



# Stability of axial free-boundary hyperplanes in circular cones

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Received: 12 March 2025 / Accepted: 11 September 2025  
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## Abstract

Given an axially-symmetric,  $(n + 1)$ -dimensional convex cone  $\Omega \subset \mathbb{R}^{n+1}$ , we study the stability of the free-boundary minimal surface  $\Sigma$  obtained by intersecting  $\Omega$  with a  $n$ -plane that contains the axis of  $\Omega$ . In the case  $n = 2$ ,  $\Sigma$  is always unstable, as a special case of the vertex-skipping property that we recently proved in another article. Conversely, as soon as  $n \geq 3$  and  $\Omega$  has a sufficiently large aperture (depending on the dimension  $n$ ), we show that  $\Sigma$  is strictly stable. For our stability analysis, we introduce a Lipschitz flow  $\Sigma_t[f]$  of deformations of  $\Sigma$  associated with a compactly-supported, scalar deformation field  $f$ , which satisfies the key property  $\partial \Sigma_t[f] \subset \partial \Omega$  for all  $t \in \mathbb{R}$ . Then, we compute the lower-right second variation of the area of  $\Sigma$  along the flow, and ultimately show that it is positive by exploiting its connection with a functional inequality studied in the context of reaction-diffusion problems.

**Mathematics Subject Classification** Primary: 49Q05 · Secondary: 49Q10

## 1 Introduction

One of the most intriguing aspects of the theory of free-boundary minimal surfaces, and more generally of capillary surfaces, is their behaviour near the boundary of their container. A well-established regularity theory is known to hold when the free boundary of a minimal

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Communicated by A. Mondino.

G.P.Leonardi has been partially supported by: PRIN 2017TEXA3H “Gradient flows, Optimal Transport and Metric Measure Structures”; PRIN 2022PJ9EFL “Geometric Measure Theory: Structure of Singular Measures, Regularity Theory and Applications in the Calculus of Variations” (financed by European Union - Next Generation EU, Mission 4, Component 2 - CUP:E53D23005860006); Grant PID2020-118180GB-I00 “Geometric Variational Problems”. Giacomo Vianello has been supported by GNAMPA (INdAM) Project 2023: “Esistenza e proprietà fini di forme ottime”; GNAMPA (INdAM) Project 2025: “Structures of sub-Riemannian hypersurfaces in Heisenberg groups”.

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surface  $\Sigma$  meets a regular point of the boundary of its container  $\Omega$ , see [4, 8]; in this case the 90-degree contact-angle law holds true in a classical sense. More generally, Young's law for capillary surfaces [4, 20] holds. Conversely, the behaviour of  $\Sigma$  near a generic point of  $\partial\Sigma \cap \partial^s\Omega$ , where  $\partial^s\Omega$  denotes the set of singular points of  $\partial\Omega$ , is much less known. Few exceptions are represented by the studies on free-boundary minimal surfaces in 3-dimensional wedge domains, mainly due to Hildebrandt-Sauvigny [10–13], and by the analogous studies pursued in the context of capillary surfaces, see e.g. [6, 14] and the references therein.

A few years ago, Edelen-Li [5] extended the existing regularity theory by proving a very general  $\varepsilon$ -regularity theorem for free-boundary integral varifolds with bounded mean curvature in locally-almost-polyhedral domains. Subsequently, with the aim of pushing the regularity theory forward and beyond the locally almost-polyhedral class, in a couple of recent works we have shown some sort of complementary results in the framework of sets of finite perimeter. The first result is a free-boundary monotonicity formula for perimeter almost-minimizers in a non-smooth domain which satisfies a so-called *visibility condition* [15]. The second is a *vertex-skipping property* for almost-minimal boundaries in convex sets [16], which can be coupled with [5] and leads to a full, free-boundary regularity result for 2-dimensional integral varifolds with bounded mean curvature in locally almost-polyhedral 3d-domains.

Blow-up techniques are among the main tools used to retrieve information on the boundary properties of  $\Sigma$ , such as local structure, density properties, and ultimately regularity. Relying on boundary monotonicity formulas (see, e.g., [15]), one can blow-up an almost-minimal surface at one of its free-boundary points, obtaining free-boundary cones inside conical domains. This represents the free-boundary counterpart of classical monotonicity and blow-up results that hold at interior points [7, 17]. By relying on such results, we have shown in [16] that the internal boundary  $\Sigma$  of a perimeter almost-minimizer in a convex set  $\Omega \subset \mathbb{R}^3$  avoids all vertices of  $\partial\Omega$ . Here, by vertex we mean a point  $x \in \partial\Omega$  such that the tangent cone to  $\Omega$  at  $x$  does not contain full lines. It is then natural to ask whether this *vertex-skipping property* is a special feature of the 3-dimensional setting, or if instead it holds in any dimension. We remark that, in dimension 2, the property is trivially satisfied and, for this reason, this elementary case will not be discussed.

Here, we consider a  $n$ -plane in  $\mathbb{R}^{n+1}$ ,  $n \geq 2$ , containing the axis of a convex circular  $(n+1)$ -cone, and study the stability of the free-boundary minimal surface obtained by intersecting the plane with the cone. More precisely, we take  $0 < \lambda < \infty$ , then we let  $x' = (x_1, \dots, x_{n-1})$  and set

$$\Omega_\lambda := \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_n > \omega_\lambda(x', x_{n+1})\},$$

where  $\omega_\lambda(x', x_{n+1}) = \lambda \sqrt{|x'|^2 + x_{n+1}^2}$ . For future reference, we call  $\lambda$  the aperture parameter of the cone, while the angle  $\alpha_\lambda \in (0, \pi)$  such that  $\lambda = \cot(\alpha_\lambda/2)$  will be referred to as the *aperture* of the cone. Then, we define

$$\Sigma = \{x \in \Omega_\lambda : x_{n+1} = 0\}.$$

Section 3 contains the main results of the paper. Specifically, Theorem 3.4 shows that, when  $n = 2$ ,  $\Sigma$  is unstable in  $\Omega_\lambda$  for every  $0 < \lambda < \infty$ . We highlight that this is exactly the behaviour one should expect according to the vertex-skipping property proved in [16]. More interestingly, Theorem 3.5 establishes that, when  $n \geq 3$ , there exists a threshold aperture parameter  $0 < \lambda^* < \infty$  depending on the dimension  $n$ , such that, for all  $0 < \lambda \leq \lambda^*$  (i.e., for all apertures  $\alpha \in [\alpha_{\lambda^*}, \pi)$ ), the  $n$ -plane  $\Sigma$  is a strictly stable free-boundary minimal surface in  $\Omega_\lambda$ . The proof of this theorem relies on two main tools. The first is the construction

of a suitable Lipschitz flow of deformations that preserves the free-boundary condition and allows a direct writing of first and second variation (in)equalities, see Section 2. The second is a variational inequality, called *Kato's inequality*, which is the key to show stability under appropriate conditions on the dimension  $n$  and the aperture of  $\Omega_\lambda$ , see Lemma 3.6. We point out that such inequality emerges in fairly different contexts, from spectral theory [9] to reaction-diffusion problems [3].

The stability versus instability properties of  $\Sigma$  exhibit a close relationship with analogous properties of minimal surfaces within cones. In particular, a result of Morgan [18] implies that the free-boundary  $\partial\Sigma$  is an area-minimizing  $(n-1)$ -surface in  $\partial\Omega_\lambda$  as soon as  $n \geq 4$  and  $\lambda$  is small enough. Similar results are also available for free-boundary problems of Alt-Caffarelli type [1], where the dimension  $n = 3$  is critical for the free-boundary to contain the vertex of  $\Omega_\lambda$ , again when  $\lambda$  is small enough.

Going back to [18], we note that its main result is based on a calibration argument that holds precisely when  $n \geq 4$ . For this reason, we expect that our strict stability result can be improved to a full area-minimizing result at least when  $n \geq 4$ . However, the same kind of analysis in the case  $n = 3$  seems a harder open problem, as it is not fully clear whether  $\partial\Sigma$  is unstable, or at least not area-minimizing, in  $\partial\Omega_\lambda$ . Our Theorem 3.5 seems to support the conjecture that  $\Sigma$  is locally area-minimizing also when  $n = 3$ . The issue of the area-minimization property when  $n \geq 3$  will be addressed in a forthcoming work.

## 2 Notations and basic constructions

Let  $n \geq 2$  be a given integer. Given  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ , we define  $\tilde{x} = (x, t) \in \mathbb{R}^{n+1}$  and  $x' \in \mathbb{R}^{n-1}$  such that  $x = (x', x_n)$ , with a slight abuse of notation. We identify  $x$  with  $(x, 0) \in \mathbb{R}^{n+1}$  and  $x'$  with  $(x', 0, 0) \in \mathbb{R}^{n+1}$ , whenever this does not create confusion. We denote by  $\tilde{B}_r(\tilde{x})$  the open ball of radius  $r > 0$  centered at  $\tilde{x}$ , and we set  $\tilde{B}_r := \tilde{B}_r(0)$ . Given a differentiable function  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , we denote the partial derivative of  $f$  with respect to  $x_i$  as  $\partial_i f$ .

For any  $1 \leq i \leq n+1$ , we denote by  $p_i$  the hyperplane of  $\mathbb{R}^{n+1}$  of equation  $x_i = 0$ , and use the same notation for the orthogonal projection of  $\mathbb{R}^{n+1}$  onto this hyperplane with a slight abuse of notation.

### 2.1 Foliated a Lipschitz epigraph

Given a Lipschitz function  $\omega = \omega(x', t) : \mathbb{R}^n \rightarrow \mathbb{R}$ , we consider the Lipschitz epigraph

$$\Omega = \{\tilde{x} = (x', x_n, t) \in \mathbb{R}^{n+1} : x_n > \omega(x', t)\}. \quad (2.1)$$

We also let  $\Sigma := \Omega \cap p_{n+1}$  and set  $\partial\Sigma = \overline{\Sigma} \cap \partial\Omega$ . Given  $x = (x', x_n) \in \overline{\Sigma}$ , we define the parametric curve  $\Gamma_x : \mathbb{R} \rightarrow \mathbb{R}^{n+1}$  as

$$\Gamma_x(t) := (x', x_n + \omega(x', t) - \omega(x', 0), t) \quad (2.2)$$

and, with a slight abuse of notation, we identify  $\Gamma_x$  with  $\Gamma_x(\mathbb{R})$ .

The next lemma collects some key properties of the family  $\{\Gamma_x : x \in \overline{\Sigma}\}$ .

**Lemma 2.1** *The family of parametric curves  $\{\Gamma_x : x \in \overline{\Sigma}\}$  defines a foliation of  $\overline{\Omega}$ . More precisely, the following properties hold:*

- (i)  $\Gamma_x(0) = x$  and  $\Gamma_x \subset \overline{\Omega}$ , for all  $x \in \overline{\Sigma}$ ;
- (ii) if  $x \neq y \in \overline{\Sigma}$  then  $\Gamma_x \cap \Gamma_y = \emptyset$ ;
- (iii)  $\Gamma_x(t) \in \partial\Omega$  for some  $t \in \mathbb{R}$  if and only if  $\Gamma_x \subset \partial\Omega$ .
- (iv) the map  $(x, t) \mapsto \Gamma_x(t)$  is Lipschitz.

**Proof** Property (i) directly follows from the definition of  $\Gamma_x$  and by observing that  $x \in \overline{\Sigma}$  implies  $x_n \geq \omega(x', 0)$ , hence

$$x_n + \omega(x', t) - \omega(x', 0) \geq \omega(x', t).$$

Then, concerning property (ii), if  $\Gamma_x(t) = \Gamma_y(u)$  for some  $x, y \in \overline{\Sigma}$  and some  $t, u \in \mathbb{R}$ , by definition we must have  $t = u$ ,  $x' = y'$ , and

$$x_n + (\omega(x', t) - \omega(x', 0)) = y_n + (\omega(y', u) - \omega(y', 0)),$$

which implies  $x_n = y_n$  and hence  $x = y$ , which proves (ii). The less obvious implication of (iii) is the only if part: if  $\Gamma_x(t) \in \partial\Omega$  for some  $t \in \mathbb{R}$ , then

$$x_n + \omega(x', t) - \omega(x', 0) = \omega(x', t),$$

i.e.,  $x_n = \omega(x', 0)$ . Consequently,

$$\Gamma_x(u) = (x', \omega(x', 0) + \omega(x', u) - \omega(x', 0), u) = (x', \omega(x', u), u) \in \partial\Omega$$

for all  $u \in \mathbb{R}$ . Finally, the proof of (iv) easily follows from the estimate

$$\begin{aligned} |\Gamma_x(t) - \Gamma_y(u)| &\leq |x' - y'| + |x_n - y_n| + |\omega(x', t) - \omega(y', u)| \\ &\quad + |\omega(x', 0) - \omega(y', 0)| + |t - u| \\ &\leq (1 + 2\text{Lip}(\omega))(|x' - y'| + |x_n - y_n| + |t - u|). \end{aligned}$$

□

## 2.2 A flow of compact deformations for $\Sigma$ .

Let now  $f : \overline{\Sigma} \rightarrow \mathbb{R}$  be a Lipschitz function whose support  $\text{spt}(f)$  is compact in  $\overline{\Sigma}$ . We define the flow associated to  $f$  as

$$\Phi[f](x, t) := \Gamma_x(tf(x)) \quad (2.3)$$

for all  $(x, t) \in \overline{\Sigma} \times \mathbb{R}$ . The map  $\Phi[f]$  satisfies  $\Phi[f](x, t) = x$  and, thanks to Lemma 2.1, it is Lipschitz and one-to-one. When  $\Phi[f](\cdot, t)$  is differentiable at  $x$  (which is true for almost all  $x \in \Sigma$ ), we denote by

$$D\Phi[f](x, t) = (\partial_1 \Phi[f](x, t), \dots, \partial_n \Phi[f](x, t))$$

the Jacobian matrix of  $\Phi[f](\cdot, t)$  at  $x$ , and by  $D\Phi[f](x, t)^T$  its transpose. We also define

$$S[f](x, t) := D\Phi[f](x, t)^T \cdot D\Phi[f](x, t)$$

and

$$J[f](x, t) := \sqrt{\det(S[f](x, t))},$$

and note for future reference that

$$J[f](x, 0) = 1 \quad \forall x \in \Sigma. \quad (2.4)$$

Then, we define

$$\Sigma_t[f] = \Phi[f](\Sigma, t)$$

and

$$A[f](t) := \mathcal{H}^n(\Phi[f](\Sigma \cap \text{spt}(f), t)) = \int_{\text{spt}(f)} J[f](x, t) dx, \quad (2.5)$$

where the second identity follows from the Area Formula (see, e.g., [2, Theorem 2.91]). The function  $A[f](t)$  represents the area of the compact portion of the hyperplane  $\Sigma$  deformed via the flow map  $\Phi[f](\cdot, t)$ . The minimality/stability of  $\Sigma$  is related to the asymptotic properties of  $A[f](t)$  when  $t \rightarrow 0$ . In general, we cannot expect that  $A[f](t)$  is (twice) differentiable at  $t = 0$ , hence it will not be possible to compute the first and second variations of the area in a classical sense. However, we can test the local minimality of  $\Sigma$  by taking lower right variations, i.e.,

$$\underline{\partial}_t A[f](0^+) := \liminf_{t \rightarrow 0^+} \frac{A[f](t) - A[f](0)}{t}$$

and, assuming  $\underline{\partial}_t A[f](0^+) = 0$ ,

$$\underline{\partial}_t^2 A[f](0^+) := 2 \liminf_{t \rightarrow 0^+} \frac{A[f](t) - A[f](0)}{t^2}.$$

Therefore, a first-order necessary condition for local minimality is

$$\underline{\partial}_t A[f](0^+) \geq 0, \quad \forall f \in C_c^{0,1}(\overline{\Sigma}), \quad (2.6)$$

with strict inequality only if the local minimality is strict. Then, a second-order necessary condition is

$$\underline{\partial}_t A[f](0^+) = 0 \implies \underline{\partial}_t^2 A[f](0^+) \geq 0, \quad (2.7)$$

for all  $f \in C_c^{0,1}(\overline{\Sigma})$ .

### 3 Stability of $\Sigma$ in $\Omega_\lambda$

We recall that  $\Omega_\lambda$  is the  $(n+1)$ -dimensional circular cone defined as the epigraph of the function

$$\omega_\lambda(x', x_{n+1}) := \lambda \sqrt{|x'|^2 + x_{n+1}^2},$$

with  $\lambda > 0$  a fixed parameter defining the aperture of the cone. In this case, for all  $f \in C_c^{0,1}(\overline{\Sigma})$ , the one-parameter flow associated with  $f$  is given by

$$\begin{aligned} \Phi[f](x, t) &:= (x', \omega_\lambda(x', tf(x)) + x_n - \omega_\lambda(x', 0), t f(x)) \\ &= \left( x', \lambda \sqrt{|x'|^2 + t^2 f(x)^2} + x_n - \lambda |x'|, t f(x) \right). \end{aligned}$$

It is immediate to check that  $\Phi[f](x, t)$  is differentiable on the set

$$\Delta_f := \{x \in \text{spt}(f) : x \neq 0, f \text{ is differentiable at } x\} \cup (\Sigma \setminus \text{spt}(f)).$$

In particular, since  $f$  is Lipschitz, we infer that

$$\mathcal{H}^n(\Sigma \setminus \Delta_f) = 0.$$

For almost all  $x \in \Delta_f$  and for  $1 \leq i \leq n-1$ , we have

$$\begin{aligned} \partial_i \Phi[f](x, t) &= e_i + [t \partial_i f(x) \partial_{n+1} \omega_\lambda(x', tf(x)) + \partial_i \omega_\lambda(x', tf(x)) - \partial_i \omega_\lambda(x', 0)] e_n \\ &\quad + t \partial_i f(x) e_{n+1} \\ &= e_i + \lambda \left( \frac{t^2 f(x) \partial_i f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} + \frac{x_i}{\sqrt{|x'|^2 + t^2 f(x)^2}} - \frac{x_i}{|x|} \right) e_n + t \partial_i f(x) e_{n+1} \end{aligned} \quad (3.1)$$

while, if  $i = n$ ,

$$\begin{aligned} \partial_n \Phi[f](x, t) &= (t \partial_n f(x) \partial_{n+1} \omega_\lambda(x', tf(x)) + 1) e_n + t \partial_n f(x) e_{n+1} \\ &= \left( \lambda \frac{t^2 f(x) \partial_n f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} + 1 \right) e_n + t \partial_n f(x) e_{n+1}. \end{aligned} \quad (3.2)$$

We introduce the following notations:

$$\alpha_i(x, t) := \begin{cases} \lambda \left( \frac{t^2 f(x) \partial_i f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} + \frac{x_i}{\sqrt{|x'|^2 + t^2 f(x)^2}} - \frac{x_i}{|x|} \right) & \text{if } i = 1, \dots, n-1 \\ \lambda \frac{t^2 f(x) \partial_n f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} & \text{if } i = n, \end{cases} \quad (3.3)$$

$$\beta_i(x, t) := t \partial_i f(x), \quad \forall i = 1, \dots, n. \quad (3.4)$$

We thus have

$$\partial_i \Phi[f](x, t) = e_i + \alpha_i(x, t) e_n + \beta_i(x, t) e_{n+1}, \quad \forall i = 1, \dots, n. \quad (3.5)$$

In the next lemma we compute the Jacobian  $J[f](x, t)$  of the flow map.

**Lemma 3.1** *Given  $f \in C_c^{0,1}(\overline{\Sigma})$ , for all  $t \in \mathbb{R}$ , we have*

$$J[f](x, t)^2 = 1 + t^2 \left( |\nabla f(x)|^2 + \frac{2\lambda f(x) \partial_n f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} \right) + R[f](x, t), \quad (3.6)$$

where  $R[f](x, t)$  is such that

$$\text{spt}(R[f](t, \cdot)) \subset \text{spt}(f), \quad \lim_{t \rightarrow 0} \frac{R[f](x, t)}{t^2} = 0, \quad \text{a.e. in } \Sigma, \quad \left| \frac{R[f](x, t)}{t^2} \right| \leq C, \quad (3.7)$$

where  $C \geq 0$  is a constant depending on  $\text{Lip}(f)$  only.

**Proof** By Lemma 3.2 below, we have

$$J[f]^2 = (1 + \alpha_n)^2 \left( 1 + \sum_{i=1}^{n-1} \beta_i^2 \right) + \beta_n^2 \left( 1 + \sum_{i=1}^{n-1} \alpha_i^2 \right) - 2(1 + \alpha_n) \beta_n \sum_{i=1}^{n-1} \alpha_i \beta_i. \quad (3.8)$$

where  $\alpha_i, \beta_i$  are defined in (3.3) and (3.4). We observe that, for every  $1 \leq i \leq n-1$ ,

$$\lim_{t \rightarrow 0^+} \alpha_i(t) = 0 \quad \text{for } x \neq 0. \quad (3.9)$$

In addition,

$$\lim_{t \rightarrow 0} \frac{\alpha_n[f](t)}{t} = 0 \quad \text{for } x \neq 0, \quad \lim_{t \rightarrow 0} \beta_i[f](t) = 0 \quad \text{for all } x, \quad (3.10)$$

and

$$\frac{\|\alpha_n[f](t)\|_{L^\infty(\Sigma)}}{t} \leq \lambda \|\partial_n f\|_{L^\infty(\Sigma)}, \quad \frac{\|\beta_i[f](t)\|_{L^\infty(\Sigma)}}{t} \leq \|\partial_i f\|_{L^\infty(\Sigma)}. \quad (3.11)$$

Let us define

$$R[f] := \alpha_n^2 \left( 1 + \sum_{i=1}^{n-1} \beta_i^2 \right) + 2\alpha_n \sum_{i=1}^{n-1} \beta_i^2 + \beta_n^2 \sum_{i=1}^{n-1} \alpha_i^2 - 2(1 + \alpha_n) \beta_n \sum_{i=1}^{n-1} \alpha_i \beta_i.$$

By the definition of  $\alpha_i, \beta_i$  provided in (3.3), it is evident that  $\alpha_i$  and  $\beta_i$  vanish as  $f$  vanishes, and this implies that  $\text{spt}(R[f](t, \cdot)) \subset \text{spt}(f)$ . Owing to (3.9), (3.10), we infer that

$$\lim_{t \rightarrow 0} \frac{R[f](x, t)}{t^2} = 0, \quad \text{a.e. in } \Sigma,$$

while (3.11) yields

$$\left| \frac{R[f](x, t)}{t^2} \right| \leq C, \quad C \text{ constant depending on } \text{Lip}(f).$$

This proves the validity of (3.7). On the other hand

$$J[f]^2 - R[f] = 1 + 2\alpha_n + \sum_{i=1}^n \beta_i^2 = 1 + t^2 \left( |\nabla f(x)|^2 + \frac{2\lambda f(x) \partial_n f(x)}{\sqrt{|x'|^2 + t^2 f(x)^2}} \right),$$

that is precisely what we claimed.  $\square$

Before proving our technical Lemma 3.2, we introduce some convenient notation. We let  $\mu = (\mu_1, \dots, \mu_k)$  be a multi-index with  $\mu_\ell \in \{1, \dots, n+1\}$ , then we set  $e_\mu = e_{\mu_1} \wedge \dots \wedge e_{\mu_k}$  and, given  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, n+1\}$ , we define

$$e_{\mu,i}^j = e_{\mu_1} \wedge \dots \wedge e_{\mu_{i-1}} \wedge e_j \wedge e_{\mu_{i+1}} \wedge \dots \wedge e_{\mu_k}.$$

We have the following

**Lemma 3.2** Assume that  $\partial_i \Phi[f]$  are given by (3.5), for  $1 \leq i \leq n$ . Then  $J[f]^2$  is expressed by formula (3.8).

**Proof** For  $1 \leq i \leq n$ , let  $v_i := \partial_i \Phi[f]$ . Well-known results of multi-linear algebra [19, Chapter 2] guarantee that

$$J[f]^2 = |v_1 \wedge \dots \wedge v_n|^2, \quad (3.12)$$

where  $|\cdot|$  denotes the standard Euclidean norm for multivectors. Let us set  $\mu = (1, 2, \dots, n-1)$ . By the definition of  $v_n$  and the fact that  $e_{\mu,i}^n \wedge e_{n+1} = -e_{\mu,i}^{n+1} \wedge e_n$ , we get

$$\begin{aligned} v_1 \wedge \dots \wedge v_n &= (1 + \alpha_n) e_\mu \wedge e_n + \beta_n e_\mu \wedge e_{n+1} + \\ &\quad + \beta_n \sum_{i=1}^{n-1} \alpha_i e_{\mu,i}^n \wedge e_{n+1} + (1 + \alpha_n) \sum_{i=1}^{n-1} \beta_i e_{\mu,i}^{n+1} \wedge e_n \\ &= (1 + \alpha_n) e_\mu \wedge e_n + \beta_n e_\mu \wedge e_{n+1} + \sum_{i=1}^{n-1} [\beta_n \alpha_i - (1 + \alpha_n) \beta_i] e_{\mu,i}^n \wedge e_{n+1}. \end{aligned} \quad (3.13)$$

By the orthonormality of the multivectors  $e_\mu \wedge e_n$ ,  $e_\mu \wedge e_{n+1}$ , and  $e_{\mu,i}^n \wedge e_{n+1}$  for  $i = 1, \dots, n-1$ , we obtain

$$|v_1 \wedge \dots \wedge v_n|^2 = (1 + \alpha_n)^2 \left( 1 + \sum_{i=1}^{n-1} \beta_i^2 \right) + \beta_n^2 \left( 1 + \sum_{i=1}^{n-1} \alpha_i^2 \right) - 2(1 + \alpha_n) \beta_n \sum_{i=1}^{n-1} \alpha_i \beta_i.$$

Then (3.8) immediately follows from (3.12).  $\square$

We observe that (3.6) explicitly depends on  $t^2$ . This implies the vanishing of the first (lower) variation of the area,  $\partial_t A[f](0^+) = 0$ , and the fact that the second variation can be computed as the first variation of the area with respect to the parameter  $s = t^2$ . To this end, given  $s > 0$  we conveniently set  $A[f](s) = A[f](\sqrt{s})$  and  $\mathcal{J}[f](x, s) = J[f](x, \sqrt{s})$ . The following theorem holds.

**Theorem 3.3** *The following identity holds:*

$$\partial_s \mathcal{A}[f](0^+) = \frac{1}{2} \left( \int_{\Sigma} |\nabla f(x)|^2 dx - \lambda \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|} dx' \right), \quad \forall f \in C_c^{0,1}(\overline{\Sigma}). \quad (3.14)$$

**Proof** Owing to (2.4), we have  $\mathcal{J}[f](0) \equiv 1$ , hence

$$\begin{aligned} \partial_s \mathcal{A}[f](0^+) &= \liminf_{s \rightarrow 0^+} \int_{\text{spt}(f)} s^{-1} (\mathcal{J}[f](x, s) - \mathcal{J}[f](x, 0)) dx \\ &= \liminf_{s \rightarrow 0^+} \int_{\text{spt}(f)} s^{-1} (\mathcal{J}[f](x, s) - 1) dx. \end{aligned} \quad (3.15)$$

Now, (3.6) guarantees that

$$\mathcal{J}[f](x, s)^2 = 1 + s G, \quad \text{where } G := \left( |\nabla f(x)|^2 + \frac{2\lambda f(x) \partial_n f(x)}{\sqrt{|x'|^2 + s f(x)^2}} + \frac{R[f](x, \sqrt{s})}{s} \right).$$

Thus by (3.15) we infer

$$\int_{\text{spt}(f)} s^{-1} (\mathcal{J}[f](x, s) - 1) dx = \mathcal{I}_s^1 + \mathcal{I}_s^2,$$

where

$$\mathcal{I}_s^1 := \frac{1}{s} \int_{\text{spt}(f)} \sqrt{1 + sG} - \left( 1 + s \frac{G}{2} \right) dx, \quad \mathcal{I}_s^2 := \int_{\text{spt}(f)} \frac{G}{2} dx.$$

An elementary computation yields

$$\sqrt{1 + sG} - \left( 1 + s \frac{G}{2} \right) = \frac{-s^2 G^2 / 4}{\sqrt{1 + sG} + \left( 1 + s \frac{G}{2} \right)},$$

consequently

$$|\mathcal{I}_s^1| \leq \frac{1}{4} \int_{\text{spt}(f)} \frac{sG^2}{\left| \sqrt{1 + sG} + \left( 1 + s \frac{G}{2} \right) \right|} dx.$$

Now it is immediate to observe that  $G = 0$  whenever  $x \notin \text{spt}(f)$ , and, as  $s \rightarrow 0^+$ ,

$$\sqrt{s} G \rightarrow 0 \quad \text{a.e. in } \Sigma, \quad |\sqrt{s} G| \leq C \quad \text{for some } C > 0 \text{ depending on } \text{Lip}(f) \text{ only.}$$

By Dominated Convergence, we get

$$\lim_{s \rightarrow 0^+} \mathcal{I}_s^1 = 0.$$

Let us now study the limit as  $s \rightarrow 0^+$  of

$$\mathcal{I}_s^2 = \int_{\Sigma} \left( \frac{1}{2} |\nabla f(x)|^2 + \frac{\lambda f(x) \partial_n f(x)}{\sqrt{|x'|^2 + s f(x)^2}} + \frac{1}{2} \frac{R[f](x, \sqrt{s})}{s} \right) dx$$

By (3.7) and Dominated Convergence, we have

$$\int_{\Sigma} \frac{R[f](x, \sqrt{s})}{s} dx \rightarrow 0, \quad \text{as } s \rightarrow 0^+.$$

On the other hand, denoting by  $g(x') := f(x', \lambda|x'|)^2$ , an integration by parts yields

$$\begin{aligned} \int_{\Sigma} \frac{\lambda f(x) \partial_n f(x)}{\sqrt{|x'|^2 + s f(x)^2}} dx &= \frac{\lambda}{s} \int_{\Sigma} \frac{s f(x) \partial_n f(x)}{\sqrt{|x'|^2 + s f(x)^2}} dx \\ &= \frac{\lambda}{s} \int_{\Sigma} \partial_n \sqrt{|x'|^2 + s f(x)^2} dx \\ &= \frac{\lambda}{s} \int_{\mathbb{R}^{n-1}} \left( \int_{\lambda|x'|}^{+\infty} \partial_n \sqrt{|x'|^2 + s f(x)^2} dx_n \right) dx' \\ &= \frac{\lambda}{s} \int_{\mathbb{R}^{n-1}} |x'| - \sqrt{|x'|^2 + s g(x')} dx' \\ &= -\lambda \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'| + \sqrt{|x'|^2 + s g(x')}} dx' \end{aligned}$$

Now, the integrand

$$\frac{g(x')}{|x'| + \sqrt{|x'|^2 + s g(x')}}$$

is positive and monotonically non-increasing in  $s$ , for any  $x' \neq 0$ . Then, by Beppo Levi's Theorem, we obtain

$$\lim_{s \rightarrow 0^+} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'| + \sqrt{|x'|^2 + s g(x')}} dx' = \frac{1}{2} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'|} dx'.$$

This implies that

$$\lim_{s \rightarrow 0^+} \mathcal{I}_s^2 = \frac{1}{2} \int_{\Sigma} |\nabla f(x)|^2 dx - \frac{\lambda}{2} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'|} dx',$$

and the proof is concluded.  $\square$

Theorem 3.3 allows us to test in which cases  $\Sigma$  is stable. Indeed, by (3.14), the first variation of the area vanishes, hence the stability condition can be stated in terms of the lower-right second variation:

$$\underline{\mathcal{A}}_I^2 A[f](0^+) = 2 \underline{\partial}_s \mathcal{A}[f](0^+) \geq 0,$$

which is equivalent to the functional inequality

$$\int_{\Sigma} |\nabla f(x)|^2 dx \geq \lambda \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|} dx'. \quad (3.16)$$

### 3.1 The case $\dim \Sigma = 2$

In this case, we see that  $\Sigma$  is always unstable (we proved a more general version of this result in [16]).

**Theorem 3.4** *Let  $n = 2$ . Then, for all  $f \in C_c^{0,1}(\overline{\Sigma})$  with the property that  $f(0) \neq 0$ , we have*

$$\underline{\partial}_S \mathcal{A}[f](0^+) < 0.$$

*In particular, this shows that  $\Sigma$  is unstable.*

**Proof** When  $n = 2$  the inequality (3.16) becomes

$$\int_{\Sigma} |\nabla f(x)|^2 dx \geq \lambda \int_{\mathbb{R}} \frac{f(x_1, \lambda|x_1|)^2}{|x_1|} dx_1. \quad (3.17)$$

We now observe that, for all  $f \in C_c^{0,1}(\overline{\Sigma})$ , the left-hand side of (3.17) is always finite while the right-hand side is finite only if  $f(0) = 0$ . Thus (3.17) fails for all  $f \in C_c^{0,1}(\overline{\Sigma})$  such that  $f(0) \neq 0$ .  $\square$

### 3.2 The case $\dim \Sigma \geq 3$

In this case, the aperture of the cone  $\Omega_\lambda$  plays a role. Indeed, we can prove that there exists a threshold aperture parameter  $\lambda^* > 0$ , such that for larger apertures (i.e., for  $0 < \lambda \leq \lambda^*$ ), the hyperplane  $\Sigma$  becomes strictly stable.

**Theorem 3.5** *Let  $n \geq 3$ . Then there exists  $\lambda^* = \lambda^*(n) > 0$  such that, if  $0 < \lambda \leq \lambda^*$ , we have*

$$\underline{\partial}_S \mathcal{A}[f](0^+) \geq 0 \quad \text{for all } f \in C_c^{0,1}(\overline{\Sigma}) \quad (3.18)$$

*and the inequality is strict whenever  $f$  is not identically zero, which means that  $\Sigma$  is strictly stable.*

To prove Theorem 3.5, we need the following result.

**Lemma 3.6** *Let  $n \geq 3$ . Then*

$$\int_{\Sigma} |\nabla f(x)|^2 dx \geq \frac{K_n}{(1 + \lambda)^2} \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|} dx', \quad \text{for all } f \in H^1(\Sigma), \quad (3.19)$$

where  $K_n = 2\Gamma^2(n/4)\Gamma^{-2}((n-2)/4)$  and  $\Gamma(u)$  is Euler's Gamma function.

Inequality (3.19) is known as *Kato's Inequality*. A proof of a stronger version of (3.19), valid when  $\Sigma$  is a half-space (i.e. for  $\lambda = 0$ ), is provided in [3, Theorem 1.4]. Although our domain  $\Sigma$  is not a half-space, our version directly follows from inequality (9) in the same work. We also mention that a former proof of Kato's Inequality was given by Herbst in [9].

**Proof of Lemma 3.6** Let us denote by  $\mathbb{R}_+^n$  the half-space of points  $x \in \mathbb{R}^n$  with  $x_n > 0$ . Inequality (9) of [3] states that

$$\int_{\mathbb{R}_+^n} |\nabla g(x)|^2 dx \geq K_n \int_{\mathbb{R}^{n-1}} \frac{g(x', 0)^2}{|x'|} dx', \quad \text{for all } g \in H^1(\mathbb{R}_+^n). \quad (3.20)$$

Given  $f \in H^1(\Sigma)$ , let us set

$$T(x) := (x', x_n + \lambda|x'|), \quad g(x) := f(T(x)).$$

We have  $g \in H^1(\mathbb{R}_+^n)$ . Since  $g(x', 0) = f(x', \lambda|x'|)$ , we get

$$\int_{\mathbb{R}^{n-1}} \frac{g(x', 0)^2}{|x'|} dx' = \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|} dx'. \quad (3.21)$$

We observe that

$$\nabla g(x) = \nabla f(T(x)) + \lambda \partial_n f(T(x)) \left( \frac{x'}{|x'|}, 0 \right),$$

and thus

$$|\nabla g(x)|^2 \leq (1 + \lambda)^2 |\nabla f(T(x))|^2. \quad (3.22)$$

Combining (3.20), (3.21) and (3.22), and noting that  $\det DT \equiv 1$ , by the change of variable formula we get

$$\begin{aligned} (1 + \lambda)^2 \int_{\Sigma} |\nabla f(x)|^2 dx &= (1 + \lambda)^2 \int_{\mathbb{R}_+^n} |\nabla f(T(x))|^2 dx \\ &\geq \int_{\mathbb{R}_+^n} |\nabla g(x)|^2 dx \\ &\geq K_n \int_{\mathbb{R}^{n-1}} \frac{f(x', \lambda|x'|)^2}{|x'|} dx' \end{aligned}$$

and conclude the proof.  $\square$

**Proof of Theorem 3.5** By Lemma 3.6, the validity of (3.16), for all  $f \in C_c^{0,1}(\overline{\Sigma})$ , is guaranteed by the condition

$$0 < \lambda \leq \frac{K_n}{(1 + \lambda)^2}. \quad (3.23)$$

Since the function  $\lambda \mapsto \lambda(1 + \lambda)^2$  is monotonically increasing from 0 to  $+\infty$ , there exists a unique  $\lambda^* > 0$  such that

$$\lambda^*(1 + \lambda^*)^2 = K_n$$

and, consequently, (3.23) is satisfied if and only if  $0 < \lambda \leq \lambda^*$ , as wanted.  $\square$

**Funding** Open access funding provided by Università degli Studi di Padova within the CRUI-CARE Agreement.

**Data Availability** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

## Declarations

**Conflicts of Interest** The authors declare that they have no financial interests or conflicts of interest related to the subject matter or materials discussed in this paper.

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