SMALL NOISE ASYMPTOTIC EXPANSIONS FOR STOCHASTIC PDE'S.I THE CASE OF A DISSIPATIVE POLYNOMIALLY BOUNDED NON LINEARITY.

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Abstract. We study a reaction-diffusion evolution equation perturbed by a Gaussian noise. Here the leading operator is the infinitesimal generator of a C_0-semigroup of strictly negative type, the nonlinear term has at most polynomial growth and is such that the whole system is dissipative.

The corresponding Itô stochastic equation describes a process on a Hilbert space with dissipative nonlinear drift and a Gaussian noise.

Under smoothness assumptions on the non-linearity, asymptotics to all orders in a small parameter in front of the noise are given, with uniform estimates on the remainders. Applications to nonlinear SPDEs with a linear term in the drift given by a Laplacian in a bounded domain are included. As a particular example we consider the small noise asymptotic expansions for the stochastic FitzHugh-Nagumo equations of neurobiology around deterministic solutions.

1. Introduction

In many problems of natural sciences and engineerings, modeling of dynamical systems by non linear deterministic partial differential equations (PDEs) is heavily used. This is, e.g. , the case for the equations of classical hydrodynamics and more generally classical field theory, as well as for the equations used in the description of certain neurodynamical processes, see, e.g. , [AFS08], [CM07].

Due to the uncertainty concerning stochastic influences on the systems (e.g. by additive *random forcing*) an addition of stochastic terms in the equations describing such systems is appropriate.

This generates the necessity to study stochastic partial differential equations.

The problem of the study of a deterministic evolution equation of first order in time and finite dimensional state spaces perturbed by an additive Gaussian noise and the associated small noise expansions has been discussed by several authors. Roughly speaking, the work concerning this

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problem concerns either individual solutions or expectations of functionals of the solution process. For the first category let us mention e.g. [Wat87, Tuc08], [IW89].

Work concerning the second category uses methods which go back to Donsker's school, see e.g. [Sch66, AM09, AS99, AFP09] and references therein.

To the latter category belongs also the Laplace method for infinite dimensional integrals, see e.g. [AM09, AS99, AL05, MT97, Sim05, IK08], and to the *semiclassical expansions for Wiener type integrals*, see e.g. [AM09]. These expansions go beyond the large deviations estimates which are on the other hand valid without smoothness assumption on the drift, see e.g. [DZ98, DS89] and [DPZ92, DPZ96, DZ98].

Astonishing enough corresponding work for the case of evolving systems with infinite dimensional state space, i.e. for SPDEs, is much more sparse, see however e.g. [CF06, ARS00, RT00, CF06].

R. Marcus studied in [Mar78, Mar74] problems of this type in the case of globally Lipschitz non-linear terms.

Our present paper extends the latter work in the direction of dropping the global Lipschitz condition and allowing for non linearities of at most polynomial growth and of dissipative character, like the ones occurring in the case of the stochastic FitzHugh-Nagumo equation studied in stochastic neurodynamics, see [BM08] and references therein.

More precisely our paper considers a system the deterministic part of which corresponds to a non linear PDEs of the semilinear type with an (unbounded) linear term and a nonlinearity which is smooth and at most polynomially growing at infinity.

This deterministic PDEs is perturbed additively by a space-time noise term of the Gaussian type, with a small coefficient ε in front of it. Mathematically the problem can be looked upon as described by a stochastic differential equation with a a small parameter in front of the Wiener process.

Since we allow for polynomial growth of the non linear part of the drift, in order to assure existence and uniqueness of mild solutions we assume that the total drift term is dissipative.

In turn this is assured by assumptions on the linear drift term and by one sided dissipative type conditions on the non linear term.

The study of such equations was from the very beginning influenced by motivations from areas like quantum field theory (such as stochastic quantization equation, see [AR91, Alb, DPT00, JLM85, Par88] and the references therein, the Ginzburg-Landau equation of classical statistical mechanics (the equation describing growth of surfaces in solid state physics), biology (e.g. in the study of stochastic neurodynamics e.g. [Wat87, BM08]) and economics (e.g. interest rate models [ALM04, FLT09]).

In many problems it is interesting to know how the solutions of the perturbed problem depend on the small parameter ε describing the random forcing. E.g. in connection with FitzHugh-Nagumo models of neurodynamics this has been discussed heuristically by Tuckwell, see [Tuc88a, Tuc88b, Tuc92], see also e.g. [LAGE00] for motivations in connection with the description of phenomena in the study of epilepsy and [AC07] for connections with problems of syncronization in neuronal systems.

The exploitation of the dissipativity permits to compensate for the lack of a global Lipschitz condition on the non linear drift.

Our aim is to provide asymptotic expansions of the solution to all orders in the perturbation parameter ε, with explicit expressions both for the expansion coefficients and the remainder.

The technique used is general and also covers the case of deterministic *forcing terms*.

An application to models of stochastic FitzHugh-Nagumo dynamics on networks, used for the description of biological neurons will be given in a subsequent paper [ADPM].

In a further paper we shall extend our method to study other SPDEs having other types of non linear polynomially growing drifts.

2. Outline of the paper

Let us consider the following deterministic problem:

$$\begin{cases} du_0(t) = [Au_0(t) + F(u_0(t))]dt, & t \in [0, +\infty) \\ u_0(0) = u^0, \quad u^0 \in D(F), \end{cases} \tag{2.1}$$

where A is a linear operator on a separable Hilbert space H which generates a C_0-semigroup of strict negative type. The term F is a *smooth* nonlinear, quasi-m-dissipative mapping from the domain $D(F) \subset H$ (dense in H) with values in H; this means that there exists $\omega \in \mathbb{R}$ such that $(F - \omega I)$ is m-dissipative in the sense of [DPZ96] (pag. 73), with (at most) polynomial growth at infinity (and satisfying some further assumptions which will be specified in Hypothesis 3.1) below, $D(F)$ is the domain of F, dense in H. Existence and uniqueness of solutions for equation (2.1) is discussed in Proposition 3.7 below.

Our aim is to study a stochastic (white noise) perturbation of (2.1) and to write its (unique) solution as an expansion in powers of a parameter $\varepsilon > 0$, which controls the strength of the noise, as ε goes to zero. More precisely, we are concerned with the following stochastic Cauchy problem on the Hilbert space H:

$$\begin{cases} du(t) = [Au(t) + F(u(t))]dt + \varepsilon\sqrt{Q}dW(t), & t \in [0, +\infty) \\ u(0) = u^0, \quad u^0 \in D(F) \end{cases} \tag{2.2}$$

where A and F are as described above, W is a cylindrical Wiener process on H, Q is a positive trace class linear operator from H to H and $\varepsilon > 0$ is the parameter which determines the magnitude of the stochastic perturbation. A unique solution of problem (2.2) can be shown to exist exploiting as in [BM08] results on stochastic differential equations contained, e.g., in [DPZ92, DPZ96]. Our purpose is to show that the solution of equation (2.2), which will be denoted by $u = u(t), t \in [0, +\infty)$, can be written as

$$u(t) = u_0(t) + \varepsilon u_1(t) + \cdots + \varepsilon^n u_n(t) + R_n(t, \varepsilon) ,$$

where n depends on the differentiability order of F. The function $u_0(t)$ solves the associated deterministic problem (2.1), $u_1(t)$ is the stochastic process which solves the following linear stochastic (non-autonomous) equation

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(u_0(t))[u_1(t)]]dt + \sqrt{Q}dW(t), & t \in [0, +\infty) \\ u_1(0) = 0, \end{cases} \tag{2.3}$$

while for each $k = 2, \ldots, n$, $u_k(t)$ solves the following non-homogeneous linear differential equation with stochastic coefficients

$$\begin{cases} du_k(t) = [Au_k(t) + \nabla F(u_0(t))[u_k(t)]]\, dt + \Phi_k(t)dt \\ u_k(0) = 0 \end{cases} \tag{2.4}$$

$\Phi_k(t)$ is a stochastic process which depends on $u_1(t), \ldots, u_{k-1}(t)$ and the Fréchet derivatives of F up to order k, see Section 4 for details.

The paper is organized as follows. In Section 3 we recall standard results for the solution of equations of types (2.1), (2.2), (2.3) and (2.4). Section 4 is devoted to the study of some properties of the nonlinear term F, in particular the n-th remainder of its Taylor expansion, while Section 5 is concerned with the proof of the main result which gives the asymptotic behaviour of the Taylor remainder $R_n(t,\varepsilon)$ derived in previous section. We conclude with some remarks on applications of the results, in particular concerning the FitzHugh-Nagumo equation.

3. Assumptions and Basic Estimates

Before recalling some known results on problems of the type (2.1),(2.2), (2.3), (2.4), we begin by presenting our notation and assumptions. We are concerned with a real separable Hilbert space H, on which there are given a linear operator $A : D(A) \subset H \to H$, a nonlinear operator $F : D(F) \subset H \to H$ with dense domain in H and a bounded linear operator Q on H. Moreover, we are given a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions, i.e., the probability space is complete, $\mathcal{F}$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous. $|\cdot|_H$ the norm of any element of H, by $\|\cdot\|_{\mathcal{L}}(H)$ the norm of any linear operator on H and by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm of any linear operator on H. Further, for any trace-class linear operator Q, we will denote by $\mathrm{Tr}Q$ its trace; if f is any mapping on H which is Fréchet differentiable up to order n, $n \in \mathbb{N}$, we will denote by $f^{(i)}$, $i = 1, \dots, n$ its i-th Fréchet differential and by $D(f^i)$ its domain (for a short survey on Fréchet differentiable mapping we refer to Section 4). Finally, we will denote by $\mathcal{L}^p(\Omega; C[0,T]; H)$ the space of continuous and adapted processes taking value in H such that the following norm is finite

$$|||u|||^p = \mathbb{E} \sup_{t\in[0,T]} |u(t)|_H^p < \infty.$$

Hypothesis 3.1.

(1) *The operator $A : D(A) \subset H \to H$ generates an analytic semigroup e^{tA}, $t \geq 0$, on H of strict negative type such that*

$$\left\|e^{tA}\right\|_{\mathcal{L}(H)} \leq e^{-\omega t}, \quad t \geq 0$$

with ω a strictly positive.

(2) *The mapping $F : D(F) \subset H \to H$ is continuous, nonlinear, Fréchet differentiable up to order n for some positive integer n and there exist positive real numbers η, γ and a natural number $m > 0$ such that:*

$$\langle F(x) - F(y) - \eta(x-y), x-y\rangle < 0$$
$$|F(x)|_H \leq \gamma\left(1 + |x|_H^m\right), \quad x, y \in D(F).$$

(3) *For some $n \in \mathbb{N}$ and any $x \in D(F^{(i)}), i = 1, \dots, n$, there exist positive real constants γ_i, $i = 1, \dots, n$ such that*

$$\|F^{(i)}(x)\|_{\mathcal{L}(H)} \leq \gamma_i(1 + |x|_H^{m-i}), \quad \text{with } m \text{ as in (2)}$$

(4) *The constants ω, η satisfy the inequality $\omega - \eta > 0$; this implies that the term $A + F$ is $m-$dissipative in the sense of* [DPZ92], [DPZ96] *(Pag.73).*

(5) *The term W is a cylindrical Wiener process (in the sense of, e.g.,* [DPZ92, DPZ96]*)*

(6) *Q is a positive linear bounded operator on H of trace class, that is $\mathrm{Tr}Q < \infty$.*

Remark 3.2. Let us give an example of a mapping F satisfying the above hypothesis (in view of the application to stochastic neural models). Let $H = L^2(\Lambda)$ with $\Lambda \subset \mathbb{R}^n$, bounded and open; let F be a multinomial of degree m, i.e. a mapping of the form $F(u) = g_m(u)$, where $g_m(u)$ is a polynomial of degree m, that is, $g_m(u) = a_0 + a_1 u + \cdots + a_m u^m$. Then it is easy to prove that $D(F) = L^{2m}(\Lambda) \subseteq L^2(\Lambda)$ and (by using Hölder inequality) $D(\nabla F^{(i)}) = L^{2i}(\Lambda)$. Clearly, $\nabla^{(i)} F$ satysfies Hypothesis 3.1. Further, in the case $g_3(u) = -u(u-1)(u-\xi), 0 < \xi < 1$ the corresponding mapping F coincides with the non linear term of the FitzHugh-Nagumo equation (see Remark 5.4 below).

We recall the notion of mild solution for the deterministic and stochastic problems (2.1), (2.2); next we recall the definition of stochastic convolution and we list some of its properties.

Definition 3.3. *Let $u^0 \in D(F)$; we say that the function $u_0 : [0,\infty] \to H$ is a mild solution of equation* (2.1) *if it is continuous (in t), with values in H and it satisfies:*

$$u_0(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u_0(s))ds, \quad t \in [0,+\infty), \tag{3.1}$$

with the integral existing in the sense of Bochner integrals on Hilbert spaces.

Definition 3.4. *A predictable H-valued process $u := (u(t))_{t\geq 0}$ is said to be a mild solution to* (2.2) *if for arbitrary $t \geq 0$ we have:*

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u(s))ds + \varepsilon \int_0^t e^{(t-s)A}\sqrt{Q}dW(s), \quad \mathbb{P} - a.s.$$

Moreover $W_A(t) := \int_0^t e^{(t-s)A}\sqrt{Q}dW(s)$ is called stochastic convolution and under our hypothesis it is a well defined mean square continuous $\mathcal{F}_t-$adapted Gaussian process (see e.g. [DPZ92, Th. 5.2, Pag.119]*).*

The first integral on the right hand side is defined pathwise in the Bochner sense, $\mathbb{P}$-almost surely.

Let us recall a fundamental property of the stochastic convolution which will be used later:

Proposition 3.5. *Under Hypothesis 3.1 the stochastic convolution $W_A(t)$ is $\mathbb{P}-$almost surely continuous for $t \in [0,+\infty)$ and it verifies the following estimate*

$$\mathbb{E}\left[\sup_{t\geq 0} |W_A(t)|_H^{2m}\right] \leq C \tag{3.2}$$

for some positive constant C and every $m \in \mathbb{N}$.

Proof. The proof is based on Burkholder-Davis-Gundy's inequality (BDG); in fact we have:

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\geq 0} |W_A(t)|_H^{2m}\right] &= \mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t e^{(t-s)A}\sqrt{Q}dW(s)\right|^{2m}\right] \\
&\leq C_0\mathbb{E}\left[\sup_{t\geq 0}\left|\int_0^t \|e^{(t-s)A}\sqrt{Q}\|_{HS}^{2m} ds\right|\right] \\
&\leq C_0 \sup_{t\geq 0}\int_0^t \|e^{(t-s)A}\|_{\mathcal{L}(H)}^{2m}(\mathrm{Tr}Q)^{2m}ds \\
&< \infty,
\end{aligned} \tag{3.3}$$

for some constant $C_0 > 0$ (where in the first inequality we used the (DGB)-inequality and in the latter inequality we used the hypothesis on the trace of Q (condition (6) in Hypothesis 3.1). □

Remark 3.6. *As the proof above shows, we could drop the assumptions on Q provided we assume more on the semigroup $(e^{tA})_{t\geq 0}$, namely that $(e^{tA})_{t\geq 0}$ be Hilbert-Schmidt for all $t > 0$, because in the latter case the bound in the last step in* (3.3) *can be replaced by the bound*

$$\int_0^t \|e^{(t-s)A}\|_{HS}^{2m}\|Q\|_{\mathcal{L}(H)}^{2m}ds.$$

A known result for the semilinear deterministic equation (2.1) associated to A and F is given below.

Proposition 3.7. *Under Hypothesis 3.1 there exists a unique mild solution $u_0 = u_0(t), t \in [0, \infty)$ of the deterministic problem* (2.1) *such that*

$$|u_0(t)|_H \leq e^{-2(\omega-\eta)t}|u^0|_H, \quad t \geq 0. \tag{3.4}$$

Proof. The proof of existence and uniqueness can be found e.g. in [DPZ92, Theorem 7.13, pag. 203], while estimate (3.4) is a direct consequence of the application of Gronwall's lemma to the following inequality

$$\begin{aligned}\frac{d}{dt}|u_0(t)|_H^2 &= 2\langle Au_0(t), u_0(t)\rangle dt + 2\langle F(u_0(t)), u_0(t)\rangle \\ &\leq -2(\omega-\eta)|u_0(t)|^2.\end{aligned}$$

□

Proposition 3.8. *Assume that A and F satisfy Hypothesis 3.1. Then for any $u^0 \in D(F)$ and $t > 0$, there exists a unique mild solution $u = (u(t))_{t\geq 0}$ of equation* (2.2) *which belongs to the space $\mathcal{L}^p(\Omega; C([0,T]; H))$, for any $T > 0$, i.e. such that*

$$\mathbb{E} \sup_{t\in[0,T]} |u(t)|_H^p < +\infty, \tag{3.5}$$

for any $p \in [2, \infty)$.

Proof. For the existence and uniqueness of the solution see e.g. [DPZ92, Theorem 7.13, pag. 203]. Hence we only have to prove estimate (3.5). Let $z(t) := u(t) - W_A(t)$; then it is not difficult to show that $z(t)$ is the unique solution of the following deterministic equation:

$$\begin{cases} z'(t) = Az(t) + F(z(t) + W_A(t)) \\ z(0) = u^0 \end{cases}$$

with $z'(t) := \frac{d}{dz}z(t)$.

With no loss of generality (because of inclusion results for L^p-spaces with respect to bounded measures) we can assume that $p = 2a$, $a \in \mathbb{N}$. Now combining conditions (1) and (2) in Hypothesis 3.1 and reacalling Newton's formula we have:

$$\begin{aligned}\frac{d}{dt}|z(t)|_H^{2a} &= 2a\langle z'(t), z(t)\rangle |z(t)|_H^{2a-2} = 2a\langle Az(t) + F(z(t) + W_A(t)), z(t)\rangle |z(t)|_H^{2a-2} \\ &\leq -2a\omega|z(t)|_H^{2a} + 2a\langle F(z(t) + W_A(t)), z(t)\rangle |z(t)|_H^{2a-2} \\ &\leq -2a(\omega-\eta)|z(t)|_H^{2a} + 2a|F(W_A(t))|_H |z(t)|_H^{2a-1} \\ &\leq -2a(\omega-\eta)|z(t)|_H^{2a} + 2a \cdot \frac{C_a}{\xi}|F(W_A(t))|_H^{2a} + C_a 2a\xi |z(t)|^{2a}\end{aligned} \tag{3.6}$$

for some constant $C_a > 0$ and a sufficiently small $\xi > 0$ such that $-2a(\omega - \eta) + 2a \cdot \xi C_a < 0$. Applying the previous inequality and Gronwall's lemma we get:

$$|z(t)|_H^{2a} \leq e^{(-2a(\omega-\eta)+\xi C_a 2a)t}|u^0|_H^{2a} + \frac{2aC_a}{\xi}\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a} ds.$$

Then there exists a positive constant C such that:

$$(3.7)\quad |u(t)|_H^{2a} \le C\left(e^{(-2a(\omega-\eta)+\xi C_a 2a)t}|u^0|_H^{2a} + 2a\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a}ds + |W_A(t)|_H^{2a}\right).$$

Since by condition (2) in Hypothesis 3.1, F has (at most) polynomial growth at infinity, for any $a \in \mathbb{N}$ we have:

$$|F(W_A(t))|_H^{2a} \le C_{a,m}(1+|W_A(t)|^m)^{2a} \le C_{a,m}(1+|W_A(t)|_H^{2am}),$$

for some positive constant $C_{a,m}$ depending on m and a. Moreover, we recall that, by Proposition 3.5 it holds that

$$\mathbb{E}\left[\sup_{t\ge 0}|W_A(t)|_H^{2am}\right] \le C'_{a,m},$$

where $C'_{a,m}$ is again a positive constant depending on m and a; hence

$$(3.8)\quad \mathbb{E}\left[\sup_{t\ge0}\int_0^t e^{-2a(\omega-\eta)(t-s)}|F(W_A(s))|_H^{2a}ds\right] \le \tilde{C}\mathbb{E}\left[\sup_{t\ge0}\int_0^t e^{-2a(\omega-\eta)(t-s)}(1+|W_A(t)|_H^{2am})ds\right]$$
$$\le \tilde{C}\mathbb{E}\left[\sup_{t\ge0}\int_0^t e^{-2a(\omega-\eta)(t-s)}ds + C'_{a,m}\int_0^t e^{-2a(\omega-\eta)}ds\right] \le \bar{C}$$

for some positive constant $\tilde{C}, \bar{C}$ depending on a and m. Consequently, putting togheter inequalities (**??**) and (3.8), we obtain

$$\mathbb{E}\sup_{t\ge0}|u(t)|_H^{2a} \le C|u^0|_H^{2a} + \bar{C},$$

so that the thesis follows.

□

4. Properties of the non-linear term F and Taylor expansions

In this section we study the non-linear term F in order to write its Taylor expansion around the solution $u_0(t)$ of (3.1) with respect to an increment given in terms of powers of ε. In order to do that we recall some basic properties of Fréchet differentiable functions.

Let X and V be two real Banach spaces. For a mapping $F : X \to V$ the Gâteaux differential at $x \in X$ in the direction $h \in X$ is defined as

$$\nabla F(x)[h] = \lim_{s\to0}\frac{F(x+sh)-F(x)}{s},$$

whenever the limit exists in the topology of V, see , e.g., [LL72, Pag.12].

We notice that if $\nabla F(x)[h]$ exists in a neighborhood of $x_0 \in X$ and is continuous in x at x_0 and also continuous in h at $h = 0$, then $\nabla F(x)[h]$ is linear in h (see, e.g., [LL72, Problem 1.61, Pag. 15]). If $\nabla F(x_0)[h]$ has this property for all $x_0 \in X_0 \subseteq X$ and all $h \in X$ we shall say that F belongs to the space $\mathcal{G}^1(X_0;V)$. If F is continuous from X to V and $F \in \mathcal{G}^1(X_0;V)$ and one has $F(x+h) = F(x) + \nabla F(x)[h] + R(x,h)$, for any $x \in X_0$ with:

$$(4.1)\qquad \lim_{|h|_X\to0}\frac{|R(x,h)|_V}{|h|_X} = 0$$

with $|\cdot|_V$ and $|\cdot|_X$ denoting respectively the norm in V and X, then the map $h \to \nabla F(x)[h]$ is a bounded linear operator from X_0 to V, and $\nabla F(x)[h]$ is, by definition, the unique Fréchet differential of F at $x \in X_0$ with increment h. $R(x,h)$ is called the remainder of this Fréchet differential, while the operator sending h into $\nabla F(x)[h]$ is then called the Fréchet derivative of F at x and is usually denoted by $F'(x)$ (see,e.g., [LL72, Pagg. 15/16, Problem 1.6.2 and Lemma 1.6.3]). We have then $\nabla F(x)[h] = F'(x) \cdot h$, with the symbol $\cdot$ denoting the action of the linear bounded operator $F'(x)$ on h.

$F'(x)$ is also called the gradient of F at x (see, e.g., [LL72, Pag.15]) and it coincides with the Gâteaux derivative of F at x. We shall denote by $\mathcal{F}^{(1)}(X_0, V)$ the subset of $\mathcal{G}^1(X_0, V)$ such that the Fréchet derivative exists at any point of X_0. Similarly we introduce the Fréchet derivative $F''(x)$ of F' at $x \in X$. This is a bounded linear map from a subset $D(F')$ of X into $B(X, V)$ ($B(X, V)$ being the space of bounded linear operators from X to V). One has thus $F'' \in B(X, B(X, V))$. If we choose $h, k \in X$ then $F''(x) \cdot k \in B(X, V)$ and $(F''(x) \cdot k) \cdot h \in V$. The latter is also written $F''(x)\ h\ k$ or $F''(x)[h, k]$. $F''(x)[h, k]$ is bilinear in h, k, for any given $x \in D(F'')$, it can be identified with the Gâteaux differential $\nabla^{(2)}F(x)[h, k]$ of $\nabla F(x)[h]$ in the direction k, the latter looked upon as a map from X to $B(X, V)$.

Similarly one defines the j-th Fréchet derivative $F^{(j)}(x)$ and the j-th Gâteaux derivative $\nabla F^{(j)}(x)[h_1, \dots, h_j]$. $F^{(j)}(x)$ acts j-linearly on $h_1, \dots, h_j$ with $h_i \in X$ for any $i = 1, \dots, j$. Let X_0 be an open subset of X and consider the space $\mathcal{F}^{(j)}(X_0, V)$ of maps F from X to V such that $F^{(j)}(x)$ exists at all $x \in X_0$ and is uniformly continuous on X_0. The following Taylor formula holds for any $x, h \in X$ for which $F(h)$ and $F(x + h)$ are well defined (i.e. h and $x + h$ are elements of $D(F)$), and $j = 1, \dots, n + 1$ with $x \in \cap_{i=1}^{n} F^{(j)}(X_0, V)$ such that

$$F(x+h) = F(x) + \nabla F(x)[h] + \frac{1}{2}\nabla^{(2)}F(x)[h,h] + \cdots + \frac{1}{n!}\nabla^{(n)}F(x)\underbrace{[h,\dots,h]}_{n-terms} + R^{(n)}(x;h) \tag{4.2}$$

where $\left|R^{(n)}(x;h)\right|_X \leq C_{x,n} \cdot |h|_X^n$ for some constant $C_{x,n}$ depending only on x and n, see, e.g., [KF61, Theorem X.1.2 Chapter].

Now let us consider the case $X = H$, with H being the same Hilbert space appearing in problem (2.1). Let F be as in Hypothesis 3.1 and set $X_0 = D(F)$. Let us define for $1 \geq \varepsilon > 0$ the function $h(t)$:

$$h(t) = \sum_{k=1}^{n} \varepsilon^k u_k(t),$$

where the functions $u_k(t)$ are p-mean integrable continuous stochastic processes in $\mathcal{L}^p(\Omega, C([0,T]; H))$, defined on the whole interval $[0, T]$ for $p \in [2, \infty)$. Then using the above Taylor formula we have

$$\begin{aligned} F(u_0(t) + h(t)) = F(u_0(t)) + \nabla F(u_0(t))[h(t)] + \frac{1}{2}\nabla F^{(2)}[h(t), h(t)] + \cdots \\ + \frac{1}{n!}\nabla^{(n)}F(x)\underbrace{[h(t),\dots,h(t)]}_{n-terms} + R^{(n)}(u_0(t); h(t)) \end{aligned} \tag{4.3}$$

and, recalling that for any $j = 1, \dots, n$, $\nabla^{(j)}F(u_0(t))$ is multlinear, we have

$$\frac{1}{j!}\nabla^{(j)}F(u_0(t))\underbrace{[h(t),\dots,h(t)]}_{jterms} = \frac{1}{j!}\sum_{k_1+\cdots+k_j=j}^{nj} \varepsilon^k \nabla^{(i)}F(u_0(t))[u_{k_1}(t), \dots, u_{k_j}(t)].$$

We notice that any derivative in the right member is multiplied by the parameter ε raised to a power between j and nj.

Taking into account the above equality we can rewrite (4.3) as

$$
\begin{aligned}
F(u_0(t)+h(t)) = {} & F(u_0(t)) + \sum_{k=1}^{n} \varepsilon^k \nabla F(u_0(t))[u_k(t)] \\
& + \sum_{j_1+j_2=2}^{n} \frac{\varepsilon^{j_1+j_2}}{2!} \nabla^{(2)} F(u_0(t))[u_{j_1}(t), u_{j_2}(t)] + \dots \\
& + \sum_{j_1+\dots+j_k=k}^{n} \frac{\varepsilon^{j_1+\dots+j_k}}{k!} \nabla^{(k)} F(u_0(t))[u_{j_1}(t), \dots, u_{j_k}(t)] + \dots \\
& + \frac{\varepsilon^n}{n!} \nabla^{(n)} F(u_0(t))[u_1(t), \dots, u_1(t)] + R_1^{(n)}(u_0(t); h(t), \varepsilon),
\end{aligned}
\tag{4.4}
$$

where the quantity $R_1^{(n)}(u_0(t); h(t), \varepsilon)$ is given in terms of the derivatives of f with the parameter ε raised to powers greater than n and in terms of the n-th remainder $R^{(n)}(u_0(t); h(t))$ in the Taylor expansion of the map F (as stated in Equation (4.2)). Namely, we have:

$$
\begin{aligned}
R_1^{(n)}(u_0(t); h(t), \varepsilon) = {} & \sum_{j=2}^{n} \sum_{i_1+\ldots+i_j=n+1}^{nj} \varepsilon^{i_1+\dots+i_j} \\
& \frac{1}{j!} \nabla^{(j)} F(u_0(t))[u_{i_1}(t), \dots, u_{i_j}(t)] + R^{(n)}(u_0(t); h(t)),
\end{aligned}
\tag{4.5}
$$

$R^{(n)}(u_0(t); h(t))$ being as in (4.2). In this way equation (4.4) can be rearranged as

$$
\begin{aligned}
F(u_0(t)+h(t)) = {} & F(u_0(t)) + \sum_{j=2}^{n} \sum_{i_1+\dots+i_j=j}^{n} \varepsilon^{i_1+\dots+i_j} \\
& \frac{1}{j!} \nabla^{(j)} F(u_0(t))[u_{i_1}(t), \dots, u_{i_j}(t)] + R_1^{(n)}(u_0(t); h(t), \varepsilon).
\end{aligned}
\tag{4.6}
$$

Lemma 4.1. *Let $R_1^{(n)}$ be as in formula (4.5), then for all $p \in [2, \infty)$ there exists a constant $C > 0$, depending on $|u_0|_H, \dots, |u_n|_H, \nabla F^{(1)}, \dots, \nabla F^{(n)}, p, n$, such that:*

$$
\mathbb{E} \sup_{t \in [0,+\infty)} |R_1^{(n)}(u_0(t); h(t), \varepsilon)|_H^p \le C \varepsilon^{p(n+1)}
$$

for all $0 < \varepsilon \le 1$.

Proof. Since:

$$
\begin{aligned}
R_1^{(n)}(u_0(t); h(t), \varepsilon) = {} & \sum_{j=2}^{n} \sum_{i_1+\ldots+i_j=n+1}^{nj} \varepsilon^{i_1+\dots+i_j} \\
& \frac{1}{j!} \nabla^{(j)} F(u_0(t))[u_{i_1}(t), \dots, u_{i_j}(t)] + R^{(n)}(u_0(t); h(t)),
\end{aligned}
$$

then for $\varepsilon \in (0,1]$ we have

$$\begin{aligned}
|R_1^{(n)}(u_0(t);h(t),\varepsilon)|_H^p &\le C_{n,p}^1 \varepsilon^{(n+1)p} \left[\left(\max_{j=1,\dots,n} \|\nabla^{(j)} F(u_0(t))\|_{\mathcal{L}(H)} \right)^p \left(\sum_{i=1}^n |u_i(t)|_H^p \right) \right] \\
&\quad + C_{n,p}^2 \left| R^{(n)}\left(u_0(t);h(t)\right) \right|_H^p \\
&\le C_{n,p}^{(1)} \varepsilon^{(n+1)p} \max_{j=1,\dots,n} \left[\gamma_j^p (1+|u_0(t)|^{m-j})^p \right] \left(\sum_{i=1}^n |u_i(t)|_H^p \right) + C_{n,p}^{(2)} |R^{(n)}(u_0(t);h(t))|_H^p \\
&\le \tilde{C}_n \varepsilon^{(n+1)p} + C_{n,p}^{(2)} |R^{(n)}(u_0(t);h(t))|_H^p
\end{aligned} \tag{4.7}$$

where $C_{n,p}^{(1)}, C_{n,p}^{(2)}$ are constants depending only on n, p and $\tilde{C}_n$ is a suitable positive constant depending on $p, n, \max_{j=1,\dots,n} \left[\gamma_j^p (1+|u_0(t)|^{m-j})^p\right]$ (γ_i being the constants appearing in Hypothesis 3.1, condition (3)) and $|u_i(t)|_H^p$, $i = 1,\dots,n$. We notice that the above inequality follows by recalling that the deterministic function $u_0(t)$ is bounded (in the $H-$norm) (see Proposition 3.7).

Now by the bound on $R^{(n)}$ in equation (4.2) we have that

$$|R^{(n)}(u_0(t);h(t))|_H^p \le \hat{C}_n |h(t)|_H^{(n+1)p}$$

with $\hat{C}_n$ depending on $u_0(t)$ and n but independent of $h(t)$. Since $h(t) = \sum_{k=1}^n \varepsilon^k u_k(t)$, then:

$$|R^{(n)}(u_0(t);h(t))|_H^p \le \varepsilon^{(n+1)p} \hat{C}_{n,p}(|u_1(t)|_H, \dots, |u_n(t)|_H) \tag{4.8}$$

with $\hat{C}_{n,p} = \hat{C}_{n,p}(|u_1(t)|_H, \dots, |u_n(t)|_H)$ independent of ε.

Hence by (4.7) and (4.8) we have that

$$\mathbb{E} \sup_{t\in[0,T]} |R_1^{(n)}(u_0(t);h(t),\varepsilon)|_H^p \le C_n' \varepsilon^{n+1},$$

where $C_n' := C_n'(p, \nabla^{(1)}F, \dots, \nabla^{(n)}F, |u_0|_H, \dots, |u_n|_H)$ is independent of ε.

□

As we said before, we want to expand the solution of equation (2.2) around $u_0(t)$, that is we want to write $u(t)$ as:

$$u(t) = u_0(t) + \varepsilon u_1(t) + \dots + \varepsilon^n u_n(t) + R_n(t,\varepsilon), \tag{4.9}$$

(with the term $R_n(t,\varepsilon) = O(\varepsilon^{n+1})$), for any $t \ge 0$), where the processes $(u_i(t))_{t\ge0}, i = 1\dots,n$ can be found by using the Taylor expansion of F around $u_0(t)$ and *matching terms* in the equation for u. Given predictable $H-$valued stochastic processes $v_0(t),\dots,v_n(t)$ let us use the notation:

$$\Phi_k(t)\left[v_0(t), v_1(t), \dots, v_k(t)\right] := \sum_{j=2}^k \sum_{\substack{i_1,\dots,i_j \in 1,\dots,n \\ \sum_{l=1}^j i_l = k}} \nabla^{(j)} F(v_0(t))[v_{i_1}(t), \dots, v_{i_j}(t)]. \tag{4.10}$$

With the above notation $u_1(t),\dots,u_n(t)$ occurring in (4.9) satisfy the following equations:

$$\begin{aligned}
du_1(t) &= [Au_1(t) + \nabla F(u_0(t))[u_1(t)]]dt + \sqrt{Q}dW(t),\ u_1(0) = 0\ , \\
du_2(t) &= [Au_2(t) + \nabla F(u_0(t))[u_2(t)]]dt + \Phi_2(t)dt,\ u_2(0) = 0\ , \\
&\dots \\
du_k(t) &= [Au_k(t) + \nabla F(u_0(t))[u_k(t)]]dt + \Phi_k(t)dt,\ u_k(0) = 0\ ,
\end{aligned} \tag{4.11}$$

with

$$\Phi_k(t) := \Phi_k(t)\,[u_0(t),\dots,u_{k-1}(t)]\,. \tag{4.12}$$

Notice that while $u_1(t)$ is the solution of a linear stochastic differential equation (with time dependent drift operator $A+\nabla F(u_0(t))$), the processes $u_2,\dots,u_n$ are solutions of non-homogenous differential equations with random coefficients whose meaning is given below.

Definition 4.2. *Let $2 \le k \le n$, then a predictable H-valued stochastic process $u_k = u_k(t)\ , t \ge 0$ is a solution of problem* (2.4) *(i.e.* (4.11) *) if almost surely it satisfies the following integral equation*

$$u_k(t) = \int_0^t e^{(t-s)A}\nabla F(u_0(s))[u_k(s)]ds + \int_0^t \Phi_k(s)ds, \qquad t \ge 0\ ,\ 2 \le k \le n,$$

with Φ_k as in (4.10).

In the following result we estimate the norm of Φ_k in H by means of the norms of the Gâteaux derivatives of F and the norms of $v_j(t)$, $j = 1,\dots,k-1$, where $v_j(t)$ are H-valued stochastic processes.

Lemma 4.3. *Let us fix $2 \le k \le n$ and let $v_0(t), v_1(t),\dots,v_{k-1}(t)$ be H-valued stochastic processes. Then $\Phi_k(t)\,[v_0(t),\dots,v_{k-1}(t)]$ as in* (4.10) *satisfies the following inequality*

$$|\Phi_k(t)\,[v_0(t),\dots,v_{k-1}(t)]|_H \le C|v_0(t)|_H k^3(k + |v_1(t)|_H^{k-1} + \dots + |v_{k-1}(t)|_H^{k-1}),$$

where C is some positive constants depending on k and the constant γ_j, $j = 2,\dots,k$ introduced in Hypothesis 3.1.

Proof. We have

$$\begin{aligned} |\Phi_k(t)\,[v_0(t),\dots,v_{k-1}(t)]|_H &= \left| \sum_{j=2}^{k} \sum_{\substack{i_1,\dots,i_j \in \mathbb{N} \\ \sum_{l=1}^j i_l = k}} \frac{\nabla^{(j)}F(v_0(t))[v_{i_1}(t),\dots,v_{i_j}(t)]}{j!} \right|_H \\ &\le \sum_{j=2}^{k} \sum_{\substack{i_1,\dots,i_j \in \mathbb{N} \\ \sum_{l=1}^j i_l = k}} \left| \frac{\nabla^{(j)}F(v_0(t))[v_{i_1}(t),\dots,v_{i_j}(t)]}{j!} \right|_H \end{aligned} \tag{4.13}$$

and using assumption (3) in Hypothesis 3.1, we get

$$
\begin{aligned}
|\Phi_k(t)|_H &\le \sum_{j=2}^{k} \sum_{\substack{i_1,\dots,i_j\in\mathbb{N}\\ \sum_{l=1}^{j} i_l=k}} \frac{1}{j!}\|\nabla F^{(j)}(v_0(t))\|_{\mathcal{L}(H^j,H))} \prod_{l=1}^{j} |v_{i_l}(t)|_H \\
&\le \sum_{j=2}^{k} \frac{1}{j!}\gamma_j(1+|v_0(t)|_H)^{m-j} \sum_{\substack{i_1,\dots,i_j\in\mathbb{N}\\ \sum_{l=1}^{j} i_l=k}} \sum_{l=1}^{j} |v_{i_l}(t)|_H^j \\
&\le \sum_{j=2}^{k} \frac{1}{j!}\gamma_j(1+|v_0(t)|_H)^{m-j} \sum_{\substack{i_1,\dots,i_j\in\mathbb{N}\\ \sum_{l=1}^{j} i_l=k}} \left(j+\sum_{l=1}^{k-1} |v_l(t)|_H^{k-1}\right) \qquad (4.14)\\
&\le \sum_{j=2}^{k} \frac{1}{j!}\gamma_j(1+|v_0(t)|_H)^{m-j} k^2 \left(k+\sum_{l=1}^{k-1} |v_l(t)|_H^{k-1}\right) \\
&\le C(1+|v_0(t)|_H^{m-2})k^2 \left(k+\sum_{l=1}^{k-1} |v_l(t)|_H^{k-1}\right),
\end{aligned}
$$

for some positive constant C, from which the assertion in Lemma (4.3) follows.

□

Remark 4.4. *Notice that by Lemma 4.3, if $v_1,\dots,v_{k-1}$ are p-mean ($p\in[2,\infty)$), integrable continuous stochastic processes then the same holds for Φ_k.*

5. Main results

Proposition 5.1. *Under Hypothesis 3.1 the following stochastic differential equation:*

$$
\begin{cases} du_1(t) = [Au_1(t) + \nabla F(u_0(t))[u_1(t)]]dt + \sqrt{Q}dW(t), & t\in[0,+\infty) \\ u_1(0)=0, \end{cases} \qquad (5.1)
$$

has a unique mild solution such that for any $p\ge 2$ it satisfies the following estimate:

$$
\mathbb{E}\sup_{t\in[0,T]} |u_1(t)|_H^p < +\infty, \qquad \text{for any } T>0. \qquad (5.2)
$$

Proof. First we show uniqueness. Let us suppose that $w_1(t)$ and $w_2(t)$ are two solutions of (5.1). Then by Itô's formula we have:

$$
\begin{aligned}
d|w_1(t)-w_2(t)|^2 &= \langle A(w_1(t)-w_2(t)), w_1(t)-w_2(t)\rangle\, dt \\
&\quad + \langle \nabla F(u_0(t))[w_1(t)-w_2(t)], w_1(t)-w_2(t)\rangle\, dt,
\end{aligned}
$$

so that, by the dissipativity condition on A and the estimate on ∇F in Hypothesis 3.1, (3), we have

$$
d|w_1(t)-w_2(t)|^2 \le -\omega|w_1(t)-w_2(t)|^2 + \gamma_1(1+|u_0|_H^{m-1})|w_1(t)-w_2(t)|^2.
$$

Now uniqueness follows by applying Gronwall's lemma.

As far as existence is concerned, we proceed by a fixed point argument. We introduce the mapping Γ from $\Xi_{p,T}$ (see. eq. (**??**)) into itself defined by

$$\Gamma(w(t)) := \int_0^t e^{(t-s)A} \nabla F(u_0(s)))[w(s)]ds + W_A(t).$$

We are going to prove that there exists $\tilde{T} > 0$ such that Γ is a contraction on $\mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$. In fact, for any $v, w \in \Xi_{p,T}$ we have, for any $0 \le t \le \tilde{T}$:

$$\begin{aligned}
\|\Gamma(v(t)) - \Gamma(w(t))\|^p &= \mathbb{E} \sup_{t\in[0,\tilde{T}]} \left| \int_0^t e^{(t-s)A} \nabla F(u_0(s)))[v(s) - w(s)]ds \right|_H^p \\
\le &\mathbb{E} \sup_{[0,\tilde{T}]} \int_0^t \|e^{(t-s)A}\|^p_{\mathcal{L}(H)} \left| \nabla F(u_0(s))[v(s) - w(s)] \right|_H^p ds \\
&\le \mathbb{E} \sup_{[0,\tilde{T}]} |\nabla F(u_0(s))[v(s) - w(s)]|^p_H \int_0^{\tilde{T}} \|e^{(\tilde{T}-s)A}\|^p_{\mathcal{L}(H)} ds \\
&\le \mathbb{E} \sup_{[0,\tilde{T}]} |v(s) - w(s)|^p_H \gamma_1^p (1 + |u_0(t)|^{m-1})^p \frac{1}{\omega p} \left(1 - e^{-\omega p \tilde{T}}\right) \\
&\le \frac{1}{\omega p} \gamma_1^p (1 + |u_0|^{m-1})^p \|v - w\|^p \left(1 - e^{-\omega p \tilde{T}}\right),
\end{aligned}$$

where in the third inequality we used conditions (3) in Hypothesis 3.1. Then if $\tilde{T}$ is sufficiently small (depending on ω, p, γ_1, u_0), we see that Γ is a contraction on $\mathcal{L}^p(\Omega; C([0, T]; H))$.

By considering the map Γ on intervals $[0, \tilde{T}], [\tilde{T}, 2\tilde{T}], \dots$ we have that Γ is a contraction on $\mathcal{L}^p(\Omega; C([0, T]; H))$, hence we have existence and uniqueness of the solution for inequality (5.1) in the space $\mathcal{L}^p(\Omega; C([0, T]; H))$ for all $p \in [2, \infty)$. Let us now consider estimate (5.5). By condition (3) in Hypothesis 3.1 we have for all points in the probability space and $p = 2a$ with $a \in \mathbb{N}$:

$$\begin{aligned}
\frac{d}{dt}|u_1(t)|^{2a}_H &= 2a\langle Au_1(t), u_1(t)\rangle |u_1(t)|^{2a-2}_H + 2a\langle \nabla F(u_0(t))[u_1(t)], u_1(t)\rangle |u_1(t)|^{2a-2}_H \\
&\quad + 2a\langle W_A(t), u_1(t)\rangle |u_1(t)|^{2a-2}_H \\
&\le -2a\omega |u_1(t)|^{2a}_H + 2a\gamma(1 + |u_0|^{m-1})|u_1(t)|^{2a}_H + 2a\langle W_A(t), u_1(t)\rangle |u_1(t)|^{2a-1}_H \\
&\le -2a\tilde{\omega}|u_1(t)|^{2a}_H + C_a |W_A(t)|^{2a}_H,
\end{aligned} \tag{5.3}$$

where $\tilde{\omega} := \omega - \gamma(1 + |u_0|)$. By Proposition 3.5 we have that:

$$\mathbb{E} \sup_{t\in[0,T]} |W_A(t)|^{2a}_H \le C'_a, \qquad T > 0,$$

so that integrating on $[0, \tilde{T}]$, taking the expectation of both members in the inequality and applying Gronwall's lemma to (5.3) we obtain:

$$\mathbb{E}\left[\sup_{t\in[0,T]} |u_1(t)|^{2a}_H \right] \le C'_a e^{-2\, a\, \tilde{\omega} T} < C_a$$

where C_a is a positive constant (independent of T) and (5.2) follows.

□

Theorem 5.2. *Let us fix $2 \leq k \leq n$, assume that Hypothesis 3.1 holds, and let u_1 be the solution of the problem (2.3), suppose moreover that u_j is the unique mild solution of the following Abstract Cauchy Problem (ACP):*

$$\text{(ACP(j))} \qquad \begin{aligned} &du_j(t) = [Au_j(t) + \nabla F(u_0(t))[u_j(t)]]dt + \Phi_j(t)\,[u_0(t), \dots, u_{j-1}(t)]\,dt \\ &u_j(0) = 0 \end{aligned}$$

for $j = 2, \dots, k-1$ satisfying:

$$\mathbb{E} \sup_{t \in [0,T]} |u_j(t)|_H^p < +\infty, \qquad T > 0; \tag{5.4}$$

then there exists a unique mild solution $u_k(t)$ of the following non-homogeneous linear differential equation with stochastic coefficients (in the sense of Def. 4.2) :

$$\text{(ACP(k))} \qquad \begin{aligned} &du_k(t) = [Au_k(t) + \nabla F(u_0(t))[u_k(t)]]dt + \Phi_k(t)dt \quad t \in [0, +\infty) \\ &u_k(0) = 0 \end{aligned}$$

and for any $p \in [2, \infty)$ it satisfies the following estimates, for any $T > 0$:

$$\mathbb{E} \sup_{t \in [0,T]} |u_k(t)|_H^p < +\infty. \tag{5.5}$$

Proof. We proceed by a fixed point argument, where the contraction is given by

$$\Gamma(y(t)) := \int_0^t e^{(t-s)A} \nabla F(u_0(t))[y(t)]ds + \int_0^t e^{(t-s)A} \Phi_k(s)ds$$

on $\mathcal{L}^p(\Omega; C([0,T]; H))$. In fact, arguing as in Proposition 5.1, we see that for $\tilde{T} \in [0,T]$ sufficiently small, Γ is a contraction on $\mathcal{L}^p(\Omega; C([0,\tilde{T}]; H))$, so that existence and uniqueness of the solution for ACP(k) follows for every $p \in [2, \infty)$. Let us consider estimate (5.5). By condition (4) in Hypothesis 3.1 we have, for $p = 2a$ with $a \in \mathbb{N}$ (and all points in the probability space) :

$$\begin{aligned} \frac{d}{dt} |u_k(t)|_H^{2a} &= 2a\langle Au_k(t), u_k(t)\rangle |u_k(t)|_H^{2a-2} + 2a\langle \nabla F(u_0(t))[u_k(t)], u_k(t)\rangle |u_k(t)|_H^{2a-2} \\ &\quad + 2a\langle \Phi_k(t), u_k(t)\rangle |u_k(t)|_H^{2a-2} \\ &\leq -2a\omega |u_k(t)|_H^{2a} + 2a\gamma(1 + |u_0|)|u_k(t)|_H^{2a} + 2a|\Phi_k(t)|_H |u_k(t)|_H^{2a-1} \\ &\leq -2a\tilde{\omega}|u_k(t)|_H^{2a} + C_a |\Phi_k(t)|_H^{2a}, \end{aligned} \tag{5.6}$$

where $\tilde{\omega} := \omega - \gamma(1 + |u_0|)$ as in the proof of Prop (5.1). By the assumption (5.4) made on $u_1(t), \dots, u_j(t)$ and Lemma 4.3 we have that:

$$\mathbb{E} \sup_{t \in [0,T]} |\Phi_k(t)|_H^{2a} \leq C'_a, \qquad T > 0,$$

so that taking the expectation of inequality (5.6) and applying Gronwall's lemma (similarly as in the proof of Proposition 5.1) we obtain:

$$\mathbb{E} \sup_{t \in [0,T]} |u_k(t)|_H^{2a} \leq C'_a e^{-2\,a\,\tilde{\omega} T} < C_a$$

where C_a is a positive constant (which can be taken independent of T) and the thesis follows. □

We are now able to state the main result of this section:

Theorem 5.3. *Under Hypothesis 3.1 the solution $u(t)$ of (2.2) can be expanded in powers of $\varepsilon > 0$ in the following form*

$$u(t) = u_0(t) + \varepsilon u_1(t) + \cdots + \varepsilon^n u_n(t) + R_n(t,\varepsilon), \quad n \in \mathbb{N},$$

where u_1 is the solution of

$$\begin{aligned} du_1(t) &= [Au_1(t) + \nabla F(u_0(t))[u_1(t)]]dt + \sqrt{Q}dW(t) \\ u_1(0) &= 0, \end{aligned}$$

while u_k, $k = 2, \dots, n$ is the solution of

$$\text{(ACP}(k)) \qquad \begin{cases} du_k(t) &= [Au_k(t) + \nabla F(u_0(t))u_k(t)]dt + \Phi_k(t)dt, \\ u_k(0) &= 0 \end{cases}$$

and $R_n(t,\varepsilon)$,

$$\begin{aligned} R_n(t,\varepsilon) &:= u(t) - u_0(t) - \sum_{k=1}^n \varepsilon^k u_k(t) = \\ &= \int_0^t e^{(t-s)A} \left(F(u(s)) - F(u_0(s)) - \sum_{k=1}^n \varepsilon^k \nabla F(u_0(s))[u_k(s)] - \sum_{k=2}^n \Phi_k(s) \right) ds, \end{aligned} \tag{5.7}$$

verifies the following inequality

$$\mathbb{E} \sup_{t\in[0,T]} |R_n(t,\varepsilon)|_H^p \le C_p \varepsilon^{n+1},$$

with a constant $C_p > 0$.

Proof. Let us define $R_n(t,\varepsilon)$, $n \in \mathbb{N}$, as stated in the theorem. Since by construction

- $u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A} F(u(s))ds + \varepsilon W_A(t)$
- $u_0(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A} F(u_0(s))ds$
- $u_1(t) = \int_0^t e^{(t-s)A} \nabla F(u_0(s))[u_1(s)]ds + W_A(t)$
- $u_k(t) = \int_0^t e^{(t-s)A} \nabla F(u_0(s))[u_k(s)]ds + \int_0^t e^{(t-s)A} \Phi_k(s)ds$ for $k = 2, \dots, n$

with $\Phi_k(s) := \Phi_k\left[u_0(s), \dots, u_{k-1}(s)\right]$ defined in (4.12), we have

$$R_n(t,\varepsilon) = \int_0^t e^{(t-s)A} \left(F(u(s)) - F(u_0(s)) - \sum_{k=1}^n \varepsilon^k \nabla F(u_0(s))[u_k(s)] - \sum_{k=2}^n \Phi_k(s) \right) ds.$$

Recalling that $R_1^{(n)}(u_0(s); h(s), \varepsilon) = F(u(s)) - F(u_0(s)) - \sum_{k=1}^n \varepsilon^k \nabla F(u_0(s))[u_k(s)] - \sum_{k=2}^n \Phi_k(s)$ we get:

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} R_n(t,\varepsilon)_H^p &\le \mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t e^{(t-s)A} R_1^{(n)}(u_0(s); h(s), \varepsilon) ds \right|_H^p \\ &\le \mathbb{E} \sup_{t\in[0,T]} \int_0^t \| e^{(t-s)A} \|_{\mathcal{L}(H,H)}^p |R_1^{(n)}(u_0(s); h(s), \varepsilon)|_H^p ds \\ &\le \mathbb{E} \sup_{t\in[0,T]} |R_1^{(n)}(u_0(t); h(t), \varepsilon)|_H^p \int_0^t e^{-\omega(t-s)p} ds \\ &\le C_{np} \varepsilon^{p(n+1)} \mathbb{E} \sup_{t\in[0,T]} |R_1^{(n)}(u_0(t); h(t), \varepsilon)|_H^p, \end{aligned} \tag{5.8}$$

for some positive constant C_{np} (depending on n, p, but not on ε), where in the second and third inequality we have used the contraction property of the semigroup generated by A. Now recalling Lemma 4.1 the inequality in Theorem 5.3 follows.

□

Remark 5.4. Our results apply in particular to stochastic PDEs describing the FitzHugh-Nagumo equation with a Gaussian noise perturbation, as those studied e.g., in [Tuc88a, Tuc88b, Tuc92] and [BM08].

The reference equation is given by (see [BM08, Equation (1.1)])

$$\begin{cases} \partial_t v(t,x) = \partial_x(c(x)\partial_x v(t,x)) - p(x)v(t,x) - w(t,x) + f(v(t,x)) + \varepsilon\dot{\beta}_1(t,x) \\ \partial_t w(t,x) = \gamma v(t,x) - \alpha w(t,x) + \varepsilon\dot{\beta}_2(t,x) \\ \partial_x v(t,0) = \partial_x v(t,1) = 0 \\ v(0,x) = v_0(x) \quad w(0,x) = w_0(x) \end{cases} \tag{5.9}$$

with the parameter $\varepsilon > 0$ in front of the noise, where u, w are real valued random variables, α, γ are strictly positive phenomenological constants and c, p are strictly positive smooth functions on $[0,1]$. Moreover, the initial values v_0, w_0 are in $C([0,1])$. The nonlinear term is of the form $f(v) = -v(v-1)(v-\xi)$ where $\xi \in (0,1)$. Finally β_1, β_2 are independent Q_i-Brownian motions with values in $L^2(0,1)$, with Q_i positive trace class commuting operators, commuting also with A_0, A_0 being defined below. The above equation can be rewritten in the form of an infinite dimensional stochastic evolution equation on the space

$$H := L^2(0,1) \times L^2(0,1) \tag{5.10}$$

by introducing the following operators:

$$\begin{aligned} &A_0 := \partial_x c(x) \partial_x, \\ &D(A_0) := \left\{ u \in H^2(0,1) : v_x(0) = v_x(1) \right\}, \end{aligned}$$

and

$$A = \begin{pmatrix} A_0 - p & -I \\ \gamma I & \alpha I \end{pmatrix},$$

with domain $D(A) := D(A_0) \times L^2(0,1)$, and

$$F\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -v(v-1)(v-\xi) \\ 0 \end{pmatrix},$$

In this way, equation (5.9) can be rewritten as

$$\begin{cases} du(t) = Au(t) + F(u(t))dt + \sqrt{Q}dW(t) \\ u(0) = u_0 \end{cases}$$

with A and F satisfying Hypothesis 3.1 when $\xi^2 - \xi + 1 \le 3\min_{x\in[0,1]} p(x)$ (cfr. [BM08]).

Then by Theorem 5.3 we get an asymptotic expansion in powers of $\varepsilon > 0$ of the solution, in terms of solutions of the corresponding deterministic FitzHugh-Nagumo equation and the solution of a system of (explicit) linear (non homogeneous) stochastic equations. The expansion holds for all orders in $\varepsilon > 0$. The remainders are estimated according to Theorem 5.3. We can use these results to carry through a discussion similar to the one made by Tuckwell [Tuc08, Tuc92] in the case where $Q = (Q_1, Q_2)$ with Q_i the identity. Tuckwell, in particular, has made expansions up to second order in ε for the mean and the variance of the solution process $u = (u(t))_{t\geq 0}$ (see [Tuc08, Tuc92]), proving in particular that one has enhancement (respectively reduction) of the mean according to whether the expansion is around which stable point of the stationary deterministic equation.

In a future work [ADPM] we shall apply these results to the case of networks of FitzHugh-Nagumo neurons. Moreover in the second part of the present work we shall study asymptotic expansions for the case where the dissipativity condition is replaced by other conditions on the non Lipschitz drift term.

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