Asymptotic convergence results for a system of partial differential equations with hysteresis

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Abstract

A partial differential equation motivated by electromagnetic field equations in ferromagnetic media is considered with a relaxed rate dependent constitutive relation. It is shown that the solutions converge to the unique solution of the limit parabolic problem with a rate independent Preisach hysteresis constitutive operator as the relaxation parameter tends to zero.

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Key words: partial differential equations, hysteresis, asymptotic convergence, Preisach operator.

1 Introduction

The aim of this paper is to study the following system of partial differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}(\alpha\, u + \beta\, w) - \triangle u = f \\ w = \overline{\mathcal{F}}\left(u - \gamma\, \dfrac{\partial w}{\partial t}\right) \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\overline{\mathcal{F}}$ is a continuous rate independent invertible hysteresis operator, f is a given function, γ, α and β are given positive constants.

This system can be obtained by coupling the Maxwell equations, the Ohm law and a constitutive relation between the magnetic field and the magnetic induction, provided we neglect the displacement current. A detailed derivation will be given in Section 3 below. The meaning of the parameter γ is to take into account in the constitutive relation also a rate dependent component of the memory. A similar system has been considered recently in [1] in the context of soil hydrology, with γ fixed and with a more general form of the elliptic part. The main goal of this paper, instead, is to investigate

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the behaviour of the solution as $\gamma \to 0$. Our main result consists in proving that the solutions to (1.1) converge as $\gamma \to 0$ to the (unique) solution (see [5]) of the system

$$\begin{cases} \dfrac{\partial}{\partial t}(\alpha\, u + \beta\, w) - \triangle u = f \\ w = \overline{\mathcal{F}}(u) \end{cases} \tag{1.2}$$

as an extension of the results contained in Chapter 4 of [4]. For γ positive, the second equation in (1.1) defines a constitutive operator $S : \mathbb{R} \times \mathcal{C}^0([0,T]) \to \mathcal{C}^1([0,T])$ which with each $u \in \mathcal{C}^0([0,T])$ and each initial condition $w^0 \in \mathbb{R}$ associates $w = S(w^0, u)$. Then (1.1) has the form

$$\frac{\partial}{\partial t}(\alpha\, u + \beta\, S(w^0, u)) - \triangle u = f. \tag{1.3}$$

The regularizing properties of S enable us to solve the problem by means of a simple application of the Banach contraction mapping principle. The passage to the limit as $\gamma \to 0$ is achieved in several steps, using in particular a lemma constructed ad hoc which allows us to pass to the limit in the nonlinear hysteresis term.

The outline of the paper is the following: after some remarks concerning Preisach operators (Section 2), we explain the physical interpretation of our model system in Section 3. Then we present in Section 4 the existence and uniqueness result while Section 5 is devoted to the asymptotic convergence of the solution as $\gamma \to 0$.

2 The Preisach operator

We describe the ferromagnetic behaviour using the Preisach model proposed in 1935 (see [16]). Mathematical aspects of this model were investigated by Krasnosel'skiĭ and Pokrovskiĭ (see [7], [8], and [9]). The model has been also studied in connection with partial differential equations by Visintin (see for example [17], [18]). The monograph of Mayergoyz ([15]) is mainly devoted to its modeling aspects.

Here we use the one-parametric representation of the Preisach operator which goes back to [10]. The starting point of our theory is the so called *play operator*. This operator constitutes the simplest example of continuous hysteresis operator in the space of continuous functions; it has been introduced in [9] but we can also find equivalent definitions in [2] and [18]; for its extension to less regular inputs, see also [12] and [13].

Let $r > 0$ be a given parameter. For a given input function $u \in \mathcal{C}^0([0,T])$ and initial condition $x^0 \in [-r, r]$, we define the output $\xi = \mathcal{P}_r(x^0, u) \in \mathcal{C}^0([0,T]) \cap BV(0,T)$ of the *play operator*

$$\mathcal{P}_r : [-r, r] \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T]) \cap BV(0,T)$$

as the solution of the variational inequality in Stieltjes integral form

$$\begin{cases} \displaystyle\int_0^T (u(t) - \xi(t) - y(t))\, d\xi(t) \geq 0 & \forall\, y \in \mathcal{C}^0([0,T]), \quad \displaystyle\max_{0 \le t \le T} |y(t)| \leq r, \\ |u(t) - \xi(t)| \leq r & \forall\, t \in [0,T], \\ \xi(0) = u(0) - x^0. \end{cases} \tag{2.1}$$

Let us consider now the whole family of play operators $\mathcal{P}_r$ parameterized by $r > 0$, which can be interpreted as a *memory variable*. Accordingly, we introduce the *hysteresis memory state space*

$$\Lambda := \{\lambda : \mathbb{R}_+ \to \mathbb{R} : \, |\lambda(r) - \lambda(s)| \leq |r - s| \;\; \forall\, r, s \in \mathbb{R}_+ : \lim_{r \to +\infty} \lambda(r) = 0\},$$

together with its subspaces

$$\Lambda_K = \{\lambda \in \Lambda : \, \lambda(r) = 0 \text{ for } r \geq K\}, \qquad \Lambda_\infty = \bigcup_{K > 0} \Lambda_K. \tag{2.2}$$

For $\lambda \in \Lambda,\; u \in \mathcal{C}^0([0,T])$ and $r > 0$ we set

$$\wp_r[\lambda, u] := \mathcal{P}_r(x_r^0, u) \qquad \wp_0[\lambda, u] := u,$$

where x_r^0 is given by the formula

$$x_r^0 := \min\{r, \max\{-r, u(0) - \lambda(r)\}\}.$$

It turns out that

$$\wp_r : \Lambda \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$$

is Lipschitz continuous in the sense that, for every $u, v \in \mathcal{C}^0([0,T]),\; \lambda, \mu \in \Lambda$ and $r > 0$ we have

$$||\wp_r[\lambda, u] - \wp_r[\mu, v]||_{\mathcal{C}^0([0,T])} \leq \max\{|\lambda(r) - \mu(r)|, ||u - v||_{\mathcal{C}^0([0,T])}\}. \tag{2.3}$$

Moreover, if $\lambda \in \Lambda_R$ and $||u||_{\mathcal{C}^0([0,T])} \leq R$, then $\wp_r[\lambda, u](t) = 0$ for all $r \geq R$ and $t \in [0,T]$. For more details, see Sections II.3, II.4 of [11].
Now we introduce the *Preisach plane* as follows

$$\mathscr{P} := \{(r, v) \in \mathbb{R}^2 : \, r > 0\}$$

and consider a function $\varphi \in L^1_{\text{loc}}(\mathscr{P})$ such that there exists $\beta_1 \in L^1_{\text{loc}}(0, \infty)$ with

$$0 \leq \varphi(r, v) \leq \beta_1(r) \qquad \text{for a.e. } (r, v) \in \mathscr{P}.$$

We set

$$g(r, v) := \int_0^v \varphi(r, z)\, dz \qquad \text{for } (r, v) \in \mathscr{P}$$

and for $R > 0$, we put $b_1(R) := \displaystyle\int_0^R \beta_1(r)\, dr.$
Then the *Preisach operator*

$$\mathcal{W} : \Lambda_\infty \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$$

generated by the function g is defined by the formula

$$\mathcal{W}[\lambda, u](t) := \int_0^\infty g(r, \wp_r[\lambda, u](t))\, dr, \tag{2.4}$$

for any given $\lambda \in \Lambda_\infty$, $u \in \mathcal{C}^0([0,T])$ and $t \in [0,T]$. The equivalence of this definition and the classical one in [15], [18], e.g., is proved in [10].
Using the Lipschitz continuity (2.3) of the operator $\wp_r$, it is easy to prove that also $\mathcal{W}$ is locally Lipschitz continuous, in the sense that, for any given $R > 0$, for every $\lambda, \mu \in \Lambda_R$ and $u, v \in \mathcal{C}^0([0,T])$ with $||u||_{\mathcal{C}^0([0,T])}, ||v||_{\mathcal{C}^0([0,T])} \leq R$, we have

$$||\mathcal{W}[\lambda, u] - \mathcal{W}[\mu, v]||_{\mathcal{C}^0([0,T])} \leq \int_0^R |\lambda(r) - \mu(r)|\, \beta_1(r)\, dr + b_1(R)\, ||u - v||_{\mathcal{C}^0([0,T])}.$$

The first result on the inverse Preisach operator was proved in [3]. We make use of the following formulation proved in [11], Section II.3.

Theorem 2.1. *Let $\lambda \in \Lambda_\infty$ and $b > 0$ be given. Then the operator $b\,I + \mathcal{W}[\lambda, \cdot] : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ is invertible and its inverse is Lipschitz continuous.*

Finally we have the following local monotonicity result for the Preisach operator $\mathcal{W}$.

Theorem 2.2. *Consider $b \geq 0$, $R > 0$, $\lambda \in \Lambda_R$ and $u \in W^{1,1}(0,T)$ be given such that $||u||_{\mathcal{C}^0([0,T])} \leq R$. Put $w := b\,u + \mathcal{W}[\lambda, u]$. Then*

$$b \left(\frac{\partial u}{\partial t}(t)\right)^2 \leq \frac{\partial w}{\partial t}(t)\, \frac{\partial u}{\partial t}(t) \leq (b + b_1(R)) \left(\frac{\partial u}{\partial t}(t)\right)^2.$$

As we are dealing with partial differential equations, we should consider both the input and the initial memory configuration λ that additionally depend on x. If for instance $\lambda(x, \cdot)$ belongs to Λ_∞ and $u(x, \cdot)$ belongs to $\mathcal{C}^0([0,T])$ for (almost) every x, then we define

$$\overline{\mathcal{W}}[\lambda, u](x,t) := \mathcal{W}[\lambda(x,\cdot), u(x,\cdot)](t) := \int_0^\infty g(r, \wp_r[\lambda(x,\cdot), u(x,\cdot)](t))\, dr. \qquad (2.5)$$

3 Physical interpretation of the model system (1.1)

Let a ferromagnetic material occupy a bounded region $\mathscr{D} \subset \mathbb{R}^3$; we set $\mathscr{D}_T := \mathscr{D} \times (0,T)$ for a fixed $T > 0$, and we assume that the body is surrounded by vacuum. We denote by $\vec{g}$ a prescribed electromotive force; then Ohm's law reads

$$\vec{J} = \sigma\,(\vec{E} + \vec{g}) \qquad \text{in } \mathscr{D},$$

where σ is the electric conductivity, $\vec{J}$ is the electric current density and $\vec{E}$ is the electric field; we also prescribe $\vec{J} = 0$ outside $\mathscr{D}$.
In $\mathscr{D}$, we consider the Ampère and the Faraday laws in the form

$$c\,\nabla \times \vec{H} = 4\pi\,\vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{in } \mathscr{D}_T,$$

$$c\,\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{in } \mathscr{D}_T,$$

where c is the speed of light in vacuum, $\vec{H}$ is the magnetic field, $\vec{D}$ is the electric displacement and $\vec{B}$ is the magnetic induction.

In case of a ferromagnetic metal, σ is very large, hence we can assume

$$4\,\pi\,|\vec{J}| \gg \left|\frac{\partial \vec{D}}{\partial t}\right| \qquad \text{in } \mathscr{D},$$

provided that the field $\vec{g}$ does not vary too rapidly.

Then we neglect the displacement current $\dfrac{\partial \vec{D}}{\partial t}$ in Ampère's law; this is the so-called *eddy current approximation*. By coupling this reduced law with Faraday's and Ohm's laws, in Gauss units we get

$$4\,\pi\,\sigma\,\frac{\partial \vec{B}}{\partial t} + c^2\,\nabla\times\nabla\times\vec{H} = 4\,\pi\,c\,\sigma\,\nabla\times\vec{g} \qquad \text{in } \mathscr{D}_T. \tag{3.1}$$

We consider the constitutive equation between $\vec{H}$ and $\vec{B}$ in the form $\vec{B} = \vec{H} + 4\,\pi\,\vec{M}$, where $\vec{M}$ is the magnetization, so we can rewrite (3.1) as

$$4\,\pi\,\sigma\,\frac{\partial}{\partial t}(\vec{H} + 4\,\pi\,\vec{M}) + c^2\,\nabla\times\nabla\times\vec{H} = 4\,\pi\,c\,\sigma\,\nabla\times\vec{g} \qquad \text{in } \mathscr{D}_T\,.$$

For more details on this topics, we refer to a classical text of electromagnetism, for example [6].

We now reduce this system to a scalar one describing *planar waves*. More precisely, let Ω be a domain of $\mathbb{R}^2$. We assume (using the orthogonal Cartesian coordinates x, y, z) that $\vec{H}$ is parallel to the $z-$axis and only depends on the coordinates x, y, i.e.

$$\vec{H} = (0, 0, H(x, y)).$$

Then

$$\nabla\times\nabla\times\vec{H} = (0, 0, -\triangle_{x,y}H) \qquad \left(\triangle_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{3.2}$$

We also assume that

$$\vec{M} = (0, 0, M(x, y)), \qquad \nabla\times\vec{g} = (0, 0, \tilde{f});$$

then equation (3.1) is reduced to a scalar equation

$$\frac{4\,\pi\,\sigma}{c^2}\left[\frac{\partial}{\partial t}(H + 4\,\pi\,M)\right] - \triangle_{x,y}H = f := \frac{4\,\pi\,\sigma}{c}\,\tilde{f}. \tag{3.3}$$

The purely rate independent hysteretic constitutive relation between H and M is considered in the form

$$M = \overline{\mathcal{W}}(H), \tag{3.4}$$

where $\overline{\mathcal{W}}$ is a Preisach operator. Since $\overline{\mathcal{W}}$ itself is in typical cases not invertible, we introduce a new variable $V = M + \delta\, H$ with some $\delta \in (0, 1/4\pi)$ to be specified below, and rewrite (3.3), (3.4) as

$$\begin{cases} \dfrac{4\,\pi\,\sigma}{c^2}\,\dfrac{\partial}{\partial t}\,[(1-4\,\pi\,\delta)\,H + 4\,\pi\,V] - \triangle_{x,y} H = f \\ V = (\delta\, I + \overline{\mathcal{W}})(H), \end{cases} \tag{3.5}$$

which is precisely (1.2) with $\alpha = \dfrac{4\,\pi\,\sigma}{c^2}\,(1-4\,\pi\,\delta)$, $\beta = \dfrac{16\,\pi^2\,\sigma}{c^2}$, $u = H$, $w = V$ and $\overline{\mathcal{F}} = \delta\, I + \overline{\mathcal{W}}$. The rate dependent relaxed constitutive law leading to (1.1) reads

$$V = (\delta\, I + \overline{\mathcal{W}})\left(H - \gamma\,\frac{\partial V}{\partial t}\right). \tag{3.6}$$

4 Existence and uniqueness

In the setting (1.1) or (1.2), the space dimension is not relevant. We therefore consider an open bounded set of Lipschitz class $\Omega \subset \mathbb{R}^N$, $N \geq 1$, set $Q := \Omega \times (0,T)$, and fix an initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_K) \qquad \text{for some } K > 0\,, \tag{4.1}$$

where Λ_K is introduced in (2.2).
Let $\mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ be the Fréchet space of strongly measurable functions $\Omega \to \mathcal{C}^0([0,T])$, i.e. the space of functions $v : \Omega \to \mathcal{C}^0([0,T])$ such that there exists a sequence v_n of simple functions with $v_n \to v$ in $\mathcal{C}^0([0,T])$ a.e. in Ω.
We fix a constant $b_{\mathcal{F}} > 0$ and introduce the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ in the following way

$$\overline{\mathcal{F}}(u)(x,t) := \mathcal{F}(u(x,\cdot))(t) := b_{\mathcal{F}}\, u(x,t) + \mathcal{W}[\lambda(x,\cdot), u(x,\cdot)](t); \tag{4.2}$$

here $\mathcal{W}$ is the scalar Preisach operator defined in (2.4).
Now Theorem 2.1 yields that $\mathcal{F}$ is invertible and its inverse is a Lipschitz continuous operator in $\mathcal{C}^0([0,T])$. Let us set $\mathcal{G} = \mathcal{F}^{-1}$ and let $L_{\mathcal{G}}$ be the Lipschitz constant of the operator $\mathcal{G}$.
At this point we introduce the operator

$$\overline{\mathcal{G}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \qquad \overline{\mathcal{G}} := \overline{\mathcal{F}}^{-1}. \tag{4.3}$$

It turns out that

$$\overline{\mathcal{G}}(w)(x,t) := \mathcal{G}(w(x,\cdot))(t) \qquad \forall\, w \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]));\tag{4.4}$$

it follows from Theorem 2.1 that $\overline{\mathcal{G}}$ is Lipschitz continuous in the following sense

$$||\overline{\mathcal{G}}(u_1)(x,\cdot) - \overline{\mathcal{G}}(u_2)(x,\cdot)||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{G}}\, ||u_1(x,\cdot) - u_2(x,\cdot)||_{\mathcal{C}^0([0,T])}$$

for any $u_1, u_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$, a.e. in Ω.

Moreover Theorem 2.2 entails that there exist two constants $c_{\mathcal{F}}$ and $C_{\mathcal{F}}$ such that

$$c_{\mathcal{F}} \left(\frac{\partial u}{\partial t}\right)^2 \leq \frac{\partial \overline{\mathcal{F}}(u)}{\partial t} \frac{\partial u}{\partial t} \leq C_{\mathcal{F}} \left(\frac{\partial u}{\partial t}\right)^2 \qquad \text{a.e. in } Q. \tag{4.5}$$

On the other hand, (4.5) entails

$$c_{\mathcal{G}} \left(\frac{\partial w}{\partial t}\right)^2 \leq \frac{\partial \overline{\mathcal{G}}(w)}{\partial t} \frac{\partial w}{\partial t} \leq C_{\mathcal{G}} \left(\frac{\partial w}{\partial t}\right)^2 \quad \text{a.e. in } Q, \text{ with } C_{\mathcal{G}} = \frac{1}{c_{\mathcal{F}}}, \;\; c_{\mathcal{G}} = \frac{1}{C_{\mathcal{F}}}. \tag{4.6}$$

Consider now system (1.1) with homogeneous Dirichlet boundary conditions and set $V := H_0^1(\Omega)$. We first state the existence and uniqueness result.

Theorem 4.1. *(Existence and uniqueness)*
Let α, β, γ be given positive constants. Suppose that the following assumptions on the data

$$f \in L^2(Q), \;\; u^0 \in V, \;\; w^0 \in L^2(\Omega)$$

hold. Then (1.1) *with homogeneous Dirichlet boundary conditions and initial conditions*

$$u(x,0) = u^0(x), \qquad w(x,0) = w^0(x), \tag{4.7}$$

admits a unique solution

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,2}(\Omega)), \; w \in L^2(\Omega;\mathcal{C}^1([0,T])).$$

Proof. The proof is divided into two steps.

• STEP 1: THE SOLUTION OPERATOR S. We neglect for the moment the dependence on the space parameter x within the constitutive relation

$$\gamma \frac{\partial w}{\partial t} + \overline{\mathcal{G}}(w) = u. \tag{4.8}$$

This means that we deal here with the following problem: for a given $u \in \mathcal{C}^0([0,T])$, find $w \in \mathcal{C}^1([0,T])$ such that

$$\begin{cases} \gamma \dfrac{dw}{dt} + \mathcal{G}(w) = u \\ w(0) = w^0 \end{cases} \qquad \text{in } [0,T]. \tag{4.9}$$

Clearly problem (4.9) admits a unique solution $w \in \mathcal{C}^1([0,T])$, for every $u \in \mathcal{C}^0([0,T])$, due to the Lipschitz continuity of $\mathcal{G}$. In this manner we can define the solution operator

$$S : \mathcal{C}^0([0,T]) \to \mathcal{C}^1([0,T]) : \;\; u \mapsto w.$$

Let us show now that S is Lipschitz continuous in the sense that we prove that there exists a constant L_S such that

$$||S(u_1) - S(u_2)||_{\mathcal{C}^1([0,t])} \leq \; L_S \, ||u_1 - u_2||_{\mathcal{C}^0([0,t])}, \;\; \forall \, u_1, u_2 \in \mathcal{C}^0([0,t]), \;\; \forall \, t \in [0,T]. \tag{4.10}$$

Let us consider $u_1, u_2 \in \mathcal{C}^0([0,T])$ and let $w_1, w_2 \in \mathcal{C}^1([0,T])$ be such that $w_i = S(u_i)$, $i = 1,2$. The initial data are fixed, that is, $w_1(0) = w_2(0) = w^0$. For any $t \in [0,T]$ we have

$$\left|\frac{dw_1}{dt}(t) - \frac{dw_2}{dt}(t)\right| \leq \frac{1}{\gamma}\,|u_1(t) - u_2(t)| + \frac{L_{\mathcal{G}}}{\gamma} \max_{0\leq\tau\leq t} |w_1(\tau) - w_2(\tau)|$$

$$\leq \frac{1}{\gamma}\,|u_1(t) - u_2(t)| + \frac{L_{\mathcal{G}}}{\gamma} \int_0^t \left|\frac{dw_1}{dt} - \frac{dw_2}{dt}\right|(\tau)\,d\tau.$$

Hence, by Gronwall's argument,

$$\int_0^t \left|\frac{dw_1}{dt} - \frac{dw_2}{dt}\right|(\tau)\,d\tau \leq \frac{1}{\gamma}\int_0^t e^{\frac{L_{\mathcal{G}}}{\gamma}(t-\tau)}\,|u_1(\tau) - u_2(\tau)|\,d\tau,$$

which yields

$$\left|\frac{dw_1}{dt}(t) - \frac{dw_2}{dt}(t)\right| \leq \frac{1}{\gamma}\,e^{\frac{L_{\mathcal{G}}}{\gamma}T}\,||u_1 - u_2||_{\mathcal{C}^0([0,t])}$$

for every $t \in [0,T]$. Hence (4.10) holds with $L_S = \left(\frac{1}{\gamma} + \frac{1}{L_{\mathcal{G}}}\right) e^{\frac{L_{\mathcal{G}}}{\gamma}T}$.

We easily extend this estimate to the space dependent problem

$$\begin{cases} \gamma\,\dfrac{\partial w}{\partial t} + \overline{\mathcal{G}}(w) = u \\ w(\cdot,0) = w^0(\cdot) \end{cases} \qquad \text{a.e. in } Q, \tag{4.11}$$

with given functions $u \in L^2(\Omega;\mathcal{C}^0([0,T]))$, $w^0 \in L^2(\Omega)$. It immediately follows from (4.10) that the solution mapping

$$\bar{S} : L^2(\Omega;\mathcal{C}^0([0,T])) \to L^2(\Omega;\mathcal{C}^1([0,T])) : \qquad u \mapsto w \tag{4.12}$$

associated with (4.11) is well defined and Lipschitz continuous, with Lipschitz constant L_S.

STEP 2: FIXED POINT. Our model problem can be rewritten now as

$$\frac{\partial}{\partial t}(\alpha\,u + \beta\,\bar{S}(u)) - \triangle u = f \tag{4.13}$$

with $u(\cdot,0) = u^0(\cdot)$ and homogeneous Dirichlet boundary conditions. The unique solution will be found by the Banach contraction mapping principle.

Let us fix $z \in H^1(0,T;L^2(\Omega))$; then $z \in L^2(\Omega;\mathcal{C}^0([0,T]))$ and therefore $\bar{S}(z)$ is well-defined and belongs to $L^2(\Omega;\mathcal{C}^1([0,T]))$. Instead of (4.13), we consider the equation

$$\frac{\partial}{\partial t}(\alpha\,u + \beta\,\bar{S}(z)) - \triangle u = f \tag{4.14}$$

which is nothing but the linear heat equation. As $f \in L^2(Q)$, this means that (4.14) admits a unique solution $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;V)$.

We now introduce the set

$$\tilde{B} = \{z \in H^1(0,T;L^2(\Omega)) : \, z(\cdot,0) = u^0(\cdot)\}$$

and the operator

$$\tilde{J} : \tilde{B} \to \tilde{B} : \qquad z \mapsto u,$$

which with every $z \in \tilde{B}$ associates the solution $u \in \tilde{B}$ of (4.14). In order to prove that $\tilde{J}$ is a contraction, consider now two elements $z_1, z_2 \in \tilde{B}$, and set $u_1 := \tilde{J}(z_1)$, $u_2 := \tilde{J}(z_2)$. Then we have

$$\frac{\partial}{\partial t}(\alpha\,(u_1 - u_2) + \beta\,(\bar{S}(z_1) - \bar{S}(z_2))) - \triangle(u_1 - u_2) = 0.$$

We test this equation by $\dfrac{\partial}{\partial t}(u_1 - u_2)$ and obtain

$$\begin{aligned} &\alpha \int_\Omega \left|\frac{\partial}{\partial t}(u_1 - u_2)\right|^2 (x,t)\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla(u_1 - u_2)|^2(x,t)\,dx \\ &\le \frac{\alpha}{2}\int_\Omega \left|\frac{\partial}{\partial t}(u_1 - u_2)\right|^2 (x,t)\,dx + \frac{L_S^2\,\beta^2}{2\,\alpha}\int_\Omega \max_{0\le\tau\le t} |z_1 - z_2|^2(x,\tau)\,dx, \end{aligned}$$

where L_S is the Lispchitz constant of the operator $\bar{S}$. This implies that

$$\begin{aligned} &\int_\Omega \left|\frac{\partial}{\partial t}(u_1 - u_2)\right|^2 (x,t)\,dx + \frac{1}{\alpha}\frac{d}{dt}\int_\Omega |\nabla(u_1 - u_2)|^2(x,t)\,dx \\ &\le \frac{L_S^2\,\beta^2\,t}{\alpha^2}\int_0^t \int_\Omega \left|\frac{\partial}{\partial t}(z_1 - z_2)\right|^2 (x,\tau)\,dx\,dt. \end{aligned} \tag{4.15}$$

We set $\theta := \dfrac{L_S^2\,\beta^2}{\alpha^2}$ and we introduce the following equivalent norm on $H^1(0,T;L^2(\Omega))$

$$|||\eta||| = \left(||\eta(0)||^2_{L^2(\Omega)} + \int_0^T e^{-\theta\,t^2}\left|\left|\frac{\partial \eta}{\partial t}\right|\right|^2_{L^2(\Omega)}(t)\,dt\right)^{1/2} \qquad \forall\,\eta \in H^1(0,T;L^2(\Omega)).$$

If now we multiply (4.15) by $e^{-\theta\,t^2}$ and integrate over $t \in (0,T)$, we obtain that

$$|||u_1 - u_2||| \le \frac{1}{2}\,|||z_1 - z_2|||$$

and thus $\tilde{J}$ is a contraction on the closed subset $\tilde{B}$ of $H^1(0,T;L^2(\Omega))$, which yields the existence and uniqueness of the solution $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty(0,T;V)$. $\square$

5 Asymptotic convergence

In this section we investigate the behaviour of the solution of our model problem if the parameter γ goes to zero. We prove the following theorem.

Theorem 5.1. *Under the assumptions of Theorem* 4.1, *let* (u_γ, w_γ) *be the unique solution of* (1.1) *corresponding to* $\gamma > 0$ *with initial conditions* (4.7) *and homogeneous Dirichlet boundary conditions. Then there exists*

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,2}(\Omega))$$

such that

$$\begin{aligned} u_\gamma &\to u && \text{strongly in } L^2(\Omega;\mathcal{C}^0([0,T])) \\ w_\gamma &\to \overline{\mathcal{F}}(u) && \text{strongly in } L^2(\Omega;\mathcal{C}^0([0,T])) \end{aligned}$$

as $\gamma \to 0$, *and* u *is the unique solution of the equation*

$$\frac{\partial}{\partial t}(\alpha\, u + \beta\, \overline{\mathcal{F}}(u)) - \triangle u = f \tag{5.1}$$

with initial condition $u(x,0) = u^0(x)$ *and homogeneous Dirichlet boundary condition.*

Proof. The regularity of u_γ and w_γ allows us to differentiate (4.11) in time and obtain

$$\gamma\, \frac{\partial^2 w_\gamma}{\partial t^2} + \frac{\partial \overline{\mathcal{G}}(w_\gamma)}{\partial t} = \frac{\partial u_\gamma}{\partial t} \qquad \text{a.e.} \tag{5.2}$$

In the series of estimates below, we denote by $C_1, C_2, \ldots$ any positive constant depending only on the data of the problem, but independent of γ.

We now test the first equation of (1.1) by $\dfrac{\partial u_\gamma}{\partial t}$ and (5.2) by $\beta\, \dfrac{\partial w_\gamma}{\partial t}$. This yields

$$\int_\Omega \left(\alpha \left(\frac{\partial u_\gamma}{\partial t} \right)^2 + \beta\, \frac{\partial u_\gamma}{\partial t}\, \frac{\partial w_\gamma}{\partial t} \right) dx + \frac{1}{2}\, \frac{d}{dt} \int_\Omega |\nabla u_\gamma|^2\, dx = \int_\Omega \left(f\, \frac{\partial u_\gamma}{\partial t} \right) dx \tag{5.3}$$

and

$$\beta\, \frac{\gamma}{2}\, \frac{d}{dt} \int_\Omega \left(\frac{\partial w_\gamma}{\partial t} \right)^2 dx + \beta \int_\Omega \frac{\partial \overline{\mathcal{G}}(w_\gamma)}{\partial t}\, \frac{\partial w_\gamma}{\partial t}\, dx = \beta \int_\Omega \frac{\partial u_\gamma}{\partial t}\, \frac{\partial w_\gamma}{\partial t}\, dx. \tag{5.4}$$

Summing up (5.3), (5.4) and using (4.6), we obtain

$$\frac{\alpha}{2} \int_\Omega \left| \frac{\partial u_\gamma}{\partial t} \right|^2 dx + \frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla u_\gamma|^2\, dx + c_{\mathcal{G}}\, \beta \int_\Omega \left| \frac{\partial w_\gamma}{\partial t} \right|^2 dx + \beta\, \frac{\gamma}{2}\frac{d}{dt} \int_\Omega \left| \frac{\partial w_\gamma}{\partial t} \right|^2 dx \le C_1.$$

This allows us to obtain the following estimates

$$\begin{cases} ||u_\gamma||_{H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)} \le C_2, & \text{(5.5a)} \\ ||w_\gamma||_{H^1(0,T;L^2(\Omega))} \le C_3, & \text{(5.5b)} \\ \sqrt{\gamma}\, \left|\left| \dfrac{\partial w_\gamma}{\partial t} \right|\right|_{L^\infty(0,T;L^2(\Omega))} \le C_4, & \text{(5.5c)} \end{cases}$$

and, by comparison, $||\triangle u_\gamma||_{L^2(Q)} \leq C_5$. This entails that there exists a function u and a sequence $\gamma_n \to 0$ such that

$$u_{\gamma_n} \to u \text{ weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,2}(\Omega)).$$

On the other hand, by interpolation and after a suitable choice of representatives, we deduce that (see [14], Chapter 4)

$$H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \subset L^2(\Omega;\mathcal{C}^0([0,T]))$$

with continuous and compact injection; this ensures that

$$u_{\gamma_n} \to u \text{ strongly in } L^2(\Omega;\mathcal{C}^0([0,T])),$$

in particular (passing to subsequences if necessary),

$$u_{\gamma_n} \to u \text{ uniformly in } [0,T], \text{ a.e. in } \Omega. \tag{5.6}$$

On the other hand, the constitutive relation (4.8) yields

$$||u_\gamma - \overline{\mathcal{G}}(w_\gamma)||_{L^\infty(0,T;L^2(\Omega))} \leq \gamma \left|\left|\frac{\partial w_\gamma}{\partial t}\right|\right|_{L^\infty(0,T;L^2(\Omega))}$$

and this, together with (5.5c), entails that

$$u_\gamma - \overline{\mathcal{G}}(w_\gamma) \to 0 \qquad \text{strongly in } L^\infty(0,T;L^2(\Omega)) \text{ as } \gamma \to 0.$$

From now on, we keep the sequence $\gamma_n \to 0$ fixed as in (5.6). Our aim is now to show that there exists a function w such that

$$w_{\gamma_n} \to w \text{ uniformly in } [0,T], \text{ a.e. in } \Omega. \tag{5.7}$$

In fact, this will allow us to pass to the limit in the nonlinear hysteresis term. We show that (5.7) is obtained from (5.6) by using the following lemma:

Lemma 5.2. *Consider a sequence of functions* $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{C}^0([0,T])$ *such that*

$$||u_n - u||_{\mathcal{C}^0([0,T])} \to 0 \qquad \text{as } n \to \infty.$$

Let $0 < a_n \leq \alpha_n(t) \leq b_n$ *be measurable functions, with* $\lim_{n\to\infty} b_n = 0$. *Finally let* $\{v_n\}_{n\in\mathbb{N}}$ *be solutions of the following Cauchy problem*

$$\begin{cases} \alpha_n(t)\,\dfrac{dv_n}{dt}(t) + v_n(t) = u_n(t), \\ v_n(0) = u_n(0). \end{cases}$$

Then

$$||v_n - u||_{\mathcal{C}^0([0,T])} \to 0 \qquad \text{as } n \to \infty.$$

Proof. Put $\beta_n(t) = \dfrac{1}{\alpha_n(t)}$. Then

$$v_n(t) = e^{-\int_0^t \beta_n(\tau)\,d\tau}\, u_n(0) + \int_0^t \beta_n(s)\, e^{-\int_s^t \beta_n(\tau)\,d\tau}\, u_n(s)\, ds$$

hence, for all $t \in [0,T]$, we get

$$v_n(t) - u_n(t) = e^{-\int_0^t \beta_n(\tau)\,d\tau}\, (u_n(0) - u_n(t)) + \int_0^t \beta_n(s)\, e^{-\int_s^t \beta_n(\tau)\,d\tau}\, (u_n(s) - u_n(t))\, ds.$$

Let now $\varepsilon > 0$ be given. Using the Ascoli-Arzelà theorem, we find $\delta > 0$ *independent of* n such that

$$|t_1 - t_2| < \delta \Rightarrow |u_n(t_1) - u_n(t_2)| < \varepsilon.$$

For $t \in [0,\delta]$ we have

$$|v_n(t) - u_n(t)| \le \varepsilon \left(e^{-\int_0^s \beta_n(\tau)\,d\tau} + \int_0^t \beta_n(s)\, e^{-\int_s^t \beta_n(\tau)\,d\tau} ds \right) = \varepsilon.$$

Let now $t > \delta$, and let

$$C = \sup\{|u_n(t_1) - u_n(t_2)|,\ t_1, t_2 \in [0,T], n \in \mathbb{N}\}.$$

Then

$$\begin{aligned} |v_n(t) - u_n(t)| \le & C e^{-\int_0^t \beta_n(\tau)\,d\tau} + \varepsilon \int_{t-\delta}^t \beta_n(s) e^{-\int_s^t \beta_n(\tau)\,d\tau}\, ds + C \int_0^{t-\delta} \beta_n(s) e^{-\int_s^t \beta_n(\tau)\,d\tau}\, ds \\ = & \varepsilon \left(1 - e^{-\int_{t-\delta}^t \beta_n(\tau)\,d\tau} \right) + C\, e^{-\int_{t-\delta}^t \beta_n(\tau)\,d\tau} \le \varepsilon + C\, e^{-\frac{\delta}{b_n}}, \end{aligned}$$

and thus Lemma 5.2 follows. □

Let $\Omega' \subset \Omega$ be a set of full measure ($\text{meas}(\Omega \setminus \Omega') = 0$) such that, by virtue of (5.6), $u_{\gamma_n}(x,\cdot) \to u(x,\cdot)$ converges uniformly for all $x \in \Omega'$. Keeping now $x \in \Omega'$ fixed, set

$$u_\gamma(x,\cdot) := \tilde{u}_\gamma(\cdot), \qquad w_\gamma(x,\cdot) := \tilde{w}_\gamma(\cdot).$$

We recall from (4.2) that

$$\mathcal{F}(v(x,\cdot))(t) = \overline{\mathcal{F}}(v)(x,t) \qquad \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])).$$

Our idea is to apply Lemma 5.2 to the Cauchy problem

$$\begin{cases} \tilde{w}_\gamma = \mathcal{F}\left(\tilde{u}_\gamma - \gamma \dfrac{d\tilde{w}_\gamma}{dt} \right), \\ \tilde{w}_\gamma(0) = \mathcal{F}(\tilde{u}_\gamma)(0), \end{cases} \tag{5.8}$$

which we rewrite as

$$\begin{cases} \gamma \dfrac{d\tilde{w}_\gamma}{dt} + \tilde{v}_\gamma = \tilde{u}_\gamma, \\ \tilde{w}_\gamma = \mathcal{F}(\tilde{v}_\gamma), \\ \tilde{w}_\gamma(0) = \mathcal{F}(\tilde{u}_\gamma)(0). \end{cases} \tag{5.9}$$

We now set

$$\alpha_\gamma(t) = \begin{cases} \gamma \, \dfrac{d\mathcal{F}(\tilde v_\gamma)}{dt}(t) \Big/ \dfrac{d\tilde v_\gamma}{dt}(t) & \text{if } \dfrac{d\tilde v_\gamma}{dt} \neq 0 \\ \gamma \, c_{\mathcal{F}} & \text{if } \dfrac{d\tilde v_\gamma}{dt} = 0. \end{cases}$$

From (4.5) we obtain that

$$0 < \gamma \, c_{\mathcal{F}} \leq \alpha_\gamma(t) \leq \gamma \, C_{\mathcal{F}}.$$

Hence, system (5.9) can be rewritten in the form

$$\begin{cases} \alpha_\gamma(t) \, \dfrac{d\tilde v_\gamma}{dt}(t) + \tilde v_\gamma(t) = \tilde u_\gamma(t), \\ \tilde v_\gamma(0) = \tilde u_\gamma(0). \end{cases}$$

We have that

$$\tilde u_{\gamma_n} \to \tilde u \ \text{ uniformly in } \mathcal{C}^0([0,T]) \ \text{ as } \gamma_n \to 0\,,$$

hence by Lemma 5.2,

$$\tilde v_{\gamma_n} \to \tilde u \ \text{ uniformly in } \mathcal{C}^0([0,T]) \ \text{ as } \gamma_n \to 0\,.$$

This in turn entails that

$$\tilde w_{\gamma_n} \to \mathcal{F}(\tilde u) \ \text{ uniformly in } \mathcal{C}^0([0,T]) \ \text{ as } \gamma_n \to 0\,.$$

Since $x \in \Omega'$ has been chosen arbitrarily, we obtain

$$w_{\gamma_n} \to \overline{\mathcal{F}}(u) \ \text{ uniformly in } \mathcal{C}^0([0,T])\,, \text{ a.e. in } \Omega \ \text{ as } \gamma_n \to 0\,.$$

We thus checked that u is a solution of (5.1) with the required boundary and initial condition. Since this solution is unique by the argument of [5], we conclude that u_γ converges to u independently of how γ tends to 0. This completes the proof of Theorem 5.1. $\square$

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