a periodically identified null coordinate. Up to two loops, we show that the computation of operator dimensions, using an effective Hamiltonian technique derived from renormalized perturbation theory and a twisted Bethe ansatz which is a simple generalization of the Beisert-Dippel-Staudacher [76] long range spin chain, agree with each other and also agree with a computation of the analogous quantity in the string theory. We compute the spectrum at three loop order using the twisted Bethe ansatz and find a disagreement with the string spectrum very similar to the known one in the near BMN limit of $\mathcal{N}=4$ super-Yang-Mills theory [70, 73, 76]. We show that, as in the parent $\mathcal{N}=4$ case, this disagreement can be resolved by adding a conjectured "dressing factor" [77] to the twisted *string* Bethe ansatz for the string sigma model on the orbifolded background $AdS_5 \times S^5/Z_M$.

There is recent proposal [120] for a phase factor in the Bethe ansatz [85] of planar $\mathcal{N}=4$ gauge theory, that would be non-perturbatively related to a recently conjectured crossing-symmetric phase factor for perturbative string theory on $AdS_5 \times S^5$. This proposal lends support to an exact interpretation of the AdS/CFT correspondence, and suggests a promising solution to the longstanding discrepancies between gauge and string theory.

It would be interesting to analyze the disagreement from another possible point of view. In general, calculations are done or in the planar limit (*i.e.* by sending to infinity the rank N of the gauge group) and taking a finite value L for the length of the chain this means to end with a set of discrete equations - or by sending first L to infinity (thermodynamic limit) and keeping finite N which implies integral equations and string loops. It would be interesting to proceed in the calculation of two impurity operators spectrum in the $\mathcal{N}=2$ QGT [92] in which the non planar calculations simplify [90] [2] - by maintaining finite both the size of the length and the rank of the gauge group, to see if these limits commute.

[2]The main observation is that the existence of a positive definite, discrete light-cone momentum greatly simplifies the operator mixing problem.

Appendix A

Computation of $I(k_1, k_2, k_3)$

In this Appendix we shall compute the integral $I(k_1, k_2, k_3)$ appearing in eq.(1.34)

$$\begin{aligned} I(k_1,k_2,k_3) = & \frac{2^{2k_1k_2+2k_2k_3+2k_1k_3}}{(2\pi)^3} \int_0^{2\pi} d\tau_1 d\tau_2 d\tau_3 \left[\sin^2\left(\frac{\tau_1-\tau_2}{2}\right)\right]^{k_1k_2} \\ & \left[\sin^2\left(\frac{\tau_2-\tau_3}{2}\right)\right]^{k_2k_3} \left[\sin^2\left(\frac{\tau_1-\tau_3}{2}\right)\right]^{k_1k_3} \end{aligned} \tag{A.1}$$

Introducing the variables

$$x = \frac{\tau_1-\tau_2}{2} \quad , \quad y = \frac{\tau_3-\tau_1}{2}$$

the integral over τ_1, τ_2 and τ_3 can be written as

$$I = -\frac{4^{k_1k_2+k_2k_3+k_1k_3+1}}{2\pi^3} \int_0^{2\pi} d\tau_1 \int_{\frac{\tau_1}{2}}^{\frac{\tau_1}{2}-\pi} dx \int_{-\frac{\tau_1}{2}}^{\pi-\frac{\tau_1}{2}} dy \left[\sin^2 x\right]^{k_1k_2} \left[\sin^2 y\right]^{k_1k_3} \left[\sin^2(x+y)\right]^{k_2k_3}$$

With a suitable shift of the integration variables we obtain

$$\begin{aligned} I = & \frac{4^{k_1k_2+k_2k_3+k_1k_3}}{\pi^2} \int_0^\pi dx \int_0^\pi dy \left[\sin x\right]^{2k_1k_2} \left[\sin y\right]^{2k_1k_3} \left[\sin^2(x+y)\right]^{k_2k_3} \\ = & \frac{4^{k_1k_2+k_2k_3}}{\pi^2} \int_0^\pi dx \int_0^\pi dy \left[\sin x\right]^{2k_1k_2} \left[\sin y\right]^{2k_1k_3} \left[1-e^{2i(x+y)}\right]^{k_1k_3} \left[1-e^{-2i(x+y)}\right]^{k_2k_3} \\ = & \frac{4^{k_1k_2+k_1k_3}}{\pi^2} \int_0^\pi dx \int_0^\pi dy \left[\sin x\right]^{2k_1k_2} \left[\sin y\right]^{2k_1k_3} \sum_{n,m=0}^{\infty} \frac{\Gamma(n-k_2k_3)\Gamma(m-k_2k_3)}{n!m!\Gamma^2(-k_2k_3)} e^{2i(x+y)(n-m)} \end{aligned} \tag{A.2}$$

Integrating over x and y we have

$$\begin{aligned} I = \sum_{n,m=0}^{\infty} & \frac{\Gamma(n-a_3)\Gamma(m-a_3)}{n!m!\Gamma(1+a_1+n-m)\Gamma(1+a_1-n+m)\Gamma(1+a_2+n-m)\Gamma(1+a_2-n+m)} \\ & \frac{\Gamma(1+2a_1)\Gamma(1+2a_2)}{\Gamma^2(-a_3)} \end{aligned} \tag{A.3}$$

where $a_1 = k_1k_2$, $a_2 = k_2k_3$ and $a_3 = k_1k_3$. Shifting $m \to m - n$ in the sum over m we have

$$
\begin{aligned}
I &= \sum_{n,m=0}^{\infty} \frac{\Gamma(n-a_2)\Gamma(n+m+a_2)\Gamma(1+2a_1)\Gamma(1+2a_3)}{n!(n+m)!\Gamma^2(-a_2)\Gamma(1+a_1+m)\Gamma(1+a_1-m)\Gamma(1+a_3+m)\Gamma(1+a_3-m)} \\
&+ \sum_{n=0}^{\infty}\sum_{m=-n}^{0} \frac{\Gamma(n-a_2)\Gamma(n+m+a_2)\Gamma(1+2a_1)\Gamma(1+2a_3)}{n!(n+m)!\Gamma^2(-a_2)\Gamma(1+a_1+m)\Gamma(1+a_1-m)\Gamma(1+a_3+m)\Gamma(1+a_3-m)} \\
&- \frac{\Gamma(1+2a_1)\Gamma(1+2a_3)}{\Gamma^2(1+a_1)\Gamma^2(1+a_3)}\ {}_2F_1(-a_2,-a_2;1;1)
\end{aligned}
\tag{A.4}
$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ is the Hypergeometric function. Changing the sign of the integer m in the second term of the previous equation and noting that

$$
\sum_{n=0}^{\infty}\sum_{m=0}^{n} = \sum_{n=m}^{\infty}\sum_{m=0}^{\infty}
$$

we find

$$
\begin{aligned}
I &= 2\sum_{m=0}^{\infty} \frac{\Gamma(m-a_2)\Gamma(1+2a_1)\Gamma(1+2a_3)}{\Gamma(-a_2)\Gamma(1+a_1+m)\Gamma(1+a_1-m)\Gamma(1+a_3+m)\Gamma(1+a_3-m)} \\
&\quad {}_2F_1(m-a_2,-a_2;m+1;1) - \frac{\Gamma(1+2a_1)\Gamma(1+2a_3)}{\Gamma^2(1+a_1)\Gamma^2(1+a_3)}\ {}_2F_1(-a_2,-a_2;1;1)
\end{aligned}
\tag{A.5}
$$

It is not difficult to show that the sum over m can be extended to negative values so that we find

$$
\begin{aligned}
I = \quad & -\frac{\Gamma(1+2a_1)\Gamma(1+2a_2)\Gamma(1+2a_3)}{\Gamma(1-a_1)\Gamma(a_1)\Gamma(1-a_2)\Gamma(a_2)\Gamma(1-a_3)\Gamma(a_3)} \\
& \sum_{m=-\infty}^{\infty} \frac{\Gamma(m-a_1)\Gamma(m-a_2)\Gamma(m-a_3)}{\Gamma(1+m+a_1)\Gamma(1+m+a_2)\Gamma(1+m+a_3)}
\end{aligned}
\tag{A.6}
$$

The series in the second line of the right-hand side of (A.6) is convergent for $1 + a_1 + a_2 + a_3 > 0$. To sum over m we use a standard procedure. Consider the path in Fig.A.1

Defining

$$
S = \sum_{m=-\infty}^{\infty} \frac{\Gamma(m-a_1)\Gamma(m-a_2)\Gamma(m-a_3)}{\Gamma(1+m+a_1)\Gamma(1+m+a_2)\Gamma(1+m+a_3)} \equiv \sum_{m=-\infty}^{\infty} f(m) \tag{A.7}
$$

we can write

$$
\oint_C \pi \mathrm{cotg}\pi z f(z) dz = \sum_{m=-N}^{N} f(m) + \tilde{S} \tag{A.8}
$$

where $\tilde{S}$ is the sum of the residues of $\pi\mathrm{cotg}\pi z f(z)$ evaluated in the poles of $f(z)$. In the limit $N \to \infty$ the left-hand side of the previous equation vanishes reducing S to

$$
S = \quad -\frac{\Gamma(1+a_1+a_2+a_3)}{\Gamma(1+a_1)\Gamma(1+a_2)\Gamma(1+a_3)\Gamma(1+a_1+a_2)\Gamma(1+a_1+a_3)\Gamma(1+a_2+a_3)}
$$

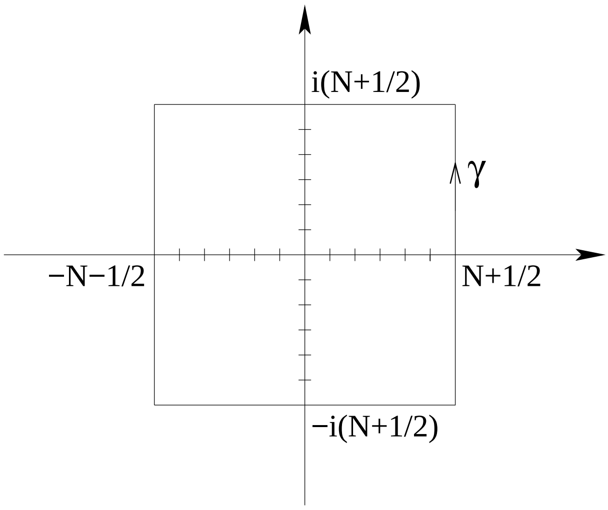


Figure A.1: Contour C.

$$\left[\frac{\pi^3 \cos^2 \pi a_1}{\sin \pi a_1 \sin \pi(a_1 - a_2) \sin \pi(a_1 - a_3)} + \text{cycl.}\right] \tag{A.9}$$

So that I becomes

$$I = \frac{\Gamma(1 + a_1 + a_2 + a_3)\Gamma(1 + 2a_1)\Gamma(1 + 2a_2)\Gamma(1 + 2a_3)}{\Gamma(1 + a_1)\Gamma(1 + a_2)\Gamma(1 + a_3)\Gamma(1 + a_1 + a_2)\Gamma(1 + a_2 + a_3)\Gamma(1 + a_1 + a_3)} \tag{A.10}$$

Appendix B

Neumann coefficients

Exact formulas for the Neumann coefficients V^{rs} and X^{rs} appearing in (2.51) were computed in [23][1]. The indices r, s take values from 1-3 and indicate wich Fock space the oscillators act in. The 3-string coefficients V^{rs}_{mn}, X^{rs}_{mn} are given in terms of the 6-string Neumann coefficients $N^{r,\pm s}_{nm}$

$$N^{r,\pm r}_{nm} = \begin{cases} \frac{1}{3(n\pm m)}(-1)^n(A_nB_m \pm B_nA_m), & m+n\,\text{even},\ m\neq n \\ 0, & m+n\,\text{odd} \end{cases} \tag{B.1}$$

$$N^{r,\pm(r+\sigma)}_{nm} = \left\{ \begin{array}{ll} \frac{1}{6(n\pm\sigma m)}(-1)^{n+1}(A_nB_m \pm \sigma B_nA_m), & m+n\,\text{even},\ m\neq n \\ \sigma\frac{\sqrt{3}}{6(n\pm\sigma m)}(A_nB_m \mp \sigma B_nA_m), & m+n\,\text{odd} \end{array} \right] . \tag{B.2}$$

where in $N^{r,\pm(r+\sigma)}$, $\sigma=\pm1$, and $r+\sigma$ is taken modulo 3 to be between 1 and 3. In (B.2) A_n, B_n are defined for $n\geq 0$ through

$$\begin{aligned} \left(\frac{1+ix}{1-ix}\right)^{1/3} &= \sum_{n\,\text{even}} A_nx^n + i\sum_{m\,\text{odd}} A_mx^m \\ \left(\frac{1+ix}{1-ix}\right)^{2/3} &= \sum_{n\,\text{even}} B_nx^n + i\sum_{m\,\text{odd}} B_mx^m\,. \end{aligned} \tag{B.3}$$

The 3-string matter Neumann coefficients V^{rs}_{nm} are then given by

$$\begin{aligned} V^{rs}_{nm} &= -\sqrt{mn}(N^{r,s}_{nm} + N^{r,-s}_{nm}), \quad m\neq n,\ \text{and}\, m,n\neq 0 \\ V^{rr}_{nn} &= -\frac{1}{3}\left[2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - (-1)^n - A_n^2\right], \quad n\neq 0 \\ V^{r,r+\sigma}_{nn} &= \frac{1}{2}\left[(-1)^n - V^{rr}_{nn}\right], \quad n\neq 0 \\ V^{rs}_{0n} &= -\sqrt{2n}\left(N^{r,s}_{0n} + N^{r,-s}_{0n}\right), \quad n\neq 0 \\ V^{rr}_{00} &= \ln(27/16) \end{aligned} \tag{B.4}$$

[1]In some references signs and factors in the Neumann coefficients may be slightly different. We follow here the choices of [105].

The ghost Neumann coefficients $X^{rs}_{mn}, m \geq 0, n > 0$ are given by

$$
\begin{aligned}
X^{rr}_{mn} &= m\left(-N^{r,r}_{nm} + N^{r,-r}_{nm}\right), \qquad n \neq m \\
X^{r(r\pm1)}_{mn} &= m\left(\pm N^{r,r\mp1}_{nm} \mp N^{r,-(r\mp1)}_{nm}\right), \qquad n \neq m \\
X^{rr}_{nn} &= \frac{1}{3}\left[-(-1)^n - A_n^2 + 2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - 2(-1)^n A_n B_n\right] \\
X^{r(r\pm1)}_{nn} &= -\frac{1}{2}(-1)^n - \frac{1}{2}X^{rr}_{nn}
\end{aligned}
\tag{B.5}
$$

The Neumann coefficients satisfy a cyclic symmetry under $r \to r+1, s \to s+1$, corresponding to the geometric symmetry of rotating the vertex. Furthermore, they are symmetric under the exchange $r \leftrightarrow s, n \leftrightarrow m$ and satisfy the twist symmetry associated with reflection of the strings [2]

$$
\begin{aligned}
V^{rs}_{nm} &= (-1)^{n+m}V^{sr}_{nm} \\
X^{rs}_{nm} &= (-1)^{n+m}X^{sr}_{nm}\,.
\end{aligned}
\tag{B.6}
$$

[2] The relations among the Neumann coefficients are not just these ones, they satisfy a remarkable integrability property underlying Witten's string field theory, the Toda lattice (dTL) hierarchy [113].

Appendix C

Level truncation method

As a specific example of the level truncation method explained in Section 2.2.3 let us derive explicitly the four tachyon amplitude for $L = 2$ in the time-dependent case. At this level of truncation and with the change of coordinates (2.58), the matrices $\tilde{V^{11}}$ and $\tilde{X^{11}}$ in (2.49) become the 2×2 matrices

$$\tilde{V^{11}} = \begin{pmatrix} V^{11}_{11}\sigma & V^{11}_{12}\sigma^{\frac{3}{2}} \\ V^{11}_{21}\sigma^{\frac{3}{2}} & V^{11}_{22}\sigma^2 \end{pmatrix}, \qquad \tilde{X^{11}} = \begin{pmatrix} X^{11}_{11}\sigma & X^{11}_{12}\sigma^{\frac{3}{2}} \\ X^{11}_{21}\sigma^{\frac{3}{2}} & X^{11}_{22}\sigma^2 \end{pmatrix} \tag{C.1}$$

and analog forms for all the objects contained in (2.52) may be written. Expanding the determinant and the exponential in (2.49) in powers of σ up to σ^2 one gets

$$\begin{aligned} A_4(p_1,p_2,p_3,p_4) \quad = \quad & \frac{\lambda_c^2 g^2}{2}\lambda_c^{\frac{2}{3}(\sum_{i=1}^4 p_i^2+p_1\cdot p_2+p_3\cdot p_4)}\delta(\sum_i p_i)\int_0^1 \frac{d\sigma}{\sigma^2}\sigma^{-\frac{1}{2}[(p_1+p_2)^2+(p_3+p_4)^2]} \\ & \Big\{1 - b_1(p_1-p_2)(p_3-p_4)\sigma + \frac{1}{2}\left[b_2 + b_3\left((p_1-p_2)^2+(p_3-p_4)^2\right)\right. \\ & \left.+b_4(p_1-p_2)^2(p_3-p_4)^2 + b_5(p_1+p_2)(p_3+p_4)\right]\sigma^2 + O(\sigma^3)\Big\} \end{aligned} \tag{C.2}$$

where

$$\begin{aligned} b_1 &= (V^{12}_{01})^2, & b_2 &= 26(V^{11}_{11})^2 - 2(X^{11}_{11})^2, & b_3 &= (V^{12}_{01})^2 V^{11}_{11}, \\ b_4 &= (V^{12}_{01})^4, & b_5 &= 18(V^{12}_{02})^2. \end{aligned} \tag{C.3}$$

To get the quartic term in the tachyon effective action on has to subtruct the contribution from the tachyon in the propagator, that corresponds to the σ^0 power -the constant term 1- in (C.2). Since, as already noticed, for a four point amplitude only even powers of σ need to be considered, one is left with the coefficient of the σ^2 term in the sum. Performing the integral over σ, one finally gets the formula for the quartic term in the CSFT tachyon effective action (3.2) in the time-dependent case

$$\begin{aligned} A_4^{L=2}(p_1,p_2,p_3,p_4) = \quad & \lambda_c^2 g^2 \int \prod_{i=1}^n (2\pi dp_i)\delta(\sum_i p_i)\phi(p_i)\frac{\lambda_c^{\frac{2}{3}(\sum_{i=1}^4 p_i^2+p_1\cdot p_2+p_3\cdot p_4)}}{1-(p_1+p_2)^2} \\ & \left[\frac{b_2}{4} + b_3 p_1(p_2-p_1) + b_4 p_2 p_4(p_2-p_1)(p_4-p_3) + b_5 p_2 p_4\right] \end{aligned} \tag{C.4}$$

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