PROBABILISTIC REPRESENTATIONS FOR THE SOLUTION OF HIGHER ORDER DIFFERENTIAL EQUATIONS

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Abstract. A probabilistic representation for the solution of the partial differential equation $\frac{\partial}{\partial t}u(t,x) = -\alpha\Delta^2 u(t,x)$, $\alpha \in \mathbb{C}_+$, is constructed in terms of the expectation with respect to the measure associated to a complex-valued stochastic process.

Key words: Partial differential equations, probabilistic representation of solutions of PDEs, stochastic processes.

AMS classification : 35C15, 35G10, 35K30, 60G50, 60J35.

1. Introduction

The connection between the solution of parabolic equations associated to second-order elliptic operators and the theory of stochastic processes is a largely studied topic [17]. The main instance is the *Feynman-Kac formula*, providing a representation of the solution of the heat equation with potential $V \in C_\infty(\mathbb{R}^d)$ (the continuous functions vanishing at infinity)

$$\left\{\begin{array}{ll} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) - V(x)u(t,x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d \\ u(0,x) = u_0(x) & \end{array}\right. \tag{1}$$

in terms of an integral with respect to the measure of the Wiener process, the mathematical model of the Brownian motion [31]:

$$u(t,x) = \int_{C_t} e^{-\int_0^t V(\omega(s)+x)ds} u_0(\omega(t)+x)dW(\omega). \tag{2}$$

If the Laplacian in Eq (1) is replaced with an higher order differential operator, i.e. if we consider a Cauchy problem of the form

$$\left\{\begin{array}{ll} \frac{\partial}{\partial t}u(t,x) = (-1)^{N+1}\Delta^N u(t,x) - V(x)u(t,x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & \end{array}\right. \tag{3}$$

with $N \in \mathbb{N}$, $N \geq 2$, then a formula analogous to (2), giving the solution of (3) in terms of the expectation with respect to the measure associated to a Markov process, is lacking. In fact, such a formula cannot be proved for semigroups whose generator does not satisfy the maximum principle, as in the case of Δ^N with $N > 1$ [45]. In other words it is not possible to find a stochastic process X_t which plays for the parabolic equation (3) the same role that the Wiener process plays for the heat equation.

We would like to point out that the problem of the probabilistic representation of the solution of the Cauchy problem (3) presents some similarities with the problem of the mathematical definition of Feynman path integrals (see [1, 2, 40, 30] for a discussion of this topic). Indeed in both cases it is not possible to implement an integration theory of Lebesgue type in terms of a bounded variation measure on a space of continuous paths [14]. An analogous of the Feynman-Kac formula for the parabolic equation (3), namely an equation of the form:

$$u(t,x)=\int_{\omega(0)=x} e^{-\int_0^t V(\omega(s))ds}u_0(\omega(t))dP(\omega), \tag{4}$$

(where P should be some "measure" on a space of "paths" $\omega : [0,t] \to \mathbb{R}$) can be obtained only under some restrictions on u_0 and V and by giving up a traditional integration theory in the Lebesgue sense with respect to a bounded variation measure on a space of (real) continuous paths.

In the mathematical literature two main approaches have been proposed. The first one [32, 26] realizes formula (4) in terms of the expectation with respect to a *signed* measure on a space of paths on the interval $[0,t]$. Indeed V. Yu. Krylov in 1960 [32] and K. Hochberg in 1978 [26] proposed a representation for the solution of the parabolic equation associated to an even order differential operator

$$\begin{cases} \frac{\partial u}{\partial t} = (-1)^{N+1}\frac{\partial^{2N} u}{\partial x^{2N}} & -\infty < x < \infty, 0 \leq t < \infty, \\ u(0,x) = u_0(x) \end{cases}$$

in terms of a *signed measure with infinite total variation* P_{2N} on the space Ω of measurable functions $\omega : [0,\infty) \to \mathbb{R}$, called "paths". By a standard technique [39], the measure is defined on cylinder subsets $I_k := \{\omega \in \Omega : \omega(t_j) \in [a_j, b_j], j = 1, \dots k\}$, $0 < t_1 < t_2 < \dots t_k$, by the formula:

$$P_{2N}(I_k) = \int_{a_1}^{b_1} \dots \int_{a_k}^{b_k} \prod_{j=0}^{k-1} G_{2N}(t_{j+1}-t_j, x_{j+1}-x_j)dx_{j+1}, \tag{5}$$

where $x_0 \equiv 0$, $t_0 \equiv 0$ and $G_{2n}(t,x)$ is the Green function, i.e. the fundamental solution of equation (5), namely:

$$G_{2N}(t,x) = \frac{1}{2\pi}\int e^{ix\xi}e^{-\xi^{2N}t}d\xi.$$

The measure P_{2N} satisfies the Markov property, since the Chapman-Kolmogorov equation

$$G_{2N}(t+s,x,y) = \int G_{2N}(t,x,z)G_{2N}(s,z,y)dz$$

holds, however P_{2N} is a signed measure as $G_{2N}(t,x)$ is a signed transition density (actually the analysis of the asymptotic behavior of $G_{2N}(t,x)$ as $x \to \infty$ shows that it changes sign an infinite number of times [26]). P_{2N} is countably additive on the σ-algebra generated by cylinder sets I_k, however, contrary to the case of the Wiener measure, P_{2N} cannot be extended to a σ-additive bounded measure on the σ-algebra generated by all cylinder sets, as it would have infinite total variation [32, 26].

It is worthwhile to mention that an analogous of the arc-sine law [26, 28], of the central limit theorem [27] and of Ito formula and Ito stochastic calculus [26, 41] have been proved for the (finite additive) signed measure P_{2N}. Moreover, a Feynman-Kac formula has been proved [32, 26, 28], for the representation of the solution of

$$\frac{\partial u}{\partial t} = (-1)^{N+1}\frac{\partial^{2N}u}{\partial x^{2N}} - Vu \qquad -\infty < x < \infty, 0 \leq t < \infty,$$

where V is a bounded piecewise continuous function and for an initial datum $u_0 \in C^{2N}$. In this case the solution of the Cauchy problem is given by

$$u(t,x) = \mathbb{E}_x[e^{-\int_0^t V(\omega(s)ds}u_0(\omega(t)],$$

where the expectation is meant as limit of finite dimensional cylindrical approximations [6]:

$$u(t,x) = \lim_{k\to\infty}\int_{\mathbb{R}^k} e^{-\sum_{j=1}^k V(x_{j-1})(t_j-t_{j-1})}\Pi_{j=1}^k G_{2N}(t_j-t_{j-1}, x_j - x_{j-1}) \\ u_0(x_k)dx_1\cdots dx_k,$$

with $x_0 \equiv x$.

We also mention the work by D. Levin and T. Lyons [36] on rough paths, conjecturing that the signed measure P_{2N} could exist on the quotient space of equivalence classes of paths corresponding to different parametrization of the same path.

A different approach is based on the construction of a stochastic process on a space of *complex paths*. In this case the integration is performed with respect to a well defined positive probability measure on a complex space. One of the first results was given by T. Funaki [19], who constructed a complex stochastic process $\{X_t\}_{t\geq 0}$ by composing two independent Brownian motions $\{B(t)\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$ in the following way

$$X_t := \begin{cases} B(w(t)) & \text{if } w(t) \geq 0 \\ iB(-w(t)) & \text{if } w(t) < 0 \end{cases}$$

and proving that, for a suitable class of analytic initial datum u_0, the solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{8}\frac{\partial^4 u}{\partial x^4} & -\infty < x < \infty, 0 < t < \infty, \\ u(0,x) = u_0(x) \end{cases} \tag{6}$$

is given by the expectation

$$u(t,x) = \mathbb{E}[u_0(x + X_t)]. \tag{7}$$

In fact the result can be generalized to partial differential equations of order 2^n, by multiple iterations of suitable processes [19, 29, 42]. These results are also related to Bochner subordination [9].

There are also similarities between the Funaki's process $\{X_t\}$ and the "iterated Brownian motion" [11], but the latter is not connected to the probabilistic representation of the solution of a partial differential equation with regular coefficients. In fact the processes constructed by iterating copies of independent BMs (or other process) are associated to higher order PDE of particular form, where the initial datum plays a particular role and enters also in the differential equation [5].

Complex valued processes, connected to PDE of the form (3) have been also proposed by other authors by means of different techniques. In [37, 12, 13] K. Burdzy and A. Madrecki consider the fourth degree heat-type equation (6) and construct a probabilistic representation for its solution in terms of a stable probabilistic Borel measure m on the space $\Omega = C([0,t], \mathbb{C}^\infty)$ of continuous mappings on $[0,t]$ with values in the set $\mathbb{C}^\infty$ of complex valued sequences, endowed with the product topology. In this setting a Feynman-Kac type formula is proved, for the fourth order heat equation with linear potential

$$\frac{\partial u}{\partial t} = \frac{1}{8}\frac{\partial^4 u}{\partial x^4} + (iax + b)u.$$

Another probabilistic approach is presented by P. Sainty in Ref. [43], where a representation for the solution of $\frac{\partial}{\partial t}u(t,x) = \frac{\partial^n}{\partial x^n}u(t,x)$, is given in terms of the expectation with respect to a particular complex valued process $X_{[n]}(t)$, $t \geq 0$, called "Brownian motion of order n". It is worthwhile also to mention a completely different approach proposed by R. Léandre [33], which has some analogies with the mathematical realization of Feynman path integrals by means of white noise calculus [25]. Indeed Léandre has recently constructed a "probabilistic representation" of the solution of the Cauchy problem (3) not as an integral with respect to a measure but as an infinite dimensional distribution on the Connes space [34, 35]

We eventually mention another probabilistic approach to the equation $\Delta^k u = 0$ described in [22].

The present paper presents the construction of an alternative complex-valued stochastic process generalizing Funaki's result [19] and a corresponding probabilistic representation for the solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -\frac{\alpha}{8}\frac{\partial^4}{\partial x^4}u(x,t) + V(t,x)u(t,x) & -\infty < x < \infty,\ 0 \leq t < \infty, \\ u(x,0) = u_0(x), \end{cases} \tag{8}$$

with $\alpha \in \mathbb{C}_+$ and for V, u_0 satisfying a set a conditions.

The paper is organized as follows. Section 2 presents the construction of a complex random variable z_t^α and the representation of the solution of equation (8) with $V \equiv 0$ in terms of the expectation with respect to the probability measure associated with z_t^α. Section 3 presents the proof of a Feynman-Kac type formula for the solution of equation (8) in the cases where V is linear in the space variables and presents an explicit time dependence.

2. A COMPLEX VALUED RANDOM VARIABLE ASSOCIATED TO THE 4-ORDER HEAT-TYPE EQUATION

In the present section we construct a probabilistic representation for the solution of equation (8) in the case where $V = 0$, namely

$$\frac{\partial u}{\partial t}(x,t) = -\frac{\alpha}{8}\frac{\partial^4 u}{\partial x^4}(x,t) \qquad -\infty < x < \infty,\ 0 \leq t < \infty,$$

An equation of this form, as mentioned in the introduction, has been studied by several authors by means of different techniques [19, 32, 26, 43]. In this section we show that the results in [19, 32, 26] can be seen as particular cases of a general theory presented in [10, 21, 23, 24], connecting the solution of parabolic problems with the solution of related hyperbolic problems.

Given a Banach space X and a strongly continuous group of operators $\{T_A(t)\}_{t\in\mathbb{R}}$ on X with generator A, it is possible to construct the holomorphic semigroup $e^{t\frac{A^2}{2}}$ with generator $A^2/2$ in terms of a Gaussian expectation of the group $T(t)$:

$$e^{t\frac{A^2}{2}}f=\mathbb{E}_{N(0,t)}[T(s)f]=(2\pi t)^{-1/2}\int_{-\infty}^{\infty}e^{-s^2/2t}T(s)fds,\qquad f\in X. \tag{9}$$

More generally, given a polynomial $P(A)$ in A with complex coefficients, whose leading term has the form $c_{2m}A^{2m}$, with $(-1)^{m+1}Re(c_{2m})>0$, then $P(A)$ generates an holomorphic semigroup on X. Its action on a vector $f\in X$ belonging to the domain of A^{2m} is given by

$$e^{tP(A)}f=\int_{-\infty}^{\infty}\hat{g}_t(s)T_A(s)fds,\qquad f\in X \tag{10}$$

with $\hat{g}_t(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-is\xi}e^{tP(i\xi)}d\xi$ (see [23]).

Theorem 1. *Let us consider the Cauchy problem*

$$\begin{cases}\frac{\partial u}{\partial t}(x,t)=-\frac{\alpha}{8}\frac{\partial^4 u}{\partial x^4}(x,t) & -\infty<x<\infty,\ 0\leq t<\infty,\\ u(x,0)=u_0(x),\end{cases} \tag{11}$$

with an initial datum $u_0\in L^2(\mathbb{R})$ satisfying the following properties:

1. *u_0 can be extended to an entire function on the complex plane $\mathbb{C}$, denoted again with u_0,*
2. *for any $h\in\mathbb{R}^+$, $e^{-h|z|^2}|u_0(z)|$ is a bounded function on $\mathbb{C}$.*

Then the solution is given by

$$\begin{aligned}&u(t,x)\\&=\int_0^{\infty}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\frac{e^{-\frac{y^2}{2s}}}{\sqrt{2\pi s}}\big(u_0(x+\alpha^{1/4}e^{i\pi/4}y)+u_0(x+\alpha^{1/4}e^{-i\pi/4}y)\big)dyds\end{aligned} \tag{12}$$

Proof: Let us consider equation (10) in the case where X is the Hilbert space $L^2(\mathbb{R})$, $A=i\frac{\Delta}{2}$ and $P(x)=x^2/2$. One gets the following representation

$$e^{-\frac{t}{8}\Delta^2}=\int_{-\infty}^{\infty}e^{is\frac{\Delta}{2}}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}ds, \tag{13}$$

giving the semigroup $e^{-\frac{t}{8}\Delta^2}$ in terms of a Gaussian expectation, with respect to the time variable $s\in\mathbb{R}$, of the Schrödinger group $e^{is\frac{\Delta}{2}}$.

Given an initial datum $u_0 \in L^2(\mathbb{R})$ satisfying the hypothesis of the theorem one can write the following chain of equalities

$$\begin{aligned} e^{-\frac{t}{8}\Delta^2}u_0(x) &= \int_{-\infty}^{\infty}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}e^{is\frac{\Delta}{2}}u_0(x)ds \\ &= \int_0^{\infty}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}(e^{is\frac{\Delta}{2}}u_0(x)+e^{-is\frac{\Delta}{2}}u_0(x))ds \\ &= \int_0^{\infty}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\Big(\frac{e^{i\frac{y^2}{2s}}}{\sqrt{2\pi is}}u_0(x+y)+\frac{e^{-i\frac{y^2}{2s}}}{\sqrt{-2\pi is}}u_0(x+y)\Big)dyds \\ &= \int_0^{\infty}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\frac{e^{-\frac{y^2}{2s}}}{\sqrt{2\pi s}}(u_0(x+e^{i\pi/4}y)+u_0(x+e^{-i\pi/4}y))dyds, \end{aligned}$$

where the latter line is the result of a rotation of the integration path in the complex y-plane. By an analogous reasoning, the formula in the case where α is a complex constant is:

$$\begin{aligned} &e^{-\alpha\frac{t}{8}\Delta^2}u_0(x) \\ &= \int_0^{\infty}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\frac{e^{-\frac{y^2}{2s}}}{\sqrt{2\pi s}}\big(u_0(x+\alpha^{1/4}e^{i\pi/4}y)+u_0(x+\alpha^{1/4}e^{-i\pi/4}y)\big)dyds \end{aligned}$$

□

The Funaki formula (7) for the solution of (11) in the case where $\alpha = -1$ can be written in the following form

$$u(t,x) = \int_0^{\infty}\frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\frac{e^{-\frac{z^2}{2y}}}{\sqrt{2\pi y}}(u_0(x+z)+u_0(x+iz))dzdy \tag{14}$$

and can be obtained as a special case of equation (12).

Equation (12) can be written in terms of the expectation w.r.t. the measure associated to a complex random variable z_t^α:

$$e^{-\alpha\frac{t}{8}\Delta^2}u_0(x) = \mathbb{E}[u_0(x+z_t^\alpha)], \tag{15}$$

where z_t^α has the following distribution

$$P(z_t^\alpha \in A) = \int_0^{\infty}\frac{e^{-s^2/2t}}{\sqrt{2\pi t}}\int_{\mathbb{R}}\frac{e^{-\frac{y^2}{2s}}}{\sqrt{2\pi s}}\big(\chi_A(\alpha^{1/4}e^{i\pi/4}y)+\chi_A(\alpha^{1/4}e^{-i\pi/4}y)\big)dyds, \tag{16}$$

A being a Borel subset of the complex plane and χ_A being its characteristic function. Clearly the measure is concentrated on two rays of the complex plane $\alpha^{1/4}e^{i\pi/4}\mathbb{R}$ and $\alpha^{1/4}e^{-i\pi/4}\mathbb{R}$.

One can easily verify that random variable z_t^α has the following properties:

- $z_t^\alpha \sim t^{1/4} z_1^\alpha$,
- $\mathbb{E}[(z_t^\alpha)^k] = 0$, $k = 1, 2, 3$,
- $\mathbb{E}[(z_t^\alpha)^4] = -3t$,
- $\mathbb{E}[|z_t^\alpha|^2] = 2|\alpha t|^{1/2} \int_0^\infty \frac{e^{-s^2/2}}{\sqrt{2\pi}} s ds < +\infty$
- $\mathbb{E}[e^{i\lambda z_t^\alpha}] = e^{-\frac{t}{8}\alpha\lambda^4}$.

Moreover formula (15) can be written in Funaki's notation (see Equations (6) and (7)). Indeed let us consider two independent Brownian motions $\{B(t)\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$, and define the process $\{X_t^\alpha\}_{t\geq 0}$ as:

$$X_t^\alpha := \begin{cases} e^{i\pi/4}\alpha^{1/4} B(w(t)) & \text{if } w(t) \geq 0 \\ e^{-i\pi/4}\alpha^{1/4} B(-w(t)) & \text{if } w(t) < 0 \end{cases} \tag{17}$$

Then equation (12) can then be written in the following form:

$$e^{-\alpha \frac{t}{8}\Delta^2} u_0(x) = \mathbb{E}[u_0(x + X_t^\alpha)]. \tag{18}$$

Remark 1. *Analogous results can be obtained also in the case where* Δ^2 *is replaced with higher powers of the Laplacian, namely* $\Delta^4, \Delta^8, ..., \Delta^{2^n}$. *It is sufficient to iterate* $n-$ *times formula* (9). *One obtains a formula with multiple Gaussian integrations, similar to the one proposed for instance in* [29]. *As in the Funaki approach, the probability measure of the complex random variable can also been obtained by composing three independent Brownian motions in a suitable way. In fact any even power of the Laplacian can be handled by means of the general formula* (10), *but a probabilistic interpretation in terms of the composition of several independent Brownian motions is not always possible. For instance, in the case where one considers* Δ^6, *equation* (10) *gives the following result:*

$$e^{-\frac{1}{2^6}\Delta^6} f(x) = \int_{-\infty}^{\infty} \hat{g}_t(s) e^{is\frac{\Delta}{2}} u_0(x) ds,$$

with

$$\hat{g}_t(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-is\xi} e^{-t\xi^6} d\xi.$$

We shall not further develop these formulae here, but we shall only focus on the case of Δ^2.

3. Feynman-Kac type formulae

The process $\{X_t^\alpha\}_{t\geq 0}$ appearing in (18) provides a probabilistic representation for the solution of equation (11) (under suitable analyticity assumptions on the initial datum f). On the other hand it has not independent increments, so it does not naturally give rise to generalizations of formula (18) to the case where equation (11) contains also a potential V, i.e. to a Feynman-Kac formula of the form:

$$u(t,x)=\mathbb{E}[u_0(x+X^\alpha_t)e^{-\int_0^t V(x+X^\alpha_s)ds}].$$

Indeed, by applying formally the Trotter product formula, one gets:

$$\begin{aligned} e^{-t(\frac{\Delta^2}{8}+V)}u_0(x) &= \lim_{n\to\infty}\left(e^{-\frac{t}{n}\frac{\Delta^2}{8}}e^{-\frac{t}{n}V}\right)^n u_0(x)\\ &= \lim_{n\to\infty}\mathbb{E}[e^{-\frac{t}{n}\sum_{k=1}^n V(x+\sum_{j=1}^k z_j^\alpha(t/n))}u_0(x+\sum_{j=1}^n z_j^\alpha(t/n))]\end{aligned}$$

where $z_j^\alpha(t/n)$, $j=1,...n$, is a family on n i.i.d random variables, distributed as $z^\alpha_{t/n}$ (see equation (16)). In the latter line one would be tempted to interpret the r.v $z_j^\alpha(t/n)$ as the independent increments of a complex valued stochastic process, different from $\{X^\alpha_t\}_{t\geq 0}$, i.e. to interpret the limit as the cylindrical approximations of an integral with respect to the measure associated to a complex valued stochastic process with independent increments.

In fact such a process cannot exist, as its construction would be possible provided the weak convergence of the sequence of complex random variables $\sum_{j=1}^n z_j^\alpha(t/n)$ as $n\to\infty$. By using the scaling properties of the random variable $z^\alpha(t)$, this is equivalent to the weak convergence of the sequence $n^{-1/4}\sum_{j=1}^n z_j^\alpha(1)$. As z_j^α are independent identically distributed complex random variables with finite covariance, then the sequence $n^{-1/2}\sum_{j=1}^n z_j^\alpha(1)$ has a Gaussian limit. Consequently, the sequence $n^{-1/4}\sum_{j=1}^n z_j^\alpha(1)$ cannot converge weakly, as erroneously stated in [43].

The present section is devoted to the proof, for a suitable class of continuous functions V and initial datum u_0, of a Feynman-Kac formula representing the solution of the Cauchy problem (8) as the limit of a sequence of finite dimensional approximations:

$$u(t,x)=\lim_{n\to\infty}\mathbb{E}[e^{-\frac{t}{n}\sum_{k=1}^n V(x+\sum_{j=1}^k z_j^\alpha(t/n))}u_0(x+\sum_{j=1}^n z_j^\alpha(t/n))] \qquad (19)$$

where $z_j^\alpha(t/n)$, $j=1,...n$, is a family on n i.i.d random variables, distributed as $z^\alpha_{t/n}$.

The implementation of formula (19) presents some technical problems, which do not appear in the proof of the classical Feynman-Kac formula (for the heat equation with potential). The first one is the definition of the integrals involved. In fact, since the random variables $z^\alpha(t/n)$ are complex valued, the function V must be extended to an entire function of the complex plane. We cannot require that it is bounded on $\mathbb{C}$, otherwise we could consider only the trivial case. Consequently we shall integrate unbounded function and in principle the convergence of the integrals has to be checked. In fact, for a large class of potentials, the integrals are not absolutely convergent and have to be defined in a suitable way.

The second problem concerns the proof that the integral (19) represents the solution of the Cauchy problem (8). Even if the second line of (19) recalls Trotter's product, this formula cannot be directly applied since it holds in $L^2(\mathbb{R})$, while formula (12) holds for an initial datum f belonging to a different class of vectors (i.e. those satisfying analyticity and slow growing conditions).

The problem of the proof of a Feynman-Kac type formula for equations of the form (8) has been analyzed in [37], where the case with linear V is handled, and in [20], but a detailed proof for a sufficiently large class of potentials V is still lacking. We generalized this results to the case where V is linear in the space variable and presents an explicit time dependence.

Theorem 2. *Let u_0 be of the form $u_0(x) = \int_{\mathbb{R}} e^{ixy} d\mu_0(y)$, where μ_0 is a complex bounded variation measure on $\mathbb{R}$ such that for any $\lambda \in \mathbb{R}$ the following holds:*

$$\int_{\mathbb{R}} e^{\lambda|y|} d|\mu_0|(y) < \infty. \tag{20}$$

Let $a : \mathbb{R} \to \mathbb{C}$ be a continuous function. Then the solution of the Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -\frac{\alpha}{8}\frac{\partial^4}{\partial x^4}u(x,t) - ia(t)xu(t,x) & -\infty < x < \infty,\ 0 \le t < \infty, \\ u(x,0) = u_0(x), \end{cases} \tag{21}$$

is given by formula (19).

Proof: Under the stated assumptions, the finite dimensional integrals appearing in formula (19) assume the following form:

$$e^{-ix\int_0^t a(t-s)ds}\mathbb{E}[e^{-i\frac{t}{n}\sum_{k=1}^n a(t-\frac{kt}{n})\sum_{j=1}^k z_j^\alpha(t/n)}\int_{\mathbb{R}} e^{ixy}e^{iy\sum_{j=1}^n z_j^\alpha(t/n)}d\mu_0(y)],$$

where $z_j^\alpha(t/n)$, $j = 1, ...n$, is a family on n i.i.d. random variables, distributed as $z^\alpha_{t/n}$. Now by applying the Fubini theorem, which holds because of condition (20), the latter is equal to

$$e^{-ix\int_0^t a(s)ds}\int_{\mathbb{R}} e^{ixy}\mathbb{E}[e^{-i\sum_{j=1}^n z_j^\alpha(t/n)\left(-y+t/n\sum_{k=j}^n a(t-\frac{kt}{n})\right)}]d\mu_0(y)$$

$$= e^{-ix\int_0^t a(s)ds}\int_{\mathbb{R}} e^{ixy}e^{-\frac{\alpha}{8}\frac{t}{n}\sum_{j=1}^n(-y+t/n\sum_{k=j}^n a(t-\frac{kt}{n}))^4}]d\mu_0(y).$$

By dominated convergence theorem, the limit as $n\to\infty$ of the last line is equal to

$$e^{-ix\int_0^t a(s)ds}\int_{\mathbb{R}} e^{ixy}e^{-\frac{\alpha}{8}\int_0^t(-y+\int_s^t a(t-u)du)^4 ds}]d\mu_0(y),$$

which is the solution of Cauchy problem (21), as one can easily verify by direct calculation. □

4. Conclusions

In this paper we have proposed the construction of a particular probabilistic representation for the solution of the equation $\dot{u} = -\alpha\Delta^2 u + V$ in terms of a Feynman-Kac type formula. The class of potentials V which can be handled by requiring that the probabilistic integrals are defined in Lebesgue sense, i.e. as absolutely convergent integrals, is rather restricted because of the complex nature of the process. A generalization of these results to more general potentials requires the implementation of an integration technique, in infinite dimensions, of a different type, by relaxing the absolute convergence of the integrals, as in the cases handled for instance in [40] concerning the functional integral representation for the solution of Schrödinger equations. This problem will be handled in a forthcoming paper.

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