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ORIGINAL PAPER

Orders and polytropes: matrix algebras from valuations

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Abstract

We apply tropical geometry to study matrix algebras over a field with valuation. Using the shapes of min-max convexity, known as polytropes, we revisit the graduated orders introduced by Plesken and Zassenhaus. These are classified by the polytrope region. We advance the ideal theory of graduated orders by introducing their ideal class polytropes. This article emphasizes examples and computations. It offers first steps in the geometric combinatorics of endomorphism rings of configurations in affine buildings.

Keywords Graduated orders · Polytropes · Tropical geometry

1 Introduction

Let K be a field with a surjective discrete valuation $\mathrm{val} : K \to \mathbb{Z} \cup \{\infty\}$. We fix $p \in K$ satisfying $\mathrm{val}(p) = 1$. The *valuation ring* $\mathcal{O}_K$ is the set of elements in K with non-negative valuation. This is a local ring with maximal ideal $\langle p \rangle = \{x \in \mathcal{O}_K :$

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$\mathrm{val}(x) > 0\}$. In our examples, $K = \mathbb{Q}$ is the field of rational numbers, with the p-adic valuation for some prime p.

We write $K^{d\times d}$ for the ring of $d \times d$ matrices with entries in K. The map val is applied coordinatewise to matrices and vectors. For example, if $K = \mathbb{Q}$ with $p = 2$, then the vector $x = (8/7, 5/12, 17)$ has $\mathrm{val}(x) = (3, -2, 0)$. In what follows, we often take $X = (x_{ij})$ to be a $d \times d$ matrix with nonzero entries in K. In this case, $\mathrm{val}(X) = (\mathrm{val}(x_{ij}))$ is a matrix in $\mathbb{Z}^{d\times d}$.

Fix any square matrix $M = (m_{ij})$ in $\mathbb{Z}^{d\times d}$. This paper revolves around the interplay between the following two objects associated with M, one algebraic and the other geometric:

1. the set $\Lambda_M = \{X \in K^{d\times d} : \mathrm{val}(X) \geq M\}$, an $\mathcal{O}_K$-lattice in the vector space $K^{d\times d}$;
2. the set $Q_M = \{u \in \mathbb{R}^d/\mathbb{R}\mathbf{1} : u_i - u_j \leq m_{ij} \text{ for } 1 \leq i, j \leq d\}$, where $\mathbf{1} = (1, \ldots, 1)$.

This interplay is strongest and most interesting when Λ_M is closed under multiplication. In this case, Λ_M is a non-commutative ring of matrices. Such a ring is called an *order* in $K^{d\times d}$. The quotient space $\mathbb{R}^d/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^{d-1}$ is the usual setting for tropical geometry (Joswig 2022; Maclagan and Sturmfels 2015). Note that Q_M is a convex polytope in that space. It is also tropically convex, for both the min-plus algebra and the max-plus algebra. Following (Joswig and Kulas 2010; Tran 2017), we use the term *polytrope* for Q_M.

Example 1 For $d = 4$, fix the matrix with diagonal entries 0 and off-diagonal entries 1:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{1}$$

The polytrope Q_M is the set of solutions to the 12 inequalities $u_i - u_j \leq 1$ for $i \neq j$. It is the 3-dimensional polytope shown in Fig. 1. Namely, Q_M is a *rhombic dodecahedron*, with 14 vertices, 24 edges and 12 facets. The vertices are the images in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ of the 14 vectors in $\{0, 1\}^4 \backslash \{\mathbf{0}, \mathbf{1}\}$. Vertices e_i are blue, vertices $e_i + e_j$ are yellow, and vertices $e_i + e_j + e_k$ are red.

The order Λ_M consists of all 4×4 matrices with entries in the valuation ring $\mathcal{O}_K$ whose off-diagonal elements lie in the maximal ideal $\langle p \rangle$. We shall see in Theorem 16 that the blue and red vertices encode the injective modules and the projective modules of Λ_M respectively.

The connection between algebra, geometry and combinatorics we present was pioneered by Plesken and Zassenhaus. Our primary source on their work is the book (Plesken 1983). One objective of this article is to give an exposition of their results using the framework of tropical geometry (Joswig 2022; Maclagan and Sturmfels 2015). But we also present a range of new results. Our presentation is organized as follows.

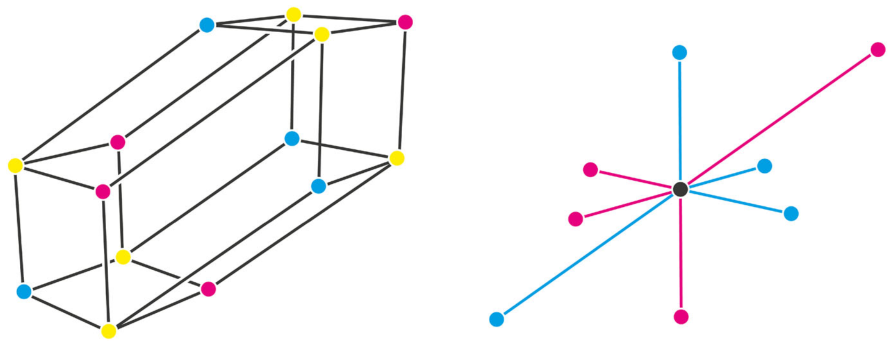

Fig. 1 The polytrope Q_M on the left is a rhombic dodecahedron. The four blue vertices and the four red vertices, highlighted on the right, will play a special role for the order Λ_M (colour figure online)

Section 2 concerns graduated orders in $K^{d\times d}$. In Propositions 6 and 7 we present linear inequalities that characterize these orders and the lattices they act on. These inequalities play an important role in tropical convexity, to be explained in Sect. 3. Theorem 10 gives a tropical matrix formula for the Plesken-Zassenhaus order of a collection of diagonal lattices.

In Sect. 4 we introduce polytrope regions. These are convex cones and polyhedra whose integer points represent graduated orders. Section 5 is concerned with (fractional) ideals in an order Λ_M. These are parametrized by the ideal class polytrope $\mathcal{Q}_M$. In Sect. 6 we turn to Bruhat-Tits buildings and their chambers. While the present study is restricted to Plesken-Zassenhaus orders arising from one single apartment, it sets the stage for a general theory.

Several results in this article were found by computations. The codes and all data are made available at https://mathrepo.mis.mpg.de/OrdersPolytropes/index.html.

2 Graduated orders

By a *lattice* in K^d we mean a free $\mathcal{O}_K$-submodule of rank d. Two lattices L and L' are equivalent if $L' = p^n L$ for some $n \in \mathbb{Z}$. We write $[L] = \{p^n L : n \in \mathbb{Z}\}$ for the equivalence class of L. An *order* in $K^{d\times d}$ is a lattice in the d^2-dimensional vector space $K^{d\times d}$ that is also a ring. Thus, every order contains the identity matrix. An order Λ is *maximal* if it is not properly contained in any other order. One example of a maximal order is the matrix ring

$$\mathcal{O}_K^{d\times d} \;:=\; \{X \in K^{d\times d} : \mathrm{val}(x_{ij}) \geq 0 \text{ for all } 1 \leq i, j \leq d\}.$$

This is spanned as an $\mathcal{O}_K$-lattice by the matrix units E_{ij} where $1 \leq i, j \leq d$. It is multiplicatively closed because $E_{ij}E_{jk} = E_{ik}$. We begin with some standard facts found in Plesken (1983). The first is a natural bijection between lattice classes $[L]$ in K^d and maximal orders in $K^{d\times d}$.

Proposition 2 *Any order Λ in $K^{d\times d}$ is contained in the endomorphism ring of a lattice $L \subset K^d$. The maximal orders in $K^{d\times d}$ are exactly the endomorphism rings of lattices L:*

$$\mathrm{End}_{\mathcal{O}_K}(L) := \{X \in K^{d\times d} : XL \subseteq L\}.$$

Two lattices L and L' in K^d are equivalent if and only if $\mathrm{End}_{\mathcal{O}_K}(L) = \mathrm{End}_{\mathcal{O}_K}(L')$.

Proof Let $\Lambda = \bigoplus_{j=1}^{d^2} \mathcal{O}_K X_j$ be an order in $K^{d\times d}$. If we apply the matrices X_j to the *standard lattice* $L_0 = \mathcal{O}_K^d = \bigoplus_{i=1}^{d} \mathcal{O}_K e_i$, then we obtain the following lattice in K^d:

$$L := \sum_{j=1}^{d^2} X_j L_0 = \sum_{i=1}^{d}\sum_{j=1}^{d^2} \mathcal{O}_K X_j\, e_i.$$

Since Λ is multiplicatively closed, we have $X_j L \subseteq L$ for all j. Therefore $\Lambda \subseteq \mathrm{End}_{\mathcal{O}_K}(L)$.

Endomorphism rings of lattices are orders. Indeed, if $L = gL_0$ for $g \in \mathrm{GL}_d(K)$, then

$$\mathrm{End}_{\mathcal{O}_K}(L) = g\,\mathrm{End}_{\mathcal{O}_K}(L_0)g^{-1} = g\,\mathcal{O}_K^{d\times d} g^{-1}. \tag{2}$$

This is a ring, and it is spanned as an $\mathcal{O}_K$-lattice by $\{gE_{ij}g^{-1} : 1 \le i, j \le d\}$. This allows to conclude that the maximal orders are exactly the endomorphism rings of lattices. □

For $u \in \mathbb{Z}^d$ we abbreviate $g_u = \mathrm{diag}(p^{u_1}, p^{u_2}, \ldots, p^{u_d})$. This diagonal matrix transforms the standard lattice $\mathcal{O}_K^d$ to $L_u = g_u\mathcal{O}_K^d$. The endomorphism ring $\mathrm{End}_{\mathcal{O}_K}(L_u)$ is the maximal order in (2). Let $M(u)$ denote the $d \times d$ matrix whose entry in position (i, j) equals $u_i - u_j$.

Lemma 3 *The endomorphism ring of the lattice L_u is given by valuation inequalities:*

$$\mathrm{End}_{\mathcal{O}_K}(L_u) = \Lambda_{M(u)} = \{X \in K^{d\times d} : \mathrm{val}(X) \ge M(u)\}. \tag{3}$$

Proof The elements of $\mathrm{End}_{\mathcal{O}_K}(L_u)$ are the matrices $X = g_u Y g_u^{-1}$ where $Y \in \mathcal{O}_K^{d\times d}$. Writing $X = (x_{ij})$ and $Y = (y_{ij})$, the equation $X = g_u Y g_u^{-1}$ means that $x_{ij} = p^{u_i - u_j} y_{ij}$ for all i, j. The condition $\mathrm{val}(y_{ij}) \ge 0$ is equivalent to $\mathrm{val}(x_{ij}) \ge u_i - u_j$. Taking the conjunction over all (i, j), we conclude that $\mathrm{val}(Y) \ge 0$ is equivalent to the desired inequality $\mathrm{val}(X) \ge M(u)$. □

The matrices $M(u)$ are characterized by the following two properties. All diagonal entries are zero and the tropical rank is one, cf. (Maclagan and Sturmfels 2015, Section 5.3). What happens if we replace $M(u)$ in (3) by an arbitrary matrix $M \in \mathbb{Z}^{d\times d}$? Then we get the set Λ_M from the Introduction.

Remark 4 For any matrix $M \in \mathbb{Z}^{d\times d}$, the set Λ_M is a lattice in $K^{d\times d}$. It is generated as an $\mathcal{O}_K$-module by the matrices $p^{m_{ij}} E_{ij}$ for $1 \le i, j \le d$. The lattice Λ_M may not be an order.

Write $\mathbb{Z}_0^{d\times d}$ for the set of integer matrices M with zeros on the diagonal, i.e. $m_{ii} = 0$ for all i. If M lies in $\mathbb{Z}_0^{d\times d}$ then Λ_M contains the identity matrix, but may still not be an order.

Example 5 Let $K = \mathbb{Q}$ with the p-adic valuation, for some prime $p \ge 5$. For $d = 3$, set

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{so} \quad X^2 = \begin{bmatrix} 2+p & 2+p & 1+2p \\ 3 & 3 & 2+p \\ 3 & 3 & 2+p \end{bmatrix}.$$

Since $\mathrm{val}(X) = M$ and $\mathrm{val}(X^2) = 0$, we have $X \in \Lambda_M$ but $X^2 \notin \Lambda_M$. So Λ_M is not an order.

The inequalities derived in the next two propositions are the main points of this section. These results are due to Plesken (1983). He states them in Plesken (1983, Definition II.2) and (Plesken, 1983, Definition II.4). The orders Λ_M in Proposition 6 are called *graduated orders* in (Plesken, 1983, Remark II.4). They are also known as *tiled orders* (Dokuchaev et al. 2017; Jategaonkar 1974), *split orders* (Shemanske 2010) or *monomial orders* (Yang and Chia-Fu 2015). A graduated order Λ_M is in *standard form* if $M \ge 0$ and $m_{ij} + m_{ji} > 0$ for $i \ne j$.

Proposition 6 *Given $M = (m_{ij})$ in $\mathbb{Z}_0^{d\times d}$, the lattice Λ_M is an order in $K^{d\times d}$ if and only if*

$$m_{ij} + m_{jk} \ge m_{ik} \quad \textit{for all} \quad 1 \le i, j, k \le d. \tag{4}$$

Proof To prove the if direction, we assume (4). Our hypothesis $m_{ii} = 0$ ensures that Λ_M contains the identity matrix, so Λ_M has a multiplicative unit. Suppose $X, Y \in \Lambda_M$. Then the (i, k) entry of XY equals $\sum_{j=1}^{d} x_{ij} y_{jk}$. This is a scalar in K whose valuation is at least $m_{ij} + m_{jk}$ for some index j. Hence it is greater than or equal to m_{ik} since (4) holds.

For the only-if direction, suppose $m_{ij} + m_{jk} < m_{ik}$. Then $X = p^{m_{ij}} E_{ij}$ and $Y = p^{m_{jk}} E_{jk}$ are in Λ_M. However, $XY = p^{m_{ij}+m_{jk}} E_{ik}$ is not in Λ_M because its entry in position (i, k) has valuation less than m_{ik}. Hence Λ_M is not multiplicatively closed, so it is not an order. □

Fix M that satisfies (4). The graduated order Λ_M is an $\mathcal{O}_K$-subalgebra of $K^{d\times d}$. It is therefore natural to ask which lattices in K^d are Λ_M-stable.

Proposition 7 *A lattice L is stable under Λ_M if and only if $L = L_u$ with $u \in \mathbb{Z}^d$ that satisfies*

$$u_i - u_j \le m_{ij} \quad \text{for} \quad 1 \le i, j \le d. \tag{5}$$

Moreover, if $u, u' \in \mathbb{Z}^d$ satisfy (5), then the diagonal lattices L_u and $L_{u'}$ are isomorphic as Λ_M-modules if and only if they are equivalent, i.e. $u = u'$ in the quotient space $\mathbb{R}^d/\mathbb{R}\mathbf{1}$.

Proof Fix a lattice L and let $u = (u_1, \ldots, u_d)$ be defined by $u_i = \min\{\mathrm{val}(b_i) : b \in L\}$. Then $L \subseteq L_u$ because every $b \in L$ is an $\mathcal{O}_K$-linear combination of the standard basis of L_u, namely $b = \sum_{i=1}^d b_i e_i = \sum_{i=1}^d (b_i\, p^{-u_i})\, p^{u_i} e_i$. Suppose that L is Λ_M-stable. Since $m_{ii} = 0$, we have $E_{ii} \in \Lambda_M$. Hence $E_{ii}\, b = b_i e_i \in L$ for every $b \in L$. This implies $L_u \subseteq L$ and hence $L = L_u$. Applying $p^{m_{ij}} E_{ij} \in \Lambda_M$ to $p^{u_j} e_j \in L_u$, we see that $p^{m_{ij}+u_j} e_i$ lies in L_u, and this implies $m_{ij} + u_j \geq u_i$. Hence (5) holds. Conversely, suppose that (5) holds. Then the generator $p^{m_{ij}} E_{ij}$ of Λ_M maps each basis vector $p^{u_k} e_k$ of L_u either to zero (if $j \neq k$), or to $p^{m_{ik}+u_k} e_i \in L_u$. This proves the first assertion.

For the second assertion, let $u, u' \in \mathbb{Z}^d$ satisfy (5). Since multiplication by $\alpha \in K^*$ is an isomorphism of $\mathcal{O}_K$-modules, the if-direction is clear. Conversely, if L_u and $L_{u'}$ are isomorphic, then there exists $g \in \mathrm{GL}_d(K)$ such that $L_{u'} = g L_u$ and $gX = Xg$ for all $X \in \Lambda_M$. Pick $s \in \mathbb{Z}_{>0}$ such that $p^s \mathcal{O}_K^{d\times d} \subset \Lambda_M$. Then g commutes with every matrix in $p^s \mathcal{O}_K^{d\times d}$. This implies that g is central in $\mathcal{O}_K^{d\times d}$, and therefore g is a multiple of the identity matrix. □

3 Bi-tropical convexity

We now develop the relationship between graduated orders and tropical mathematics (Joswig 2022; Maclagan and Sturmfels 2015). Both the *min-plus algebra* $(\mathbb{R}, \underline{\oplus}, \odot)$ and the *max-plus algebra* $(\mathbb{R}, \overline{\oplus}, \odot)$ will be used. Its arithmetic operations are the minimum, maximum, and classical addition of real numbers:

$$a \,\underline{\oplus}\, b \;=\; \min(a,b)\,, \quad a \,\overline{\oplus}\, b \;=\; \max(a,b)\,, \quad a \odot b \;=\; a+b \quad \text{for} \quad a, b \in \mathbb{R}.$$

If M and N are real matrices, and the number of columns of M equals the number of rows of N, then we write $M \,\underline{\odot}\, N$ and $M \,\overline{\odot}\, N$ for their respective matrix products in these algebras.

Example 8 Consider the 2×2 matrices $M = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We find that

$$M \,\underline{\odot}\, M = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \; M \,\underline{\odot}\, N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; N \,\underline{\odot}\, M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; N \,\underline{\odot}\, N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M \,\overline{\odot}\, M = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}, \; M \,\overline{\odot}\, N = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, \; N \,\overline{\odot}\, M = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \; N \,\overline{\odot}\, N = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

There are two flavors of tropical convexity (Maclagan and Sturmfels 2015, Section 5.2). A subset of $\mathbb{R}^d$ is *min-convex* if it is closed under linear combinations in the min-plus algebra, and *max-convex* if the same holds for the max-plus algebra. Thus convex sets are images of matrices under linear maps.

We are especially interested in bi-tropical convexity in the ambient space $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. This is ubiquitous in (Joswig, 2022, Section 5.4) and (Maclagan and Sturmfels, 2015). Joswig (2022, Section 1.4) calls it the *tropical projective torus*. At a later stage, we also work in the corresponding matrix space $\mathbb{R}^{d\times d}/\mathbb{R}\mathbf{1}$.

Let $\mathbb{R}_0^{d\times d}$ denote the space of real $d \times d$ matrices with zeros on the diagonal, which is a real (d^2-d)-dimensional vector space with lattice $\mathbb{Z}_0^{d\times d}$. For $M = (m_{ij})$ in $\mathbb{R}_0^{d\times d}$, we define

$$Q_M \;=\; \left\{ u \in \mathbb{R}^d/\mathbb{R}\mathbf{1} \,:\, u_i - u_j \;\le\; m_{ij} \;\text{ for } 1 \le i, j \le d \right\}. \tag{6}$$

Such a set is known as a *polytrope* in tropical geometry (Joswig and Kulas 2010; Maclagan and Sturmfels 2015). Other communities use the terms *alcoved polytope* and *weighted digraph polytope*. We note that Q_M is both min-convex and max-convex (Joswig 2022, Proposition 5.30) and, being a polytope, it is also classically convex.

Using tropical arithmetic, the linear inequalities in (4) can be written concisely as

$$M \odot M \;=\; M. \tag{7}$$

Thus, M is *min-plus idempotent*. This holds for M in Example 8. Joswig's book (Joswig 2022, Section 3.3) uses the term *Kleene star* for matrices $M \in \mathbb{R}_0^{d\times d}$ with (7). Propositions 6 and 7 imply:

Corollary 9 *The lattice Λ_M is an order in $K^{d\times d}$ if and only if (7) holds. In this case, the integer points u in the polytrope Q_M are in bijection with the isomorphism classes of Λ_M-lattices L_u. Here, by a Λ_M-*lattice *we mean a Λ_M-module that is also a lattice in K^d.*

Let $\Gamma = \{L_1, \ldots, L_n\}$ be a finite set of lattices in K^d, which might be taken up to equivalence. The intersection of two orders in $K^{d\times d}$ is again an order. Hence the intersection

$$\mathrm{PZ}(\Gamma) \;=\; \mathrm{End}_{\mathcal{O}_K}(L_1) \,\cap\, \cdots \,\cap\, \mathrm{End}_{\mathcal{O}_K}(L_n) \tag{8}$$

is an order in $K^{d\times d}$. We call $\mathrm{PZ}(\Gamma)$ the *Plesken-Zassenhaus order* of the configuration Γ.

In the following we assume that each L_i is a *diagonal lattice*, i.e. $L_i = L_{u^{(i)}}$ for $u^{(i)} \in \mathbb{Z}^d$. Our next result involves a curious mix of max-plus algebra and min-plus algebra.

Theorem 10 *Let $\Gamma = \{L_{u^{(1)}}, \ldots, L_{u^{(n)}}\}$ be any configuration of diagonal lattices in K^d. Then its Plesken-Zassenhaus order $PZ(\Gamma)$ coincides with the graduated order Λ_M where*

$$M \;=\; M(u^{(1)}) \,\overline{\oplus}\, M(u^{(2)}) \,\overline{\oplus}\, \cdots \,\overline{\oplus}\, M(u^{(n)}). \tag{9}$$

This max-plus sum of tropical rank one matrices is min-plus idempotent, i.e. (4) and (7) hold.

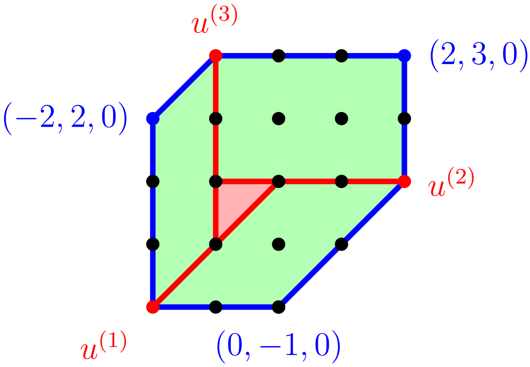

Fig. 2 A polytrope with three min-plus vertices (blue) and three max-plus vertices (red) (colour figure online)

Proof We regard Γ as a configuration in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. By construction, M is the entrywise smallest matrix such that Γ is, contained in the polytrope Q_M. From (Joswig, 2022, Lemma 3.25) the matrix M is a Kleene star, that is (4) and (7) hold. The intersection in (8) is defined by the conjunction of the n inequalities $\mathrm{val}(X) \geq M(u^{(i)})$, which is equivalent to $\mathrm{val}(X) \geq M$. □

Example 11 For $d = n = 3$, fix $u^{(1)} = (-2,-1,0)$, $u^{(2)} = (2,1,0)$, $u^{(3)} = (-1,3,0)$ in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. The configuration $\Gamma = \{u^{(1)}, u^{(2)}, u^{(3)}\}$ consists of the three red points in Fig. 2. The red diagram is their min-plus convex hull. This tropical triangle consists of a classical triangle together with three red line segments connected to Γ. This red min-plus triangle is not convex. The green shaded hexagon is the polytrope spanned by Γ. By (Joswig, 2022, Remark 5.33), this is the geodesic convex hull of Γ. It equals Q_M where M is computed by (9):

$$M = (u^{(1)})^t \odot (-u^{(1)}) \;\overline{\oplus}\; (u^{(2)})^t \odot (-u^{(2)}) \;\overline{\oplus}\; (u^{(3)})^t \odot (-u^{(3)}) = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}.$$

The polytrope Q_M is both a min-plus triangle and a max-plus triangle. Its min-plus vertices, shown in blue, are equal in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ to the columns of M. Its max-plus vertices, shown in red, are the points $u^{(i)}$. These are equal in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ to the columns of $-M^t$; cf. Theorem 16. Moreover, the three green cells correspond to the collection of homothety classes of lattices contained in $u^{(i)} \underline{\oplus}\, u^{(j)}$ and containing $u^{(i)} \overline{\oplus}\, u^{(j)}$, for each choice of $i \neq j$.

Remark 12 All lattices L_u for $u \in Q_M$ are indecomposable as Λ_M-modules, cf. (Plesken, 1983). This is no longer true if $\mathbb{R}$ is enlarged to the tropical numbers $\mathbb{R} \cup \{\infty\}$. The combinatorial theory of polytropes in (Joswig, 2022) is set up for this extension, and it indeed makes sense to study orders Λ_M with $m_{ij} = \infty$. While we do not pursue this here, our approach would extend to that setting.

Example 13 Set $d = 4$. The rhombic dodecahedron in Example 1 was called the *pyrope* in (Joswig and Kulas, 2010, Figure 4) and can be seen as the unit ball with respect to the tropical metric, cf. (Cohen et al. 2004, §3.3). This Q_M is a tropical tetrahedron for both min-convexity and max-convexity. The respective vertices are shown in red and blue in Fig. 1. We have $\Lambda_M = \mathrm{PZ}(\Gamma)$ where Γ is either set of four vertices. The Λ_M-lattices L_u correspond to the 15 integer points in Q_M.

4 Polytrope regions

We next introduce a cone that parametrizes all graduated orders Λ_M. Following (Tran, 2017), the *polytrope region* $\mathcal{P}_d$ is the set of all min-plus idempotent matrices $M \in \mathbb{R}_0^{d\times d}$. Thus, $\mathcal{P}_d$ is the (d^2-d)-dimensional convex polyhedral cone defined by the linear inequalities in (4). The equations $m_{ik} = m_{ij} + m_{jk}$ define the cycle space of the complete bidirected graph $\mathcal{K}_d$. This is the lineality space of $\mathcal{P}_d$. Modulo this $(d-1)$-dimensional space, the polytrope region $\mathcal{P}_d$ is a pointed cone of dimension $(d-1)^2$. We view it as a polytope of dimension d^2-2d. Each inequality $m_{ik} \le m_{ij} + m_{jk}$ is facet-defining, so the number of facets of $\mathcal{P}_d$ is $d(d-1)(d-2)$.

It is interesting but difficult to list the vertices of $\mathcal{P}_d$ and to explore the face lattice. The same problem was studied by Avis (1980) for the *metric cone*, which is the restriction of $\mathcal{P}_d$ to the subspace of symmetric matrices in $\mathbb{R}_0^{d\times d}$. A website maintained by Antoine Deza (2021) reports that the number of rays of the metric cone equals $3, 7, 25, 296, 55226, 119269588$ for $d = 3, 4, 5, 6, 7, 8$. We here initiate the census for the polytrope region. The following tables report the size of the orbit, the number of incident facets, and a representative matrix $[m_{ij}]$. Here orbit and representatives refer to the natural action of the symmetric group S_d on $\mathcal{P}_d$. The matrices $[m_{ij}]$ in $\mathbb{Z}_0^{3\times 3}$ are written in the vectorized format $[m_{12}m_{13}m_{21}m_{23}m_{31}m_{32}]$.

Proposition 14 *The polytope $\mathcal{P}_3$ is a bypramid, with f-vector* $(5, 9, 6)$. *Its five vertices are*

$$3\,,\ 4\,[001100] \quad \text{and} \quad 2\,,\ 3\,[001110].$$

The polytope $\mathcal{P}_4$ has the f-vector $(37, 327, 1140, 1902, 1680, 808, 204, 24)$. *Its* 37 *vertices are*

12, 10[111011001001]	6, 12 [111011001000]	12, 14[011011001000]
3, 16 [011011000000]	4, 18[111000000000].	

The corresponding polytropes Q_M are pyramid, tetrahedron, triangle, segment, and segment. The 15*-dimensional polytope $\mathcal{P}_5$ has* 2333 *vertices in* 33 *symmetry classes. These classes are*

5, 48 [00000000000000001111]	10, 18 [00001001211121111100]	10, 42 [00000000000011101110]
20, 15 [00002012323231012201]	20, 21 [00001000110021112111]	20, 39 [00000000000011101111]
24, 20 [00001001210122111110]	24, 30 [00001000110011101111]	30, 24 [00001000110121111110]
30, 30 [00000000110011111111]	30, 30 [00000000110111111110]	30, 36 [00000000110011001111]
40, 18 [00002000221222212212]	60, 18 [00001000210122112110]	60, 18 [00001001210122121100]
60, 22 [00001000110122111110]	60, 27 [00001000110011102111]	60, 29 [00000000110011102211]
60, 33 [00000000110011101111]	120, 16 [00001001220132122110]	120, 17 [00001001210122122110]
120, 18 [00001001210122112110]	120, 18 [00001001210122122210]	120, 18 [00001001210222122110]
120, 18 [00001001220132213210]	120, 19 [00001000210022103221]	120, 19 [00001000210122122110]
120, 19 [00001001210122212210]	120, 22 [00001000110021102221]	120, 22 [00001000110122121110]
120, 23 [00001000110021102211]	120, 23 [00001000110021102222]	120, 25 [00001000110011102211]

Proof This was found by computations with `Polymake` (Gawrilow and Joswig 2000); see our `mathrepo` site. □

Remark 15 The integer matrices M in the polytrope region $\mathcal{P}_d$ represent the graduated orders $\Lambda_M \subset K^{d\times d}$. The data above enables us to sample from these orders. A variant of $\mathcal{P}_d$ that assumes nonnegativity constraints was studied in (Deza et al. 2002), which offers additional data. We also refer to (Dokuchaev et al. 2017) for a study of the cone of polytropes from the perspective of semiring theory.

Our next result relates the structure of a polytrope Q_M to that of its graduated order Λ_M.

Theorem 16 *Let $M \in \mathcal{P}_d$ be in standard form. The $(d-1)$-dimensional polytrope Q_M is both a min-plus simplex and a max-plus simplex. The min-plus vertices u are the columns of M. They represent precisely those modules L_u over the order Λ_M that are projective. The max-plus vertices v are the columns of $-M^t$, and they represent the injective Λ_M-modules L_v.*

Proof Thanks to (Joswig and Kulas, 2010, Theorem 7), full-dimensional polytropes are tropical simplices, with vertices given by the columns of the defining matrix M. We know from bi-tropical convexity (Joswig 2022, Proposition 5.30) that Q_M is both min-convex and max-convex, so it is a simplex in both ways. This duality corresponds to swapping M with its negative transpose $-M^t$. Note its appearence in (Maclagan and Sturmfels, 2015, Theorem 5.2.21). The connection to projective and injective modules appears in parts (v) and (vii) of (Plesken, 1983, Remark II.4). For completeness, we sketch a proof.

Recall that a module is projective if and only if it is a direct summand of a free module. Let $m^{(1)}, \ldots, m^{(d)}$ denote the columns of M. The lattice associated to the j-th column equals

$$L_{m^{(j)}} \;=\; \left\{ x \in K^d \;:\; \mathrm{val}(x_i) \geq m_{ij} \;\text{ for } i = 1, \ldots, d \right\}.$$

Taking the direct sum of these d lattices gives the following identification of $\mathcal{O}_K$-modules:

$$\Lambda_M \;=\; L_{m^{(1)}} \oplus L_{m^{(2)}} \oplus \cdots \oplus L_{m^{(d)}}. \tag{10}$$

We see that $L_{m^{(j)}}$ is a direct summand of the free rank one module Λ_M, so it is projective.

Conversely, let P be any indecomposable projective Λ_M-module. Then $P \oplus Q \cong \Lambda_M^r$ for some module Q and some $r \in \mathbb{Z}_{>0}$. The module Λ_M^r decomposes into $r \cdot d$ indecomposables, found by aggregating r copies of (10). By the Krull-Schmidt Theorem, such decompositions are unique up to isomorphism, and hence P is isomorphic to $L_{m^{(j)}}$ for some j.

A Λ_M-module P is projective if and only if $\mathrm{Hom}_{\mathcal{O}_K}(P, \mathcal{O}_K)$ is an injective Λ_M-module, but now with the action on the right. The decomposition (10) dualizes gracefully. We derive the assertion for injective modules by similarly dualizing all steps in the argument above. □

In relation to Theorem 16 we remark that the columns and negative rows of M also have a natural interpretation as potentials in combinatorial optimization; cf. (Joswig 2022, Theorem 3.26).

Example 17 The columns of the matrix M in Example 1 are the negated unit vectors $-e_i$. The columns of $-M^t$ are the unit vectors e_i. Our color coding in Fig. 1 exhibits the two structures of Q_M as a tropical tetrahedron in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$. The four red points are the min-plus vertices, giving the projective Λ_M-modules. The four blue points are the max-plus vertices.

Given a min-plus idempotent matrix $M \in \mathcal{P}_d$, its *truncated polytrope region* is

$$\mathcal{P}_d(M) \;=\; \{N \in \mathcal{P}_d \;:\; N \leq M\}. \tag{11}$$

This polytope has dimension $d^2 - d$ if M is in the interior of $\mathcal{P}_d$. It parametrizes all subpolytropes of Q_M, i.e. all the polytropes Q_N contained in Q_M, as the following lemma shows.

Lemma 18 *Given matrices M in $\mathcal{P}_d$ and N in $\mathbb{R}_0^{d\times d}$ such that $Q_N \subseteq Q_M$, there exists a matrix C in the truncated polytrope region $\mathcal{P}_d(M)$ such that $Q_N = Q_C$.*

Proof For each choice of i and j, we define $c_{ij} = \max\{u_i - u_j : u \in Q_N\}$. The matrix $C = (c_{ij})$ lives in $\mathbb{R}_0^{d\times d}$ and has the property that $Q_N = Q_C$. Moreover, since Q_N is contained in Q_M, we have $C \leq M$. The fact that $C \odot C = C$ follows from the definition of the c_{ij}'s and (4). In particular, C belongs to the truncated polytrope region $\mathcal{P}_d(M)$. □

On the algebraic side, $\mathcal{P}_d(M)$ parametrizes all $\mathcal{O}_K$-orders Λ_N that contain the given order Λ_M. Here M and N are assumed to be integer matrices. In particular, the integer points u in Q_M correspond to maximal orders $\Lambda_{M(u)} = \mathrm{End}_{\mathcal{O}_K}(L_u)$ that contain Λ_M; cf. Proposition 2.

Example 19 Let M be the $d \times d$ matrix with entries 0 on the diagonal and 1 off the diagonal. Thus Q_M is the pyrope (Joswig and Kulas 2010, § 3). We consider two cases: the hexagon ($d = 3$) and Example 1 ($d = 4$). The truncated polytrope region $\mathcal{P}_d(M)$ classifies subpolytropes of Q_M.
d=3: The 6-dimensional polytope $\mathcal{P}_3(M)$ has the f-vector (36, 132, 199, 151, 60, 12). Its 36 vertices come in ten symmetry classes. We list the corresponding 3×3 matrices:

1, 6 [1,1,1,1,1,1] 2, 6 $[1,\frac{1}{2},\frac{1}{2},1,1,\frac{1}{2}]$ 3, 8 [0,–1, 0,–1, 1, 1] 3, 8 [1, 0,–1,–1, 0, 1] 3, 8 [1,0,1,1,0,1]
3, 6 [1,1,1,1,0,0] 3, 6 [0,1,1,1,1,0] 6, 7 [0, –1, 1, 0, 1, 1] 6, 7 [1, 1, 1, 1, 0, 1] 6, 6 [0,0,1,1,1,0]

These polytropes are shown in red in Fig. 3. Our classification into S_3-orbits is finer than that from symmetries of the hexagon Q_M, which leads to only eight orbits. For us, this classification is more natural because it reflects algebraic properties of orders. It distinguishes min-plus vertices from max-plus vertices of Q_M. The polytope $\mathcal{P}_3(M)$ has 41 integer points, so there are 41 orders containing Λ_M. In addition to 34 integer

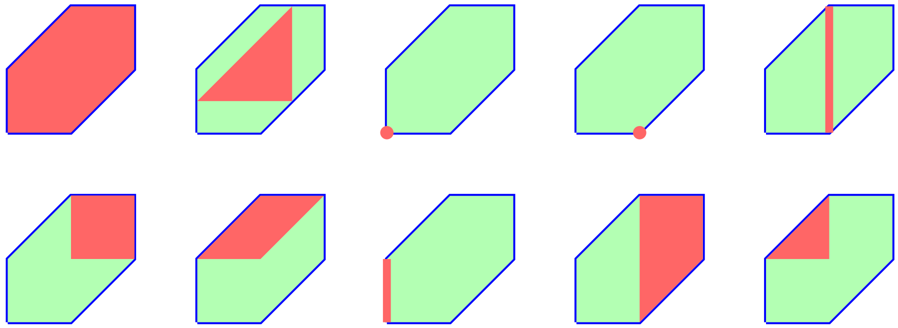

Fig. 3 The regular hexagon has 36 extreme subpolytropes in ten symmetry classes

vertices, there are seven interior integer points, namely [0, 0, 0, 0, 0, 0] and six like [0, 0, 0, 0, 1, 1], not seen in Fig. 3.
d=4: The truncated polytrope region $\mathcal{P}_4(M)$ for (1) is 12-dimensional. Its f-vector is

$$(961, 17426, 103780, 304328, 517293, 549723, 377520, 168720, 48417, 8620, 894, 48).$$

The 961 vertices come in 65 orbits under the S_4-action. Among the simple vertices we find:

1, 12	[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	8, 12	$[1, 1, 1, 1, \frac{1}{2}, 1, 1, 1, \frac{1}{2}, 1, \frac{1}{2}, 1]$
4, 27	$[1, 1, 1, -1, 0, 0, -1, 0, 0, -1, 0, 0]$	4, 27	$[-1, -1, -1, 1, 0, 0, 1, 0, 0, 1, 0, 0]$

The list of all vertices, and much more, is made available at our `mathrepo` site. Such data sets can be useful for computational studies of $\mathcal{O}_K$-orders in $K^{d\times d}$.

5 Ideals

To better understand the order Λ_M for $M \in \mathcal{P}_d$, we study its (fractional) ideals. By an *ideal* of Λ_M we mean an additive subgroup I of Λ_M such that $\Lambda_M I \subseteq I$ and $I\Lambda_M \subseteq I$. A *fractional ideal* of Λ_M is a (two sided) Λ_M-submodule J of $K^{d\times d}$ such that $\alpha J \subset \Lambda_M$ for some $\alpha \in K^*$.

Example 20 Fix $X \in K^{d\times d}$ and consider the two-sided Λ_M-module

$$\langle X\rangle = \Lambda_M X \Lambda_M = \left\{AXB : A, B \in \Lambda_M\right\}.$$

This is an ideal when $X \in \Lambda_M$. If $X \notin \Lambda_M$ then $\alpha X \in \Lambda_M$ for some $\alpha \in K^*$. Hence, $\langle X\rangle$ is a fractional ideal. These are the *principal (fractional) ideals* of Λ_M.

For all that follows, we assume that $M \in \mathcal{P}_d$ is an integer matrix in standard form.

Proposition 21 *The nonzero fractional ideals of the order Λ_M are the sets of the form*

$$I_N = \left\{ X \in K^{d\times d} :\ \mathrm{val}(X) \geq N \right\}, \tag{12}$$

where $N = (n_{ij})$ *is any matrix in* $\mathbb{Z}^{d\times d}$ *with* $N \odot M = M \odot N = N$*. This is equivalent to*

$$n_{ik} \le n_{ij} + m_{jk} \quad \text{and} \quad n_{ik} \le m_{ij} + n_{jk} \qquad \text{for} \quad 1 \le i, j, k \le d. \tag{13}$$

Proof The result appears in (viii) from (Plesken, 1983, Remark II.4). The min-plus matrix identity $N \odot M = N$ is equivalent to $n_{ik} \le n_{ij} + m_{jk}$ because $m_{jj} = 0$. □

Remark 22 If N has zeros on its diagonal and satisfies (4) then $I_N = \Lambda_N$ is an order, as before. However, among all lattices in $K^{d\times d}$, ideals are more general than orders. In particular, we generally have $n_{ii} \neq 0$ for the matrices N in (12). A fractional ideal I_N is an ideal in Λ_M if and only if $N \ge M$. If this holds then the polytrope Q_N is contained in Q_M.

Example 23 The Jacobson radical of the order Λ_M is the ideal $\mathrm{Jac}(\Lambda_M) = I_{M+\mathrm{Id}_d}$. Here Id_d is the identity matrix. The quotient of Λ_M by its Jacobson radical is the product of residue fields $\Lambda_M/\mathrm{Jac}(\Lambda_M) \cong (\mathcal{O}_K/\langle p\rangle)^d$. See (i) in (Plesken, 1983, Remark II.4) for more details.

Let $\mathcal{Q}_M$ denote the set of matrices N in $\mathbb{R}^{d\times d}$ that satisfy the inequalities in (13). These inequalities are bounds on differences of matrix entries in N. We can thus regard $\mathcal{Q}_M$ as a polytrope in $\mathbb{R}^{d\times d}/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = \sum_{i,j=1}^{d} E_{ij}$. The matrices N parameterizing the fractional ideals I_N of Λ_M (up to scaling) are the integer points of $\mathcal{Q}_M$. One checks directly that $\mathcal{Q}_M$ is closed under both addition and multiplication of matrices in the min-plus algebra. Its product $\odot$ represents the multiplication of fractional ideals as the following proposition shows.

Proposition 24 *If* $M \in \mathcal{P}_d$ *is in standard form and* $N, N' \in \mathcal{Q}_M$ *then* $I_N I_{N'} = I_{N \odot N'}$.

Proof Let $X \in I_N, Y \in I_{N'}$. The inequalities $\mathrm{val}(X) \ge N$, $\mathrm{val}(Y) \ge N'$ imply $\mathrm{val}(XY) \ge \mathrm{val}(X) \odot \mathrm{val}(Y) \ge N \odot N'$ and so $XY \in I_{N \odot N'}$. This gives the inclusion $I_N I_{N'} \subseteq I_{N \odot N'}$. Let $u_{ij} = \min_{1\le k\le d}(n_{ik} + n'_{kj})$ be the (i, j) entry of $N \odot N'$. For the inclusion $I_{N \odot N'} \subseteq I_N I_{N'}$, it suffices to show that $p^{u_{ij}} E_{ij}$ is in $I_N I_{N'}$ for all i, j. Fix i, j and let k satisfy $u_{ij} = n_{ik} + n'_{kj}$. The matrices $p^{n_{ik}} E_{ik}$ and $p^{n'_{kj}} E_{kj}$ are in I_N and $I_{N'}$. Their product $p^{u_{ij}} E_{ij}$ is in $I_N I_{N'}$. □

We call $\mathcal{Q}_M$ the *ideal class polytrope* of M. The min-plus semigroup $(\mathcal{Q}_M, \odot)$ plays the role of the ideal class group in number theory. Its neutral element is the given matrix M.

Example 25 Fix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{P}_2$. The polytrope $\mathcal{Q}_M$ is the octahedron with vertices

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \in \mathbb{Z}^{2\times 2}/\mathbb{Z}\mathbf{1}.$$

This octahedron contains 19 integer points N. These are in bijection with the equivalence classes of fractional ideals I_N in the order Λ_M. The midpoint of $\mathcal{Q}_M$ corresponds to the Jacobson radical $I_{M+\mathrm{Id}_2}$. The remaining 12 integer points are the midpoints of the edges.

One may ask whether the ideal class semigroup $(\mathcal{Q}_M, \odot)$ is actually a group. To address this question, we define the *pseudo-inverse* of a fractional ideal I in the order Λ_M as follows:

$$(\Lambda_M : I) \;=\; \{\, X \in K^{d\times d} : XI \subseteq \Lambda_M \text{ and } IX \subseteq \Lambda_M\}.$$

Lemma 26 *The pseudo-inverse of a fractional ideal in Λ_M is a fractional ideal in Λ_M.*

Proof Let $A \in \Lambda_M$ and $X \in (\Lambda_M : I)$, so that $XI, IX \subseteq \Lambda_M$. Since I is a fractional ideal, we have $AI \subseteq I$ and $IA \subseteq I$. From these inclusions we deduce that XAI, IXA, AXI, IAX are all subsets of Λ_M. This implies $XA, AX \in (\Lambda_M : I)$. Hence $(\Lambda_M : I)$ is a fractional ideal. □

Proposition 27 *Let $M \in \mathcal{P}_d$ in standard form and $N \in \mathcal{Q}_M$. Then $(\Lambda_M : I_N) = I_{N'}$ where*

$$n'_{ij} \;=\; \max_{1\le \ell\le d}\big(\max(m_{\ell j} - n_{\ell i}, m_{i\ell} - n_{j\ell})\big) \qquad \text{for } 1 \le i, j \le d. \tag{14}$$

Proof By Proposition 21 and Lemma 26, there exists $N' \in \mathcal{Q}_M$ such that $I_{N'} = (\Lambda_M : I_N)$. Then $I_{N'} I_N \subseteq \Lambda_M$ and $I_N I_{N'} \subseteq \Lambda_M$, and $I_{N'}$ is the largest fractional ideal with this property. These two conditions are equivalent to $p^{n'_{ij}} E_{ij} I_N \subseteq \Lambda_M$ and $p^{n'_{ij}} I_N E_{ij} \subseteq \Lambda_M$ for all i, j. The first condition holds if and only if $n'_{ij} + n_{j\ell} \ge m_{i\ell}$ for all ℓ. The second condition holds if and only if $n_{\ell i} + n'_{ij} \ge m_{\ell j}$ for all ℓ. The smallest solution $N' = (n'_{ij})$ is given by (14). □

Passing from ideals to their matrices, we also call N' the *pseudo-inverse* of N in $\mathcal{Q}_M$.

Example 28 Let $d = 2$ and M as in Example 25. The 19 ideal classes N in $\mathcal{Q}_M$ have only three distinct pseudo-inverses: $N' \in \left\{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\right\}$. For most ideal classes N, we have $N \odot N' \neq M$ and $N' \odot N \neq M$. This means that most N do not have an inverse in $(\mathcal{Q}_M, \odot)$. In particular, the ideal class polytrope $\mathcal{Q}_M$ is a semigroup but not a group.

The semigroup $\mathcal{Q}_M$ has the neutral element M and each ideal class $N \in \mathcal{Q}_M$ has a pseudo-inverse N' given by the formula (14). With this data, we define the *ideal class group*

$$\mathcal{G}_M \;=\; \big\{ N \in \mathcal{Q}_M \;:\; N \odot N' \;=\; N' \odot N \;=\; M \big\}.$$

This is the maximal subgroup of the semigroup $\mathcal{Q}_M$. It would be interesting to understand how M determines the structure of $\mathcal{G}_M$. Note that $\mathcal{G}_M = \left\{\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\right\}$ in Example 28.

Example 29 Here are three examples of ideal class groups of graduated orders:

$$\begin{array}{lll} M_2 = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] & M_3 = \left[\begin{smallmatrix}0&1&1\\1&0&1\\1&1&0\end{smallmatrix}\right] & M_4 = \left[\begin{smallmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{smallmatrix}\right] \\ \mathcal{G}_{M_2} \cong \mathbb{Z}/2\mathbb{Z} & \mathcal{G}_{M_3} \cong \mathbb{Z}/6\mathbb{Z} & \mathcal{G}_{M_4} \cong S_4 \end{array}$$

The isomorphism types of these groups were computed using GAP; the code is at our mathrepo site. We do not know how this list continues for pyropes (Joswig and Kulas 2010, §3) in higher dimensions.

We end this section with a conjecture about the geometry of $\mathcal{G}_M$ inside $\mathcal{Q}_M$.

Conjecture 30 For any integer matrix M in the polytrope region $\mathcal{P}_d$, the elements in the ideal class polytrope $\mathcal{G}_M$ are among the classical vertices of the ideal class group $\mathcal{Q}_M$.

6 Towards the building

Affine buildings (Abramenko and Brown 2008; Zhang 2021) provide a natural setting for orders and min-max convexity. The objects we discussed in this paper so far are associated to one apartment in this building, namely, that corresponding to the diagonal lattices. The aim of this section is to present this perspective and to lay the foundation for a general theory that goes beyond one apartment.

Definition 31 The *affine building* $\mathcal{B}_d(K)$ is an infinite simplicial complex. Its vertices are the equivalence classes $[L]$ of lattices in K^d. A configuration $\{[L_1],\dots,[L_s]\}$ is a simplex in $\mathcal{B}_d(K)$ if and only if, up to some permutation, there exist representatives $\tilde{L}_i \in [L_i]$ satisfying $\tilde{L}_1 \supset \tilde{L}_2 \supset \cdots \supset \tilde{L}_s \supset p\tilde{L}_1$. The maximal simplices $\{[L_1],\dots,[L_d]\}$ are called *chambers*. The *standard chamber* C_0 is given by the diagonal lattices $L_i = L_{(\mathbf{1}_{i-1},\mathbf{0}_{d-i+1})} = L_{(1,\dots,1,0,\dots,0)}$.

Given a basis $\{b_1,\dots,b_d\}$ of K^d, the *apartment* defined by this basis is the set of classes $[L]$ of all lattices $L = \bigoplus_{i=1}^d p^{u_i}\mathcal{O}_K b_i$ where $u_1,\dots,u_d$ range over $\mathbb{Z}$. Hence the apartment is

$$\left\{\left[p^{u_1}\mathcal{O}_K b_1 \oplus \cdots \oplus p^{u_d}\mathcal{O}_K b_d\right] : u_1,\dots,u_d \in \mathbb{Z}\right\} \;=\; \left\{[gL_u] : u \in \mathbb{Z}^d\right\},$$

where $g \in \mathrm{GL}_d(K)$ is the matrix with columns $b_1,\dots,b_d$. The *standard apartment* is the one associated with the standard basis $(e_1,\dots,e_d)$ of K^d. The vertices of the standard apartment are the diagonal lattice classes $[L_u]$ for $u \in \mathbb{Z}^d$. We identify this set of vertices with $\mathbb{Z}^n/\mathbb{Z}\mathbf{1}$.

The general linear group $\mathrm{GL}_d(K)$ acts on the building $\mathcal{B}_d(K)$. This action preserves the simplicial complex structure. In fact, the action is transitive on lattice classes, on apartments and also on the chambers. The stabilizer of the standard lattice L_0 is the subgroup

$$\mathrm{GL}_d(\mathcal{O}_K) \;=\; \{\, g \in \mathcal{O}_K^{d\times d} \;:\; \mathrm{val}(\det(g)) = 0 \,\} \;\subset\; \mathrm{GL}_d(K).$$

Starting from the standard chamber C_0, there exist reflections $s_0, s_1, \dots, s_{d-1}$ in $\mathrm{GL}_d(K)$ that map C_0 to the d adjacent chambers in the standard apartment. For $i \geq 1$, define s_i by

$$s_i(e_i) = e_{i+1},\quad s_i(e_{i+1}) = e_i \;\text{ and }\; s_i(e_j) = e_j \text{ when } j \neq i, i+1.$$

The map s_0 is defined by $s_0(e_i) = e_i$ for $i = 2, \dots, d-1$ and $s_0(e_d) = pe_1$, $s_0(e_1) = p^{-1}e_d$. The reflections $s_0, \dots, s_{d-1}$ are Coxeter generators for the *affine Weyl group* $W = \langle s_0, \dots, s_{d-1}\rangle$. The group W acts regularly on the chambers C in the standard apartment (Bourbaki 2002, § 1.5, Thm. 2): for every C there is a unique $w \in W$ such that $C = wC_0$. The elements of W are the matrices $h_\sigma g_u$ where $h_\sigma = (1_{i=\sigma(j)})_{i,j}$ for $\sigma \in S_d$, and $u \in \mathbb{Z}^d$ with $u_1 + \cdots + u_d = 0$. Thus W is the semi-direct product of S_d and the group of diagonal matrices g_u whose exponents sum to 0.

Our primary object of interest is the Plesken-Zassenhaus order $\mathrm{PZ}(\Gamma)$ of a finite configuration Γ in the affine building $\mathcal{B}_d(K)$. This is the intersection (8) of endomorphism rings. In this paper we studied the case when Γ lies in one apartment. In Theorem 10 we showed that $\mathrm{PZ}(\Gamma) = \Lambda_M$ where M is the matrix in $\mathcal{P}_d$ that encodes the min-max convex hull of Γ. This was used in Sections 4 and 5 to elucidate combinatorial and algebraic structures in $\mathrm{PZ}(\Gamma)$. A subsequent project will extend our results to arbitrary configurations Γ in $\mathcal{B}_d(K)$.

We conclude this article with configurations given by two chambers C, C' in $\mathcal{B}_d(K)$. We are interested in the their order $\mathrm{PZ}(C \cup C')$. A fundamental fact about buildings states that any two chambers C, C' lie in a common apartment, cf. (Bourbaki, 2002), (Abramenko and Brown, 2008). Also, since the affine Weyl group W acts regularly on the chambers of the standard apartment, we can then reduce to the case where the two chambers in question are C_0 and wC_0 for some $w = h_\sigma g_u \in W$.

Example 32 The standard chamber C_0 is encoded by $M_0 = \sum_{1\le i<j\le d} E_{ij}$. The polytrope Q_{M_0} is a simplex. The order $\mathrm{PZ}(C_0) = \Lambda_{M_0}$ consists of all $X \in \mathcal{O}_K^{d\times d}$ with $x_{ij} \in \langle p\rangle$ for $i < j$.

Let $D_u = \mathrm{val}(g_u)$ denote the tropical diagonal matrix with $u_1, \dots, u_d$ on the diagonal and $+\infty$ elsewhere. We also write $P_\sigma := \mathrm{val}(h_\sigma)$ for the tropical permutation matrix given by σ.

Proposition 33 *We have* $\mathrm{PZ}(C_0 \cup h_\sigma g_u C_0) = \Lambda_{M^{\sigma,u}}$ *where the matrix* $M^{\sigma,u}$ *equals*

$$M^{\sigma,u} = M_0 \mathbin{\overline{\oplus}} \left(P_\sigma \mathbin{\underline{\odot}} D_u \mathbin{\underline{\odot}} M_0 \mathbin{\underline{\odot}} D_{-u} \mathbin{\underline{\odot}} P_{\sigma^{-1}}\right).$$

Proof We have $\mathrm{PZ}(C_0 \cup h_\sigma g_u C_0) = \mathrm{PZ}(C_0) \cap \mathrm{PZ}(h_\sigma g_u C_0)$ and $\mathrm{PZ}(C_0) = \Lambda_{M_0}$ from Example 32. Suppose that $M \in \mathbb{Z}_0^{d\times d}$ satisfies $\mathrm{PZ}(h_\sigma g_u C_0) = \Lambda_M$. By Theorem 10, the order $\Lambda_{M_0 \overline{\oplus} M}$ is equal to $\mathrm{PZ}(C_0 \cup h_\sigma g_u C_0)$. To determine M, notice that $\mathrm{PZ}(wC_0) = h_\sigma g_u \mathrm{PZ}(C_0) g_{-u} h_{\sigma^{-1}}$. This implies the stated formula $M = P_\sigma \mathbin{\underline{\odot}} D_u \mathbin{\underline{\odot}} M_0 \mathbin{\underline{\odot}} D_{-u} \mathbin{\underline{\odot}} P_{\sigma^{-1}}$. □

We may ask for invariants of the orders $\mathrm{PZ}(C_0 \cup wC_0)$ in terms of $w \in W$. Clearly, not all polytropes in an apartment arise as the min-max convex hull of two chambers. Which graduated orders are of the form $\mathrm{PZ}(C_0 \cup wC_0)$? Which other elements w' in the affine Weyl group W give rise to the same Plesken-Zassenhaus order $\mathrm{PZ}(C_0 \cup wC_0)$ up to isomorphism?

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