The data that supports the findings of this study are available within the article.

On the Herglotz variational problem

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A geometric approach to the Herglotz problem is developed, based on the bundle of affine scalars on the configuration manifold of the given system. The environment, originally introduced to formalize the gauge structure of Lagrangian Mechanics[1], provides the natural setting for the representation of the Herglotz functional as well as for the study of its extremals. Various aspects of the problem are considered: the lagrangian approach, leading to a generalization of the Poincaré-Cartan algorithm; the parametric approach, involving the introduction of an appropriate *super-Lagrangian*; the corresponding hamiltonian and super-hamiltonian counterparts; the relationship between the Herglotz problem and a constrained variational problem; the evaluation of the abnormality index[2] of the resulting extremals; the gauge structure of the theory and the consequent existence of Herglotz's functionals gauge-equivalent to ordinary action functionals.

Keywords: Variational principles in Physics, Herglotz variational problem, Gauge structure of Lagrangian Mechanics

INTRODUCTION

In 1930 Herglotz proposed a variational principle in which the Lagrangian $L(t, q^i, u, \dot{q}^i)$ involved in the definition of the action functional can also depend on the instantaneous value $u(t) = \int_{t_0}^{t} L dt$ of the action itself[3].

The original idea was published in 1979 in Herglotz's collected works[4]. In this connection see also *The Herglotz Lectures on Contact Transformations and Hamiltonian Systems*, published by Guenther and others in 1996[5].

In recent years, Herglotz's variational principle and its application have attracted renewed interest: in 2002, Georgieva, Guenther and Bodurov discussed the subject, obtaining some interesting results in the direction of the first and second Noether theorem[6–8]. In the Conference on Geometry, Integrability and Quantization (Varna, June 2010) Georgieva presented a list of problems admitting a variational formulation in the sense of Herglotz[9].

In 2014 Donchev used Herglotz's principle to obtain variational formulations of the Bôcher equation, the nonlinear Schrödinger equation and other equations of Mathematical Physics[10].

In 2017 Santos, Martins and Torres[11], Garra, Taverna and Torres[12] and Tavares, Almeida and Torres[13] worked on Herglotz's problem considering time delay and fractional derivatives.

In 2020 Zhang summarized the state of art on the subject, discussing various related problems, like Noether symmetries and conservation laws, as well as possible generalizations of the principle to time-delay dynamics, fractional dynamics and time-scale dynamics[14].

In 2023 de Leon, Lainz and Muñoz-Lecanda formulated Herglotz's principle in the geometric environment of contact manifolds, following an approach based on Pontryagin's optimality principle[15].

In this paper we propose a revisitation of the Herglotz problem in the geometrical setup introduced some years ago for a gauge-invariant formulation of Lagrangian Mechanics[1,16].

The construction, briefly reviewed in Sections I, III, involves the introduction of a principal fibre bundle $P \xrightarrow{\pi} \mathcal{V}_{n+1}$ over the configuration space time, referred to as the bundle of *affine scalars*.

Strictly associated with P are two principal bundles over the velocity space, respectively called the *lagrangian* and the *co-lagrangian* bundle, as well as two principal bundles over the phase space, called the *hamiltonian* and the *co-hamiltonian* bundle.

The resulting environment is applied to the study of the Herglotz problem, interpreting the Lagrangian as the function involved in the representation of a map $\dot{u} = L(t, q^i, u, \dot{q}^i)$ of the co-lagrangian bundle into the first-jet space $j_1(P, \mathbb{R})$.

Various approaches to the study of the extremals of the resulting action functional are developed. Those more closely related to the lagrangian setup are presented in Section II: a direct method, viewed as a natural generalization of the Poincaré-Cartan formalism, and a parametric method, based on the conversion of the Herglotz problem into a free variational problem involving a function $\tilde{L}$ over the tangent space $T(P)$, called the *super-Lagrangian*.

The argument is completed by an analysis of the behaviour of the formalism under arbitrary transformations of the fibred coordinates in the manifold P, and by a comparison of the parametric approach with the method of Lagrange multipliers.

The hamiltonian and super-hamiltonian aspects of the problem are presented in Section III, through a revisitation of the Legendre transformation, suitably adapted to the context in study. The results are compared with those obtained applying Pontryagin's maximum principle[2,17–19] to the Herglotz problem, viewed as a constrained variational problem. The abnormality index of the extremals is explicitly evaluated.

The invariance properties of the Herglotz functional are discussed in Section IV, proving the existence of a group of gauge transformations isomorphic to the group of diffeomorphisms of the bundle $P \to \mathcal{V}_{n+1}$ fibred over the identity map of $\mathcal{V}_{n+1}$. The result is applied to the characterization of the class of Herglotz Lagrangians gauge-equivalent to ordinary ones, i.e. to Lagrangians giving rise to evolution equations not involving the variable u.

I. PRELIMINARIES

A geometric framework for the description of the gauge aspects of Classical Mechanics has been described in Ref.[1] For ease of reading, a brief outline of the topic is reported below.
(i) The basic environment is a double fibration $P \xrightarrow{\pi} \mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, where:

- $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, called the configuration space-time, is an $(n+1)$-dimensional fiber bundle, referred to fibred coordinates $t, q^1, \ldots, q^n$, with t representing *absolute time*;
- $P \xrightarrow{\pi} \mathcal{V}_{n+1}$ is a principal fiber bundle with structural group $(\mathbb{R}, +)$, called the bundle of *affine scalars* over $\mathcal{V}_{n+1}$, diffeomorphic, in a non canonical way, to the cartesian product $\mathcal{V}_{n+1} \times \mathbb{R}$.

Unless otherwise specified, P will be referred to coordinates $t, q^1, \ldots, q^n, u$, with u representing a *trivialization* of P. In this way, the fundamental vector field of P coincides with the field $\frac{\partial}{\partial u}$, while the group of allowed coordinate transformations reads

$$\bar{t} = t, \quad \bar{q}^k = \bar{q}^k(t, q^1, \ldots, q^n), \quad \bar{u} = u + f(t, q^1, \ldots, q^n), \tag{1}$$

with $\det \frac{\partial(\bar{q}^1, \ldots, \bar{q}^n)}{\partial(q^1, \ldots, q^n)} \neq 0$. The coordinates in the tangent space $T(P)$ will be denoted by t, q^i, u, t', q'^i, u'.

(ii) The (pull-back of the) absolute time function provides a fibration $P \xrightarrow{t} \mathbb{R}$. Closely related with the latter are three significant bundles, namely:

- the first-jet space $j_1(P, \mathbb{R})$, referred to jet-coordinates $t, q^i, u, \dot{q}^i, \dot{u}$;
- the *lagrangian bundle* $\mathfrak{L}(\mathcal{V}_{n+1})$, quotient of $j_1(P, \mathbb{R})$ with respect to the 1-parameter group of diffeomorphisms generated by the field $\frac{\partial}{\partial u}$, referred to coordinates $t, q^i, \dot{q}^i, \dot{u}$;
- the *co-lagrangian bundle* $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$, quotient of $j_1(P, \mathbb{R})$ with respect to the 1-parameter group generated by the field $\frac{\partial}{\partial \dot{u}}$, referred to coordinates $t, q^i, u, \dot{q}^i$.

Denoting by $j_1(\mathcal{V}_{n+1})$ the first-jet space $j_1(\mathcal{V}_{n+1}, \mathbb{R})$, referred to coordinates $t, q^i, \dot{q}^i$, the previous definitions give rise to the commutative diagram[1]

$$\begin{array}{ccc} j_1(P, \mathbb{R}) & \xrightarrow{\pi} & \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathfrak{L}(\mathcal{V}_{n+1}) & \xrightarrow{\pi} & j_1(\mathcal{V}_{n+1}) \end{array}$$

in which all arrows denote principal fibrations. Given a function f, either in $\mathscr{F}(\mathcal{V}_{n+1})$ or in $\mathscr{F}(P)$, the symbolic time derivative $\frac{df}{dt}$ will be denoted by $\dot{f}$.

Further significant aspects are:

- the identification of $j_1(P, \mathbb{R})$ with the hyperplane $t' = 1$ in the tangent space $T(P)$, and the consequent existence of a fibration $\nu : T_+(P) \to j_1(P, \mathbb{R})$ of the open submanifold $T_+(P) = \{X \,|\, X \in T(P), \langle X, dt \rangle > 0\}$ over the first-jet space $j_1(P, \mathbb{R})$, based on the prescription $\nu(X) = \frac{X}{\langle X, dt \rangle}$ and described in coordinates as $\nu^*(\dot{q}^i) = \frac{q'^i}{t'}$, $\nu^*(\dot{u}) = \frac{u'}{t'}$;
- the existence of an anti-derivation d_v of the Grassmann algebra over $j_1(P, \mathbb{R})$, known

as the *fiber differential* [1,20], uniquely defined by the prescriptions

$$d_v f := \frac{\partial f}{\partial \dot q^k}\big(dq^k - \dot q^k\,dt\big) + \frac{\partial f}{\partial \dot u}\big(du - \dot u\,dt\big) \qquad \forall\ f \in \mathscr{F}(j_1(P,\mathbb{R}));$$

$$d_v \cdot d + d \cdot d_v = -dt \wedge d.$$

For brevity, the notation $\omega^k := dq^k - \dot q^k\,dt$ will often be used.

(iii) In ordinary Lagrangian Mechanics, the invariance of Lagrange's equations under arbitrary transformations $L' = L + \dot f$, $f \in \mathscr{F}(\mathcal{V}_{n+1})$ is conveniently accounted for [1] by interpreting the Lagrangian as the representation of a section $\dot u = L(t, q^i, \dot q^i)$ of the bundle $\mathfrak{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ or, what is the same, of a section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P,\mathbb{R})$ *equivariant* with respect to the action of the 1-parameter group of diffeomorphisms generated by the field $\frac{\partial}{\partial u}$.

The same approach is also suited to a geometrization of the Herglotz problem: to this end one has simply to drop the equivariance requirement, extending the analysis to the totality of sections $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P,\mathbb{R})$ described in coordinates as $\dot u = L(t, q^i, u, \dot q^i)$.

The function L involved in the representation of ℓ is still called a Lagrangian: more specifically an *ordinary Lagrangian* in the equivariant case, corresponding to $\frac{\partial L}{\partial u} = 0$, and a *Herglotz Lagrangian* in the opposite case.

Under a change $\bar u = u + f(t, q^i)$ of the trivialization of P, the representation of ℓ transforms into $\dot{\bar u} = L(t, q^i, \bar u - f, \dot q^i) + \dot f := \bar L(t, q^i, \bar u, \dot q^i)$. When L is an ordinary Lagrangian, $\bar L$ is also an ordinary Lagrangian, gauge-equivalent to L in the usual sense.

The section ℓ is called *regular* if and only if the associated Lagrangian $L = \ell^*(\dot u)$ satisfies the gauge invariant condition $\det\left(\frac{\partial^2 L}{\partial \dot q^i\,\partial \dot q^j}\right) \neq 0$.

(iv) After these preliminaries, let us focus on the formulation of the Herglotz problem. To this end, in addition to the section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P,\mathbb{R})$, we introduce another section $\sigma : \mathcal{V}_{n+1} \to P$, described in coordinates as $u = s(t, q^i)$. By means of the pair (ℓ, σ), every curve $\gamma : q^i = q^i(t)$, $t_0 \le t \le t_1$ in $\mathcal{V}_{n+1}$ can be lifted to a curve $\hat\gamma : q^i = q^i(t), u = u(t)$ in P, with the function $u(t)$ determined by the ordinary differential equation

$$\frac{du}{dt} = L\left(t, q^i(t), u, \frac{dq^i}{dt}\right) \tag{2a}$$

with initial data $u(t_0) = s(t_0, q^i(t_0))$ or, equivalently, by the integral equation

$$u(t) = u(t_0) + \int_{t_0}^{t} L\left(t, q^i(t), u, \frac{dq^i}{dt}\right) dt. \tag{2b}$$

The functional I assigning to each curve γ in $\mathcal{V}_{n+1}$ the difference $I[\gamma] = u(t_1) - u(t_0)$ is called the *action integral* determined by the pair ℓ, σ. When the section ℓ is ordinary, and only then, the value of $I[\gamma]$ is independent of the choice of σ.

In any case, given the necessary input, the Herglotz problem consists in the determination of the extremals $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ of the functional I with respect to deformations of γ preserving the endpoints $\gamma(t_0)$, $\gamma(t_1)$, and therefore also the initial value $u(t_0)$.

The issue is clearly equivalent to identifying the extremals of the functional

$$\hat{I}\,[\hat{\gamma}] = \int_{t_0}^{t_1} L\left(t, q^i, u, \dot{q}^i\right) dt$$

within the family of curves $\hat{\gamma} : [t_0, t_1] \to P$ satisfying the non-holonomic constraint $\dot{u} = L\left(t, q^i, u, \dot{q}^i\right)$ and joining the initial point $\hat{\gamma}(t_0) = \sigma(\gamma(t_0))$ with a final point $\hat{\gamma}(t_1)$ varying along the fiber $\pi^{-1}(\gamma(t_1))$.

The latter is not a variational problems with fixed endpoints. However, since its aim is to characterize the curves along which the first variation of the difference $\hat{I}[\hat{\gamma}] = u(t_1) - u(t_0)$ is zero, it can be approached with the standard techniques of constrained variational calculus, requiring the vanishing of the first variation $\delta \hat{I}[\hat{\gamma}]$ with respect to the totality of admissible infinitesimal deformations null at both endpoints.

We will return to this aspect in Subsections II B and III B. The analysis will show that the fact that the vanishing of $\delta u(t_1)$ is not due to the assignment of the value $u(t_1)$ but to the requirement of stationarity of $\hat{I}[\hat{\gamma}]$ does not affect the characterization of the extremals, but their *normality*, assigning them an abnormality index equal to 1.

II. THE LAGRANGIAN SETUP

A. The direct approach

(i) Following the scheme outlined in Section I, we start with an analysis of the functional $I[\gamma] = u(t_1) - u(t_0)$ acting on curves $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$, with the function $u(t)$ given by eq. (2b). Every finite deformation $\gamma_\xi : q^i = \varphi^i(t, \xi)$ of the curve γ gives rise to a variation

$$u(t, \xi) = u(t_0) + \int_{t_0}^{t} L\left(t, \varphi^i(t, \xi), u(t, \xi), \frac{\partial \varphi^i}{\partial t}\right) dt.$$

In particular, given an infinitesimal deformation $X = X^i(t) \left(\frac{\partial}{\partial q^i}\right)_\gamma$, the corresponding variation $U := \left.\frac{\partial u(t,\xi)}{\partial \xi}\right|_{\xi=0}$ obeys the evolution equation

$$\frac{dU}{dt} = \frac{\partial L}{\partial q^k}\,X^k + \frac{\partial L}{\partial \dot q^k}\,\frac{dX^k}{dt} + \frac{\partial L}{\partial u}\,U. \tag{3}$$

The latter entails the identification

$$U(t_1) - U(t_0) = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial q^k}\,X^k + \frac{\partial L}{\partial \dot q^k}\,\frac{dX^k}{dt} + \frac{\partial L}{\partial u}\,U\right)dt = \left.\frac{dI[\gamma_\xi]}{d\xi}\right|_{\xi=0} \tag{4}$$

characterizing the extremals of the Herglotz functional as curves $\gamma : q^i = q^i(t)$ along which the solution of eq. (3) with initial value $U(t_0) = 0$ satisfies $U(t_1) = 0$ for all infinitesimal deformations $X = X^i(t)\left(\frac{\partial}{\partial q^i}\right)_\gamma$ vanishing at the endpoints.

To speed up the subsequent analysis, in addition to the function $u(t)$, along each curve γ we introduce the auxiliary function $g(t) = \exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial u}\,dt\right)$. We have then the identities

$$\frac{dg}{dt} = -g\,\frac{\partial L}{\partial u}\,, \qquad \frac{d}{dt}\,(g\,U) = -g\,\frac{\partial L}{\partial u}\,U + g\,\frac{dU}{dt} = g\left(\frac{\partial L}{\partial q^k}\,X^k + \frac{\partial L}{\partial \dot q^k}\,\dot X^k\right), \tag{5}$$

whence also

$$\begin{aligned} g(t_1)\,U(t_1) &= \int_{t_0}^{t_1} g\left(\frac{\partial L}{\partial q^k}\,X^k + \frac{\partial L}{\partial \dot q^k}\,\dot X^k\right)dt = \\ &= \int_{t_0}^{t_1}\left[g\,\frac{\partial L}{\partial q^k} - \frac{d}{dt}\left(g\,\frac{\partial L}{\partial \dot q^k}\right)\right]X^k dt = \int_{t_0}^{t_1} g\left(\frac{\partial L}{\partial q^k} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot q^k} + \frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^k}\right)X^k dt. \end{aligned}$$

Being $g(t) \neq 0$, a necessary and sufficient condition for the vanishing of $U(t_1)$ for all $X^i(t)$ null at the endpoints is the validity of the Herglotz equation

$$\frac{d}{dt}\,\frac{\partial L}{\partial \dot q^k} - \frac{\partial L}{\partial q^k} - \frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^k} = 0. \tag{6}$$

Parenthetically we observe that, along any extremal curve, eq. (6) yields the relation

$$\frac{d}{dt}\left[g\left(U - \frac{\partial L}{\partial \dot q^k}\,X^k\right)\right] = g\left(\frac{\partial L}{\partial q^k}\,X^k + \cancel{\frac{\partial L}{\partial \dot q^k}\,\dot X^k} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot q^k}\,X^k - \cancel{\frac{\partial L}{\partial \dot q^k}\,\dot X^k} + \frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^k}\,x^k\right) = 0.$$

which, together with $U(t_0) = X^i(t_0)$, provides the equality

$$U = \frac{\partial L}{\partial \dot q^k}\,X^k. \tag{7}$$

Remark 1. *If we introduce the quantities* $\pi_i := g\,\frac{\partial L}{\partial \dot q^i}$, $\pi_0 := g\left(L - \dot q^i\,\frac{\partial L}{\partial \dot q^i}\right)$, *eqs. (2a), (5), (6) yield the evolution equations*

$$\frac{d\pi_i}{dt} = -g\,\frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^i} + g\,\frac{d}{dt}\,\frac{\partial L}{\partial \dot q^i} \;=\; g\,\frac{\partial L}{\partial q^i} \tag{8a}$$

$$\begin{aligned}\frac{d\pi_0}{dt} &= -g\,\frac{\partial L}{\partial u}\left(\cancel{L} - \frac{\partial L}{\partial \dot q^i}\dot q^i\right) + g\left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^i}\,\dot q^i + \cancel{\frac{\partial L}{\partial \dot q^i}}\,\ddot q^i + \cancel{\frac{\partial L}{\partial u}}\,L - \frac{d}{dt}\,\frac{\partial L}{\partial \dot q^i}\,\dot q^i - \cancel{\frac{\partial L}{\partial \dot q^i}}\,\ddot q^i\right)\\ &= g\left[\frac{\partial L}{\partial t} - \left(\frac{d}{dt}\,\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^i}\right)\dot q^i\right] = g\,\frac{\partial L}{\partial t}\,,\end{aligned} \tag{8b}$$

pointing out a correlation between cyclicality of the variables q^i, t *and conservation laws.*

More generally, eqs. (8a,b) may be combined into a Noether-type theorem. To this end, let $X = X^0\,\frac{\partial}{\partial t} + X^i\,\frac{\partial}{\partial q^i}$ *(with* $X^0 =$ *const.) be a vector field over* $\mathcal{V}_{n+1}$ *satisfying the condition*

$$X^0\,\frac{\partial L}{\partial t} + X^i\,\frac{\partial L}{\partial q^i} + \dot X^i\,\frac{\partial L}{\partial \dot q^i} \;= 0. \tag{9}$$

Then, on account of eqs. (8), (9), the quantity $X^i\pi_i + X^0\pi_0 = g\left[X^i\,\frac{\partial L}{\partial \dot q^i} + X^0\left(L - \dot q^i\,\frac{\partial L}{\partial \dot q^i}\right)\right]$ *obeys the conservation law*

$$\frac{d}{dt}\big(X^i\pi_i + X^0\pi_0\big) = \dot X^i\pi_i + X^i g\,\frac{\partial L}{\partial q^i} + X^0 g\,\frac{\partial L}{\partial t} = g\left(\dot X^i\,\frac{\partial L}{\partial \dot q^i} + X^i\,\frac{\partial L}{\partial q^i} + X^0\,\frac{\partial L}{\partial t}\right) = 0.$$

It is worth noticing that, due to the presence of the term $g = \exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial u}\,dt\right)$, *the conserved quantity thus obtained are not functions on the manifold* $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$ *but* functionals, *to be evaluated along the evolution of the system. Therefore, they do not represent first integrals in the strict sense of the term.*

(ii) An explicitly gauge-invariant approach to eq. (6) is obtained by assigning a primary role to the section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$.

Denoting by $\varphi = \dot u - L(t, q^i, u, \dot q^i)$ the corresponding trivialization of $j_1(P, \mathbb{R})$, the fiber differential $d_v\varphi = du - \dot u\,dt - \frac{\partial L}{\partial \dot q^k}\,\omega^k$, pulled back to $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$, determines a 1-form

$$\vartheta = \ell^*\big(d_v\varphi\big) = du - L dt - \frac{\partial L}{\partial \dot q^k}\,\omega^k, \tag{10}$$

whose opposite may be viewed as the natural generalization of the Poincaré-Cartan 1-form associated with L.[21]

Theorem 1. *Let* $\mathcal{I}$ *denote the differential ideal generated by the* 1*-form* (10)*. Then, under the regularity assumption* $\det\left(\frac{\partial^2 L}{\partial \dot q^i\,\partial \dot q^j}\right) \neq 0$, *the integral lines of the characteristic distribution associated with* $\mathcal{I}$ *are (lifts of) solutions of the Herglotz equation.*

Proof. The ideal $\mathcal{I}$ is identical to the ideal generated by ϑ itself and by the 2-form

$$\Omega := \frac{\partial}{\partial u} \rfloor \left(\vartheta \wedge d\vartheta\right) = d\vartheta - \vartheta \wedge \mathscr{L}_{\partial/\partial u}\vartheta =$$
$$= \left[\left(\frac{\partial L}{\partial q^k} + \frac{\partial L}{\partial u}\frac{\partial L}{\partial \dot q^k}\right)dt - d\left(\frac{\partial L}{\partial \dot q^k}\right) + \frac{\partial^2 L}{\partial \dot q^k\,\partial u}\,\vartheta\right] \wedge \omega^k. \tag{11}$$

The associated characteristic distribution includes the totality of vector fields $Z \in \mathcal{D}^1(\mathfrak{L}^{(c)}(\mathcal{V}_{n+1}))$ satisfying the conditions

$$Z \rfloor \vartheta = Z \rfloor \left(du - L dt - \frac{\partial L}{\partial \dot q^k}\,\omega^k\right) = 0 \tag{12a}$$

$$Z \rfloor \Omega = Z \rfloor \left\{\left[\left(\frac{\partial L}{\partial q^k} + \frac{\partial L}{\partial u}\frac{\partial L}{\partial \dot q^k}\right)dt - d\left(\frac{\partial L}{\partial \dot q^k}\right) + \frac{\partial^2 L}{\partial \dot q^k\,\partial u}\,\vartheta\right] \wedge \omega^k\right\} = \alpha\vartheta. \tag{12b}$$

Eqs. (11), (12) and the regularity assumption imply the relations

$$\frac{\partial}{\partial u} \rfloor \Omega = 0 \tag{13a}$$

$$\alpha = \frac{\partial}{\partial u} \rfloor \left(Z \rfloor \Omega\right) = -Z \rfloor \left(\frac{\partial}{\partial u} \rfloor \Omega\right) = 0 \qquad \Longrightarrow \qquad Z \rfloor \Omega = 0 \tag{13b}$$

$$0 = \frac{\partial}{\partial \dot q^r} \rfloor \left(Z \rfloor \Omega\right) = -Z \rfloor \left(\frac{\partial}{\partial \dot q^r} \rfloor \Omega\right) = -\frac{\partial^2 L}{\partial \dot q^r\,\partial \dot q^k}\, Z \rfloor \omega^k \qquad \Longrightarrow \qquad Z \rfloor \omega^k = 0 \tag{13c}$$

$$Z(u) = Z(t)L \tag{13d}$$

$$Z(t)\left(\frac{\partial L}{\partial q^k} + \frac{\partial L}{\partial u}\frac{\partial L}{\partial \dot q^k}\right) - Z\left(\frac{\partial L}{\partial \dot q^k}\right) = 0. \tag{13e}$$

A straightforward check shows that every non-zero solution of eqs. (13c,d,e) satisfies $Z(t) \neq 0$. Up to a multiplicative factor, we can therefore set $Z(t) = 1$. With this choice, eqs. (13c,d,e) reduce to

$$Z(u) = L, \qquad Z(q^k) = \dot q^k, \qquad Z\left(\frac{\partial L}{\partial \dot q^k}\right) - \frac{\partial L}{\partial q^k} - \frac{\partial L}{\partial u}\frac{\partial L}{\partial \dot q^k} = 0,$$

whence the thesis. The vector field $Z = \frac{\partial}{\partial t} + \dot q^i \frac{\partial}{\partial q^i} + L\frac{\partial}{\partial u} + Z^i \frac{\partial}{\partial \dot q^i}$ thus obtained is called the *dynamical flow* associated with ϑ. □

(iii) The content of Theorem 1 is further highlighted by noting that the differential ideal generated by the 1-form ϑ is also generated by any multiple of it. This is related to the fact that, in principle, the trivialization $\varphi \in \mathscr{F}(j_1(P,\mathbb{R}))$ associated with the section ℓ may be described in *any* coordinate system: the restricted choice adopted so far, based on the identification of u with a trivialization of the bundle $P \to \mathcal{V}_{n+1}$, although significant in many respects (e.g. in the definition $L = \ell^*(\dot{u})$ of the Lagrangian) is not mandatory at all.

For example, starting with a coordinate system t, q^i, u of the restricted type and given a function $\bar{u} = \bar{u}(t, q^i, u)$ satisfying $\frac{\partial \bar{u}}{\partial u} \neq 0$, we may adopt $t, q^i, \bar{u}$ as fibred coordinates in P, and extend them to coordinates $t, q^i, \bar{u}, \dot{q}^i$ in $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$ and $t, q^i, \bar{u}, \dot{q}^i, \dot{\bar{u}}$ in $j_1(P, \mathbb{R})$, with $\dot{\bar{u}} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}}{\partial q^k}\,\dot{q}^k + \frac{\partial \bar{u}}{\partial u}\,\dot{u}$.

The description of the section ℓ takes then the form $\dot{\bar{u}} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}}{\partial q^k}\,\dot{q}^k + \frac{\partial \bar{u}}{\partial u}\,L := \bar{L}$, with $\bar{L}$ expressed in terms of the variables $t, q^i, \bar{u}, \dot{q}^i$ through the relation $u = u(t, q^i, \bar{u})$ implicitly defined by $\bar{u} = \bar{u}(t, q^i, u)$.

Recalling the representation $\varphi = \dot{u} - L$ of the trivialization of $j_1(P, \mathbb{R})$ induced by ℓ we have then the relation

$$\dot{\bar{u}} - \bar{L} \,=\, \frac{\partial \bar{u}}{\partial u}\,(\dot{u} - L) \,=\, \frac{\partial \bar{u}}{\partial u}\,\varphi\,.$$

According to our conventions, $\bar{L}$ is not a Lagrangian, but merely a function on the manifold $j_1(P, \mathbb{R})$. Nevertheless, the equality

$$\ell^*\Big[d_v(\dot{\bar{u}} - \bar{L})\Big] \,=\, \ell^*\left[\frac{\partial \bar{u}}{\partial u}\,d_v\varphi\right] = \frac{\partial \bar{u}}{\partial u}\,\vartheta$$

indicates that the 1-forms ϑ and $\ell^*\Big[d_v(\dot{\bar{u}} - \bar{L})\Big] = d\bar{u} - \bar{L}dt - \frac{\partial \bar{L}}{\partial \dot{q}^k}\,\omega^k$ determine the same differential ideal, and therefore also the same characteristic distribution in $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$.

Proceeding as in the proof of Theorem 1, with L and u replaced by $\bar{L}$ and $\bar{u}$, we conclude:

Proposition 1. *Independently of the choice of the coordinate $\bar{u}$ along the fibres of P, given any section $\ell\colon \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$, the extremals of the Herglotz functional are determined by the differential equations*

$$\frac{d\bar{u}}{dt} = \bar{L}\,, \qquad \frac{d}{dt}\,\frac{\partial \bar{L}}{\partial \dot{q}^k} - \frac{\partial \bar{L}}{\partial q^k} - \frac{\partial \bar{L}}{\partial \bar{u}}\,\frac{\partial \bar{L}}{\partial \dot{q}^k} = 0,$$

with $\bar{L}(t, q^i, \bar{u}, \dot{q}^i) = \ell^*(\dot{\bar{u}})$.

Proposition 1 is essentially a restatement of the fact that, in its original formulation, the Herglotz problem does not impose any condition on the function u, except the requirement $\frac{du}{dt} = L$. As such, it does not add anything to what was already known at the outset, but is rather a test of self-consistency of the geometric setup.

In what follows we stick to the choice of u as a trivialization of P, thus preserving the definition $L = \ell^*(\dot{u})$ of the Lagrangian and the consequent distinction between ordinary Lagrangians and Herglotz ones. Different choices, when needed, will be explicitly declared.

B. The super-Lagrangian

(i) The central role of the trivialization $\varphi = \dot{u} - L$ is further emphasized making use of the projection $\nu(X) = \frac{X}{\langle X, dt\rangle}$ of the open submanifold $T_+(P) \subset T(P)$ onto the first-jet space $j_1(P, \mathbb{R})$, described in coordinates as $\nu^*(\dot{q}^i) = \frac{q'^i}{t'}$, $\nu^*(\dot{u}) = \frac{u'}{t'}$.

To analyse this aspect, we lift the opposite $-\varphi$ to a function $\tilde{L} := \nu^*(-\varphi) \in \mathscr{F}(T_+(P))$, homogeneous of degree 0 in the fiber variables, described in coordinates as

$$\tilde{L} = L\left(t, q^i, u, \frac{q'^i}{t'}\right) - \frac{u'}{t'}\,. \tag{14}$$

The role of $\tilde{L}$, henceforth called the *super-Lagrangian*, is clarified by the following

Theorem 2. *The solutions of the Herglotz problem based on the pair of sections (ℓ, σ) are projections of solutions $t = t(\tau), q^i = q^i(\tau), u = u(\tau)$ of the Lagrange equations determined by the super-Lagrangian (14), completed with the initial and boundary data $t(\tau_0) = t_0$, $t(\tau_1) = t_1$, $q^i(\tau_0) = q^i(t_0)$, $q^i(\tau_1) = q^i(t_1)$, $u(\tau_0) = u_0$, $\tilde{L}_{|\tau=\tau_0} = 0$.*

Proof. The Lagrange equations generated by $\tilde{L}$ read

$$0 = \frac{d}{d\tau}\frac{\partial \tilde{L}}{\partial u'} - \frac{\partial \tilde{L}}{\partial u} = -\frac{d}{d\tau}\left(\frac{1}{t'}\right) - \frac{\partial L}{\partial u}\,, \tag{15a}$$

$$0 = \frac{d}{d\tau}\frac{\partial \tilde{L}}{\partial q'^k} - \frac{\partial \tilde{L}}{\partial q^k} = \frac{d}{d\tau}\left(\frac{1}{t'}\frac{\partial L}{\partial \dot{q}^k}\right) - \frac{\partial L}{\partial q^k} = \frac{1}{t'}\frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial u}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\,, \tag{15b}$$

$$0 = \frac{d}{d\tau}\frac{\partial \tilde{L}}{\partial t'} - \frac{\partial \tilde{L}}{\partial t} = \frac{d}{d\tau}\left(-\frac{q'^i}{t'^2}\frac{\partial L}{\partial \dot{q}^i} + \frac{u'}{t'^2}\right) - \frac{\partial L}{\partial t}\,. \tag{15c}$$

On account of the identity $\frac{d}{d\tau} = t'\frac{d}{dt}$, eq. (15a) entails the relation

$$t'\frac{d}{dt}\frac{1}{t'} = -\frac{\partial L}{\partial u} \qquad \Longrightarrow \qquad \frac{1}{t'} = \frac{d\tau}{dt} = A\, e^{-\int_{t_0}^{t}\frac{\partial L}{\partial u}dt}\,,$$

with the constant A determined by the conditions $t(\tau_0) = t_0$, $t(\tau_1) = t_1$.

In particular, no matter how the interval $[\tau_0, \tau_1]$ is chosen, the inequality $t(\tau_0) < t(\tau_1)$, together with the positivity of the exponential, implies the positivity of both A and t'. This ensures the self-consistency of the algorithm, i.e. the fact that the lifts of the solutions belong to the submanifold $T_+(P)$, as well as the possibility of using t rather than τ as a parameter along the curves.

Written in terms of t, eq. (15b) becomes identical to the Herglotz equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial u}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} = 0.$$

Moreover, in view of eqs. (15a,b), taking the non vanishing of t' and the vanishing of $\frac{\partial \tilde L}{\partial \tau}$ into account, eq. (15c) may be replaced by the equation

$$0 = u'\left(\frac{d}{d\tau}\frac{\partial \tilde L}{\partial u'} - \frac{\partial \tilde L}{\partial u}\right) + q'^k\left(\frac{d}{d\tau}\frac{\partial \tilde L}{\partial q'^k} - \frac{\partial \tilde L}{\partial q^k}\right) + t'\left(\frac{d}{d\tau}\frac{\partial \tilde L}{\partial t'} - \frac{\partial \tilde L}{\partial t}\right) = \frac{d\tilde H}{d\tau}, \tag{16}$$

$\tilde H = u'\frac{\partial \tilde L}{\partial u'} + q'^k\frac{\partial \tilde L}{\partial q'^k} + t'\frac{\partial \tilde L}{\partial t'} - \tilde L$ denoting the super-Hamiltonian associated with $\tilde L$.

On the other hand, the homogeneity of degree 0 of $\tilde L$ with respect to the variables t', q'^k, u' entails the equality $\tilde H = -\tilde L$. Eq. (16) is therefore equivalent to the conservation law $\frac{d\tilde L}{d\tau} = 0$. The latter, together with the initial condition $\tilde L_{|\tau=\tau_0} = 0$, implies the validity of the relation

$$\tilde L(t, q^i, u, t', q'^i, u') = L - \frac{u'}{t'} = L - \frac{du}{dt} = 0$$

along the whole evolution.

Summing up, we conclude that the Lagrange equations (15), together with the prescribed initial and boundary data, reproduce the content of the Herglotz algorithm. □

Theorem 2, together with Hamilton's principle, indicates that the super-Lagrangian approach transforms the Herglotz problem into a *free* variational problem for the action integral $\int_{\tau_0}^{\tau_1} \tilde L\, d\tau$, with the difference $\dot u - L$ converted into a first integral and the non-holonomic constraint $\dot u = L$ reduced to a condition on the initial data.

The same conclusion holds if, chosen t as the independent variable, we include the function $\tau(t)$ among the unknowns. In this way, using the relation $\frac{1}{t'} = \frac{d\tau}{dt} := \dot\tau$, the action integral reads

$$\int_{\tau_0}^{\tau_1}\left[L\left(t, q^i, u, \frac{q'^i}{t'}\right) - \frac{u'}{t'}\right]d\tau \;=\; \int_{t_0}^{t_1}(L - \dot u)\,\dot\tau\, dt. \tag{17}$$

The extremals of the latter, with $q^i(t_0), q^i(t_1), \tau(t_0), \tau(t_1)$ and $u(t_0)$ fixed, are determined by the Euler-Lagrange equations

$$\frac{d}{dt}(L - \dot u) = 0, \qquad -\frac{d\dot\tau}{dt} - \dot\tau\,\frac{\partial L}{\partial u} = 0, \qquad \frac{d}{dt}\left(\dot\tau\,\frac{\partial L}{\partial \dot q^k}\right) - \dot\tau\,\frac{\partial L}{\partial q^k} = 0, \tag{18}$$

clearly equivalent to eqs. (15), due to the equality $\frac{d}{d\tau} = t'\frac{d}{dt}$.

(ii) The conversion of the Herglotz problem into a free variational problem through the introduction of an auxiliary unknown closely resembles the method of Lagrange multipliers.

This aspect is made explicit by observing that, by a mere change of notation ($1-\lambda$ in place of λ), the action functional involved in Lagrange's method can be converted into the gauge-equivalent one $\int_{t_0}^{t_1}\big[L+(1-\lambda)(\dot u-L)-\dot u\big]dt=\int_{t_0}^{t_1}\lambda(L-\dot u)dt$, identical to the functional (17), up to the formal substitution $\lambda=\dot\tau$.

The resulting Euler-Lagrange equations

$$L-\dot u=0\,,\qquad -\frac{d\lambda}{dt}-\lambda\,\frac{\partial L}{\partial u}=0\,,\qquad \frac{d}{dt}\left(\lambda\,\frac{\partial L}{\partial\dot q^k}\right)-\lambda\,\frac{\partial L}{\partial q^k}=0$$

are then equivalent to eqs. (18), except for a minor difference related to the interpretation of the multiplier: with the identification $\lambda=\dot\tau$, the infinitesimal deformation $\delta\lambda$ is the time-derivative $-\frac{d\delta\tau}{dt}$ of a parent deformation null at the endpoints, and is therefore subject to the condition $\int_{t_0}^{t_1}\delta\lambda=\delta\tau(t_1)-\delta\tau(t_0)=0$. Due to this fact, the vanishing of $\int_{t_0}^{t_1}(\dot u-L)\,\delta\lambda\,dt$ for all admissible $\delta\lambda$'s is ensured by the condition $\dot u-L=\text{const}$.

Conversely, if λ is regarded as an independent object — as in Lagrange's method — the vanishing of $\int_{t_0}^{t_1}(\dot u-L)\,\delta\lambda\,dt$ for arbitrary $\delta\lambda$ requires the stronger condition $\dot u-L=0$.

This dissymmetry is reflected in the fact that, while in the super-lagrangian approach all the unknowns are uniquely determined, in the method of Lagrange multipliers the function $\lambda(t)$ is determined only up to a multiplicative factor.

As we shall see, all these peculiarities are formally accounted for by the result, established in Section III B, that the extremals of the Herglotz functional are *abnormal* in the sense of variational calculus.[2]

III. THE HAMILTONIAN SETUP

A. The direct approach

(i) Paralleling the discussion in Section I, we now focus on the bundles associated with the fibration $P\to\mathcal{V}_{n+1}$. They include[1]

- the first-jet space $j_1(P,\mathcal{V}_{n+1})$, referred to jet-coordinates t,q^i,u,p_0,p_i;
- the *hamiltonian bundle* $\mathcal{H}(\mathcal{V}_{n+1})$, quotient of $j_1(P,\mathcal{V}_{n+1})$ with respect to the 1-parameter group of diffeomorphisms generated by the field $\frac{\partial}{\partial u}$, referred to coordinates t,q^i,p_0,p_i;

- the *co-hamiltonian bundle* $\mathcal{H}^{(c)}(\mathcal{V}_{n+1})$, quotient of $j_1(P, \mathcal{V}_{n+1})$ with respect to the 1-parameter group generated by the field $\frac{\partial}{\partial p_0}$, referred to coordinates t, q^i, u, p_i;

- the *phase space* $\Pi(\mathcal{V}_{n+1})$, at the same time quotient of $\mathcal{H}(\mathcal{V}_{n+1})$ with respect to the 1-parameter group generated by the field $\frac{\partial}{\partial p_0}$ and quotient of $\mathcal{H}^{(c)}(\mathcal{V}_{n+1})$ with respect to the group generated by the field $\frac{\partial}{\partial u}$, referred to coordinates t, q^i, p_i.

The situation is summarized in the commutative diagram

$$\begin{array}{ccc} j_1(P, \mathcal{V}_{n+1}) & \longrightarrow & \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}(\mathcal{V}_{n+1}) & \longrightarrow & \Pi(\mathcal{V}_{n+1}) \end{array}$$

in which all arrows denote principal fibrations.

(ii) Every regular section ℓ determines a *Legendre diffeomorphism* $j_1(P, \mathbb{R}) \xrightarrow{\psi} j_1(P, \mathcal{V}_{n+1})$, uniquely defined by the requirement that the pull-back of the Liouville 1-form of $j_1(P, \mathcal{V}_{n+1})$ coincides with the fiber-differential $d_v(\dot{u} - L)$ of the trivialization of $j_1(P, \mathbb{R})$ induced by ℓ.

In coordinates, this entails the equality

$$du - \dot{u}\, dt - \frac{\partial L}{\partial \dot{q}^k}\big(dq^k - \dot{q}^k dt\big) = \psi^*\big(du - p_0 dt - p_k dq^k\big),$$

corresponding to the representation

$$p_0 = \dot{u} - \frac{\partial L}{\partial \dot{q}^k}\, \dot{q}^k, \qquad p_k = \frac{\partial L}{\partial \dot{q}^k}\,. \tag{19}$$

The diffeomorphism ψ induces a diffeomorphism $\hat{\psi} : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to \mathcal{H}^{(c)}(\mathcal{V}_{n+1})$, expressed in coordinates as $p_k = \frac{\partial L}{\partial \dot{q}^k}$ and satisfying the fibred diagram

$$\begin{array}{ccc} j_1(P, \mathbb{R}) & \xrightarrow{\psi} & j_1(P, \mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) & \xrightarrow{\hat{\psi}} & \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \end{array}$$

Through the latter, every regular section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$ determines a section $h := \psi \cdot \ell \cdot \hat{\psi}^{-1} : \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$, locally represented as $p_0 = -H(t, q^i, u, p_i)$.

The function H, called the Hamiltonian, satisfies the relation

$$H = -h^*(p_0) = -\big(\hat{\psi}^{-1}\big)^* \cdot \ell^* \cdot \psi^*(p_0) = -\big(\hat{\psi}^{-1}\big)^*\left(L - \frac{\partial L}{\partial \dot{q}^k}\, \dot{q}^k\right) = p_k\, \dot{q}^k - L, \tag{20}$$

with all expressions at the right-hand side evaluated in terms of t, q^i, u, p_i through the inverse diffeomorphism $\hat{\psi}^{-1} : \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \to \mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$.

Eq. (20) entails the identity

$$dH = -(\hat{\psi}^{-1})^* \left(dL - \frac{\partial L}{\partial \dot{q}^k} d\dot{q}^k - \dot{q}^k d \frac{\partial L}{\partial \dot{q}^k} \right) = -(\hat{\psi}^{-1})^* \left(\frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial q^k} dq^k + \frac{\partial L}{\partial u} du - \dot{q}^k dp_k \right),$$

mathematically equivalent to the relations

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \qquad \frac{\partial H}{\partial q^k} = -\frac{\partial L}{\partial q^k}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \qquad \frac{\partial H}{\partial u} = -\frac{\partial L}{\partial u}. \tag{21}$$

In particular, according the last equality, the vanishing of $\frac{\partial L}{\partial u}$ implies the vanishing of $\frac{\partial H}{\partial u}$, i.e. the equivariance of the section $h : \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$ under the action of the field $\frac{\partial}{\partial u}$. Consequently, every ordinary lagrangian section induces a section $p_0 = -H(t, q^i, p_i)$ of the phase space $\Pi(\mathcal{V}_{n+1})$ into the hamiltonian bundle $\mathcal{H}(\mathcal{V}_{n+1})$ [1].

(iii) In view of eqs. (19), (21), the Herglotz equation (6) can be written in the form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial u}. \tag{22}$$

The equation $\frac{du}{dt} = L$ can be similarly reformulated as

$$\frac{du}{dt} = -H + p_i \dot{q}^i = -H + p_i \frac{\partial H}{\partial p^i}. \tag{23}$$

Eqs. (22), (23), completed by the relations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \tag{24}$$

provide the hamiltonian counterpart of the Herglotz algorithm. In the resulting framework, the Hamiltonian obeys the Jacobi-type evolution equation

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q^i} \frac{dq^i}{dt} + \frac{\partial H}{\partial p_i} \frac{dp_i}{dt} + \frac{\partial H}{\partial u} \frac{du}{dt} = \frac{\partial H}{\partial t} - H \frac{\partial H}{\partial u}.$$

B. The super-Hamiltonian

(i) An alternative formulation of the Herglotz problem in hamiltonian terms comes from the use of the Legendre transformation $T(P) \to T^*(P)$ induced by the super-Lagrangian (14).

Resuming the notation t, q^i, u, t', q'^i, u' for the coordinates in $T(P)$ and referring the cotangent space $T^*(P)$ to coordinates t, q^i, u, y_0, y_i, y_u, the transformation reads

$$y_u = \frac{\partial \tilde{L}}{\partial u'} = -\frac{1}{t'}, \qquad y_i = \frac{\partial \tilde{L}}{\partial q'^i} = \frac{1}{t'} \frac{\partial L}{\partial \dot{q}^i}, \qquad y_0 = \frac{\partial \tilde{L}}{\partial t'} = -\frac{1}{t'} \left(\frac{\partial L}{\partial \dot{q}^k} \frac{q'^k}{t'} - \frac{u'}{t'} \right), \tag{25a}$$

completed by the expression

$$\tilde{H} = u'\,\frac{\partial \tilde{L}}{\partial u'} + q'^k\,\frac{\partial \tilde{L}}{\partial q'^k} + t'\,\frac{\partial \tilde{L}}{\partial t'} - \tilde{L} = -\tilde{L} \tag{25b}$$

for the super-Hamiltonian.[22]

Eqs. (14), (19), (20), (25a,b) yield the relations

$$\frac{y_i}{y_u} = -\frac{\partial L}{\partial \dot{q}^i} = -p_i\,, \tag{26a}$$

$$\frac{y_0}{y_u} = \frac{\partial L}{\partial \dot{q}^k}\,\dot{q}^k - \frac{u'}{t'} = H + L - \frac{u'}{t'} = H + \tilde{L} = H - \tilde{H}. \tag{26b}$$

From these, expressing everything in coordinates, we get the representation

$$\tilde{H} = H\left(t, q^i, u, -\frac{y_i}{y_u}\right) - \frac{y_0}{y_u}\,. \tag{27}$$

In view of eqs. (26), (27) it is easily seen that the Hamilton equations

$$\frac{dt}{d\tau} = \frac{\partial \tilde{H}}{\partial y_0} = -\frac{1}{y_u}\,, \quad \frac{dq^i}{d\tau} = \frac{\partial \tilde{H}}{\partial y_i} = -\frac{1}{y_u}\,\frac{\partial H}{\partial p_i}\,, \quad \frac{dy_u}{d\tau} = -\frac{\partial \tilde{H}}{\partial u}\,, \quad \frac{dy_i}{d\tau} = -\frac{\partial \tilde{H}}{\partial q^i} \tag{28}$$

reproduce the content of eqs. (22), (24). In a similar way, the equation

$$\frac{du}{d\tau} = \frac{\partial \tilde{H}}{\partial y_u} = \frac{\partial H}{\partial p_i}\,\frac{y_i}{y_u^2} + \frac{y_0}{y_u^2} = \frac{1}{y_u}\left(-\frac{\partial H}{\partial p_i}\,p_i + H - \tilde{H}\right),$$

together with the first integral $\frac{d\tilde{H}}{dt} = 0$, ensures the validity of eq. (23) along any extremal satisfying the initial requirement $\tilde{H}_{|\tau=\tau_0} = -\tilde{L}_{|\tau=\tau_0} = 0$.

(ii) It is worth remarking the close analogy between the correspondences $L \to \tilde{L}$ and $H \to \tilde{H}$ given by eqs. (14), (27), as well as between the equalities $\dot{q}^i = \frac{q'^i}{t'}$, $\dot{u} = \frac{u'}{t'}$ and $p_i = -\frac{y_i}{y_u}$, $p_0 = -\frac{y_0}{y_u}$. The reason relies on the fact that, exactly as it happens with the imbedding $j_1(P,\mathbb{R}) \to T(P)$, the first-jet space $j_1(P,\mathcal{V}_{n+1})$ can be viewed as a submanifold of the cotangent space $T^*(P)$, namely as the affine subbundle formed by the totality of 1-forms ω satisfying $y_u(\omega) = \left\langle \omega, \frac{\partial}{\partial u}\right\rangle = 1$.

This allows to set up a fibration $\nu : T^*_-(P) \to j_1(P,\mathcal{V}_{n+1})$ of the open submanifold $T^*_-(P) = \left\{\omega \in T^*(P), y_u(\omega) < 0\right\}$ onto $j_1(P,\mathcal{V}_{n+1})$, described in coordinates as[23]

$$p_i = -\frac{y_i}{y_u}\,, \quad p_0 = -\frac{y_0}{y_u}\,.$$

Given any section $h : \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$ locally represented as $p_0 = -H(t, q^i, u, p_i)$, we can then lift the associated trivialization $p_0 + H$ to a function $\tilde{H} = \nu^*(p_0 + H) = -\frac{y_0}{y_u} + H\left(t, q^i, u, -\frac{y_i}{y_u}\right) \in \mathscr{F}(T^*_-(P))$.

In this way, if $h = \psi \cdot \ell \cdot \hat{\psi}^{-1}$ is the hamiltonian section associated with the lagrangian section ℓ by the Legendre diffeomorphism (19), the function $\tilde{H}$ determined by h coincides with the super-Hamiltonian (27) induced by $\tilde{L}$ through the Legendre transformation (25).

(iii) Just as the super-lagrangian approach is related to the method of Lagrange multipliers, the super-hamiltonian one reminds the content of another classical tool for the solution of constrained variational problems, known as Pontryagin's *maximum principle*.[17–19]

To analyse this aspect, we denote by $V^*(P)$ the bundle of *virtual* 1-*forms* over P, referred to local coordinates t, q^i, u, y_i, y_u and identified with the quotient of the cotangent space $T^*(P)$ with respect to the equivalence relation $\omega \sim \omega' \iff \omega - \omega' \propto dt$.[2]

We recall that the quotient map $\pi(\omega) = [\omega]$ makes $T^*(P)$ a principal fibre bundle over $V^*(P)$, with fundamental vector field $\frac{\partial}{\partial y_0}$. We then exploit the fact that the extremals of the Herglotz functional are in 1–1 correspondence with the solutions of the Hamilton equations (28) belonging to the submanifold $\tilde{H} = 0$ and that, by eq. (27), a representation of this submanifold is provided by the section $h_P : V^*(P) \to T^*(P)$ described in coordinates as

$$y_0 = y_u\, H\left(t, q^i, u, -\frac{y_i}{y_u}\right) := -H_P\big(t, q^i, u, y_i, y_u\big). \tag{29}$$

On account of Maupertuis' least action principle, denoted by $\Theta = y_0 dt + y_i dq^i + y_u du$ the Liouville 1-form of $T^*(P)$, the aforesaid solutions are then extremals of the functional $\int_{\tilde{\gamma}} \Theta$ evaluated on curves $\tilde{\gamma} = \tilde{\gamma}(t)$ belonging to the submanifold $h_P\big(V^*(P)\big)$, i.e. of the functional $\int_{\tilde{\gamma}} h_P^*(\Theta) = \int_{\tilde{\gamma}} -H_P\, dt + y_i dq^i + y_u du$ on the manifold $V^*(P)$.

These curves, parameterized in terms of t, are solutions of the Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H_P}{\partial y_i}, \qquad \frac{du}{dt} = \frac{\partial H_P}{\partial y_u}, \qquad \frac{dy_i}{dt} = -\frac{\partial H_P}{\partial q^i}, \qquad \frac{dy_u}{dt} = -\frac{\partial H_P}{\partial u} \tag{30}$$

determined by the Hamiltonian H_P.

The same situation occurs in the study of the Hamilton-Jacobi equation: formulated in terms of the Hamiltonian H_P, the latter reads

$$\frac{\partial S}{\partial t} + H_P\left(t, q^i, u, \frac{\partial S}{\partial q^i}, \frac{\partial S}{\partial u}\right) = 0, \tag{31}$$

$S(t, q^i, u)$ denoting Hamilton's principal function.

If we adopt the super-Hamiltonian $\tilde{H}$ and focus on solutions with constant energy $\tilde{H} = 0$ we must instead consider the equation

$$\tilde{H}\left(t, q^i, u, \frac{\partial S}{\partial t}, \frac{\partial S}{\partial q^i}, \frac{\partial S}{\partial u}\right) = 0, \tag{32}$$

with S representing now Hamilton's characteristic function. But, on account of eq. (27), eq. (32) can be written as

$$\frac{1}{\frac{\partial S}{\partial u}}\left[\frac{\partial S}{\partial t} + H_P\left(t, q^i, u, \frac{\partial S}{\partial q^i}, \frac{\partial S}{\partial u}\right)\right] = 0$$

which, by the non vanishing of $\frac{\partial S}{\partial u} = y_u$, is identical to eq. (31). Therefore, both procedures lead to the same differential equation, the only difference being in the interpretation or, more simply, in the denomination of the unknown function.

The relationship between the above discussion and Pontryagin's maximum principle can be highlighted as follows: in Pontryagin's approach, the solution of the Herglotz problem is converted into the search for the extremals of the functional

$$I[\tilde{\gamma}] = \int_{t_0}^{t_1}\left[L + y_i\left(\frac{dq^i}{dt} - \dot{q}^i\right) + y_u\left(\frac{du}{dt} - L\right)\right]dt$$

in the independent variables $t, q^i, u, \dot{q}^i, y_i, y_u$, with y_i, y_u playing the role of "multipliers", needed in order to incorporate the relations $\frac{dq^i}{dt} = \dot{q}^i$, $\frac{du}{dt} = L$ in the extremality conditions.

Writing $y_u + 1$ in place of y_u (a mere change of notation, similar to the one made in Section II B), subtracting a total time derivative $\frac{du}{dt}$ and introducing the function

$$\mathfrak{H} := y_i\,\dot{q}^i + y_u\,L, \tag{33}$$

the expression for $I[\tilde{\gamma}]$ takes the form

$$I[\tilde{\gamma}] = \int_{t_0}^{t_1}\left(-\mathfrak{H} + y_i\,\frac{dq^i}{dt} + y_u\,\frac{du}{dt}\right)dt \;=\; \int_{\tilde{\gamma}} -\mathfrak{H}\,dt + y_i dq^i + y_u du. \tag{34}$$

The extremals of the functional (34) satisfy the equations

$$\frac{\partial \mathfrak{H}}{\partial \dot{q}^i} = y_i + y_u\,\frac{\partial L}{\partial \dot{q}^i} = 0 \tag{35}$$

$$\frac{dq^i}{dt} = \frac{\partial \mathfrak{H}}{\partial y_i}, \qquad \frac{du}{dt} = \frac{\partial \mathfrak{H}}{\partial y_u}, \qquad \frac{dy_i}{dt} = -\frac{\partial \mathfrak{H}}{\partial q^i}, \qquad \frac{dy_u}{dt} = -\frac{\partial \mathfrak{H}}{\partial u}\,. \tag{36}$$

Eqs. (35), formally identical to eqs. (26a), can be solved with respect to the $\dot{q}^i$'s, giving rise to expressions of the form $\dot{q}^i = \dot{q}^i(t, q^i, y_i, y_u)$. Inserting these in eq. (33) and comparing with eqs. (20), (29) we obtain the identification

$$\mathfrak{H}\big(t, q^i, u, \dot{q}^i(t, q^i, y_i, y_u), y_i, y_u\big) = -y_u\left(\frac{\partial L}{\partial \dot{q}^i}\,\dot{q}^i - L\right) = -y_u H = H_P,$$

proving the complete equivalence between eqs. (30) and eqs. (36).

(iv) A noteworthy aspect of the Herglotz problem, viewed as a constrained variational problem, is the *abnormality* of its extremals. To analyse this aspect, referring to [2] for notation and terminology, we note that, in the case in study, the controls z^i are the variables $\dot{q}^i$ themselves, while the representation of the constraints reads $\dot{q}^i = \psi^i(t, q^i, \dot{q}^i)$, $\dot{u} = \psi^u(t, q^i, u, \dot{q}^i)$, with $\psi^i = \dot{q}^i$ and $\psi^u = L$. We then recall that the abnormality index of a kinematically admissible curve $\hat{\gamma} : q^i = q^i(t)$, $u = u(t)$ coincides with the dimension of the vector space spanned by the solutions of the linear homogeneous system

$$\rho_k\,\frac{\partial \psi^k}{\partial \dot{q}^i} + \rho_u\,\frac{\partial \psi^u}{\partial \dot{q}^i} = \rho_i + \rho_u\,\frac{\partial L}{\partial \dot{q}^i} = 0 \tag{37a}$$

$$\frac{d\rho_i}{dt} + \rho_k\,\frac{\partial \psi^k}{\partial q^i} + \rho_u\,\frac{\partial \psi^u}{\partial q^i} = \frac{d\rho_i}{dt} + \rho_u\,\frac{\partial L}{\partial q^i} = 0 \tag{37b}$$

$$\frac{d\rho_u}{dt} + \rho_k\,\frac{\partial \psi^k}{\partial u} + \rho_u\,\frac{\partial \psi^u}{\partial u} = \frac{d\rho_u}{dt} + \rho_u\,\frac{\partial L}{\partial u} = 0 \tag{37c}$$

in the unknowns $\rho_i(t)$, $\rho_u(t)$.

Eq. (37c) admits the solution $\rho_u = A\,e^{-\int_{t_0}^{t} \frac{\partial L}{\partial u}\,dt}$ ($A = \text{const.}$). Moreover, on account of eq. (37c), eqs. (37a,b) can be replaced by the pair of relations

$$\frac{\rho_i}{\rho_u} = -\frac{\partial L}{\partial \dot{q}^i}\,, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = -\frac{d}{dt}\frac{\rho_i}{\rho_u} = -\frac{1}{\rho_u}\frac{d\rho_i}{dt} + \frac{\rho_i}{\rho_u^2}\frac{d\rho_u}{dt} = \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial u}$$

The first of these uniquely determines the ratio $\frac{\rho_i}{\rho_u}$, while the second one holds identically along any extremal, as a consequence of the Herglotz equation.

The space of solutions of the system (37) is therefore 1-dimensional, thus assigning the value 1 to the abnormality index of the extremals of the Herglotz functional.[24]

This feature is reflected in the fact — easily verifiable — that eqs. (35), (36) do not determine the functions $y_i(t)$, $y_u(t)$ uniquely, but only up to a common multiplicative factor. The same argument explains the indeterminacy of the multiplier λ pointed out in Section II B.

IV. GAUGE STRUCTURE OF THE HERGLOTZ FUNCTIONAL

A. Dynamically equivalent lagrangian sections

As a final topic, we discuss the possibility that different sections $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$ determine the same extremal curves in $\mathcal{V}_{n+1}$. To this end, let $P \xrightarrow{\kappa} P$ denote a diffeomorphism of the manifold P fibred over the identity map $\mathcal{V}_{n+1} \xrightarrow{id} \mathcal{V}_{n+1}$, described in coordinates as $\kappa^*(u) = G(t, q^i, u)$, with $\frac{\partial G}{\partial u} > 0$.[25]

The inverse diffeomorphism κ^{-1} is similarly described by $(\kappa^{-1})^*(u) = N(t, q^i, u)$, with G and N satisfying the condition

$$u = G\big(t, q^i, N(t, q^i, u)\big) = N\big(t, q^i, G(t, q^i, u)\big) . \tag{38}$$

The map κ (as well as κ^{-1}) can be raised to a bundle diffeomorphism

$$\begin{array}{ccc} \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) & \xrightarrow{\kappa} & \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ j_1(\mathcal{V}_{n+1}) & = & j_1(\mathcal{V}_{n+1}) \end{array}$$

indicated by the same symbol κ, and to a diffeomorphism $\delta\kappa : j_1(P, \mathbb{R}) \to j_1(P, \mathbb{R})$, fibred on the previous one and described in coordinates as

$$(\delta\kappa)^*(u) = G(t, q^i, u) , \qquad (\delta\kappa)^*(\dot{u}) = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot{q}^k + \frac{\partial G}{\partial u}\,\dot{u} := \dot{G} .$$

Given a regular section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$, let $L(t, q^i, u, \dot{q}^i) = \ell^*(\dot{u})$ denote the corresponding Lagrangian. Then, the composite map $\ell' = \delta\kappa \cdot \ell \cdot \kappa^{-1} : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$ is itself a regular section, with Lagrangian $\ell'^*(\dot{u}) = L'(t, q^i, u, \dot{q}^i)$ given by the equation

$$L' = (\kappa^{-1})^* \cdot \ell^* \cdot (\delta\kappa)^*(\dot{u}) = (\kappa^{-1})^* \cdot \ell^*(\dot{G}) = (\kappa^{-1})^* \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot{q}^k + \frac{\partial G}{\partial u}\,L \right), \tag{39a}$$

more conveniently written as

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot{q}^k + \frac{\partial G}{\partial u}\,L = \kappa^*(L'). \tag{39b}$$

Returning to the original formulation of the Herglotz problem — which, in addition to the section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$, involves a second section $\sigma : \mathcal{V}_{n+1} \to P$, necessary to specify the initial value $u(t_0)$ — we can now state

Theorem 3. *The diffeomorphism $\kappa : P \to P$ transforms the extremals $\hat{\gamma} : [t_0, t_1] \to P$ of the Herglotz functional determined by the pair of sections ℓ, σ into extremals $\hat{\gamma}' = \kappa \cdot \hat{\gamma}$ of the analogous functional determined by the pair $\ell' = \delta\kappa \cdot \ell \cdot \kappa^{-1}$, $\sigma' = \kappa \cdot \sigma$.*

Proof. Preserving the notation $G = \kappa^*(u)$, we adopt t, q^i and $\bar{u} = G(t, q^i, u)$ as fiber coordinates in P, and transform consequently the coordinates in $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$ and in $j_1(P, \mathbb{R})$. In this way, denoting by $h : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to \mathbb{R}^{2n+2}$ and $h' : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to \mathbb{R}^{2n+2}$ the coordinate maps determined by the functions $(t, q^i, u, \dot{q}^i)$ and $(t, q^i, \bar{u}, \dot{q}^i)$ — and also, by abuse of language, the coordinate maps $P \to \mathbb{R}^{n+2}$ determined by the functions (t, q^i, u) and $(t, q^i, \bar{u})$ in P — we have the relation $h' = h \cdot \kappa$, mathematically equivalent to $\kappa = h^{-1} \cdot h'$.

In the coordinate system h', the section ℓ is represented by the pull-back

$$\bar{L} := \ell^*(\dot{\bar{u}}) = \ell^*\left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot{q}^k + \frac{\partial G}{\partial u}\,\dot{u}\right) = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot{q}^k + \frac{\partial G}{\partial u}\,L.$$

Together with eq. (39b), the last expression entails the equality

$$\bar{L} = \kappa^*(L') = L' \cdot \kappa = L' \cdot h^{-1} \cdot h' \implies \bar{L} \cdot h'^{-1} = L' \cdot h^{-1},$$

indicating that the function $\bar{L}$ depends on the variables $t, q^i, \bar{u}, \dot{q}^i$ exactly in the same way as L' depends on $t, q^i, u, \dot{q}^i$.

According to Proposition 1 this means that, up to the exchange $\bar{u} \leftrightarrow u$, the Herglotz equations for the unknowns $t, q^i, \bar{u}$ derivable from $\bar{L}$ are identical to the equations for the unknowns t, q^i, u derivable from L'.

Therefore, if $\hat{\gamma}' : q^i = r^i(t), u = s(t)$ is an extremal of the functional $\int_{t_0}^{t_1} L' dt$ subject to the constraint $\dot{u} = L'$, the curve $\hat{\gamma} : q^i = r^i(t), \bar{u} = s(t)$ is an extremal of the functional $\int_{t_0}^{t_1} \bar{L} dt$ subject to the constraint $\dot{\bar{u}} = \bar{L}$, i.e. a solution of the Herglotz problem determined by the section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathbb{R})$, described in the coordinate system h'.

This entails the identification $h \cdot \hat{\gamma}' = h' \cdot \hat{\gamma}$, whence $\hat{\gamma}' = h^{-1} \cdot h' \cdot \hat{\gamma} = \kappa \cdot \hat{\gamma}$ and therefore also $\hat{\gamma}'(t_0) = \kappa \cdot \hat{\gamma}(t_0) = \kappa \cdot \sigma(t_0, q^i(t_0)) = \sigma'(t_0, q^i(t_0))$, with $\sigma' = \kappa \cdot \sigma$. □

Since the diffeomorphism $\kappa : P \to P$ is fibred over $\mathcal{V}_{n+1}$, the projections on $\mathcal{V}_{n+1}$ of the curves $\hat{\gamma}$ and $\hat{\gamma}' = \kappa \cdot \hat{\gamma}$ coincide: as far as the Herglotz problem is concerned, the pairs (ℓ, σ) and $(\ell', \sigma') = (\delta\kappa \cdot \ell \cdot \kappa^{-1}, \kappa \cdot \sigma)$ determine the same solutions.

In the stated circumstance, the Lagrangians $L = \ell^*(\dot{u})$ and $L' = \ell'^*(\dot{u})$ are said to be *gauge-equivalent.* In the Herglotz framework, the gauge transformations are therefore in 1–1 correspondence with the fibred diffeomorphism of P.

B. Reducible dynamical flows

(i) As established in Section II A, for a non-singular Lagrangian the extremals of the Herglotz functional are integral lines of the dynamical flow $Z = \frac{\partial}{\partial t} + \dot q^i \frac{\partial}{\partial q^i} + L\,\frac{\partial}{\partial u} + Z^i \frac{\partial}{\partial \dot q^i}$ uniquely determined by the Herglotz equations

$$Z\left(\frac{\partial L}{\partial \dot q^k}\right) - \frac{\partial L}{\partial q^k} - \frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^k} = 0\,. \tag{40}$$

A dynamical flow p-related to a vector field on $j_1(\mathcal{V}_{n+1})$, i.e. satisfying $\frac{\partial Z^i}{\partial u} = 0$ is said to be *reducible*. In this special situation, the determination of the unknowns $q^k(t)$ is decoupled from the evaluation of $u(t)$, and constitutes a dynamic problem in the ordinary sense.

Several examples of reducible flows may be found in Georgieva.[9] Others can easily be envisaged, taking advantage of the fact that every dynamical flow determined by a Lagrangian gauge-equivalent to an ordinary one is automatically reducible.

A geometric approach to the subject can be based on the following observations:

a) for non-singular Lagrangians, the fields $\frac{\partial}{\partial u}, Z$ span the characteristic distribution $\mathcal{D}$ associated with the ideal $\mathcal{I}(\Omega)$ generated by the 2-form (11);

b) the vanishing of $\frac{\partial Z^i}{\partial u}$ is mathematically equivalent to $\left[\frac{\partial}{\partial u}, Z\right] = \frac{\partial L}{\partial u}\frac{\partial}{\partial u}$, i.e. to the complete integrability of the distribution $\mathcal{D}$;

c) eq. (40) entails the relation

$$\begin{aligned} Z\left(\frac{\partial^2 L}{\partial u\,\partial \dot q^k}\right) &= \frac{\partial}{\partial u}\left(Z\left(\frac{\partial L}{\partial \dot q^k}\right)\right) - \left[\frac{\partial}{\partial u}, Z\right]\left(\frac{\partial L}{\partial \dot q^k}\right) = \\ &= \frac{\partial}{\partial u}\left(\frac{\partial L}{\partial q^k} + \frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^k}\right) - \frac{\partial L}{\partial u}\,\frac{\partial^2 L}{\partial u\,\partial \dot q^k} - \frac{\partial Z^r}{\partial u}\,\frac{\partial^2 L}{\partial \dot q^k\,\partial \dot q^r}\,. \end{aligned}$$

The vanishing of $\frac{\partial Z^i}{\partial u}$ is therefore equivalent to the request

$$Z\left(\frac{\partial^2 L}{\partial u\,\partial \dot q^k}\right) - \frac{\partial^2 L}{\partial u\,\partial q^k} - \frac{\partial^2 L}{\partial u^2}\,\frac{\partial L}{\partial \dot q^k} = 0\,.$$

By Frobenius Theorem, statement b) ensures that a sufficient condition for reducibility is the differential nature of the ideal $\mathcal{I}(\Omega)$. We will return to this point in Appendix.

At the moment, we focus on statement c). Its content is enhanced by the following

Lemma 1. *A necessary and sufficient condition for a function $g \in \mathscr{F}(\mathfrak{L}^{(c)}(\mathcal{V}_{n+1}))$ to admit a local representation of the form $g = Z(f)$, where $f = f(t, q^i, u)$ is the pull-back of a function on P, is the validity of the relations*

$$\frac{\partial^2 g}{\partial \dot q^k\,\partial \dot q^r} = a_u(t, q^i, u)\,\frac{\partial^2 L}{\partial \dot q^k\,\partial \dot q^r} \tag{41}$$

$$Z\left(\frac{\partial g}{\partial \dot q^k}\right) - \frac{\partial g}{\partial q^k} - \frac{\partial g}{\partial u}\,\frac{\partial L}{\partial \dot q^k} = 0\,. \tag{42}$$

Proof. Necessity: if $g = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k}\,\dot q^k + \frac{\partial f}{\partial u}\,L$, eq. (41) holds identically. Moreover

$$Z\left(\frac{\partial g}{\partial \dot q^k}\right) = Z\left(\frac{\partial f}{\partial q^k} + \frac{\partial f}{\partial u}\,\frac{\partial L}{\partial \dot q^k}\right) = Z\left(\frac{\partial f}{\partial q^k}\right) + Z\left(\frac{\partial f}{\partial u}\right)\frac{\partial L}{\partial \dot q^k} + \frac{\partial f}{\partial u}\,Z\left(\frac{\partial L}{\partial \dot q^k}\right).$$

But, by elementary calculations

$$Z\left(\frac{\partial f}{\partial q^k}\right) = \frac{\partial}{\partial q^k}\,Z(f) - \left[\frac{\partial}{\partial q^k}\,, Z\right](f) = \frac{\partial}{\partial q^k}\,Z(f) - \frac{\partial L}{\partial q^k}\,\frac{\partial f}{\partial u} = \frac{\partial g}{\partial q^k} - \frac{\partial L}{\partial q^k}\,\frac{\partial f}{\partial u}$$

$$Z\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\,Z(f) - \left[\frac{\partial}{\partial u}\,, Z\right](f) = \frac{\partial}{\partial u}\,Z(f) - \frac{\partial L}{\partial u}\,\frac{\partial f}{\partial u} = \frac{\partial g}{\partial u} - \frac{\partial L}{\partial u}\,\frac{\partial f}{\partial u}\,,$$

whence, recalling eq. (40) and the identification $g = Z(f)$

$$Z\left(\frac{\partial g}{\partial \dot q^k}\right) = \frac{\partial g}{\partial q^k} - \cancel{\frac{\partial f}{\partial u}\,\frac{\partial L}{\partial \dot q^k}} + \frac{\partial g}{\partial u}\,\frac{\partial L}{\partial \dot q^k} - \cancel{\frac{\partial f}{\partial u}\,\frac{\partial L}{\partial u}\,\frac{\partial L}{\partial \dot q^k}} + \cancel{\frac{\partial f}{\partial u}\,Z\left(\frac{\partial L}{\partial \dot q^k}\right)}.$$

Sufficiency: on account of eqs. (41), (42), g admits the representation

$$g = a_u(t, q^i, u)\,L + a_0(t, q^i, u) + a_i(t, q^i, u)\dot q^i, \tag{43}$$

with the functions a_u, a_0, a_i satisfying the condition

$$\begin{aligned} &Z\left(a_u\,\frac{\partial L}{\partial \dot q^k} + a_k\right) - \frac{\partial a_u}{\partial q^k}\,L - a_u\,\frac{\partial L}{\partial q^k} - \frac{\partial a_0}{\partial q^k} - \frac{\partial a_i}{\partial q^k}\,\dot q^i - \\ &- \left(\frac{\partial a_u}{\partial u}\,L + a_u\,\frac{\partial L}{\partial u} + \frac{\partial a_0}{\partial u} + \frac{\partial a_i}{\partial u}\dot q^i\right)\frac{\partial L}{\partial \dot q^k} = 0. \end{aligned} \tag{44}$$

In view of eq. (40), trough elementary calculations eq. (44) reduces to

$$\begin{aligned} &\left[\frac{\partial a_u}{\partial t} - \frac{\partial a_0}{\partial u} + \left(\frac{\partial a_u}{\partial q^i} - \frac{\partial a_i}{\partial u}\right)\dot q^i\right]\frac{\partial L}{\partial \dot q^k} + \\ &+ \left(\frac{\partial a_k}{\partial u} - \frac{\partial a_u}{\partial q^k}\right)L + \left(\frac{\partial a_k}{\partial t} - \frac{\partial a_0}{\partial q^k}\right) + \left(\frac{\partial a_k}{\partial q^i} - \frac{\partial a_i}{\partial q^k}\right)\dot q^i = 0. \end{aligned} \tag{45}$$

Deriving with respect to $\dot q^s$ yields

$$\left[\frac{\partial a_u}{\partial t}-\frac{\partial a_0}{\partial u}+\left(\frac{\partial a_u}{\partial q^i}-\frac{\partial a_i}{\partial u}\right)\dot q^i\right]\frac{\partial^2 L}{\partial \dot q^k\,\partial \dot q^s}+$$
$$+\left(\frac{\partial a_u}{\partial q^s}-\frac{\partial a_s}{\partial u}\right)\frac{\partial L}{\partial \dot q^k}+\left(\frac{\partial a_k}{\partial u}-\frac{\partial a_u}{\partial q^k}\right)\frac{\partial L}{\partial \dot q^s}+\frac{\partial a_k}{\partial q^s}-\frac{\partial a_s}{\partial q^k}=0.$$

Separating the symmetric and the antisymmetric parts entails the relations

$$\frac{\partial a_u}{\partial t}-\frac{\partial a_0}{\partial u}+\left(\frac{\partial a_u}{\partial q^i}-\frac{\partial a_i}{\partial u}\right)\dot q^i=0 \quad\Longrightarrow\quad \frac{\partial a_u}{\partial t}-\frac{\partial a_0}{\partial u}=\frac{\partial a_u}{\partial q^i}-\frac{\partial a_i}{\partial u}=0$$
$$\cancel{\left(\frac{\partial a_u}{\partial q^s}-\frac{\partial a_s}{\partial u}\right)}\frac{\partial L}{\partial \dot q^k}+\cancel{\left(\frac{\partial a_k}{\partial u}-\frac{\partial a_u}{\partial q^k}\right)}\frac{\partial L}{\partial \dot q^s}+\frac{\partial a_k}{\partial q^s}-\frac{\partial a_s}{\partial q^k}=0.$$

In view of these, the original equation (45) reduces to $\frac{\partial a_k}{\partial t}-\frac{\partial a_0}{\partial q^k}=0$. Summing up we conclude that, under the stated assumptions, the 1-form $\omega=a_0\,dt+a_i\,dq^i+a_u\,du$ is closed. As such, it admits the local representation $\omega=df \iff a_0=\frac{\partial f}{\partial t},\ a_i=\frac{\partial f}{\partial q^i},\ a_u=\frac{\partial f}{\partial u}$, with $f=f(t,q^i,u)$. Eq. (43) then takes the form

$$g=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial q^i}\,\dot q^i+\frac{\partial f}{\partial u}\,L=Z(f),$$

as required. □

(ii) Returning to the problem of reducibility we now state:

Theorem 4. *The validity of a local relation of the form*

$$\frac{\partial L}{\partial u}=Z(f), \tag{46}$$

with $f=f(t,q^i,u)$ is necessary and sufficient for the Lagrangian L to be gauge-equivalent to an ordinary Lagrangian $L_C(t,q^i,\dot q^i)$.

Proof. Setting $f=-\log\left(\frac{\partial G}{\partial u}\right)$ we have the equality

$$\frac{\partial}{\partial u}\,Z(G)=Z\!\left(\frac{\partial G}{\partial u}\right)+\left[\frac{\partial}{\partial u},Z\right](G)=Z\!\left(\frac{\partial G}{\partial u}\right)+\frac{\partial L}{\partial u}\frac{\partial G}{\partial u}=e^{-f}\left(\frac{\partial L}{\partial u}-Z(f)\right),$$

showing that eq. (46) is mathematically equivalent to the relation $\frac{\partial}{\partial u}\,Z(G)=0$.

But, on account of eq. (39b), under the assumption $\frac{\partial G}{\partial u}>0$ — here identically satisfied by construction — the Lagrangian L is gauge-equivalent to the Lagrangian $(\kappa^{-1})^*\big(Z(G)\big)$ which, for $\frac{\partial}{\partial u}\,Z(G)=0$, coincides with $Z(G)$ itself. The conclusion then follows by setting $L_C=Z(G)$. □

Remark 2. *The content of Theorem 4 is confirmed by the fact that, as a consequence of eq. (46), the Herglotz equation (6) takes the form* $Z\Big(\frac{\partial L}{\partial \dot q^k}\Big) - \frac{\partial L}{\partial q^k} - Z(f)\,\frac{\partial L}{\partial \dot q^k} = 0$, *mathematically equivalent to*

$$Z\left(\frac{\partial}{\partial \dot q^k}\left(e^{-f}\,L\right)\right) - e^{-f}\,\frac{\partial L}{\partial q^k} = 0. \tag{47}$$

Restoring the notation $e^{-f} = \frac{\partial G}{\partial u}$, $Z(G) = L_C(t, q^i, \dot q^i)$ *and employing the relations*

$$Z\left(\frac{\partial G}{\partial q^k}\right) = \frac{\partial}{\partial q^k}\,Z(G) - \left[\frac{\partial}{\partial q^k}\,, Z\right](G) = \frac{\partial L_C}{\partial q^k} - \frac{\partial L}{\partial q^k}\,\frac{\partial G}{\partial u}$$

$$\frac{\partial}{\partial \dot q^k}\left(\frac{\partial G}{\partial u}\,L\right) = \frac{\partial}{\partial \dot q^k}\left(Z(G) - \frac{\partial G}{\partial q^r}\,\dot q^r\right) = \frac{\partial L_C}{\partial \dot q^k} - \frac{\partial G}{\partial q^k}\,,$$

eq. (47) may be rewritten in the form

$$0 = Z\left[\frac{\partial}{\partial \dot q^k}\left(\frac{\partial G}{\partial u}\,L\right)\right] - \frac{\partial G}{\partial u}\,\frac{\partial L}{\partial q^k} = Z\left(\frac{\partial L_C}{\partial \dot q^k} - \frac{\partial G}{\partial q^k}\right) - \frac{\partial G}{\partial u}\,\frac{\partial L}{\partial q^k} = Z\left(\frac{\partial L_C}{\partial \dot q^k}\right) - \frac{\partial L_C}{\partial q^k}\,,$$

showing that the functions $q^k(t)$ *are solutions of the Lagrange equations determined by the ordinary Lagrangian* L_C.

The previous discussion is summarized in the following

Proposition 2. *Given a Lagrangian* $L(t, q^i, u, \dot q^i)$, *the condition*

$$Z\left(\frac{\partial^2 L}{\partial u\,\partial \dot q^k}\right) - \frac{\partial^2 L}{\partial u\,\partial q^k} - \frac{\partial^2 L}{\partial u^2}\,\frac{\partial L}{\partial \dot q^k} = 0 \tag{48a}$$

is necessary and sufficient for the associated dynamical flow to be reducible. In particular, L is gauge-equivalent to an ordinary Lagrangian L_C *if and only if, in addition to eq. (48a), it fulfils the additional relation*

$$\frac{\partial^3 L}{\partial u\,\partial \dot q^k\,\partial \dot q^r} \;=\; a_u\,\frac{\partial^2 L}{\partial \dot q^k\,\partial \dot q^r}\,, \tag{48b}$$

with $a_u = a_u(t, q^i, u)$.

The validity of both conditions (48a,b) is mathematically equivalent to the existence of a function $G(t, q^i, u) \in \mathscr{F}(P)$ *satisfying the equation*

$$\frac{\partial G}{\partial u} \neq 0\,, \qquad \frac{\partial}{\partial u}\left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot q^k + \frac{\partial G}{\partial u}\,L\right) = 0\,. \tag{49}$$

In terms of G, the Lagrangian L_C *gauge-equivalent to L is expressed by the relation*

$$L_C(t, q^i, u, \dot q^i) = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot q^k + \frac{\partial G}{\partial u}\,L. \tag{50}$$

Remark 3. *Eq.* (50) *includes the gauge-equivalence between ordinary Lagrangians, simply restricting the choice of the fibred diffeomorphism* $\kappa : P \to P$ *to the class of principal bundle isomorphisms of* P*, i.e. requiring the condition* $\frac{\partial G}{\partial u} = 1$*. In the stated circumstance, setting* $G = u + h(t, q^i)$ *and renaming* L_C *as* L'*, eq.* (50) *returns the familiar expression*

$$L' = L + \frac{\partial h}{\partial t} + \frac{\partial h}{\partial q^k}\,\dot q^k := L + \frac{dh}{dt}\,.$$

Another indication of the fact that the "classical" Lagrangian gauge is implicitly present in eq. (50) comes from the observation that if $G(t, q^i, u)$ *is a solution of eqs.* (49)*, any other function* $G' = G + h(t, q^i)$ *is itself a solution. The ordinary Lagrangian (50) is therefore defined up to a transformation* $L_C \to L_C + \frac{dh}{dt}$*.*

(iii) According to Proposition 2, the most general Lagrangian $L(t, q^i, u, \dot q^i)$ gauge-equivalent to an ordinary one admits the representation

$$L = \frac{1}{\frac{\partial G}{\partial u}}\left[L_C - \frac{\partial G}{\partial t} - \frac{\partial G}{\partial q^k}\,\dot q^k\right] \tag{51}$$

for arbitrary $L_C = L_C(t, q^i, \dot q^i)$ and for $G(t, q^i, u)$ satisfying the condition $\frac{\partial G}{\partial u} > 0$.

Common examples of Lagrangians of the form (51) are generated by functions $G(t, q^i, u)$ depending linearly on u, namely $G = A(t, q^i)u + B(t, q^i)$, with $A(t, q^i) \neq 0$.

From eq. (51) it can be seen that the term $B(t, q^i)$ has the effect of replacing the Lagrangian L_C with the gauge-equivalent one $L_C - \frac{dB}{dt}$, and is therefore dynamically irrelevant. Omitting it and writing A in exponential notation we have $G = ue^{-f(t,q^k)}$, whence

$$L = e^f\left(L_C + ue^{-f}\dot f\right) := \hat L(t, q^k, \dot q^k) + u\dot f, \tag{52}$$

with $\hat L = e^{f(t,q^k)}L_C$.

The family (52) includes the ordinary Lagrangians $L = \hat L$ ($\Leftrightarrow f = \text{const.}$), and most of the Herglotz Lagrangians found in the literature. For example, with the notation of Ref.[9], we have the translation table

$$L = \tfrac{1}{2}(\dot x^2 - kx^2) - au \qquad \longleftrightarrow \qquad \hat L = \tfrac{1}{2}(\dot x^2 - kx^2), \qquad f = -at,$$

$$L = \tfrac{1}{2} + \tfrac{x^{(n+1)}}{n+1} - \tfrac{2u}{t} \qquad \longleftrightarrow \qquad \hat L = \tfrac{1}{2}\,\dot x^2 + \tfrac{x^{(n+1)}}{n+1}, \qquad f = -2\log t,$$

$$L = \tfrac{1}{2}\dot x^2 - [2a(x)\dot x + b(t)]u \qquad \longleftrightarrow \qquad \hat L = \tfrac{1}{2}\,\dot x^2,\;\; f = -2\int a\,dx - \int b\,dt,$$

As established in Remark 2, every Lagrangian of the form (52) generates a set of evolution equations in $j_1(\mathcal{V}_{n+1})$ identical to those determined by the ordinary Lagrangian $L_C = e^{-f(t,q^i)}\hat{L}$ or by any other Lagrangian gauge-equivalent to it.

Infact, the choice $G = ue^{-f(t,q^k)}$ is just one of the infinitely many possibilities available. For example, setting $G(t,q^i,u) = \alpha\, g\big(\frac{u}{\alpha}\big)e^{-f(t,q^i)}$ with $g'\big(\frac{u}{\alpha}\big) \neq 0$ generates the Lagrangian

$$L = \frac{e^f}{g'}\Big[L_C + \alpha g e^{-f}\dot{f}\Big] := \frac{1}{g'\big(\frac{u}{\alpha}\big)}\Big[\hat{L} + \alpha g\left(\frac{u}{\alpha}\right)\dot{f}\Big],$$

with $\hat{L} = e^f L_C$. In particular, for $g\big(\frac{u}{\alpha}\big) = e^{u/\alpha}$ the latter reads

$$L = e^{-u/\alpha}\hat{L} + \alpha\dot{f}. \tag{53}$$

Although of little practical interest, eq. (53) highlights the fact, already ascertained on theoretical grounds, that adding a symbolic time derivative to a Lagrangian explicitly dependent on the variable u does not mean performing a gauge transformation: the Lagrangian (53) is not equivalent to $e^{-u/\alpha}\hat{L}$ but to the ordinary Lagrangian $L_C = e^{-f}\hat{L}$.

Remark 4. *In a mechanical context, the Herglotz formalism is often presented as a tool for enlarging the class of admissible Lagrangians $L = T + U$, keeping the identification $T = \frac{1}{2}\sum m_i v_i^2$, and allowing "hyper-generalized" potentials $U = U_0(t,q^i,u) + A_k(t,q^i,u)\dot{q}^k$, able to represent a wider class of lagrangian forces. The resulting expression for L is then*

$$L = \tfrac{1}{2}\, a_{ij}(t,q^i)\dot{q}^i\dot{q}^j + b_i(t,q^i,u)\dot{q}^i + c(t,q^i,u). \tag{54}$$

Actually, the effects of this generalization are more apparent than real. In fact, in order for the procedure to be dynamically significant, the Lagrangian L must generate a reducible *flow, i.e. it must fulfil the requirement (48a). On the other hand, every Lagrangian of the form* (54) *automatically fulfils eq. (48b), with $a_u = 0$. Therefore, if the associated flow is reducible, L is gauge-equivalent to — and generates the same equations of motion as — an ordinary Lagrangian.*

Appendix: Reducibility revisited

For completeness, given a non-singular section $\ell : \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P,\mathbb{R})$, we present an alternative approach to the study of reducibility, based on the theory of exterior differential systems.

To this end, we refer to eqs. (10), (11), rewritten here for the reader's convenience:

$$\vartheta = du - L dt - \frac{\partial L}{\partial \dot{q}^k}\,\omega^k, \tag{A.1}$$

$$\Omega = d\vartheta - \vartheta \wedge \mathscr{L}_{\partial/\partial u}\vartheta. \tag{A.2}$$

Preserving the notation $\mathcal{I}(\Omega)$ for the ideal generated by the 2-form (A.2) and Z for the dynamical flow generated by ℓ, we have then the properties:

- the characteristic distribution associated with $\mathcal{I}(\Omega)$ coincides with the 2-dimensional module $\mathcal{D}$ spanned by the fields $\frac{\partial}{\partial u}, Z$;
- the complete integrability of $\mathcal{D}$ is equivalent to the reducibility of Z. In particular, a sufficient condition for Z to be reducible is the differential nature of $\mathcal{I}(\Omega)$, i.e. the existence of a 1-form λ fulfilling the requirement $d\Omega = \lambda \wedge \Omega$.

Theorem 5. *For $n > 1$, a necessary and sufficient condition for $\mathcal{I}(\Omega)$ to be a differential ideal is the existence of a function $f(t, q^i, u)$ satisfying the relation*

$$\mathscr{L}_{\partial/\partial u}\big(e^{-f}\vartheta\big) = de^{-f} \qquad \Longleftrightarrow \qquad \mathscr{L}_{\partial/\partial u}\vartheta - \frac{\partial f}{\partial u}\,\vartheta = -df. \tag{A.3}$$

Proof. Sufficiency: eq. (A.3) entails the relation

$$e^{-f}\,\Omega = e^{-f}\big(d\vartheta + \vartheta \wedge df\big) = d\big(e^{-f}\vartheta\big) \quad \Longrightarrow \quad d\big(e^{-f}\,\Omega\big) = 0 \quad \Longrightarrow \quad d\Omega = df \wedge \Omega\,.$$

Necessity: setting $d\Omega = \lambda\wedge\Omega$ and choosing a function $g(t, q^i, u, \dot{q}^i)$ satisfying the requirement $\frac{\partial g}{\partial u} = \frac{\partial}{\partial u} \,\lrcorner\, \lambda$, we have the equality

$$0 = \mathscr{L}_{\partial/\partial u}\Omega - \frac{\partial g}{\partial u}\,\Omega = \mathscr{L}_{\partial/\partial u} d\vartheta - \vartheta \wedge \mathscr{L}_{\partial/\partial u}\Big(\mathscr{L}_{\partial/\partial u}\vartheta\Big) - \frac{\partial g}{\partial u}\Big(d\vartheta - \vartheta \wedge \mathscr{L}_{\partial/\partial u}\vartheta\Big). \tag{A.4}$$

Introducing the 1-form $\omega = \mathscr{L}_{\partial/\partial u}\vartheta - \frac{\partial g}{\partial u}\,\vartheta + dg$ and employing the relations

$$\mathscr{L}_{\partial/\partial u} d\vartheta = d\omega + \frac{\partial g}{\partial u}\, d\vartheta + d\left(\frac{\partial g}{\partial u}\right) \wedge \vartheta$$

$$\vartheta \wedge \mathscr{L}_{\partial/\partial u}\vartheta = \vartheta \wedge (\omega - dg)$$

$$\vartheta \wedge \mathscr{L}_{\partial/\partial u}\Big(\mathscr{L}_{\partial/\partial u}\vartheta\Big) = \vartheta \wedge \mathscr{L}_{\partial/\partial u}\left(\omega + \frac{\partial g}{\partial u}\,\vartheta - dg\right) = \vartheta \wedge \left[\mathscr{L}_{\partial/\partial u}\omega + \frac{\partial g}{\partial u}(\omega - dg) - d\left(\frac{\partial g}{\partial u}\right)\right],$$

eq. (A.4) reduces to

$$0 = d\omega + \frac{\partial g}{\partial u}\, d\vartheta - \vartheta \wedge d\left(\frac{\partial g}{\partial u}\right) - \vartheta \wedge \left[\mathscr{L}_{\partial/\partial u}\omega + \frac{\partial g}{\partial u}(\omega - dg) - d\left(\frac{\partial g}{\partial u}\right)\right] -$$
$$- \frac{\partial g}{\partial u}\, d\vartheta + \frac{\partial g}{\partial u}\,\vartheta \wedge \big(\omega - dg\big) = d\omega - \vartheta \wedge \mathscr{L}_{\partial/\partial u}\omega. \tag{A.5}$$

From eq. (A.5) we get the pair of relations

$$\mathscr{L}_{\partial/\partial u} d\omega = \mathscr{L}_{\partial/\partial u}\vartheta \wedge \mathscr{L}_{\partial/\partial u}\omega + \vartheta \wedge \mathscr{L}_{\partial/\partial u}\Big(\mathscr{L}_{\partial/\partial u}\omega\Big)$$

$$d\vartheta \wedge \mathscr{L}_{\partial/\partial u}\omega \,-\, \vartheta \wedge \mathscr{L}_{\partial/\partial u} d\omega \,=0,$$

whence also

$$\Big(\Omega + \vartheta \wedge \mathscr{L}_{\partial/\partial u}\vartheta\Big) \wedge \mathscr{L}_{\partial/\partial u}\omega - \vartheta \wedge \mathscr{L}_{\partial/\partial u}\vartheta \wedge \mathscr{L}_{\partial/\partial u}\omega = \Omega \wedge \mathscr{L}_{\partial/\partial u}\omega \,=0. \tag{A.6}$$

Being $\operatorname{rank}\Omega = 2n$, for $n>1$ eq. (A.6) implies $\mathscr{L}_{\partial/\partial u}\omega = 0$. In view of eq. (A.5) this entails $d\omega = d\Big(\mathscr{L}_{\partial/\partial u}\vartheta - \frac{\partial g}{\partial u}\,\vartheta\Big) = 0$, thus ensuring the validity of a local representation of the form

$$\mathscr{L}_{\partial/\partial u}\vartheta - \frac{\partial g}{\partial u}\,\vartheta = -df. \tag{A.7}$$

in which, on account of the identities $\frac{\partial}{\partial \dot q^k} \lrcorner\, \vartheta \,=\, \frac{\partial}{\partial \dot q^k} \lrcorner\, \mathscr{L}_{\partial/\partial u}\vartheta \,=\, \frac{\partial}{\partial u} \lrcorner\, \mathscr{L}_{\partial/\partial u}\vartheta \,=\, 0$, the function f is subject to the conditions

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial u}\,, \qquad \frac{\partial f}{\partial \dot q^k} = 0.$$

These make eq. (A.7) identical to (A.3), thus completing the proof. $\square$

Let us now see the implications of Theorem 5. In view of eq. (A.1), the content of eq. (A.3) is expressed by the equation

$$\frac{\partial L}{\partial u}\,dt + \frac{\partial^2 L}{\partial u\,\partial \dot q^k}\,\omega^k + \frac{\partial f}{\partial u}\Big(du - Ldt - \frac{\partial L}{\partial \dot q^k}\,\omega^k\Big) = \frac{\partial f}{\partial u}\,du + \Big(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k}\,\dot q^k\Big)dt + \frac{\partial f}{\partial q^k}\,\omega^k,$$

with $f = f(t, q^i, u)$. The latter implies the relations

$$\frac{\partial L}{\partial u} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k}\,\dot q^k + L\,\frac{\partial f}{\partial u} \tag{A.8a}$$

$$\frac{\partial^2 L}{\partial u\,\partial \dot q^k} = \frac{\partial f}{\partial u}\,\frac{\partial L}{\partial \dot q^k} + \frac{\partial f}{\partial q^k}\,, \tag{A.8b}$$

the second of which is identically satisfied as a consequence of the first.

Eq. (A.3) is therefore equivalent to the single condition (A.8a) which, as proved in Theorem 4, is necessary and sufficient for the existence of an ordinary Lagrangian gauge equivalence to L. Summing up, we conclude

Proposition 3. *The following statements are equivalent:*

- *the ideal $\mathcal{I}(\Omega)$ generated by the 2-form* (A.2) *is a differential ideal;*
- *there exists a function $f(t, q^i, u)$ satisfying eq. (A.8a);*
- *there exists a function $G(t, q^i, u)$ satisfying $\frac{\partial G}{\partial u} \neq 0$ and $\frac{\partial}{\partial u}\Big(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\dot{q}^k + \frac{\partial G}{\partial u}L\Big) = 0$;*
- *the Lagrangian L is gauge-equivalent to an ordinary Lagrangian.*

Remark 5. *Proposition 3 does not exclude the existence of reducible dynamical flows generated by Lagrangians not gauge-equivalent to ordinary ones: it simply restricts this possibility to the case in which the characteristic distribution associated with the ideal $\mathcal{I}(\Omega)$ is completely integrable, without $\mathcal{I}(\Omega)$ being a differential ideal.*

REFERENCES

[a]The Author E.P. acknowledges support by the Italian National Group of Mathematical Physics (GNFM-INdAM).

[1]E. Massa, E. Pagani, and P. Lorenzoni, Transp. Theory Stat. Phys. **29**, 69 (2000).

[2]E. Massa, D. Bruno, G. Luria, and E. Pagani, Int. J. Geom. Methods Mod. Phys. **12**, 1550061 (2015).

[3]G. Herglotz, *Berührungstranformationen, Lectures at the University of Göttingen* (Univ. Göttingen, 1930).

[4]G. Herglotz, *Gesammelte Schriften Göttingen*, edited by H. Schwerdtfeger (Vandenhoeck & Ruprecht, 1979).

[5]R. B. Guenther, J. A. Gottsch, and C. M. Guenther, *The Herglotz Lectures on Contact Transformations and Hamiltonian Systems* (Juliusz Center for Nonlinear Studies, Toruń, 1996).

[6]B. Georgieva, Topological Methods in Nonlinear Analysis **20**, 261 (2002).

[7]B. Georgieva, R. Guenther, and T. Bodurov, Journal of Mathematical Physics **44**, 3911 (2003).

[8]B. Georgieva and R. Guenther, Topological Methods in Nonlinear Analysis **26**, 307 (2005).

[9]B. Georgieva, XIII Int. Conf. Geometry, Integrability and Quantization, Varna, Bulgaria (2011).

[10]V. Donchev, J. Math. Phys. **55**, 1 (2014).

[11] P. S. Santos, N. Martins, and D. F. M. Torres, Discrete and Continuous Dynamical Systems – Series A (2015).

[12] R. Garra, S. Taverna, and D. F. M. Torres, Chaos, Solitons & Fractals **102** (2017).

[13] D. Tavares, R. Almeida, and D. F. M. Torres, Discrete and Continuous Dynamical Systems – Series S (2017).

[14] Y. Zhang, Transactions of Nanjing University of Aeronautics & Astronautics , 1 (2020).

[15] M. de León, M. Lainz, and M. C. Muñoz-Lecanda, Journal of Nonlinear Science **33:9**, 1 (2023).

[16] E. Massa, S. Vignolo, and D. Bruno, J. Phys. A: Math. Gen. **35**, 6713 (2002).

[17] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Process* (Interscience, New York, 1962).

[18] I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Prentice-Hall, Englewood Cliffs, NJ, 1963).

[19] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory* (Wiley, New York, 1966).

[20] E. Massa and E. Pagani, Ann. Inst. Henri Poincaré **61** (1994).

[21] Indeed, as shown in[1], when L is an ordinary Lagrangian, the 1-form (10) identifies a connection in the principal fibre bundle $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$, whose representation in $j_1(\mathcal{V}_{n+1})$ is the opposite $-\Big(L\,dt + \frac{\partial L}{\partial \dot q^k}\,\omega^k\Big)$ of the Poincaré-Cartan 1-form of L.

[22] Notice that, consistently with eq. (25a), the coordinate function y_u is invariant under the group of transformations (1).

[23] The minus sign reflects the fact that the fiber coordinates in $j_1(P, \mathcal{V}_{n+1})$ are defined according to the identification $\omega = du - p_0(\omega)\,dt - p_i(\omega)\,dq^i$.

[24] It is worth noticing that, although abnormal, the extremals of the Herglotz functional satisfy the property of *ordinariness*[2], ensuring that every infinitesimal deformation vanishing at the endpoints is tangent to a finite deformation with fixed endpoints . The conclusion follows easily from the discussion in Section II A. The details are left to the Reader.

[25] As in the rest of the paper, the coordinate function u is assumed to be a trivialization of the principal bundle $P \to \mathcal{V}_{n+1}$. The diffeomorphism κ is a principal bundle isomorphism if and only if the associated function G satisfies $\frac{\partial G}{\partial u} = 1$.