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PhD Thesis In Mathematical Physics

Cycle XXXV

The Resolvent Algebra Perspective on Point Interactions - A First Glance

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Contents

Introduction

Since its early days, pioneers like P. Jordan, J. von Neumann and E. P. Wigner seemingly perceived the suitability of an algebraic formulation of quantum mechanics ([1]). Such an intent was triggered by the will of finding, on the one hand, a more phenomenological justification to some of the tools already in use and, on the other hand, of disregarding physically unjustified statements as well as mathematically untenable formalisms. Initial efforts had very little impact on the formal developments rapidly taking place, however, from the sixties, *algebraic quantum mechanics* started drawing attention of part of the mathematical physicists community, due to the R. Haag and D. Kastler's researches ([2]). A complete review of algebraic quantum mechanics goes beyond the scopes of this thesis, nevertheless, it is worth recalling the following axiom.

Axiom I (Algebra of Observables) ([3]): A physical system[1] $\mathscr{S}$ gets singled out by a pair $\left(\mathfrak{A}, \mathfrak{S}\left(\mathfrak{A}\right)\right)$, where $\mathfrak{A}$ is a unital C$^*-$algebra, $\mathfrak{S}\left(\mathfrak{A}\right)$ the set of all positive and normalized linear functionals ω on $\mathfrak{A}$. (Bounded) observables of $\mathscr{S}$ are to be found within the self-adjoint subset of $\mathfrak{A}$, while $\omega\left(a\right)$, should $a \in \mathfrak{A}$ be an observable, represents the *expectation value of a in the state ω.* $\square$

Such an axiom does not distinguish between classical and quantum systems, the difference lying in the commutativity of the algebra: non-commutative algebras take care of quantum mechanical modeling ([4]). Of course, assigning to a given physical system $\mathscr{S}$ the corresponding C$^*-$algebra of observables is an open problem. One of the first historical attempts was the *Weyl algebra* $\mathrm{CCR}\left(X, \sigma\right)$[2]: widely spread and useful in different contexts, shortcomings about its capability of covering interesting physical phenomenology exist. As a matter of fact, the following proposition holds.

Proposition ([5]): Let $\left(H_\lambda = H_0 + \lambda V, \mathcal{D}_{H_\lambda}\right)$ be a self-adjoint Hamiltonian on $L^2\left(\mathbb{R}\right)$, where H_0 is the free Hamiltonian. If $V \in L^1\left(\mathbb{R}\right) \cap L^\infty\left(\mathbb{R}\right)$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$, the automorphism $\alpha_t^\lambda\left(\cdot\right) = e^{itH_\lambda}\left(\cdot\right) e^{-itH_\lambda}$ of $\mathfrak{B}\left(L^2\left(\mathbb{R}\right)\right)$ is not an automorphism of $\mathrm{CCR}\left(\mathbb{R}^2, \sigma\right)$ unless $V = 0$. $\blacksquare$

Furthermore, in regular representations, natural observables, as bounded functions of Hamiltonians, do not belong to $\mathrm{CCR}\left(X, \sigma\right)$; these drawbacks[3] led D. Buchholz and H. Grundling to introduce a novel C$^*-$algebra, the *resolvent algebra* ([6]), as candidate for canonical quantum mechanical modeling. Such an algebra has proved capable of accommodating various interesting dynamics, for both the finite and the infinite dimensional case ([7], [8]). This thesis aims at taking a step in that direction: dealing with *point-like interactions* within the resolvent algebra setting.

[1] *Physical system* should be understood as *mathematical-physical model*.
[2] $\left(X, \sigma\right)$ is the symplectic vector space whose corresponding Weyl algebra is $\mathrm{CCR}\left(X, \sigma\right)$.
[3] Consult [6] for further insight.

The interest for Dirac delta potentials lies in their ubiquity in applications, as they represent good candidates to approximate unknown, short-scaled interactions. For example, the *Gross-Pitaevskii equation* results from a two-body delta interaction model within a Hartree-Fock description of a Bose gas; moreover, certain field theories ([9]) produce point-like potential models in the non-relativistic limit. An extensive mathematical literature has been dedicated to these singular potentials: [10] is a very well-known reference as well as the vast production authored by Gianfausto dell'Antonio and colleagues ([11], [12] to mention few).

Structure of the Thesis

Chapter 1 - 4

These are review chapters, in which main structural results about the resolvent algebra are recalled. The sources consulted are [6] and [13].

Chapter 5

This chapter contains the first original result about point interactions: given a quantum non-relativistic spinless particle moving on the real line and undergoing one up to countably many different *fixed-center* delta interactions, it is proved that $\left[e^{itH}\pi_S\left(a\right)e^{-itH}\right] \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$ for all $a \in \mathcal{R}\left(\mathbb{R}^2,\sigma\right)$ and $\left(H - i\lambda\mathbb{1}\right)^{-1} \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$, for all $\lambda \in \mathbb{R}\setminus\{0\}$, where π_S is the Schrödinger representation of the resolvent algebra $\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$ of the case and H is alternatively given by

- $-\frac{d^2}{dx^2} + \alpha\delta\left(x - x_0\right)$, $\quad \alpha \in \mathbb{R}\setminus\{0\}$, $x_0 \in \mathbb{R}$,
- $-\frac{d^2}{dx^2} + \sum_{i=1}^{N} \alpha_i\delta\left(x - x_i\right)$, $\quad \alpha_i \in \mathbb{R}\setminus\{0\}$, $x_i \in \mathbb{R} :\, x_i \neq x_j$, $i, j \in \{1,\ldots,N\}$,
- $-\frac{d^2}{dx^2} + \sum_{i=1}^{\infty} \alpha_i\delta\left(x - x_i\right)$, $\quad \{\alpha_i\}_i \in l^1\left(\mathbb{N}\right)\setminus\{0\}$, $x_i \in \mathbb{R} :\, x_i \neq x_j$, $i, j \in \mathbb{N}$.

The claimed results are proved by showing that $\left\|e^{-itH} - e^{-itH_\epsilon}\right\| \underset{\epsilon\downarrow 0}{\longrightarrow} 0$, where $\{H_\epsilon\}_\epsilon$ is a proper family of self-adjoint Hamiltonians on $L^2\left(\mathbb{R}\right)$, whose corresponding unitary time propagator belongs to $\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$, for all $t \in \mathbb{R}$ and $\epsilon > 0$. Finally, a C$^*-$dynamical system has also been singled out.

Chapter 6

This chapter addresses the problem of a one-dimensional n-body system, made up of distinguishable non-relativistic spinless particles interacting via a *two-body* delta potential. The affiliation to the resolvent algebra of the case $\mathcal{R}\left(\mathbb{R}^{2n},\sigma\right)$ of the Hamiltonian corresponding to such a system is proved: given

$$H = -\sum_{i=1}^{n} \frac{1}{2m_i}\frac{\partial^2}{\partial x_i^2} - g\sum_{1\leq i<j\leq n} \delta\left(x_i - x_j\right) \equiv H_0 - g\sum_{1\leq i<j\leq n}\delta\left(x_i - x_j\right)$$

it is shown that $\left(H - i\lambda\mathbb{1}\right)^{-1} \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^{2n},\sigma\right)\right]$ for all $\lambda \in \mathbb{R}\setminus\{0\}$ (see Corollary 6.8.1). An approximation procedure has been invoked in this case too: $\left\|\left(H - i\lambda\mathbb{1}\right)^{-1} - \left(H_\epsilon - i\lambda\mathbb{1}\right)^{-1}\right\| \underset{\epsilon\downarrow 0}{\longrightarrow} 0$ for all $\lambda \in \mathbb{R}\setminus\{0\}$, where $\{H_\epsilon\}_\epsilon$ is a proper family of self-adjoint Hamiltonians on $L^2\left(\mathbb{R}^n\right)$ such that $\left(H_\epsilon - i\lambda\mathbb{1}\right)^{-1} \in \mathcal{R}\left(\mathbb{R}^{2n},\sigma\right)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$ and $\epsilon > 0$.

Part I

The Resolvent Algebra

Chapter 1

Introduction and First Results

Definition 1.1. Let (X, σ) be a symplectic vector space. Given the set of symbols $\mathcal{G} \doteq \left\{R(\lambda, f) \mid \lambda \in \mathbb{R} \setminus \{0\}, f \in X\right\}$, let $\tilde{\mathcal{R}}_0(X, \sigma)$ be the corresponding freely generated algebra. Let, then, L be the following list of relations.

1. $R(\lambda, 0) = -\frac{i}{\lambda}\mathbb{1}$,
2. $R(\lambda, f)^* = R(-\lambda, f)$,
3. $\nu R(\nu\lambda, \nu f) = R(\lambda, f)$,
4. $R(\lambda, f) - R(\mu, f) = i(\mu - \lambda) R(\lambda, f) R(\mu, f)$,
5. $\left[R(\lambda, f), R(\mu, g)\right] = i\sigma(f, g) R(\lambda, f) R(\mu, g)^2 R(\lambda, f)$,
6. $R(\lambda, f) R(\mu, g) = R(\lambda + \mu, f + g)\left[R(\lambda, f) + R(\mu, g) + i\sigma(f, g) R(\lambda, f)^2 R(\mu, g)\right]$,

where $f, g \in X$, $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ and, in 6., $\lambda + \mu \neq 0$. Denoted by $\mathcal{I}_L$ the corresponding ideal generated in $\tilde{\mathcal{R}}_0(X, \sigma)$, the quotient $\mathcal{R}_0(X, \sigma) \doteq \tilde{\mathcal{R}}_0(X, \sigma) / \mathcal{I}_L$ is a unital $^*-$ algebra called **pre-resolvent algebra over** (X, σ). □

Remark 1.1. *1. $\mathcal{R}_0(X, \sigma)$ is non-trivial, since non-trivial representations exist: if π is the Fock representation of $CCR(X, \sigma)$ and $\left(\phi_\pi(f), \mathcal{D}_{\phi_\pi(f)}\right)$ is the corresponding self-adjoint bosonic field, a representation of $\mathcal{R}_0(X, \sigma)$ can be constructed from $\pi\left[R(\lambda, f)\right] = \left[\phi_\pi(f) - i\lambda\mathbb{1}\right]^{-1}$, for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $f \in X$.*

2. To build a C^*- algebra out of $\mathcal{R}_0(X, \sigma)$, a C^*-norm is required. The following proposition serves the purpose.

□

Proposition 1.1. *Let (X, σ) be a symplectic vector space and let $\mathcal{R}_0(X, \sigma)$ be the corresponding pre-resolvent algebra.*

1. *If $\mathcal{H}_0$ is a Hilbert space and $\pi_0 : \mathcal{R}_0(X, \sigma) \longrightarrow \mathfrak{B}(\mathcal{H}_0)$ is a bounded representation of $\mathcal{R}_0(X, \sigma)$, for all $R(\lambda, f)$,*
$$\left\|\pi_0\left[R(\lambda, f)\right]\right\| \leq \frac{1}{|\lambda|}.$$
In other terms, $\forall a \in \mathcal{R}_0(X, \sigma), \exists c_a > 0$ such that $\left\|\pi(a)\right\| \leq c_a$, for all the bounded representations π of $\mathcal{R}_0(X, \sigma)$.

2. *If ω is a positive, linear functional on $\mathcal{R}_0(X,\sigma)$, the corresponding GNS-representation is bounded.*

Proof. 1. $\mathfrak{B}(\mathcal{H})$ is a C$^*-$algebra with respect to the operator norm, therefore $\|bb^*\| = \|b\|^2$, $\forall b \in \mathfrak{B}(\mathcal{H})$. Then, given $\lambda \in \mathbb{R}\setminus\{0\}$

$$\begin{aligned}2|\lambda|\Big\|\pi_0\left[R(\lambda,f)\right]\Big\|^2 &= 2|\lambda|\Big\|\pi_0\left[R(\lambda,f)\right]\pi_0\left[R(\lambda,f)\right]^*\Big\| = \Big\|\pi_0\left[2\lambda R(\lambda,f)\,R(\lambda,f)^*\right]\Big\| = \\ &= \Big\|\pi_0\left[R(\lambda,f)-R(\lambda,f)^*\right]\Big\| = \Big\|\pi_0\left[R(\lambda,f)\right]-\pi_0\left[R(\lambda,f)\right]^*\Big\| \le \\ &\le 2\Big\|\pi_0\left[R(\lambda,f)\right]\Big\| \implies |\lambda|\Big\|\pi_0\left[R(\lambda,f)\right]\Big\| < 1.\end{aligned}$$

The estimate is independent on the particular representation considered, therefore $|\lambda|^{-1}$ may, at most, depend on $R(\lambda,f)$; in other words, $\forall a \in \mathcal{R}_0(X,\sigma)$, $\exists\, c_a \in \mathbb{R}^+$ such that $\big\|\pi(a)\big\| \le c_a$ for all the bounded representations π.

2. Given $\psi \in {}^{\mathcal{R}_0(X,\sigma)}/_{\mathcal{N}_\omega}$ with $\mathcal{N}_\omega = \big\{a \in \mathcal{R}_0(X,\sigma) \mid \omega(aa^*) = 0\big\}$, then

$$\begin{aligned}\Big\|\pi_\omega\left[R(\lambda,f)\right]\psi\Big\|^2 &= \Big\langle \pi_\omega\left[R(\lambda,f)\right]\psi, \pi_\omega\left[R(\lambda,f)\right]\psi\Big\rangle = \Big\langle \psi, \pi_\omega\left[R(\lambda,f)\right]^*\pi_\omega\left[R(\lambda,f)\right]\psi\Big\rangle = \\ &= \Big\langle \psi, \pi_\omega\left[R(\lambda,f)^*R(\lambda,f)\right]\psi\Big\rangle.\end{aligned}$$

The first resolvent formula $R(\lambda,f)^*R(\lambda,f) = (2i\lambda)^{-1}\left[R(\lambda,f)^* - R(\lambda,f)\right]$ allows for

$$\begin{aligned}\Big\|\pi_\omega\left[R(\lambda,f)\right]\psi\Big\|^2 &= \left|\frac{1}{2i\lambda}\Big\langle \psi, \Big\{\pi_\omega\left[R(\lambda,f)\right]^* - \pi_\omega\left[R(\lambda,f)\right]\Big\}\psi\Big\rangle\right| = \\ &= \frac{1}{2|\lambda|}\left|\Big\langle \psi, \pi_\omega\left[R(\lambda,f)\right]^*\psi\Big\rangle - \Big\langle \psi, \pi_\omega\left[R(\lambda,f)\right]\psi\Big\rangle\right| \le \\ &\le \frac{1}{|\lambda|}\Big\|\pi_\omega\left[R(\lambda,f)\right]\psi\Big\|\cdot\|\psi\|,\end{aligned}$$

i.e. the representation is bounded.

■

Lemma 1.2. *Let (X,σ) be a symplectic vector space and let $\mathcal{R}_0(X,\sigma)$ be the corresponding pre-resolvent algebra. Denoted by $\mathfrak{S}$ the set of all positive and normalized states on $\mathcal{R}_0(X,\sigma)$, the map*

$$\tilde{\|\cdot\|} : a \in \mathcal{R}_0(X,\sigma) \longmapsto \tilde{\|a\|} \doteq \sup_{\omega\in\mathfrak{S}} \big\|\pi_\omega(a)\big\| \in \mathbb{R}_0^+$$

is a C$^*-$seminorm on $\mathcal{R}_0(X,\sigma)$. ■

Definition 1.2. Let (X,σ) be a symplectic vector space. The corresponding **resolvent algebra** is $\mathcal{R}(X,\sigma) = \overline{{}^{\mathcal{R}_0(X,\sigma)}/_{\ker\tilde{\|\cdot\|}}}$. □

Remark 1.2. *Once* $\ker\tilde{\|\cdot\|}$ *is factored out,* $\tilde{\|\cdot\|}$ *becomes a norm; it will be denoted by* $\|\cdot\|$ *in the following.* □

Proposition 1.3. *Let (X,σ) be a symplectic vector space. For all $\lambda,\mu \in \mathbb{R}\setminus\{0\}$, $f,g \in X$*

1. $\big[R(\lambda,f), R(\mu,f)\big] = 0$*, implying that $R(\lambda,f)$ is normal in $\mathcal{R}(X,\sigma)$;*

2. *$R(\lambda,f)\,R(\mu,g)^2\,R(\lambda,f)=R(\mu,g)\,R(\lambda,f)^2\,R(\mu,g)$;*

3. *$\left\|R(\lambda,f)\right\|=|\lambda|^{-1}$;*

4. *$R(\lambda,f)$ is analytic in λ, i.e. the Neumann series*

$$R(\lambda,f)=\sum_{n=0}^{\infty}(\lambda_0-\lambda)^n\,i^n R(\lambda_0,f)^{n+1}$$

absolutely converges whenever $0<|\lambda_0-\lambda|<|\lambda_0|$ and $\lambda_0\in\mathbb{R}\setminus\{0\}$;

5. *if $T\in Sp(X,\sigma)$ is a symplectic transformation, then $R(\lambda,Tf)\equiv\alpha\left[R(\lambda,f)\right]$ extends to an automorphism of $\mathcal{R}(X,\sigma)$.*

Proof. 1. $\sigma(f,f)=0$, $\forall f\in X$ immediately gives $\left[R(\lambda,f),R(\mu,f)\right]=0$. Then, $R(-\lambda,f)=R(\lambda,f)^*$ implies $R(\lambda,f)\,R(\lambda,f)^*=R(\lambda,f)^*\,R(\lambda,f)$, i.e. $R(\lambda,f)$ is normal in $\mathcal{R}(X,\sigma)$, for all $\lambda\in\mathbb{R}\setminus\{0\}$ and $f\in X$.

2. Definition 1.1 gives

$$\begin{aligned}\left[R(\lambda,f),R(\mu,g)\right]&=i\sigma(f,g)\,R(\lambda,f)\,R(\mu,g)^2\,R(\lambda,f)\\ \left[R(\mu,g),R(\lambda,f)\right]&=i\sigma(g,f)\,R(\mu,g)\,R(\lambda,f)^2\,R(\mu,g)\,.\end{aligned}$$

By summing both expressions, the anti-commutativity of the commutator map provides the result.

3. The foregoing proposition already gives $\left\|R(\lambda,f)\right\|\le|\lambda|^{-1}$; concerning the reverse inequality, since $\left\|R(\lambda,f)\right\|=\sup_{\omega\in\mathfrak{S}}\left\|\pi_\omega\left[R(\lambda,f)\right]\right\|$, denoted by π the Fock representation of $\mathcal{R}(X,\sigma)$,

$$\left\|R(\lambda,f)\right\|\ge\left\|\pi\left[R(\lambda,f)\right]\right\|=\left\|\left[\phi_\pi(f)-i\lambda\right]^{-1}\right\|=\sup_{t\in\sigma\left(\phi_\pi(f)\right)=\mathbb{R}}\left|\frac{1}{t-i\lambda}\right|=\left|\frac{1}{\lambda}\right|.$$

4. Given $\lambda,\lambda_0\in\mathbb{R}\setminus\{0\}$ such that $\lambda\neq\lambda_0$,

$$\begin{aligned}&R(\lambda,f)-R(\lambda_0,f)=i(\lambda_0-\lambda)\,R(\lambda,f)\,R(\lambda_0,f)\implies\\ &\qquad\implies R(\lambda,f)=R(\lambda_0,f)\left[\mathbb{1}-i(\lambda_0-\lambda)\,R(\lambda_0,f)\right]^{-1}.\end{aligned}$$

If $\left\|\left[\mathbb{1}-i(\lambda_0-\lambda)\,R(\lambda_0,f)\right]^{-1}\right\|<1$, $\left[\mathbb{1}-i(\lambda_0-\lambda)\,R(\lambda_0,f)\right]^{-1}$ would be the sum of a geometric series and, since $\left\|R(\lambda_0,f)\right\|=|\lambda_0|^{-1}$, this is the case whenever $|\lambda_0-\lambda|<|\lambda_0|$, therefore

$$R(\lambda,f)=R(\lambda_0,f)\sum_{n=0}^{\infty}i^n(\lambda_0-\lambda)^n\,R(\lambda_0,f)^n\,.$$

5. The rule $\alpha\left[R(\lambda,f)\right]\doteq R(\lambda,Tf)$ automorphically maps generating elements of $\mathcal{R}_0(X,\sigma)$ into themselves. Moreover, since T is symplectic, the ideal resulting from the defining relations gets unchanged, hence such an α extends to an automorphism of $\mathcal{R}_0(X,\sigma)$. Finally, $\mathfrak{S}$ is invariant under α, therefore the C*−norm is preserved.

■

Remark 1.3. *As a matter of fact the foregoing Neumann series absolutely converges* $\forall\lambda \in \mathbb{C}$ *such that* $|\lambda - \lambda_0| < |\lambda_0|$, *as long as* λ *is not purely imaginary. Infact,* $R(\lambda, f)$ *can be analytically continued to* $\mathbb{C} \setminus i\mathbb{R}$ *and* $\mathcal{R}(X, \sigma)$ *can be defined as the enveloping* C^*- *algebra of the unital* $^*-$ *algebra* $\mathfrak{R}_0(X, \sigma)$, *constructed out of* $\left\{R(z, f) \mid z \in \mathbb{C} \setminus i\mathbb{R},\ f \in X\right\}$ *and*

1. $R(z, 0) = -\frac{i}{z}\mathbb{1}$;
2. $R(z, f)^* = R(-\bar{z}, f)$;
3. $\nu R(\nu z, \nu f) = R(\nu, f), \quad \forall\nu \in \mathbb{R} \setminus \{0\}$;
4. $R(z_1, f) - R(z_2, g) = i(z_2 - z_1) R(z_1, f) R(z_2, g)$;
5. $\left[R(z_1, f), R(z_2, g)\right] = i\sigma(f, g) R(z_1, f) R(z_2, g)$;
6. $R(z_1, f) R(z_2, g) = R(z_1 + z_2, f + g)\left[R(z_1, f) + R(z_2, g) + i\sigma(f, g) R(z_1, f)^2 R\left(z_g, g\right)\right]$

with $z_1, z_2, z \in \mathbb{C} \setminus i\mathbb{R}$ *and, in* 6., $z_1 + z_2 \notin i\mathbb{R}$, *as in definition 1.1.* □

Proposition 1.4. *Let* (X, σ) *be a symplectic vector space and let* $\mathcal{R}(X, \sigma)$ *be the corresponding resolvent algebra.* $\forall\lambda \in \mathbb{R} \setminus \{0\}, \forall f \in X \setminus \{0\}$,

$$\left[R(\lambda, f) \mathcal{R}(X, \sigma)\right] = \left[\mathcal{R}(X, \sigma) R(\lambda, f)\right] = \left[\mathcal{R}(X, \sigma) R(\lambda, f) \mathcal{R}(X, \sigma)\right]$$

is a proper two-sided closed ideal ($[\cdots]$ *denotes the topological closure of the linear span of the argument). Moreover, given* $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus \{0\}$ *and* $f_1, \ldots, f_n \in X \setminus \{0\}$,

$$\bigcap_{i=1}^{n} \left[R(\lambda_i, f_i) \mathcal{R}(X, \sigma)\right] = \left[R(\lambda_1, f_1) \cdots R(\lambda_n, f_n) \mathcal{R}(X, \sigma)\right].$$

Proof. The equality $\left[R(\lambda, f) \mathcal{R}(X, \sigma)\right] = \left[\mathcal{R}(X, \sigma) R(\lambda, f)\right]$ is first proved by showing the double inclusion. The linear span of $\left\{\prod_{i=1}^{k} R(\lambda_i, f_i) \mid \lambda_i \in \mathbb{R} \setminus \{0\},\ f_i \in X,\ k \in \mathbb{N}\right\}$ is dense in $\mathcal{R}(X, \sigma)$, hence the same holds true for $\langle\left\{R(\lambda, f) \prod_{i=1}^{k} R(\lambda_i, f_i) \mid \lambda_i \in \mathbb{R} \setminus \{0\},\ f_i \in X,\ k \in \mathbb{N}\right\}\rangle$ in $\left[R(\lambda, f) \mathcal{R}(X, \sigma)\right]$. However,

$$R(\lambda, f) R(\mu, g) = \left[R(\mu, g) + i\sigma(f, g) R(\lambda, f) R(\mu, g)^2\right] R(\lambda, f),$$

therefore, moving $R(\lambda, f)$ to the right of each $R(\lambda, f) \prod_{i=1}^{k} R(\lambda_i, f_i)$ implies that the linear span of $\left\{R(\lambda, f) \prod_{i=1}^{k} R(\lambda_i, f_i) \mid \lambda_i \in \mathbb{R} \setminus \{0\},\ f_i \in X \setminus \{0\}, k \in \mathbb{N}\right\}$ is dense in $\left[\mathcal{R}(X, \sigma) R(\lambda, f)\right]$ too, i.e. $\left[R(\lambda, f) \mathcal{R}(X, \sigma)\right] \subseteq \left[\mathcal{R}(X, \sigma) R(\lambda, f)\right]$. The inverse inclusion is proved analogously. Finally, to prove that such an ideal is proper, *reductio ad absurdum* is referred to. Let $e \in \left[\mathcal{R}(X, \sigma) R(\lambda, f)\right]$ be; there exists a sequence $\{a_n\}_n \in \mathcal{R}(X, \sigma)$ such that $a_n R(\lambda, f)$ converges to e. Given the Fock representation π, let $\phi_\pi(f) = i\mathbb{1} + \pi\left[R(1, f)\right]^{-1}$ be the associated self-adjoint field operator, with spectral measure P. $\forall\psi_k \in P\left([k, k+1]\right)\mathcal{H}_\pi$ of unit norm, $k \in \mathbb{Z}$,

$$\left\|\pi\left[R(\lambda, f)\right]\psi_k\right\| \leq \sup_{t \in [k, k+1]}\left|\frac{1}{i\lambda - t}\right| = \frac{1}{\sqrt{\lambda^2 + k^2}}.$$

Since $\pi\left[a_n R(\lambda, f)\right] \underset{n}{\longrightarrow} \mathbb{1}$ in $\mathfrak{B}(\mathcal{H}_\pi)$, $\forall\epsilon\in(0,1)$, there exists $N\in\mathbb{N}$ such that, if $n>N$,

$$\begin{aligned}1-\epsilon &\leq \left\|\pi\left[a_n R(\lambda, f)\right]\psi_k\right\| = \left\|\pi(a_n)\,\pi\left[R(\lambda, f)\right]\psi_k\right\| \leq \\ &\leq \left\|\pi(a_n)\right\| \frac{1}{\sqrt{\lambda^2+k^2}}, \quad \forall k.\end{aligned}$$

This would imply $(1-\epsilon)\sqrt{\lambda^2+k^2} \leq \left\|\pi(a_n)\right\|$, $\forall k$, contradicting $\pi(a_n)\in\mathfrak{B}(\mathcal{H})$.
Finally, let $\lambda_1,\lambda_2\in\mathbb{R}\setminus\{0\}$, $f_1,f_2\in X\setminus\{0\}$ be arbitrary.

$$\left[R(\lambda_1,f_1)\,R(\lambda_2,f_2)\,\mathcal{R}(X,\sigma)\right] \subseteq \left[R(\lambda_1,f_1)\,\mathcal{R}(X,\sigma)\right] \cap \left[R(\lambda_2,f_2)\,\mathcal{R}(X,\sigma)\right]$$

is trivial, hence interest is focused on the inverse inclusion. Let $a\in\left[R(\lambda_1,f_1)\,\mathcal{R}(X,\sigma)\right]\cap\left[R(\lambda_2,f_2)\,\mathcal{R}(X,\sigma)\right]$; there exists $\{b_n\}_n$ in $\mathcal{R}(X,\sigma)$ such that $R(\lambda_1,f_1)\,b_n \underset{n}{\longrightarrow} a$ and a $N\in\mathbb{N}$ such that

$$R(\lambda_1,f_1)\,b_n\in\left[R(\lambda_2,f_2)\,\mathcal{R}(X,\sigma)\right],\ \forall n>N.$$

Let $\{e_i\}$ be an approximate identity for $\left[R(\lambda_2,f_2)\,\mathcal{R}(X,\sigma)\right]$; it is possible to construct a sequence $\{e_{i_n}\}$ such that $\left\{R(\lambda_1,f_1)\,b_n e_{i_n}\right\}_n$ converges to a. Since $e_{i_n}\in\left[R(\lambda_2,f_2)\,\mathcal{R}(X,\sigma)\right]$, for all n, there exists a $f_n\in\mathcal{R}(X,\sigma)$ such that $\left\|f_n R(\lambda_2,f_2)-e_{i_n}\right\|$ is arbitrarily small. Consequently, the sequence $\{b_n f_n\}_n$ shows up and $R(\lambda_1,f_1)\,b_n f_n R(\lambda_2,f_2) \underset{n}{\longrightarrow} a$, i.e. $a\in\left[R(\lambda_1,f_1)\,R(\lambda_2,f_2)\,\mathcal{R}(X,\sigma)\right]$. ■

Chapter 2

(Regular) Representation Theory

Proposition 2.1. *Let (X, σ) be a symplectic vector space and let $\lambda \in \mathbb{R} \setminus \{0\}$, $f \in X \setminus \{0\}$ be.*

1. *Given a representation π of $\mathcal{R}(X, \sigma)$ such that $\ker\left\{\pi\left[R(\lambda, f)\right]\right\} \neq \{\mathbb{0}\}$, $\ker\left\{\pi\left[R(\lambda, f)\right]\right\}$ reduces $\pi\left[\mathcal{R}(X, \sigma)\right]$. Consequently, there exists a unique orthogonal decomposition $\pi = \pi_1 \oplus \pi_2$ such that $\pi_1\left[R(\lambda, f)\right] = \mathbb{0}$ and $\pi_2\left[R(\lambda, f)\right]$ is invertible.*
2. *If π is a non-degenerate representation of $\mathcal{R}(X, \sigma)$, then*
$$P_f = s - \lim_{\lambda \to \infty} i\lambda\pi\left[R(\lambda, f)\right]$$
exists and defines a central projection of $\left\{\pi\left[\mathcal{R}(X, \sigma)\right]\right\}''$; in particular, it is the range projection of $\pi\left[R(\lambda, f)\right]$ and of the ideal $\pi\left(\left[\mathcal{R}(X, \sigma) R(\lambda, f)\right]\right)$ as well.
3. *If π is a factorial representation of $\mathcal{R}(X, \sigma)$, P_f is either $\mathbb{0}$ or $\mathbb{1}$.*
4. *There exists $\omega \in \mathfrak{S}\left(\mathcal{R}(X, \sigma)\right)$ such that $R(\lambda, f) \in \ker \omega$; vice versa, if ω is such that $R(\lambda, f) \in \ker \omega$, $R(\lambda, f) \in \ker \pi_\omega$.*

Proof. 1. Set $K = \ker\left\{\pi\left[R(\lambda, f)\right]\right\}$, by observing that

$$\pi\left[R(\lambda, f) R(\mu, g)\right] = \pi\left[R(\mu, g) R(\lambda, f) + i\sigma(f, g) R(\lambda, f) R(\mu, g)^2 R(\lambda, f)\right] =$$
$$= \pi\left[R(\mu, g) + i\sigma(f, g) R(\lambda, f) R(\mu, g)^2\right] \pi\left[R(\lambda, f)\right],$$

$\pi\left[R(\lambda, f) R(\mu, g)\right] K = \{\mathbb{0}\} \implies \pi\left[R(\mu, g)\right] K \subseteq K$, $\forall \mu \in \mathbb{R} \setminus \{0\}$ and $g \in X$. Since $R(\mu, g)^* = R(-\mu, g)$, K reduces $\pi\left[\mathcal{R}(X, \sigma)\right]$. By setting $\pi_1 = \pi_{|_K}$ and $\pi_2 = \pi_{|_{K^\perp}}$, the result follows.

2. According to 1., $\pi = \pi_1 \oplus \pi_2$, where $\pi_1\left[R(\lambda, f)\right] = \mathbb{0}$ and $\ker\left\{\pi_2\left[R(\lambda, f)\right]\right\} = \{\mathbb{0}\}$, therefore, by 2. of proposition 2.2,
$$s - \lim_{\lambda \to \infty} i\lambda\pi\left[R(\lambda, f)\right] = \mathbb{1}_{K^\perp} \equiv P_f,$$
trivially a projection of $\left\{\pi\left[\mathcal{R}(X, \sigma)\right]\right\}''$. $P_f \in \left\{\pi\left[\mathcal{R}(X, \sigma)\right]\right\}''' \equiv \left\{\pi\left[\mathcal{R}(X, \sigma)\right]\right\}'$ implies it is a central projection. Moreover, since $\ker\left\{\pi_2\left[R(X, \sigma)\right]\right\} = \{\mathbb{0}\}$, $\pi_2\left[R(X, \sigma)\right]$

is a bijection onto its image, this last being obligatorily dense. Denoted its closure by $K^{\perp}$, it is $P_f \equiv P_{K^{\perp}}$, therefore

$$\pi\left(\left[R(\lambda,f)\,\mathcal{R}(X,\sigma)\right]\right)\mathcal{H} = \pi_2\left(\left[R(\lambda,f)\,\mathcal{R}(X,\sigma)\right]\right)K^{\perp}$$

and this manifold is dense.

3. If π is factorial, its centre is trivial.

4. It has already been established that $\left[R(\lambda,f)\,\mathcal{R}(X,\sigma)\right]$ is a two-sided closed ideal of $\mathcal{R}(X,\sigma)$. Any state ω of $\mathcal{R}(X,\sigma)/\left[R(\lambda,f)\,\mathcal{R}(X,\sigma)\right]$ can then be lifted to a state of $\mathcal{R}(X,\sigma)$ with $R(\lambda,f) \in \left[R(\lambda,f)\,\mathcal{R}(X,\sigma)\right]$ in its kernel. Vice versa, if ω is such that $R(\lambda,f) \in \ker\omega$, $R(\lambda,f)^* \in \ker\omega$ too, hence, the first resolvent formula gives $R(\lambda,f)\,R(\lambda,f)^* \in \ker\omega$, i.e. $R(\lambda,f) \in \mathcal{N}_\omega \equiv \left\{a \in \mathcal{R}(X,\sigma)\,\middle|\,\omega(aa^*) = 0\right\}$. Since $\mathcal{N}_\omega$ is an ideal, $\left[R(\lambda,f)\,\mathcal{R}(X,\sigma)\right] \subset \ker\omega$ and, as $\left[R(\lambda,f)\,\mathcal{R}(X,\sigma)\right]$ is two-sided, it is contained in $\ker\pi_\omega$.

■

Proposition 2.2. *Let (X,σ) be a symplectic vector space. Given $f,h \in X$, let π be a representation of $\mathcal{R}(X,\sigma)$ such that* $\ker\left\{\pi\left[R(1,f)\right]\right\} = \{\mathbb{O}\} = \ker\left\{\pi\left[R(1,h)\right]\right\}$.

1. $\left(\phi_\pi(f) \doteq \left\{\pi\left[R(1,f)\right]\right\}^{-1} + i\mathbb{1}, \mathcal{D}_{\phi_\pi(f)}\right)$ *is self-adjoint and* $\pi\left[R(\lambda,f)\right]\mathcal{D}_{\phi_\pi(h)} \subseteq \mathcal{D}_{\phi_\pi(h)}$.

2. *Denoted by $\mathcal{H}_\pi$ the representation Hilbert space,* $\lim_{\lambda\to\infty} i\lambda\pi\left[R(\lambda,f)\right]\psi = \psi$, $\forall\psi \in \mathcal{H}_\pi$.

3. $\lim_{\mu\to 0} i\pi\left[R(1,\mu f)\right]\psi = \psi$, $\forall\psi \in \mathcal{H}_\pi$.

4. $\mathcal{D} = \pi\left[R(1,f)\,R(1,h)\right]\mathcal{H}_\pi$ *is a joint dense domain for* $\left(\phi_\pi(f), \mathcal{D}_{\phi_\pi(f)}\right)$ *and* $\left(\phi_\pi(h), \mathcal{D}_{\phi_\pi(h)}\right)$; *moreover,* $\left[\phi_\pi(f), \phi_\pi(h)\right] = i\sigma(f,h)\,\mathbb{1}$ *on* $\mathcal{D}$.

5. $\ker\left\{\pi\left[R(1,\nu f + h)\right]\right\} = \{\mathbb{O}\}$ *for all* $\nu \in \mathbb{R}$. $\left(\phi_\pi(\nu f + h), \mathcal{D}_{\phi_\pi(\nu f + h)}\right)$ *is then well-defined and essentially self-adjoint on $\mathcal{D}$; moreover,* $\phi_\pi(\nu f + h) = \nu\phi_\pi(f) + \phi_\pi(h)$ *on* $\mathcal{D}$.

6. $\phi_\pi(f)\,\pi\left[R(\lambda,f)\right] = \pi\left[R(\lambda,f)\right]\phi_\pi(f) = i\lambda\pi\left[R(\lambda,f)\right] + \mathbb{1}$ *on* $\mathcal{D}_{\phi_\pi(f)}$.

7. $\left[\phi_\pi(f), \pi\left[R(\lambda,h)\right]\right] = i\sigma(f,h)\,\pi\left[R(\lambda,h)^2\right]$ *on* $\mathcal{D}_{\phi_\pi(f)}$.

8. *Set* $W(f) = \exp\left[i\phi_\pi(f)\right]$, *then*

$$W(f)\,W(h) = e^{-\frac{i}{2}\sigma(f,h)}W(f+h)$$
$$W(f)\,\pi\left[R(\lambda,h)\right]W(f)^{-1} = \pi\left[R\left(\lambda + i\sigma(h,f), h\right)\right].$$

Moreover, $W(sf)\,\mathcal{D} \subseteq \mathcal{D}$ *and* $W(th)\,\mathcal{D} \subseteq \mathcal{D}$ *for* $s,t \in \mathbb{R}$.

Proof. 1. It can be shown that $\ker\left\{\pi\left[R(1,f)\right]\right\} = \{\mathbb{0}\}$ implies that $\pi\left[R(\lambda,f)\right]$ is the resolvent of $\phi_\pi(f)$, for all $\lambda \in \mathbb{R}\setminus\{0\}$, i.e. $\phi_\pi(f) = \left\{\pi\left[R(\lambda,f)\right]\right\}^{-1} + i\lambda\mathbb{1}$, for all $\lambda \in \mathbb{R}\setminus\{0\}$. Moreover

$$\begin{aligned}\phi_\pi(\mu f) &= i\mathbb{1} + \left\{\pi\left[R(1,\mu f)\right]\right\}^{-1} = i\mathbb{1} + \left\{\pi\left[\frac{1}{\mu}R\left(\frac{1}{\mu},f\right)\right]\right\}^{-1} \equiv \\ &= i\mathbb{1} + \mu\left\{\pi\left[R\left(\frac{1}{\mu},f\right)\right]\right\}^{-1} = \mu\left\{i\frac{1}{\mu}\mathbb{1} + \left\{\pi\left[R\left(\frac{1}{\mu},f\right)\right]\right\}^{-1}\right\} = \\ &= \mu\phi_\pi(f).\end{aligned}$$

Further

$$\begin{aligned}\phi_\pi(f)^* &= \left[i\mathbb{1} + \left\{\pi\left[R(1,f)\right]\right\}^{-1}\right]^* \supseteq -i\mathbb{1} + \left[\left\{\pi\left[R(1,f)\right]\right\}^{-1}\right]^* = \\ &= -i\mathbb{1} + \left[\left\{\pi\left[R(1,f)\right]\right\}^*\right]^{-1} = -i\mathbb{1} + \left\{\pi\left[R(-1,f)\right]\right\}^{-1} = \\ &= -i\mathbb{1} - \left\{\pi\left[R(1,-f)\right]\right\}^{-1} = -\phi_\pi(-f) = \phi_\pi(f),\end{aligned}$$

i.e. $\phi_\pi(f) \subseteq \left[\phi_\pi(f)\right]^*$, meaning that $\phi_\pi(f)$ is symmetric. However, by observing that

$$\mathrm{Ran}\ \left[\phi_\pi(f) \pm i\mathbb{1}\right] = \mathrm{Ran}\ \left\{\pi\left[R(\pm 1,f)\right]^{-1}\right\} = \mathcal{D}_{\pi\left[R(\pm 1,f)\right]} \equiv \mathcal{H}_\pi,$$

self-adjointness is ensured. Eventually

$$\begin{aligned}\pi\left[R(\lambda,f)\right]\mathcal{D}_{\phi_\pi(f)} &= \pi\left[R(\lambda,f)\right]\pi\left[R(1,h)\right]\mathcal{H}_\pi = \\ &= \pi\left[R(1,h)R(\lambda,f) + i\sigma(f,h)R(\lambda,f)R(1,h)^2R(\lambda,f)\right]\mathcal{H}_\pi \subseteq \\ &\subseteq \pi\left[R(1,h)\right]\mathcal{H}_\pi \equiv \mathcal{D}_{\phi_\pi(h)}.\end{aligned}$$

2. $\left(\phi_\pi(f), \mathcal{D}_{\phi_\pi(f)}\right)$ is self-adjoint, hence it admits a spectral decomposition. Consequently

$$i\lambda\pi\left[R(\lambda,f)\right] = \int_{\mathbb{R}} \frac{i\lambda}{\mu - i\lambda} dP(\mu).$$

$\left|\frac{i\lambda}{i\lambda-\mu}\right| < 1$ guarantees the applicability of the dominated convergence theorem.

3. Analogously to point 2.,

$$i\pi\left[R(1,\mu f)\right] = \int_{\mathbb{R}} \frac{i}{\mu\lambda - i} dP(\lambda) \underset{\mu\to 0}{\longrightarrow} \mathbb{1}.$$

4. Let $\mathcal{D} = \pi\left[R(1,f)R(1,h)\right]\mathcal{H}_\pi$ be.

$$\pi\left[R(1,f)R(1,h)\right]\mathcal{H}_\pi = \pi\left[R(1,f)\right]\left\{\pi\left[R(1,h)\right]\mathcal{H}_\pi\right\},$$

immediately gives $\mathcal{D} \subseteq \mathrm{Ran}\ \left(\pi\left[R\left(1,f\right)\right]\right) \equiv \mathcal{D}_{\phi_\pi(f)}$. Then, the commutation relations allows for

$$\pi\left[R\left(1,f\right)R\left(1,h\right)\right] = \pi\left\{R\left(1,h\right)\left[R\left(1,f\right)+i\sigma\left(f,h\right)R\left(1,f\right)^2R\left(1,h\right)\right]\right\} =$$
$$= \pi\left[R\left(1,h\right)\right]\pi\left[R\left(1,f\right)+i\sigma\left(f,h\right)R\left(1,f\right)^2R\left(1,h\right)\right],$$

i.e. $\mathcal{D} \subseteq \mathrm{Ran}\ \left(\pi\left[R\left(1,h\right)\right]\right) \equiv \mathcal{D}_{\phi_\pi(h)}$, implying that $\mathcal{D}$ is a joint domain for both $\left(\phi_\pi\left(f\right),\mathcal{D}_{\phi_\pi(f)}\right)$ and $\left(\phi_\pi\left(h\right),\mathcal{D}_{\phi_\pi(h)}\right)$. To prove its density, given $\psi \in H_\pi$ arbitrary,

$$\lim_{\mu\to0}\lim_{\nu\to0}\pi\left[R\left(1,\mu f\right)R\left(1,\nu h\right)\right]\psi = \lim_{\mu\to0}\left(\pi\left[R\left(1,\mu f\right)\right]\left\{\lim_{\nu\to0}\pi\left[R\left(1,\nu h\right)\right]\psi\right\}\right) =$$
$$= -i\left\{\lim_{\mu\to0}\pi\left[R\left(1,\mu f\right)\right]\psi\right\} = -\psi$$

i.e. every $\psi \in \mathcal{H}_\pi$ admits a converging family of vectors from $\mathcal{D}$. Eventually, given $\psi \doteq \pi\left[R\left(1,f\right)R\left(1,h\right)\right]\phi$, $\phi \in \mathcal{H}_\pi$,

$$\pi\left[R\left(1,h\right)R\left(1,f\right)\right]\left[\phi_\pi\left(f\right),\phi_\pi\left(h\right)\right]\psi =$$
$$\pi\left[R\left(1,h\right)R\left(1,f\right)\right]\left[\left\{\pi\left[R\left(1,f\right)\right]\right\}^{-1},\left\{\pi\left[R\left(1,h\right)\right]\right\}^{-1}\right]\psi = \ldots =$$
$$\pi\left[R\left(1,h\right)R\left(1,f\right)-R\left(1,f\right)R\left(1,h\right)\right]\phi = \pi\left[R\left(1,h\right)R\left(1,f\right)\right]i\sigma\left(f,g\right)\psi.$$

Since $\ker\left\{\pi\left[R\left(1,h\right)R\left(1,f\right)\right]\right\} = \{\mathbb{O}\}$[1], $\left[\phi_\pi\left(f\right),\phi_\pi\left(h\right)\right] = i\sigma\left(f,g\right)\mathbb{1}$ holds on $\mathcal{D}$.

5. First,

$$\pi\left[R\left(\nu,f\right)\right] = \left[\phi_\pi\left(f\right)-i\nu\mathbb{1}\right]^{-1} = \frac{1}{\nu}\pi\left[R\left(1,\frac{1}{\nu}f\right)\right] = \frac{1}{\nu}\left[\phi_\pi\left(\frac{1}{\nu}f\right)-i\mathbb{1}\right]^{-1},$$

i.e. $\phi_\pi\left(\lambda f\right) = \lambda\phi_\pi\left(f\right), \forall\lambda \in \mathbb{R}\setminus\{0\}$ and the statement holds for $h = 0$. Then, it is first observed that

$$\pi\left[R\left(\lambda,f\right)R\left(\mu,h\right)\right] = \pi\left\{R\left(\lambda+\mu,f+g\right)\left[R\left(\lambda,f\right)+R\left(\mu,h\right)+i\sigma\left(f,h\right)R\left(\lambda,f\right)^2R\left(\mu,h\right)\right]\right\};$$

Since $K = \ker\left\{\pi\left[R\left(1,f+h\right)\right]\right\}$ is reducing for $\pi\left[\mathcal{R}\left(X,\sigma\right)\right]$, $K = \ker\pi\left[R\left(\lambda,f\right)R\left(\mu,h\right)\right]$. However, $\pi\left[R\left(\lambda,f\right)R\left(\mu,h\right)\right]$ is invertible, hence $K = \{\mathbb{O}\}$ and the right hand side square brackets term is invertible too. Quite obviously $\mathcal{D} = \mathrm{Ran}\ \pi\left[R\left(\lambda,f\right)R\left(\mu,h\right)\right] \subset$

[1] Attention has been focused on representations π such that $\ker\left\{\pi\left[R\left(\lambda,f\right)\right]\right\} = \{\mathbb{O}\}$, $\forall\lambda \in \mathbb{R}\setminus\{0\}, \forall f \in X$. Given an arbitrary ψ in $\ker\left\{\pi\left[R\left(\lambda,h\right)R\left(\mu,f\right)\right]\right\}$,

$$\pi\left[R\left(\lambda,h\right)R\left(\mu,f\right)\right]\psi = 0 = \pi\left[R\left(\lambda,h\right)\right]\left\{\pi\left[R\left(\mu,f\right)\right]\psi\right\},$$

i.e. $\pi\left[R\left(\mu,f\right)\right]\psi \in \ker\left\{\pi\left[R\left(\lambda,h\right)\right]\right\} \equiv \{\mathbb{O}\}$, i.e. $\psi = 0_H$. Arbitrariness of ψ leads to $\ker\left\{\pi\left[R\left(\lambda,h\right)R\left(\mu,f\right)\right]\right\} = \{\mathbb{O}\}$, $\forall\lambda,\mu \in \mathbb{R}\setminus\{0\}$.

$\mathcal{D}_{\phi_\pi(f+h)}$ is a core for $\left(\phi_\pi(f+h), \mathcal{D}_{\phi_\pi(f+h)}\right)$. By left multiplying both members by $\left[\phi_\pi(f+h) - i(\lambda+\mu)\mathbb{1}\right]$, it results

$$\left[\phi_\pi(f+h) - i(\lambda+\mu)\mathbb{1}\right]\pi\left[R(\lambda,f)R(\mu,h)\right] = \pi\left[R(\lambda,f) + R(\mu,h) + i\sigma(f,h)R(\lambda,f)^2 R(\mu,h)\right].$$

By applying what obtained to $\left[\phi_\pi(h) - i\mu\mathbb{1}\right]\left[\phi_\pi(f) - i\lambda\mathbb{1}\right]\psi$, $\psi \in \mathcal{D}$, one gets

$$\left[\phi_\pi(f+h) - i(\lambda+\mu)\mathbb{1}\right]\psi = \left\{\left[\phi_\pi(h) - i\mu\mathbb{1}\right] + \left[\phi_\pi(f) - i\lambda\mathbb{1}\right]\right\}\psi$$

for all $\psi \in \mathcal{D}$. The additivity of ϕ_π on $\mathcal{D}$ is this way shown.

6. The spectral resolution of $\phi_\pi(f)$ allows for

$$\begin{aligned}\phi_\pi(f)\pi\left[R(\lambda,f)\right] &= \int_{\mathbb{R}} \frac{\mu}{\mu - i\lambda} dP(\mu) = \pi\left[R(\lambda,f)\right]\phi_\pi(f) = \\ &= \int_{\mathbb{R}} \left(\frac{i\lambda - i\lambda + \mu}{\mu - i\lambda}\right) dP(\mu) = i\lambda\pi\left[R(\lambda,f)\right] + \mathbb{1}.\end{aligned}$$

7. Let $\psi \in \mathcal{D}_{\phi_\pi(f)} = \text{Ran }\left\{\pi\left[R(\lambda,f)\right]\right\}$, i.e. $\psi = \pi\left[R(\lambda,f)\right]\Phi$ for some $\Phi \in \mathcal{H}_\pi$. Then

$$\begin{aligned}\pi\left[R(\lambda,f)\right]\left[\phi_\pi(f), \pi\left[R(\lambda,h)\right]\right]\psi &= \pi\left(\left[R(\lambda,f), R(\lambda,h)\right]\right)\Phi = \\ &= \pi\left[R(\lambda,f)\right]\left\{i\sigma(f,h)\pi\left[R(\lambda,h)^2\right]\psi\right\}.\end{aligned}$$

$\ker\left\{\pi\left[R(\lambda,f)\right]\right\} = \{\mathbb{0}\}$ implies

$$\left[\phi_\pi(f), \pi\left[R(\lambda,h)\right]\right] = i\sigma(f,h)\pi\left[R(\lambda,h)^2\right]$$

on $\mathcal{D}_{\phi_\pi(f)}$.

8. The second inequality is proved first. Let $\psi, \Phi \in \tilde{\mathcal{D}} \doteq \text{span}\left\{\chi_{[-a,a]}\left[\phi_\pi(f)\right]\mathcal{H}_\pi \mid a \in \mathbb{R}^+\right\}$. Since $\left\|\left[\phi_\pi(f)^n\right]_{|_{\chi_{[-a,a]}[\phi_\pi(f)]\mathcal{H}_\pi}}\right\| \leq a^n$, $n \in \mathbb{N}$,

$$W(f)\psi \doteq \exp\left[i\phi_\pi(f)\right]\psi \equiv \sum_{n=0}^{\infty} \frac{\left[i\phi_\pi(f)\right]^n}{n!}\psi, \quad \psi \in \tilde{\mathcal{D}}$$

makes sense. Furthermore

$$\left\langle\Phi, W(f)\pi\left[R(\lambda,h)\right]W(f)^{-1}\psi\right\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\Phi, \left[i\phi_\pi(f)\right]^n\pi\left[R(\lambda,h)\right]\left\{\left[i\phi_\pi(f)\right]^*\right\}^n\psi\right\rangle$$

for all $\Phi, \psi \in \tilde{\mathcal{D}}$. By repeatedly using 7., whenever $\left|t\sigma(h,f)\right| < |\lambda|$,

$$\left\langle\Phi, W(tf)\pi\left[R(\lambda,h)\right]W(tf)^{-1}\psi\right\rangle = \left\langle\Phi, \pi\left[R\left(\lambda + it\sigma(h,f), h\right)\right]\psi\right\rangle, \ \forall\Phi,\psi \in \tilde{\mathcal{D}},\ t \in \mathbb{R}.$$

The operators involved are bounded and $\tilde{\mathcal{D}}$ is dense, hence

$$W(tf)\,\pi\left[R(\lambda,h)\right]W(tf)^{-1}=\pi\left[R\left(\lambda+it\sigma(h,f),h\right)\right].$$

Particularly, $W(f)\,\pi\left[R(\lambda,h)\right]W(f)^{-1}=\pi\left[R\left(\lambda+i\sigma(h,f),h\right)\right]$. To prove the Weyl relation, by

$$\lim_n\left(1+\frac{it}{n}\right)^{-n}=e^{it},\ \forall t\in\mathbb{R}\quad\text{and}\quad\sup_{t\in\mathbb{R}}\left|\left(1+\frac{it}{n}\right)^{-n}\right|=1,$$

one has (check [14], thm. VIII.5(d))

$$W(h)=e^{i\phi_\pi(h)}=\lim_n\left[1+\frac{i\phi_\pi(h)}{n}\right]^{-n}=\lim_n\left\{-\pi\left[iR\left(1,\frac{h}{n}\right)\right]\right\}^n$$

in the strong operator sense. Therefore

$$\begin{aligned}W(f)\,W(h)\,W(f)^{-1}&=s-\lim_n\left\{-\pi\left[iR\left(1+i\sigma\left(\frac{h}{n},f\right),\frac{h}{n}\right)\right]\right\}^n=\\&=s-\lim_n\left\{\mathbb{1}+\left(\frac{i}{n}\right)\left[\sigma(h,f)\,\mathbb{1}+\phi_\pi(h)\right]\right\}^{-n}=\\&=e^{-i\sigma(f,h)}W(h).\end{aligned}$$

Further

$$\begin{aligned}W(f+h)&=s-\lim_n\left[W\left(\frac{1}{n}f\right)W\left(\frac{e}{n}h\right)\right]^n=\\&=s-\lim_n e^{\frac{i}{2}\left(1-\frac{1}{n}\right)\sigma(f,h)}\,W(f)\,W(h)=\\&=e^{\frac{i}{2}\sigma(f,h)}W(f)\,W(h).\end{aligned}$$

Eventually, by observing that

$$W(sf)\,\mathcal{D}\equiv W(sf)\,\pi\left[R(\lambda,f)\,R(\mu,h)\right]\mathcal{H}_\pi=\pi\left[R(\lambda,f)\,R\left(\mu+i\sigma(h,sf),h\right)\right]\mathcal{H}_\pi\subset\mathcal{D}$$

[14] theorem VIII.11 ensures $\mathcal{D}$ is a core for $\left(\phi_\pi(f),\mathcal{D}_{\phi_\pi(f)}\right)$.

■

Definition 2.1. Let (X,σ) be a symplectic vector space. Fixed a Hilbert space $\mathcal{H}$, a representation $\pi:\ \mathcal{R}(X,\sigma)\longrightarrow\mathfrak{B}(\mathcal{H})$ is said **regular on** $S\subset X$ if and only if

$$\ker\,\pi\left[R(1,f)\right]=\{0\},\ \forall f\in S.$$

Analogously, a state ω of $\mathcal{R}(X,\sigma)$ is *regular on* S if and only if the associated GNS-representation π_ω is *regular on* S. A representation (state) is simply called *regular* if and only if it is regular on X. The set of all (non-degenerate) regular representations of $\mathcal{R}(X,\sigma)$ on $\mathcal{H}$ is denoted by $\mathrm{Reg}\left(\mathcal{R}(X,\sigma),\mathcal{H}\right)$, while the set of all regular states by $\mathfrak{S}_r\left(\mathcal{R}(X,\sigma)\right)$. □

Remark 2.1. *Regular representations, apart from being mathematically interesting, are of physical importance: the set of all regular representations of $\mathcal{R}(X,\sigma)$ is surely non-void, the Fock representation, repeatedly considered, being one of them. This emphasized, point 8. of theorem 2.2 establishes a link between the resolvent algebra regular representations and the Weyl algebra regular representations on a given Hilbert space.* $\square$

Corollary 2.2.1. *Given $\pi \in Reg\left(\mathcal{R}(X,\sigma), \mathcal{H}\right)$, the map*

$$\tilde{\pi}: \ \delta_f \in CCR(X,\sigma) \longmapsto \tilde{\pi}\left(\delta_f\right) \doteq \exp\left[i\phi_\pi(f)\right] \in \mathfrak{B}(\mathcal{H})$$

defines a regular $\mathcal{H}$ representation of $CCR(X,\sigma)$.

$$G: \pi \in Reg\left(\mathcal{R}(X,\sigma), \mathcal{H}\right) \longmapsto G(\pi) \doteq \tilde{\pi} \in Reg\left(CCR(X,\sigma), \mathcal{H}\right)$$

is a bijection preserving irreducibility and direct sums; its inverse $G^{-1}: \tilde{\pi} \longmapsto G^{-1}(\tilde{\pi}) \doteq \pi$ is such that

$$\pi: \ R(\lambda, f) \in \mathcal{R}(X,\sigma) \longmapsto \pi\left[R(\lambda,f)\right] = -i\int_0^{\sigma\infty} e^{-\lambda t}\tilde{\pi}\left(\delta_{-tf}\right) dt \in \mathfrak{B}(\mathcal{H}), \quad \sigma = sign\,\lambda,$$

where the integral is defined in the strong operator topology. $\blacksquare$

Remark 2.2. *G allows for the Stone-von Neumann theorem to keep on holding for the resolvent algebra too, i.e., as long as X is finite dimensional, all the irreducible, regular representations of $\mathcal{R}(X,\sigma)$ on a given Hilbert space $\mathcal{H}$ are unitarily equivalent.* $\square$

Proposition 2.3. *Let (X,σ) be a symplectic vector space.*

1. *If a representation π of $\mathcal{R}(X,\sigma)$ is faithful and factorial, it is regular.*
2. *If a representation π is regular, $\left\|\pi\left[R(\lambda,f)\right]\right\| = \left\|R(\lambda,f)\right\| = |\lambda|^{-1}$, for all $\lambda \in \mathbb{R}\setminus\{0\}$, $f \in X$.*
3. *A state ω of $\mathcal{R}(X,\sigma)$ is regular $\iff$ $\omega(a) = \lim_{\lambda\to\infty} i\lambda\omega\left[R(\lambda,f)\,a\right]$, $\forall a \in \mathcal{R}(X,\sigma)$ and $f \in X$.*

Proof. 1. Given arbitrarily $f \in X\setminus\{0\}$, proposition 2.1 gives either $P_f = \mathbb{1}$ or $P_f = \mathbb{O}$. $P_f = \mathbb{O}$ means that $f \in X\setminus\{0\}$ is such that $\pi\left[R(\lambda,f)\right] = \mathbb{O}$ for all λ, contradicting that π is faithful. Therefore, P_f has to be equal to $\mathbb{1}$. Since P_f is the orthogonal projection onto $\left\{\ker\pi\left[R(\lambda,f)\right]\right\}^{\perp}$, $\ker\pi\left[R(\lambda,f)\right] = \{\mathbb{O}\}$. The arbitrariness of $f \in X\setminus\{0\}$ proves the regularity of π.

2. A generalization of proposition 1.3, point 3..

3. ($\Longleftarrow$) Given $a \in \mathcal{R}(X,\sigma)$, denoted by π_ω the GNS-representation induced by ω,

$$\lim_{\lambda\to\infty} i\lambda\omega\left[R(\lambda,f)\,a\right] = \left\langle \Omega_\omega, s-\lim_{\lambda\to\infty} i\lambda\pi_\omega\left[R(\lambda,f)\,a\right]\Omega_\omega\right\rangle = \left\langle\Omega_\omega,\, P_f\pi_\omega[a]\,\Omega_\omega\right\rangle.$$

Since P_f is a central projection in $\left\{\pi\left[\mathcal{R}(X,\sigma)\right]\right\}''$, for all $b, c \in \mathcal{R}(X,\sigma)$,

$$\begin{aligned}\left\langle\pi_\omega(b)\,\Omega_\omega, \pi_\omega(a)\,\Omega_\omega\right\rangle \equiv \omega(b^*a) &= \lim_{\lambda\to\infty} i\lambda\omega\left(R(\lambda,f)\,b^*a\right) = \\ &= \left\langle\pi_\omega(b)\,\Omega_\omega, P_f\pi_\omega(a)\,\Omega_\omega\right\rangle \implies \\ &\implies P_f = \mathbb{1}.\end{aligned}$$

This, in turn, implies $\pi_\omega\left[R(\lambda, f)\right]$ is invertible ($P_f \equiv P_{K^\perp}$ and $K = \ker \pi\left[R(\lambda, f)\right]$) hence the regularity of π_ω follows from the arbitrariness of $f \in X \setminus \{0\}$.
($\Longrightarrow$) Trivially from what above.

$\blacksquare$

Chapter 3

Further Structure

Further structural results about resolvent algebras[1] are exposed.

Proposition 3.1. *Let (X, σ) be a symplectic vector space and let $\mathcal{R}(X, \sigma)$ be the corresponding resolvent algebra.*

1. *If $S \subset X$ is a non-trivial symplectic subspace, then*

$$\mathcal{R}(S, \sigma) \otimes \mathcal{R}\left(S^{\perp}, \sigma\right) \subset \mathcal{R}(X, \sigma),$$

   the tensor product being referred to the spatial tensor norm.

2. *Given $\{q_1, \ldots, q_k\} \subset X$ such that $\sigma\left(q_i, q_j\right) = 0, \forall i, j$, for all $F \in C_0\left(\mathbb{R}^k\right)$, there exists a unique $R_F \in \mathcal{R}(X, \sigma)$ such that, in any regular representation π, $\pi(R_F) = F\left(\phi_\pi(q_1), \ldots, \phi_\pi(q_k)\right)$.*

■

Proposition 3.2. *Let (X, σ) be a symplectic vector space and $f, h \in X \setminus \{0\}$ such that $f \notin \mathbb{R}h$. Then*

1. *$R(1, f) \notin \left[\mathcal{R}(X, \sigma) R(1, h)\right]$.*
2. *$\left\|R(1, f) - R(1, h)\right\| \geq 1$, the equality holding for $\sigma(f, h) = 0$ only.*
3. *$\mathcal{R}(X, \sigma)$ is non-separable.*

Proof. 1. The $\sigma(f, h) \neq 0$ case is first considered. Set $C = \{f\}$, since $\sigma(C, C) = 0$, $\mathfrak{S}_D \neq \emptyset$ holds, hence there exists ω such that $\omega\left[R(1, f)\right] = -i$. Moreover, $\sigma(f, h) \neq 0$ allows for $\omega\left[R(\lambda, h)\right] = 0$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$, i.e. $\omega\left[R(1, h)\right] = 0$. Proposition 2.1 implies $R(1, h) \in \ker \pi_\omega$, hence $\pi_\omega\left(\left[\mathcal{R}(X, \sigma) R(1, h)\right]\right) = \mathbb{O}$; however, since $\pi_\omega\left[R(1, f)\right] \neq \mathbb{O}$, $R(1, f) \notin \left[\mathcal{R}(X, \sigma) R(1, h)\right]$.
Regarding the $\sigma(f, h) = 0$ case, by augmenting f, h to a symplectic basis of $S \subset X$, i.e. $S = \left\langle \left\{f, p_f; h, p_h\right\} \right\rangle$, it would be $X = S \oplus S^{\perp}$ and, analogously, $S = S_1 \oplus S_2$ with $S_1 = \left\langle \left\{f, p_f\right\} \right\rangle$, $S_2 = \langle \{h, p_h\} \rangle$. Consequently

$$\mathcal{R}(S_1, \sigma) \otimes \mathcal{R}(S_2, \sigma) \otimes \mathcal{R}\left(S^{\perp}, \sigma\right) \subset \mathcal{R}(X, \sigma).$$

[1] Left out details can be found in [6].

It is now possible to choose a product state $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3 \in \mathcal{R}(S_1, \sigma) \otimes \mathcal{R}(S_2, \sigma) \otimes \mathcal{R}\left(S^{\perp}, \sigma\right)$ such that ω_1 is a Fock state of $\mathcal{R}(S_1, \sigma)$, $\omega_2\left[R(1,h)\right] = 0$ and ω_3 is regular on $\mathcal{R}\left(S^{\perp}, \sigma\right)$. By the Hanh-Banach theorem, ω is extendable to $\mathcal{R}(X, \sigma)$, preserving $\omega\left[R(1,h)\right] = 0$ and $\omega\left[R(1,f)\right] \neq 0$. The proof then proceeds as above.

2. If $\sigma(f,h) \neq 0$, $\mathfrak{S}_D \neq \emptyset$ and, given a state ω as in point 1.,

$$\left\|R(1,f) - R(1,h)\right\| \geq \left|\omega\left[R(1,f) - R(1,h)\right]\right| = |-i| = 1.$$

On the other hand, if $\sigma(f,h) = 0$, $\{f,h\}$ is again augmented to a symplectic basis such that $S = \left\langle\left\{f, p_f; h, p_h\right\}\right\rangle$. Given the Schrödinger representation π_S of $\mathcal{R}(S,\sigma) \subset \mathcal{R}(X,\sigma)$, because of the regularity,

$$\left\|R(1,f) - R(1,h)\right\| = \left\|\pi_S\left[R(1,f) - R(1,h)\right]\right\| = \sup_{\rho,\sigma \in \mathbb{R}}\left|\frac{1}{i-\rho} - \frac{1}{i-\sigma}\right| = 1.$$

3. Let $f_\xi = \xi f + (1-\xi) h$ be, with $\xi \in [0,1]$. Particularly, if $\xi \neq \zeta$, $f_\xi \neq f_\zeta$ and $\left\|R\left(1, f_\xi\right) - R\left(1, f_\zeta\right)\right\| \geq 1$. By centering an open unit ball around each $R\left(1, f_\xi\right)$, $\xi \in [0,1]$, a non-countable family of disjoint sets results, therefore, if S is a dense subset of $\mathcal{R}(X,\sigma)$, there would exist an element of its in each of these balls. Such an argument proves that S is non-countable.

■

Remark 3.1. *Proposition 3.2 allows to state that the resolvent algebra as a whole cannot be part of a C^*–dynamical system.* □

Proposition 3.3. *Let (X,σ), $\mathcal{H}$, $\pi_0 : \mathcal{R}(X,\sigma) \longrightarrow \mathfrak{B}(\mathcal{H})$ be, respectively, a finite-dimensional symplectic vector space, with symplectic basis $\{q_1, p_1; \ldots; q_n, p_n\}$, a fixed Hilbert space and an irreducible regular representation of $\mathcal{R}(X,\sigma)$ on $\mathcal{H}$.*

1. *$\pi_0\left\{\left[R(\lambda_1, p_1) R(\mu_1, q_1)\right] \cdots \left[R(\lambda_n, p_n) R(\mu_n, q_n)\right]\right\}$ is a Hilbert-Schmidt operator for all $\lambda_i, \mu_i \in \mathbb{R} \setminus \{0\}$.*

2. *There exists a unique closed two-sided ideal $\mathcal{K}$ of $\mathcal{R}(X,\sigma)$ isomorphic to $\mathfrak{B}_\infty(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$ through π_0.*

Proof. 1. Since all the regular irreducible representations of $\mathcal{R}(X,\sigma)$ are unitarily equivalent, it does not affect generality letting $\pi_0 \equiv \pi_S$ be the Schrödinger representation on $L^2(\mathbb{R}^n)$. By taking advantage of the resolvents commutation relations, straightforward computations lead to

$$\pi_S\left\{\left[R(\lambda_1, p_1) R(\mu_1, q_1)\right] \cdots \left[R(\lambda_n, p_n) R(\mu_n, q_n)\right]\right\} = \prod_{j=1}^{n}\left(i\lambda_j - Q_j\right)^{-1} \cdot \prod_{k=1}^{n}\left(i\mu_k - P_k\right)^{-1},$$

where $Q_j = \phi_{\pi_S}\left(p_j\right)$, $P_k = \phi_{\pi_S}(q_k)$ are the usual position and momentum operators on $L^2(\mathbb{R}^n)$. Now, if f and g are continuous, bounded, square integrable functions on $\mathbb{R}^n$, $f(Q_1 \cdots Q_n)\, g(P_1 \cdots P_n)$ is a Hilbert-Schmidt[2] operator.

[2] [15], thm. XI.20.

2. It is a well known fact that, if a C*− algebra acting irreducibly on a Hilbert space contains a non-trivial compact operator, then, it contains all the compact operators ([16], thm. 2.4.9). Consequently by point 1., $\mathfrak{B}_\infty(\mathcal{H}) \subset \pi_S\left[\mathcal{R}(X,\sigma)\right]$. Since π_S is faithful, $\mathcal{R}(X,\sigma)$ will contain a closed, two-sided ideal $\mathcal{K}$ isomorphic to $\mathfrak{B}_\infty(\mathcal{H})$, whose uniqueness follows from the up to unitary equivalence uniqueness of π_S.

■

Chapter 4

Outlines of Dynamics

This section will only give a taste of finite dimensional possibilities the resolvent algebra offers for quantum dynamical descriptions. Particularly, as anticipated, the Stone-von Neumann theorem holds in this regime, therefore it does not affect generality proceeding as follows.

Definition 4.1. Given $N \in \mathbb{N}$, let $\left(\mathbb{R}^{2N}, \sigma\right)$ be the canonical symplectic vector space. A self-adjoint Hamiltonian $(H, \mathcal{D}_H)$ on $L^2\left(\mathbb{R}^N\right)$ **induces a dynamics on** $\mathcal{R}\left(\mathbb{R}^{2N}, \sigma\right)$ if and only if

$$e^{itH}\pi_S(a)\, e^{-itH} \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^{2N}, \sigma\right)\right]$$

for all $a \in \mathcal{R}\left(\mathbb{R}^{2N}, \sigma\right)$. ☐

Remark 4.1. *A direct consequence of definition 4.1 is*

$$\alpha_t : a \in \mathcal{R}\left(\mathbb{R}^{2N}, \sigma\right) \longmapsto \alpha_t(a) \doteq \pi_S^{-1}\left[e^{itH}\pi_S(a)\, e^{-itH}\right] \in \mathcal{R}\left(\mathbb{R}^{2N}, \sigma\right)$$

resulting in a resolvent algebra automorphism, for all $t \in \mathbb{R}$. ☐

Proposition 4.1. *Let* $V \in C_0(\mathbb{R})$ *be a non-trivial real function. The self-adjoint Hamiltonian* $(H = H_0 + V,\, \mathcal{D}_H)$ *on* $L^2(\mathbb{R})$ *induces a dynamics on* $\mathcal{R}\left(\mathbb{R}^2, \sigma\right)$.

Proof. Since H_0 is quadratic in the momentum variable,

$$\alpha_t^{(0)} : a \in \mathcal{R}\left(\mathbb{R}^2, \sigma\right) \longmapsto \alpha_t^{(0)}(a) \doteq \pi_S^{-1}\left\{e^{itH_0}\pi_S(a)\, e^{-itH_0}\right\} \in \mathcal{R}\left(\mathbb{R}^2, \sigma\right)$$

is well-defined and equivalent to a symplectic transformation on $\mathcal{R}\left(\mathbb{R}^2, \sigma\right)$, i.e. an automorphism by proposition 1.3. Then, within the *interaction picture* framework,

$$e^{-itH} \equiv e^{-itH_0}\, \Gamma_V(t)$$

with

$$\Gamma_V(t) \equiv \mathbb{1} + \sum_{n\in\mathbb{N}} (-i)^n \int_0^t dt_n \cdots \int_0^{t_2} dt_1\, V(t_1) \cdots V(t_n) \tag{4.1}$$

for all $t \in \mathbb{R}$, the integrals are defined in the strong operator topology of $\mathfrak{B}\left(L^2(\mathbb{R})\right)$ and the sum converges uniformly. Consequently

$$\left(e^{-itH}\right)^* \pi_S(a)\, e^{-itH} = \Gamma_V(t)^* \left[e^{itH_0}\pi_S(a)\, e^{-itH_0}\right] \Gamma_V(t)$$

allows to state that checking whether H induces a dynamics on $\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$ or not amounts in showing that

$$\Gamma_V\left(t\right)^*\pi_S\left(a\right)\Gamma_V\left(t\right)\in\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right] \tag{4.2}$$

for all $a\in\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$, $t\in\mathbb{R}$. First, given $V_1,V_2\in C_0\left(\mathbb{R}\right)$, $t\in\mathbb{R}$ arbitrary,

$$\left\|\Gamma_{V_1}\left(t\right)-\Gamma_{V_2}\left(t\right)\right\|\leq\sum_{n\in\mathbb{N}}\int_0^t dt_n\cdots\int_0^{t_2}dt_1\left\|V_1\left(t_1\right)\cdots V_1\left(t_n\right)-V_2\left(t_1\right)\cdots V_2\left(t_n\right)\right\|.$$

By observing that

$$\begin{aligned}\left[V_1\left(t_1\right)\cdots V_1\left(t_n\right)-V_2\left(t_1\right)\cdots V_2\left(t_n\right)\right]=&\left[V_1\left(t_1\right)-V_2\left(t_1\right)\right]V_2\left(t_2\right)\cdots V_2\left(t_n\right)+\\&+V_1\left(t_1\right)\left[V_1\left(t_2\right)-V_1\left(t_2\right)\right]V_2\left(t_3\right)\cdots V_2\left(t_n\right)+\\&+V_1\left(t_1\right)V_1\left(t_2\right)\left[V_1\left(t_3\right)-V_1\left(t_3\right)\right]V_2\left(t_4\right)\cdots V_2\left(t_n\right)+\\&+\cdots+\\&+V_1\left(t_1\right)V_1\left(t_2\right)\cdots V_1\left(t_{n-2}\right)\left[V_1\left(t_{n-1}\right)-V_2\left(t_{n-1}\right)\right]V_2\left(t_n\right)+\\&+V_1\left(t_1\right)\cdots V_1\left(t_{n-1}\right)\left[V_1\left(t_n\right)-V_2\left(t_n\right)\right],\end{aligned}$$

it results

$$\begin{aligned}&\left\|V_1\left(t_1\right)\cdots V_1\left(t_n\right)-V_2\left(t_1\right)\cdots V_2\left(t_n\right)\right\|\leq\\&\leq\|V_1-V_2\|\left[\|V_1\|^{n-1}+\|V_1\|\|V_2\|^{n-2}+\cdots+\|V_1\|^{n-2}\|V_2\|+\|V_2\|^{n-1}\right]\\&\leq\|V_1-V_2\|\left[\|V_1\|+\|V_2\|\right]^{n-1}\equiv\frac{\|V_1-V_2\|}{\|V_1\|+\|V_2\|}\cdot\left[\|V_1\|+\|V_2\|\right]^n.\end{aligned}$$

Consequently

$$\left\|\Gamma_{V_1}\left(t\right)-\Gamma_{V_2}\left(t\right)\right\|\leq\sum_{n\in\mathbb{N}}\int_0^t dt_n\cdots\int_0^{t_2}dt_1\left\|V_1\left(t_1\right)\cdots V_1\left(t_n\right)-V_2\left(t_1\right)\cdots V_2\left(t_n\right)\right\|\leq \tag{4.3}$$

$$\leq\frac{\|V_1-V_2\|}{\|V_1\|+\|V_2\|}\left(\sum_{n\in\mathbb{N}}\frac{|t|^n}{n!}\left[\|V_1\|+\|V_2\|\right]^n\right)\equiv\left\{\frac{e^{|t|\left[\|V_1\|+\|V_2\|\right]}-1}{\|V_1\|+\|V_2\|}\right\}\|V_1-V_2\|\equiv C\left\|V_1-V_2\right\|_\infty \tag{4.4}$$

proving that

$$\Gamma_{(\cdot)}\left(t\right):\,V\in\left(C_0\left(\mathbb{R}\right),\|\cdot\|_\infty\right)\longmapsto\Gamma_V\left(t\right)\in\left(\mathfrak{B}\left(L^2\left(\mathbb{R}\right)\right),\|\cdot\|\right)$$

is a continuous function, for all $t\in\mathbb{R}$. Let then $S\equiv\left\{f\in\mathcal{S}\left(\mathbb{R}\right)\,\middle|\,\int f=0\right\}$ be[1] and let $W\in S$ be arbitrarily non-trivial. $\int_0^t ds\left[\exp\left(isH_0\right)W\exp\left(-isH_0\right)\right]\equiv\int_0^t ds\,W\left(s\right)$ is a Hilbert-Schmidt operator on $L^2\left(\mathbb{R}\right)$: its Fourier-transformed integral kernel

$$\left\{\mathfrak{F}\left[\int_0^t ds\,W\left(s\right)\right]\mathfrak{F}^{-1}\right\}\left(p,q\right)=\frac{i}{\sqrt{2\pi}}\cdot\frac{1-e^{it\left(p^2-q^2\right)}}{p^2-q^2}\cdot\tilde{W}\left(p-q\right),\quad p,q\in\mathbb{R}$$

[1] S is dense in $\left(C_0\left(\mathbb{R}\right),\|\cdot\|_\infty\right)$.

is such that

$$\int_{\mathbb{R}^2} dpdq \left| \frac{i}{\sqrt{2\pi}} \cdot \frac{1-e^{it\left(p^2-q^2\right)}}{p^2-q^2} \cdot \tilde{W}(p-q) \right|^2 = |t| \left(\int_{\mathbb{R}} dy \, \frac{\sin^2(y)}{y^2} \right) \left(\int_{\mathbb{R}} dz \, \frac{\left|\tilde{W}(z)\right|^2}{|z|} \right) < \infty,$$

$\tilde{W}$ being the Fourier transform of W. The map

$$(t_2, \ldots, t_n) \in \mathbb{R}^{n-1} \longmapsto \left[\int_0^{t_2} dt_1 \, W(t_1) \right] W(t_2) \cdots W(t_n) \in \mathfrak{B}_2\left(L^2(\mathbb{R})\right)$$

is strongly continuous and such that

$$\left\| \left[\int_0^{t_2} dt_1 \, W(t_1) \right] W(t_2) \cdots W(t_n) \right\|_2^2 \leq \frac{|t_2|}{2} \left(\int_{\mathbb{R}} dz \, \frac{\left|\tilde{W}(z)\right|^2}{|z|} \right) \|W\|^{2n-2}$$

is uniformly bounded on compact subsets of $\mathbb{R}^{n-1}$. Its integral is then again a Hilbert-Schmidt operator[2], allowing to assess that each term of the Dyson expansion, apart from the zeroth order, is compact. (4.2) is then proved for all $W \in S$, while a density argument give it for $C_0(\mathbb{R})$. ∎

Lemma 4.2. *Let $\mathcal{H}$ be a separable Hilbert space and let $F : t \in \mathbb{R}^m \longmapsto F(t) \in \mathfrak{B}_2(\mathcal{H})$ be a strongly continuous function such that $\|F(t)\|_2$, $t \in \mathbb{R}^m$ is uniformly bounded on compact subsets of $\mathbb{R}^m$. Given $K \subset \mathbb{R}^m$ compact,*

$$\hat{F} \doteq \left[\int_K dt \, F(t) \right] \in \mathfrak{B}_2(\mathcal{H}).$$

Proof. Let $\{e_n\}_n$ be a whatever orthonormal basis of $\mathcal{H}$.

$$\begin{aligned}\left\|\hat{F}\right\|_2^2 &= \left| \sum_n \int_{K\times K} dtds \, \langle F(s)\, e_n, F(t)\, e_n \rangle \right| \leq \sum_n \int_{K\times K} dsdt \, \left|\langle F(s)\, e_n, F(t)\, e_n \rangle\right| \\ &\leq \int_{K\times K} dtds \left(\sum_n \|F(s)\, e_n\| \|F(t)\, e_n\| \right) \leq \int_{K\times K} dtds \left(\sum_n \|F(s)\, e_n\|^2 \right)^{1/2} \left(\sum_n \|F(t)\, e_n\|^2 \right)^{1/2} \\ &\leq \int_{K\times K} dtds \, \|F(s)\|_2 \|F(t)\|_2 \leq \text{(uniform boundedness hypothesis)} \leq \left[\lambda^{(m)}(K)\right]^2 C^2 < \infty,\end{aligned}$$

$\lambda^{(m)}$ denoting the m-dimensional Lebesgue measure on $\mathbb{R}^m$ and $C > 0$ representing the uniform bound. ∎

Proposition 4.3. *Let $V \in C_0(\mathbb{R})$ be arbitrary. The self-adjoint Hamiltonian $(H = H_0 + V, \mathcal{D}_H)$ is affiliated to $\mathcal{R}\left(\mathbb{R}^2, \sigma\right)$.*

Proof. As already known, $(P - i\lambda\mathbb{1})^{-1} \in \pi_S\left[\mathcal{R}(X, \sigma)\right]$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$. Proposition 3.1 then gives the free Hamiltonian affiliation; in the end, since

$$(H - i\lambda\mathbb{1})^{-1} = (H_0 - i\lambda\mathbb{1})^{-1} + (H - i\lambda\mathbb{1})^{-1} V (H_0 - i\lambda\mathbb{1})^{-1}$$

for all $\lambda \in \mathbb{R}\setminus\{0\}$, the result follows from $V(H_0 - i\lambda\mathbb{1})^{-1}$ being compact for all $\lambda \in \mathbb{R}\setminus\{0\}$. ∎

[2]See Lemma 4.2.

Proposition 4.4. *Given $n \in \mathbb{N} : n \geq 2$ and $V \in C_0(\mathbb{R})$, let the self-adjoint operator $\left(H_V, \mathcal{D}_{H_V}\right)$ on $L^2(\mathbb{R}^n)$, where*

$$H_V = \sum_{i=1}^{n} \frac{P_i^2}{2m_i} + \sum_{1 \leq i < j \leq n} V\left(Q_i - Q_j\right)$$

be. It is affiliated to $\mathcal{R}\left(\mathbb{R}^{2n}, \sigma\right)$.

Proof. Let $X_k \subset X$ be a bi-dimensional sub-manifold generated by the symplectic pair (f_k, g_k), $k = 1, \ldots, n$; denoting by σ_k the restriction of σ to $X_k \times X_k$, the symplectic subspace (X_k, σ_k) results. The corresponding resolvent algebra is $\mathcal{R}(X_k, \sigma_k)$; let then π_k be its unique (up to unitary equivalence) regular and irreducible representation on $\mathcal{H}_k$. Consequently, $\pi_0 \doteq \pi_1 \otimes \cdots \otimes \pi_n$ singles out a regular irreducible representation of $\mathcal{R}(X_1, \sigma_1) \otimes \cdots \mathcal{R}(X_n, \sigma_n)$ on $\mathcal{H}_0 \doteq \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Such representation extends to a regular irreducible representation of $\mathrm{CCR}(X_1, \sigma_1) \otimes \cdots \otimes \mathrm{CCR}(X_n, \sigma_n) \simeq \mathrm{CCR}(X, \sigma)$, hence to a regular irreducible representation of $\mathcal{R}(X, \sigma)$ on $\mathcal{H}_0$ because of corollary 2.2.1. For all $k = 1, \ldots, n$, the commutative C$^*-$algebra generated by $(P_k - i\lambda\mathbb{1})^{-1}$ coincides with $C_0(P_k)$, hence[3], since $\left(P_k^2/2m_k - i\lambda\mathbb{1}\right)^{-1} \in C_0(P_k)$, $(H_{0k} - i\lambda\mathbb{1})^{-1} \in \pi_k\left[\mathcal{R}(X_k, \sigma_k)\right]$. Recalling that $C_0\left(\mathbb{R}_+^n\right) = C_0(\mathbb{R}_+) \underbrace{\otimes \cdots \otimes}_{n-\text{times}} C_0(\mathbb{R}_+)$, together with the fact that $f : (x_1, \ldots, x_n) \in \mathbb{R}_+^n \longmapsto (x_1 + \cdots + x_n - i\lambda)^{-1} \in \mathbb{C}$ belongs to $C_0\left(\mathbb{R}_+^n\right)$, since the commutative C$^*-$algebra generated by the positive self-adjoint operator H_{0k} is $C_0(H_{0k})$,

$$(H_0 - i\lambda\mathbb{1})^{-1} \equiv (H_{01} + \cdots + H_{0n} - i\lambda\mathbb{1})^{-1} \in C_0(H_{01}) \otimes \cdots \otimes C_0(H_{0n}) \subset \pi_0\left[\mathcal{R}(X, \sigma)\right]$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. Analogously, for all $k, l \in \{1, \ldots, n\} : k < l$, the commutative C$^*-$algebra generated by $\left[(Q_k - Q_l) - i\lambda\mathbb{1}\right]^{-1}$ is $C_0(Q_k - Q_l)$, hence $\sum_{k<l} V(Q_k - Q_l) \in \pi_0\left[\mathcal{R}(X, \sigma)\right]$. Consequently, set $W \equiv \sum_{k<l} V(Q_k - Q_l)$, $\left[\mathbb{1} + (H_0 - i\lambda\mathbb{1})^{-1} W\right] \in \pi_0\left[\mathcal{R}(X, \sigma)\right]$; as well known, if $\left\|(H_0 - i\lambda\mathbb{1})^{-1} W\right\| < 1$, $\left[\mathbb{1} + (H_0 - i\lambda\mathbb{1})^{-1} W\right] \in \pi_0\left[\mathcal{R}(X, \sigma)\right]$ is invertible and this is the case as long as $\lambda \in \mathbb{R} \setminus \{0\}$ is such that $\|W\| < |\lambda|$.

$$\begin{aligned}(H - i\lambda\mathbb{1})^{-1} = (H_0 - i\lambda\mathbb{1} + W)^{-1} = \left\{(H_0 - i\lambda\mathbb{1})\left[\mathbb{1} + (H_0 - i\lambda\mathbb{1})^{-1} W\right]\right\}^{-1} = \\ = \left[\mathbb{1} + (H_0 - i\lambda\mathbb{1})^{-1} W\right]^{-1} (H_0 - i\lambda\mathbb{1})^{-1} \in \pi_0\left[\mathcal{R}(X, \sigma)\right],\end{aligned}$$

meaning that, if $\lambda \in \mathbb{R} \setminus \{0\} : \|W\| < |\lambda|$, H is affiliated to $\mathcal{R}\left(\mathbb{R}^{2n}, \sigma\right)$. The Neumann series expansion allows then to extend the statement to all $\lambda \in \mathbb{R} \setminus \{0\}$. ■

[3] [6], prop. 5.1.

Part II

Point Interactions

Chapter 5

Fixed Centers

Proposition 4.1 guarantees that, for one-dimensional quantum mechanical systems, Schrödinger Hamiltonians with $C_0(\mathbb{R})$ potentials do induce dynamics on $\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$. This section, on the other hand, addresses the stability problem of $\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$ under the action of symbolic Hamiltonians[1] as

$$H=-\frac{d^2}{dx^2}+\sum_{i=1}^{N}\alpha_i\delta\left(x-x_i\right), \tag{5.1}$$

with $N\in\mathbb{N}\cup\{\infty\},\ x_i\in\mathbb{R}:\ x_i\neq x_j,\ \forall\, i,j,\ \alpha_i\in\mathbb{R}\setminus\{0\},\ \forall i$, clearly not of Schrödinger type. Definition 4.1 requires showing

$$e^{itH}\pi_S\left(a\right)e^{-itH}\in\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right],\quad \forall a\in\mathcal{R}\left(\mathbb{R}^2,\sigma\right),\ \forall t\in\mathbb{R}, \tag{5.2}$$

hence the first issue to be dealt with is the explicit construction of e^{-itH}, $t\in\mathbb{R}$ given H as in (5.1). [17] is extremely useful for the announced purpose; concretely, by observing that

$$\begin{aligned}
\Gamma_V\left(t\right)&=\mathbb{1}+\sum_{n\in\mathbb{N}}\left(-i\right)^n\int_0^t dt_n\cdots\int_0^{t_2}dt_1\,V\left(t_1\right)\cdots V\left(t_n\right)\\
&=\mathbb{1}+\sum_{n\in\mathbb{N}}\left(-i\right)^n\int_0^t dt_n\cdots\left\{\int_0^{t_3}dt_2\left[\int_0^{t_2}dt_1V\left(t_1\right)\right]V\left(t_2\right)\cdots\right\}V\left(t_n\right)\\
&=\mathbb{1}+\sum_{n\in\mathbb{N}}\left(-i\right)^n\int_0^t dt_n\,\Gamma_{V,\,(n-1)}\left(t_n\right)V\left(t_n\right)\equiv\mathbb{1}+\sum_{n\in\mathbb{N}}\left(-i\right)^n\,\Gamma_{V,\,(n)}\left(t\right),
\end{aligned}$$

for all $t\in\mathbb{R}$, i.e. by considering

$$\Gamma_{V,\,(n)}\left(t\right)=\int_0^t dt_n\,\Gamma_{V,\,(n-1)}\left(t_n\right)V\left(t_n\right),\quad t\in\mathbb{R},\,n\in\mathbb{N} \tag{5.3}$$

with $\Gamma_{V,\,(0)}\left(t\right)=\mathbb{1}$ for all $t\in\mathbb{R}$, $\Gamma_V\left(t\right)$ can be explicitly built out of the Fourier domain counterparts of $\Gamma_{V,\,(n)}\left(t\right)$, in turn defined by the integral kernels

$$K_{t,\,(1)}\left(p,q\right)=\left[\mathfrak{F}\Gamma_{V,\,(1)}\left(t\right)\mathfrak{F}^{-1}\right]\left(p,q\right)=\left[\frac{e^{it\left(p^2-q^2\right)}-1}{i\left(p^2-q^2\right)}\right]\frac{\tilde{V}\left(p-q\right)}{\sqrt{2\pi}} \tag{5.4}$$

$$K_{t,\,(n)}\left(p,q\right)=\left[\mathfrak{F}\Gamma_{V,\,(n)}\left(t\right)\mathfrak{F}^{-1}\right]\left(p,q\right)=\int_0^t dt_n\int_{\mathbb{R}}dz_{n-1}K_{t_n,\,(n-1)}\left(p,z_{n-1}\right)e^{it_n\left(z_{n-1}^2-q^2\right)}\frac{\tilde{V}\left(z_{n-1}-q\right)}{\sqrt{2\pi}}, \tag{5.5}$$

[1] In this chapter, $H_0=-\frac{d^2}{dx^2}$ is assumed.

$t, p, q \in \mathbb{R}$, $n \in \mathbb{N} : n \geq 2$, where $\mathfrak{F}$ is the Fourier-Plancherel operator. V is then allowed to be a distribution over $\mathbb{R}$ whose Fourier transform $\tilde{V}$ is a L^∞ function such that $\overline{\tilde{V}(p)} = \tilde{V}(-p)$, $p \in \mathbb{R}$. Further, [17] ensures that each $K_{t,(n)}$ is a bounded operator on $L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ and that the Dyson series $\sum_{n\in\mathbb{N}} K_{t,(n)}$ converges in the uniform norm topology; set, then, $K(t) = \mathbb{1} + \sum_{n\in\mathbb{N}} K_{t,(n)}$,

$$U(t) = e^{-itH_0}\left[\mathfrak{F}^{-1} K(t)\, \mathfrak{F}\right], \quad t \in \mathbb{R}$$

gives the rigorous unitary time evolution operator of a system governed by the symbolic Hamiltonian $H = H_0 + V$. □

5.1 One Fixed-Center Point Interaction

One spinless particle undergoing a unique point interaction placed in a fixed location of the real line is considered. Its formal Hamiltonian is

$$H = -\frac{d^2}{dx^2} + \alpha\delta(x - x_0), \quad x_0 \in \mathbb{R}, \tag{5.6}$$

where $\alpha \in \mathbb{R}\backslash\{0\}$ is the coupling constant and $x_0 \in \mathbb{R}$ is the $\delta-$location. Given $V = \alpha\delta(\cdot - x_0)$, one has

- $\tilde{V}(p) = \frac{\alpha}{\left(\sqrt{2\pi}\right)} e^{-ipx_0}$, $p \in \mathbb{R} \implies \tilde{V} \in L^\infty(\mathbb{R})$,
- $\overline{\tilde{V}(p)} = \overline{\frac{\alpha}{\left(\sqrt{2\pi}\right)} e^{-ipx_0}} \equiv \frac{\alpha}{\left(\sqrt{2\pi}\right)} e^{-i(-p)x_0} \equiv \tilde{V}(-p)$, $\quad p \in \mathbb{R}$.

Consequently,

$$K^{(\alpha)}_{t,(1)}(p, q) = \frac{\alpha}{2\pi}\left[\frac{e^{it\left(p^2-q^2\right)} - 1}{i\left(p^2 - q^2\right)}\right] e^{-i(p-q)x_0} \tag{5.7}$$

$$K^{(\alpha)}_{t,(n)}(p, q) = \frac{\alpha}{2\pi}\int_0^t dt_n \int_{\mathbb{R}} dz_{n-1}\, K^{(\alpha)}_{t_n,(n-1)}(p - z_{n-1})\, e^{it_n\left(z^2_{n-1}-q^2\right)} e^{-i(z_{n-1}-q)x_0}, \quad n \in \mathbb{N} \tag{5.8}$$

allow to build $\Gamma_V(t) \equiv \Gamma_\alpha(t)$ as described, hence the unitary time evolution operator $U_\alpha(t) = e^{-itH_0}\Gamma_\alpha(t)$ corresponding to (5.6), for all $t \in \mathbb{R}$. □

Remark 5.1. *Endowed with the unitary time evolution operator, the resolvent algebra $\mathcal{R}\left(\mathbb{R}^2, \sigma\right)$ stability remains to be proved; the following strategy is adopted: given a non-negative smooth function of compact support W*[2]*, by introducing W_ϵ as*

$$W_\epsilon : x \in \mathbb{R} \longmapsto W_\epsilon(x) \doteq \frac{1}{\epsilon} W\left(\frac{x}{\epsilon}\right) \in \mathbb{R}, \quad \epsilon > 0,$$

along with the Schrödinger Hamiltonian $(H_\epsilon = H_0 + \alpha W_\epsilon,\ \mathcal{D}_{H_0})$, Proposition 4.1 allows to claim that $\exp(-itH_\epsilon) \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^2, \sigma\right)\right]$ for all $t \in \mathbb{R}$. Therefore, should

$$\left\|U_\alpha(t) - e^{-itH_\epsilon}\right\| \underset{\epsilon\downarrow 0}{\longrightarrow} 0 \tag{5.9}$$

[2] It does not harm generality assuming $\int_{\mathbb{R}} W = 1$

hold, the stability of $\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$ *would easily follow; infact, for all* $a\in\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$, $t\in\mathbb{R}$,

$$\begin{aligned}&\left\|U_\alpha(t)^*\,\pi_S(a)\,U_\alpha(t)-e^{itH_\epsilon}\pi_S(a)\,e^{-itH_\epsilon}\right\|=\\=&\left\|U_\alpha(t)^*\,\pi_S(a)\,U_\alpha(t)-e^{itH_\epsilon}\pi_S(a)\,U_\alpha(t)+e^{itH_\epsilon}\pi_S(a)\,U_\alpha(t)-e^{itH_\epsilon}\pi_S(a)\,e^{-itH_\epsilon}\right\|\leq\\\leq&\left\|U_\alpha(t)^*-e^{itH_\epsilon}\right\|\left\|\pi_S(a)\right\|+\left\|\pi_S(a)\right\|\left\|U_\alpha(t)-e^{-itH_\epsilon}\right\|\underset{\epsilon\downarrow 0}{\longrightarrow}0.\end{aligned}$$

□

Remark 5.2. *1.* $\{W_\epsilon\}_{\epsilon>0}$ *converges to* δ *in* $\mathscr{D}'(\mathbb{R})$ *as* $\epsilon\to 0$*; infact, given a whatever compactly supported real smooth function* f *on* $\mathbb{R}$,

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}}\left[\frac{1}{\epsilon}W\left(\frac{x-x_0}{\epsilon}\right)\right]f(x)\,dx&=\int_{\mathbb{R}}W(x)\left[\lim_{\epsilon\downarrow 0}f(\epsilon x+x_0)\right]dx=\left(\int_{\mathbb{R}}W\right)f(x_0)\equiv\\&\equiv\int_{\mathbb{R}}\left[\delta(x-x_0)\right]f(x)\,dx,\end{aligned}$$

by using the Lebesgue dominated convergence theorem and $\int_{\mathbb{R}}W=1$.

2. $\left\{\mathcal{F}\left[W_\epsilon(\cdot-x_0)\right]\right\}_\epsilon$[3] *is point-wise convergent to* $\left[\left(\sqrt{2\pi}\right)^{-1}e^{-i(\cdot)x_0}\right]$ *as* $\epsilon\downarrow 0$ *for all* $x_0\in\mathbb{R}$ *and there exists* $M\in\mathbb{R}^+$ *such that* $\left|\tilde{W}_\epsilon(p)\right|\leq M$, *for all* ϵ *and* p*: straightforwardly,*

$$\lim_{\epsilon\downarrow 0}\mathcal{F}\left[W_\epsilon(\cdot-x_0)\right](p)\equiv\lim_{\epsilon\downarrow 0}\tilde{W}_\epsilon(p)=\lim_{\epsilon\downarrow 0}\frac{e^{-ipx_0}}{\sqrt{2\pi}}\int_{\mathbb{R}}W(x)\,e^{-i(\epsilon p)x}dx=\frac{e^{-ipx_0}}{\sqrt{2\pi}},\quad\forall p\in\mathbb{R},$$

by the Lebesgue dominated convergence theorem. Clearly

$$\left|\tilde{W}_\epsilon(p)\right|\leq\frac{1}{\sqrt{2\pi}}\equiv M.$$

□

Proposition 5.1. *Let* $\alpha\in\mathbb{R}\setminus\{0\}$ *be and* $W\in C_c^\infty(\mathbb{R})$ *as in Remark 5.1. For all* $t\in\mathbb{R}$,

$$\left\|U_\alpha(t)-e^{-itH_\epsilon}\right\|\underset{\epsilon\downarrow 0}{\longrightarrow}0.$$

holds.

Proof. Set

$$K^{(\epsilon)}_{t,(1)}(p,q)=\frac{\alpha}{\sqrt{2\pi}}\left[\frac{e^{it\left(p^2-q^2\right)}-1}{i\left(p^2-q^2\right)}\right]\tilde{W}_\epsilon(p-q)\tag{5.10}$$

$$K^{(\epsilon)}_{t,(n)}(p,q)=\frac{\alpha}{\sqrt{2\pi}}\int_0^t dt_n\int_{\mathbb{R}}dz_{n-1}K^{(\epsilon)}_{t_n,(n-1)}(p,z_{n-1})\,e^{it_n\left(z_{n-1}^2-q^2\right)}\tilde{W}_\epsilon(z_{n-1}-q),\quad n\in\mathbb{N}\tag{5.11}$$

[3] $\mathcal{F}$ denotes the L^1-Fourier transform operator.

it results

$$\left\| U_\alpha (t) - e^{-itH_\epsilon} \right\| =$$

$$= \left\| e^{-itH_0} \left[\mathbb{1} + \sum_{n\in\mathbb{N}} i^n \mathfrak{F}^{-1} K^{(\alpha)}_{t,\,(n)} \mathfrak{F} \right] - e^{-itH_0} \left[\mathbb{1} + \sum_{n\in\mathbb{N}} i^n \mathfrak{F}^{-1} K^{(\epsilon)}_{t,\,(n)} \mathfrak{F} \right] \right\| \leq \sum_{n\in\mathbb{N}} \left\| K^{(\alpha)}_{t,\,(n)} - K^{(\epsilon)}_{t,\,(n)} \right\|,$$

meaning that proving the claim amounts in showing

$$\left\| K^{(\alpha)}_{t,\,(n)} - K^{(\epsilon)}_{t,\,(n)} \right\| \underset{\epsilon\downarrow 0}{\longrightarrow} 0, \quad \forall n \in \mathbb{N},\ \forall t \in \mathbb{R}.$$

[17] thm. 3.4 allows for

$$\left\| K^{(\alpha)}_{t,\,(n)} - K^{(\epsilon)}_{t,\,(n)} \right\| \leq \left\{ \left(\sup_{p\in\mathbb{R}} \int_{\mathbb{R}} \left| K^{(\alpha)}_{t,\,(n)} (p,q) - K^{(\epsilon)}_{t,\,(n)} (p,q) \right| dq \right) \left(\sup_{p\in\mathbb{R}} \int_{\mathbb{R}} \left| K^{(\alpha)*}_{t,\,(n)} (p,q) - K^{(\epsilon)*}_{t,\,(n)} (p,q) \right| dq \right) \right\}^{\frac{1}{2}} < \infty, \quad n \in \mathbb{N}$$

hence the *induction principle* is going to be used.

$\boxed{k = 1}$

$$\lim_{\epsilon\downarrow 0} \int_{\mathbb{R}} \left| K^{(\alpha)}_{t,\,(1)} (p,q) - K^{(\epsilon)}_{t,\,(1)} (p,q) \right| dq = \frac{|\alpha|}{\sqrt{2\pi}} \lim_{\epsilon\downarrow 0} \int_{\mathbb{R}} \left| \frac{e^{it\left(p^2-q^2\right)} - 1}{p^2 - q^2} \right| \left| \frac{e^{-i(p-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon (p-q) \right| dq$$

is intended to be studied. By observing that

$$\frac{|\alpha|}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| \frac{e^{it\left(p^2-q^2\right)} - 1}{p^2 - q^2} \right| \left| \frac{e^{-i(p-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon (p-q) \right| dq \leq \sqrt{\frac{2}{\pi}} |\alpha|\, M \int_{\mathbb{R}} \left| \frac{e^{it\left(p^2-q^2\right)} - 1}{p^2 - q^2} \right| dq < \infty$$

because of [17] thm. 2.3, remark 5.2 and the dominated convergence theorem allow for

$$\int_{\mathbb{R}} \left| \frac{e^{it\left(p^2-q^2\right)} - 1}{p^2 - q^2} \right| \left\{ \lim_{\epsilon\downarrow 0} \left| \frac{e^{-i(p-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon (p-q) \right| \right\} dq = 0$$

Since $K_{t,\,(1)} = K^*_{t,\,(1)}$,

$$\lim_{\epsilon\downarrow 0} \int_{\mathbb{R}} \left| K^{*(\alpha)}_{t,\,(1)} (p,q) - K^{*(\epsilon)}_{t,\,(1)} (p,q) \right| dq = 0$$

holds all the same.

$\boxed{k=n}$ It is assumed the statement holds for $k \leq n-1$.

$$K^{(\alpha)}_{t,\,(n)}(p,q) - K^{(\epsilon)}_{t,\,(n)}(p,q) = \tag{5.12}$$

$$= \frac{\alpha}{\sqrt{2\pi}} \int_0^t dt_n \int_{\mathbb{R}} dz_{n-1} e^{it_n\left(z_{n-1}^2 - q^2\right)} K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1}) \left[\frac{e^{-i(z_{n-1}-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon(z_{n-1}-q)\right] + \tag{5.13}$$

$$+ \frac{\alpha}{\sqrt{2\pi}} \int_0^t dt_n \int_{\mathbb{R}} dz_{n-1} e^{it_n\left(z_{n-1}^2 - q^2\right)} \left[K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1}) - K^{(\epsilon)}_{t_n,\,(n-1)}(p, z_{n-1})\right] \tilde{W}_\epsilon(z_{n-1}-q). \tag{5.14}$$

A priori, the foregoing integrals are *double* integrals; to use them as *iterated*, Fubini theorem hypotheses have to be ascertained.

$$\int_0^t \int_{\mathbb{R}} \left|K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1})\right| \left|\frac{e^{-i(z_{n-1}-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon(z_{n-1}-q)\right| dz_{n-1} dt_n \leq$$

$$\leq 2M \int_0^t \int_{\mathbb{R}} \left|K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1})\right| dz_{n-1} dt_n < \infty$$

by the fact that $K^{(\alpha)}_{t,\,(n)} \in \mathfrak{B}\left(L^2(\mathbb{R})\right)$, $\forall n \in \mathbb{N}$, $\forall t \in \mathbb{R}$. Fubini also holds for both (5.14) and the adjoint case, as can be readily verified. Then, to compute

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \left|K^{(\alpha)}_{t,\,(n)}(p,q) - K^{(\epsilon)}_{t,\,(n)}(p,q)\right| dq,$$

the dominated convergence theorem hypotheses need to be checked out. Therefore

$$\left|\int_0^t \int_{\mathbb{R}} K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1}) \left[\frac{e^{-i(z_{n-1}-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon(z_{n-1}-q)\right] e^{it_n\left(z_{n-1}^2 - q^2\right)} dz_{n-1} dt_n\right| \leq$$

$$\leq \left|\int_{\mathbb{R}} \left[\frac{e^{-i(z_{n-1}-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon(z_{n-1}-q)\right] \int_0^t K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1}) e^{it_n\left(z_{n-1}^2 - q^2\right)} dt_n dz_{n-1}\right| \leq$$

$$\leq 2M \int_{\mathbb{R}} \left|\int_0^t K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1}) e^{it_n\left(z_{n-1}^2 - q^2\right)} dt_n\right| dz_{n-1} \equiv$$

$$\equiv (2M)\, \tilde{K}^{(\alpha)}_{t,\,(n)}(p,q),$$

i.e.

$$\int_{\mathbb{R}} \left|\int_0^t \int_{\mathbb{R}} K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1}) \left[\frac{e^{-i(z_{n-1}-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon(z_{n-1}-q)\right] e^{it_n\left(z_{n-1}^2 - q^2\right)} dz_{n-1} dt_n\right| dq \leq$$

$$\leq 2M \int_{\mathbb{R}} \tilde{K}^{(\alpha)}_{t,\,(n)}(p,q)\, dq \leq 2M \left[\sup_{p \in \mathbb{R}} \int_{\mathbb{R}} \tilde{K}^{(\alpha)}_{t,\,(n)}(p,q)\, dq\right] < \infty,$$

the estimate holding because of [17] thm. 3.4. Hence, concerning (5.13),

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \left|\int_0^t \int_{\mathbb{R}} K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1}) \left[\frac{e^{-i(z_{n-1}-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon(z_{n-1}-q)\right] e^{it_n\left(z_{n-1}^2 - q^2\right)} dz_{n-1} dt_n\right| dq \leq$$

$$\leq \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \left|K^{(\alpha)}_{t_n,\,(n-1)}(p, z_{n-1})\right| \left[\lim_{\epsilon \downarrow 0} \left|\frac{e^{-i(z_{n-1}-q)x_0}}{\sqrt{2\pi}} - \tilde{W}_\epsilon(z_{n-1}-q)\right|\right] dz_{n-1} dt_n dq = 0.$$

On the other hand, regarding (5.14), the inductive hypothesis gives

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}\left[K_{t_n,(n-1)}(p,z_{n-1})-K^{(\epsilon)}_{t_n,(n-1)}(p,z_{n-1})\right]\tilde{W}_\epsilon(z_{n-1}-q)\,e^{it_n\left(z_{n-1}^2-q^2\right)}dz_{n-1}dt_n\right|dq\leq$$

$$\leq M\int_{\mathbb{R}}\int_0^t\left\{\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}}\left|K_{t_n,(n-1)}(p,z_{n-1})-K^{(\epsilon)}_{t_n,(n-1)}(p,z_{n-1})\right|dq\right\}dt_ndz_{n-1}=0.$$

By proceeding analogously for the adjoint relations, the Schur test gives

$$\left\|K^{(\alpha)}_{t,(n)}-K^{(\epsilon)}_{t,(n)}\right\|\underset{\epsilon\downarrow 0}{\longrightarrow}0,\ \forall t\in\mathbb{R},\ \forall n\in\mathbb{N}.$$

■

Proposition 5.2. *What follows holds.*

1. $U^*_\alpha(t)\,\pi_S(a)\,U_\alpha(t)\in\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$ *for all* $a\in\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$;
2. *Denoted by* $(H_\alpha,\mathcal{D}_{H_\alpha})$ *the self-adjoint operator on* $L^2(\mathbb{R})$ *generating the one parameter family of strongly continuous unitary operators* $\left\{U_\alpha(t)\right\}_{t\in\mathbb{R}}$, $(H_\alpha,\mathcal{D}_{H_\alpha})$ *is affiliated to* $\mathcal{R}\left(\mathbb{R}^2,\sigma\right)$;
3. *The map* $\alpha_t:\ a\in\mathcal{R}(\mathbb{R}^2,\sigma)\longmapsto\alpha_t(a)\in\mathcal{R}(\mathbb{R}^2,\sigma)$, *with*

$$\alpha_t(a)\doteq\pi_S^{-1}\left[e^{itH_\alpha}\pi_S(a)e^{-itH_\alpha}\right],$$

results in an automorphism of $\mathcal{R}(\mathbb{R}^2,\sigma)$ *for all* $t\in\mathbb{R}$.

Proof.
1. Directly from Proposition 5.1 and remark 5.1.
2. It is a very well known fact that norm dynamical convergence[4] implies[5] norm resolvent convergence, therefore

$$\left\|U_\alpha(t)-e^{-itH_\epsilon}\right\|\underset{\epsilon\downarrow 0}{\longrightarrow}0,\ \forall t\in\mathbb{R}\implies\left\|(H_\alpha-i\lambda\mathbb{1})^{-1}-(H_\epsilon-i\lambda\mathbb{1})^{-1}\right\|\underset{\epsilon\downarrow 0}{\longrightarrow}0,\ \forall\lambda\in\mathbb{R}\backslash\{0\}.$$

Proposition 4.3 states that $(H_\epsilon-i\lambda\mathbb{1})^{-1}\in\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$ for all $\lambda\in\mathbb{R}\setminus\{0\}$ and $\epsilon>0$. Since $\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$ is closed with respect to the uniform norm topology, the affiliation of $(H_\alpha,\mathcal{D}_{H_\alpha})$ results.

3. Given $t\in\mathbb{R}$, the map

$$a\in\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]\longmapsto e^{itH_\alpha}ae^{-itH_\alpha}\in\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]\tag{5.15}$$

---

[4]

Definition 5.1. Let $\mathcal{H}$ be a complex Hilbert space. Given self-adjoint operators $(A_n,\mathcal{D}_{A_n})$, $(A,\mathcal{D}_A)$, A_n is **norm dynamically convergent** to A if and only if, for all $t\in\mathbb{R}$, $\left\{e^{itA_n}\right\}_n$ converges to e^{itA} with respect to the $\mathfrak{B}(\mathcal{H})$ norm.

□

[5] See [18], thm. 10.1.16.

is surely injective, by being isometric. On the other hand, given $b \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$, because of proposition 5.1, $e^{-itH_\alpha}\, b\, e^{itH_\alpha} \equiv d \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$, hence

$$e^{itH_\alpha}\, d\, e^{-itH_\alpha} = b,$$

allowing to conclude that (5.15) is surjective. The same map is obviously a homomorphism; finally, since $\pi_S : \mathcal{R}\left(\mathbb{R}^2,\sigma\right) \longrightarrow \pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$ is an isomorphism, the result follows.

■

5.1.1 Many Fixed-Centers Point Interactions

5.1.1.1 Finitely Many Fixed-Centers Point Interactions

Focus is set on the symbolic Hamiltonian

$$H = H_0 + \sum_{i=1}^{N} \alpha_i\, \delta\left(x - x_i\right), \tag{5.16}$$

with $N \in \mathbb{N}$, coupling constants $\alpha_i \in \mathbb{R}\setminus\{0\}$ and fixed-centers location $x_i \in \mathbb{R} : x_i \neq x_j$. By setting $(\alpha) \equiv (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ and $V = \sum_{i=1}^{N} \alpha_i \delta\left(\cdot - x_i\right)$,

1.
$$\tilde{V}(p) = \int_{\mathbb{R}} \left[\sum_{m=1}^{N} \alpha_m \delta(x - x_m)\right] e^{-ipx} \frac{dx}{\sqrt{2\pi}} = \sum_{m=1}^{N} \frac{\alpha_m}{\sqrt{2\pi}} e^{-ipx_m},$$

   i.e. $\tilde{V} \in L^\infty(\mathbb{R})$ and

2.
$$\tilde{V}(p) = \sum_{m=1}^{N} \frac{\alpha_m}{\sqrt{2\pi}} e^{-ipx_m} = \sum_{m=1}^{N} \frac{\alpha_m}{\sqrt{2\pi}} e^{i(-p)x_m} = \overline{\sum_{m=1}^{N} \frac{\alpha_m}{\sqrt{2\pi}} e^{-i(-p)x_m}} = \overline{\tilde{V}(-p)},$$

   for all $p \in \mathbb{R}$.

Consequently, one legitimately relies on

$$K^{(\alpha)}_{t,(1)}(p,q) = \left[\frac{e^{it(p^2-q^2)} - 1}{i(p^2-q^2)}\right] \left[\sum_{m=1}^{N} \frac{\alpha_m}{\sqrt{2\pi}} e^{-i(p-q)x_m}\right] \equiv \sum_{m=1}^{N} K^{(\alpha),m}_{t,(1)}(p,q), \tag{5.17}$$

$$K^{(\alpha)}_{t,(n)}(p,q) = \int_0^t \int_{\mathbb{R}} K^{(\alpha)}_{t_n,(n-1)}(p,z_{n-1}) \left[\sum_{m=1}^{N} \frac{\alpha_m}{\sqrt{2\pi}} e^{-i(z_{n-1}-q)x_m}\right] e^{it_n(z_{n-1}^2-q^2)} dz_{n-1} dt_n = \tag{5.18}$$

$$\equiv \sum_{m=1}^{N} K^{(\alpha),m}_{t,(n)}(p,q), \tag{5.19}$$

for all $t, p, q \in \mathbb{R}$, to build $\Gamma_{(\alpha)}(t)$ up, hence the unitary time evolution operator $U_{(\alpha)}(t)$, $t \in \mathbb{R}$.

Proposition 5.3. *Given $N \in \mathbb{N}$, let $\alpha_1, \ldots, \alpha_N \in \mathbb{R} \setminus \{0\}$ and non-negative smooth functions of compact support $W_1, \ldots, W_N \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} W_i = 1$ be. Considered the Schrödinger Hamiltonians*

$$H_\epsilon = H_0 + \sum_{i=1}^N \alpha_i W_{\epsilon,i} \equiv H_0 + W_\epsilon, \quad \epsilon > 0,$$

where $W_{\epsilon,i}(x) = \epsilon^{-1} W_i\left(x/\epsilon\right),\ x \in \mathbb{R}, i \in \{1, \ldots, N\}$, for all $t \in \mathbb{R}$,

$$\left\| U_{(\alpha)}(t) - e^{-itH_\epsilon} \right\| \underset{\epsilon \downarrow 0}{\longrightarrow} 0.$$

Proof. By using

$$K^{(\epsilon)}_{t,(1)}(p,q) = \left[\frac{e^{it(p^2-q^2)} - 1}{i(p^2-q^2)}\right] \frac{\tilde{W}_\epsilon(p-q)}{\sqrt{2\pi}} \equiv \sum_{m=1}^N K^{(\epsilon),m}_{t,(1)}(p,q) \tag{5.20}$$

$$K^{(\epsilon)}_{t,(n)}(p,q) = \int_0^t \int_{\mathbb{R}} K^{(\epsilon)}_{t_n,(n-1)}(p, z_{n-1}) \frac{\tilde{W}_\epsilon(z_{n-1}-q)}{\sqrt{2\pi}} e^{it_n(z_{n-1}^2 - q^2)}\, dz_{n-1} dt_n = \sum_{m=1}^N K^{(\epsilon),m}_{t,(n)}(p,q) \tag{5.21}$$

for all $t, p, q \in \mathbb{R}$, to build $\exp(-itH_\epsilon)$ up, $t \in \mathbb{R}$, one then has

$$\left\| U_{(\alpha)}(t) - e^{-itH_\epsilon} \right\| \leq \sum_{n \in \mathbb{N}} \left\| K^{(\alpha)}_{n,t,s} - K^{(\epsilon)}_{n,t,s} \right\| \leq \text{(by Schur test)}$$

$$\leq \sum_{n \in \mathbb{N}} \left\{ \left[\sup_p \int_{\mathbb{R}} \left| K^{(\alpha)}_{t,(n)}(p,q) - K^{(\epsilon)}_{t,(n)}(p,q) \right| dq \right] \left[\sup_p \int_{\mathbb{R}} \left| K^{(\alpha),*}_{t,(n)}(p,q) - K^{(\epsilon),*}_{t,(n)}(p,q) \right| dq \right] \right\}^{\frac{1}{2}}.$$

Therefore, by observing that

$$\int_{\mathbb{R}} \left| K^{(\alpha)}_{t,(n)}(p,q) - K^{(\epsilon)}_{t,(n)}(p,q) \right| dq = \int_{\mathbb{R}} \left| \sum_{m=1}^N \left[K^{(\alpha),m}_{t,(n)}(p,q) - K^{(\epsilon),m}_{t,(n)}(p,q) \right] \right| dq \leq \tag{5.22}$$

$$\leq \sum_{m=1}^N \int_{\mathbb{R}} \left| K^{(\alpha),m}_{t,(n)}(p,q) - K^{(\epsilon),m}_{t,(n)}(p,q) \right| dq, \tag{5.23}$$

the result is proved as in Proposition 5.1. ■

Proposition 5.4. *What follows holds.*

1. *$U^*_{(\alpha)}(t)\, \pi_S(a)\, U_{(\alpha)}(t) \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^2, \sigma\right)\right]$ for all $a \in \mathcal{R}\left(\mathbb{R}^2, \sigma\right)$;*

2. *Denoted by $\left(H_{(\alpha)}, \mathcal{D}_{H_{(\alpha)}}\right)$ the self-adjoint operator on $L^2(\mathbb{R})$ generating the one parameter family of strongly continuous unitary operators $\left\{U_{(\alpha)}(t)\right\}_{t \in \mathbb{R}}$, $\left(H_{(\alpha)}, \mathcal{D}_{H_{(\alpha)}}\right)$ is affiliated to $\mathcal{R}\left(\mathbb{R}^2, \sigma\right)$;*

3. *The map $\alpha_t :\ a \in \mathcal{R}(\mathbb{R}^2, \sigma) \longmapsto \alpha_t(a) \in \mathcal{R}(\mathbb{R}^2, \sigma)$, with*

$$\alpha_t(a) \doteq \pi_S^{-1}\left[e^{itH_{(\alpha)}} \pi_S(a) e^{-itH_{(\alpha)}}\right],$$

results in an automorphism of $\mathcal{R}(\mathbb{R}^2, \sigma)$ for all $t \in \mathbb{R}$.

Proof. The proof closely mimics that of Proposition 3.2. ■

5.1.1.2 Countably Many Fixed-Centers Point Interactions

Given $\{\alpha_i\}_{i\in\mathbb{N}} \in l^1(\mathbb{N}) \setminus \{0\}$ the symbolic Hamiltonian

$$H = -\frac{d^2}{dx^2} + \sum_{i=1}^{\infty} \alpha_i \delta(x - x_i),$$

with $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ such that $x_i \neq x_j$, for all $i \neq j$, is finally considered. Set $V = \sum_{i=1}^{\infty} \alpha_i \delta(\cdot - x_i)$,

1. $\tilde{V}(p) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{N}} \alpha_m e^{-ipx_m} = \overline{\frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{N}} \alpha_m e^{ipx_m}} = \overline{\tilde{V}(-p)}, \quad \forall p \in \mathbb{R}$ and

2. $\tilde{V} \in L^\infty(\mathbb{R})$,

therefore $\Gamma_{\{\alpha_i\}}(t)$ can be obtained via (5.4), (5.5).

Proposition 5.5. *What follows holds.*

1. *For all $t \in \mathbb{R}$, the unitary time evolution operator $U_{\{\alpha_i\}}(t) \doteq e^{-itH_0}\Gamma_{\{\alpha_i\}}(t)$ belongs to $\pi_S\left[\mathcal{R}\left(\mathbb{R}^2, \sigma\right)\right]$. Moreover, denoted by $\left(H_{\{\alpha_i\}}, \mathcal{D}_{H_{\{\alpha_i\}}}\right)$ the self-adjoint operator on $L^2(\mathbb{R})$ generating the one parameter family of unitary operators $\left\{U_{\{\alpha_i\}}(t)\right\}_{t\in\mathbb{R}}$, it is affiliated to $\mathcal{R}\left(\mathbb{R}^2, \sigma\right)$.*

2. *$U_{\{\alpha_i\}}(t)^* \pi_S(a) U_{\{\alpha_i\}}(t) \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^2, \sigma\right)\right]$ for all $a \in \mathcal{R}\left(\mathbb{R}^2, \sigma\right)$, $t \in \mathbb{R}$.*

3. *The map*

$$\alpha_t : a \in \mathcal{R}\left(\mathbb{R}^2, \sigma\right) \longmapsto \alpha_t(a) = \pi_S^{-1}\left[U_{\{\alpha_i\}}(t)^* \pi_S(a) U_{\{\alpha_i\}}(t)\right] \in \mathcal{R}\left(\mathbb{R}^2, \sigma\right)$$

is an automorphism of $\mathcal{R}\left(\mathbb{R}^2, \sigma\right)$.

Proof. Concerning 1., the result follows from [19] prop. 2, Proposition 5.3 and the fact that norm dynamical convergence implies norm resolvent convergence. 2. and 3. are proved as in Proposition 3.2. ■

Remark 5.3. *The resolvent algebra non-trivial ideal structure has already proved to be fundamental for the possibility of accommodating non-trivial quantum dynamics. The same feature is also of primary importance for the following final result to hold.* □

Proposition 5.6. *Let $\mathfrak{K}_0$ be the C^*-subalgebra of $\pi_S\left[\mathcal{R}\left(\mathbb{R}^2, \sigma\right)\right]$ generated by $\mathfrak{B}_\infty\left(L^2(\mathbb{R})\right)$ and the identity operator. $\left(\mathcal{K}_0 \equiv \pi_S^{-1}(\mathfrak{K}_0), \mathbb{R}, \beta\right)$, where*

$$\beta : t \in \mathbb{R} \longmapsto \beta_t \in Aut(\mathcal{K}_0)$$

and

$$\beta_t : a \in \mathcal{K}_0 \longmapsto \beta_t(a) \doteq \pi_S^{-1}\left[U(t)^* \pi_S(a)\, U(t)\right] \in \mathcal{K}_0,$$

$U(t) \in \mathfrak{B}\left(L^2(\mathbb{R})\right)$ propagating fixed point interactions, is a C^*-dynamical system.

Proof. First of all, it is observed that Proposition 3.3 allows for $\mathfrak{B}_\infty\left(L^2\left(\mathbb{R}\right)\right)$ to be contained in $\pi_S\left[\mathcal{R}\left(\mathbb{R}^2,\sigma\right)\right]$; then, for all $t_0\in\mathbb{R}$, $\left\|U(t)^*U(t)-U(t_0)^*U(t_0)\right\|=0$. On the other hand, given $\psi,\varphi\in L^2\left(\mathbb{R}\right)$, let the finite rank operator $T=\langle\psi,\cdot\rangle\varphi$ be. Fixed again $t_0\in\mathbb{R}$,

$$\left\|U\left(t\right)^*T-U\left(t_0\right)^*T\right\|\leq\|\psi\|\left\|U\left(t\right)^*\varphi-U\left(t_0\right)^*\varphi\right\|\underset{t\to t_0}{\longrightarrow}0.$$

Analogously,

$$\left\|TU\left(t\right)-TU\left(t_0\right)\right\|\leq\|\varphi\|\left\|U\left(t\right)^*\psi-U\left(t_0\right)^*\psi\right\|\underset{t\to t_0}{\longrightarrow}0,$$

therefore

$$\left\|U\left(t\right)^*TU\left(t\right)-U\left(t_0\right)^*TU\left(t_0\right)\right\|\leq\left\|U\left(t\right)^*T-U\left(t_0\right)^*T\right\|+\left\|TU\left(t\right)-TU\left(t_0\right)\right\|\underset{t\to t_0}{\longrightarrow}0.$$

Linearity, density and continuity arguments prove the statement. ■

Chapter 6

Two-Body Delta Potential

The second system investigated is made up of $n \in \mathbb{N} : n \geq 2$ distinguishable particles, interacting via a two-body delta potential in one spatial dimension. The symbolic Hamiltonian of the case is

$$H = -\sum_{i=1}^{n} \frac{1}{2m_i} \frac{\partial^2}{\partial x_i^2} - g \sum_{1 \leq i < j \leq n} \delta\left(x_i - x_j\right) \equiv H_0 - g \sum_{1 \leq i < j \leq n} \delta\left(x_i - x_j\right), \tag{6.1}$$

where $g \in \mathbb{R} \setminus \{0\}$ is the coupling constant and $(H_0, \mathcal{D}_{H_0})$ the free Hamiltonian. Purpose of the chapter is showing that (6.1) is affiliated to $\mathcal{R}\left(\mathbb{R}^{2n}, \sigma\right)$. The adopted strategy is briefly sketched: given a positive, even, smooth function of compact support v[1], by introducing $V \doteq v^2$ and

$$V_\epsilon : x \in \mathbb{R} \longmapsto V_\epsilon(x) \doteq \frac{1}{\epsilon} V\left(\frac{x}{\epsilon}\right) \in \mathbb{R}, \quad \epsilon > 0$$

one refers to the Schrödinger Hamiltonians

$$H_\epsilon = -\sum_{i=1}^{n} \frac{1}{2m_i} \frac{\partial^2}{\partial x_i^2} - g \sum_{1 \leq i < j \leq n} V_\epsilon\left(x_i - x_j\right) \equiv H_0 - g \sum_{1 \leq i < j \leq n} V_\epsilon^{(ij)}, \quad \epsilon > 0, \tag{6.2}$$

self-adjoint on $\mathcal{D}_{H_0}$. Because of proposition 4.4, $R_{H_\epsilon}(z) \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^{2n}, \sigma\right)\right]$ for all $\epsilon > 0,\ z \in i\mathbb{R} \setminus \{0\}$; the result then follows from H_ϵ converging to H in the norm resolvent sense. $\square$

6.1 Preliminaries

Remark 6.1. *Given $i, j \in \{1, \ldots, n\} : i < j$ generic, let the following coordinate transformation be.*

$$\begin{cases} R_{(ij)} = \frac{m_i x_i + m_j x_j}{m_i + m_j} \\ r_{(ij)} = x_i - x_j \\ y_k = x_k, \quad k \neq i, j \end{cases} \iff \begin{cases} x_i = R_{(ij)} + \left(\frac{m_j}{m_i + m_j}\right) r_{(ij)} \\ x_j = R_{(ij)} - \left(\frac{m_i}{m_i + m_j}\right) r_{(ij)} \\ x_k = y_k, \quad k \neq i, j \end{cases} \tag{6.3}$$

The corresponding jacobian is identically equal to 1, as can be easily verified. $\square$

Definition 6.1. Given (6.3), by introducing the Hilbert space

$$\chi_{(ij)} \doteq L^2\left(\mathbb{R}^n,\ dr_{(ij)} dR_{(ij)} dy_1 \ldots \widehat{dy_i} \ldots \widehat{dy_j} \ldots dy_n\right),$$

[1]It does not harm generality assuming $\int_{\mathbb{R}} v^2 = 1$.

(hats represent omission) the (unitary) operator implementing (6.3) is

$$U_{(ij)}:\ L^2\left(\mathbb{R}^n, dx_1 \dots dx_i \dots dx_j \dots dx_n\right) \longrightarrow \chi_{(ij)}. \tag{6.4}$$

□

Definition 6.2. Given $i, j \in \{1, \dots, n\} : i < j,\ \epsilon > 0$, denoted $\left(R_{(ij)}, y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_n\right) \in \mathbb{R}^{n-1}$ by $\underline{Y}_{(ij)}$, the scaling operator

$$U_\epsilon^{(ij)}:\ \tilde{\psi} \in \chi_{(ij)} \longmapsto U_\epsilon^{(ij)}\tilde{\psi} \in \chi_{(ij)}$$

is introduced, where, should $\tilde{\psi}$ be continuous, $\left(U_\epsilon^{(ij)}\tilde{\psi}\right)\left(r_{(ij)}, \underline{Y}_{(ij)}\right) \doteq \sqrt{\epsilon}\tilde{\psi}\left(\epsilon r_{(ij)}, \underline{Y}_{(ij)}\right)$. □

Remark 6.2. *By introducing the Hilbert space*

$$\chi_{(ij)}^{(red)} \doteq L^2\left(\mathbb{R}^{n-1},\ dR_{(ij)}dy_1 \dots \widehat{dy_i} \dots \widehat{dy_j} \dots dy_n\right),$$

on the one hand, $\chi_{(ij)} = L^2\left(\mathbb{R}, dr_{(ij)}\right) \otimes \chi_{(ij)}^{(red)}$, *on the other hand,* $U_\epsilon^{(ij)} \equiv u_\epsilon^{(ij)} \otimes \mathbb{1}$, *where*

$$u_\epsilon^{(ij)}:\ \varphi \in L^2\left(\mathbb{R},\ dr_{(ij)}\right) \longmapsto u_\epsilon^{(ij)}\varphi \in L^2\left(\mathbb{R}, dr_{(ij)}\right) \tag{6.5}$$

and $\left(u_\epsilon^{(ij)}\varphi\right)\left(r_{(ij)}\right) = \sqrt{\epsilon}\varphi\left(\epsilon r_{(ij)}\right)$, *should* φ *be a continuous function.* $u_\epsilon^{(ij)}$ *is well-defined and unitary because of*

$$\int_{\mathbb{R}} \left|\left(u_\epsilon^{(ij)}\varphi\right)\left(r_{(ij)}\right)\right|^2 dr_{(ij)} \equiv \int_{\mathbb{R}} \epsilon\left|\varphi\left(\epsilon r_{(ij)}\right)\right|^2 dr_{(ij)} = \left(\tilde{r}_{(ij)} = \epsilon r_{(ij)}\right) = \int_{\mathbb{R}} \left|\varphi\left(\tilde{r}_{(ij)}\right)\right|^2 d\tilde{r}_{(ij)} \equiv \|\varphi\|_2^2.$$

□

Definition 6.3. Let $v \in C_0^\infty(\mathbb{R})$ be even, positive and such that $\int_{\mathbb{R}} v^2 = 1$[2]. For all $i, j \in \{1, \dots, n\} : i < j,\ \epsilon > 0$, the bounded linear operator

$$a_\epsilon^{(ij)} \doteq (v \otimes \mathbb{1}) \frac{U_\epsilon^{(ij)}}{\sqrt{\epsilon}} U_{(ij)} \equiv \left(\frac{v u_\epsilon^{(ij)}}{\sqrt{\epsilon}} \otimes \mathbb{1}\right) U_{(ij)}:\ L^2\left(\mathbb{R}^n, dx_1 \dots dx_n\right) \longrightarrow \chi_{(ij)},$$

is introduced. □

Remark 6.3.

- *For all* $i, j \in \{1, \dots, n\} : i < j$, $V_\epsilon^{(ij)} = a_\epsilon^{(ij)*} a_\epsilon^{(ij)}$ *and, for all* $\epsilon > 0$,

$$H_\epsilon = H_0 - g \sum_{1 \le i < j \le n} V_\epsilon^{(ij)} \equiv H_0 - g \sum_{1 \le i < j \le n} a_\epsilon^{(ij)*} a_\epsilon^{(ij)}. \tag{6.6}$$

- *From now on, an interacting pair* (ij) *will be denoted by a greek index* $\sigma, \nu, \dots,$ *varying in* $\mathcal{I}$, *with* $|\mathcal{I}| = \binom{n}{2}$ = *number of interacting pairs.*

□

[2] The same letter will be used to denote the corresponding multiplication operator on $L^2\left(\mathbb{R}, dr_{(ij)}\right)$.

Definition 6.4. Let the Hilbert space $\chi = \bigoplus_\sigma \chi_\sigma$ be. Given $\epsilon > 0$, the bounded operator

$$A_\epsilon : L^2(\mathbb{R}^n, dx_1 \dots dx_n) \longrightarrow \chi$$

is defined, where

$$A_\epsilon \psi \doteq \left(\sqrt{|g|} a_\epsilon^{\sigma_1} \psi, \dots, \sqrt{|g|} a_\epsilon^{\sigma_{|\mathcal{I}|}} \psi \right) \equiv \left(A_\epsilon^{\sigma_1} \psi, \dots, A_\epsilon^{\sigma_{|\mathcal{I}|}} \psi \right)$$

for all $\psi \in L^2(\mathbb{R}^n, dx_1 \dots dx_n)$. It is further introduced the bounded operator J on χ via the position

$$J(\psi_\sigma)_{\sigma=1}^{|\mathcal{I}|} \doteq \left((\operatorname{sgn} g)\, \psi_\sigma \right)_{\sigma=1}^{|\mathcal{I}|}$$

for all $(\psi_\sigma)_\sigma \in \chi$, to eventually define $B_\epsilon = JA_\epsilon : L^2(\mathbb{R}^n, dx_1 \dots dx_n) \longrightarrow \chi$. □

Remark 6.4. *The foregoing definition allows to conclude that $g \sum_\sigma V_\epsilon^\sigma = A_\epsilon^* B_\epsilon$, for all $\epsilon > 0$, hence $H_\epsilon = H_0 - A_\epsilon^* B_\epsilon$. The Konno-Kuroda formula can then be used (see App. 1) and*

$$(H_\epsilon - z\mathbb{1})^{-1} \equiv R_{H_\epsilon}(z) = R_{H_0}(z) + \sum_{\sigma,\nu} \left[A_\epsilon^\sigma R_{H_0}(\overline{z}) \right]^* \left[\Lambda_\epsilon(z)^{-1} \right]_{\sigma\nu} \left[B_\epsilon^\nu R_{H_0}(z) \right], \tag{6.7}$$

for all $z \in \rho(H_\epsilon) \cap \rho(H_0)$, $\epsilon > 0$, where $\left[\Lambda_\epsilon(z) \right]_{\sigma\nu} = \delta_{\sigma\nu} - B_\epsilon^\sigma R_{H_0}(z) A_\epsilon^{\nu *} \in \mathfrak{B}(\chi_\sigma, \chi_\nu)$, $\sigma, \nu \in \mathcal{I}$. The entire analysis is based on the $\epsilon \downarrow 0$ behaviour of (6.7). □

6.2 The Limit of $a_\epsilon^\sigma R_{H_0}(z)$, $z < 0$

Remark 6.5. *Given U_σ as in (6.4), $U_\sigma H_0 = H_0^\sigma U_\sigma$ on $\mathcal{D}_{H_0}$, where*

$$H_0^\sigma = -\frac{1}{2\mu_\sigma} \frac{\partial^2}{\partial r_\sigma^2} - \frac{1}{2M_\sigma} \frac{\partial^2}{\partial R_\sigma^2} - \sum_{\substack{k=1 \\ k \neq i,j}}^n \frac{1}{2m_k} \frac{\partial^2}{\partial x_k^2},$$

hence

$$U_\sigma (H_0 - z\mathbb{1})^{-1} = (H_0^\sigma - z\mathbb{1})^{-1} U_\sigma, \quad z \in \rho(H_0) \equiv \rho(H_0^\sigma),$$

and

$$\begin{aligned} a_\epsilon^\sigma (H_0 - z\mathbb{1})^{-1} &= (v \otimes \mathbb{1}) \frac{U_\epsilon^\sigma}{\sqrt{\epsilon}} U_\sigma (H_0 - z\mathbb{1})^{-1} = (v \otimes \mathbb{1}) \frac{U_\epsilon^\sigma}{\sqrt{\epsilon}} (H_0^\sigma - z\mathbb{1})^{-1} U_\sigma = \\ &= T_\epsilon^\sigma(z) U_\sigma \end{aligned}$$

collecting both the z and ϵ dependence. Moreover, since $a_\epsilon^\sigma R_{H_0}(z) \in \mathfrak{B}\left(L^2(\mathbb{R}^n), \chi_\sigma\right)$, one has $T_\epsilon^\sigma(z) \in \mathfrak{B}(\chi_\sigma)$, for all $z \in \rho(H_0)$. □

Definition 6.5. Denoted by $\mathfrak{F}_{\underline{Y}_\sigma}$ the Fourier operator on $\chi_\sigma^{(red)}$, the bounded operator

$$\left(\mathbb{1} \otimes \mathfrak{F}_{\underline{Y}_\sigma} \right) T_\epsilon^\sigma(z) \left(\mathbb{1} \otimes \mathfrak{F}_{\underline{Y}_\sigma}^{-1} \right) \doteq T_{\epsilon, \underline{P}_\sigma}^\sigma(z) : L^2(\mathbb{R}, dr_\sigma) \otimes \tilde{\chi}_\sigma^{(red)} \to L^2(\mathbb{R}, dr_\sigma) \otimes \tilde{\chi}_\sigma^{(red)}$$

is introduced, where $\tilde{\chi}_\sigma^{(red)} = \mathfrak{F}_{\underline{Y}_\sigma} \chi_\sigma^{(red)}$. □

Remark 6.6. *By definition,*

$$T^{\sigma}_{\epsilon,\underline{P}_\sigma}(z) \equiv \left(\frac{vu^{\sigma}_{\epsilon}}{\sqrt{\epsilon}} \otimes \mathbb{1}\right)\left(\mathbb{1}\otimes\mathfrak{F}_{\underline{Y}_\sigma}\right)\left(H^{\sigma}_0 - z\mathbb{1}\right)^{-1}\left(\mathbb{1}\otimes\mathfrak{F}^{-1}_{\underline{Y}_\sigma}\right) \equiv$$
$$\equiv (2\mu_\sigma)\left(\frac{vu^{\sigma}_{\epsilon}}{\sqrt{\epsilon}} \otimes \mathbb{1}\right)\left[-\frac{\partial^2}{\partial r^2_\sigma} - (2\mu_\sigma)(z-Q_\sigma)\mathbb{1}\right]^{-1},$$

where $Q_\sigma = \frac{P^2_\sigma}{2M_\sigma} + \sum_{\substack{k=1\\k\neq i,j}}^{n}\frac{p^2_k}{2m_k}$. *In particular, for all* $\psi \in L^2(\mathbb{R}, dr_\sigma)\otimes\tilde{\chi}^{(red)}_\sigma$, *it is*

$$\left[T^{\sigma}_{\epsilon,\underline{P}_\sigma}(z)\psi\right](r_\sigma,\underline{P}_\sigma) = (2\mu_\sigma)v(r_\sigma)\int_{\mathbb{R}} G^{(1)}_{(2\mu_\sigma)(z-Q_\sigma)}\left(\epsilon r_\sigma - r'_\sigma\right)\psi\left(r'_\sigma,\underline{P}_\sigma\right)dr'_\sigma,$$

i.e. on $\tilde{\chi}^{(red)}_\sigma$, $T^{\sigma}_{\epsilon,\underline{P}_\sigma}(z)$ *behaves as a multiplication operator, while, on* $L^2(\mathbb{R}, dr_\sigma)$, *as an integral operator whose kernel is*

$$(2\mu_\sigma)\,v(r_\sigma)\,G^{(1)}_{(2\mu_\sigma)(z-Q_\sigma)}\left(\epsilon r_\sigma - r'_\sigma\right). \tag{6.8}$$

Definition 6.6. Given $\sigma\in\mathcal{I}$, $z<0$, let $T^{\sigma}_{0,\underline{P}_\sigma}(z):\ \psi\in L^2(\mathbb{R}, dr_\sigma)\otimes\tilde{\chi}^{(red)}_\sigma \longmapsto T^{\sigma}_{0,\underline{P}_\sigma}(z)\psi\in L^2(\mathbb{R}, dr_\sigma)\otimes\tilde{\chi}^{(red)}_\sigma$ be such that

$$\left[T^{\sigma}_{0,\underline{P}_\sigma}(z)\psi\right](r_\sigma,\underline{P}_\sigma) \doteq (2\mu_\sigma)v(r_\sigma)\int_{\mathbb{R}} G^{(1)}_{(2\mu_\sigma)(z-Q_\sigma)}\left(-r'_\sigma\right)\psi\left(r'_\sigma,\underline{P}_\sigma\right)dr'_\sigma,$$

with Q_σ as above. Correspondingly

$$\left(\mathbb{1}\otimes\mathfrak{F}^{-1}_{\underline{Y}_\sigma}\right)T^{\sigma}_{0,\underline{P}_\sigma}(z)\left(\mathbb{1}\otimes\mathfrak{F}_{\underline{Y}_\sigma}\right) \doteq T^{\sigma}_0(z):\ L^2(\mathbb{R}, dr_\sigma)\otimes\chi^{(red)}_\sigma \longrightarrow L^2(\mathbb{R}, dr_\sigma)\otimes\chi^{(red)}_\sigma \tag{6.9}$$

is introduced. □

Lemma 6.1. *Let* $\sigma\in\mathcal{I}$, $z<0$ *be. Then*

$$\lim_{\epsilon\downarrow 0}\left\|T^{\sigma}_{\epsilon}(z) - T^{\sigma}_0(z)\right\|_{\mathfrak{B}(\chi_\sigma)} = 0.$$

Proof. Directly from [20], Lemma 3.1 and Proposition 3.2. ■

Remark 6.7. *Direct consequence of Lemma 6.1, for all* $\sigma\in\mathcal{I}$, $z<0$, *is*

$$\lim_{\epsilon\downarrow 0} a^{\sigma}_{\epsilon}\left(H_0 - z\mathbb{1}\right)^{-1} = \lim_{\epsilon\downarrow 0} T^{\sigma}_{\epsilon}(z)\,U_\sigma = T^{\sigma}_0(z)\,U_\sigma \doteq S^{\sigma}(z),$$

with $S^{\sigma}(z)\in\mathfrak{B}\left(L^2(\mathbb{R}^n, dx_1\cdots dx_n),\chi_\sigma\right)$. □

6.3 $\Lambda_\epsilon(z)$-related analysis

Remark 6.8. *Given* $z\in\rho(H_0)$, $\sigma,\nu\in\mathcal{I}$, *one starts observing that*

$$\begin{aligned}\left[\Lambda_\epsilon(z)\right]_{\sigma\nu} &= \delta_{\sigma\nu} - B^{\sigma}_{\epsilon}R_{H_0}(z)A^{\nu*}_{\epsilon} =\\ &= \delta_{\sigma\nu} - B^{\sigma}_{\epsilon}R_{H_0}(z)A^{\nu*}_{\epsilon} + \delta_{\sigma\nu}B^{\sigma}_{\epsilon}R_{H_0}(z)A^{\nu*}_{\epsilon} - \delta_{\sigma\nu}B^{\sigma}_{\epsilon}R_{H_0}(z)A^{\nu*}_{\epsilon} =\\ &= \left[\mathbb{1} - B^{\sigma}_{\epsilon}R_{H_0}(z)A^{\nu*}_{\epsilon}\right]\delta_{\sigma\nu} + \left(\delta_{\sigma\nu} - \mathbb{1}\right)B^{\sigma}_{\epsilon}R_{H_0}(z)A^{\nu*}_{\epsilon} \equiv\\ &\equiv \left[\Lambda_\epsilon(z)_{diag}\right]_{\sigma\nu} + \left[\Lambda_\epsilon(z)_{off}\right]_{\sigma\nu}.\end{aligned}$$

Where it all makes sense,

$$\left[\Lambda_\epsilon(z)\right]^{-1} = \left\{\mathbb{1} + \left[\Lambda_\epsilon(z)_{diag}\right]^{-1}\left[\Lambda_\epsilon(z)_{off}\right]\right\}^{-1}\left[\Lambda_\epsilon(z)_{diag}\right]^{-1}, \tag{6.10}$$

therefore, purpose of the section is finding a range of values for $z\in\rho(H_0)$ *such that (6.10) holds.* □

6.3.1 $\Lambda_\epsilon(z)_{\mathbf{diag}}$

Remark 6.9. *Let $\sigma \in \mathcal{I}$, $\epsilon > 0$ be. Set $\phi^\sigma_\epsilon(z) = B^\sigma_\epsilon R_{H_0}(z) A^{\sigma*}_\epsilon \in \mathfrak{B}(\chi_\sigma)$, $z \in \rho(H_0)$,*

$$\phi^\sigma_\epsilon(z) = g(v \otimes \mathbb{1}) \frac{U^\sigma_\epsilon}{\sqrt{\epsilon}} U_\sigma (H_0 - z\mathbb{1})^{-1} \left[(v \otimes \mathbb{1}) \frac{U^\sigma_\epsilon}{\sqrt{\epsilon}} U_\sigma \right]^* =$$
$$= g(v \otimes \mathbb{1}) \frac{U^\sigma_\epsilon}{\sqrt{\epsilon}} (H^\sigma_0 - z\mathbb{1})^{-1} \frac{U^{\sigma*}_\epsilon}{\sqrt{\epsilon}} (v \otimes \mathbb{1}).$$

Moreover

$$\left[-\frac{1}{2\mu_\sigma} \frac{\partial^2}{\partial r^2_\sigma} - (z - Q_\sigma)\mathbb{1} \right]^{-1} = \left(\mathbb{1} \otimes \mathfrak{F}_{\underline{Y}_\sigma}\right)(H^\sigma_0 - z\mathbb{1})^{-1} \left(\mathbb{1} \otimes \mathfrak{F}^{-1}_{\underline{Y}_\sigma}\right),$$

allows for

$$\phi^\sigma_\epsilon(z) = \frac{g}{\epsilon} \left\{ (v u^\sigma_\epsilon \otimes \mathbb{1}) \left(\mathbb{1} \otimes \mathfrak{F}^{-1}_{\underline{Y}_\sigma}\right) \left[-\frac{1}{2\mu_\sigma} \frac{\partial^2}{\partial r^2_\sigma} - (z - Q_\sigma)\mathbb{1} \right]^{-1} \left(\mathbb{1} \otimes \mathfrak{F}_{\underline{Y}_\sigma}\right)(u^{\sigma*}_\epsilon v \otimes \mathbb{1}) \right\} =$$
$$= \left(\mathbb{1} \otimes \mathfrak{F}^{-1}_{\underline{Y}_\sigma}\right) \left\{ g(v \otimes \mathbb{1}) \frac{U^\sigma_\epsilon}{\sqrt{\epsilon}} \left[-\frac{1}{2\mu_\sigma} \frac{\partial^2}{\partial r^2_\sigma} - (z - Q_\sigma)\mathbb{1} \right]^{-1} \frac{U^{\sigma*}_\epsilon}{\sqrt{\epsilon}} (v \otimes \mathbb{1}) \right\} \left(\mathbb{1} \otimes \mathfrak{F}_{\underline{Y}_\sigma}\right).$$

□

Definition 6.7. Fixed $\sigma \in \mathcal{I}$, $\epsilon > 0$, $z \in \rho(H_0)$, the linear bounded operator $\phi^\sigma_{\epsilon,\underline{P}_\sigma}(z)$ on $L^2(\mathbb{R}, dr_\sigma) \otimes \tilde{\chi}^{(\mathrm{red})}_\sigma$, where

$$\phi^\sigma_{\epsilon,\underline{P}_\sigma}(z) \doteq g(v \otimes \mathbb{1}) \frac{U^\sigma_\epsilon}{\sqrt{\epsilon}} \left[-\frac{1}{2\mu_\sigma} \frac{\partial^2}{\partial r^2_\sigma} - (z - Q_\sigma)\mathbb{1} \right]^{-1} \frac{U^{\sigma*}_\epsilon}{\sqrt{\epsilon}} (v \otimes \mathbb{1})$$

and Q_σ as above, is introduced. □

Remark 6.10. *By observing that, for all $\psi, \varphi \in L^2(\mathbb{R}, dr_\sigma)$, $\epsilon > 0$,*

$$\langle \varphi, u^\sigma_\epsilon \psi \rangle = \int_{\mathbb{R}} \overline{\varphi}(x)(u^\sigma_\epsilon \psi)(x)\, dx = \int_{\mathbb{R}} \overline{\varphi}(x) \sqrt{\epsilon}\psi(\epsilon x)\, dx = \int_{\mathbb{R}} \overline{\left[\frac{1}{\sqrt{\epsilon}} \varphi\left(\frac{x'}{\epsilon}\right) \right]} \psi(x')\, dx' =$$
$$= \langle u^{\sigma*}_\epsilon \varphi, \psi \rangle,$$

for all $z < 0$, $\varphi \in L^2(\mathbb{R}, dr_\sigma) \otimes \tilde{\chi}^{(red)}_\sigma$, it results

$$\left\{ \left[-\frac{\partial^2}{\partial r^2_\sigma} - (2\mu_\sigma)(z - Q_\sigma)\mathbb{1} \right]^{-1} \frac{u^{\sigma*}_\epsilon}{\sqrt{\epsilon}} \varphi \right\}(r_\sigma, \underline{P}_\sigma) = \int_{\mathbb{R}} G^{(1)}_{[(2\mu_\sigma)(z - Q_\sigma)]}(r_\sigma, r'_\sigma) \frac{1}{\epsilon} \varphi\left(\frac{r'_\sigma}{\epsilon}, \underline{P}_\sigma\right) dr'_\sigma =$$
$$= \int_{\mathbb{R}} G^{(1)}_{[(2\mu_\sigma)(z - Q_\sigma)]}(r_\sigma, \epsilon r'_\sigma)\, \varphi(r'_\sigma, \underline{P}_\sigma)\, dr'_\sigma.$$

Consequently

$$\left[\left\{\frac{U_\epsilon}{\sqrt{\epsilon}}\left[-\frac{\partial^2}{\partial r_\sigma^2}-(2\mu_\sigma)(z-Q_\sigma)\mathbb{1}\right]^{-1}\frac{U_\epsilon^*}{\sqrt{\epsilon}}\right\}\varphi\right](r_\sigma,\underline{P}_\sigma)=\int_{\mathbb{R}}G^{(1)}_{\left[(2\mu_\sigma)(z-Q_\sigma)\right]}\left(\epsilon r_\sigma,\,\epsilon r'_\sigma\right)\varphi\left(r'_\sigma,\underline{P}_\sigma\right)dr'_\sigma=$$

$$=\int_{\mathbb{R}}\left[\int_0^\infty\frac{e^{-\frac{\left|\epsilon r_\sigma-\epsilon r'_\sigma\right|^2}{4t}+(2\mu_\sigma)(z-Q_\sigma)t}}{\sqrt{4\pi t}}dt\right]\varphi\left(r'_\sigma,\underline{P}_\sigma\right)dr'_\sigma=\epsilon\int_{\mathbb{R}}G^{(1)}_{\left[\epsilon^2(2\mu_\sigma)(z-Q_\sigma)\right]}\left(r_\sigma,r'_\sigma\right)\varphi\left(r'_\sigma,\underline{P}_\sigma\right)dr'_\sigma=$$

$$=\epsilon\left\{\left[-\frac{\partial^2}{\partial r_\sigma^2}-\epsilon^2(2\mu_\sigma)(z-Q_\sigma)\mathbb{1}\right]^{-1}\varphi\right\}(r_\sigma,\underline{P}_\sigma),$$

i.e.

$$\frac{U^\sigma_\epsilon}{\sqrt{\epsilon}}\left[-\frac{\partial^2}{\partial r_\sigma^2}-(2\mu_\sigma)(z-Q_\sigma)\mathbb{1}\right]^{-1}\frac{U^{\sigma*}_\epsilon}{\sqrt{\epsilon}}=\epsilon\left[-\frac{\partial^2}{\partial r_\sigma^2}-\epsilon^2(2\mu_\sigma)(z-Q_\sigma)\mathbb{1}\right]^{-1},$$

allowing to state that

$$\phi^\sigma_{\epsilon,\underline{P}_\sigma}(z)=g\,(v\otimes\mathbb{1})\,\frac{U^\sigma_\epsilon}{\sqrt{\epsilon}}(2\mu_\sigma)\left[-\frac{\partial^2}{\partial r_\sigma^2}-(2\mu_\sigma)(z-Q_\sigma)\mathbb{1}\right]^{-1}\frac{U^{\sigma*}_\epsilon}{\sqrt{\epsilon}}(v\otimes\mathbb{1})\equiv$$

$$\equiv(v\otimes\mathbb{1})\left\{(2\mu_\sigma)\,g\epsilon\left[-\frac{\partial^2}{\partial r_\sigma^2}-\epsilon^2(2\mu_\sigma)(z-Q_\sigma)\mathbb{1}\right]^{-1}\right\}(v\otimes\mathbb{1}).$$

□

Proposition 6.2. *For all* $\sigma\in\mathcal{I}$, $\epsilon>0$, $z<0$, $\left\|\phi^\sigma_{\epsilon,\underline{P}_\sigma}(z)\right\|_{\mathfrak{B}(\chi_\sigma)}\le\frac{\mathfrak{C}|g|}{\sqrt{|z|}}$, *where* $\mathfrak{C}=\sqrt{\left(\max_\sigma\frac{\mu_\sigma}{2}\right)}$.

Proof. Given $\eta\in L^2(\mathbb{R},dr_\sigma)$, $\xi\in\tilde{\chi}^{(\mathrm{red})}_\sigma$ arbitrary, what follows holds.

$$\left\|\phi^\sigma_{\epsilon,\underline{P}_\sigma}(z)\,\eta\otimes\xi\right\|_2^2=\int_{\mathbb{R}^n}\left|\left(g\sqrt{\frac{\mu_\sigma}{2}}\right)\frac{\xi(\underline{P}_\sigma)}{\sqrt{|z-Q_\sigma|}}v(r_\sigma)\int_{\mathbb{R}}e^{-\epsilon\sqrt{(2\mu_\sigma)|z-Q_\sigma|}|r_\sigma-r'_\sigma|}v(r'_\sigma)\,\eta(r'_\sigma)\,dr'_\sigma\right|^2dr_\sigma d\underline{P}_\sigma\le$$

$$\le\frac{g^2\mu_\sigma}{2}\int_{\mathbb{R}^n}\frac{\left|\xi(\underline{P}_\sigma)\right|^2}{|z-Q_\sigma|}v(r_\sigma)^2\left[\int_{\mathbb{R}}e^{-\epsilon\sqrt{(2\mu_\sigma)|z-Q_\sigma|}|r_\sigma-r'_\sigma|}v(r'_\sigma)\left|\eta(r'_\sigma)\right|dr'_\sigma\right]^2dr_\sigma d\underline{P}_\sigma\le$$

$$\le\frac{g^2\mu_\sigma}{2}\int_{\mathbb{R}^n}\frac{\left|\xi(\underline{P}_\sigma)\right|^2}{|z-Q_\sigma|}v(r_\sigma)^2\left[\int_{\mathbb{R}}e^{-2\epsilon\sqrt{(2\mu_\sigma)|z-Q_\sigma|}|r_\sigma-r'_\sigma|}v(r'_\sigma)^2\,dr'_\sigma\right]\left[\int_{\mathbb{R}}\left|\eta(r'_\sigma)\right|^2dr'_\sigma\right]dr_\sigma d\underline{P}_\sigma\le$$

$$\le\frac{g^2\mu_\sigma}{2}\left[\int_{\mathbb{R}^{n-1}}\frac{\left|\xi(\underline{P}_\sigma)\right|^2}{|z-Q_\sigma|}d\underline{P}_\sigma\right]\left[\int_{\mathbb{R}}\left|\eta(r'_\sigma)\right|^2dr'_\sigma\right]\left[\int_{\mathbb{R}^2}v(r_\sigma)^2\,v(r'_\sigma)^2\,e^{-2\epsilon\sqrt{(2\mu_\sigma)|z-Q_\sigma|}|r_\sigma-r'_\sigma|}dr_\sigma dr'_\sigma\right]\le$$

$$\le\frac{g^2\mu_\sigma}{2|z|}\|\eta\otimes\xi\|_2^2\le\left(\max_\sigma\mu_\sigma\right)\frac{g^2}{2|z|}\|\eta\otimes\xi\|_2^2,$$

by using Hölder inequality in passing from the second to the third line. As a consequence

$$\left\|\phi^\sigma_{\epsilon,\underline{P}_\sigma}(z)\right\|_{\mathfrak{B}(\chi_\sigma)}\le\sqrt{\left(\max_\sigma\frac{\mu_\sigma}{2}\right)\frac{g^2}{|z|}}\equiv\frac{\mathfrak{C}|g|}{\sqrt{|z|}}.$$

■

Corollary 6.2.1. *For all $\sigma \in \mathcal{I}$, $\epsilon > 0$, if $z \in \mathbb{R}^- : z < -\mathfrak{C}^2 g^2$, $\Lambda_\epsilon(z)_{diag}$ is invertible on χ_σ.*

Proof. By definition

$$\left[\Lambda_\epsilon(z)_{\text{diag}}\right]_{\sigma\sigma} = \mathbb{1} - \phi_\epsilon^\sigma(z),$$

hence $\left\|\phi_\epsilon^\sigma(z)\right\|_{\mathfrak{B}(\chi_\sigma)} < 1$ guarantees the invertibility of $\mathbb{1} - \phi_\epsilon^\sigma(z)$ on χ_σ.

$$z < -\mathfrak{C}^2 g^2 \implies \frac{\mathfrak{C}|g|}{\sqrt{|z|}} < 1 \implies \left\|\phi_\epsilon^\sigma(z)\right\|_{\mathfrak{B}(\chi_\sigma)} < 1.$$

■

Remark 6.11. *Given $\sigma \in \mathcal{I}$ and $z < -\mathfrak{C}^2 g^2$, what above allows to state that*

$$\left[\left(\Lambda_\epsilon(z)_{diag}\right)^{-1}\right]_{\sigma\sigma} = \left[\mathbb{1} - \phi_\epsilon^\sigma(z)\right]^{-1} = \sum_{n\in\mathbb{N}_0}\left[\phi_\epsilon^\sigma(z)\right]^n,$$

hence

$$\left\|\left[\left(\Lambda_\epsilon(z)_{diag}\right)^{-1}\right]_{\sigma\sigma}\right\|_{\mathfrak{B}(\chi_\sigma)} \leq \sum_{n\in\mathbb{N}_0}\left\|\phi_\epsilon^\sigma(z)\right\|^n_{\mathfrak{B}(\chi_\sigma)} = \frac{1}{1-\left\|\phi_\epsilon^\sigma(z)\right\|_{\mathfrak{B}(\chi_\sigma)}} \leq \left[1 - \frac{\mathfrak{C}|g|}{\sqrt{|z|}}\right]^{-1}.$$

□

6.3.2 Investigating $\Lambda_\epsilon(z)_{\text{diag}}$ as $\epsilon \to 0^+$

Definition 6.8. Given $\sigma \in \mathcal{I}$ and $z < 0$, the linear operator

$$\phi^\sigma_{0,\underline{P}_\sigma}(z) : \varphi \in \mathcal{D} \subseteq L^2(\mathbb{R}, dr_\sigma) \otimes \tilde{\chi}_\sigma^{(\text{red})} \longmapsto \phi^\sigma_{0,\underline{P}_\sigma}(z)\varphi \in L^2(\mathbb{R}, dr_\sigma) \otimes \tilde{\chi}_\sigma^{(\text{red})},$$

where

$$\left[\phi^\sigma_{0,\underline{P}_\sigma}(z)\varphi\right](r_\sigma, \underline{P}_\sigma) \doteq \left(g\sqrt{\frac{\mu_\sigma}{2}}\right) v(r_\sigma) \int_{\mathbb{R}} \frac{\varphi(r'_\sigma, \underline{P}_\sigma)}{\sqrt{|z - Q_\sigma|}} v(r'_\sigma)\, dr'_\sigma, \quad \varphi \in \mathcal{D},$$

is introduced. □

Lemma 6.3. *For arbitrary $\sigma \in \mathcal{I}$, $z < 0$, $\mathcal{D} \equiv L^2(\mathbb{R}, dr_\sigma) \otimes \tilde{\chi}_\sigma^{(red)}$, $\phi^\sigma_{0,\underline{P}_\sigma}(z)$ is bounded and*

$$\left\|\phi^\sigma_{\epsilon,\underline{P}_\sigma}(z) - \phi^\sigma_{0,\underline{P}_\sigma}(z)\right\|_{\mathfrak{B}(\chi_\sigma)} \xrightarrow[\epsilon\downarrow 0]{} 0.$$

Proof. Let whatever $\eta \in L^2(\mathbb{R}, dr_\sigma)$, $\xi \in \tilde{\chi}_\sigma^{(\mathrm{red})}$ be.

$$\left\| \left[\phi^\sigma_{0,\underline{P}_\sigma}(z) - \phi^\sigma_{\epsilon,\underline{P}_\sigma}(z) \right] \eta \otimes \xi \right\|_2^2 \equiv \int_{\mathbb{R}^n} \left| \left\{ \left[\phi^\sigma_{0,\underline{P}_\sigma}(z) - \phi^\sigma_{\epsilon,\underline{P}_\sigma}(z) \right] \eta \otimes \xi \right\} (r_\sigma, \underline{P}_\sigma) \right|^2 dr_\sigma d\underline{P}_\sigma =$$

$$= \int_{\mathbb{R}^n} \left| \left(g\sqrt{\frac{\mu_\sigma}{2}} \right) \frac{\xi(\underline{P}_\sigma)}{\sqrt{|z - Q_\sigma|}} \int_{\mathbb{R}} v(r_\sigma) \left[e^{-\epsilon\sqrt{(2\mu_\sigma)|z-Q_\sigma|}|r_\sigma - r'_\sigma|} - 1 \right] v(r'_\sigma)\, \eta(r'_\sigma)\, dr'_\sigma \right|^2 dr_\sigma d\underline{P}_\sigma \le$$

$$\le g^2 \left(\frac{\mu_\sigma}{2} \right) \int_{\mathbb{R}^n} \frac{|\xi(\underline{P}_\sigma)|^2}{|z - Q_\sigma|} \left[\int_{\mathbb{R}} \left| e^{-\epsilon\sqrt{(2\mu_\sigma)|z-Q_\sigma|}|r_\sigma - r'_\sigma|} - 1 \right| v(r_\sigma)\, v(r'_\sigma) \left| \eta(r'_\sigma) \right| dr'_\sigma \right]^2 dr_\sigma d\underline{P}_\sigma \le$$

$$\le \left(\frac{g^2\mu_\sigma}{2} \right) \int_{\mathbb{R}^n} \frac{|\xi(\underline{P}_\sigma)|^2}{|z - Q_\sigma|} \left\{ \left[\int_{\mathbb{R}} \left| e^{-\epsilon\sqrt{(2\mu_\sigma)|z-Q_\sigma|}|r_\sigma - r'_\sigma|} - 1 \right|^2 v(r_\sigma)^2\, v(r'_\sigma)^2\, dr'_\sigma \right] \left[\int_{\mathbb{R}} \left| \eta(r'_\sigma) \right|^2 dr'_\sigma \right] \right\} \le$$

$$\le \left(\frac{g^2\mu_\sigma}{2} \right) \left(\int_{\mathbb{R}} \left| \eta(r'_\sigma) \right|^2 dr'_\sigma \right) \int_{\mathbb{R}^{n-1}} \left\{ \frac{|\xi(\underline{P}_\sigma)|^2}{|z - Q_\sigma|} \left[\int_{\mathbb{R}^2} \left| \epsilon\sqrt{(2\mu_\sigma)|z - Q_\sigma|}|r_\sigma - r'_\sigma| \right|^2 v(r_\sigma)^2\, v(r'_\sigma)^2\, dr_\sigma dr'_\sigma \right] d\underline{P}_\sigma \right\}$$

$$\le \epsilon^2 \left(g^2\mu_\sigma^2 \right) \left[2\int_{\mathbb{R}^2} v(r_\sigma)^2\, v(r'_\sigma)^2 \left(r_\sigma^2 + r'^2_\sigma \right) dr_\sigma dr'_\sigma \right] \|\eta \otimes \xi\|_2^2,$$

by using Hölder inequality in passing from the third to the fourth line. Since

$$2\int_{\mathbb{R}^2} v(r_\sigma)^2\, v(r'_\sigma)^2 \left(r_\sigma^2 + r'^2_\sigma \right) dr_\sigma dr'_\sigma < 4\|V\|_1^2 \left(\sup_{\mathrm{supp}\, V} r_\sigma^2 \right) < \infty,$$

it results

$$\left\| \phi^\sigma_{0,\underline{P}_\sigma}(z) - \phi^\sigma_{\epsilon,\underline{P}_\sigma}(z) \right\|_{\mathfrak{B}(\chi_\sigma)} \le K\epsilon,$$

for some constant $K > 0$. ■

Lemma 6.4. *Given $z < 0$ and $\phi^\sigma_0(z) \doteq \left(\mathbb{1} \otimes \mathfrak{F}^{-1}_{\underline{Y}_\sigma} \right) \phi^\sigma_{0,\underline{P}_\sigma}(z) \left(\mathbb{1} \otimes \mathfrak{F}_{\underline{Y}_\sigma} \right)$, if $z < -\mathfrak{C}^2 g^2$, $\mathbb{1} - \phi^\sigma_0(z)$ is invertible and*

$$\lim_{\epsilon \downarrow 0} \left[\mathbb{1} - \phi^\sigma_\epsilon(z) \right]^{-1} = \left[\mathbb{1} - \phi^\sigma_0(z) \right]^{-1}.$$

Proof. Let $\eta \in L^2(\mathbb{R}, dr_\sigma)$, $\xi \in \tilde{\chi}_\sigma^{(\mathrm{red})}$ be arbitrary.

$$\left\| \phi^\sigma_{0,\underline{P}_\sigma} \eta \otimes \xi \right\|_2^2 = \int_{\mathbb{R}^n} \left| \left(g\sqrt{\frac{\mu_\sigma}{2}} \right) \frac{v(r_\sigma)}{\sqrt{|z - Q_\sigma|}} \xi(\underline{P}_\sigma) \int_{\mathbb{R}} v(r'_\sigma)\, \eta(r'_\sigma)\, dr'_\sigma \right|^2 dr_\sigma d\underline{P}_\sigma \le$$

$$\le \left(g^2 \frac{\mu_\sigma}{2} \right) \int_{\mathbb{R}^n} \frac{|\xi(\underline{P}_\sigma)|^2}{|z - Q_\sigma|} v^2(r_\sigma) \left[\int_{\mathbb{R}} \left| \eta(r'_\sigma) \right| v(r'_\sigma)\, dr'_\sigma \right]^2 dr_\sigma d\underline{P}_\sigma \le \left(g^2 \frac{\mu_\sigma}{2|z|} \right) \|\eta \otimes \xi\|_2^2.$$

As a consequence

$$\left\| \phi^\sigma_0(z) \right\|_{\mathfrak{B}(\chi_\sigma)} \equiv \left\| \phi^\sigma_{0,\underline{P}_\sigma}(z) \right\|_{\mathfrak{B}(\chi_\sigma)} \le \frac{\mathfrak{C}|g|}{\sqrt{|z|}}.$$

■

Remark 6.12. *Important and direct consequence of what above is that, as $z < -\mathfrak{C}^2 g^2$,*

$$\lim_{\epsilon \downarrow 0} \left[\left(\Lambda_\epsilon(z)_{diag} \right)_{\sigma\sigma} \right]^{-1} = \lim_{\epsilon \downarrow 0} \left[\mathbb{1} - \phi^\sigma_\epsilon(z) \right]^{-1} = \left[\mathbb{1} - \phi^\sigma_0(z) \right]^{-1} = \tag{6.11}$$

$$= \left[\left(\Lambda_0(z)_{diag} \right)_{\sigma\sigma} \right]^{-1} = \left[\left(\Lambda_0(z)_{diag} \right)^{-1} \right]_{\sigma\sigma}. \tag{6.12}$$

□

6.3.3 $\left\{\mathbb{1} + \left[\Lambda_\epsilon(z)_{\mathbf{diag}}\right]^{-1} \Lambda_\epsilon(z)_{\mathbf{off}}\right\}^{-1}$-related investigations

Proposition 6.5. *Let $\epsilon > 0$, $z < 0$ be arbitrary. (6.14) and (6.15) hold.*

Proof. First of all, let $\sigma, \nu \in \mathcal{I} : \sigma \neq \nu$ be. Then

$$\left[\Lambda_\epsilon(z)_{\text{off}}\right]_{\sigma\nu} = -B_\epsilon^\sigma R_{H_0}(z) A_\epsilon^{\nu *} = -(v \otimes \mathbb{1}) \frac{U_\epsilon^\sigma}{\sqrt{\epsilon}} U_\sigma R_{H_0}(z) \left[(v \otimes \mathbb{1}) \frac{U_\epsilon^\nu}{\sqrt{\epsilon}} U_\nu\right]^* =$$
$$= -(v \otimes \mathbb{1}) \frac{U_\epsilon^\sigma}{\sqrt{\epsilon}} U_\sigma R_{H_0}(z) U_\nu^* \frac{U_\epsilon^{\nu *}}{\sqrt{\epsilon}} (v \otimes \mathbb{1}) : \chi_\nu \longrightarrow \chi_\sigma.$$

To simplify the notation, without harming generality, $\sigma = (12)$ is assumed; for all $\psi \in \chi_\nu$, $\left[\Lambda_\epsilon(z)_{\text{off}}\right]_{\sigma\nu} \psi \in \chi_\sigma$, hence

$$\left(\left[\Lambda_\epsilon(z)_{\text{off}}\right]_{\sigma\nu} \psi\right)(r_\sigma, R_\sigma, x_3, \ldots, x_n) =$$
$$= \left\{-\left[(v \otimes \mathbb{1}) \frac{U_\epsilon^\sigma}{\sqrt{\epsilon}} U_\sigma R_{H_0}(z) U_\nu^* \frac{U_\epsilon^{\nu *}}{\sqrt{\epsilon}} (v \otimes \mathbb{1})\right] \psi\right\}(r_\sigma, R_\sigma, x_3, \ldots, x_n)$$
$$= -v(r_\sigma) \left\{\left[U_\sigma R_{H_0}(z) U_\nu^* \frac{U_\epsilon^{\nu *}}{\sqrt{\epsilon}} (v \otimes \mathbb{1})\right] \psi\right\}(\epsilon r_\sigma, R_\sigma, x_3, \ldots, x_n).$$

It is recalled that

$$U_{(12)} : \Psi \in L^2(\mathbb{R}^n, dx_1 \ldots dx_n) \longmapsto U_{(12)}\Psi \in L^2\left(\mathbb{R}^n, dr_{(12)} dR_{(12)} dx_3 \ldots dx_n\right)$$

with

$$\left(U_{(12)}\Psi\right)\left(r_{(12)}, R_{(12)}, x_3, \ldots, x_n\right) \equiv \Psi\left(R_{(12)} - \frac{m_2}{m_1 + m_2} r_{(12)}, R_{(12)} + \frac{m_1}{m_1 + m_2} r_{(12)}, x_3, \ldots, x_n\right),$$

therefore

$$
\begin{aligned}
&\left(\left[\Lambda_\epsilon\left(z\right)_{\text{off}}\right]_{\sigma\nu}\psi\right)\left(r_\sigma,R_\sigma,x_3,\ldots,x_n\right)=\\
&=-v\left(r_\sigma\right)\left[R_{H_0}\left(z\right)U_\nu^*\frac{U_\epsilon^{\nu\,*}}{\sqrt{\epsilon}}\left(v\otimes\mathbb{1}\right)\psi\right]\left(R_\sigma-\frac{m_2}{m_1+m_2}\epsilon r_\sigma,\,R_\sigma+\frac{m_1}{m_1+m_2}\epsilon r_\sigma,\,x_3,\ldots,x_n\right)\\
&=-v\left(r_\sigma\right)\int_{\mathbb{R}^n}\left\{\left[R_{H_0}\left(z\right)\right]\left(R_\sigma-\frac{\epsilon m_2}{m_1+m_2}r_\sigma,R_\sigma+\frac{\epsilon m_1}{m_1+m_2}r_\sigma,\,x_3,\ldots,x_n,x'_1,\ldots,x'_n\right)\cdot\right.\\
&\qquad\qquad\left.\cdot\left[U_\nu^*\frac{U_\epsilon^{\nu\,*}}{\sqrt{\epsilon}}\left(v\otimes\mathbb{1}\right)\psi\right]\left(x'_1,\ldots,x'_n\right)\right\}dx'_1\cdots dx'_n\\
&=-v\left(r_\sigma\right)\int_{\mathbb{R}^n}\left\{\left[R_{H_0}\left(z\right)\right]\left(R_\sigma-\frac{\epsilon m_2}{m_1+m_2}r_\sigma,R_\sigma+\frac{\epsilon m_1}{m_1+m_2}r_\sigma,\,x_3,\ldots,x_n,x'_1,\ldots,x'_n\right)\cdot\right.\\
&\qquad\qquad\left.\cdot\left[\frac{U_\epsilon^{\nu\,*}}{\sqrt{\epsilon}}\left(v\otimes\mathbb{1}\right)\psi\right]\left(x'_1,\ldots,x'_{\nu_1}-x'_{\nu_2},\ldots,\frac{m_{\nu_1}x'_{\nu_1}+m_{\nu_2}x'_{\nu_2}}{m_{\nu_1}+m_{\nu_2}},\ldots,x'_n\right)\right\}dx'_1\cdots dx'_n\\
&=-v\left(r_\sigma\right)\int_{\mathbb{R}^n}\left\{\left[R_{H_0}\left(z\right)\right]\left(R_\sigma-\frac{\epsilon m_2}{m_1+m_2}r_\sigma,R_\sigma+\frac{\epsilon m_1}{m_1+m_2}r_\sigma,\,x_3,\ldots,x_n,x'_1,\ldots,x'_n\right)\frac{1}{\epsilon}v\left(\frac{x'_{\nu_1}-x'_{\nu_2}}{\epsilon}\right)\cdot\right.\\
&\qquad\qquad\left.\cdot\psi\left(\frac{x'_{\nu_1}-x'_{\nu_2}}{\epsilon},\,\frac{m_{\nu_1}x'_{\nu_1}+m_{\nu_2}x'_{\nu_2}}{m_{\nu_1}+m_{\nu_2}},\,x'_1,\ldots,\hat{x}'_{\nu_1},\ldots,\hat{x}'_{\nu_2},\ldots,x'_n\right)\right\}dx'_1\cdots dx'_n\\
&=-v\left(r_\sigma\right)\int_{\mathbb{R}^n}\left\{\left[R_{H_0}\left(z\right)\right]\left(R_\sigma-\frac{\epsilon m_2}{m_1+m_2}r_\sigma,R_\sigma+\frac{\epsilon m_1}{m_1+m_2}r_\sigma,\,x_3,\ldots,x_n,x'_1,\ldots,x'_n\right)\frac{1}{\epsilon}v\left(\frac{x'_{\nu_1}-x'_{\nu_2}}{\epsilon}\right)\cdot\right.\\
&\qquad\qquad\left.\cdot\psi\left(\frac{x'_{\nu_1}-x'_{\nu_2}}{\epsilon},\,x'_{\nu_1}-\frac{\epsilon m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}\frac{x'_{\nu_1}-x'_{\nu_2}}{\epsilon},\,x'_1,\ldots,\hat{x}'_{\nu_1},\ldots,\hat{x}'_{\nu_2},\ldots,x'_n\right)\right\}dx'_1\cdots dx'_n\\
&=-v\left(r_\sigma\right)\int_{\mathbb{R}^{n+1}}\left\{\left[R_{H_0}\left(z\right)\right]\left(R_\sigma-\frac{\epsilon m_2}{m_1+m_2}r_\sigma,R_\sigma+\frac{\epsilon m_1}{m_1+m_2}r_\sigma,\,x_3,\ldots,x_n,x'_1,\ldots,x'_n\right)\frac{1}{\epsilon}v\left(r'_\nu\right)\cdot\right.\\
&\qquad\qquad\cdot\;\psi\left(r'_\nu,\,x'_{\nu_1}-\frac{\epsilon m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}r'_\nu,\,x'_1,\ldots,\hat{x}'_{\nu_1},\ldots,\hat{x}'_{\nu_2},\ldots,x'_n\right)\cdot\\
&\qquad\qquad\left.\cdot\;\;\delta\left(r'_\nu-\frac{x'_{\nu_1}-x'_{\nu_2}}{\epsilon}\right)\right\}dx'_1\cdots dx'_n dr'_\nu\\
&\equiv\left(\frac{1}{\epsilon}\delta\left(r'_\nu-\frac{x'_{\nu_1}-x'_{\nu_2}}{\epsilon}\right)\equiv\delta\left(\epsilon r'_\nu-\left(x'_{\nu_1}-x'_{\nu_2}\right)\right)\equiv\delta\left(x'_{\nu_1}-x'_{\nu_2}-\epsilon r'_\nu\right)\;\equiv\right.\\
&\qquad\left.\equiv\delta\left(\left(x'_{\nu_1}-\frac{\epsilon m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}r'_\nu\right)-\left(x'_{\nu_2}+\frac{\epsilon m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}r'_\nu\right)\right)\right)\equiv\\
&\equiv\int_{\mathbb{R}^{n+2}}\left\{-v\left(r_\sigma\right)\left[R_{H_0}\left(z\right)\right]\left(R_\sigma-\frac{\epsilon m_2}{m_1+m_2}r_\sigma,R_\sigma+\frac{\epsilon m_1}{m_1+m_2}r_\sigma,x_3,\ldots,x_n,x'_1,\ldots,x'_n\right)v\left(r'_\nu\right)\right.\\
&\qquad\left.\delta\left(R'_\nu-x'_{\nu_1}+\frac{\epsilon m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}r'_\nu\right)\delta\left(R'_\nu-x'_{\nu_2}-\frac{\epsilon m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}r'_\nu\right)\right\}\cdot\\
&\qquad\cdot\psi\left(r'_\nu,R'_\nu,x'_1,\ldots,\hat{x}'_{\nu_1},\ldots,\hat{x}'_{\nu_2},\ldots,x'_n\right)dr'_\nu dR'_\nu dx'_1\cdots dx'_n.
\end{aligned}
$$

Now, it is observed that $\sigma \neq \nu$ may nonetheless imply $\nu_i \in \{1,2\}$, $i=1,2$, hence the following cases are discussed.

$\boxed{\sigma = (12),\ \nu = (1\nu_2),\ \nu_2 \geq 3}$

$$
\begin{aligned}
&\left[\left(\Lambda_\epsilon(z)_{\text{off}}\right)_{\sigma\nu}\psi\right](r_\sigma, R_\sigma, x_3, \ldots, x_n) = \\
&\equiv \int_{\mathbb{R}^n}\Bigg\{-v(r_\sigma)\left[R_{H_0}(z)\right]\left(R_\sigma - \frac{\epsilon m_2}{m_1+m_2}r_\sigma, R_\sigma + \frac{\epsilon m_1}{m_1+m_2}r_\sigma, x_3, \ldots, x_n, x_1', \ldots, x_n'\right)v(r_\nu') \\
&\qquad\qquad \delta\left(R_\nu' - x_1' + \frac{\epsilon m_{\nu_2}}{m_1+m_{\nu_2}}r_\nu'\right)\delta\left(R_\nu' - x_{\nu_2}' - \frac{\epsilon m_1}{m_1+m_{\nu_2}}r_\nu'\right)\Bigg\}\cdot \\
&\qquad\qquad \cdot\psi\left(r_\nu', R_\nu', \hat{x}_1', x_2', \ldots, \hat{x}_{\nu_2}', \ldots, x_n'\right)dr_\nu' dR_\nu' dx_1'\cdots dx_n' \equiv \\
&\equiv \left(\text{by integrating with respect to } x_1',\ x_{\nu_2}'\right) \equiv \\
&\equiv \int_{\mathbb{R}^n}\Bigg\{-v(r_\sigma)\left[R_{H_0}(z)\right]\Bigg(R_\sigma - \frac{\epsilon m_2}{m_1+m_2}r_\sigma - R_\nu' - \frac{\epsilon m_{\nu_2}}{m_1+m_{\nu_2}}r_\nu',\ R_\sigma + \frac{\epsilon m_1}{m_1+m_2}r_\sigma - x_2',\ x_3 - x_3',\ \ldots, \\
&\qquad\qquad x_{\nu_2} - R_\nu' + \frac{\epsilon m_1}{m_1+m_{\nu_2}}r_\nu, \ldots, x_n - x_n'\Bigg)v(r_\nu')\Bigg\}\cdot \\
&\qquad\qquad \cdot\psi\left(r_\nu', R_\nu', \hat{x}_1', x_2', \ldots, \hat{x}_{\nu_2}', \ldots, x_n'\right)dr_\nu' dR_\nu' dx_2'\cdots d\hat{x}_{\nu_2}'\cdots dx_n'.
\end{aligned}
$$

It is clearly seen that, with respect to the tuple of variables $\underline{Y}_\nu \equiv (x_3, \ldots, \hat{x}_{\nu_2}, \ldots, x_n) \in \mathbb{R}^{n-3}$, $\left(\Lambda_\epsilon(z)_{\text{off}}\right)_{\sigma\nu}$ behaves as a convolution operator; set $\chi_\nu^- \equiv L^2\left(\mathbb{R}^{n-3}, dx_3\cdots d\hat{x}_{\nu_2}\cdots dx_n\right)$ and denoted by $\mathfrak{F}_{\underline{Y}_\nu}$ the Fourier operator on χ_ν^-, the operator

$$
\left(\mathbb{1}\otimes\mathfrak{F}_{\underline{Y}_\nu}\right)\left(\Lambda_\epsilon(z)_{\text{off}}\right)_{\sigma\nu}\left(\mathbb{1}\otimes\mathfrak{F}_{\underline{Y}_\nu}^{-1}\right) \doteq \left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu} \tag{6.13}
$$

will be multiplicative with respect to the conjugate tuple $\underline{P}_{(1\nu_2)} \equiv (p_3, \ldots, \hat{p}_{\nu_2}, \ldots, p_n)$; concerning the remaining variables, it behaves as an integral operator whose kernel is

$$
-2^{\frac{3}{2}}\sqrt{m_1 m_2 m_{\nu_2}}\,v(r_\sigma)\,G^{(3)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right)v\left(r_\nu'\right) \equiv C(m_1, m_2, m_{\nu_2})\,v(r_\sigma)\,G^{(3)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right)v\left(r_\nu'\right),
$$

and

$$
X_{\sigma\nu,\,\epsilon} = \begin{pmatrix} \sqrt{2m_1}\left[R_\sigma - R_\nu' - \epsilon\left(\frac{m_2}{m_1+m_2}r_\sigma - \frac{m_{\nu_2}}{m_1+m_{\nu_2}}r_\nu'\right)\right] \\ \sqrt{2m_2}\left[R_\sigma - x_2' + \epsilon\left(\frac{m_1}{m_1+m_2}\right)r_\sigma\right] \\ \sqrt{2m_{\nu_2}}\left[x_{\nu_2} - R_\nu' + \epsilon\left(\frac{m_1}{m_1+m_{\nu_2}}\right)r_\nu'\right] \end{pmatrix}, \qquad Q_\nu = \sum_{\substack{k=3\\k\neq\nu_2}}^{n}\frac{p_k^2}{2m_k}.
$$

Given $\eta \in L^2\left(\mathbb{R}^3, dr_\nu dR_\nu dx_2\right)$, $\xi \in \tilde{\chi}_\nu^- \equiv \mathfrak{F}_{\underline{Y}_\nu}\chi_\nu^-$ arbitrarily,

$$\left\|\left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu}\eta\otimes\xi\right\|_2^2 = C^2\int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_2}\left|\int_{\mathbb{R}^3} dr'_\nu dR'_\nu dx'_{\nu_2}\, v(r_\sigma)\, v(r'_\nu)\, G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon})\,\eta(r'_\nu,R'_\nu,x'_2)\,\xi(\underline{P}_\nu)\right|^2$$

$$= C^2\int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_2}\left|\int_{\mathbb{R}} dr'_\nu\left\{v(r_\sigma)\,v(r'_\nu)\int_{\mathbb{R}^2} dR'_\nu dx'_2\left[G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon})\,\eta(r'_\nu,R'_\nu,x'_2)\,\xi(\underline{P}_\nu)\right]\right\}\right|^2$$

$$\leq C^2\int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_2}\left[\int_{\mathbb{R}} dr'_\nu\left\{v(r_\sigma)\,v(r'_\nu)\left|\int_{\mathbb{R}^2} dR'_\nu dx'_2\, G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon})\,\eta(r'_\nu,R'_\nu,x'_2)\,\xi(\underline{P}_\nu)\right|\right\}\right]^2$$

$$\leq C^2\int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_2}\left\{\left[\int_{\mathbb{R}} dr'_\nu\, V(r_\sigma)\, V(r'_\nu)\right]\left[\int_{\mathbb{R}} dr'_\nu\left|\int_{\mathbb{R}^2} G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon})\,\eta(r'_\nu,R'_\nu,x'_2)\,\xi(\underline{P}_\nu)\right|\right]^2\right\}$$

$$\leq C^2\left\{\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^n} d\underline{P}_\nu dR_\sigma dx_{\nu_2} dr'_\nu\left[\int_{\mathbb{R}^2} dR'_\nu dx'_2\, G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon})\left|\eta(r'_\nu,R'_\nu,x'_2)\right|\left|\xi(\underline{P}_\nu)\right|\right]^2\right\}.$$

By considering the coordinate transformation

$$\begin{cases}\overline{R}'_\nu = R'_\nu + \epsilon\left(\frac{m_2}{M_\sigma}r_\sigma - \frac{m_{\nu_2}}{M_\nu}r'_\nu\right)\\ \overline{x}'_2 = x'_2 - \epsilon\left(\frac{m_1}{M_\sigma}\right)r_\sigma\\ \overline{x}_{\nu_2} = x_{\nu_2} + \epsilon\left(\frac{m_2}{M_\sigma}r_\sigma - r'_\nu\right)\end{cases},$$

what follows holds[3].

$$\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^n} d\underline{P}_\nu dR_\sigma dx_{\nu_2} dr'_\nu\left[\int_{\mathbb{R}^2} dR'_\nu dx'_2\, G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon})\left|\eta(r'_\nu,R'_\nu,x'_2)\right|\left|\xi(\underline{P}_\nu)\right|\right]^2\equiv$$

$$\equiv\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^n} d\underline{P}_\nu dR_\sigma d\overline{x}_{\nu_2} dr'_\nu\left[\int_{\mathbb{R}^2} d\overline{R}'_\nu d\overline{x}'_2\, G^{(3)}_{z-Q_\nu}\left(\sqrt{2m_1}\left(R_\sigma-\overline{R}'_\nu\right),\sqrt{2m_2}\left(R_\sigma-\overline{x}'_2\right),\sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2}-\overline{R}'_\nu\right)\right)\right.$$
$$\left.\left|\eta\left(r'_\nu,\overline{R}'_\nu-\epsilon\left(\frac{m_2}{M_\sigma}r_\sigma-\frac{m_{\nu_2}}{M_\nu}r'_\nu\right),\overline{x}'_2+\epsilon\left(\frac{m_1}{M_\sigma}\right)r_\sigma\right)\right|\left|\xi(\underline{P}_\nu)\right|\right]^2$$

$$\leq\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^n} d\underline{P}_\nu dR_\sigma d\overline{x}_{\nu_2} dr'_\nu\left[\int_{\mathbb{R}^2} d\overline{R}'_\nu d\overline{x}'_2\, G^{(3)}_{z}\left(\sqrt{2m_1}\left(R_\sigma-\overline{R}'_\nu\right),\sqrt{2m_2}\left(R_\sigma-\overline{x}'_2\right),\sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2}-\overline{R}'_\nu\right)\right)\right.$$
$$\left.\left|\eta\left(r'_\nu,\overline{R}'_\nu-\epsilon\left(\frac{m_2}{M_\sigma}r_\sigma-\frac{m_{\nu_2}}{M_\nu}r'_\nu\right),\overline{x}'_2+\epsilon\left(\frac{m_1}{M_\sigma}\right)r_\sigma\right)\right|\left|\xi(\underline{P}_\nu)\right|\right]^2$$

$$\leq\left\{\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^3} dR_\sigma d\overline{x}_{\nu_2} dr'_\nu\left[\int_{\mathbb{R}^2} d\overline{R}'_\nu d\overline{x}'_2\, G^{(3)}_{z}\left(\sqrt{2m_1}\left(R_\sigma-\overline{R}'_\nu\right),\sqrt{2m_2}\left(R_\sigma-\overline{x}'_2\right),\sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2}-\overline{R}'_\nu\right)\right)\right.\right.$$
$$\left.\left.\left|\eta\left(r'_\nu,\overline{R}'_\nu-\epsilon\left(\frac{m_2}{M_\sigma}r_\sigma-\frac{m_{\nu_2}}{M_\nu}r'_\nu\right),\overline{x}'_2+\epsilon\left(\frac{m_1}{M_\sigma}\right)r_\sigma\right)\right|\right]^2\right\}\cdot$$

$$\cdot\left(\int_{\mathbb{R}^{n-3}} d\underline{P}_\nu\left|\xi(\underline{P}_\nu)\right|^2\right)\leq\|F\|^2\|\eta\otimes\xi\|^2.$$

Eventually,

$$\left\|\left[\Lambda_\epsilon(z)_{\text{off}}\right]_{\sigma\nu}\right\|_{\mathfrak{B}(\chi_\sigma,\chi_\nu)}\leq|C|\|F\|\leq\frac{|C|}{2\sqrt{2|z|}}\leq\left(\max_i m_i^{\frac{3}{2}}\right)\frac{1}{\sqrt{|z|}},\tag{6.14}$$

[3]Appendix 4 enters the argument.

with $\sigma = (12)$, $\nu = (1\nu_2)$, $\nu_2 \geq 3$ and independently of $\epsilon > 0$. An analogous argument holds for $\sigma = (12)$, $\nu = (2\nu_2)$, $\nu_2 \geq 3$.

$\boxed{\sigma = (12),\ \nu = (\nu_1\nu_2),\ 3 \leq \nu_1 < \nu_2 \leq n}$

$$\begin{aligned}
&\left\{\left[\Lambda_\epsilon(z)_{\text{off}}\right]_{\sigma\nu}\psi\right\}(r_\sigma, R_\sigma, x_3, \ldots, x_n) = \\
&= \int_{\mathbb{R}^{n+2}} \left\{-v(r_\sigma)\left[R_{H_0}(z)\right]\left(R_\sigma - \frac{\epsilon m_2}{m_1+m_2}r_\sigma, R_\sigma + \frac{\epsilon m_1}{m_1+m_2}r_\sigma, x_3, \ldots, x_n, x_1', \ldots, x_n'\right) v(r_\nu')\right. \\
&\qquad\left.\delta\left(R_\nu' - x_{\nu_1}' + \frac{\epsilon m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}r_\nu'\right)\delta\left(R_\nu' - x_{\nu_2}' - \frac{\epsilon m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}r_\nu'\right)\right\}\cdot \\
&\qquad\cdot\psi\left(r_\nu', R_\nu', x_1', \ldots, \hat{x}_{\nu_1}', \ldots, \hat{x}_{\nu_2}', \ldots, x_n'\right) dr_\nu' dR_\nu' dx_1' \cdots dx_n' \\
&= \int_{\mathbb{R}^n} \left\{-v(r_\sigma)\cdot\right. \\
&\quad\cdot\left[R_{H_0}(z)\right]\left(R_\sigma - \frac{\epsilon m_2}{m_1+m_2}r_\sigma - x_1',\ R_\sigma + \frac{\epsilon m_1}{m_1+m_2}r_\sigma - x_2',\ x_3 - x_3', \ldots,\ x_{\nu_1} - R_\nu' - \frac{\epsilon m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}r_\nu',\right. \\
&\qquad\left.\ldots,\ x_{\nu_2} - R_\nu' + \frac{\epsilon m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}r_\nu', \ldots, x_n - x_n'\right)\cdot \\
&\quad\left.\cdot\,\psi\left(r_\nu', R_\nu', x_1', \ldots, \hat{x}_{\nu_1}', \ldots, \hat{x}_{\nu_2}', \ldots, x_n'\right)\right\} dr_\nu' dR_\nu' dx_1' \cdots d\hat{x}_{\nu_1}' \cdots d\hat{x}_{\nu_2}' \cdots dx_n.
\end{aligned}$$

Concerning the variables $\underline{Y}_\nu = (x_3, \ldots, \hat{x}_{\nu_1}, \ldots, \hat{x}_{\nu_2}, \ldots, x_n)$, $\left[\Lambda_\epsilon(z)_{\text{off}}\right]_{\sigma\nu}$ behaves as a convolution operator, therefore, by introducing

$$\chi_\nu^- \doteq L^2\left(\mathbb{R}^{n-4},\ dx_3 \ldots d\hat{x}_{\nu_1} \ldots d\hat{x}_{\nu_2} \ldots dx_n\right)$$

and the corresponding Fourier operator on it $\mathfrak{F}_{\underline{Y}_\nu}$,

$$\left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu} \doteq \left(\mathbb{1}\otimes\mathfrak{F}_{\underline{Y}_\nu}\right)\left[\Lambda_\epsilon(z)_{\text{off}}\right]_{\sigma\nu}\left(\mathbb{1}\otimes\mathfrak{F}_{\underline{Y}_\nu}^{-1}\right)$$

would be multiplicative in the conjugate variables $\underline{P}_\nu = (p_3, \ldots, \hat{p}_{\nu_1}, \ldots, \hat{p}_{\nu_2}, \ldots, p_n)$; on the other hand, on $L^2\left(\mathbb{R}^4, dr_\nu dR_\nu dx_1 dx_2\right)$, $\left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu}$ is a integral operator, with kernel

$$-4\sqrt{m_1 m_2 m_{\nu_1} m_{\nu_2}}\, v(r_\sigma)\, G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right) v\left(r_\nu'\right) \equiv C\left(m_1, m_2, m_{\nu_1}, m_{\nu_2}\right) v(r_\sigma)\, G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right) v\left(r_\nu'\right),$$

where

$$X_{\sigma\nu,\epsilon} = \begin{pmatrix} \sqrt{2m_1}\left(R_\sigma - \frac{\epsilon m_2}{m_1+m_2}r_\sigma - x_1'\right) \\ \sqrt{2m_2}\left(R_\sigma + \frac{\epsilon m_1}{m_1+m_2}r_\sigma - x_2'\right) \\ \sqrt{2m_{\nu_1}}\left(x_{\nu_1} - R_\nu' - \frac{\epsilon m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}r_\nu'\right) \\ \sqrt{2m_{\nu_2}}\left(x_{\nu_2} - R_\nu' + \frac{\epsilon m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}r_\nu'\right) \end{pmatrix}, \qquad Q_\nu = \sum_{\substack{k=3 \\ k\neq\nu_1,\nu_2}}^{n} \frac{p_k^2}{2m_k}.$$

Therefore, concerning boundedness, given arbitrarily $\eta \in L^2\left(\mathbb{R}^4, dr_\nu dR_\nu dx_1 dx_2\right)$, $\xi \in \tilde{\chi}_\nu^-$, it is as follows.

$$\left\|\left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\mathrm{off}}\right]_{\sigma\nu}\eta\otimes\xi\right\|_2^2=$$
$$=C^2\int_{\mathbb{R}^n}d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_1}dx_{\nu_2}\left|\int_{\mathbb{R}^4}dr'_\nu dR'_\nu dx'_1dx'_2\,v(r_\sigma)\,G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right)v\left(r'_\nu\right)\eta\left(r'_\nu,R'_\nu,x'_1,x'_2\right)\xi\left(\underline{P}_\nu\right)\right|^2$$
$$\leq C^2\int_{\mathbb{R}^n}d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_1}dx_{\nu_2}\left\{\int_{\mathbb{R}}dr'_\nu v(r_\sigma)\,v\left(r'_\nu\right)\left|\int_{\mathbb{R}^3}dR'_\nu dx'_1dx'_2\,G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right)\eta\left(r'_\nu,R'_\nu,x'_1,x'_2\right)\xi\left(\underline{P}_\nu\right)\right|\right\}^2$$
$$\leq\left(\text{by using Hölder inequality together with the fact that }\int_{\mathbb{R}}V=1\right)$$
$$\leq C^2\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^{n-1}}d\underline{P}_\nu dR_\sigma dx_{\nu_1}dx_{\nu_2}\int_{\mathbb{R}}dr'_\nu\left|\int_{\mathbb{R}^3}dR'_\nu dx'_1dx'_2\,G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right)\eta\left(r'_\nu,R'_\nu,x'_1,x'_2\right)\xi\left(\underline{P}_\nu\right)\right|^2.$$

The coordinate transformation

$$\begin{cases}\overline{R}'_\nu&=R'_\nu+\epsilon\left(\frac{m_1}{m_1+m_2}r_\sigma-\frac{m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}r'_\nu\right)\\ \overline{x}_{\nu_1}&=x_{\nu_1}+\epsilon\left(\frac{m_1}{m_1+m_2}r_\sigma-r'_\nu\right)\\ \overline{x}_{\nu_2}&=x_{\nu_2}+\epsilon\frac{m_1}{m_1+m_2}r_\sigma\\ \overline{x}'_1&=x'_1+\epsilon\frac{m_2}{m_1+m_2}r_\sigma\\ \overline{x}'_2&=x'_2-\epsilon\frac{m_1}{m_1+m_2}r_\sigma\end{cases}$$

allows for

$$\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^{n-1}}d\underline{P}_\nu dR_\sigma dx_{\nu_1}dx_{\nu_2}\int_{\mathbb{R}}dr'_\nu\left|\int_{\mathbb{R}^3}dR'_\nu dx'_1dx'_2\,G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right)\eta\left(r'_\nu,R'_\nu,x'_1,x'_2\right)\xi\left(\underline{P}_\nu\right)\right|^2\equiv$$
$$\equiv\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^n}d\underline{P}_\nu dR_\sigma d\overline{x}_{\nu_1}d\overline{x}_{\nu_2}dr'_\nu\left|\int_{\mathbb{R}^3}d\overline{R}'_\nu d\overline{x}'_1d\overline{x}'_2\,\left[\,\tilde{\eta}\left(r'_\nu,\overline{R}'_\nu,\overline{x}'_1,\overline{x}'_2\right)\xi\left(\underline{P}_\nu\right)\right.\right.$$
$$\left.\left.G^{(4)}_{z-Q_\nu}\left(\sqrt{2m_1}\left(R_\sigma-\overline{x}'_1\right),\sqrt{2m_2}\left(R_\sigma-\overline{x}'_2\right),\sqrt{2m_{\nu_1}}\left(\overline{x}_{\nu_1}-\overline{R}'_\nu\right),\sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2}-\overline{R}'_\nu\right)\right)\right]\right|^2$$
$$\leq\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}^n}d\underline{P}_\nu dR_\sigma d\overline{x}_{\nu_1}d\overline{x}_{\nu_2}dr'_\nu\left[\int_{\mathbb{R}^3}d\overline{R}'_\nu d\overline{x}'_1d\overline{x}'_2\left|\tilde{\eta}\left(r'_\nu,\overline{R}'_\nu,\overline{x}'_1,\overline{x}'_2\right)\xi\left(\underline{P}_\nu\right)\right|\right.$$
$$\left.G^{(4)}_{z}\left(\sqrt{2m_1}\left(R_\sigma-\overline{x}'_1\right),\sqrt{2m_2}\left(R_\sigma-\overline{x}'_2\right),\sqrt{2m_{\nu_1}}\left(\overline{x}_{\nu_1}-\overline{R}'_\nu\right),\sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2}-\overline{R}'_\nu\right)\right)\right]^2$$
$$\leq(\text{by using Hölder inequality})$$
$$\leq\left(\int_{\mathbb{R}^{n-4}}d\underline{P}_\nu\left|\xi\left(\underline{P}_\nu\right)\right|^2\right)\cdot\sup_{r_\sigma\in\mathbb{R}}\int_{\mathbb{R}}dr'_\nu\int_{\mathbb{R}^3}dR_\sigma d\overline{x}_{\nu_1}d\overline{x}_{\nu_2}\left[\int_{\mathbb{R}^3}d\overline{R}'_\nu d\overline{x}'_1d\overline{x}'_2\left|\tilde{\eta}\left(r'_\nu,\overline{R}'_\nu,\overline{x}'_1,\overline{x}'_2\right)\right|\cdot\right.$$
$$\left.\cdot G^{(4)}_{z}\left(\sqrt{2m_1}\left(R_\sigma-\overline{x}'_1\right),\sqrt{2m_2}\left(R_\sigma-\overline{x}'_2\right),\sqrt{2m_{\nu_1}}\left(\overline{x}_{\nu_1}-\overline{R}'_\nu\right),\sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2}-\overline{R}'_\nu\right)\right)\right]^2,$$

where

$$\tilde{\eta}\left(r'_\nu,\overline{R}'_\nu,\overline{x}'_1,\overline{x}'_2\right)=\eta\left(r'_\nu,\,\overline{R}'_\nu-\epsilon\left(\frac{m_1}{m_1+m_2}r_\sigma-\frac{m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}r'_\nu\right),\,\overline{x}'_1-\epsilon\frac{m_2}{m_1+m_2}r_\sigma,\,\overline{x}'_2+\epsilon\frac{m_1}{m_1+m_2}r_\sigma\right).$$

Consequently, by Appendix 4, independently of $\epsilon>0$, it results

$$\left\|\left[\Lambda_\epsilon(z)_{\mathrm{off}}\right]_{\sigma\nu}\right\|_{\mathfrak{B}(\chi_\sigma,\chi_\nu)}\leq\left(\max_i m_i^2\right)\frac{1}{\sqrt{|z|}}.\tag{6.15}$$

■

Remark 6.13. *A summary of what obtained up to this point is in order: by setting* $K \doteq \max\left[\left(\max_i m_i^{\frac{3}{2}}, \max_i m_i^2\right)\right]$,

$$\left\| \left[\Lambda_\epsilon(z)_{\mathit{off}}\right]_{\sigma\nu} \right\|_{\mathfrak{B}(\chi_\sigma,\chi_\nu)} \le \frac{K}{\sqrt{|z|}},$$

for all $\epsilon > 0$, $z < 0$, $\sigma \neq \nu$, *hence*

$$\max_{\sigma,\nu} \left\| \left[\Lambda_\epsilon(z)_{\mathit{off}}\right]_{\sigma\nu} \right\|_{\mathfrak{B}(\chi_\sigma,\chi_\nu)} \le \frac{K}{\sqrt{|z|}}.$$

By taking into account the diagonal contribution too and denoting by $\|\cdot\|_\oplus$ *the operator norm of* $\mathfrak{B}(\chi)$,

$$\begin{aligned}\left\| \left[\Lambda_\epsilon(z)_{\mathit{diag}}\right]^{-1} \left[\Lambda_\epsilon(z)_{\mathit{off}}\right] \right\|_\oplus &\le \left\| \left[\Lambda_\epsilon(z)_{\mathit{diag}}\right]^{-1} \right\|_\oplus \left\| \left[\Lambda_\epsilon(z)_{\mathit{off}}\right] \right\|_\oplus \le \\ &\le \frac{n(n-1)}{2} \cdot \left[1 - \frac{\mathfrak{C}|g|}{\sqrt{|z|}}\right]^{-1} \cdot \frac{K}{\sqrt{|z|}},\end{aligned}$$

allowing to state that, as long as $z < -\left[\frac{n(n-1)}{2}K + \mathfrak{C}|g|\right]^2 \doteq z_0$, $\left\| \left[\Lambda_\epsilon(z)_{\mathit{diag}}\right]^{-1} \left[\Lambda_\epsilon(z)_{\mathit{off}}\right] \right\|_\oplus < 1$ *and* $\left\{\mathbb{1} + \left[\Lambda_\epsilon(z)_{\mathit{diag}}\right]^{-1} \left[\Lambda_\epsilon(z)_{\mathit{off}}\right]\right\}$ *is invertible in* $\mathfrak{B}(\chi)$, *for all* $\epsilon > 0$. □

6.3.4 Computing $\Lambda_\epsilon(z)_{\mathbf{off}}$ as $\epsilon \downarrow 0$

$\boxed{\sigma = (12),\ \nu = (1\nu_2),\ 3 \le \nu_2 \le n}$

Proposition 6.6. *For all* $z < 0$, *given*

$$\left[\Lambda_{0,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu} : L^2\left(\mathbb{R}^3, dr_\nu dR_\nu dx_2\right) \otimes \tilde{\chi}_\nu^- \longrightarrow L^2\left(\mathbb{R}^3, dr_\sigma dR_\sigma dx_{\nu_2}\right) \otimes \tilde{\chi}_\nu^-$$

defined by

$$\left(\left[\Lambda_{0,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu} \psi\right)(r_\sigma, R_\sigma, x_{\nu_2}, \underline{P}_\nu) \doteq C v(r_\sigma) \int_{\mathbb{R}^3} dr'_\nu dR'_\nu dx'_2\, G^{(3)}_{z-Q_\nu}\left(X_{\sigma\nu,0}\right) v\left(r'_\nu\right) \psi\left(r'_\nu, R'_\nu, x'_2, \underline{P}_\nu\right),$$

for all $\psi \in L^2\left(\mathbb{R}^3, dr_\nu dR_\nu dx_2\right) \otimes \tilde{\chi}_\nu^-$, *where*

$$X_{\sigma\nu,0} = \begin{pmatrix} \sqrt{2m_1}\left(R_\sigma - R'_\nu\right) \\ \sqrt{2m_2}\left(R_\sigma - x'_2\right) \\ \sqrt{2m_{\nu_2}}\left(x_{\nu_2} - R'_\nu\right) \end{pmatrix}, \quad Q_\nu = \sum_{\substack{k=3 \\ k\neq\nu_2}}^n \frac{p_k^2}{2m_k}, \quad C = -(2)^{\frac{3}{2}} \sqrt{m_1 m_2 m_{\nu_2}},$$

it results

$$\lim_{\epsilon\downarrow 0} \left\| \left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu} - \left[\Lambda_{0,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu} \right\|_{\mathfrak{B}(\chi_\sigma,\chi_\nu)} = 0.$$

Proof. Let $\eta \in L^2\left(\mathbb{R}^3, dr_\nu dR_\nu x_2\right)$, $\xi \in \tilde{\chi}_\nu^-$ be arbitrary.

$$
\begin{aligned}
&\frac{1}{C^2}\left\|\left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu} - \left[\Lambda_{0,\underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu}\eta\otimes\xi\right\|_2^2 = \\
&= \int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_2 \left|\int_{\mathbb{R}^3} dr'_\nu dR'_\nu dx'_2\, v(r_\sigma)\left[G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,0})\right] v(r'_\nu)\,\eta(r'_\nu, R'_\nu, x'_2)\,\xi(\underline{P}_\nu)\right|^2 \equiv \\
&= \int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_2\, V(r_\sigma)\left|\int_{\mathbb{R}} dr'_\nu\, v(r'_\nu)\,\cdot\right. \\
&\qquad\qquad \left.\cdot\int_{\mathbb{R}^2} dR'_\nu dx'_2\left[G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,0})\right]\eta(r'_\nu, R'_\nu, x'_2)\,\xi(\underline{P}_\nu)\right|^2 \le \\
&\le \int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_2\, V(r_\sigma)\left\{\int_{\mathbb{R}} dr'_\nu\, v(r'_\nu)\,\frac{1+|r'_\nu|^{\frac{1}{2}}}{1+|r'_\nu|^{\frac{1}{2}}}\right. \\
&\qquad\qquad \left.\left|\int_{\mathbb{R}^2} dR'_\nu dx'_2\left[G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,0})\right]\eta(r'_\nu, R'_\nu, x'_2)\,\xi(\underline{P}_\nu)\right|\right\}^2 \le \\
&\le 2\int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_2\, V(r_\sigma)\left[\int_{\mathbb{R}} dr'_\nu\,(1+|r'_\nu|)\,V(r'_\nu)\right] \\
&\qquad \cdot\int_{\mathbb{R}} dr'_\nu \frac{1}{1+|r'_\nu|}\left|\int_{\mathbb{R}^2} dR'_\nu dx'_2\left[G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,0})\right]\eta(r'_\nu, R'_\nu, x'_2)\,\xi(\underline{P}_\nu)\right|^2 \le \\
&\equiv I\left(V,\frac{1}{2}\right)\int_{\mathbb{R}^n} d\underline{P}_\nu dR_\sigma dx_2 \int_{\mathbb{R}^2} dr_\sigma dr'_\nu\left[\frac{V(r_\sigma)}{1+|r'_\nu|}\right]\cdot \\
&\qquad\qquad \left[\int_{\mathbb{R}^2} dR'_\nu dx'_2\left|G^{(3)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(3)}_{z-Q_\nu}(X_{\sigma,\nu,0})\right|\left|\eta(r'_\nu, R'_\nu, x'_2)\,\xi(\underline{P}_\nu)\right|\right]^2 \le \\
&\le I\left(V,\frac{1}{2}\right)\int_{\mathbb{R}^n} d\underline{P}_\nu dR_\sigma dx_2 \int_{\mathbb{R}^2} dr_\sigma dr'_\nu\left[\frac{V(r_\sigma)}{1+|r'_\nu|}\right]\cdot \\
&\qquad\qquad \left[\int_{\mathbb{R}^2} dR'_\nu dx'_2\left|G^{(3)}_{z}(X_{\sigma\nu,\epsilon}) - G^{(3)}_{z}(X_{\sigma,\nu,0})\right|\left|\eta(r'_\nu, R'_\nu, x'_2)\,\xi(\underline{P}_\nu)\right|\right]^2 \le \\
&\equiv I\left(V,\frac{1}{2}\right)\left[\int_{\mathbb{R}^{n-3}} d\underline{P}_\nu\,|\xi(\underline{P}_\nu)|^2\right]\left\{\int_{\mathbb{R}^2} dr_\sigma dr'_\nu\left[\frac{V(r_\sigma)}{1+|r'_\nu|}\right]\cdot\right. \\
&\qquad\qquad \left.\cdot\int_{\mathbb{R}^2} dR_\sigma dx_2\left[\int_{\mathbb{R}^2} dR'_\nu dx'_2\left|G^{(3)}_{z}(X_{\sigma\nu,\epsilon}) - G^{(3)}_{z}(X_{\sigma\nu,0})\right|\left|\eta(r'_\nu, R'_\nu, x'_2)\right|\right]^2\right\}
\end{aligned}
$$

by introducing

$$
I\left(V,\frac{1}{2}\right) = 2\int_{\mathbb{R}} dr\,\left(1+|r|\right)V(r) < \infty.
$$

Let then $F_{r_\sigma, r'_\nu,\epsilon} : L^2\left(\mathbb{R}^2, dR_\nu dx_2\right) \longrightarrow L^2\left(\mathbb{R}^2, dR_\sigma dx_{\nu_2}\right)$ be defined by

$$
\left[F_{r_\sigma, r'_\nu,\epsilon}\varphi\right](R_\sigma, x_{\nu_2}) \doteq \int_{\mathbb{R}^2} dR'_\nu dx'_2\,\left[G^{(3)}_z(X_{\sigma\nu,\epsilon}) - G^{(3)}_z(X_{\sigma\nu,0})\right]\varphi(R'_\nu, x'_2)\,.
$$

The Schur test is going to be used the ascertain whether it is bounded or not. By introducing

$$
\tilde{X}_{\sigma\nu,\epsilon} = \begin{pmatrix} \sqrt{2m_1}\left[R_\sigma - R'_\nu - \epsilon\left(\frac{m_2}{m_1+m_2}r_\sigma - \frac{m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}r'_\nu\right)\right] \\ \sqrt{2m_2}\,(R_\sigma - x'_2) \\ \sqrt{2m_{\nu_2}}\,(x_{\nu_2} - R'_\nu) \end{pmatrix},
$$

the following procedure is adopted.

$$\int_{\mathbb{R}^2} dR'_\nu dx'_2 \left| G_z^{(3)}\left(X_{\sigma\nu,\epsilon}\right) - G_z^{(3)}\left(X_{\sigma\nu,0}\right)\right| \leq \int_{\mathbb{R}^2} dR'_\nu dx'_2 \left| G_z^{(3)}\left(X_{\sigma\nu,\epsilon}\right) - G_z^{(3)}\left(\tilde{X}_{\sigma\nu,\epsilon}\right)\right| + \\ + \int_{\mathbb{R}^2} dR'_\nu dx'_2 \left| G_z^{(3)}\left(\tilde{X}_{\sigma\nu,\epsilon}\right) - G_z^{(3)}\left(X_{\sigma\nu,0}\right)\right| \equiv \boxed{A} + \boxed{B}.$$

By observing that

$$\left| G_z^{(3)}\left(X_{\sigma\nu,\epsilon}\right) - G_z^{(3)}\left(\tilde{X}_{\sigma\nu,\epsilon}\right)\right| = \\ \left| \int_0^\infty \left\{ e^{-\frac{\left\{\sqrt{2m_1}\left[R_\sigma - R'_\nu - \epsilon\left(\frac{m_2}{m_1+m_2} r_\sigma - \frac{m_{\nu 2}}{m_{\nu 1}+m_{\nu 2}} r'_\nu\right)\right]\right\}^2 + \left\{\sqrt{2m_2}\left(R_\sigma - x'_2 + \epsilon\frac{m_1}{m_1+m_2} r_\sigma\right)\right\}^2 + \left\{\sqrt{2m_{\nu 2}}\left[x_{\nu 2} - R'_\nu + \epsilon\left(\frac{m_{\nu 1}}{m_{\nu 1}+m_{\nu 2}}\right) r'_\nu\right]\right\}^2}{4t} + zt} \right.\right. \\ \left.\left. - e^{-\frac{\left\{\sqrt{2m_1}\left[R_\sigma - R'_\nu - \epsilon\left(\frac{m_2}{m_1+m_2} r_\sigma - \frac{m_{\nu 2}}{m_{\nu 1}+m_{\nu 2}} r'_\nu\right)\right]\right\}^2 + \left\{\sqrt{2m_2}\left(R_\sigma - x'_2\right)\right\}^2 + \left\{\sqrt{2m_{\nu 2}}\left(x_{\nu 2} - R'_\nu\right)\right\}^2}{4t} + zt} \right\} \frac{dt}{(4\pi t)^{\frac{3}{2}}} \right| \leq \\ \leq \left| \int_0^\infty \left[e^{-\frac{\left[\sqrt{2m_2}\left(R_\sigma - x'_2 + \epsilon\frac{m_1}{m_1+m_2} r_\sigma\right)\right]^2 + \left\{\sqrt{2m_{\nu 2}}\left[x_{\nu 2} - R'_\nu + \epsilon\left(\frac{m_{\nu 1}}{m_{\nu 1}+m_{\nu 2}}\right) r'_\nu\right]\right\}^2}{4t} + zt} + \right.\right. \\ \left.\left. - e^{-\frac{\left[\sqrt{2m_2}\left(R_\sigma - x'_2\right)\right]^2 + \left[\sqrt{2m_{\nu 2}}\left(x_{\nu 2} - R'_\nu\right)\right]^2}{4t} + zt} \right] \frac{dt}{(4\pi t)^{\frac{3}{2}}} \right| \leq \\ \leq \left| G_z^{(3)}\left(0,\, \sqrt{2m_2}\left[R_\sigma - x'_2 + \epsilon\left(\frac{m_1}{m_1+m_2}\right) r_\sigma\right], \sqrt{2m_{\nu_2}}\left(x_{\nu_2} - R'_\nu\right)\right) + \right. \\ \left. - G_z^{(3)}\left(0,\, \sqrt{2m_2}\left(R_\sigma - x'_2\right), \sqrt{2m_{\nu_2}}\left(x_{\nu_2} - R'_\nu\right)\right) \right|,$$

regarding $\boxed{A}$, one has

$$\int_{\mathbb{R}^2} dR'_\nu dx'_2 \left| G_z^{(3)}\left(X_{\sigma\nu,\epsilon}\right) - G_z^{(3)}\left(\tilde{X}_{\sigma\nu,\epsilon}\right)\right| \leq$$

$$\leq \int_{\mathbb{R}^2} dR'_\nu dx'_2 \left| G_z^{(3)}\left(0,\, \sqrt{2m_2}\left[R_\sigma - x'_2 + \epsilon\left(\frac{m_1}{m_1+m_2}\right) r_\sigma\right], \sqrt{2m_{\nu_2}}\left(x_{\nu_2} - R'_\nu\right)\right) + \right.$$

$$\left. - G_z^{(3)}\left(0,\, \sqrt{2m_2}\left(R_\sigma - x'_2\right), \sqrt{2m_{\nu_2}}\left(x_{\nu_2} - R'_\nu\right)\right)\right|,$$

$$\leq \int_{\mathbb{R}^2} dR'_\nu dx'_2 \left| \int_0^\infty \left[e^{-\frac{\left[\sqrt{2m_2}\left(R_\sigma - x'_2 + \epsilon\frac{m_1}{m_1+m_2} r_\sigma\right)\right]^2 + \left\{\sqrt{2m_{\nu_2}}\left[x_{\nu_2} - R'_\nu + \epsilon\left(\frac{m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}\right) r'_\nu\right]\right\}^2}{4t} + zt} + \right.\right.$$

$$\left.\left. - e^{-\frac{\left[\sqrt{2m_2}\left(R_\sigma - x'_2\right)\right]^2 + \left[\sqrt{2m_{\nu_2}}\left(x_{\nu_2} - R'_\nu\right)\right]^2}{4t} + zt} \right] \frac{dt}{(4\pi t)^{\frac{3}{2}}} \right|.$$

By using

$$\begin{cases} \overline{x}'_2 & = \sqrt{2m_2}\, x'_2 \\ \overline{R}'_\nu & = \sqrt{2m_{\nu_2}}\, R'_\nu \end{cases}$$

one obtains

$$\boxed{A} \leq \int_{\mathbb{R}^2} \frac{d\overline{x}'_2 d\overline{R}'_\nu}{\sqrt{2m_2}\sqrt{2m_{\nu_2}}} \left| \int_0^\infty \left\{ e^{-\frac{\left[\overline{x}'_2 - \sqrt{2m_2}\left(R_\sigma + \epsilon\frac{m_1}{m_1+m_2} r_\sigma\right)\right]^2 + \left[\overline{R}'_\nu - \sqrt{2m_{\nu_2}}\left(x_{\nu_2} + \epsilon\frac{m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}} r'_\nu\right)\right]^2}{4t} + zt} + \right.\right.$$

$$\left.\left. - e^{-\frac{\left(\overline{x}'_2 - \sqrt{2m_2} R_\sigma\right)^2 + \left(\overline{R}'_\nu - \sqrt{2m_{\nu_2}} x_{\nu_2}\right)^2}{4t} + zt} \right\} \frac{dt}{(4\pi t)^{\frac{3}{2}}} \right| \equiv$$

$$\equiv \left(\begin{cases} x'_2 & = \overline{x}'_2 - \sqrt{2m_2} R_\sigma \\ R'_\nu & = \overline{R}'_\nu - \sqrt{2m_{\nu_2}} x_{\nu_2} \end{cases} \right)$$

$$\equiv \int_{\mathbb{R}^2} \frac{dx'_2 dR'_\nu}{2\sqrt{m_2 m_{\nu_2}}} \left| G_z^{(3)}\left(0, x'_2 - \sqrt{2m_2}\left(\frac{\epsilon m_1}{m_1+m_2}\right) r_\sigma,\, R'_\nu - \sqrt{2m_{\nu_2}}\left(\frac{\epsilon m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}\right) r'_\nu\right) + \right.$$

$$\left. - G_z^{(3)}\left(0, x'_2, R'_\nu\right) \right|.$$

What obtained structurally coincides with what reported in [20], Proposition 4.5; since similar arguments hold true for $\boxed{B}$ all the same, $F_{r_\sigma, r'_\nu, \epsilon}$ is a bounded operator and

$$\lim_{\epsilon \downarrow 0} \left\| \left[\Lambda_{\epsilon, \underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu} - \left[\Lambda_{0, \underline{P}_\nu}(z)_{\text{off}}\right]_{\sigma\nu} \right\|_{\mathfrak{B}(\chi_\sigma, \chi_\nu)} = 0.$$

■

Remark 6.14. *An analogously proven result holds for* $\sigma = (12)$, $\nu = (2\nu_2)$, $\nu_2 \geq 3$. □

$\boxed{\sigma = (12), \nu = (\nu_1\nu_2),\, 3 \le \nu_1 < \nu_2 \le n}$

Proposition 6.7. *For all $z<0$, given*

$$\left[\Lambda_{0,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu} : L^2\left(\mathbb{R}^4, dx_1dx_2dR_\nu dr_\nu\right)\otimes\tilde{\chi}_\nu^- \longrightarrow L^2\left(\mathbb{R}^4, dr_\sigma dR_\sigma dx_{\nu_1}dx_{\nu_2}\right)\otimes\tilde{\chi}_\nu^-$$

defined by

$$\left(\left[\Lambda_{0,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu}\psi\right)(r_\sigma,R_\sigma,x_{\nu_1},x_{\nu_2}) \doteq Cv(r_\sigma)\int_{\mathbb{R}^4} dr'_\nu dR'_\nu dx'_1dx'_2\, G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,0})\, v(r'_\nu)\,\psi(r'_\nu,R'_\nu,x'_1,x'_2),$$

with $\psi\in L^2\left(\mathbb{R}^4, dx_1dx_2dR_\nu dr_\nu\right)\otimes\tilde{\chi}_\nu^-$, where

$$X_{\sigma\nu,0} = \begin{pmatrix}\sqrt{2m_1}\,(R_\sigma - x'_1)\\ \sqrt{2m_2}\,(R_\sigma - x'_2)\\ \sqrt{2m_{\nu_1}}\,(x_{\nu_1}-R'_\nu)\\ \sqrt{2m_{\nu_2}}\,(x_{\nu_2}-R'_\nu)\end{pmatrix},\quad Q_\nu = \sum_{\substack{k=3\\ k\neq\nu_1,\nu_2}}^{n}\frac{p_k^2}{2m_k},\quad C = -4\sqrt{m_1m_2m_{\nu_1}m_{\nu_2}},$$

one has

$$\lim_{\epsilon\downarrow 0}\left\|\left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu} - \left[\Lambda_{0,\underline{P}_\nu}(z)_{\mathit{off}}\right]_{\sigma\nu}\right\|_{\mathfrak{B}(\chi_\sigma,\chi_\nu)} = 0.$$

Proof. Let $\eta\in L^2\left(\mathbb{R}^4, dx_1dx_2dr_\nu dR_\nu\right), \xi\in\tilde{\chi}_\nu^-$ be arbitrary.

$$\begin{aligned}
&\left\|\left\{\left[\Lambda_{\epsilon,\underline{P}_\nu}(z)_{\mathrm{off}}\right]_{\sigma\nu} - \left[\Lambda_{0,\underline{P}_\nu}(z)_{\mathrm{off}}\right]_{\sigma\nu}\right\}\eta\otimes\xi\right\|_2^2 = \\
&= \int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_1}dx_{\nu_2}\left|C\int_{\mathbb{R}^4} dx'_1dx'_2dr'_\nu dR'_\nu\left\{v(r_\sigma)\,v(r'_\nu)\left[G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,0})\right]\right.\right.\\
&\qquad\left.\left.\eta(x'_1,x'_2,r'_\nu,R'_\nu)\,\xi(\underline{P}_\nu)\right\}\right|^2\\
&= C^2\int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_1}dx_{\nu_2}\, V(r_\sigma)\left|\int_{\mathbb{R}^4} dx'_1dx'_2dr'_\nu dR'_\nu\left\{v(r'_\nu)\left[G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,0})\right]\right.\right.\\
&\qquad\left.\left.\eta(x'_1,x'_2,r'_\nu,R'_\nu)\,\xi(\underline{P}_\nu)\right\}\right|^2\\
&\le C^2\int_{\mathbb{R}^n} d\underline{P}_\nu dr_\sigma dR_\sigma dx_{\nu_1}dx_{\nu_2}\, V(r_\sigma)\left\{\int_{\mathbb{R}} dr'_\nu v(r'_\nu)\,\frac{1+|r'_\nu|^{\frac12}}{1+|r'_\nu|^{\frac12}}\right.\\
&\qquad\left.\left|\int_{\mathbb{R}^3} dR'_\nu dx'_2dx'_1\left[G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,0})\right]\eta(x'_1,x'_2,r'_\nu,R'_\nu)\,\xi(\underline{P}_\nu)\right|\right\}^2\\
&\le C^2 I\left(V,\frac12\right)\int_{\mathbb{R}^{n-1}} d\underline{P}_\nu dR_\sigma dx_{\nu_1}dx_{\nu_2}\left\{\int_{\mathbb{R}^2} dr'_\nu dr_\sigma\frac{V(r_\sigma)}{1+|r'_\nu|}\cdot\right.\\
&\qquad\left.\cdot\left|\int_{\mathbb{R}^3} dR'_\nu dx'_1dx'_2\left[G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,\epsilon}) - G^{(4)}_{z-Q_\nu}(X_{\sigma\nu,0})\right]\eta(x'_1,x'_2,r'_\nu,R'_\nu)\,\xi(\underline{P}_\nu)\right|^2\right\}\\
&\le C^2 I\left(V,\frac12\right)\left(\int_{\mathbb{R}^{n-4}} d\underline{P}_\nu|\xi(\underline{P}_\nu)|^2\right)\left\{\int_{\mathbb{R}^2} dr'_\nu dr_\sigma\frac{V(r_\sigma)}{1+|r'_\nu|}\cdot\right.\\
&\qquad\left.\cdot\int_{\mathbb{R}^3} dR_\sigma dx_{\nu_1}dx_{\nu_2}\left|\int_{\mathbb{R}^3} dR'_\nu dx'_1dx'_2\left[G^{(4)}_{z}(X_{\sigma\nu,\epsilon}) - G^{(4)}_{z}(X_{\sigma\nu,0})\right]\eta(x'_1,x'_2,r'_\nu,R'_\nu)\right|^2\right\}
\end{aligned}$$

It is then considered the linear map $K:\ \psi \in L^2\left(\mathbb{R}^3, dx_1 dx_2 dR_\nu\right) \longmapsto K\psi \in L^2\left(\mathbb{R}^3, dx_{\nu_1} dx_{\nu_2} dR_\sigma\right)$ defined by

$$\left(K\psi\right)\left(x_{\nu_1}, x_{\nu_2}, R_\sigma\right) = \int_{\mathbb{R}^3} dR'_\nu dx'_1 dx'_2 \left[G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,0}\right)\right]\psi\left(x'_1, x'_2, R'_\nu\right).$$

By introducing the point

$$\tilde{X}_{\sigma\nu,\epsilon} = \begin{pmatrix} \sqrt{2m_1}\left(x'_1 - R_\sigma\right) \\ \sqrt{2m_2}\left(x'_2 - R_\sigma\right) \\ \sqrt{2m_{\nu_1}}\left(R'_\nu - x_{\nu_1}\right) \\ \sqrt{2m_{\nu_2}}\left[R'_\nu - x_{\nu_2} - \epsilon\left(\frac{m_{\nu_1}}{m_{\nu_1}+m_{\nu_2}}\right) r'_\nu\right] \end{pmatrix},$$

to check whether K is bounded or not, the Schur test is referred to again.

$$\begin{aligned}
&\int_{\mathbb{R}^3} dR'_\nu dx'_1 dx'_2 \left|G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z-Q_\nu}\left(X_{\sigma\nu,0}\right)\right| \leq \int_{\mathbb{R}^3} dR'_\nu dx'_1 dx'_2 \left|G^{(4)}_{z}\left(X_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z}\left(X_{\sigma\nu,0}\right)\right| \equiv \\
&= \int_{\mathbb{R}^3} dR'_\nu dx'_1 dx'_2 \left|G^{(4)}_{z}\left(X_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z}\left(\tilde{X}_{\sigma\nu,\epsilon}\right) + G^{(4)}_{z}\left(\tilde{X}_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z}\left(X_{\sigma\nu,0}\right)\right| \leq \\
&\leq \int_{\mathbb{R}^3} dR'_\nu dx'_1 dx'_2 \left|G^{(4)}_{z}\left(X_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z}\left(\tilde{X}_{\sigma\nu,\epsilon}\right)\right| + \int_{\mathbb{R}^3} dR'_\nu dx'_1 dx'_2 \left|G^{(4)}_{z}\left(\tilde{X}_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z}\left(X_{\sigma\nu,0}\right)\right| \equiv \\
&\equiv \boxed{A} + \boxed{B}.
\end{aligned}$$

Then

$$\begin{aligned}
\boxed{A} &= \int_{\mathbb{R}^3} dR'_\nu dx'_1 dx'_2 \left|G^{(4)}_{z}\left(X_{\sigma\nu,\epsilon}\right) - G^{(4)}_{z}\left(\tilde{X}_{\sigma\nu,\epsilon}\right)\right| \leq \\
&\leq \int_{\mathbb{R}^3} \frac{dx'_1 dx'_2 dR'_\nu}{\sqrt{6 m_1 m_2 m_{\nu_1}}} \left|G^{(4)}_{z}\left(x'_1 + \epsilon\left(\frac{\sqrt{2m_1}m_2}{m_1+m_2}\right) r_\sigma,\ x'_2 - \epsilon\left(\frac{\sqrt{2m_2}m_1}{m_1+m_2}\right) r_\sigma, R'_\nu + \epsilon\left(\frac{\sqrt{2m_{\nu_1}}m_{\nu_2}}{m_{\nu_1}+m_{\nu_2}}\right) r'_\nu, 0\right) + \right. \\
&\qquad\qquad \left. - G^{(4)}_{z}\left(x'_1, x'_2, R'_\nu, 0\right)\right|,
\end{aligned}$$

by having respectively used the coordinate transformations

$$\begin{cases} \overline{x}'_1 = \sqrt{2m_1}x'_1 \\ \overline{x}'_2 = \sqrt{2m_2}x'_2 \\ \overline{R}'_\nu = \sqrt{2m_{\nu_1}}R'_\nu \end{cases} \quad \text{and} \quad \begin{cases} x'_1 = \overline{x}'_1 - \sqrt{2m_1}R_\sigma \\ x'_2 = \overline{x}'_2 - \sqrt{2m_2}R_\sigma \\ R'_\nu = \overline{R}'_\nu - \sqrt{2m_{\nu_1}}x_{\nu_1} \end{cases}.$$

From this point on, it is possible to proceed as in [20], Proposition 4.8, to eventually state that K is a bounded operator and

$$\lim_{\epsilon\downarrow 0}\left\|\left[\Lambda_{\epsilon,\underline{P}_\nu}\left(z\right)_{\text{off}}\right]_{\sigma\nu} - \left[\Lambda_{0,\underline{P}_\nu}\left(z\right)_{\text{off}}\right]_{\sigma\nu}\right\|_{\mathfrak{B}(\chi_\sigma,\chi_\nu)} = 0$$

holds, since analogous arguments apply to $\boxed{B}$ too. ∎

Corollary 6.7.1. *For all* $z < z_0 \doteq -\left[\frac{n(n-1)}{2}K + \mathfrak{C}|g|\right]^2$, *set* $\Lambda_0(z) \doteq \Lambda_0(z)_{diag} + \Lambda_0(z)_{off}$,

$$\lim_{\epsilon\downarrow 0} \Lambda_\epsilon(z)^{-1} = \Lambda_0(z)^{-1} \equiv \left\{\mathbb{1} + \left[\Lambda_0(z)_{diag}\right]^{-1}\Lambda_0(z)_{off}\right\}^{-1}\left[\Lambda_0(z)_{diag}\right]^{-1}$$

in $\mathfrak{B}(\chi)$. *Consequently*

$$\lim_{\epsilon\downarrow 0}(H_\epsilon - z\mathbb{1})^{-1} = R_{H_0}(z) + \sum_{\sigma,\nu\in\mathcal{I}}\left[S^{(\sigma)}(z)\right]^*\left[\Lambda_0(z)^{-1}\right]_{\sigma\nu}\left[J^{(\nu)}S^{(\nu)}(z)\right] \doteq R(z).$$

■

Remark 6.15. *By recalling the self-adjoint operator* $(H, \mathcal{D}_H)$ *introduced in Appendix 3, [20], Appendix C allows to state that* H_ϵ *converges to* H *in the strong resolvent sense, as* $\epsilon \downarrow 0$. *Consequently, as long as* $z < z_0$, $R_H(z) = (H - z\mathbb{1})^{-1} = R(z)$, *i.e. if* $z < z_0$,

$$\left\|R_H(z) - R_{H_\epsilon}(z)\right\| \underset{\epsilon\downarrow 0}{\longrightarrow} 0.$$

□

Proposition 6.8. $H_\epsilon \underset{\epsilon\downarrow 0}{\longrightarrow} H$ *in the norm resolvent sense.*

Proof. Let $z \in (-\infty, z_0)$ be arbitrary and $\delta > 0$ such that $\delta < |z|$. Let then $\omega_\pm \doteq z \pm i\delta$ be; H_ϵ, H are self-adjoint operators, for all $\epsilon > 0$, hence $\omega_\pm \in \rho(H_\epsilon) \cap \rho(H)$ for all $\epsilon > 0$ and $R_{H_\epsilon}(\omega_\pm) - R_H(\omega_\pm)$ makes sense. Eventually, the Neumann series expansion allows for

$$\begin{aligned}\left\|R_{H_\epsilon}(\omega_\pm) - R_H(\omega_\pm)\right\| &\leq \left\|\sum_{n\in\mathbb{N}_0}(\omega_\pm - z)^n R_{H_\epsilon}(z)^{n+1} - \sum_{n\in\mathbb{N}_0}(\omega_\pm - z)^n R_H(z)^{n+1}\right\| \leq \\ &\leq \sum_{n\in\mathbb{N}_0}\delta^n\left\|R_{H_\epsilon}(z)^{n+1} - R_H(z)^{n+1}\right\| \underset{\epsilon\downarrow 0}{\longrightarrow} 0,\end{aligned}$$

because of the n^{th}-power function continuity. Repeating the process, the result holds for all $z \in \mathbb{C}\setminus\mathbb{R}$. ■

Corollary 6.8.1. *The self-adjoint operator* $(H, \mathcal{D}_H)$ *is affiliated to* $\mathcal{R}\left(\mathbb{R}^{2n}, \sigma\right)$.

Proof. By Proposition 6.8, $\left\|R_{H_\epsilon}(z) - R_H(z)\right\| \underset{\epsilon\downarrow 0}{\longrightarrow} 0$ for all $z \in i\mathbb{R}\setminus\{0\}$, i.e., because of proposition 4.4, $(H - i\lambda\mathbb{1})^{-1} \in \pi_S\left[\mathcal{R}\left(\mathbb{R}^{2n}, \sigma\right)\right]$ for all $\lambda \in \mathbb{R}\setminus\{0\}$. ■

Part III

Appendix

1 - The Konno-Kuroda Formula

Proposition 6.9. *Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces. Let $(H_0, \mathcal{D}_{H_0})$ be self-adjoint operator on $\mathcal{H}$ and let $A : \mathcal{H} \longrightarrow \mathcal{K}$ be a bounded operator. Given the self-adjoint operator $\left(H_g \doteq H_0 - gA^*A, \mathcal{D}_{H_0}\right)$ on $\mathcal{H}$, $g \in \mathbb{R} \setminus \{0\}$, for all $z \in \rho(H_0) \cap \rho\left(H_g\right)$,*

1. $\left[\mathbb{1}_{\mathcal{K}} - \phi(z)\right]^{-1} = \mathbb{1}_{\mathcal{K}} + M(z)$, *with* $\phi(z) = gAR_{H_0}(z)A^*$ *and* $M(z) = gAR_{H_g}(z)A^*$,

2. $R_{H_g}(z) = R_{H_0}(z) + gR_{H_0}(z)A^*\left[\mathbb{1}_{\mathcal{K}} - \phi(z)\right]^{-1} AR_{H_0}(z)$.

Proof. It is first observed that A^*A is a bounded self-adjoint operator on $\mathcal{H}$, hence the same holds for $V \equiv gA^*A$. Particularly, for all $x \in \mathcal{D}_{H_0}$,

$$\|Vx\| \leq \|V\|\|X\| \leq \epsilon\|H_0x\| + \|V\|\|x\|$$

for all $\epsilon \in \mathbb{R}_0^+$; consequently, $\left(H_g, \mathcal{D}_{H_0}\right)$ is self-adjoint by the Kato-Rellich theorem. Let then $z \in \rho(H_0) \cap \rho\left(H_g\right)$ be arbitrary.

1. By direct inspection, the second resolvent formula allows for

$$\left[\mathbb{1}_{\mathcal{K}} + \phi(z)\right]\left[\mathbb{1}_{\mathcal{K}} + M(z)\right] = \mathbb{1}_{\mathcal{K}} = \left[\mathbb{1}_{\mathcal{K}} + M(z)\right]\left[\mathbb{1}_{\mathcal{K}} + \phi(z)\right].$$

2.

$$\begin{aligned}
R_{H_g}(z) &= R_{H_0}(z) + \left[R_{H_g}(z) - R_{H_0}(z)\right] = \text{(by the second resolvent formula)} \\
&= R_{H_0}(z) + gR_{H_0}(z)A^*AR_{H_g}(z) + gR_{H_0}(z)A^*AR_{H_0}(z) - gR_{H_0}(z)A^*AR_{H_0}(z) = \\
&= R_{H_0}(z) + gR_{H_0}(z)A^*AR_{H_0}(z) + gR_{H_0}(z)A^*A\left[R_{H_g}(z) - R_{H_0}(z)\right] = \\
&= \text{(by the second resolvent formula again)} = \\
&= R_{H_0}(z) + gR_{H_0}(z)A^*AR_{H_0}(z) + g^2R_{H_0}(z)A^*AR_{H_g}(z)A^*AR_{H_0}(z) = \\
&= R_{H_0}(z) + gR_{H_0}(z)A^*\left[\mathbb{1}_{\mathcal{K}} + M(z)\right]AR_{H_0}(z) \equiv \\
&= R_{H_0}(z) + gR_{H_0}(z)A^*\left[\mathbb{1}_{\mathcal{K}} - \phi(z)\right]^{-1}AR_{H_0}(z).
\end{aligned}$$

Such an expression allows to compute the resolvent of $\left(H_g, \mathcal{D}_{H_0}\right)$ at $z \in \rho(H_0) \cap \rho\left(H_g\right)$ only in terms of $R_{H_0}(z)$, A and A^*.

■

Corollary 6.9.1. *Given $n \in \mathbb{N}$, let $\mathcal{H}, \mathcal{K}_i$, $i = 1, \ldots, n$ be complex Hilbert spaces. Let $(H_0, \mathcal{D}_{H_0})$ be a self-adjoint operator on $\mathcal{H}$ and let $A_i : \mathcal{H} \longrightarrow \mathcal{K}_i$, $i = 1, \ldots, n$, be bounded operators. Given $g \in \mathbb{R} \setminus \{0\}$ and considered the self-adjoint operator $\left(H_g = H_0 - g\sum_{i=1}^n A_i^*A_i, \mathcal{D}_{H_0}\right)$ on $\mathcal{H}$, for all $z \in \rho(H_0) \cap \left(H_g\right)$,*

$$R_{H_g}(z) = R_{H_0}(z) + g\sum_{i,j=1}^n R_{H_0}(z)A_j^*\left[\Lambda(z)^{-1}\right]_{ji} A_iR_{H_0}(z) \tag{6.16}$$

where $\Lambda(z)_{ji} \doteq \delta_{ji} - gA_jR_{H_0}(z)A_i^* : \mathcal{K}_i \longrightarrow \mathcal{K}_j$.

Proof. By introducing the Hilbert space $\mathcal{K} \doteq \bigoplus_i \mathcal{K}_i$, let $A : \mathcal{H} \longrightarrow \mathcal{K}$ be the bounded operator such that $A\psi \doteq (A_i\psi)_{i=1}^n$, for all $\psi \in \mathcal{H}$. For all $\psi \in \mathcal{D}_{H_0}$,

$$H_g\psi = H_0\psi - g\sum_{i=1}^{n} A_i^* A_i\psi \equiv H_0\psi - gA^*A\psi,$$

and Proposition 6.9 can be applied. Straightforward computations allow to get formula (6.16). ■

2 - A Trace Operator

Lemma 6.10. *Let $\psi \in C_c^\infty\left(\mathbb{R}^{n+1} \simeq \mathbb{R} \times \mathbb{R}^n\right)$ be a real function. For all $x \in \mathbb{R}$, $\psi_x \in C_c^\infty\left(\mathbb{R}^n\right)$, where*

$$\psi_x : \boldsymbol{y} \in \mathbb{R}^n \longmapsto \psi_x\left(\boldsymbol{y}\right) \doteq \psi\left(x, \boldsymbol{y}\right) \in \mathbb{R}.$$

Proof. Let $K \equiv \operatorname{supp} \psi$ be and let $\pi_{\mathbb{R}^n}(K)$ be the $\mathbb{R}^n$ projection of K; by definition of product topology, $\pi_{\mathbb{R}^n}(K)$ is compact in $\mathbb{R}^n$. If $\mathbf{y} \in \mathbb{R}^n : \mathbf{y} \notin \pi_{\mathbb{R}^n}(K)$, $(x, \mathbf{y}) \notin K$, i.e. $\psi_x(\mathbf{y}) = 0$.
Given whatever $\mathbf{y} \in \mathbb{R}^n$, let $\{\mathbf{y}_n\}_n \subset \mathbb{R}^n$ be such that $\mathbf{y}_n \underset{n}{\longrightarrow} \mathbf{y}$; ψ_x is continuous at $\mathbf{y}$ if and only if $\lim\limits_n \psi_x(\mathbf{y}_n) = \psi_x(\mathbf{y})$. However, $(x, \mathbf{y}_n) \underset{n}{\longrightarrow} (x, \mathbf{y})$, hence the continuity of ψ implies $\psi_x(\mathbf{y}_n) \equiv \psi(x, \mathbf{y}) \underset{n}{\longrightarrow} \psi(x, \mathbf{y}) \equiv \psi_x(\mathbf{y})$; in other words, ψ_x is at least a continuous function of compact support.
Let then $\mathbf{y} \in \mathbb{R}^n$ be arbitrary; for all $j = 1, \ldots, n$, what follows holds.

$$\frac{\psi_x\left(\mathbf{y} + t\mathbf{e}_j\right) - \psi_x(\mathbf{y})}{t} = \frac{\psi\left(x, \mathbf{y} + t\mathbf{e}_j\right) - \psi(x, \mathbf{y})}{t} \underset{t \to 0}{\longrightarrow} \left(\frac{\partial \psi}{\partial \mathbf{f}_{j+1}}\right)(x, \mathbf{y})$$

i.e.

$$\left(\frac{\partial \psi_x}{\partial \mathbf{e}_j}\right)(\mathbf{y}) = \left(\frac{\partial \psi}{\partial \mathbf{f}_{j+1}}\right)(x, \mathbf{y}),$$

where

$$\mathbf{e}_j = \left(\underbrace{0, \ldots, \underbrace{1}_{j-th}, 0, \ldots, 0}_{n}\right), \quad \mathbf{f}_{j+1} = \left(0, \mathbf{e}_j\right).$$

The continuity of $\left(\partial_j \psi_x\right)$ is proved as above, hence induction gives $\psi_x \in C_c^\infty\left(\mathbb{R}^n\right)$. ∎

Remark 6.16. *Given ψ_x as above, $\psi_x \in C_c^\infty\left(\mathbb{R}^n\right)$ implies $\psi_x \in L^2\left(\mathbb{R}^n\right)$, hence*

$$\varphi : x \in \mathbb{R} \longmapsto \varphi(x) \doteq \int_{\mathbb{R}^n} \left|\psi_x(\boldsymbol{y})\right|^2 \, d\lambda^{(n)}(\boldsymbol{y}) \equiv \int_{\mathbb{R}^n} \psi(x, \boldsymbol{y})^2 \, d\lambda^{(n)}(\boldsymbol{y}) \in \mathbb{R}_0^+ \tag{6.17}$$

is well-defined. Particularly, set $K' = \pi_{\mathbb{R}^n}(K)$,

$$\varphi : x \in \mathbb{R} \longmapsto \varphi(x) \equiv \int_{K'} \psi_x^2(\boldsymbol{y}) \, d\lambda^{(n)}(\boldsymbol{y}).$$

□

Lemma 6.11. *Let $\psi \in C_c^\infty\left(\mathbb{R}^{n+1} \simeq \mathbb{R} \times \mathbb{R}^n\right)$ be a real function and φ as in (6.17), $\varphi \in C_c^\infty(\mathbb{R})$.*

Proof. Let $\pi_{\mathbb{R}}(K)$ be the compact projection of $K \equiv \operatorname{supp}\psi$ on the real line; if $x \notin \pi_{\mathbb{R}}(K)$, $(x, \mathbf{y}) \notin K$ for all $\mathbf{y} \in \mathbb{R}^n$, hence $\psi_x^2(\mathbf{y}) = 0$ and, correspondingly, $\varphi(x) = 0$.
Given arbitrarily $x \in \mathbb{R}$, let $\{x_n\}_n \subset \mathbb{R}$ be such that it converges to x. Since ψ^2 is continuous of compact support, there exists $C > 0$ such that $\left|\psi^2_{x_{(m)}(\mathbf{y})}\right| \leq C$ for all $\mathbf{y} \in \mathbb{R}^n$, $m \in \mathbb{N}$. The dominated convergence theorem then implies

$$\begin{aligned}\lim_m \varphi(x_m) = \lim_m \int_{K'} \psi^2_{x_m}(\mathbf{y})\, d\lambda^{(n)}(\mathbf{y}) = \int_{K'} \lim_m \left[\psi^2_{x_m}(\mathbf{y})\right] d\lambda^{(n)}(\mathbf{y}) = \\ = \int_{K'} \psi_x^2(\mathbf{y})\, d\lambda^{(n)}(\mathbf{y}) = \varphi(x),\end{aligned}$$

i.e., the arbitrarity of $x \in \mathbb{R}$ gives that φ is a continuous function of compact support.
$\left(\partial\psi_x^2\right) \in C_c^\infty(\mathbb{R}^{1+n})$, hence uniformly bounded over K, giving

$$\varphi'(x) = \int_{K'} \left(\frac{\partial\psi^2}{\partial x}\right)(x, \mathbf{y})\, d\lambda^{(n)}(\mathbf{y}), \quad \forall x \in \mathbb{R}.$$

By repeating the process, φ' is a continuous function of compact support; inductively, $\varphi \in C_0^\infty(\mathbb{R})$. ■

Lemma 6.12. *Let $\psi \in C_c^\infty\left(\mathbb{R}^{1+n}\right)$ be a complex function.*

$$\sup_{r\in\mathbb{R}} \int_{\mathbb{R}^n} \left|\psi(r, \boldsymbol{x})\right|^2 d\lambda^{(n)}(\boldsymbol{x}) \leq \|\psi\|^2_{H^1\left(\mathbb{R}^{n+1}\right)}$$

holds.

Proof. Let $r \in \mathbb{R}$ be. Trivially,

$$\int_{\mathbb{R}^n} \left|\psi(r, \mathbf{x})\right|^2 d\lambda^{(n)}(\mathbf{x}) = \int_{\mathbb{R}^n} \psi_R^2(r, \mathbf{x})\, d\lambda^{(n)}(\mathbf{x}) + \int_{\mathbb{R}^n} \psi_I^2(r, \mathbf{x})\, d\lambda^{(n)}(\mathbf{x}),$$

where $\psi_R \equiv \Re\psi$ and $\psi_I \equiv \Im\psi$. Then

$$\begin{aligned}&\int_{\mathbb{R}^n} \psi_i^2(r, \mathbf{x})\, d\lambda^{(n)}(\mathbf{x}) \equiv \varphi(r) = \int_{-\infty}^{r} \varphi'(s)\, d\lambda^{(1)}(s) \leq \int_{\mathbb{R}} \left|\varphi'(s)\right| d\lambda^{(1)}(s) \equiv \\ &\equiv \int_{\mathbb{R}} \left|\left[\frac{d}{dr} \int_{\mathbb{R}^n} \psi_i^2(r, \mathbf{x})\, d\lambda^{(n)}(\mathbf{x})\right](s)\right| d\lambda^{(1)}(s) \leq \\ &\leq 2 \int_{\mathbb{R}^{n+1}} \left|\psi_i(\mathbf{y})\left(\partial_r\psi_i\right)(\mathbf{y})\right| d\lambda^{(n+1)}(\mathbf{y}) \leq (\text{by using Hölder inequality}) \leq 2\|\psi_i\|_2 \|\partial_r\psi_i\|_2 \leq \\ &\leq \|\psi_i\|_2^2 + \left\|\left(\partial_r\psi\right)_i\right\|_2^2\end{aligned}$$

with $i = R, I$, therefore

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\psi(r, \mathbf{x})\right|^2 d\lambda^{(n)}(\mathbf{x}) \leq \left(\|\psi_R\|_2^2 + \|\psi_I\|_2^2\right) + \left(\left\|\left(\partial_r\psi\right)_R\right\|_2^2 + \left\|\left(\partial_r\psi\right)_I\right\|_2^2\right) \equiv \\ \equiv \|\psi\|_2^2 + \|\partial_r\psi\|_2^2 \leq \|\psi\|_2^2 + \left\|\|\nabla\psi\|\right\|_2^2 \equiv \|\psi\|^2_{H^1\left(\mathbb{R}^{n+1}\right)}.\end{aligned}$$

The left hand side of the foregoing inequality is independent of $r \in \mathbb{R}$, hence

$$\sup_r \int_{\mathbb{R}^n} \left|\psi(r, \mathbf{x})\right|^2 d\lambda^{(n)}(\mathbf{x}) \leq \|\psi\|^2_{H^1\left(\mathbb{R}^{n+1}\right)}.$$

■

Proposition 6.13. *The map* $\tilde{\tau}_0 : \psi \in C_c^\infty\left(\mathbb{R}^{1+n}\right) \longmapsto \tilde{\tau}_0\psi \in L^2\left(\mathbb{R}^n\right)$, *with* $\left(\tilde{\tau}_0\psi\right)(\boldsymbol{x}) = \psi(0,\boldsymbol{x})$, *for all* $\boldsymbol{x} \in \mathbb{R}^n$, *results in a linear, densely defined bounded operator from* $H^1\left(\mathbb{R}^{n+1}\right)$ *to* $L^2\left(\mathbb{R}^n\right)$.

Proof. $\tilde{\tau}_0$ is clearly well-defined, linear and densely defined. Concerning boundedness, one has

$$\begin{aligned}\left\|\tilde{\tau}_0\psi\right\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left|\left(\tilde{\tau}_0\psi\right)(\mathbf{x})\right|^2 d\lambda^{(n)}(\mathbf{x}) = \int_{\mathbb{R}^n} \left|\psi\left(0,\mathbf{x}\right)\right|^2 d\lambda^{(n)}(\mathbf{x}) \leq \\ &\leq \sup_{r\in\mathbb{R}} \int_{\mathbb{R}^n} \left|\psi\left(r,\mathbf{x}\right)\right|^2 d\lambda^{(n)}(\mathbf{x}) \leq \left\|\psi\right\|^2_{H^1\left(\mathbb{R}^{n+1}\right)}.\end{aligned}$$

■

Remark 6.17. *The foregoing proposition allows for a bounded, norm-preserving extension* τ_0 *of* $\tilde{\tau}_0$ *to all* $H^1\left(\mathbb{R}^{n+1}\right)$. □

Definition 6.9. Given $n \in \mathbb{N}$ and $i, j \in \{1, \ldots, n\} :\ i < j$, by considering $U_{(ij)}$ as in (6.4), $\tau_{(ij)} \doteq \tau_0 U_{(ij)}$ denotes the corresponding **trace operator**. □

3 - Two-body Delta Interactions - Quadratic Form Investigation

Proposition 6.14. *Given $n \in \mathbb{N} : n \geq 2$, let $m_1, \ldots, m_n \in \mathbb{R}^+$ be and correspondingly $a_j = (2m_j)^{-1}$, $j = 1, \ldots, n$. Consider, further, $g \in \mathbb{R} \setminus \{0\}$; the map $(t, \mathcal{D}_t)$, such that $\mathcal{D}_t = H^1(\mathbb{R}^n)$ and $t : (\varphi, \psi) \in \mathcal{D}_t \times \mathcal{D}_t \longmapsto t(\varphi, \psi) \in \mathbb{C}$ with*

$$t(\varphi, \psi) = \sum_{i=1}^{n} a_j \int_{\mathbb{R}^n} \left[\overline{\frac{\partial \varphi(\boldsymbol{x})}{\partial x_j}} \frac{\partial \psi(\boldsymbol{x})}{\partial x_j} \right] d\lambda^{(n)}(\boldsymbol{x}) - g \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^{n-1}} \left[\overline{\left(\tau_{(ij)}\varphi\right)}(\boldsymbol{x}) \left(\tau_{(ij)}\psi\right)(\boldsymbol{x}) \right] d\lambda^{(n-1)}(\boldsymbol{x})$$

results in a sesquilinear, densely defined, hermitian, lower semi-bounded, closed form on $L^2(\mathbb{R}^n)$.

Proof. Clearly $H^1(\mathbb{R}^n)$ is dense in $\left(L^2(\mathbb{R}^n), \|\cdot\|_{L^2(\mathbb{R}^n)}\right)$. Now, given $\varphi, \psi \in H^1(\mathbb{R}^n)$,

$$\begin{aligned} t(\varphi, \psi) &= \sum_{i=1}^{n} a_j \int_{\mathbb{R}^n} \overline{(\partial_j \varphi)} (\partial_j \psi) - g \sum_{i<j} \int_{\mathbb{R}^{n-1}} \overline{(\tau_{(ij)}\varphi)} (\tau_{(ij)}\psi) = \\ &= \overline{\sum_{i=1}^{n} a_j \int_{\mathbb{R}^n} \overline{(\partial_j \psi)} (\partial_j \varphi) - g \sum_{i<j} \int_{\mathbb{R}^{n-1}} \overline{(\tau_{(ij)}\psi)} (\tau_{(ij)}\varphi)} = \overline{t(\psi, \varphi)}, \end{aligned}$$

i.e. t is hermitian. To prove it is lower semi-bounded, let $\psi \in H^1(\mathbb{R}^n)$ be.

$$\begin{aligned} q_t(\psi) &= \sum_{j=1}^{n} a_j \int_{\mathbb{R}^n} \left|\partial_j \psi\right|^2 - g \sum_{i<j} \int_{\mathbb{R}^{n-1}} \left|\tau_{(ij)}\psi\right|^2 \geq \left(a \equiv \min\{a_1, \ldots, a_n\}\right) \\ &\geq a \int_{\mathbb{R}^n} \|\nabla \psi\|^2 - g \sum_{i<j} \left\|\tau_{(ij)}\psi\right\|^2_{L^2\left(\mathbb{R}^{n-1}\right)}. \end{aligned}$$

If $g < 0$, then

$$q_t(\psi) \geq a \int_{\mathbb{R}^n} \|\nabla \psi\|^2 + |g| \sum_{i<j} \left\|\tau_{(ij)}\psi\right\|^2_{L^2\left(\mathbb{R}^{n-1}\right)} \geq 0 = 0 \, \|\psi\|^2_{H^1(\mathbb{R}^n)}.$$

On the other hand, if $g > 0$, one starts observing that, given $\mu > 0$, there exists $C_\mu > 0$ such that $\left\|\tau_{(ij)}\psi\right\| \leq \mu \big\| \|\nabla \psi\| \big\| + C_\mu \|\psi\|$ for all $i, j \in \{1, \ldots, n\} : i < j$. Consequently

$$g \sum_{i<j} \left\|\tau_{(ij)}\psi\right\|^2 \leq gn(n-1) \left[\mu^2 \big\|\|\nabla \psi\|\big\|^2 + C_\mu^2 \|\psi\|^2_{H^1(\mathbb{R}^n)}\right]$$

and

$$q_t(\psi) \geq \left[a - gn(n-1)\mu^2\right] \big\|\|\nabla \psi\|\big\|^2 - gn(n-1) C_\mu^2 \|\psi\|^2_{H^1(\mathbb{R}^n)}.$$

Therefore, by choosing $\mu = \sqrt{a\left[gn(n-1)\right]^{-1}}$, one obtains

$$q_t(\psi) \geq \left[-gn(n-1)C_\mu^2\right]\|\psi\|_{H^1}^2 \equiv m(g,n,a)\|\psi\|_{H^1}^2\,,$$

allowing to state that, for all $g \in \mathbb{R}\setminus\{0\}$, t is lower semi-bounded. Eventually, given $\psi \in H^1(\mathbb{R}^n)$, what follows holds.

- $\boxed{g<0}$

$$\begin{aligned}
q_t(\psi) &= \sum_{i=1}^n a_j\left\|\partial_j\psi\right\|_{L^2(\mathbb{R}^n)}^2 + |g|\sum_{i<j}\left\|\tau_{(ij)}\psi\right\|_{L^2\left(\mathbb{R}^{n-1}\right)}^2 \leq \left(A \equiv \max_j a_j, \quad K = \max_{(ij)}\left\|\tau_{(ij)}\right\|^2\right)\\
&\leq A\sum_{j=1}^n\left\|\partial_j\psi\right\|_{L^2(\mathbb{R}^n)}^2 + \frac{|g|\,n(n-1)K}{2}\|\psi\|_{H^1(\mathbb{R}^n)}^2 \leq \left(B \equiv \max\left\{A,\ \frac{|g|\,n(n-1)K}{2}\right\}\right)\\
&\leq (2B)\,\|\psi\|_{H^1(\mathbb{R}^n)}^2\,.
\end{aligned}$$

However, by recalling that $a = \min_j a_j$, one has

$$\begin{aligned}
\|\psi\|_{H^1(\mathbb{R}^n)}^2 &= \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \sum_{i=1}^n\|\partial_i\psi\|_{L^2(\mathbb{R}^n)}^2 \leq \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{a}\left[\sum_{i=1}^n a_i\|\partial_i\psi\|_{L^2(\mathbb{R}^n)}^2\right] \leq\\
&\leq \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{a}\left[\sum_{i=1}^n a_i\|\partial_i\psi\|_{L^2(\mathbb{R}^n)}^2 - g\sum_{i<j}\left\|\tau_{(ij)}\psi\right\|_{L^2\left(\mathbb{R}^{n-1}\right)}^2\right] \leq\\
&\leq \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{a}\,q_t\left(\psi\right).
\end{aligned}$$

- $\boxed{g>0}$

$$\begin{aligned}
q_t(\psi) &= \sum_{j=1}^n a_j\left\|\partial_j\psi\right\|_{L^2(\mathbb{R}^n)}^2 - g\sum_{i<j}\left\|\tau_{(ij)}\psi\right\|_{L^2\left(\mathbb{R}^{n-1}\right)}^2 \leq\\
&\leq \sum_{j=1}^n a_j\left\|\partial_j\psi\right\|_{L^2(\mathbb{R}^n)}^2 \leq A\,\|\psi\|_{H^1(\mathbb{R}^n)}^2\,,
\end{aligned}$$

Then,

$$\begin{aligned}
\|\psi\|_{H^1(\mathbb{R}^n)}^2 &= \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \sum_{j=1}^n\left\|\partial_j\psi\right\|_{L^2(\mathbb{R}^n)}^2 \leq \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{a}\sum_{j=1}^n a_j\left\|\partial_j\psi\right\|_{L^2(\mathbb{R}^n)}^2 \equiv\\
&\equiv \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{a}\left[q_t\left(\psi\right) + g\sum_{i<j}\left\|\tau_{(ij)}\psi\right\|_{L^2\left(\mathbb{R}^{n-1}\right)}^2\right] \leq\\
&\leq \|\psi\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{a}\,q_t\left(\psi\right) + \frac{n(n-1)K}{2a}\|\psi\|_{H^1(\mathbb{R}^n)}^2
\end{aligned}$$

leading to $\left[1-\frac{Kn(n-1)}{2a}\right]\|\psi\|_{H^1(\mathbb{R}^n)}^2 \leq \|\psi\|_{L^2\left((\mathbb{R}^n)\right)}^2 + \frac{1}{a}\,q_t(\psi)$.

■

Remark 6.18. *The foregoing proposition guarantees the existence of a unique self-adjoint operator $(H,\mathcal{D}_H)$ on $L^2(\mathbb{R}^n)$, to be understood as the Hamiltonian associated to the system considered in chapter 6, whose corresponding sesquilinear form is $(t,\mathcal{D}_t)$.* □

4 - Boundedness Results

Proposition 6.15. *Let* $F : \psi \in L^2\left(\mathbb{R}^2, d\overline{R}_\nu d\overline{x}_2\right) \longmapsto F\psi \in L^2\left(\mathbb{R}^2, dR_\sigma d\overline{x}_{\nu_2}\right)$ *be the linear operator defined via the position*

$$[F\psi]\left(R_\sigma, \overline{x}_{\nu_2}\right) \equiv \int_{\mathbb{R}^2} d\overline{R}'_\nu d\overline{x}'_2 \Bigg[G_z^{(3)}\left(\sqrt{2m_1}\left(R_\sigma - \overline{R}'_\nu,\right), \sqrt{2m_2}\left(R_\sigma - \overline{x}'_2\right), \sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2} - \overline{R}'_\nu\right)\right) \psi\left(\overline{R}'_\nu, \overline{x}'_2\right)\Bigg]$$

with $z < 0$. *Then,* F *is bounded.*

Proof. The Schur test will be employed.

$$\begin{aligned}
&G_z^{(3)}\left(\sqrt{2m_1}\left(R_\sigma - \overline{R}'_\nu,\right), \sqrt{2m_2}\left(R_\sigma - \overline{x}'_2\right), \sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2} - \overline{R}'_\nu\right)\right) = \\
&= \int_0^\infty e^{-\frac{2m_1\left(R_\sigma - \overline{R}'_\nu\right)^2 + 2m_2\left(R_\sigma - \overline{x}'_2\right)^2 + 2m_{\nu_2}\left(\overline{x}_{\nu_2} - \overline{R}'_\nu\right)^2}{4t} + zt} \frac{dt}{(4\pi t)^{\frac{3}{2}}} \leq \left(m \equiv \min\left(m_1, m_2, m_{\nu_2}\right)\right) \leq \\
&\leq \int_0^\infty e^{-\frac{2m\left(R_\sigma - \overline{R}'_\nu\right)^2 + 2m\left(R_\sigma - \overline{x}'_2\right)^2 + 2m\left(\overline{x}_{\nu_2} - \overline{R}'_\nu\right)^2}{4t} + zt} \frac{dt}{(4\pi t)^{\frac{3}{2}}} \equiv \\
&\equiv \int_0^\infty e^{-\frac{\left(x - x'\right)^2 + \left(x - y'\right)^2 + \left(y - x'\right)^2}{4t} + zt} \frac{dt}{(4\pi t)^{\frac{3}{2}}} \equiv K\left(x, y; x', y'\right),
\end{aligned}$$

by having set

$$\begin{cases} x &= \sqrt{2m}\, R_\sigma \\ x' &= \sqrt{2m}\, \overline{R}'_\nu \\ y' &= \sqrt{2m}\, \overline{x}'_2 \\ y &= \sqrt{2m}\, \overline{x}_{\nu_2} \end{cases}.$$

Trivially, $K\left(x, y;\, x', y'\right) = K\left(x', y';\, x, y\right)$. On the other hand

$$\int_{\mathbb{R}^2} K\left(x, y;\, x', y'\right) dx'dy' = \int_{\mathbb{R}^2} \frac{e^{-\alpha\sqrt{(x-x')^2 + (y-x')^2 + (x-y')^2}}}{4\pi\sqrt{\left(x - x'\right)^2 + \left(y - x'\right)^2 + \left(x - y'\right)^2}} dx'dy',$$

i.e. it does not exist whenever $x = y = x' = y'$. Since $\left\{(x, y) \in \mathbb{R}^2 | x = y\right\}$ is a set of $\lambda^{(2)}-$measure zero, let $x \neq y$ and $\alpha \equiv \sqrt{|z|}$ be. The coordinate transformation

$$\begin{cases} \overline{x}' = x' - \frac{x+y}{2} \\ \overline{y}' = \frac{y' - x}{\sqrt{2}} \end{cases} \iff \begin{cases} x' = \overline{x}' + \frac{x+y}{2} \\ y' = \sqrt{2}\overline{y}' + x \end{cases},$$

that gives $dx'dy' = \sqrt{2}d\overline{x}'d\overline{y}'$ and $\sqrt{(x-x')^2+(y-x')^2+(x-y')^2} = \sqrt{2\left[\overline{x}'^2+\overline{y}'^2+\frac{(y-x)^2}{4}\right]}$, allows for

$$\int_{\mathbb{R}^2} \left|K\left(x,y;x',y'\right)\right| dx'dy' = \int_{\mathbb{R}^2} \frac{e^{-\sqrt{2}\alpha\sqrt{\overline{x}'^2+\overline{y}'^2+\frac{(y-x)^2}{4}}}}{\sqrt{\overline{x}'^2+\overline{y}'^2+\frac{(y-x)^2}{4}}}\frac{dx'dy'}{4\pi} = \text{(by integrating in polar coordinates)} =$$

$$= \frac{1}{4\pi}\int_0^\infty\int_0^{2\pi} \frac{e^{-\alpha\sqrt{2}\sqrt{\rho^2+\frac{(y-x)^2}{4}}}}{\sqrt{\rho^2+\frac{(y-x)^2}{4}}}\rho d\rho d\theta \equiv \frac{1}{2}\int_0^\infty \frac{e^{-\alpha\sqrt{2}\sqrt{\rho^2+\frac{(y-x)^2}{4}}}}{\sqrt{\rho^2+\frac{(y-x)^2}{4}}}\rho d\rho <$$

$$< \frac{1}{2}\int_0^\infty e^{-\alpha\sqrt{2}\rho}d\rho = \frac{1}{2\sqrt{2|z|}}.$$

In the end, $\|F\| \leq \frac{1}{2\sqrt{2|z|}}$, by the independence on $(x,y)\in\mathbb{R}^2 : x\neq y$ of the right hand side. ■

Proposition 6.16. *Let* $B : \varphi \in L^2\left(\mathbb{R}^3,\, d\overline{R}_\nu d\overline{x}_1 d\overline{x}_2\right) \longmapsto B\varphi \in L^2\left(\mathbb{R}^3, dR_\sigma d\overline{x}_{\nu_1} d\overline{x}_{\nu_2}\right)$ *be the linear operator defined by*

$$[B\varphi]\left(R_\sigma,\overline{x}_{\nu_1},\overline{x}_{\nu_2}\right) =$$
$$= \int_{\mathbb{R}^3} d\overline{R}'_\nu d\overline{x}'_1 d\overline{x}'_2 \left[G_z^{(4)}\left(\sqrt{2m_1}\left(R_\sigma-\overline{x}'_1\right),\, \sqrt{2m_2}\left(R_\sigma-\overline{x}'_2\right),\, \sqrt{2m_{\nu_1}}\left(\overline{x}_{\nu_1}-\overline{R}'_\nu\right),\, \sqrt{2m_{\nu_2}}\left(\overline{x}_{\nu_2}-\overline{R}'_\nu\right)\right)\right.$$
$$\left.\varphi\left(\overline{R}'_\nu,\, \overline{x}'_1,\, \overline{x}'_2\right)\right]$$

for all $z<0$. *B is a bounded operator.*

Proof. By proceeding as in proposition 6.15, it does not harm generality focusing on

$$K\left(x,y,w;\, x',y',w'\right) = G_z^{(4)}\left(x-x',y-x',w-y',w-w'\right).$$

Then

$$\int_{\mathbb{R}^3} dx'dy'dw' \left|G_z^{(4)}\left(x-x',y-x',w-y',w-w'\right)\right| =$$
$$= \int_{\mathbb{R}^3} dx'dy'dw' \int_0^\infty \frac{dt}{(4\pi t)^2} \exp\left\{-\frac{(x-x')^2+(y-x')^2+(w-y')^2+(w-w')^2}{4t}+zt\right\}.$$

The following coordinate transformation is considered

$$\begin{cases} \overline{x}' &= x' - \frac{x+y}{2} \\ \overline{y}' &= \frac{y'-w}{\sqrt{2}} \\ \overline{w}' &= \frac{w'-w}{\sqrt{2}} \end{cases} \iff \begin{cases} x' &= \overline{x}' + \frac{x+y}{2} \\ y' &= \sqrt{2}\overline{y}' + w \\ w' &= \sqrt{2}\overline{w}' + w \end{cases}.$$

On the one hand, $dx'dy'dw' = 2d\overline{x}'d\overline{y}'d\overline{w}'$ and

$$\left(x-x'\right)^2+\left(y-x'\right)^2+\left(w-y'\right)^2+\left(w-w'\right)^2 = 2\left[\overline{x}'^2+\overline{y}'^2+\overline{w}'^2+\frac{(y-x)^2}{4}\right],$$

on the other hand

$$\int_{\mathbb{R}^3} dx'dy'dw' \int_0^\infty \frac{dt}{(4\pi t)^2} \exp\left\{-\frac{\left(x-x'\right)^2+\left(y-x'\right)^2+\left(w-y'\right)^2+\left(w-w'\right)^2}{4t}+zt\right\}=$$

$$=\int_{\mathbb{R}^3} d\overline{x}'d\overline{y}'d\overline{w}' \int_0^\infty \frac{dt}{8\pi^2t^2}\, \exp\left\{-\frac{\overline{x}'^{\,2}+\overline{y}'^{\,2}+\overline{w}'^{\,2}+\frac{(y-x)^2}{4}}{2t}+zt\right\}=$$

$$=\int_0^\infty \rho^2 d\rho \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty \frac{dt}{8\pi^2t^2}\, \exp\left\{-\frac{\rho^2}{2t}\right\}\, \exp\left\{-\frac{(y-x)^2}{8t}\right\} \exp\left\{zt\right\} \leq$$

$$\leq \int_0^\infty \frac{dt}{2\pi t^2}\, e^{zt} \left[\int_0^\infty d\rho\, \rho^2 \exp\left\{-\frac{\rho^2}{2t}\right\}\right] = \frac{\sqrt{\pi}}{4} \int_0^\infty \frac{dt}{2\pi}\, \frac{e^{zt}}{t^2}\, (2t)^{\frac{3}{2}} = \frac{1}{2\sqrt{2|z|}} < \infty$$

holds. The Schur test then states that $\|B\| \leq \left(2\sqrt{2|z|}\right)^{-1}$. ■

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