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Global and local $\mathbb{Q}$-algebrization problems in real algebraic geometry

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To papà

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Introduction

State of the art & open problems

The algebrization problem: state of the art. The description of geometric objects in the simplest possible terms is one of the goals of geometry, particularly real algebraic geometry. Indeed, the need to simplify the description of smooth manifolds and even topological spaces with singularities, and to find increasingly rich structures on them, has contributed significantly to the development of real algebraic geometry.

In 1936, Whitney [Whi36] proved that every smooth manifold M of dimension d can be smoothly embedded in $\mathbb{R}^{2d+1}$ and every smooth embedding $\psi : M \to \mathbb{R}^{2d+1}$ can be approximated by an arbitrarily close smooth embedding $\phi : M \to \mathbb{R}^{2d+1}$ whose image $M' := \phi(M)$ is a real analytic submanifold of $\mathbb{R}^{2d+1}$. It follows that M can be described both globally and locally by means of real analytic equations in some Euclidean space. Indeed, one first identifies M with $M' \subset \mathbb{R}^{2d+1}$ via ϕ and then observes that M' is the set of solutions of finitely many global real analytic equations defined on the whole $\mathbb{R}^{2d+1}$ (by Cartan's Theorem B) and, locally at each of its points, M' is the set of solutions of good real analytic equations, where "good" means "with linearly independent gradients". In particular, M has a real analytic structure. At this point it is natural to ask whether M admits a real algebraic structure obtained by requiring that the previous real analytic equations describing M' are polynomial equations with coefficients in $\mathbb{R}$.

In his groundbreaking paper [Nas52] published in 1952, Nash proved that the answer is affirmative in the sense that, if the smooth manifold M is compact, then we can assume that the real analytic submanifold $M' \subset \mathbb{R}^{2d+1}$ approximating $\psi(M)$ is actually a union of nonsingular connected components of a real algebraic subset of $\mathbb{R}^{2d+1}$. Nash conjectured that M' can be chosen to be a whole nonsingular real algebraic subset of $\mathbb{R}^{2d+1}$, a so-called algebraic model of M. In 1957, Wallace [Wal57] tried to verify Nash conjecture but his proof was not correct, so his main improvement was is that $M' \subset \mathbb{R}^n$ can be chosen algebraic if M is the boundary of a compact $\mathscr{C}^\infty$ manifold with boundary. However, his attempt was very important: for the first time cobordism theory came into play. Finally, in 1973, Tognoli [Tog73] proved this conjecture to be true by improving Nash approximation techniques and deeply applying cobordism theory, in particular nonsingular algebraic representatives of cobordism classes found by Milnor [Mil65]. Quoting from page 4 of [BCR98]:

"*A systematic study of real algebraic varieties started seriously only in 1973 after Tognoli's surprising discovery (based on earlier work of John Nash) that every compact smooth manifold is diffeomorphic to a nonsingular real algebraic set.*"

The latter assertion is the so-called Nash-Tognoli theorem. For a detailed proof, we refer the reader to [BCR98, Section 14.1]. See also the surveys [DL19, Section 1] and [Kol17, Section 1] on Nash's work, recently written by De Lellis and Kollár, for other fine presentations of this crucial result.

There is a wide literature devoted to improvements and extensions of Nash-Tognoli theorem. A remarkable result is a relative version with respect to a finite set of smooth submanifolds in general position proved by Akbulut and King [AK81a]. For this topic, we refer the reader to the books [AK92, Chapter II], [BCR98, Chapter 14], [Man14, §6], the survey [Kol17, Section 2], the recent papers [Ben22; GT17; Kuc11] and the numerous references therein.

The problem of making topological spaces algebraic has also been studied in the singular case.

In 1981, Akbulut and King [AK81b] obtained a complete description of the topology of real algebraic sets with isolated singularities. Their idea is to consider the class $\mathcal{T}$ of compact topological spaces V that admits a topological desingularization in the following sense: there exist a closed smooth manifold M, a finite family $\mathcal{M} = \{M_1, \dots, M_\ell\}$ of pairwise disjoint subsets of M and a finite set $S = \{p_1, \dots, p_m\}$ with $m \geq \ell$ such that each M_i is a finite union of smooth hypersurfaces of M in general position and the quotient topological space obtained from $M \sqcup S$ by blowing down each M_i to p_i is homeomorphic to V. The topological data $(M, \mathcal{M}; S)$ represents a topological desingularization of V. By Hironaka's desingularization theorem [Hir64] (see also [BM97; Vil89; Kol07]), the class $\mathcal{T}$ includes all the compact real algebraic sets with isolated singularities. Now, the strategy of Akbulut and King is first to make algebraic the topological data $(M, \mathcal{M}; S)$, obtaining real algebraic data $(M', \mathcal{M}'; S)$ by mean of algebraic approximation techniques a là Nash-Tognoli, and then to blow down these algebraic data obtaining a real algebraic set homeomorphic to V. As Alexandrov's compactification of a real algebraic set can be made algebraic, it follows that a (not necessarily compact) topological space V is homeomorphic to a real algebraic set with isolated singularities if and only if it can be obtained from topological data $(M, \mathcal{M}; S)$ such as the above by considering $M \sqcup S$, blowing down some of the M_i to points of S and removing the remaining M_i (see [AK81b, Section 4]). As a consequence, a noncompact smooth manifold admits an algebraic model if and only if it is diffeomorphic to the interior of a compact smooth manifold with non-empty boundary (see [AK81b, Corollary 4.3]).

In addition to the case of isolated singularities, singular topological spaces admitting algebraic models have been deeply studied in small dimension, that is, in dimension ≤ 3. In 1981, Benedetti and Dedò [BD81] completely characterized those triangulable compact topological spaces of dimension 2 admitting algebraic models by means of a unique condition on a local invariant: the evenness of the Euler characteristic of the link. Later on, in 1992, Akbulut and King [AK92] characterized those triangulable compact topological spaces of dimension 3 admitting an algebraic model by means of five independent local topological invariants at each point, one of those is the evenness of the Euler characteristic of the link. For this very remarkable characterization, the construction previously developed for the case of isolated singularities in [AK81b] is deeply improved by the so-called "Resolution Tower technique". As one may expect, the study of triangulable compact topological spaces becomes more and more difficult as the dimension increases and already in dimension 4 a complete characterization seems unreachable. Indeed, in 2000, McCrory and Parusiński [MP00] proved that there are at least $2^{43} - 43$ local independent characteristic numbers which vanish for algebraic sets of dimension 4. Last result then directed the research on the characterization of those triangulable topological spaces admitting an algebraic structure on the subclass of compact Nash sets. In 2017, Ghiloni and Tancredi [GT17] proved that a compact Nash set is semialgebraically homeomorphic to an algebraic set if and only if it is asymmetrically cobordant to a point, last condition is then conjectured by the authors to be always satisfied.

Another classical topic of study is the algebrization of germs of analytic sets. In 1986 Kucharz [Kuc86] proved that if $(V,0) \subset (\mathbb{R}^n,0)$ is the germ of a coherent analytic set with an isolated singularity at 0, then $(V,0)$ is analytically equivalent to an algebraic set germ $(V',0) \subset (\mathbb{R}^n,0)$. However, it is well known since [Whi65] that in general germs of analytic sets are not even diffeomorphic to germs of algebraic sets. Nevertheless, by decreasing the expected regularity on the homeomorphism, the situation is different. In 1984 Mostowski [Mos84] proved that every analytic set germ $(V,0) \subset (\mathbb{R}^n,0)$ is homeomorphic to an algebraic set germ $(V',0) \subset (\mathbb{R}^m,0)$. In the same year Bochnak and Kucharz [BK84] proved that the algebraic set germ $(V',0) \subset (\mathbb{R}^m,0)$ can be chosen with $m = n$. A remarkable result used in the proof of latter theorem, which is an improvement of Artin-Mazur theorem [AM65; BCR98, Theorem 8.4.4], ensures that every Nash set germ $(V,0) \subset (\mathbb{R}^n,0)$ is Nash diffeomorphic to an algebraic set germ $(V',0) \subset (\mathbb{R}^n,0)$.

How to simplify the equations of algebraic sets? Results & open problems. Now it is natural to ask whether the description of a geometric object admitting a real algebraic structure can be further simplified by requiring that the coefficients of the describing polynomial equations belong to the smallest possible subfield K of $\mathbb{R}$. Here the final goal is $K = \mathbb{Q}$, the field of rational numbers which is the smallest subfield of $\mathbb{R}$.

The answer is affirmative if K is the field $\overline{\mathbb{Q}}^r$ of real algebraic numbers, the smallest real closed field. This is due to three of the most important results in semialgebraic and Nash geometry. Let V be a real algebraic subset of $\mathbb{R}^n$. For short we often omit the adjective "real", saying that V is an algebraic subset of $\mathbb{R}^n$ or $V \subset \mathbb{R}^n$ is an algebraic set. Choose a description of V:

$$V = \{x \in \mathbb{R}^n : f_1(a,x) = \ldots = f_\ell(a,x) = 0\}$$

for some polynomials $f_i \in \mathbb{Z}[y_1,\ldots,y_m,x_1,\ldots,x_n]$, where $a = (a_1,\ldots,a_m) \in \mathbb{R}^m$. Note that $a \in \mathbb{R}^m$ is the vector of all coefficients (ordered in some way) that appear in a fixed polynomial system $f_1 = 0,\ldots,f_\ell = 0$ in $\mathbb{R}^n$ whose set of solutions is V. Define

$$X := \{(b,x) \in (\overline{\mathbb{Q}}^r)^{m+n} : f_1(b,x) = \ldots = f_\ell(b,x) = 0\}$$

and denote by $\pi : X \to (\overline{\mathbb{Q}}^r)^m$ the projection $(b,x) \mapsto b$. By Hardt's trivialization theorem [Har80], there exists a finite semialgebraic partition $\{M_i\}_{i=1}^p$ of $(\overline{\mathbb{Q}}^r)^m$ and, for each $i \in \{1,\ldots,p\}$, an algebraic subset F_i of $(\overline{\mathbb{Q}}^r)^n$ and a semialgebraic homeomorphism $h_i : M_i \times F_i \to X \cap \pi^{-1}(M_i)$ compatible with π. By the Tarski-Seidenberg principle [BCR98, Chapter 5], we can extend coefficients from $\overline{\mathbb{Q}}^r$ to $\mathbb{R}$, obtaining a semialgebraic partition $\{(M_i)_\mathbb{R}\}_{i=1}^p$ of $\mathbb{R}^m$ and semialgebraic homeomorphisms $(h_i)_\mathbb{R} : (M_i)_\mathbb{R} \times (F_i)_\mathbb{R} \to X_\mathbb{R} \cap \pi_\mathbb{R}^{-1}((M_i)_\mathbb{R})$ compatible with $\pi_\mathbb{R}$, where

$\pi_{\mathbb{R}} : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is the projection $(b, x) \mapsto b$. It follows that a belongs to $(M_j)_{\mathbb{R}}$ for a unique $j \in \{1, \ldots, p\}$, and hence V is semialgebraically homeomorphic to $(F_j)_{\mathbb{R}}$. Note that $(F_j)_{\mathbb{R}}$ is an algebraic subset of $\mathbb{R}^n$ that admits a global description as the set of solutions of finitely many polynomial equations with coefficients in $\overline{\mathbb{Q}}^r$. In [CS92, Theorem A] Coste and Shiota proved a version of Hardt's trivialization theorem for Nash manifolds. As a consequence, if the algebraic set $V \subset \mathbb{R}^n$ is nonsingular, we can assume that the algebraic set $F_j \subset (\overline{\mathbb{Q}}^r)^n$ is nonsingular and V is Nash diffeomorphic to $(F_j)_{\mathbb{R}}$. Now $(F_j)_{\mathbb{R}}$ is a nonsingular algebraic subset of $\mathbb{R}^n$ that, in addition to the above global description, admits local descriptions as the sets of solutions of good polynomial equations with coefficients in $\overline{\mathbb{Q}}^r$.

Observe that in previous paragraph $\mathbb{R}$ can be safely substituted with any real closed field R. This means that we always have a transfer principle, both for algebraic sets and Nash manifolds, from any real closed field R to $\overline{\mathbb{Q}}^r$, the smallest one which is contained in any other real closed field.

An alternative positive answer in the case K is the field $\overline{\mathbb{Q}}^r$ is given by Parusiński and Rond [PR20] by means of Zariski equisingularity, that is, they construct a Zariski equisingular deformation $a' = (a'_1, \ldots, a'_m) \in (\overline{\mathbb{Q}}^r)^m$ of the coefficients $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ in the same strata M_i given by Hardt's trivialization theorem as above. As a consequence, they are able to find a subanalytic and arc-analytic homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(V) = V' := \{x \in \mathbb{R}^n : f_1(a', x) = \ldots = f_\ell(a', x) = 0\}$. In addition, Zariski equisingularity allows the authors to prove an analogous result in the case of algebraic sets $V \subset \mathbb{C}^n$ with respect to the subfield $K = \overline{\mathbb{Q}}$ of algebraic numbers.

As $\mathbb{Q}$ is not a real closed field, none of the above results by Hardt, Tarski-Seidenberg, Coste-Shiota and Parusiński-Rond are available in general in the case $K = \mathbb{Q}$. A partial answer is given by Parusiński and Rond in [PR20, Theorem 11 & Remark 13], indeed they prove that if the field extension of $\mathbb{Q}$ obtained by adding the coefficients $a_1, \ldots, a_m$ is purely transcendental, the above algebraic set $V' \subset \mathbb{R}^n$, produced by a Zariski deformation of the coefficients $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$, can be found in such a way that $a' = (a'_1, \ldots, a'_m) \in \mathbb{Q}^m$. The reason why this approach does not provide a complete solution of the case $K = \mathbb{Q}$ is that the above Zariski equisingular deformation of the coefficients $a = (a_1, \ldots, a_m)$ preserves the polynomial relations over $\mathbb{Q}$ satisfied by $a_1, \ldots, a_m$. In general, we have the following open problem.

Open Problem 1 ([Par21, Open problem 1, p. 199]). *Is every real algebraic set homeomorphic to a real algebraic set defined by polynomial equations with rational coefficients?*

Let us focus on the case of analytic set germs $(V, 0) \subset (\mathbb{R}^n, 0)$. By [Par21, Theorem 4.4.5], which is a direct consequence of [BK84; Mos84], we may assume that $(V, 0)$ is an algebraic set germ. As in the case of algebraic sets, both Hardt's trivialization theorem and Zariski equisingularity led to positive answers with $K = \overline{\mathbb{Q}}^r$. In [Ron18], Rond constructs a Zariski equisingular deformation $a' = (a'_1, \ldots, a'_m) \in (\overline{\mathbb{Q}}^r)^m$ of the coefficients $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ in such a way that the $\overline{Q}^r$-algebraic set germ $(V', 0) \subset (\mathbb{R}^n, 0)$ is homeomorphic to the starting algebraic set germ $(V, 0) \subset (\mathbb{R}^n, 0)$, as above. In particular, the homeomorphism can be found subanalytic and arc-analytic. As in the case of Zarisky deformations of algebraic sets, Rond proves an analogous theorem also for complex analytic singularities $(V, 0) \subset (\mathbb{C}^n, 0)$ with respect to the subfield $K = \overline{\mathbb{Q}}$. A remarkable example by Teissier [Tei90] proves that there are germs of algebraic singularities $(V, 0) \subset (\mathbb{C}^n, 0)$ (in his example defined over $\mathbb{Q}(\sqrt{5})$) which are not Whitney equisingular to any algebraic singularity $(V', 0) \subset (\mathbb{C}^n, 0)$ defined by polynomial equations with coefficients in $\mathbb{Q}$. The obstruction on existence of such algebraic singularities described by equations over $\mathbb{Q}$ comes from the property of Whitney equisingularity to preserve the angles. In general, we have the following open problem.

Open Problem 2 ([Par21, Open problem 2, p. 200]). *Is every real analytic set germ homeomorphic to a set germ defined by polynomial equations with rational coefficients?*

Since Zariski equisingularity preserves algebraic relations over $\mathbb{Q}$ on coefficients, it seems to be too rigid to give general answers to [Par21, Open problems 1 & 2], thus our starting idea was to go back to the algebraic approximation techniques developed by Nash, Tognoli and Akbulut-King in order to adapt them to get algebraic sets described by polynomial equations with rational coefficients. The implementation of this starting idea is not an easy task for several reasons. First, we have to decide what meaning to give to the concept of real algebraic set "defined over $\mathbb{Q}$". Indeed, unlike the complex case, this concept is rather subtle and lends itself to several natural interpretations.

Main results

Algebraic geometry over subfields. Previous ambiguity on defining algebraic sets over subfields is deeply investigated in recent work [FG] by Fernando and Ghiloni. Here we specify some of their results in order to present the main theorems of [GS23; Sav23].

Let R be a real closed field. Fix $n \in \mathbb{N} \setminus \{0\}$. For short we denote the rings of polynomials $\mathbb{Q}[x_1, \dots, x_n]$ and $R[x_1, \dots, x_n]$ as $\mathbb{Q}[x]$ and $R[x]$, respectively. Consider $\mathbb{Q}[x]$ as a subset of $R[x]$. Let V be a subset of R^n. As in [FG], we say that V is a $\mathbb{Q}$*-algebraic subset of* R^n, or $V \subset R^n$ is a $\mathbb{Q}$*-algebraic set*, if there exist polynomials $p_1, \dots, p_\ell \in \mathbb{Q}[x]$ such that

$$V = \{x \in R^n : p_1(x) = \ldots = p_\ell(x) = 0\}.$$

If $V \subset R^n$ is a $\mathbb{Q}$-algebraic set then it is also a (real) algebraic set, so we can speak about the dimension $\dim(V)$ of V, the set $\operatorname{Reg}(V)$ of regular points of V and the set $\operatorname{Sing}(V) = V \setminus \operatorname{Reg}(V)$ of singular points of V. Moreover, as with all algebraic sets, $V \subset R^n$ is said to be nonsingular if $\operatorname{Reg}(V) = V$. Our standard reference for real algebraic, semialgebraic and Nash geometry is [BCR98]. In particular, the reader can find details on the dimension, regular points and singular points of algebraic sets in [BCR98, Sections 2.8 & 3.3].

Observe that there are algebraic sets that are not $\mathbb{Q}$-algebraic, as shown by $V := \{\sqrt{2}\} \subset R$.

Pick a point $a = (a_1, \dots, a_n) \in R^n$ and consider again the set $V \subset R^n$. We denote by $\mathfrak{n}_a$ the maximal ideal $(x_1 - a_1, \dots, x_n - a_n)R[x]$ of $R[x]$ and by $\mathcal{I}_{\mathbb{Q}}(V)$ the vanishing ideal of V in $\mathbb{Q}[x]$, that is,

$$\mathcal{I}_{\mathbb{Q}}(V) := \{p \in \mathbb{Q}[x] : p(x) = 0, \forall x \in V\}.$$

The following notion of $R|\mathbb{Q}$-regular point was defined in [FG] and corresponds to Definition 1.5.1 below with the following substitutions "L"="E":= R and "K"= $\mathbb{Q}$.

Definition 1. Let $V \subset R^n$ be a $\mathbb{Q}$-algebraic set and let $a \in V$. We define the *$R|\mathbb{Q}$-local ring $\mathcal{R}^{R|\mathbb{Q}}_{V,a}$ of V at a* as

$$\mathcal{R}^{R|\mathbb{Q}}_{V,a} := R[x]_{\mathfrak{n}_a}/(\mathcal{I}_{\mathbb{Q}}(V)R[x]_{\mathfrak{n}_a}).$$

We say that a is a *$R|\mathbb{Q}$-regular point of V* if $\mathcal{R}^{R|\mathbb{Q}}_{V,a}$ is a regular local ring of dimension $\dim(V)$. We denote by $\operatorname{Reg}^{R|\mathbb{Q}}(V)$ the set of all $R|\mathbb{Q}$-regular points of V.

As was shown in [FG], the set $\operatorname{Reg}^{R|\mathbb{Q}}(V)$ is a non-empty Zariski open subset of $\operatorname{Reg}(V)$. Moreover, the following $R|\mathbb{Q}$-Jacobian criterion holds true: a point a of the $\mathbb{Q}$-algebraic set $V \subset R^n$ is $R|\mathbb{Q}$-nonsingular if and only if there exist an Euclidean open neighborhood U of a in R^n and polynomials $p_1, \dots, p_{n-d} \in \mathcal{I}_{\mathbb{Q}}(V)$, where $d := \dim(V)$, such that $V \cap U = \{x \in U : p_1(x) = \ldots = p_{n-d}(x) = 0\}$ and the gradients $\nabla p_1(a), \dots, \nabla p_{n-d}(a)$ are linearly independent.

It may happen that $\operatorname{Reg}^{R|\mathbb{Q}}(V)$ is strictly contained in $\operatorname{Reg}(V)$. For instance, the $\mathbb{Q}$-algebraic line $V := \{x_1 + \sqrt[3]{2}x_2 = 0\} = \{x_1^3 + 2x_2^3 = 0\} \subset R^2$ is nonsingular (as an algebraic set), but $(0, 0)$ is not $R|\mathbb{Q}$-nonsingular.

Let us introduce the concepts of $\mathbb{Q}$-determined and $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets that are the protagonists of the main results in [GS23; Sav23]. Next definition corresponds to Definition 1.6.1 below with the following substitution "K"= $\mathbb{Q}$.

Definition 2. Let $V \subset R^n$ be a $\mathbb{Q}$-algebraic set. We say that V is *$\mathbb{Q}$-determined* if $\operatorname{Reg}^{R|\mathbb{Q}}(V) = \operatorname{Reg}(V)$. If in addition V is nonsingular, that is, $\operatorname{Reg}^{R|\mathbb{Q}}(V) = \operatorname{Reg}(V) = V$, then we say that V is *$\mathbb{Q}$-nonsingular*.

Nash-Tognoli theorem 'over $\mathbb{Q}$'. Here fix $\mathbb{R}$, the field of real numbers, as the ground real closed field. Equip each real vector space $\mathbb{R}^n$ with the usual Euclidean topology, and each subset of $\mathbb{R}^n$ with the corresponding relative topology. Let N be a subset of $\mathbb{R}^n$. Denote by $\mathscr{C}^0_{\mathrm{w}}(N, \mathbb{R}^m)$ the set $\mathscr{C}^0(N, \mathbb{R}^m)$ of all continuous maps from N to $\mathbb{R}^m$, equipped with the usual compact-open topology (also called weak $\mathscr{C}^0$ topology). Suppose now that N is a Nash submanifold of $\mathbb{R}^n$. Let $\mathscr{C}^\infty(N, \mathbb{R}^m)$ be the set of all $\mathscr{C}^\infty$ maps from N to $\mathbb{R}^m$, and let $\mathcal{N}(N, \mathbb{R}^m)$ be the subset of $\mathscr{C}^\infty(N, \mathbb{R}^m)$ consisting of all Nash maps from N to $\mathbb{R}^m$. Denote by $\mathscr{C}^\infty_{\mathrm{w}}(M, \mathbb{R}^m)$ the set $\mathscr{C}^\infty(M, \mathbb{R}^m)$ equipped with the usual weak $\mathscr{C}^\infty$ topology, see [Hir94, §2, Section 1]. Equip $\mathcal{N}(M, \mathbb{R}^m)$ with the relative topology induced by $\mathscr{C}^\infty_{\mathrm{w}}(M, \mathbb{R}^m)$, and denote by $\mathcal{N}_{\mathrm{w}}(M, \mathbb{R}^m)$ the corresponding topological space.

Recall that the nonsingular locus $\operatorname{Reg}(V)$ of an algebraic set $V \subset \mathbb{R}^n$ is a Nash submanifold of $\mathbb{R}^n$, and $V \subset \mathbb{R}^n$ is said to have isolated singularities if $\operatorname{Sing}(V)$ is finite.

Given a subset T of $\mathbb{R}^m$, we define $T(\mathbb{Q})$ as the intersection $T \cap \mathbb{Q}^m$. If $m \geq n$, we identify $\mathbb{R}^n$ with the real vector subspace $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$. Thus, we can write $\mathbb{R}^n \subset \mathbb{R}^m$ and a subset of $\mathbb{R}^n$ is also a subset of $\mathbb{R}^m$.

We gave a first answer to [Par21, Open problem 1, p. 199] in [GS23] in the case of compact smooth manifolds, so in particular of compact nonsingular algebraic sets, with the following version of Nash-Tognoli theorem. Next result corresponds to a simplified version of Theorem 3.2.2 below after the application of Lemma 2.1.8 below.

THEOREM 3 (Nash-Tognoli theorem 'over $\mathbb{Q}$'). *Let M be a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$ of dimension d. Set $m := \max\{n, 2d+1\}$. Then, for every neighborhood $\mathcal{V}$ of the inclusion map $M \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^\infty_{\mathrm{w}}(M, \mathbb{R}^m)$, there exists a $\mathscr{C}^\infty$ embedding $\psi : M \to \mathbb{R}^m$ such that $\psi \in \mathcal{V}$ and $\psi(M)$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^m$.*

A previous version of Theorem 3 can be found in [BT92, Theorem 0.1].

Relative Nash-Tognoli theorem 'over $\mathbb{Q}$'. A natural question is whether relative versions of Theorem 3 and [Par21, Open problem 1, p. 199] hold for compact smooth manifolds and nonsingular algebraic sets, respectively. Let us clarify what we mean by 'a relative version of [Par21, Open problem 1, p. 199]'.

RELATIVE NONSINGULAR $\mathbb{Q}$-ALGEBRIZATION PROBLEM: *Is every nonsingular real algebraic set V, with nonsingular algebraic subsets $\{V_i\}_{i=1}^\ell$, in general position, homeomorphic to a nonsingular algebraic set V', with nonsingular algebraic subsets $\{V_i'\}_{i=1}^\ell$, in general position, all defined by polynomial equations with rational coefficients such that the homeomorphism sends each V_i to V_i'?*

Next result corresponds to the main $\mathbb{Q}$-algebrization theorem of [Sav23] in the compact case (see Theorem 4.1.4 below).

THEOREM 4 (Relative Nash-Tognoli theorem 'over $\mathbb{Q}$'). *Let M be a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$ of dimension d and let $\{M_i\}_{i=1}^\ell$ be a finite family of $\mathscr{C}^\infty$ submanifolds of M in general position. Set $m := \max\{n, 2d+1\}$. Then, for every neighborhood $\mathcal{U}$ of the inclusion map $\iota : M \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^\infty_{\mathrm{w}}(M, \mathbb{R}^m)$ and for every neighborhood $\mathcal{U}_i$ of the inclusion map $\iota|_{M_i} : M_i \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^\infty_{\mathrm{w}}(M_i, \mathbb{R}^m)$, for $i \in \{1, \dots, \ell\}$, there exist a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $M' \subset \mathbb{R}^m$, a family $\{M_i'\}_{i=1}^\ell$ of $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of M' in general position and a $\mathscr{C}^\infty$ diffeomorphism $h : M \to M'$ which simultaneously takes each M_i to M_i' such that, if $\jmath : M' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map, then $\jmath \circ h \in \mathcal{U}$ and $\jmath \circ h|_{M_i} \in \mathcal{U}_i$ for $i \in \{1, \dots, \ell\}$.*

If in addition M and each M_i are compact Nash manifolds, then we can assume $h : M \to M'$ is a Nash diffeomorphism and h extends to a semialgebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$.

A deep preparatory result used in the proof of Theorem 4 is the complete description 'over $\mathbb{Q}$' of the $\mathbb{Z}/2\mathbb{Z}$-homology of real embedded Grasmannians obtained by an explicit desingularization technique in [Sav23] inspired by Zelevinski paper [Zel83] (see Theorems 2.3.4 & 2.4.10 below). In addition, by algebraic compactification, resolution of singularities and our Akbulut-King blowing down lemma 'over $\mathbb{Q}$' (see Lemma 3.3.3 below) we have the following general answer to the RELATIVE NONSINGULAR $\mathbb{Q}$-ALGEBRIZATION PROBLEM originally proved in [Sav23] (see Theorem 4.1.6 below).

THEOREM 5. *Let V be a nonsingular algebraic subset of $\mathbb{R}^n$ of dimension d and let $\{V_i\}_{i=1}^\ell$ be a finite family of nonsingular algebraic subsets of V in general position.*

Set $m := n+2d+3$. Then, for every neighborhood $\mathcal{U}$ of the inclusion map $\iota : V \hookrightarrow \mathbb{R}^m$ in $\mathcal{N}_{\mathrm{w}}(V, \mathbb{R}^m)$ and for every neighborhood $\mathcal{U}_i$ of the inclusion map $\iota|_{V_i} : V_i \hookrightarrow \mathbb{R}^m$ in $\mathcal{N}_{\mathrm{w}}(V_i, \mathbb{R}^m)$ for $i \in \{1, \dots, \ell\}$, there exist a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $V' \subset \mathbb{R}^m$, a family $\{V_i'\}_{i=1}^{\ell}$ of $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of V' in general position and a Nash diffeomorphism $h : V \to V'$ which simultaneously takes each V_i to V_i' such that, if $\jmath : V' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map, then $\jmath \circ h \in \mathcal{U}$ and $\jmath \circ h|_{M_i} \in \mathcal{U}_i$ for $i \in \{1, \dots, \ell\}$. Moreover, h extends to a semialgebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$.

$\mathbb{Q}$-Algebrization of Nash manifolds over real closed fields. We provide an answer to [Par21, Open problem 1, p. 199] also in case the ground field is any real closed field R. Namely, we prove the following result (see Theorem 4.2.2 below).

THEOREM 6. *Let $M \subset R^n$ be a Nash manifold of dimension d. Then, there exists a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $M' \subset R^m$ and a Nash diffeomorphism $h : M \to M'$, for some $m \in \mathbb{N}$ with $m \geq n$. In particular, $M' \subset R^m$ can be chosen in such a way that $\dim(\mathrm{Zcl}_{R^m}(M'(\mathbb{Q}))) \geq d - 1$.*

Last result suggests that also other $\mathbb{Q}$-algebrization results we proved over $\mathbb{R}$ in [GS23; Sav23] may be extended to Nash manifolds and real algebraic sets over any real closed field. This will be one of the topics of future investigations by the author beyond this thesis.

$\mathbb{Q}$-Algebrization of isolated singularities. Next theorem is the main result of [GS23] (see Theorem 4.3.6 below). In the compact case (see Theorem 4.3.4 below), there is an improvement on the estimate of m, indeed we may set $m = n + 2d + 3$.

THEOREM 7. *Let $V \subset \mathbb{R}^n$ be an algebraic set with isolated singularities. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ with the following properties:*

(i) *$V' \subset \mathbb{R}^m$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set.*
(ii) *$\phi(\mathrm{Reg}(V)) = \mathrm{Reg}(V')$ and $\phi| : \mathrm{Reg}(V) \to \mathrm{Reg}(V')$ is a Nash diffeomorphism. In particular, V' is $\mathbb{Q}$-nonsingular if V is nonsingular.*

More precisely, the following is true. Denote by d the dimension of V and set $m := n + 2d + 4$. Choose a neighborhood $\mathcal{U}$ of the inclusion map $V \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^0_{\mathrm{w}}(V, \mathbb{R}^m)$, and a neighborhood $\mathcal{V}$ of the inclusion map $\mathrm{Reg}(V) \hookrightarrow \mathbb{R}^m$ in $\mathcal{N}_{\mathrm{w}}(\mathrm{Reg}(V), \mathbb{R}^m)$. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ that have both the preceding properties (i) *and* (ii) *and the following:*

(iii) *The Zariski closure of $V'(\mathbb{Q})$ in $\mathbb{R}^m$ has dimension at least $d - 1$.*
(iv) *ϕ extends to a semialgebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$.*
(v) *ϕ fixes $\mathrm{Sing}(V) \cap \mathbb{Q}^n$, that is, $\phi(x) = x$ for all $x \in \mathrm{Sing}(V) \cap \mathbb{Q}^n$. In particular, $V'(\mathbb{Q})$ contains $\mathrm{Sing}(V) \cap \mathbb{Q}^n$.*
(vi) *If $\jmath : V' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map, then $\jmath \circ \phi \in \mathcal{U}$ and $(\jmath \circ \phi)|_{\mathrm{Reg}(V)} \in \mathcal{V}$.*

If we are willing to lose properties (v) & (vi), we can find a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic model V' of V with an improvement on the estimate of m, namely, we can choose $m = 2d + 4$.

THEOREM 8. *Let $V \subset \mathbb{R}^n$ be an algebraic set with isolated singularities of dimension d. Set $m := 2d+4$. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ having properties* (i)-(iv) *of Theorem* 7.

As above, if V is compact, the estimate of m in Theorem 8 can be further improved to $m = 2d+3$.

Another consequence of Theorem 7 is the following $\mathbb{Q}$-algebrization result of germs of algebraic isolated singularities (see Theorem 4.4.1 below).

THEOREM 9. *Let $(V,0) \subset (\mathbb{R}^n, 0)$ be the germ of an isolated algebraic singularity. Then there exist a germ of an isolated algebraic singularity $(V',0) \subset (\mathbb{R}^m, 0)$, semialgebraic nieghborhoods U of 0 in $\mathbb{R}^n$ and U' of 0 in $\mathbb{R}^m$ and a semialgebraic homeomorphism $\phi : V \cap U \to V' \cap U'$, with the following properties:*

(i) *$V' \subset \mathbb{R}^m$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set.*
(ii) *$\phi(\mathrm{Reg}(V) \cap U) = \mathrm{Reg}(V') \cap U'$ and $\phi| : \mathrm{Reg}(V) \cap U \to \mathrm{Reg}(V') \cap U'$ is a Nash diffeomorphism.*

Observe that Theorem 9 constitutes a general answer of [Par21, Open problem 2., p. 200] in the case of algebraic sets germs with an isolated singularity.

On the degree of global smoothing mappings. In 2018, Bierstone and Parusiński [BP18] proved that every subanalytic set can be globally smoothed, both in an embedded and a non-embedded way (see [BP18, Thms. 1.1 and 1.2]). In Appendix B we introduce a evenness criterion for non-embedded global smoothings of closed subanalytic sets only depending on the global topology of the set. The criterion (see Theorem B.2.3 below) reads as follows:

THEOREM 10 ([Sav22, Theorem 4]). *Let X be a closed subanalytic subset of $\mathbb{R}^n$, let $\Gamma := (X', \varphi, U)$ be a global smoothing section of $X \subset \mathbb{R}^n$ and let W be a connected component of U. If W has a nonbounding equator in X then the degree of Γ over W is even.*

As a consequence, we have new examples of subanalytic sets (actually, semialgebraic sets) only admitting even-to-one global smoothing sections (see Examples B.2.6 below).

Structure of the thesis

Here we briefly summarize the structure of the thesis. We recall that at the beginning of each chapter and appendix the reader will find an abstract in which we present the technical content section by section.

Chapter 1: Algebraic geometry over subfields. In this first chapter we review the recent paper [FG] of Fernando and Ghiloni. We present the results in full generality, as stated and proved in their work, both for sake of clarity with respect to the reference and because we need in some particular points field extensions other than $\mathbb{R}|\mathbb{Q}$, which is the main case of study of this thesis. Here we introduce the general notions and properties we use in next chapters: we define K-algebraic sets of C^n and R^n, where C denotes an algebraically closed field of characteristic zero, R a real closed field and K a subfield of C or R, respectively, and we study the global algebraic and geometric properties via Galois theory and commutative algebra. It is evident from Fernando and Ghiloni's results that the really interesting case of algebraic geometry over subfields is the one in which the ground field R is closed real and K is not. Then we introduce and study new notions of regularity for K-algebraic sets with respect to a subfield K of a real closed field R developed by Fernando and Ghiloni. This notion leads to the definitions of K-determined and K-nonsingular K-algebraic sets, which are the main characters of next chapters. Latter notion of K-nonsingular K-algebraic set is in turn the right notion to obtain K-generic projection results and to separate 'K-algebraically' the irreducible components of K-nonsingular K-algebraic sets. Another fundamental result of [FG] is the equivalence between their new notion of $R|K$-regularity with a K-version of the Jacobian criterion. Up to Section 1.6 the author's contribution reduces to organize those notions and results originally introduced in [FG] that are necessary to next chapters and to add clarifying examples or remarks. In Section 1.6, the author characterizes $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic sets via Galois theory and introduces a new class of $\mathbb{Q}$-algebraic sets which is called "defined over $\mathbb{Q}$ in $\mathcal{R}(R^n)$", where $\mathcal{R}(R^n)$ denotes the ring of regular functions of R^n, and studies the relations of this notion with previous ones in full generality. Then, the author characterized the notion of algebraic set defined over $\mathbb{Q}$ in $\mathcal{R}(R^n)$ in the case of nonsingular algebraic sets and in the case of hypersurfaces. A general complete characterization via Galois theory is still unknown but above particular cases suggest a possible answer to be verified with further investigations beyond this thesis.

Chapter 2: $\mathbb{Q}$-Nonsingular $\mathbb{Q}$-algebraic sets. In this chapter we restrict to the field extension $\mathbb{R}|\mathbb{Q}$ and we study many properties described by $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets. We extend classical notions as regular maps and projective closure of real algebraic sets 'over $\mathbb{Q}$' and we derive similar properties with respect to classical ones. Then, we present fundamental examples of $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets, as real embedded Grassmannians $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ and some classical bundles over Grassmannians. The example of $\mathbb{G}_{m,n}$ is particularly interesting since, via an explicit desingularization of embedded Schubert varieties, we are able to generate the $\mathbb{Z}/2\mathbb{Z}$-homology group of $\mathbb{G}_{m,n}$. The main consequence of these examples is that the cobordism group of compact smooth manifolds is generated by $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets. Then, we prove a variant 'over $\mathbb{Q}$' of a the well known equivalence between algebraic homology classes and algebraic bordism classes. Then, these results exploit their role in constructing relative bordisms over $\mathbb{Q}$ à la Akbulut-King that become crucial in the proof of above Theorem 4. The results of this chapter are originally proved in [GS23; Sav23].

Chapter 3: $\mathbb{Q}$-algebraic approximations à la Akbulut-King. In this chapter we extend approximation techniques introduced by Nash, Tognoli and then further improved by Akbulut and King. We introduce the notion of $\mathbb{Q}$-nice and $\mathbb{Q}$-approximable pairs (P, L), with $L \subset P$, that are the most general classes of $\mathbb{Q}$-algebraic sets $L \subset \mathbb{R}^n$ for which we can approximate smooth functions vanishing on P by polynomial ones with rational coefficients vanishing on L. Latter result is the so-called relative Weierstrass approximation theorem that we specify 'over $\mathbb{Q}$'. We introduce fundamental examples of $\mathbb{Q}$-nice $\mathbb{Q}$-algebraic sets that play their role in our main results in Chapter 4. We further improve relative Weierstrass approximation theorem with $\mathbb{Q}$-regular functions by controlling also the behaviour at infinity. Then, the main $\mathbb{Q}$-algebrization results of this chapter are versions 'over $\mathbb{Q}$' of the Nash-Tognoli theorem and a relative Nash-Tognoli theorem with respect to a finite set of hypersurfaces in general position. Lastly, we develop a new version 'over $\mathbb{Q}$' with approximation of the Akbulut-King blowing down lemma, that is, if we have a $\mathbb{Q}$-regular map $p : A \to Y$ between $\mathbb{Q}$-algebraic sets $A, X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, with $A \subset X$, then the topological adjunction space $X \cup_p Y$ is homeomorphic to a $\mathbb{Q}$-algebraic set. We can also preserve $\mathbb{R}|\mathbb{Q}$-regular points of $X \setminus A$. The results of this chapters are originally proved in [GS23; Sav23].

Chapter 4: $\mathbb{Q}$-Algebrization results. This chapter collects all the $\mathbb{Q}$-algebrization theorems already mentioned in section Main Results of this Introduction. We present a version from the smooth category to the Nash one of Baro-Fernando-Ruiz approximation results in [BFR14]. These theorems apply in the approximation of relative diffeomorphisms between Nash manifolds with Nash submanifolds in general position with Nash diffeomorphisms. Again, latter result applies as well in each of the proof of main results in this chapter. We deeply discuss our answers of [Par21, Open problems 1 & 2, pp. 199-200] originally proved in [GS23; Sav23]. We highlight that our $\mathbb{Q}$-determined and $\mathbb{Q}$-nonsingular models can be produced with some control on their rational points, indeed we prove that we can always find an "hypersurface of rational points" contained in the nonsingular locus. One of the main future challenges will be to extend our results in small dimension, that is, in dimension ≤ 3, by finding $\mathbb{Q}$-algebraic models with dense rational points.

Appendices A & B. In Appendix A we provide explicit proves of our versions of Baro-Fernando-Ruiz approximation results introduced in Section 4.1 and originally proved in [GS23]. Appendix B corresponds integrally to [Sav22] in which we prove an evenness criterion for the degree of global smoothings of subanalytic sets only depending on the global topology of the set, as already mentioned in section Main Results.

CHAPTER 1

Algebraic geometry over subfields

Abstract. In this chapter we develop the algebraic geometry over subfields, in particular over subfiends of algebraically closed fields of characteristic zero and real closed fields. Let $L|K$ be a field extension. In section 1.1 we introduce and study in general the notion of K-algebraic subset of L^n. In Sections 1.2 and 1.3 we study the geometry of K-algebraic subsets of L^n via Galois theory when L is algebraically closed or real closed, respectively. In the remaining part of the chapter, R is assumed to be a real closed field. In Section 1.4 we introduce and characterize algebraic subsets of R^n defined over K: they are those K-algebraic sets of R^n behaving like K-algebraic subsets of L^n, with L algebraically closed. In Section 1.5 we introduce and characterize the notions of $E|K$-regular and $E|K$-singular points of a K-algebraic subset of R^n, with respect to a field extension $R|E|K$. In Section 1.6 we define and study the main class of K-algebraic subsets of R^n that will appear in next chapters, that is, K-determined K-algebraic sets. We investigate the relations between above notions for an algebraic subset of R^n to be 'defined over K'.

The main reference for this chapter is [FG], whereas Section 1.6 is a generalization of results originally proved in [GS23, Section 1].

1.1. K-Algebraic sets

Throughout this section $L|K$ denotes a field extension.

Here we develop the basic notions of algebraic geometry in L^n defined by polynomial equations whose coefficients lie in K. We will specify later on interesting cases of extensions of fields.

Let us fix some notation. Let $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and let $L|K$ be a field extension. Denote by $L[x] := L[x_1, \dots, x_n]$ and $K[x] := K[x_1, \dots, x_n]$ the polynomial rings in n variables with coefficients in L and K, respectively. Consider $K[x] \subset L[x]$ and $K^n \subset L^n$. Fix $F \subset K[x]$ and $S \subset L^n$, define:

$$\mathcal{Z}_L(F) := \{x \in L^n \mid f(x) = 0, \forall f \in F\},$$
$$\mathcal{I}_K(S) := \{f \in K[x] \mid f(x) = 0, \forall x \in S\}.$$

Observe that $\mathcal{Z}_L(F)$ is an algebraic subset of L^n and $\mathcal{I}_K(S)$ is an ideal of $K[x]$ since it coincides with $\mathcal{I}_L(S) \cap K[x]$, where $\mathcal{I}_L(S) := \{f \in L[x] \mid f(x) = 0, \forall x \in S\}$ denotes the usual vanishing ideal of S in $L[x]$. To shorten the notations, if $F = \{f_1, \dots, f_s\} \subset L[x]$ for some $s \in \mathbb{N}$, we write $\mathcal{Z}_L(F) := \mathcal{Z}_L(f_1, \dots, f_s)$.

Let us introduce the main notion of this subsection.

Definition 1.1.1. Let X be a subset of L^n. We say that X is a *K-algebraic subset of* L^n, or equivalently $X \subset L^n$ is a *K-algebraic set*, if $X = \mathcal{Z}_L(F)$ for some $F \subset K[x]$. ■

Observe that, when $K = L$, the family of K-algebraic subsets of L^n coincides with the family of algebraic subsets of L^n. More in general, for every extension $L|K$, the family of all K-algebraic subsets of L^n constitutes a topology strictly coarser than the usual Zariski topology of L^n. We will refer to this topology as the *K-Zariski topology of* L^n. Being coarser that the Zariski topology of L^n, the K-Zariski topology of L^n is Noetherian as well.

Remark 1.1.2.
(i) Let $X \subset L^n$ be a K-algebraic set. By Noetherianity of the K-Zariski topology of L^n there are finitely many polynomials $f_1, \dots, f_s \in K[x]$ such that $X = \mathcal{Z}_L(f_1, \dots, f_s)$. If in addition the field L is (formally) real, then $X = \mathcal{Z}_L(f_1^2 + \dots + f_s^2)$.
(ii) Consider the extension of fields $L|H|K$. Then, the K-Zariski topology of H^n coincides with the relative topology of H^n induced by the K-Zariski topology of L^n. ■

K-irreducibility and K-dimension. Let us introduce the notions of K-irreducibility and of K-irreducible components of a K-algebraic set.

Definition 1.1.3. Let $X \subset L^n$ be a K-algebraic set. We say that $X \subset L^n$ is *K-reducible* if it is reducible with respect the K-Zariski topology of L^n, that is, if there exist K-algebraic sets $X_1, X_2 \subset L^n$ such that $X_1, X_2 \subsetneq X$ and $X_1 \cup X_2 = X$. We say that $X \subset L^n$ is *K-irreducible* if it is not K-reducible. ■

We collect some fundamental properties of K-irreducibility miming classical ones of the usual Zariski topology.

Lemma 1.1.4. *Let $X \subset L^n$ be a K-algebraic set. The following properties hold:*

(i) *X is K-irreducible if and only if $\mathcal{I}_K(X)$ is a prime ideal of $K[x]$.*
(ii) *Let $Y_1, \dots, Y_r \subset L^n$ be K-algebraic sets such that $X \subset \bigcup_{i=1}^r Y_j$. If X is K-irreducible, then $X \subset Y_i$ for some $j \in \{1, \dots, r\}$.*
(iii) *There are finitely many K-irreducible K-algebraic subsets $X_1, \dots, X_r$ of L^n, uniquely determined by X, such that $X_i \not\subset \bigcup_{j=\{1,\dots,r\}\setminus\{i\}} X_j$ for every $i \in \{1, \dots, r\}$ and $X = \bigcup_{i=1}^r X_i$. The X_i's are called the* K-irreducible components *of X.*

Proof. Standard arguments work.

(i) Suppose X is K-reducible, that is, there are K-algebraic sets $X_1, X_2 \subset L^n$ such that $X = X_1 \cup X_2$ and $X_i \subsetneq X$ for each $i \in \{1, 2\}$. Then, there are $f_i \in \mathcal{I}_K(X_i) \setminus \mathcal{I}_K(X)$ for $i \in \{1, 2\}$, so that $\mathcal{I}_K(X)$ is not prime since $f_1 f_2 \in \mathcal{I}_K(X)$. On the other hand, suppose $\mathcal{I}_K(X)$ is not prime, that is: there are $f_1, f_2 \in K[x]$ such that $f_i \notin \mathcal{I}_K(X)$ for $i \in \{1, 2\}$, but $f_1 f_2 \in \mathcal{I}_K(X)$. Consider $X_i = X \cap \mathcal{Z}_L(f_i)$ for $i \in \{1, 2\}$, then $X_i \subsetneq X$ and $X = X_1 \cup X_2$.

(ii) Suppose $X \not\subset Y_j$ for every $j = \{1, \dots, r\}$, in particular $r \geq 2$. Let $s \in \{1, \dots, r\}$ be the maximum j such that $X \not\subset \bigcup_{i=1}^j Y_i$, thus $X \subset \bigcup_{i=1}^{s+1} Y_i$. Hence, define $X_1 := X \cap (\bigcup_{i=1}^s Y_i)$ and $X_2 := X \cap Y_{s+1}$. Observe that $X_1, X_2 \subsetneq X$, and

$X = X \cap (\bigcup_{i=1}^{s+1} Y_i) = (X \cap (\bigcup_{i=1}^{s} Y_i)) \cup (X \cap Y_{s+1}) = X_1 \cup X_2$, which is impossible since X is K-irreducible.

(iii) Since the K-Zariski topology of L^n is Noetherian, we directly get the result by [Har77, Proposition 1.5]. $\square$

If $K = L$, then K-irreducibility of a K-algebraic set X coincides with the usual notion of irreducibility of X as an algebraic set. In the same way, if $K = L$, K-irreducible components of a K-algebraic set X coincide with the usual irreducible components of X.

Definition 1.1.5. Let $S \subset L^n$. We denote by $\mathrm{Zcl}_{L^n}^K(S)$ the *K-Zariski closure of S (in L^n)*, that is the closure of S with respect to the K-Zariski topology of L^n. If $K = L$, we write $\mathrm{Zcl}_{L^n}(S)$ instead of $\mathrm{Zcl}_{L^n}^L(S)$. ■

Usual properties of classical Zariski closure extend to K-Zariski closure, for instance $\mathrm{Zcl}_{L^n}^K(S) = \mathcal{Z}_L(\mathcal{I}_K(S))$. Hence, $\mathcal{I}_K(S) \subset \mathcal{I}_L(S)$ implies $\mathrm{Zcl}_{L^n}(S) \subset \mathrm{Zcl}_{L^n}^K(S)$. Thus, $\mathrm{Zcl}_{L^n}^K(\mathrm{Zcl}_{L^n}(S)) = \mathrm{Zcl}_{L^n}^K(S)$ and $\mathcal{I}_K(S) = \mathcal{I}_K(\mathrm{Zcl}_{L^n}(S)) = \mathcal{I}_K(\mathrm{Zcl}_{L^n}^K(S))$. As above, if $K = L$, then $\mathrm{Zcl}_{L^n}(S)$ is the usual Zariski closure of S in L^n.

Let us introduce the notion of (algebraic) K-dimension of a subset S of L^n.

Definition 1.1.6. Let $S \subset L^n$. We define the *K-dimension* $\dim_K(S)$ *of S (in L^n)* as the Krull dimension of the ring $K[x]/\mathcal{I}_K(S)$. ■

Observe that $\dim_K(S) = \dim_K(\mathrm{Zcl}_{L^n}^K(S))$ and, if $K = L$, then $\dim_K(S)$ is exactly the usual dimension of $\mathrm{Zcl}_{L^n}(S)$ in L^n. Let us collect some other important properties of K-dimension of K-algebraic sets.

Lemma 1.1.7. *Let $X, Y \subset L^n$ be K-algebraic sets. Suppose that $Y \subsetneq X$ and X is K-irreducible, then $\dim_K(Y) < \dim_K(X)$.*

Proof. A standard argument works. Suppose $Y \subset X$, X is K-irreducible and $\dim_K(Y) = \dim_K(X)$, let us prove that $X = Y$. Let $\mathfrak{p}$ be a prime ideal such that $\mathrm{ht}(\mathcal{I}_K(X)) = \mathrm{ht}(\mathcal{I}_K(Y)) = \mathrm{ht}(\mathfrak{p})$. Since X is K-irreducible, $\mathcal{I}_K(X)$ is prime, thus $\mathcal{I}_K(X) = \mathcal{I}_K(Y) = \mathfrak{p}$, that is $X = \mathcal{Z}_L(\mathcal{I}_K(X)) = \mathcal{Z}_L(\mathcal{I}_K(Y)) = Y$. $\square$

In what follows, $L|H|K$ denotes an extension of fields.

Lemma 1.1.8. *Let $Y \subset L^n$ be a H-algebraic set and let $X := \mathrm{Zcl}_{L^n}^K(Y)$ be the K-Zariski closure of X in L^n. If Y is H-irreducible, then X is K-irreducible.*

Proof. Observe that $\mathcal{I}_K(X) = \mathcal{I}_K(Y) = \mathcal{I}_H(Y) \cap K[x]$ is prime since $\mathcal{I}_H(Y)$ is so. Thus, the thesis follows by Lemma 1.1.4(i). $\square$

Remark 1.1.9. The converse implication in Lemma 1.1.8 is false in general. Consider the extension $R|\mathbb{Q}$, where R denotes any real closed field. Let $L = H = R$ and $K = \mathbb{Q}$. Consider the $\mathbb{Q}$-algebraic subset $X = Y := \{-\sqrt{2}, \sqrt{2}\}$ of R. Observe that X is $\mathbb{Q}$-irreducible, on the contrary, X is reducible as an algebraic subset of R. ■

Lemma 1.1.10. *Let $Y \subset L^n$ be a H-algebraic set and let $Y_1, \dots, Y_r$ be its H-irreducible components. Let $X := \mathrm{Zcl}_{L^n}^K(Y)$ and $X_i := \mathrm{Zcl}_{L^n}^K(Y_i)$ for every $i \in \{1, \dots, r\}$. Then:*

(i) *There exist a subset* $\{i_1,\dots,i_s\}$ *of* $\{1,\dots,r\}$ *such that* $X_{i_1},\dots,X_{i_s}$ *are the* K*-irreducible components of* X*.*
(ii) *If* $\dim_K(X_i)=\dim_K(X_j)$ *for every* $i,j\in\{1,\dots,r\}$*, then there exists a surjective map* $\eta:\{1,\dots,r\}\to\{i_1,\dots,i_s\}$ *such that* $X_i=X_{\eta(i)}$ *for every* $i\in\{1,\dots,r\}$*.*

Proof. As $X=\operatorname{Zcl}^K_{L^n}(\bigcup_{i=1}^r Y_i)=\bigcup_{i=1}^r \operatorname{Zcl}^K_{L^n}(Y_i)=\bigcup_{i=1}^r X_i$, there exists a subset $\{i_1,\dots,i_s\}$ of $\{1,\dots,r\}$ of minimal cardinality such that $\bigcup_{j=1}^s X_{i_j}=X$. Thus, Lemmas 1.1.4(iii) & 1.1.8 ensure that $X_{i_1},\dots,X_{i_s}$ are the K-irreducible components of X.

Suppose that $\dim_K(X_i)=\dim_K(X_j)$ for every $i,j\in\{1,\dots,r\}$. An application of Lemmas 1.1.4(ii) & 1.1.7 ensures that for every $i\in\{1,\dots,r\}$ there exists $j\in\{1,\dots,s\}$ such that $X_i=X_{i_j}$. If $X_i=X_{i_k}$ for some $h\in\{1,\dots,s\}\setminus\{j\}$, then $X_{i_j}=X_{i_k}$, which is impossible by minimality of $\{1,\dots,s\}$ such that $X=\bigcup_{j=1}^s X_{i_s}$. Then, define the map $\eta:\{1,\dots,r\}\to\{i_1,\dots,i_s\}$ such that $\eta(i)$ is the (unique) $i_j\in\{i_1,\dots,i_s\}$ such that $X_i=X_{i_j}$. Observe that $\eta(i_j)=i_j$ for every $j\in\{1,\dots,s\}$, thus η is surjective. $\square$

Remark 1.1.11. The assumption '$\dim_K(X_i)=\dim_K(X_j)$ for every $i,j\in\{1,\dots,r\}$' in Lemma 1.1.10(ii) can not be omitted. Indeed, we can produce a counterexample as follows. Let $L=H=R$ be any real closed field and $K=\mathbb{Q}$. Consider the algebraic set $Y=Y_1\cup Y_2\subset R^2$, where $Y_1:=\{(-\sqrt{2},0)\}$ and $Y_2:=\{(x_1,x_2)\in R^2\,|\,x_1=\sqrt{2}\}$. Observe that Y_1 and Y_2 are irreducible algebraic subsets of R^2. Let $X_1:=\operatorname{Zcl}^{\mathbb{Q}}_{L^n}(Y_1)=\{(-\sqrt{2},0),(\sqrt{2},0)\}$ and $X_2=\operatorname{Zcl}^{\mathbb{Q}}_{L^n}(Y_2)=\{(x_1,x_2)\in R^2\,|\,x_1^2=2\}$, thus $X:=\operatorname{Zcl}^{\mathbb{Q}}_{L^n}(Y)=X_1\cup X_2=X_2$. Thus, X is $\mathbb{Q}$-irreducible but $X_1\subsetneq X_2$. ■

Field extension and extension of coefficients. Let $L|K$ be a field extension and let $\mathcal{B}:=\{u_j\}_{j\in J}$ be a basis of L as a K-vector space. Let $X\subset L^n$ be an algebraic set, not necessarily K-algebraic. We call $X(K):=X\cap K^n$ the K*-locus of* X. Now we will focus on the relation between the K-locus $X(K)$ of X as an algebraic subset of K^n and the algebraic subset X of L^n. An interesting case of study will be when $X\subset L^n$ is K-algebraic.

Lemma 1.1.12. *Let* $X\subset L^n$ *be an algebraic set. Then,* $X(K)\subset K^n$ *is an algebraic set.*

Proof. Let $f_1,\dots,f_r\in L[x]$ be such that $X=\mathcal{Z}_L(f_1,\dots,f_r)\subset L^n$. By [FG, Lemma 2.2.1], each f_i can be uniquely written as $f_i=\sum_{j\in J}u_jf_{ij}$, where $\mathcal{B}:=\{u_j\}_{j\in J}$ is the chosen basis of L as a K-vector space, $f_{ij}\in K[x]$ for every $j\in J$ and f_{ij} is non-null only for finitely many $j\in J$. Observe that $x\in X(K)$ if and only if $0=f_i(x)=\sum_{j\in J}u_jf_{ij}(x)$ for every $i\in\{1,\dots,r\}$. Thus, $X(K):=X\cap K^n=\mathcal{Z}_K(\{f_{ij}\}_{i\in\{1,\dots,r\},j\in J})$ is an algebraic subset of K^n. $\square$

Corollary 1.1.13. *Let* $S\subset K^n$*. Then,* $\operatorname{Zcl}_{K^n}(S)\subset\operatorname{Zcl}_{L^n}(S)$ *and* $\operatorname{Zcl}_{L^n}(S)=\operatorname{Zcl}_{L^n}(\operatorname{Zcl}_{K^n}(S))$*.*

Proof. Let $X:=\operatorname{Zcl}_{L^n}(S)$. By Lemma 1.1.12, $X(K)=X\cap K^n$ is a K-algebraic set, thus $\operatorname{Zcl}_{K^n}(S)\subset X(K)\subset X$ and $X:=\operatorname{Zcl}_{L^n}(S)\subset\operatorname{Zcl}_{L^n}(\operatorname{Zcl}_{K^n}(S))\subset\operatorname{Zcl}_{L^n}(X)=X$. $\square$

Let $X \subset L^n$ be a K-algebraic set. We focus on the comparison between $\dim_K(X)$ and $\dim_L(X)$.

LEMMA 1.1.14. *Let $X \subset L^n$ be a K-algebraic set such that $\mathcal{I}_L(X) = \mathcal{I}_K(X)L[x]$. Then,* $\dim_K(X) = \dim_L(X)$.

PROOF. If $\mathcal{I}_K(X) = (0)$, then $X = L^n$ and $\dim_K(X) = n = \dim_L(X)$. Suppose $\mathcal{I}_K(X) \neq (0)$ and let $r := \operatorname{ht}(\mathcal{I}_K(X)) \geq 1$. Noether's normalization theorem (see [GP08, Theorem 3.4.1]) ensures the existence of monic polynomials $f_i \in K[x_{i+1}, \dots, x_n][x_i] \cap \mathcal{I}_K(X) \subset L[x_{i+1}, \dots, x_n][x_i] \cap \mathcal{I}_L(X)$ for every $i \in \{1, \dots, r\}$ and $\mathcal{I}_K(X) \cap K[x_{r+1}, \dots, x_n] = (0)$. Thus, $\mathcal{I}_L(X)L[x_{r+1}, \dots, x_n] = (0)$ by [FG, Lemma 2.2.5]. As a consequence, we have two finite injective homomorphisms:

$$K[x_{r+1}, \dots, x_n] \hookrightarrow K[x]/\mathcal{I}_K(X),$$
$$L[x_{r+1}, \dots, x_n] \hookrightarrow L[x]/\mathcal{I}_L(X).$$

Hence, by [Eis95, Axiom D3, p. 219], the following holds:

$$\begin{aligned} \dim_K(X) &:= \dim(K[x]/\mathcal{I}_K(X)) = \dim(K[x_{r+1}, \dots, x_n]) = n - r \\ &= \dim(L[x_{r+1}, \dots, x_n]) = \dim(L[x]/\mathcal{I}_L(X)) =: \dim_L(X). \end{aligned}$$

□

PROPOSITION 1.1.15. *Let $Y \subset K^n$ be an algebraic set and let $X := \operatorname{Zcl}_{L^n}(Y) \subset L^n$. The following properties are satisfied:*

(i) $\mathcal{I}_L(X) = \mathcal{I}_K(Y)L[x]$ *and* $\mathcal{I}_K(X) = \mathcal{I}_K(Y)$.
(ii) $X(K) = Y$.
(iii) *Let $Y_1, \dots, Y_s$ be the irreducible components of $Y \subset K^n$ and let $X_i := \operatorname{Zcl}_{L^n}(Y_i)$ for every $i \in \{1, \dots, s\}$. Then, $X_1, \dots, X_s$ are the irreducible components of $X \subset L^n$. In particular, $X \subset L^n$ is irreducible if and only if Y is so.*
(iv) $\dim_L(X) = \dim_K(X) = \dim_K(Y)$.

PROOF. Let $g_1, \dots, g_r \in \mathcal{I}_K(X)$ such that $\mathcal{I}_K(X) = (g_1, \dots, g_r)$. Since $X = \operatorname{Zcl}_{L^n}(Y)$ we get that $\mathcal{I}_K(Y)L[x] = (g_1, \dots, g_r)L[x] \subset \mathcal{I}_L(X)$. Let us prove the converse inclusion. Let $f \in \mathcal{I}_L(X)$ and let $\mathcal{B} := \{u_j\}_{j \in J}$ a basis of L as a K-vector space. By [FG, Lemma 2.2.1], there are unique $\{f_j\}_{j \in J} \subset K[x]$ such that $f = \sum_{j \in J} u_j f_j$ and f_j is nonzero only for finitely many $j \in J$. Since $Y \subset X$, the polynomial $f \in \mathcal{I}_L(Y)$. In particular, $0 = f(x) = \sum_{j \in J} u_j f_j(x)$ for every $x \in Y$, thus $f_j \in \mathcal{I}_K(Y)$ for every $j \in J$. This proves that $\mathcal{I}_L(X) \subset \mathcal{I}_K(Y)L[x]$. Since $Y \subset X := \operatorname{Zcl}_{L^n}(Y) \subset \operatorname{Zcl}^K_{L^n}(Y)$ we get that $\operatorname{Zcl}^K_{L^n}(X) = \operatorname{Zcl}_{L^n}(Y)$, thus $\mathcal{I}_K(X) = \mathcal{I}_K(Y)$. This proves (i).

Clearly, $Y \subset X(K)$, let us prove the converse inclusion. Since $Y \subset X(K) \subset X$, we have $\mathcal{I}_K(X) \subset \mathcal{I}_K(X(K)) \subset \mathcal{I}_K(Y)$. By (i), we also have that $\mathcal{I}_K(Y) = \mathcal{I}_K(X)$. This proves that $X(K) = \mathcal{Z}_K(\mathcal{I}_K(X(K))) = \mathcal{Z}_K(\mathcal{I}_K(Y)) = Y$. This proves (ii).

Assume at first that Y is irreducible, let us prove that X is irreducible as well. Suppose this is not the case, so there are algebraic sets $Z_1, Z_2 \subset L^n$ such that $Z_1, Z_2 \subsetneq X$ and $X = Z_1 \cup Z_2$. By (ii), $Y = X \cap K^n = (Z_1 \cap K^n) \cup (Z_2 \cap K^n)$ and by Lemma 1.1.12 $Z_1 \cap K^n$ and $Z_2 \cap K^n$ are algebraic subsets of K^n. Observe that $Z_1 \cap K^n, Z_2 \cap K^n \subsetneq Y$, otherwise, if say $Z_1 \cap K^n = Y$, then $\operatorname{Zcl}_{L^n}(Z_1 \cap$

$K^n) = \mathrm{Zcl}_{L^n}(Y) =: X$, which is not the case since $Z_1, Z_2 \subsetneq X$. This leads to the contradiction that Y is reducible. Let us complete the proof of (iii). Let $Y_1, \dots, Y_s$ be the irreducible components of Y and let $X_i := \mathrm{Zcl}_{L^n}(Y_i)$ for every $i \in \{1, \dots, s\}$. We proved that X_i is irreducible since Y_i is so, for every $i \in \{1, \dots, s\}$. Observe that

$$X = \mathrm{Zcl}_{L^n}(Y) = \mathrm{Zcl}_{L^n}(\bigcup_{i=1}^{s} Y_i) = \bigcup_{i=1}^{s} \mathrm{Zcl}_{L^n}(Y_i) = \bigcup_{i=1}^{s} X_i$$

and $X_i \not\subset \bigcup_{j \neq i} X_j$ since $Y_i \not\subset \bigcup_{j \neq i} Y_j$, for every $i \in \{1, \dots, s\}$. Indeed, if $X_i \subset \bigcup_{j \neq i} X_j$ for some $i \in \{1, \dots, s\}$, then $Y_i = X_i \cap K^n \subset (\bigcup_{j \neq i} X_j) \cap K^n = \bigcup_{j \neq i}(X_j \cap K^n) = \bigcup_{j \neq i} Y_j$ by (ii), which is a contradiction.

By (i), an application of Lemma 1.1.14 gives exactly the equality in (iv). □

Corollary 1.1.16. *Let $X \subset L^n$ be an algebraic set. Then:*

(i) $\dim_K(X(K)) \leq \dim_L(X)$.
(ii) *If $X \subset L^n$ is irreducible and $\dim_K(X(K)) = \dim_L(X)$, then $X(K) \subset K^n$ is irreducible and $X = \mathrm{Zcl}_{L^n}(X(K))$.*

Proof. As $X(K) \subset X$, then $\mathrm{Zcl}_{L^n}(X(K)) \subset X$ and

$$\dim_K(X(K)) = \dim_L(\mathrm{Zcl}_{L^n}(X(K))) \leq \dim_L(X).$$

By Proposition 1.1.15(iv), if $\dim_K(X(K)) = \dim_L(X)$ we get that $\dim_L(X) = \dim_L(\mathrm{Zcl}_{L^n}(X(K)))$, thus $X = \mathrm{Zcl}_{L^n}(X(K))$ since $X \subset L^n$ is irreducible. □

Remark 1.1.17. In the statement of Corollary 1.1.16(i) the inequality can be strict. Consider K to be a real closed field and $L := K[i]$ be its algebraic closure, then $X := \{x_1^2 + \dots + x_n^2 + 1 = 0\} \subset L^n$ is an hypersurface but $X(K) = \varnothing$. ■

Here we restrict to the case in which both L and K are either real closed fields or algebraically closed fields in order to derive interactions between previous notions and extension of coefficients. The reason why we restrict to those cases is the property of the theories of real closed fields and algebraically closed fields to be model complete. For more details about extension of coefficients and model theoretical properties of those theories we refer to [BCR98, §5] and [Mar02, §3] for real closed fields and algebrically closed fields, respectively.

Definition 1.1.18. Suppose that both L and K are either real closed fields or algebraically closed fields. Let $Y \subset K^n$ be an algebraic set and let $f_1, \dots, f_r \in K[x]$ such that $Y = \mathcal{Z}_K(f_1, \dots, f_r)$. We say that $Y_L := \mathcal{Z}_L(f_1, \dots, f_r) \subset L^n$ is the *extension of coefficients of Y to L.* ■

Observe that, by model completeness of the theory of real closed fields and algebraically closed fields, the algebraic set Y_L in Definition 1.1.18 only depends on Y, so the definition is well posed.

Proposition 1.1.19. *Suppose that both L and K are either real closed fields or algebraically closed fields. Let $Y \subset K^n$ be an algebraic set and let $Y_L \subset L^n$ be the extension of coefficients of Y to L. The following properties are satisfied:*

(i) $Y_L = \mathrm{Zcl}_{L^n}(Y) = \mathrm{Zcl}^K_{L^n}(Y)$.
(ii) $\mathcal{I}_L(Y_L) = \mathcal{I}_K(Y)L[x]$, $\mathcal{I}_K(Y_L) = \mathcal{I}_K(Y)$ *and* $Y_L(K) := Y_L \cap K^n = Y$.

(iii) *The family of irreducible components of $Y_L \subset L^n$ coincides with the family of K-irreducible components of Y_L. In particular, $Y_L \subset L^n$ is irreducible if and only if Y_L is K-irreducible, or equivalently if $Y \subset K^n$ is irreducible.*
(iv) $\dim_L(Y_L) = \dim_K(Y_L) = \dim_K(Y)$.

Proof. Let $X := \mathrm{Zcl}_{L^n}(Y)$ and let $g_1, \dots, g_r \in K[x]$ such that $\mathcal{I}_K(Y) = (g_1, \dots, g_r)$. By Proposition 1.1.15(i), $\mathcal{I}_L(X) = (g_1, \dots, g_r)L[x]$, hence we have $X = \mathcal{Z}_L(g_1, \dots, g_r) = (\mathcal{Z}_K(g_1, \dots, g_r))_L = Y_L$. Moreover,

$$Y_L = X := \mathrm{Zcl}_{L^n}(Y) \subset \mathrm{Zcl}^K_{L^n}(Y) = \mathcal{Z}_L(\mathcal{I}_K(Y)) = (\mathcal{Z}_K(\mathcal{I}_K(Y)))_L = Y_L.$$

Thus, $Y_L = \mathrm{Zcl}_{L^n}(Y) = \mathrm{Zcl}^K_{L^n}(Y)$, and (i) is proved.

Observe that (ii) & (iv) directly derive from (i) and Proposition 1.1.15(i)(ii)(iv).

Let $Y_1, \dots, Y_s$ be the irreducible components of $Y \subset K^n$ and let $X_i := \mathrm{Zcl}_{L^n}(Y_i)$, for every $i \in \{1, \dots, s\}$. By (i) and Proposition 1.1.15(iii) $X_1, \dots,$ X_s are the irreducible components of $Y_L = \mathrm{Zcl}_{L^n}(Y) \subset L^n$. Recall that $X_i := \mathrm{Zcl}_{L^n}(Y_i) = (Y_i)_L$ by (i), for every $i \in \{1, \dots, s\}$. Hence, by Lemma 1.1.10(i), up to reordering, there is $t \in \{1, \dots, s\}$ such that $X_1, \dots, X_t$ are the K-irreducible components of Y_L. Since $Y_L(K) = Y = \bigcup_{i=1}^s Y_i$ and $Y_i = (Y_i)_L(K) = X_i(K)$ by (ii), we get that $t = s$, as desired. □

Corollary 1.1.20. *Suppose that both L and K are either real closed fields or algebraically closed fields. Let $X \subset L^n$ be a K-algebraic set and recall that $X(K) := X \cap K^n$. The following properties are satisfied:*

(i) $X = \mathrm{Zcl}_{L^n}(X(K)) = \mathrm{Zcl}^K_{L^n}(X(K))$.
(ii) $\mathcal{I}_L(X) = \mathcal{I}_K(X(K))L[x]$, $\mathcal{I}_K(X) = \mathcal{I}_K(X(K))$ *and* $X = (X(K))_L$.
(iii) *Let $X_1, \dots, X_s$ be the irreducible components of $X \subset L^n$. Then $X_1, \dots, X_s$ are the K-irreducible components of X and $X_1(K), \dots, X_s(K)$ are the irreducible components of $X(K) \subset K^n$. In particular, $X \subset L^n$ is irreducible if and only if $X \subset L^n$ is K-irreducible, or equivalently if $X(K) \subset K^n$ is irreducible.*
(iv) $\dim_L(X) = \dim_K(X) = \dim_K(X(K))$.

Proof. Since $X \subset L^n$ is K-algebraic it suffices to apply Proposition 1.1.19 to $Y := X(K)$. □

Remark 1.1.21. We saw that the assumption of both K and L to be real closed fields or algebraically closed fields was crucial to define (uniquely) the extension Y_L of an algebraic set $Y \subset K^n$. However, even though we do not require this definition to be unique, so to depend on the choice of the equations defining Y, we see that previous Corollary 1.1.20 is false in general. Suppose L is any real closed field and $K = \mathbb{Q}$, then Fermat's Last Theorem implies that the $\mathbb{Q}$-algebraic curve $F_k \subset L^2$ described by the polynomial equation $x_1^{2k} + x_2^{2k} = 2^k$ has no rational points for $k \geq 3$, whereas F_k is a curve in L^2. In other words, $F_k(\mathbb{Q}) = \varnothing$ but $\mathrm{Zcl}_{L^2}(F_k(\mathbb{Q})) = \varnothing \subsetneq F_k$, for $k \geq 3$, contradicting every item of Corollary 1.1.20. ■

1.2. Galois completion & complex K-algebraic sets

Along this section $C|\overline{K}|K$ denotes an extension of fields in which C is algebraically closed, $\overline{K}$ is the algebraic closure of K (in C) and G denotes the Galois group $G(C:K)$. Observe that, as $\overline{K}$ is the algebraic closure of K, the full Galois group $G(\overline{K}:K)$ of K is isomorphic to $G(C:K)/G(C:\overline{K})$.

Complex Galois completion. Here we study the K-Zariski closure of an algebraic set $X\subset C^n$. Next lemma highlights that X should be (at least) $\overline{K}$-algebraic to obtain meaningful results on its K-Zariski closure.

LEMMA 1.2.1. *Let $X\subset C^n$ be an irreducible algebraic set and let $T:=\operatorname{Zcl}^K_{C^n}(X)$. If X is not a $\overline{K}$-algebraic subset of C^n, then $\dim_C(T)<\dim_C(X)$.*

PROOF. Let $Z:=\operatorname{Zcl}^{\overline{K}}_{C^n}(X)$. Since X is not $\overline{K}$-algebraic, we have that $X\subsetneq Z\subset T$. Observe that $\mathcal{I}_{\overline{K}}(Z)=\mathcal{I}_{\overline{K}}(X)=\mathcal{I}_C(X)\cap\overline{K}[x]$, thus $\mathcal{I}_{\overline{K}}(Z)$ is prime since $\mathcal{I}_C(X)$ is so. This proves that $Z\subset C^n$ is a $\overline{K}$-irreducible $\overline{K}$-algebraic set. By Corollary 1.1.20(iii), $Z\subset C^n$ is also irreducible and, by Lemma 1.1.7, we have that $\dim_C(X)<\dim_C(Z)\leq\dim_C(T)$, as desired. □

From now on we focus on the K-Zariski closure of $\overline{K}$-algebraic subsets of C^n. More in detail, we characterize the K-Zariski closure of a $\overline{K}$-algebraic set $X\subset C^n$ by means of the Galois group G and we provide an algorithm to compute them involving precise (finite) Galois subextensions of $\overline{K}|K$.

Let us fix some notation. Let $\psi:C\to C$ be an automorphism of fields. Denote by $\psi_n:C^n\to C^n$ the isomorphism of ($\mathbb{Q}$-vector spaces) and $\widehat{\psi}:C[x]\to C[x]$ the isomorphism of rings defined by

$$\psi_n(z_1,\dots,z_n):=(\psi(z_1),\dots,\psi(z_n)),$$
$$\widehat{\psi}\Big(\sum_\nu a_\nu x^\nu\Big):=\sum_\nu\psi(a_\nu)x^\nu.$$

DEFINITION 1.2.2. Let $S\subset C^n$. We say that $\bigcup_{\psi\in G}\psi_n(S)\subset C^n$ is the *Galois completion of $S\subset C^n$ (with respect to the field extension $C|K$).* ■

ALGORITHM 1.2.3. *The algorithm works as follows:*

(0) Input: *Fix a $\overline{K}$-algebraic set $X\subset C^n$.*
(1) *Choose $g_1,\dots,g_r\in\overline{K}[x]$ such that $X=\mathcal{Z}_C(g_1,\dots,g_r)$.*
(2) *Choose any finite Galois subextension $E|K$ of $\overline{K}|K$ such that E contains all the coefficients of the polynomials $g_1,\dots,g_r$. Set $G':=G(E:K)$.*
(3) *For every $\sigma\in G'$, let $\Phi_\sigma:C\to C$ be an automorphism of fields extending σ, that is $\Phi_\sigma|_E=\sigma$. Define $g_i^\sigma:=\widehat{\Phi}_\sigma(g_i)\in E[x]$ and $Z^\sigma:=\mathcal{Z}_C(g_1^\sigma,\dots,g_r^\sigma)\subset C^n$.*
(4) Output: *Consider the $\overline{K}$-algebraic set $T:=\bigcup_{\sigma\in G'}Z^\sigma$.*

REMARK 1.2.4. In Algorithm 1.2.3(3), an extension ψ_σ of σ always exists, for every $\sigma\in G'$, by [FG, Lemma 2.2.15]. Observe that the choices in Algorithm 1.2.3(1)(2)(3) imply that, in principle, the $\overline{K}$-algebraic set $T\subset C^n$ obtained as the output of the algorithm is not uniquely determined by X. ■

Next result shows that the output $T := \bigcup_{\sigma\in G'} Z^\sigma$ of Algorithm 1.2.3 is actually the K-Zariski closure of the $\overline{K}$-algebraic set $X \subset C^n$ chosen as the input. Furthermore, it provides a procedure to compute generators of $\mathcal{I}_K(T)$ from a finite set of polynomials in $\overline{K}[x]$ whose common solution set is X.

THEOREM 1.2.5 (Galois completion & K-Zariski closure). *Let $X \subset C^n$ be a $\overline{K}$-algebraic set and let $T \subset C^n$ be a $\overline{K}$-algebraic set obtained as an output of Algorithm 1.2.3. The following properties hold:*

(i) *$\psi_n(X) = \mathcal{Z}_C(\widehat{\psi}(g_1), \dots, \widehat{\psi}(g_r))$ and $\mathcal{I}_C(\psi_n(X)) = \widehat{\psi}(\mathcal{I}_C(X))$, for every $\psi \in G$. In particular, $Z^\sigma = \Phi_{\sigma,n}(X)$ and $\mathcal{I}_C(Z^\sigma) = \widehat{\Phi}_\sigma(\mathcal{I}_C(X))$, for every $\sigma \in G'$, where $\Phi_{\sigma,n} := (\Phi_\sigma)_n$.*

(ii) *$\{\widehat{\psi}(g_i) \in E[x] \,|\, \psi \in G\} = \{\widehat{\Phi}_\sigma(g_i) \in E[x] \,|\, \sigma \in G'\}$ for every $i \in \{1, \dots, r\}$ and $T = \bigcup_{\sigma\in G'} \Phi_{\sigma,n}(X) = \bigcup_{\psi\in G} \psi_n(X)$ is a $\overline{K}$-algebraic subset of C^n. In particular, T is the Galois completion of $X \subset C^n$.*

(iii) *Let $\mathfrak{H} \subset (g_1, \dots, g_r)\overline{K}[x]$ be the set of all products of the form $\prod_{\sigma\in G'} h_\sigma$, where $h_\sigma \in \{g_1^\sigma, \dots, g_r^\sigma\}$ for every $\sigma \in G'$. Then, $T = \mathcal{Z}_C(\mathfrak{H})$ and $\widehat{\psi}(\mathfrak{H}) = \mathfrak{H}$, for every $\psi \in G$.*

(iv) *Denote by d the order of G'. For every $h \in \mathfrak{H}$, define*

$$P_h(t) := \prod_{\tau\in G'} (t - h^\tau) = t^d + \sum_{j=1}^{d} (-1)^j q_{hj} t^{d-j} \in E[x][t],$$

for some $q_{hj} \in E[x]$, for every $j \in \{1, \dots, d\}$. Then, $\mathfrak{B} := \{q_{hj} \in E[x] \,|\, h \in \mathfrak{H}, j \in \{1, \dots, d\}\} \subset K[x]$ and $T = \mathcal{Z}_C(\mathfrak{B})$. In particular, T is a K-algebraic set.

(v) *$T = \mathrm{Zcl}^K_{C^n}(X) = \mathcal{Z}_C((g_1, \dots, g_r)\overline{K}[x] \cap K[x])$.*

(vi) *$\mathcal{I}_K(T) = \mathcal{I}_K(X) = \sqrt{\mathfrak{B}K[x]}$, $\mathcal{I}_C(T) = \mathcal{I}_K(X)C[x]$ and*

$$\dim_C(X) = \dim_C(T) = \dim_K(T) = \dim_K(X).$$

PROOF. (i) Observe that $\psi(g_i(x)) = \widehat{\psi}(g_i)(\psi_n(x))$ for every $x \in C^n$ and $i \in \{1, \dots, r\}$. Thus $\psi_n(x) \in \mathcal{Z}_C(\widehat{\psi}(g_1), \dots, \widehat{\psi}(g_r))$ if and only if $x \in \mathcal{Z}_C(g_1, \dots, g_r) = X$. In addition, by Hilbert's Nullstellensatz,

$$\begin{aligned}\mathcal{I}_C(\psi_n(X)) &= \mathcal{I}_C(\mathcal{Z}_C(\widehat{\psi}(g_1), \dots, \widehat{\psi}(g_r))) = \sqrt{(\widehat{\psi}(g_1), \dots, \widehat{\psi}(g_r))C[x]} \\ &= \widehat{\psi}(\sqrt{(g_1, \dots, g_r)C[x]}) = \widehat{\psi}(\mathcal{I}_C(\mathcal{Z}_C(g_1, \dots, g_r))) = \widehat{\psi}(\mathcal{I}_C(X)),\end{aligned}$$

since $\widehat{\psi} : C[x] \to C[x]$ is an automorphism. Second part of (i) follows by substituting ψ with Φ_σ.

(ii) Since $E|K$ is a Galois extension, each automorphism $\psi \in G$ restricts to an automorphism $\sigma := \psi|_E \in G'$, conversely, by [FG, Lemma 2.2.15], each $\sigma \in G'$ extends to an automorphism $\psi_\sigma \in G$ such that $\psi_\sigma|_E = \sigma$. This proves that $\{\widehat{\psi}(g_i) \in E[x] \,|\, \psi \in G\} = \{\widehat{\Phi}_\sigma(g_i) \in E[x] \,|\, \sigma \in G'\} = \{g_i^\sigma \in E[x] \,|\, \sigma \in G'\}$, for every $i \in \{1, \dots, r\}$. Observe that $\{\psi_n(X) \subset C^n \,|\, \psi \in G\} = \{\Phi_{\sigma,n}(X) \subset C^n \,|\, \sigma \in G'\}$ is a finite set by (i), thus $T = \bigcup_{\sigma\in G'} Z^\sigma = \bigcup_{\sigma\in G'} \Phi_{\sigma,n}(X) = \bigcup_{\psi\in G} \psi_n(X)$ is a $\overline{K}$-algebraic subset of C^n.

(iii) Since $Z^\sigma = \mathcal{Z}_C(g_1^\sigma, \dots, g_r^\sigma) = \mathcal{Z}_C((g_1^\sigma, \dots, g_r^\sigma)\overline{K}[x])$, we have

$$T = \bigcup_{\sigma \in G'} Z^\sigma = \bigcup_{\sigma \in G'} \mathcal{Z}_C((g_1^\sigma, \dots, g_r^\sigma)\overline{K}[x]) = \mathcal{Z}_C(\prod_{\sigma \in G'} (g_1^\sigma, \dots, g_r^\sigma)\overline{K}[x]).$$

Observe that $\mathfrak{H}$ generates the ideal $\prod_{\sigma \in G'}(g_1^\sigma, \dots, g_r^\sigma)\overline{K}[x]$ of $\overline{K}[x]$, thus $T = \mathcal{Z}_C(\mathfrak{H})$. Let us prove that $\psi(\mathfrak{H}) = \mathfrak{H}$ for every $\psi \in G$. Let $e \in G'$ be the identity automorphism of E, thus $g_i^e = g_i$ for every $i \in \{1, \dots, r\}$ and $\prod_{\sigma \in G'} h_\sigma \in (g_1, \dots, g_r)\overline{K}[x]$. Moreover, by the considerations of (ii), we observe that for every $\psi \in G$ and $\sigma \in G'$ there is some $\tau := (\psi \circ \Phi_\sigma)|_E \in G'$ such that $\widehat{\psi}(g_i^\sigma) = (\widehat{\psi} \circ \widehat{\Phi}_\sigma)(g_i) = g_i^\tau$ for every $i \in \{1, \dots, r\}$. This proves that $\widehat{\psi}(\mathfrak{H}) = \mathfrak{H}$ fir every $\phi \in G$.

(iv) Fix $\sigma \in G'$. Denote by σ^* the unique automorphism of $E[x, t]$ extending σ satisfying $\sigma^*(t) = t$ and $\sigma^*(x_i) = x_i$ for every $i \in \{1, \dots, n\}$.

Let us prove that $\sigma^*(P_h) = P_h$ for every $h \in \mathfrak{H}$. Since the composition by σ produces an automorphism of G', that is the map $\tau \mapsto \sigma \circ \tau$, we have that

$$\sigma^*(P_h) = \prod_{\tau \in G'} (t - \sigma^*(h^\tau)) = \prod_{\tau \in G'} (t - h^{\sigma \circ \tau}) = P_h.$$

As a consequence, $P_h = \prod_{\tau \in G'}(t - h^\tau) \in K[x][t]$, for every $h \in \mathfrak{H}$.

Observe that the coefficients of P_h with respect to the terms t^{d-j} belongs to the ideal $\mathfrak{H}\overline{K}[x]$, with $j \in \{1, \dots, d\}$, thus the (finite) set $\mathfrak{B} \subset \mathfrak{H}\overline{K}[x] \cap K[x] \subset (g_1, \dots, g_r)\overline{K}[x] \cap K[x]$. In addition, since $P_h(h) = 0$, we get that $h^d \in \mathfrak{B}\overline{K}[x]$, for every $h \in \mathfrak{H}$. Thus, we conclude that $T = \mathcal{Z}_C(\mathfrak{H}) = \mathcal{Z}_C(\mathfrak{B}) \subset C^n$ is a K-algebraic set.

(v) First we prove that $T = \mathrm{Zcl}^K_{C^n}(X)$. Observe that we only have to prove $T \subset \mathrm{Zcl}^K_{C^n}(X)$, indeed the converse inclusion follows by observing that X is contained in the K-algebraic set T. Let $f_1, \dots, f_s \in K[x]$ such that $\mathrm{Zcl}^K_{C^n}(X) = \mathcal{Z}_C(f_1, \dots, f_s)$. If $x \in X$, then $0 = \psi(f_j(x)) = f_j(\psi_n(x))$, for every $\psi \in G$ and $j \in \{1, \dots, s\}$. Thus, $T = \bigcup_{\psi \in G} \psi(X) \subset \mathrm{Zcl}^K_{C^n}(X)$.

Let us prove that $T = \mathcal{Z}_C((g_1, \dots, g_r)\overline{K}[x] \cap K[x])$. Since $g_1, \dots, g_r \in \mathcal{I}_{\overline{K}}(X)$ and $\mathfrak{B} \subset (g_1, \dots, g_r)\overline{K}[x] \cap K[x]$, we deduce that

$$\begin{aligned} T = \mathrm{Zcl}^K_{C^n}(X) &= \mathcal{Z}_C(\mathcal{I}_K(X)) = \mathcal{Z}_C(\mathcal{I}_{\overline{K}}(X) \cap K[x]) \\ &= \mathcal{Z}_C((g_1, \dots, g_r)\overline{K}[x] \cap K[x]) \subset \mathcal{Z}_C(\mathfrak{B}) = T, \end{aligned}$$

as desired.

(vi) By (v) we have that $T = \mathrm{Zcl}^K_{C^n}(X)$, thus $\mathcal{I}_K(T) = \mathcal{I}_K(X)$. Define $\mathfrak{a} := \sqrt{\mathfrak{B}K[x]}$. By [Bou03, §V, Section 15, Proposition 5] and Hilbert's Nullstellensatz, since $T = \mathcal{Z}_C(\mathfrak{B}) = \mathcal{Z}_C(\mathfrak{a})$, we have that $\mathcal{I}_C(T) = \mathfrak{a}C[x]$. By [FG, Corollary 2.2.2], we have that

$$\mathcal{I}_K(T) = \mathcal{I}_C(T) \cap K[x] = \mathfrak{a}C[x] \cap K[x] = \mathfrak{a}.$$

As a consequence,

$$\mathcal{I}_C(T) = \mathfrak{a}C[x] = \mathcal{I}_K(T)C[x] = \mathcal{I}_K(X)C[x].$$

Thus, by (i), (ii) and Lemma 1.1.14, we obtain that $\dim_C(X) = \dim_C(T) = \dim_K(T) = \dim_K(X)$, as required. $\square$

Let us collect some direct consequences of Theorem 1.2.5. Let $X \subset C^n$ be a $\overline{K}$-algebraic set. We say that X is *G-invariant* if $\psi_n(X) = X$, for every $\psi \in G$.

COROLLARY 1.2.6. *Let $X \subset C^n$ be a $\overline{K}$-algebraic set. Then, X is K-algebraic if and only if X is G-invariant.*

COROLLARY 1.2.7. *Let $X \subset C^n$ be a $\overline{K}$-algebraic set. Apply Algorithm 1.2.3 with input X and denote by $\{Z^\sigma\}_{\sigma \in G'}$ a family of $\overline{K}$-algebraic subsets of C^n obtained in Algorithm 1.2.3*(3)*. Then, for every $\sigma \in G'$, the following properties are satisfied:*

(i) $\dim_C(Z^\sigma) = \dim_C(X)$.
(ii) *$Z^\sigma \subset C^n$ is irreducible if and only if $X \subset C^n$ is so.*

PROOF. Since $\mathcal{I}_C(Z^\sigma) = \widehat{\Phi}_\sigma(\mathcal{I}_C(X))$, by Theorem1.2.5(i), and $\widehat{\Phi}_\sigma$ is an isomorphism of rings, we deduce that $C[x]/\mathcal{I}_C(Z^\sigma)$ and $C[x]/\widehat{\Phi}_\sigma(\mathcal{I}_C(X))$ are isomorphic and $\mathcal{I}_C(Z^\sigma)$ is prime if and only if $\mathcal{I}_C(X)$ is so. Thus, both (i) & (ii) follow. $\square$

COROLLARY 1.2.8. *Let $X \subset C^n$ be a $\overline{K}$-algebraic set. Then,* $\dim_C(X) = \dim_{\overline{K}}(X) = \dim_K(X)$.

PROOF. Apply Corollary 1.1.20(iv) and Theorem 1.2.5(vi). $\square$

COROLLARY 1.2.9. *Let $L|H$ be an algebraic extension of fields and let $X \subset L^n$ be an algebraic set. Then,* $\dim_L(X) = \dim_H(X)$.

PROOF. Since $L|H$ be an algebraic extension, we have that $\overline{H} = \overline{L}$. Let $Z := \mathrm{Zcl}_{\overline{L}^n}(X) = \mathrm{Zcl}_{\overline{H}^n}(X)$. By Proposition 1.1.15(iv), we have that $\dim_L(X) = \dim_{\overline{H}}(Z)$. As $Z \subset \overline{H}^n$ is $\overline{H}$-algebraic, Corollary 1.2.8 ensures that $\dim_{\overline{H}}(Z) = \dim_H(Z)$. Since $\mathcal{I}_H(Z) = \mathcal{I}_H(X)$, also $\dim_H(Z) = \dim_H(X)$, thus we conclude that $\dim_L(X) = \dim_{\overline{H}}(Z) = \dim_H(Z) = \dim_H(X)$, as desired. $\square$

Simultaneous Galois completion. Observe that, if needed, Algorithm 1.2.3 can be applied also for finite families of $\overline{K}$-algebraic sets. Let us write down the explicit algorithm.

ALGORITHM 1.2.10. *The algorithm for finite families works as follows:*

(0) Input: *Fix a finite family of $\overline{K}$-algebraic subsets $X_1, \dots, X_s$ of C^n. Let $X := \bigcup_{i=1}^s X_i$.*
(1) *Choose $g_{i1}, \dots, g_{ir_i} \in \overline{K}[x]$ such that $Z_i = \mathcal{Z}_C(g_{i1}, \dots, g_{ir_i})$, for every $i \in \{1, \dots, s\}$.*
(2) *Choose any finite Galois subextension $E|K$ of $\overline{K}|K$ such that E contains all the coefficients of the polynomials $g_{i1}, \dots, g_{ir_i}$ for every $i \in \{1, \dots, s\}$. Set $G' := G(E : K)$.*
(3) *For every $\sigma \in G'$, let $\Phi_\sigma : C \to C$ be an automorphism of fields extending σ, that is $\Phi_\sigma|_E = \sigma$. Define $g_{ij}^\sigma := \widehat{\Phi}_\sigma(g_{ij}) \in E[x]$ and $Z_i^\sigma := \mathcal{Z}_C(g_{i1}^\sigma, \dots, g_{ir_i}^\sigma) \subset C^n$ for every $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, r_i\}$.*
(4) Output: *Consider the finite family of $\overline{K}$-algebraic (actually K-algebraic by Theorem* 1.2.5(iv)*) subsets $T_1, \dots, T_s$ of C^n defined as $T_i := \bigcup_{\sigma \in G'} Z_i^\sigma$. Let $T := \bigcup_{i=1}^s T_i$.*

REMARK 1.2.11. Observe that an output $T = \bigcup_{i=1}^{s} T_i \subset C^n$ of Algorithm 1.2.10 with input $X_1, \dots, X_s$ coincides with an output of Algorithm 1.2.3 with input $X = \bigcup_{i=1}^{s} X_i$. If, in particular, $X_1, \dots, X_s$ are the $\overline{K}$-irreducible components of $X \subset C^n$ and $T_i \not\subset T_i$ for every $i, j \in \{1, \dots, s\}$ with $i \leq j$, then $T_1, \dots, T_s$ are the K-irreducible components of T. ■

Galois presentation of a 'complex' K-algebraic set. Let $X \subset C^n$ be a K-algebraic set. Our aim is to detect a minimal algebraic set $Y \subset C^n$ whose K-closure is X. Such algebraic subset Y of X generating a *Galois presentation of* X is, in general, non unique, as explained below.

LEMMA 1.2.12. *Let $X \subset C^n$ be a K-irreducible K-algebraic set and let $Y \subset C^n$ be an irreducible component of X. Then:*

(i) *$Y \subset C^n$ is a $\overline{K}$-irreducible component of X.*
(ii) *Let G' be a finite Galois group and let $\{Z^\sigma\}_{\sigma \in G'}$ be a family of algebraic subsets of C^n such that $T := \bigcup_{\sigma \in G'} Z^\sigma$ is an output of Algorithm* 1.2.3 *with input $Y \subset C^n$. Then, $X = T$.*
(iii) *The family $\{Z^\sigma\}_{\sigma \in G'}$ coincides with the family of irreducible components of X. In particular, all the irreducible components of X have the same dimension.*

PROOF. As $X \subset C^n$ is a K-algebraic set, in particular $X \subset C^n$ is a $\overline{K}$-algebraic set, thus Corollary 1.1.20(iii) ensures that $Y \subset C^n$ is a $\overline{K}$-irreducible $\overline{K}$-algebraic set. This proves (i). Let $Y_1 \subset C^n$ be an irreducible component of X of dimension $d := \dim_C(X)$. By (i), $Y_1 \subset C^n$ is a $\overline{K}$-algebraic set, thus let G'_1 be a finite Galois group and let $\{Z_1^\sigma\}_{\sigma \in G'_1}$ be a family of algebraic subsets of C^n such that $T_1 := \bigcup_{\sigma \in G'_1} Z_1^\sigma$ is an output of Algorithm 1.2.3 with input $Y_1 \subset C^n$. Recall that, by Theorem 1.2.5(iv)(vi), $T_1 \subset C^n$ is K-algebraic and $\dim_C(T_1) = \dim_K(T_1) = \dim_K(Y_1) = \dim_C(Y_1) = d$. Moreover, $X \subset C^n$ is a K-irreducible K-algebraic set of dimension $\dim_K(X) = \dim_C(X) = d$, by Corollary 1.2.8. As $Y_1 \subset X$, by Theorem 1.2.5(v), $T_1 \subset X$, that is $X = T_1$. This implies that there exists $\sigma \in G'_1$ such that $Y = Z_1^\sigma$, thus, by Corollary 1.2.7(i), we get that $Y \subset C^n$ has dimension d and we can apply again Algorithm 1.2.3 with input Y. This proves both (ii) & (iii). □

Preceding lemma allows us to introduce the definition of a Galois presentation of a K-algebraic subset of C^n.

DEFINITION 1.2.13. Let $X \subset C^n$ be a K-algebraic set. Let $(X_1, \dots, X_s)$ be the K-irreducible components of X listed in some order. Let $Y_i \subset C^n$ be an irreducible component of X_i for every $i \in \{1, \dots, s\}$. Let G' be a finite Galois group and $\{Z_i^\sigma\}_{\sigma \in G'}$ be a family of algebraic subsets of C^n such that $X_1 = \bigcup_{\sigma \in G'} Z_1^\sigma, \dots, X_s = \bigcup_{\sigma \in G'} Z_s^\sigma$ and let $X = \bigcup_{i=1}^{s} X_i$ are outputs of Algorithm 1.2.10 with input $Y_1, \dots, Y_s \subset C^n$. We call the tuple

$$(Y_1, \dots, Y_s; G'; \{Z_1^\sigma\}_{\sigma \in G'}, \dots, \{Z_s^\sigma\}_{\sigma \in G'})$$

a *Galois presentation of* $X \subset C^n$ and $(Y_1, \dots, Y_s)$ the *start* of the presentation. To shorten the notation we will refer to $X = \bigcup_{i=1}^{s} \bigcup_{\sigma \in G'} Z_i^\sigma$ as a Galois presentation of $X \subset C^n$ with start $(Y_1, \dots, Y_s) := (Z_1^e, \dots, Z_s^e)$, where $e \in G'$ denotes the identity. ■

Complexification. *Throughout this subsection, R is a real closed field, $i := \sqrt{-1}$ and $C := R[i]$ is the algebraic closure of R.* Observe that the extension $C|R$ is algebraic of degree 2, thus the Galois group $G(C : R)$ has order 2 and is generated by the conjugation involution φ which fixes R pointwise and maps i to $-i$. That is, $\varphi : C \to C$ is defined as $\varphi(x + iy) := x - iy$. As above, define $\varphi_n : C^n \to C^n$ such that

$$\begin{aligned}\varphi_n(x_1 + iy_1, \dots, x_n + iy_n) &:= (\varphi(x_1 + iy_1), \dots, \varphi(x_n + iy_n)) \\ &= (x_1 - iy_1, \dots, x_n - iy_n).\end{aligned}$$

Now we see that the application of Algorithm 1.2.3, specialized to the field extension $C|R$, concides with the usual complexification of algebraic subsets of R^n.

LEMMA 1.2.14. *If $\mathfrak{a}$ is an ideal of $C[x]$, then*

$$\mathcal{Z}_C(\mathfrak{a} \cap R[x]) = \mathcal{Z}_C(\mathfrak{a}) \cup \varphi_n(\mathcal{Z}_C(\mathfrak{a})), \tag{1.2.1}$$

$$\mathcal{Z}_R(\mathfrak{a} \cap R[x]) = \mathcal{Z}_C(\mathfrak{a}) \cap R^n. \tag{1.2.2}$$

PROOF. Equality (1.2.1) derives from Theorem 1.2.5(ii)(v). In addition, applying (1.2.1) we get

$$\begin{aligned}\mathcal{Z}_R(\mathfrak{a} \cap R[x]) &= \mathcal{Z}_C(\mathfrak{a} \cap R[x]) \cap R^n = (\mathcal{Z}_C(\mathfrak{a}) \cup \varphi_n(\mathcal{Z}_C(\mathfrak{a}))) \cap R^n = \\ &= (\mathcal{Z}_C(\mathfrak{a}) \cap R^n) \cup (\varphi_n(\mathcal{Z}_C(\mathfrak{a}) \cap R^n)) = \mathcal{Z}_C(\mathfrak{a}) \cap R^n,\end{aligned}$$

as required in (1.2.2). □

Let $S \subset R^n$ be an algebraic set. The Zariski closure $\operatorname{Zcl}_{C^n}(S)$ of S in C^n is called the *complexification of* S. Let $T \subset C^n$ be an algebraic set. Recall that, as a consequence of Lemma 1.1.12, $T(R) := T \cap R^n$ is an algebraic subset of R^n. In addition, by Proposition 1.1.15(ii), we have that $(\operatorname{Zcl}_{C^n}(S))(R) := \operatorname{Zcl}_{C^n}(S) \cap R^n = S$ for every algebraic set $S \subset R^n$.

PROPOSITION 1.2.15. *Let $T \subset C^n$ be an algebraic set and let $S := T(R) \subset R^n$. The following are equivalent:*

(i) *T is the complexification of S.*
(ii) *$\mathcal{I}_C(T) = \mathcal{I}_R(S)C[x]$.*
(iii) *$\mathcal{I}_R(T) = \mathcal{I}_R(S)$.*
(iv) *$\mathcal{I}_R(T)$ is a real ideal of $R[x]$.*

Moreover, if $T \subset C^n$ is irreducible, preceding conditions are equivalent to the following one:

(v) *$\dim_C(T) = \dim_R(S)$.*

PROOF. Implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) follow by Proposition 1.1.15(i) and [FG, Corollary 2.2.2], respectively.

(iii) $\Longleftrightarrow$ (iv) Since $T \subset C^n$ is algebraic and $\mathcal{I}_R(T) = \mathcal{I}_C(T) \cap R[x]$, then equation (1.2.1) ensures that $\mathcal{Z}_R(\mathcal{I}_R(T)) = S$. Thus, by the Real Nullstellensatz (see [BCR98, Theorem 4.1.4]), the ideal $\mathcal{I}_R(T)$ is real if and only if it coincides with $\mathcal{I}_R(S)$.

(iv) $\Longrightarrow$ (ii) Assume that $\mathcal{I}_R(T)$ is real, that is assume that $\mathcal{I}_R(T) = \mathcal{I}_R(S)$. Let $g_1, \dots, g_r \in C[x]$ such that $(g_1, \dots, g_r) = \mathcal{I}_C(T)$. Write uniquely each polynomial as $g_j := a_j + ib_j$, with $a_j, b_j \in R[x]$ and define the polynomial $\overline{g}_j := a_j - ib_j$, for every $j \in \{1, \ldots, r\}$. Observe that $g_j \overline{g}_j = a_j^2 + b_j^2 \in \mathcal{I}_C(T) \cap R[x] = \mathcal{I}_R(T)$ for every $j \in \{1, \ldots, r\}$, so $a_j, b_j \in \mathcal{I}_R(T)$ for every $j \in \{1, \ldots, r\}$, since $\mathcal{I}_R(T)$ is real. Thus, $(a_1, b_1, \ldots, a_r, b_r)C[x] = \mathcal{I}_C(T)$ and $\mathcal{I}_C(T) \subset \mathcal{I}_R(S)C[x]$. On the other hand, $\mathcal{I}_R(S) = \mathcal{I}_R(T) \subset \mathcal{I}_C(T)$, thus $\mathcal{I}_R(S)C[x] \subset \mathcal{I}_C(T)$. This proves that $\mathcal{I}_R(S)C[x] = \mathcal{I}_C(T)$.

(ii) $\Longrightarrow$ (i) By Proposition 1.1.15(i), we have that $\mathrm{Zcl}_{C^n}(S) = \mathcal{Z}_C(\mathcal{I}_R(S))$. By assumption, $\mathcal{I}_C(T) = \mathcal{I}_R(S)C[x]$, thus $T = \mathcal{Z}_C(\mathcal{I}_C(T)) = \mathcal{Z}_C(\mathcal{I}_R(S)) = \mathrm{Zcl}_{C^n}(S)$.

(i) $\Longrightarrow$ (v) This follows directly from Proposition 1.1.15(iv), even when $T \subset C^n$ is reducible.

(v) $\Longrightarrow$ (i) Assume in addition that $T \subset C^n$ is irreducible. By assumption $\dim_C(T) = \dim_R(S)$. Let $Z := \mathrm{Zcl}_{C^n}(S)$, then $Z \subset T$ because $S \subset T$. Observe that Proposition 1.1.15(iv) ensures that $\dim_C(Z) = \dim_R(S)$, so $\dim_C(T) = \dim_C(Z)$. As T is supposed to be irreducible, $Z \subset T$ and $\dim_C(T) = \dim_C(Z)$ imply that $Z = T$, as desired. $\square$

COROLLARY 1.2.16. *Let $T \subset C^n$ be an algebraic set and let $S := T(R) \subset R^n$. Then:*

(i) $\mathcal{Z}_R(\mathcal{I}_R(T)) = \mathcal{Z}_R(\mathcal{I}_R(S)) = S$.
(ii) *If the ideal $\mathcal{I}_R(T)$ of $R[x]$ is real, then $\mathcal{I}_R(T) = \mathcal{I}_R(S)$ and $\dim_R(S) = \dim_C(T)$.*
(iii) *If the ideal $\mathcal{I}_R(T)$ of $R[x]$ is non-real, then $\mathcal{I}_R(T) \subsetneq \mathcal{I}_R(S)$. If in addition $T \subset C^n$ is irreducible, then $\dim_R(S) < \dim_C(T)$.*

PROOF. Since $\mathcal{I}_R(T) = \mathcal{I}_C(T) \cap R[x]$, $T \subset C^n$ is an algebraic set and $S = T(R) \subset R^n$ is an algebraic set, (i) follows directly from equation (1.2.2) of Lemma 1.2.14. Observe that equivalence (iii) $\Longleftrightarrow$ (iv) of Proposition 1.2.15 ensures that, if the ideal $\mathcal{I}_R(T)$ of $R[x]$ is real, then $\mathcal{I}_R(T) = \mathcal{I}_R(S)$ and, on the contrary, if $\mathcal{I}_R(T)$ of $R[x]$ is non-real, then $\mathcal{I}_R(T) \subsetneq \mathcal{I}_R(S)$. If $\mathcal{I}_R(T) = \mathcal{I}_R(S)$, then T is the complexification of S and $\dim_C(T) = \dim_R(S)$ by equivalences of Proposition 1.2.15. If $T \subset C^n$ is irreducible and $\mathcal{I}_R(T) \subsetneq \mathcal{I}_R(S)$, then Corollary 1.1.16(i) and equivalence (iv) $\Longleftrightarrow$ (v) of Proposition 1.2.15 ensure that $\dim_R(S) < \dim_C(T)$, as required. $\square$

1.3. Galois completions & real K-algebraic sets

Throughout this section R denotes a real closed field, $i = \sqrt{-1}$ and $C := R[i]$ the algebraic closure of R. Let $R|K$ be an extension of fields and endow K with the order $\leq$ induced by R. Denote by $\overline{K}$ the algebraic closure of K, thus $C|\overline{K}|K$ is an extension of fields. Denote by $\overline{K}^r := \overline{K} \cap R$ the real closure of K, thus $R|\overline{K}^r|K$ is an extension of fields as well. The crucial case of study for next chapters is $K = \mathbb{Q}$. In that case $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers and $\overline{\mathbb{Q}}^r$ the field of real algebraic numbers.

By Definition 1.1.1, $Y \subset R^n$ is a *K-algebraic set* if $Y = \mathcal{Z}_R(F)$ for some $F \subset K[x]$.

Dimension and subfields. Let $Y \subset R^n$ be a K-algebraic set, here we prove that $\dim_R(Y) = \dim_K(Y)$. Thus, when the ground field is real closed, the subfield K does not play any role in our notions of dimension of $Y \subset R^n$. Recall that we already proved a similar result when the ground field is algebraically closed in Corollary 1.2.8.

THEOREM 1.3.1. *Let $Y \subset R^n$ be a K-algebraic set. Then,* $\dim_R(Y) = \dim_K(Y)$.

PROOF. By Corollary 1.1.20(iv), we have
$$\dim_R(Y) = \dim_{\overline{K}^r}(Y) = \dim_{\overline{K}^r}(Y(\overline{K}^r)).$$
By Corollary 1.2.9, we also have that $\dim_{\overline{K}^r}(Y(\overline{K}^r)) = \dim_K(Y(\overline{K}^r))$. In addition, $\mathcal{I}_{\overline{K}^r}(Y) = \mathcal{I}_{\overline{K}^r}(Y(\overline{K}^r))$, by Corollary 1.1.20(ii). Thus:
$$\mathcal{I}_K(Y) = \mathcal{I}_{\overline{K}^r}(Y) \cap K[x] = \mathcal{I}_{\overline{K}^r}(Y(\overline{K}^r)) \cap K[x] = \mathcal{I}_K(Y(\overline{K}^r)),$$
that is, $\dim_K(Y) = \dim_K(Y(\overline{K}^r))$. Thus, we conclude that
$$\dim_R(Y) = \dim_{\overline{K}^r}(Y) = \dim_{\overline{K}^r}(Y(\overline{K}^r)) = \dim_K(Y(\overline{K}^r)) = \dim_K(Y),$$
as desired. $\square$

NOTATION 1.3.2. *Let $Y \subset R^n$ be a K-algebraic set. By Theorem 1.3.1, we have that* $\dim_R(Y) = \dim_K(Y)$*, thus in this section we will denote by* $\dim(Y) := \dim_R(Y) = \dim_K(Y)$. ■

Real and complex Galois completions. Let $Y \subset R^n$ be a $\overline{K}^r$-algebraic set and let $Z := \operatorname{Zcl}_{C^n}(Y) \subset C^n$ be its complexification. By Corollary 1.1.20(ii) and Proposition 1.2.15(ii), we have that $\mathcal{I}_R(Y) = \mathcal{I}_{\overline{K}^r}(Y)R[x]$ and $\mathcal{I}_C(Z) = \mathcal{I}_R(Y)C[x] = (\mathcal{I}_{\overline{K}^r}(Y)R[x])C[x] = \mathcal{I}_{\overline{K}^r}(Y)C[x]$. Thus,
$$\mathcal{I}_C(Z) = \mathcal{I}_{\overline{K}^r}(Y)C[x], \tag{1.3.1}$$
$Z \subset C^n$ is $\overline{K}$-algebraic and we can consider the Galois completion of Z as in Definition 1.2.2.

DEFINITION 1.3.3. Let $Y \subset R^n$ be a $\overline{K}^r$-algebraic set and let $Z := \operatorname{Zcl}_{C^n}(Y) \subset C^n$ be its complexification. Denote by $T \subset C^n$ the Galois completion of Z and $T(R) := T \cap R^n$. We call $T \subset C^n$ the *complex Galois completion of* $Y \subset R^n$ and $T(R) \subset R^n$ the *real Galois completion of* $Y \subset R^n$ *(with respect to the field extension* $C|K$*).* ■

In next theorem we deduce, as for the complex case (see Theorem 1.2.5), the relations between Galois completions, both real and complex, and K-Zariski closures of $Y \subset R^n$ and $Z \subset C^n$, respectively. Here we apply Algorithm 1.2.3 to $X = Z$ and we refer to previous notations of Section 1.2. Let $G := G(C : K)$.

THEOREM 1.3.4 (Galois completions & K-Zariski closures). *Let $Y \subset R^n$ be a $\overline{K}^r$-algebraic set and let $Z := \operatorname{Zcl}_{C^n}(Y) \subset C^n$ be the Zariski closure of Y in C^n. Let $T \subset C^n$ and $T(R) \subset R^n$ be the complex and real Galois completions of $Y \subset R^n$, respectively. Let $E|K$ be a finite Galois extension that contains all the coefficients of polynomials $g_1, \dots, g_r \in \overline{K}[x]$ such that $Z = \mathcal{Z}_C(g_1, \dots, g_r)$. Observe that we can actually choose $g_1, \dots, g_r \in \overline{K}^r[x]$ such that $(g_1, \dots, g_r) = \mathcal{I}_{\overline{K}^r}(Y)$ by (1.3.1). Denote by $G' := G(E : K)$. Then:*

(i) $T = \bigcup_{\psi \in G} \psi_n(Z)$, $\mathcal{I}_C(\psi_n(Z)) = \widehat{\psi}(\mathcal{I}_C(Z))$, $T(R) = \bigcup_{\psi \in G}(\psi_n(Z) \cap R^n)$ *and*

$$\dim(\psi_n(Z) \cap R^n) \leq \dim_C(\psi_n(Z)) = \dim_C(Z) = \dim(Y)$$

for every $\psi \in G$.

(ii) $T = \bigcup_{\psi \in G} Z^\sigma$, $\mathcal{I}_C(Z^\sigma) = \widehat{\Phi}_\sigma(\mathcal{I}_C(Z))$, $T(R) = \bigcup_{\psi \in G} Z^\sigma(R)$ *and*

$$\dim(Z^\sigma(R)) \leq \dim_C(Z^\sigma) = \dim_C(Z) = \dim(Y)$$

for every $\sigma \in G'$. *Let* $e \in G'$ *denote the identity automorphism. Since* $Z^e = Z$ *and* $Z^e(R) = Z(R) = Y$, *then* $\dim_C(T) = \dim_C(Z) = \dim(Y) = \dim(T(R))$.

(iii) *Let* $\mathfrak{H} \subset (g_1, \ldots, g_r)\overline{K}[x]$ *be the set of all products of the form* $\prod_{\sigma \in G'} h_\sigma$, *where* $h_\sigma \in \{g_1^\sigma, \ldots, g_r^\sigma\}$ *for every* $\sigma \in G'$. *Denote by* d *the order of* G'. *For every* $h \in \mathfrak{H}$, *define*

$$P_h(t) := \prod_{\tau \in G'} (t - h^\tau) = t^d + \sum_{j=1}^{d} (-1)^j q_{hj} t^{d-j} \in E[x][t]$$

for some $q_{hj} \in E[x]$, *for every* $j \in \{1, \ldots, d\}$. *Then,* $\mathfrak{B} := \{q_{hj} \in E[x] \mid h \in \mathfrak{H}, j \in \{1, \ldots, s\}\} \subset K[x]$ *and* $T = \mathcal{Z}_C(\mathfrak{B})$.

(iv) $T = \mathrm{Zcl}^K_{C^n}(Y)$ *and* $T(R) = \mathrm{Zcl}^K_{R^n}(Y)$. *In particular,* $T \subset C^n$ *and* $T(R) \subset R^n$ *are* K*-algebraic sets.*

(v) $\mathcal{I}_K(T) = \mathcal{I}_K(T(R)) = \mathcal{I}_K(Y) = \sqrt{\mathfrak{B}K[x]}$, $\mathcal{I}_C(T) = \mathcal{I}_K(Y)C[x]$ *and* $\mathcal{I}_R(T(R)) = \sqrt[r]{\mathcal{I}_K(Y)R[x]}$.

(vi) $\mathrm{Zcl}_{C^n}(T(R)) \subset T$ *and* $\mathcal{I}_R(T) = \mathcal{I}_K(T(R))R[x] = \mathcal{I}_R(T(R)) \cap \mathfrak{b}$, *where* $\mathfrak{b}$ *denotes the intersection of all the non-real ideals* $\mathcal{I}_R(Z^\sigma)$ *of* $R[x]$ *such that* $\sigma \in G'$ *and* $\dim(Z^\sigma(R)) < \dim_C(Z^\sigma)$. *In particular,* $\mathfrak{b}$ *is a radical ideal of* $R[x]$, $\mathcal{Z}_R(\mathfrak{b}) \subsetneq T(R)$ *and* $\dim(\mathcal{Z}_R(\mathfrak{b})) < \dim(T(R)) = \dim(Y)$.

Proof. (i)&(ii) By Definition 1.2.2 and Theorem 1.2.5(i), we directly obtain that $T = \bigcup_{\psi \in G} \psi_n(Z)$ and $\mathcal{I}_C(\psi_n(Z)) = \widehat{\psi}(\mathcal{I}_C(Z))$ for every $\psi \in G$. In particular, $T(R) := T \cap R^n = \bigcup_{\psi \in G}(\psi_n(Z) \cap R^n)$. By Proposition 1.1.15(iv) and Corollary 1.1.16(i), we have that $\dim(Y) = \dim_C(Z)$ and $\dim(\psi_n(Z) \cap R^n) \leq \dim_C(\psi_n(Z))$ for every $\psi \in G$. In addition, by Theorem 1.2.5(i), we have that $\dim_C(Z) = \dim_C(\psi_n(Z))$ for every $\psi \in G$. Hence:

$$\dim(\psi_n(Z \cap R^n)) \leq \dim_C(\psi_n(Z)) = \dim_C(Z) = \dim(Y).$$

Since every automorphism $\sigma \in G'$ extends to an automorphism $\Phi_\sigma \in G$, by [FG, Lemma 2.2.15], we have that $T = \bigcup_{\sigma \in G'} Z^\sigma$, $\mathcal{I}_C(Z^\sigma) = \widehat{\Phi}_\sigma(\mathcal{I}_C(Z))$ for every $\sigma \in G'$, $T(R) = \bigcup_{\psi \in G} Z^\sigma(R)$ and $\dim(Z^\sigma(R)) \leq \dim_C(Z^\sigma) = \dim_C(Z) = \dim(Y)$. Let $e \in G'$ be the identity automorphism. By Proposition 1.1.15(ii), we have that $Z^e(R) = Z(R) = Y$. Hence,

$$\begin{aligned} \dim_C(T) &= \max_{\sigma \in G'}\{\dim_C(Z^\sigma)\} = \dim_C(Z) = \dim(Y) \\ &= \max_{\sigma \in G'}\{\dim_C(Z^\sigma(R))\} = \dim(T(R)). \end{aligned}$$

(iii) Follows directly from Theorem 1.2.5(iii)(iv).

(iv) By Theorem 1.2.5(v), we obtain that

$$T = \mathrm{Zcl}^K_{C^n}(Z) = \mathrm{Zcl}^K_{C^n}(\mathrm{Zcl}_{C^n}(Y)) = \mathrm{Zcl}^K_{C^n}(Y).$$

In addition, since $T = \mathcal{Z}_C(\mathcal{I}_K(Y))$, then:

$$T(R) := T \cap R^n = \mathcal{Z}_C(\mathcal{I}_K(Y)) \cap R^n = \mathcal{Z}_R(\mathcal{I}_K(Y)) = \mathrm{Zcl}^K_{R^n}(Y).$$

(v) Since $T = \mathrm{Zcl}^K_{C^n}(Z) = \mathrm{Zcl}^K_{C^n}(Y)$, we have $\mathcal{I}_K(T) = \mathcal{I}_K(Z) = \mathcal{I}_K(Y)$. Since $T(R) = \mathrm{Zcl}^K_{R^n}(Y)$, then $\mathcal{I}_K(T(R)) = \mathcal{I}_K(Y)$. By Theorem 1.2.5(vi), we have that $\mathcal{I}_C(T) = \mathcal{I}_K(Z)C[x] = \mathcal{I}_K(Y)C[x]$. In addition, since $T(R) = \mathcal{Z}_R(\mathcal{I}_K(Y))$, the Real Nullstellensatz ensures that $\mathcal{I}_R(T(R)) = \sqrt[r]{\mathcal{I}_K(Y)R[x]}$.

(vi) As $T(R) \subset T$, we have $\mathrm{Zcl}_{C^n}(T(R)) \subset \mathrm{Zcl}_{C^n}(T) = T$. In addition, by (v) and [FG, Corollary 2.2.2], we have

$$\begin{aligned}\mathcal{I}_R(T) &= \mathcal{I}_C(T) \cap R[x] = (\mathcal{I}_K(Y)C[x]) \cap R[x] = (\mathcal{I}_K(Y)R[x]C[x]) \cap R[x] \\ &= \mathcal{I}_K(Y)R[x] = \mathcal{I}_K(T(R))R[x].\end{aligned}$$

Denote by $G'^* := \{\sigma \in G' \mid \dim(Z^\sigma(R)) < \dim_C(Z^\sigma) = \dim_C(T)\}$ and $\mathfrak{b} := \bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma)$ if $G'^* \neq \varnothing$, or $\mathfrak{b} = R[x]$ if $G'^* = \varnothing$. The ideal $\mathfrak{b}$ of $R[x]$ is radical since $\mathcal{I}_R(Z^\sigma)$ is so for every $\sigma \in G'$, indeed $\sqrt{\mathfrak{b}} = \sqrt{\bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma)} = \bigcap_{\sigma\in G'^*} \sqrt{\mathcal{I}_R(Z^\sigma)} = \bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma) = \mathfrak{b}$. By Corollary 1.2.16, the ideal $\mathcal{I}_R(Z^\sigma)$ is non real, $\mathcal{I}_R(Z^\sigma) \subsetneq \mathcal{I}_R(Z^\sigma(R))$, for every $\sigma \in G'^*$, and $\mathcal{I}_R(Z^\sigma) = \mathcal{I}_R(Z^\sigma(R))$ for every $\sigma \in G' \setminus G'^*$. Then:

$$\begin{aligned}\mathcal{Z}_R(\mathfrak{b}) &= \mathcal{Z}_R\Big(\bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma)\Big) = \bigcup_{\sigma\in G'^*} \mathcal{Z}_R(\mathcal{I}_R(Z^\sigma(R))) \\ &= \bigcup_{\sigma\in G'^*} Z^\sigma(R) \subset T(R) \subset T.\end{aligned}$$

Thus, $\dim(\mathcal{Z}_R(\mathfrak{b})) = \max_{\sigma\in G'^*}\{\dim(Z^\sigma(R))\} < \dim_C(T) = \dim(T(R))$, so $\mathcal{Z}_R(\mathfrak{b}) \subsetneq T(R)$, and

$$\begin{aligned}\mathcal{I}_R(T(R)) \cap \mathfrak{b} &= \mathfrak{b} \cap \mathcal{I}_R\Big(\bigcup_{\sigma\in G'} Z^\sigma(R)\Big) = \mathfrak{b} \cap \Big(\bigcap_{\sigma\in G'} \mathcal{I}_R(Z^\sigma(R))\Big) \\ &= \bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma) \cap \Big(\bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma(R)) \cap \bigcap_{\sigma\in G'\setminus G'^*} \mathcal{I}_R(Z^\sigma(R))\Big) \\ &= \Big(\bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma) \cap \bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma(R))\Big) \cap \bigcap_{\sigma\in G'\setminus G'^*} \mathcal{I}_R(Z^\sigma(R)) \\ &= \bigcap_{\sigma\in G'^*} \mathcal{I}_R(Z^\sigma) \cap \bigcap_{\sigma\in G'\setminus G'^*} \mathcal{I}_R(Z^\sigma) \\ &= \bigcap_{\sigma\in G'} \mathcal{I}_R(Z^\sigma) = \mathcal{I}_R\Big(\bigcup_{\sigma\in G'} Z^\sigma\Big) = \mathcal{I}_R(T),\end{aligned}$$

as required. □

Observe that the inclusion $\mathrm{Zcl}_{C^n}(T^r) \subset T$ of Theorem 1.3.4(iv) may be a strict inclusion, as explained by next example.

EXAMPLE 1.3.5. Consider the field $K := \mathbb{Q}$. Let $Y \subset R$ be the $\overline{\mathbb{Q}}^r$-algebraic set $\sqrt[4]{2}$. Consider the finite Galois extension $E := \mathbb{Q}(i, \sqrt[4]{2})$ of $\mathbb{Q}$, observe that $\deg(E : \mathbb{Q}) = \deg(\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}) \cdot \deg(\mathbb{Q}(i) : \mathbb{Q}) = 4 \cdot 2 = 8$. Observe that $G := G(E : \mathbb{Q})$ is completely determined by all the possible choices of permutations of the (complex) roots of $x^4 - 2$ which are compatible with the field operations of E. Then, an application of Algorithm 1.2.3 with input $Z := \mathrm{Zcl}_{C^n}(Y)$ gives the $\mathbb{Q}$-algebraic set $T := \{\sqrt[4]{2}, -\sqrt[4]{2}, i\sqrt[4]{2}, -i\sqrt[4]{2}\} = \{x \in C \,|\, x^4 - 2 = 0\} \subset C$ as an output. Then, $T(R) = \{x \in C \,|\, x^4 - 2 = 0\} \cap R = \{\sqrt[4]{2}, -\sqrt[4]{2}\} = \{x \in R \,|\, x^2 - \sqrt{2} = 0\}$ and $\mathrm{Zcl}_{C^n}(T(R)) = \{x \in C \,|\, x^2 - \sqrt{2} = 0\} = T(R) \subsetneq T$. ∎

Simultaneous Galois completion. Let $(Y_1, \dots, Y_s)$ such that $Y_i \subset R^n$ is a $\overline{K}^r$-algebraic set for every $i \in \{1, \dots, s\}$. We associate to $(Y_1, \dots, Y_s)$ the s-tuple $(Z_1, \dots, Z_s)$ such that $Z_i \subset C^n$ is the Zariski closure of Y_i in C^n for every $i \in \{1, \dots, s\}$. Observe that each Z_i is a $\overline{K}^r$-algebraic subset of C^n, thus in particular it is a $\overline{K}$-algebraic subset of C^n. For every $i \in \{1, \dots, s\}$, choose $g_{i1}, \dots, g_{ir_i} \in \overline{K}[x]$ such that $Z_i = \mathcal{Z}_C(g_{i1}, \dots, g_{ir_i})$. Observe that, by Propositions 1.1.19(ii) & 1.2.15(ii), we can actually choose $g_{i1}, \dots, g_{ir_i} \in \overline{K}^r[x]$ such that $Z_i = \mathcal{Z}_C(g_{i1}, \dots, g_{ir_i})$. Choose any finite Galois subextension $E|K$ of $\overline{K}|K$ such that E contains all the coefficients of the polynomials $g_{i1}, \dots, g_{ir_i}$ for every $i \in \{1, \dots, s\}$. Set $G' := G(E : K)$. Then, an application of Algorithm 1.2.10 with input $(Z_1, \dots, Z_s)$ gives an output $(T_1, \dots, T_s)$, here $T_i \subset C^n$ is the K-Zariski closure of Z_i in C^n. Since $Z_i := \mathrm{Zcl}_{C^n}(Y_i)$, we have that $T_i \subset C^n$ is the complex Galois completion of $Y_i \subset R^n$ for every $i \in \{1, \dots, s\}$. Let $T_i(R) := T_i \cap R^n \subset R^n$ be the real Galois completion of $Y_i \subset R^n$ for every $i \in \{1, \dots, s\}$. Consider the s-tuples $(T_1, \dots, T_s)$ and $(T_1(R), \dots, T_s(R))$. We call $(T_1, \dots, T_s)$ the *complex Galois completion of* $(Y_1, \dots, Y_s)$ and $(T_1(R), \dots, T_s(R))$ the *real Galois completion of* $(Y_1, \dots, Y_s)$ *(with respect to the field extension $C|K$).*

Galois presentation of a 'real' K-algebraic set and bad points. Let $X \subset R^n$ be a K-algebraic set. As in Section 1.2, our aim is to detect a minimal algebraic set $Y \subset R^n$ whose K-closure is $X \subset \mathbb{R}^n$. Such algebraic subset Y of X generating a *Galois presentation of* X is, in general, non unique, as explained below. With respect to the complex case, the possible lack of dimension of the irreducible components of the K-Zariski closure $\mathrm{Zcl}^K_{C^n}(X) \subset C^n$ of $X \subset R^n$ when intersected with R^n (see Theorem 1.3.4) forces us to develop a more sophisticated description.

LEMMA 1.3.6. *Let $X \subset R^n$ be a K-irreducible K-algebraic set of dimension d and let $Y \subset R^n$ be an irreducible component of X of dimension d. Then:*

(i) *$Y \subset R^n$ is a $\overline{K}^r$-irreducible component of X.*
(ii) *Let $Z \subset C^n$ be the complexification of Y. Let G' be a finite Galois group and let $\{Z^\sigma\}_{\sigma \in G'}$ be a family of algebraic subsets of C^n such that $T := \bigcup_{\sigma \in G'} Z^\sigma$ is an output of Algorithm 1.2.3 with input $Z \subset C^n$. Thus, T and $T(R)$ are the complex and real Galois completion of Y, respectively. Then: $T(R) = \bigcup_{\sigma \in G'} Z^\sigma(R) = \mathrm{Zcl}^K_{R^n}(Y) = X$ and $T = \bigcup_{\sigma \in G'} Z^\sigma = \mathrm{Zcl}^K_{C^n}(X)$.*
(iii) *The family $\{Z^\sigma\}_{\sigma \in G'}$ coincides with the family of all irreducible components of $T = \mathrm{Zcl}^K_{C^n}(X)$. In particular, all the irreducible components of T have dimension d.*
(iv) *$Z^\sigma \subset C^n$ is a $\overline{K}$-algebraic set and $Z^\sigma(R) \subset R^n$ is a $\overline{K}^r$-algebraic set, for every $\sigma \in G'$.*

PROOF. (i) As $X \subset R^n$ is a K-algebraic set and Y is one of its irreducible components, Corollary 1.1.20(iii) ensures that $Y \subset R^n$ is a $\overline{K}$-irreducible $\overline{K}$-algebraic set.

(ii) By Theorem 1.3.4(ii)(iv), we have $T(R) = \mathrm{Zcl}^K_{R^n}(Y) \subset X$ and $\dim(T(R)) = \dim(Y) = d$. Thus, Theorem 1.3.1 ensures that $\dim_K(T(R)) = \dim(T(R)) = d = \dim(X) = \dim_K(X)$ and $T(R) = X$ by Lemma 1.1.7. Again, by Theorem 1.3.4(ii)(iv) we have that $T = \bigcup_{\sigma \in G'} Z^\sigma = \mathrm{Zcl}^K_{C^n}(Y)$. In addition, $\mathrm{Zcl}^K_{C^n}(Y) = \mathrm{Zcl}^K_{C^n}(\mathrm{Zcl}_{R^n}(Y)) = \mathrm{Zcl}^K_{C^n}(X)$, thus $T = \mathrm{Zcl}^K_{C^n}(X)$.

(iii) Since Y is irreducible, then $Z := \mathrm{Zcl}_{C^n}(Y)$ is irreducible as well by Proposition 1.1.15(iii). Since $T = \bigcup_{\sigma \in G'} Z^\sigma$, Corollary 1.2.7(ii) ensures that $\{Z^\sigma\}_{\sigma \in G'}$ is the family of the irreducible components of T.

(iv) Let $\sigma \in G := G(E : K)$. By Theorem 1.3.4(ii), Z^σ is a $\overline{K}$-algebraic subset of C^n. By Corollary 1.1.13, $Z^\sigma(R) := Z^\sigma \cap R^n$ is an algebraic subset of R^n. Let us check that it is actually a $\overline{K}^r$-algebraic subset of R^n. Let $h_1, \dots, h_s \in \overline{K}[x]$ such that $\mathcal{Z}_C(h_1, \dots, h_s) = Z^\sigma$. Write $h_j := a_j + ib_j$, for some $a_j, b_j \in \overline{K}^r[x]$, for every $j \in \{1, \dots, s\}$. Then $Z^\sigma(R) = \mathcal{Z}_R(a_1, b_1, \dots, a_s, b_s)$, as desired. $\square$

Preceding lemma allows us to introduce the definition of a Galois presentation of a K-algebraic subset of R^n.

Definition 1.3.7. Let $X \subset R^n$ be a K-algebraic set. Let $(X_1, \dots, X_s)$ be the K-irreducible components of X listed in some order and let $d_i := \dim(X_i)$ for every $i \in \{1, \dots, s\}$. Let $Y_i \subset R^n$ be an irreducible component of X_i of dimension d_i and denote by $Z_i \subset C^n$ the complexification of $Y_i \subset R^n$, for every $i \in \{1, \dots, s\}$. Let G' be a finite Galois group and let $\{Z_i^\sigma\}_{\sigma \in G'}$ be a family of algebraic subsets of R^n such that $X_1 = \bigcup_{\sigma \in G'} Z_1^\sigma, \dots, X_s = \bigcup_{\sigma \in G'} Z_s^\sigma$ and $X = \bigcup_{i=1}^s X_i$ are outputs of Algorithm 1.2.10 with input $Z_1, \dots, Z_s \subset C^n$. We call the tuple

$$(Y_1, \dots, Y_s; G'; \{Z_1^\sigma\}_{\sigma \in G'}, \dots, \{Z_s^\sigma\}_{\sigma \in G'})$$

a *Galois presentation of* $X \subset R^n$ and $(Y_1, \dots, Y_s)$ the *start* of the presentation. To shorten the notation we will refer to $X = \bigcup_{i=1}^s \bigcup_{\sigma \in G'} Z_i^\sigma(R)$ as a Galois presentation of $X \subset R^n$ with start $(Y_1, \dots, Y_s) := (Z_1^e(R), \dots, Z_s^e(R))$, where $e \in G'$ denotes the identity. ∎

Let us introduce the notion of K-bad point of a K-algebraic set $X \subset R^n$. This concept will be useful in Section 1.5 in order to characterize what we will call '$R|K$-regular and $R|K$-singular points' of $X \subset R^n$.

Definition 1.3.8. Let $X \subset R^n$ be a K-algebraic set of dimension d and let $X_1, \dots, X_s$ be the K-irreducible components of X. Denote by $d_i := \dim(X_i)$, $T_i := \mathrm{Zcl}^K_{C^n}(X_i)$, $\{T_{i1}, \dots, T_{is_i}\}$ the family of irreducible components of $T_i \subset C^n$ and $J_i := \{j \in \{1, \dots, s_i\} \mid \dim(T_{ij}(R)) < d_i\}$, for every $i = 1, \dots, s$. Define:

$$B_K(X_i) := \bigcup_{j \in J_i} T_{ij}(R) \quad \text{and} \quad B_K(X) := \bigcup_{i=1}^s B_K(X_i) = \bigcup_{i=1}^s \bigcup_{j \in J_i} T_{ij}(R).$$

We say that a point $x \in B_K(X_i)$ is a *K-bad point of* X_i and $B_K(X_i)$ is the *K-bad set of* X_i. Analogously, we say that a point $x \in B_K(X)$ is a *K-bad point of* X and $B_K(X)$ is the *K-bad set of* X. ∎

Remark 1.3.9. Suppose K is a real closed field, $X \subset R^n$ be a K-algebraic set of dimension d and $X_1, \dots, X_s$ are the K-irreducible components of X. Since K is real closed, by Corollary 1.1.20(ii), we have $\mathcal{I}_R(X_i) = \mathcal{I}_K(X_i)R[x] = (f_{i1}, \dots, f_{is_i})R[x]$

for some $f_{i1},\dots,f_{is_i}\in K[x]$ and for every $i\in\{1,\dots,s\}$. Observe that $\overline{K}:=K[i]$ is a finite extension of K containing all the coefficients of the polynomials $\{f_{ij}\}_{i\in\{1,\dots,s\},j\in\{1,\dots,s_i\}}$, thus we can fix $G:=\{e,\sigma\}$, where e is the identity automorphism and $\sigma:\overline{K}\to\overline{K}$ denotes the conjugation $\sigma(a+ib):=a-ib$, for every $x=a+ib\in\overline{K}:=K[i]$. Denote by $\Phi_\sigma:C\to C$ an automoprhism extending σ. Let $f\in\overline{K}[x]$, then there are $a,b\in K[x]$ such that $f=a+ib$, then $\widehat{\Phi}_\sigma(f)=a-ib$. Hence:

$$\begin{aligned}Z_i^\sigma(R)&=\mathcal{Z}_C(\widehat{\Phi}_\sigma(f_{i1}),\dots,\widehat{\Phi}_\sigma(f_{is_i}))\cap R^n=\mathcal{Z}_C(f_{i1},\dots,f_{is_i})\cap R^n\\&=\mathcal{Z}_R(f_{i1},\dots,f_{is_i})=X_i\end{aligned}$$

for every $i\in\{1,\dots,s\}$ and $j\in\{1,\dots,s_i\}$. Then, $\dim(Z_i^\sigma(R))=\dim(X_i)$ for every $i=\{1,\dots,s\}$, $B_K(X_i)=\varnothing$ for every $i\in\{1,\dots,s\}$ and $B_K(X)=\varnothing$. ∎

1.4. Algebraic sets defined over K

Let R be a real closed field, let K be a (formally) real field and denote by $\leq$ its ordering. Denote by $C:=R[i]$ the algebraic closure of R. We say that R *contains* K if $R|K$ is an extension of fields and the ordering of R extends $\leq$.

The aim of this section is to study and characterize those K-algebraic subsets of R^n that behave as K-algebraic subsets of C^n. Recall that, by [Bou03, §V, Section 15, Proposition 5] and Hilbert's Nullstellensatz, for every radical ideal $\mathfrak{a}$ of $K[x]$ the following *zero property in* C^n is satisfied:

$$\mathcal{I}_C(\mathcal{Z}_C(\mathfrak{a}))=\mathfrak{a}C[x].$$

In particular, if $X\subset C^n$ is a K-algebraic set, then $\mathcal{I}_K(X)$ is radical, so it has the zero property in C^n. On the contrary, if $X\subset R^n$ is a K-algebraic set, it is non always guaranteed that $\mathcal{I}_R(\mathcal{Z}_R(\mathcal{I}_K(X)))=\mathcal{I}_K(X)R[x]$.

EXAMPLE 1.4.1. Consider $K=\mathbb{Q}$, R be any real closed field and $X:=\{\sqrt[3]{2}\}\subset R$. Observe that X is a $\mathbb{Q}$-algebraic set, indeed $X=\{\sqrt[3]{2}\}=\{x^3=2\}$. However, $\mathcal{I}_\mathbb{Q}(X)=(x^3-2)$, since x^3-2 it is the minimal polynomial of $\sqrt[3]{2}$, $\mathcal{I}_R(X)=(x-\sqrt[3]{2})$, by [BCR98, Theorem 4.5.1], but $\mathcal{I}_\mathbb{Q}(X)R[x]=(x^3-2)R[x]\subsetneq(x-\sqrt[3]{2})=\mathcal{I}_R(X)$. ∎

In order to characterize those ideals $\mathfrak{a}$ of $K[x]$ having the *zero property in* R^n, that is satisfying $\mathcal{I}_R(\mathcal{Z}_R(\mathfrak{a}))=\mathfrak{a}R[x]$, we start by introducing the concept of reliable family of polynomials in $K[x]$. Observe that

DEFINITION 1.4.2. Let $\{f_1,\dots,f_r\}\subset K[x]$. We say that the family of polynomials $\{f_1,\dots,f_r\}$ in $K[x]$ is K*-reliable* if

$$\mathcal{I}_R(\mathcal{Z}_R(f_1,\dots,f_r))=(f_1,\dots,f_r)R[x]$$

for every real closed field R containing K. If $K=\mathbb{Q}$ and $\{f_1,\dots,f_r\}$ in $\mathbb{Q}[x]$ is $\mathbb{Q}$-*reliable*, we will say for short that $\{f_1,\dots,f_r\}$ is *reliable*. ∎

LEMMA 1.4.3. *Let $\{f_1,\dots,f_r\}\subset K[x]$. The following conditions are equivalent:*

(i) *The family $\{f_1,\dots,f_r\}$ is K-reliable.*
(ii) *There exists a real closed field R containing K such that:*

$$\mathcal{I}_R(\mathcal{Z}_R(f_1,\dots,f_r))=(f_1,\dots,f_r)R[x].$$

(iii) *For each real closed field R containing K the ideal $(f_1,\dots,f_r)R[x]$ is real.*
(iv) *There exists a real closed field R containing K such that $(f_1,\dots,f_r)R[x]$ is a real ideal of $R[x]$.*

PROOF. Recall that an ideal $\mathfrak{a}\subset R[x]$ is real if and only if $\mathcal{I}_R(\mathcal{Z}_R(\mathfrak{a}))=\mathfrak{a}$ by Real Nullstellensatz [BCR98, Theorem 4.1.4], thus equivalences (i) $\Longleftrightarrow$ (iii) and (ii) $\Longleftrightarrow$ (iv) are clear. Also implication (i) $\Longrightarrow$ (ii) is clear, let us prove the converse implication.

(ii) $\Longrightarrow$ (i) As K is contained in R, we deduce that $R|\overline{K}^r|K$ is an extension of fields. We first show that

$$\mathcal{I}_{\overline{K}^r}(\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r))=(f_1,\dots,f_r)\overline{K}^r[x]. \tag{1.4.1}$$

Since the inclusion '$\supset$' is always satisfied, we are only left to verify the converse inclusion '$\subset$'. Let $f\in\mathcal{I}_{\overline{K}^r}(\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r))$, that is $\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r)\subset\mathcal{Z}_{\overline{K}^r}(f)$. By extending the coefficients to R we get that

$$\mathcal{Z}_R(f_1,\dots,f_r)=(\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r))_R\subset(\mathcal{Z}_{\overline{K}^r}(f))_R=\mathcal{Z}_R(f),$$

that is, $f\in\mathcal{I}_R(\mathcal{Z}_R(f_1,\dots,f_r))=(f_1,\dots,f_r)R[x]$, by (ii). Thus, by [FG, Corollary 2.2.2], we have that $f\in(f_1,\dots,f_k)\overline{K}^r[x]$. This concludes the proof of (1.4.1).

Let R_1 be a real closed field that contains K. Let us check that

$$\mathcal{I}_{R_1}(\mathcal{Z}_{R_1}(f_1,\dots,f_r))=(f_1,\dots,f_r)R_1[x]. \tag{1.4.2}$$

Denote by $C_1:=R_1[i]$ the algebraic closure of R_1. Observe that $C|\overline{K}|K$ is a field extension, thus Corollary 1.1.13 and (1.4.1) ensure that

$$\begin{aligned}\mathrm{Zcl}_{C_1^n}(\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r))&=\mathrm{Zcl}_{C_1^n}(\mathrm{Zcl}_{\overline{K}^n}(\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r)))\\&=\mathrm{Zcl}_{C_1^n}(\mathcal{Z}_{\overline{K}}(f_1,\dots,f_r))\\&=(\mathcal{Z}_{\overline{K}}(f_1,\dots,f_r))_{C_1}\\&=\mathcal{Z}_{C_1}(f_1,\dots,f_r).\end{aligned} \tag{1.4.3}$$

In addition, we have

$$\mathcal{I}_{C_1}(\mathcal{Z}_{\overline{K}}(f_1,\dots,f_r))=(f_1,\dots,f_r)C_1[x]. \tag{1.4.4}$$

Indeed, by (1.4.1), $(f_1,\dots,f_r)\overline{K}^r[x]$ is a radical ideal of $\overline{K}^r[x]$, so [Bou03, §V, Section 15, Proposition 5], Hilbert's Nullstellensatz and (1.4.2) ensure that

$$\begin{aligned}\mathcal{I}_{C_1}(\mathcal{Z}_{\overline{K}}(f_1,\dots,f_r))=\mathcal{I}_{C_1}(\mathcal{Z}_{C_1}(f_1,\dots,f_r))&=(f_1,\dots,f_r)\overline{K}^r[x]C_1[x]\\&=(f_1,\dots,f_r)C_1[x].\end{aligned}$$

Let $f\in\mathcal{I}_{R_1}(\mathcal{Z}_{R_1}(f_1,\dots,f_r))$, then

$$\begin{aligned}\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r)\subset(\mathcal{Z}_{\overline{K}^r}(f_1,\dots,f_r))_{R_1}&=\mathcal{Z}_{R_1}(f_1,\dots,f_r)\\&\subset\mathcal{Z}_{R_1}(f)\subset\mathcal{Z}_{C_1}(f).\end{aligned}$$

Hence, (1.4.3) ensures that $\mathcal{Z}_{C_1}(f_1,\dots,f_r)\subset\mathcal{Z}_{C_1}(f)$, that is,

$$f\in\mathcal{I}_{C_1}(\mathcal{Z}_{C_1}(f_1,\dots,f_r)).$$

By (1.4.4) and Hilbert's Nullstellensatz, the ideal $(f_1,\dots,f_r)C_1[x]$ of $C_1[x]$ is radical and $f\in\sqrt{(f_1,\dots,f_r)C_1[x]}=(f_1,\dots,f_r)C_1[x]$. Hence, by [FG, Corollary 2.2.2], we deduce that $f\in(f_1,\dots,f_r)R_1[x]$, that is,

$$\mathcal{I}_{R_1}(\mathcal{Z}_{R_1}(f_1,\dots,f_r))=(f_1,\dots,f_r)R_1[x].$$

This proves that the family $\{f_1,\dots,f_r\}$ of $K[x]$ is K-reliable. $\square$

Let us introduce the notion of real algebraic sets defined over K.

DEFINITION 1.4.4. Let R be a real closed field that contains K and let $X\subset R^n$ be an algebraic set. We say that X is *defined over* K if there are polynomials $f_1,\dots,f_r\in K[x]$ such that $\mathcal{I}_R(X)=(f_1,\dots,f_r)R[x]$. $\blacksquare$

REMARK 1.4.5. Let $X\subset R^n$ be an algebraic set defined over K, that is there are polynomials $f_1,\dots,f_r\in K[x]$ such that $\mathcal{I}_R(X)=(f_1,\dots,f_r)R[x]$, so in particular X is K-algebraic since $X=\mathcal{Z}_R(f_1,\dots,f_r)$. On the other hand, Example 1.4.1 shows that K-algebraic subsets of R^n are not defined over K in general. $\blacksquare$

Next result characterizes algebraic subsets X of R^n which are defined over K.

THEOREM 1.4.6. *Let R be a real closed field that contains K and let $X\subset R^n$ be a K-algebraic set. The following conditions are equivalent:*

(i) *$X\subset R^n$ is defined over K.*
(ii) *$\mathcal{I}_R(X)=\mathcal{I}_K(X)R[x]$.*
(iii) *The complex Galois completion T of $X\subset R^n$ coincides with the complexification of $X\subset R^n$, that is $\operatorname{Zcl}^K_{C^n}(X)=\operatorname{Zcl}_{C^n}(X)$.*
(iv) *$\mathcal{I}_R(X)=\mathcal{I}_R(T)$.*
(v) *There exists a K-reliable family $\{f_1,\dots,f_r\}\subset K[x]$ such that $X=\mathcal{Z}_R(f_1,\dots,f_r)$.*
(vi) *$X(\overline{K}^r)\subset(\overline{K}^r)^n$ is defined over K.*

PROOF. (i)$\iff$(ii) Implication (ii)$\Longrightarrow$(i) follows by choosing $f_1,\dots,f_r\in K[x]$ such that $(f_1,\dots,f_r)=\mathcal{I}_K(X)$. Let us prove the converse implication (i)$\Longrightarrow$(ii). Let $\{f_1,\dots,f_r\}\subset K[x]$ be a such that $\mathcal{I}_R(X)=(f_1,\dots,f_r)R[x]$, then $\mathcal{I}_K(X)=(f_1,\dots,f_r)R[x]\cap K[x]=(f_1,\dots,f_r)$, thus (ii) follows.

(ii)$\iff$(iii)$\iff$(iv) Let $T(R):=T\cap R^n$ be the real Galois completion of $X\subset R^n$. By Theorem 1.3.4(iv)(vi) we have that $T(R)=\operatorname{Zcl}^K_{R^n}(X)=X$ and $\mathcal{I}_R(T)=\mathcal{I}_K(T(R))R[x]=\mathcal{I}_K(X)R[x]$. By Proposition 1.2.15, T is the complexification of $T(R)$ if and only if $\mathcal{I}_R(X)=\mathcal{I}_R(T)$. Thus, $\mathcal{I}_R(X)=\mathcal{I}_K(X)R[x]$ if and only if $\mathcal{I}_R(X)=\mathcal{I}_R(T)$, or equivalently if and only if $T=\operatorname{Zcl}_{C^n}(X)$.

(i)$\iff$(v) Implication (i)$\Longrightarrow$(v) follows directly from (iv)$\Longrightarrow$(i) of Lemma 1.4.3. Let us prove the converse implication (v)$\Longrightarrow$(i). Assume $X=\mathcal{Z}_R(f_1,\dots,f_r)$ for some K-reliable family $\{f_1,\dots,f_r\}\subset K[x]$, then $\mathcal{I}_R(X)=\mathcal{I}_R(\mathcal{Z}_R(f_1,\dots,f_r))=(f_1,\dots,f_r)R[x]$, that is, X is defined over K.

(ii)$\Longrightarrow$(vi) Assume that $\mathcal{I}_R(X)=\mathcal{I}_K(X)R[x]$. By Proposition 1.1.19(i), we have that $X=\operatorname{Zcl}_{R^n}(X(\overline{K}^r))=\operatorname{Zcl}^K_{R^n}(X(\overline{K}^r))$, thus

$$X=\operatorname{Zcl}_{R^n}(X(\overline{K}^r))\subset\operatorname{Zcl}^{\overline{K}^r}_{R^n}(X(\overline{K}^r))\subset\operatorname{Zcl}^K_{R^n}(X(\overline{K}^r))=X.$$

Hence, we get $X=\operatorname{Zcl}^{\overline{K}^r}_{R^n}(X(\overline{K}^r))$. By Proposition 1.1.19(ii) we also have that $\mathcal{I}_{\overline{K}^r}(X)=\mathcal{I}_{\overline{K}^r}(X(\overline{K}^r))$, thus $\mathcal{I}_K(X(\overline{K}^r))=\mathcal{I}_{\overline{K}^r}(X(\overline{K}^r))\cap K[x]=\mathcal{I}_{\overline{K}^r}(X)\cap K[x]=\mathcal{I}_K(X)$. By [FG, Corollary 2.2.2], we have

$$\begin{aligned}\mathcal{I}_{\overline{K}^r}(X(\overline{K}^r))&=\mathcal{I}_{\overline{K}^r}(X)=\mathcal{I}_R(X)\cap\overline{K}^r[x]=\mathcal{I}_K(X)R[x]\cap\overline{K}^r[x]\\&=\mathcal{I}_K(X)\overline{K}^r[x]=\mathcal{I}_K(X(\overline{K}^r))\overline{K}^r[x],\end{aligned}$$

so, by (ii)$\Longrightarrow$(i) applied to $R := \overline{K}^r$, we get that $X(\overline{K}^r) \subset (\overline{K}^r)^n$ is defined over K.

(vi)$\Longrightarrow$(i) Assume that $X(\overline{K}^r) \subset (\overline{K}^r)^n$ is defined over K, that is, there are $\{f_1, \dots, f_r\} \subset K[x]$ such that $\mathcal{I}_{\overline{K}^r}(X(\overline{K}^r)) = (f_1, \dots, f_r)\overline{K}^r[x]$. By Proposition 1.1.19 (ii), we have that

$$\mathcal{I}_R(X) = \mathcal{I}_{\overline{K}^r}(X(\overline{K}^r))R[x] = ((f_1, \dots, f_r)\overline{K}^r[x])R[x] = (f_1, \dots, f_r)R[x],$$

as desired. □

Corollary 1.4.7. *Let $\{f_1, \dots, f_r\} \subset K[x]$ be a K-reliable family and let R be a real closed field that contains K. Then,*

$$\mathcal{Z}_C(f_1, \dots, f_r) = \mathrm{Zcl}_{C^n}(\mathcal{Z}_R(f_1, \dots, f_r)) = \mathrm{Zcl}^K_{C^n}(\mathcal{Z}_R(f_1, \dots, f_r)).$$

Proof. Since $\{f_1, \dots, f_r\} \subset K[x]$ is a K-reliable family, Propositions 1.1.15(i) & 1.2.15(ii) ensure that:

$$\begin{aligned}\mathcal{I}_K(\mathcal{Z}_R(f_1, \dots, f_r))C[x] &= \mathcal{I}_C(\mathrm{Zcl}_{C^n}(\mathcal{Z}_R(f_1, \dots, f_r)))\\ &= \mathcal{I}_R(\mathcal{Z}_R(f_1, \dots, f_r))C[x]\\ &= ((f_1, \dots, f_r)R[x])C[x] = (f_1, \dots, f_r)C[x].\end{aligned}$$

Hence the thesis follows. □

Real K-algebraic sets vs real algebraic sets defined over K. The aim of this subsection is to produce a way of determining whether a K-algebraic set $X \subset R^n$ is actually defined over K. Next result produces some equivalences for a K-irreducible K-algebraic set $X \subset R^n$ to be defined over K.

Theorem 1.4.8. *Let $X \subset R^n$ be a K-irreducible K-algebraic set of dimension d and let $X = \bigcup_{\sigma \in G'} Z^\sigma(R)$ be a Galois presentation of X (see Definition 1.3.7). The following conditions are equivalent:*

(i) *$X \subset R^n$ is defined over K.*
(ii) *$\mathcal{I}_R(Z^\sigma)$ is a real ideal of $R[x]$ for every $\sigma \in G'$.*
(iii) *$Z^\sigma \subset C^n$ is the complexification of $Z^\sigma(R)$ for every $\sigma \in G'$.*
(iv) *$\dim(Z^\sigma(R)) = d$ for every $\sigma \in G'$.*

Proof. Let Y be the start of the Galois presentation $X = \bigcup_{\sigma \in G'} Z^\sigma(R)$ of X, let $T := \bigcup_{\sigma \in G'} Z^\sigma$ be the complex Galois completion for $Y \subset R^n$ and let $T(R)$ be the real Galois completion of $Y \subset R^n$. Recall that, by Lemma 1.3.6, we have that $T = \mathrm{Zcl}^K_{C^n}(X)$ and $T(R) = X$.

(i)$\Longrightarrow$(ii) Assume that $X \subset R^n$ is defined over K. Theorem 1.4.6 ensures that $\mathcal{I}_R(X) = \mathcal{I}_R(T)$ and $T \subset C^n$ is the complexification of X, that is $T = \mathrm{Zcl}^K_{C^n}(X) = \mathrm{Zcl}_{C^n}(X)$. Since $Y \subset R^n$ is irreducible, its complexification $Z = \mathrm{Zcl}_{C^n}(Y) \subset C^n$ is irreducible as well by Proposition 1.2.15(iii), that is, the ideal $\mathcal{I}_C(Z)$ of $C[x]$ is prime. As a consequence, by Theorem 1.3.4(ii), we have that $\mathcal{I}_C(Z^\sigma)$ is prime for every $\sigma \in G'$, thus $\mathcal{I}_R(Z^\sigma) = \mathcal{I}_C(Z^\sigma) \cap R[x]$ is a prime ideal of $R[x]$ for every $\sigma \in G'$. Since $T := \bigcup_{\sigma \in G'} Z^\sigma$ and $\mathcal{I}_R(X) = \mathcal{I}_R(T)$, we have that $\mathcal{I}_C(T) = \bigcap_{\sigma \in G'} \mathcal{I}_C(Z^\sigma)$

and

$$\begin{aligned}\mathcal{I}_R(X) = \mathcal{I}_R(T) &= (\bigcap_{\sigma\in G'} \mathcal{I}_C(Z^\sigma)) \cap R[x] \\ &= \bigcap_{\sigma\in G'} (\mathcal{I}_C(Z^\sigma) \cap R[x]) = \bigcap_{\sigma\in G'} \mathcal{I}_R(Z^\sigma). \qquad (1.4.5)\end{aligned}$$

Thus, (1.4.5) and [AM69, Proposition 1.11], the minimal prime ideas of $\mathcal{I}_R(X)$ in $R[x]$ is a subfamily of $\{\mathcal{I}_R(Z^\sigma)\}_{\sigma\in G'}$. As $\mathcal{I}_R(X)$ is a real ideal of $R[x]$, each minimal prime ideal associated to $\mathcal{I}_R(X)$ is real as well by [BCR98, Lemma 4.1.5]. Define

$$F_0 := \{\sigma \in G' \,|\, \mathcal{I}_R(Z^\sigma) \,\text{is a real ideal of}\, R[x]\}$$

and choose a subset F of F_0 such that $\{\mathcal{I}_R(Z^\sigma)\}_{\sigma\in F} = \{\mathcal{I}_R(Z^\sigma)\}_{\sigma\in F_0}$ but $\mathcal{I}_R(Z^\sigma) \neq \mathcal{I}_R(Z^\tau)$ for every $\sigma, \tau \in F$ with $\sigma \neq \tau$. In particular,

$$\mathcal{I}_R(X) = \bigcap_{\sigma\in F} \mathcal{I}_R(Z^\sigma).$$

Now we prove that $F_0 = G'$. Suppose that there is some $\tau \in G' \setminus F_0$, that is, there is some $\tau \in G'$ such that $\mathcal{I}_R(Z^\tau)$ is not a real ideal of $R[x]$. Then, a fortiori, $\mathcal{I}_R(Z^\tau)$ is not a minimal prime ideal of $\mathcal{I}_R(X)$ and $\bigcap_{\sigma\in F} \mathcal{I}_R(Z^\sigma) \subset \mathcal{I}_R(Z^\tau)$. By [AM69, Proposition 1.11], there exists $\sigma \in F$ such that $\mathcal{I}_R(Z^\sigma) \subset \mathcal{I}_R(Z^\tau)$. Thus, Proposition 1.2.15 ensures that

$$\mathcal{I}_C(Z^\sigma) = \mathcal{I}_R(Z^\sigma)C[x] \subset \mathcal{I}_R(Z^\tau)C[x] \subset \mathcal{I}_C(Z^\tau),$$

so $Z^\tau \subset Z^\sigma$. By Theorem 1.2.5(i) we have that $\dim_C(Z^\sigma) = d = \dim_C(Z^\tau)$, that is, $Z^\sigma = Z^\tau$, since $Z^\sigma \subset C^n$ is irreducible. This leads to the contradiction that $\mathcal{I}_R(Z^\tau) = \mathcal{I}_R(Z^\sigma)$ is a real ideal of $R[x]$. This proves that $F_0 = G'$, as desired.

(ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) Since Z^σ is irreducible and $\dim_C(Z^\sigma) = d$ for every $\sigma \in G'$, previous equivalences follow directly from equivalences of Proposition 1.2.15 and Corollary 1.2.16(ii)(iii).

(iv) $\Longrightarrow$ (i) Assume that $\dim(Z^\sigma(R)) = d = \dim_C(Z^\sigma)$ for every $\sigma \in G'$. By Proposition 1.2.15, we have that $\mathcal{I}_R(Z^\sigma(R)) = \mathcal{I}_R(Z^\sigma)$, for every $\sigma \in G'$. Since $X = T(R) = \bigcup_{\sigma\in G'} Z^\sigma(R)$, we have that

$$\begin{aligned}\mathcal{I}_R(X) = \mathcal{I}_R(T(R)) &= \mathcal{I}_R(\bigcup_{\sigma\in G'} Z^\sigma(R)) \\ &= \bigcap_{\sigma\in G'} \mathcal{I}_R(Z^\sigma(R)) = \bigcap_{\sigma\in G'} \mathcal{I}_R(Z^\sigma) = \mathcal{I}_R(T).\end{aligned}$$

Hence, Theorem 1.2.5(vi) ensures that $\mathcal{I}_R(T) = \mathcal{I}_K(T(R))R[x] = \mathcal{I}_K(X)R[x]$, so we get that $\mathcal{I}_R(X) = \mathcal{I}_K(X)R[x]$. By equivalence (i) $\Longleftrightarrow$ (ii) of Theorem 1.4.6 we conclude that X is defined over K. $\square$

Next result reduces the problem of understanding whether a K-algebraic set $X \subset R^n$ is defined over K by looking at its K-irreducible components.

THEOREM 1.4.9. *Let $X \subset R^n$ be a K-algebraic set and let $X_1, \ldots, X_s$ be the K-irreducible components of X. Then, $X \subset R^n$ is defined over K if and only if $X_i \subset R^n$ is defined over K for every $i \in \{1, \ldots, s\}$.*

PROOF. Suppose that $X_i \subset R^n$ is defined over K for every $i \in \{1,\dots,s\}$. By Theorem 1.4.6 we have that $\mathcal{I}_R(X_i) = \mathcal{I}_K(X_i)R[x] = \mathcal{I}_K(X_i) \otimes_K R$ for every $i \in \{1,\dots,s\}$, then

$$\begin{aligned}\mathcal{I}_R(X) = \bigcap_{\sigma\in G'} \mathcal{I}_R(X_i)R[x] &= \bigcap_{\sigma\in G'} (\mathcal{I}_K(X_i) \otimes_K R) = \\ &= \Big(\bigcap_{\sigma\in G'} (\mathcal{I}_K(X_i)\Big) \otimes_K R = \mathcal{I}_K(X) \otimes_K R = \mathcal{I}_K(X)R[x].\end{aligned}$$

Let us prove the converse implication. Here we adapt the technique adopted in the proof of implication (i) $\Longrightarrow$ (ii) of Theorem 1.4.8 to the K-reducible setting. Assume that $X \subset R^n$ is defined over K, that is, $\mathcal{I}_R(X) = \mathcal{I}_K(X)R[x]$. Let Y_i be an irreducible component of X_i of dimension $\dim(X_i)$ and let $Z_i := \mathrm{Zcl}_{C^n}(Y_i)$, for every $i \in \{1,\dots,s\}$. Let $X = \bigcup_{i=1}^s \bigcup_{\sigma\in G'} Z^\sigma(R)$ be a Galois presentation of $X \subset R^n$ with start $(Y_1,\dots,Y_s)$ (see Definition 1.3.7). By Theorem 1.3.4 for every $\sigma \in G'$ there is a ring automorphism $\widehat{\Phi}_\sigma : C[x] \to C[x]$ such that $\widehat{\Phi}_\sigma(\mathcal{I}_C(Z_i)) = \mathcal{I}_C(Z_i^\sigma)$ for every $i \in \{1,\dots,s\}$. In particular, if T and $T(R)$ denotes the complex and real Galois completions of $\bigcup_{i=1}^s Y_i \subset R^n$, respectively, then

$$\begin{aligned} T &= \mathrm{Zcl}^K_{C^n}(\bigcup_{i=1}^s Y_i) = \bigcup_{i=1}^s \mathrm{Zcl}^K_{C^n}(Y_i) = \bigcup_{i=1}^s T_i = \bigcup_{i=1}^s \bigcup_{\sigma\in G'} Z_i^\sigma, \\ T(R) &= \mathrm{Zcl}^K_{R^n}(\bigcup_{i=1}^s Y_i) = \bigcup_{i=1}^s \mathrm{Zcl}^K_{R^n}(Y_i) = \bigcup_{i=1}^s T_i(R) = \bigcup_{i=1}^s X_i = X,\end{aligned}$$

where T_i and $T_i(R)$ denote the complex and real Galois completions of $Y_i \subset R^n$, respectively, for every $i \in \{1,\dots,s\}$. By Theorem 1.3.4(vi), we have that $\mathcal{I}_R(T) = \mathcal{I}_K(T^r)R[x] = \mathcal{I}_K(X)R[x]$. In addition, $\mathcal{I}_R(X) = \mathcal{I}_R(T)$ since $\mathcal{I}_R(X) = \mathcal{I}_K(X)R[x]$, thus

$$\mathcal{I}_R(X) = \bigcap_{i=1}^s \bigcap_{\sigma\in G'} \mathcal{I}_R(Z_i^\sigma).$$

Since $Y_i \subset R^n$ is irreducible, $Z_i^\sigma \subset C^n$ is irreducible as well, that is, $\mathcal{I}_C(Z_i^\sigma)$ is a prime ideal of $C[x]$, thus $\mathcal{I}_R(Z_i^\sigma) = \mathcal{I}_C(Z_i^\sigma) \cap R[x]$ is prime as well, for every $i \in \{1,\dots,s\}$ and $\sigma \in G'$. Since $\mathcal{I}_R(X)$ is a real ideal of $R[x]$, each minimal prime ideal associated to $\mathcal{I}_R(X)$ is a real ideal of $R[x]$. Define

$$F_0 := \{(i,\sigma) \in \{1,\dots,s\} \times G' \,|\, \mathcal{I}_R(Z_i^\sigma) \text{ is a real ideal of } R[x]\}$$

and choose a subset F of F_0 such that $\{\mathcal{I}_R(Z_i^\sigma)\}_{(i,\sigma)\in F} = \{\mathcal{I}_R(Z_i^\sigma)\}_{(i,\sigma)\in F_0}$ and $\mathcal{I}_R(Z_i^\sigma) \neq \mathcal{I}_R(Z_j^\tau)$, for every $(i,\sigma),(j,\tau) \in F$ with $(i,\sigma) \neq (j,\tau)$. In particular,

$$\mathcal{I}_R(X) = \bigcap_{(i,\sigma)\in F} \mathcal{I}_R(Z_i^\sigma).$$

Here we prove that $F_0 = \{1,\dots,s\} \times G'$. Suppose that there is $(j,\tau) \in \{1,\dots,s\} \times G'$ such that $\mathcal{I}_R(Z_j^\tau)$ is not a real ideal of $R[x]$, then it is not a minimal ideal of $\mathcal{I}_R(X)$ and

$$\bigcap_{(i,\sigma)\in F} \mathcal{I}_R(Z_i^\sigma) \subset \mathcal{I}_R(Z_j^\tau).$$

Thus, there is some $(i,\sigma) \in F$ such that $\mathcal{I}_R(Z_i^\sigma) \subset \mathcal{I}_R(Z_j^\tau)$ and Proposition 1.2.15 ensures that

$$\mathcal{I}_C(Z_i^\sigma) = \mathcal{I}_R(Z_i^\sigma)C[x] \subset \mathcal{I}_R(Z_j^\tau)C[x] \subset \mathcal{I}_C(Z_j^\tau).$$

Observe that $i \neq j$. Otherwise, since $Z_i^\tau \subset Z_i^\sigma$, Z_i^σ is irreducible and $\dim_C(Z_i^\sigma) = \dim(X_i) = \dim_C(Z_i^\tau)$, we get that $Z_i^\tau = Z_i^\sigma$, but this leads to a contradiction since $\mathcal{I}_R(Z_i^\sigma)$ is a real ideal of $R[x]$, whereas $\mathcal{I}_R(Z_i^\tau)$ is not. So, assume $i \neq j$. Set $\nu := \tau^{-1} \circ \sigma \in G'$, then

$$\mathcal{I}_C(Z_i^\nu) = \widetilde{\Phi}_{\tau^{-1}}(\mathcal{I}_C(Z_i^\sigma)) \subset \widetilde{\Phi}_{\tau^{-1}}(\mathcal{I}_C(Z_j^\tau)) = \mathcal{I}_C(Z_j),$$

so $Z_j \subset Z_i^\nu$ and $Y_j = Z_j(R) \subset Z_i^\nu(R) \subset T_i(R) = X_i$. Thus, by Lemma 1.3.6, we have that $X_j = \mathrm{Zcl}^K_{R^n}(Y_i) \subset X_i$, which is impossible.

This completes the proof that $F_0 = \{1,\dots,s\} \times G'$, that is, $\mathcal{I}_R(Z_i^\sigma)$ is a prime ideal of $R[x]$ for every $(i,\sigma) \in \{1,\dots,s\} \times G'$. Hence, implication (ii) $\Longrightarrow$ (i) of Theorem 1.4.8 ensures that $X_i \subset R^n$ is defined over K for every $i \in \{1,\dots,s\}$. $\square$

Latter two theorems have the following direct consequence.

Corollary 1.4.10. *Let $X \subset R^n$ be a K-algebraic set, let $X_1,\dots,X_s$ be the K-irreducible components of X and let $X = \bigcup_{i=1}^s \bigcup_{\sigma\in G'} Z^\sigma(R)$ be a Galois presentation of $X \subset R^n$. The following conditions are equivalent:*

(i) *$X \subset R^n$ is defined over K.*
(ii) *$\mathcal{I}_R(Z_i^\sigma)$ is a real ideal of $R[x]$ for every $i \in \{1,\dots,s\}$ and $\sigma \in G'$.*
(iii) *Z_i^σ is the complexification of $Z_i^\sigma(R)$ for every $i \in \{1,\dots,s\}$ and $\sigma \in G'$.*
(iv) *$\dim(Z_i^\sigma(R)) = \dim(X_i)$ for every $i \in \{1,\dots,s\}$ and $\sigma \in G'$.*

In particular, if $X \subset R^n$ is defined over K, then $B_K(X) = \varnothing$.

1.5. Regular and singular points of K-algebraic sets

Throughout this section, $L|E|K$ denotes a field extension in which L is either real closed or algebraically closed. Denote by $\overline{E}^\bullet$ the algebraic closure of E in L, thus if L is algebraically closed, then $\overline{E}^\bullet = \overline{E}$, otherwise if L is real closed, then $\overline{E}^\bullet = \overline{E}^r$. We will frequently use Corollary 1.2.8 and Theorem 1.3.1 without explicit mentions.

Let $a := (a_1,\dots,a_n) \in L^n$, let $\mathrm{ev}_a : E[x] \to L$ defined as $\mathrm{ev}_a(f) := f(a)$ be the evaluation homomorphism and let $\mathfrak{n}_{E,a} := \{f \in E[x] \,|\, f(a) = 0\}$ be the kernel of ev_a. Then, $\mathfrak{n}_{E,a}$ is a prime ideal of $E[x]$ and $\mathfrak{n}_{E,a} = (0)$ if and only if $a_1,\dots,a_n$ are algebraically independent over E. In addition, $\mathfrak{n}_{E,a}$ is maximal if and only if $E[a_1,\dots,a_n] = \mathrm{ev}_a(E[x])/\mathfrak{n}_{E,a}$ has dimension 0. By [Mat80, (14G) Corollary 1, p. 91], we have that the latter condition is equivalent to the fact that $a_1,\dots,a_n$ are algebraic over E, that is:

$$\mathfrak{n}_{E,a} \text{ is a maximal ideal of } E[x] \text{ if and only if } a \in (\overline{E}^\bullet)^n.$$

Latter condition shows that a definition of local dimension of a K-algebraic set of $X \subset L^n$ at $a \in X$ with respect to the field extension $E|K$ will be defined only at a point $a \in X(\overline{E}^\bullet) := X \cap (\overline{E}^\bullet)^n$.

Let us introduce the fundamental definition of $E|K$-regular points.

DEFINITION 1.5.1 ($E|K$-regular points). Let $X \subset L^n$ be a K-algebraic set and let $a \in X(\overline{E}^\bullet)$. We define the *$E|K$-local ring $\mathcal{R}^{E|K}_{X,a}$ of X at a* as

$$\mathcal{R}^{E|K}_{X,a} := E[x]_{\mathfrak{n}_{E,a}}/(\mathcal{I}_K(X)E[x]_{\mathfrak{n}_{E,a}}),$$

where $\mathfrak{n}_{E,a} := \{f \in E[x] \,|\, f(a) = 0\}$ denotes the maximal ideal of polynomials in $E[x]$ vanishing at $a \in (\overline{E}^\bullet)^n$. We say that a is a *$E|K$-regular point of X of dimension e* if $\mathcal{R}^{E|K}_{X,a}$ is a regular local ring of dimension e. If a is a $E|K$-regular point of X of dimension $d := \dim_K(X)$, we say that a is a *$E|K$-regular point of X*. We denote by $\mathrm{Reg}^{E|K}(X, e)$ the set of $E|K$-regular point of X of dimension e and by $\mathrm{Reg}^{E|K}(X) := \mathrm{Reg}^{E|K}(X, d)$ the set of $E|K$-regular point of X. We define $\mathrm{Sing}^{E|K}(X) := X(\overline{E}^\bullet) \setminus \mathrm{Reg}^{E|K}(X)$ and we say that $a \in \mathrm{Sing}^{E|K}(X)$ is an *$E|K$-singular point of X.*

If $E = K$, we simplify the notations by omitting 'E', that is:

$$\mathcal{R}^{K}_{X,a} := \mathcal{R}^{K|K}_{X_a} = K[x]_{\mathfrak{n}_{K,a}}/\mathcal{I}_K(X)K[x]_{\mathfrak{n}_{K,a}}$$

is the *K-local ring of X at a*, a *K-regular point of X (of dimension e)* is a $K|K$-regular point of X (of dimension e), $\mathrm{Reg}^K(X, e) := \mathrm{Reg}^{K|K}(X, e)$, $\mathrm{Reg}^K(X) := \mathrm{Reg}^{K|K}(X)$, $\mathrm{Sing}^K(X) := \mathrm{Sing}^{K|K}(X)$ and a *K-singular point of X* is a $K|K$-singular point of X. ∎

REMARK 1.5.2. Let $X \subset L^n$ be a K-algebraic set of dimension d.

(i) If $L = E = K$, the notions of Definition 1.5.1 reduces to the classical ones. Indeed, for every $a \in (\overline{E}^\bullet)^n = L^n$, we have that $\mathcal{R}^K_{X,a} = \mathcal{R}^L_{X,a} = \mathcal{R}_{X,a}$, thus a L-regular point of X (of dimension e) is a regular point of X (of dimension e), a L-singular point of X is a L-singular point of X, $\mathrm{Reg}^L(X, e) = \mathrm{Reg}(X, e)$, $\mathrm{Reg}^L(X) = \mathrm{Reg}(X)$ and $\mathrm{Sing}^K(X) = \mathrm{Sing}(X)$ are the usual sets of regular and singular points of an algebraic set $X \subset L^n$. For explicit definitions of those classical objects we refer to [BCR98, Section 3.3].

(ii) Observe that, for every field extension $L|E|K$, we have the usual inclusion of regular points of $X \subset L^n$ of dimension $e < d := \dim_K(X)$ in the set of singular ones, that is, $\mathrm{Reg}^{E|K}(X, e) \subset \mathrm{Sing}^{E|K}(X)$ for every $e < d$. ∎

An important case of study is $E = K$. Localizations of the ring $K[x]$, completion of the residue field $K[x]_{\mathfrak{n}_{K,a}}$ and a K-Jacobian criterion are useful tools developed in [FG, Sections 4.1 & 4.2] in order to study properties of K-regular points of a K-algebraic set $X \subset L^n$ and their relations with global properties of $X \subset L^n$. An important result which clarifies the relation between K-regular points and K-irreducible components of a K-algebraic set $X \subset L^n$ is [FG, Corollary 4.2.2]. In Section 1.6 we prove a similar statement in the case $E = L$ which will be crucial in next chapters.

Regular and $R|K$-regular points. *Let $X \subset L^n$ be a K-algebraic set.* The question we are going to answer is whether the polynomial ring $K[x]/\mathcal{I}_K(X)$ provides enough information to understand the properties of X also as a L-algebraic set. If L is algebraically closed we know that $\mathcal{I}_L(X) = \mathcal{I}_K(X)L[x]$, thus the ideal $\mathcal{I}_K(X)$ provides full information about regular and singular points of X. On the other hand, if L is real closed, we saw that equality $\mathcal{I}_L(X) = \mathcal{I}_K(X)L[x]$ holds true if and only if X is defined over K (see Theorem 1.4.6), thus we can not expect in general that the ideal $\mathcal{I}_K(X)$ provides full information about regular and singular points of X. The aim of this section is to study the set of those regular points of $X \subset L^n$ that are described by the ideal $\mathcal{I}_K(X)$. *Thus, throughout this section $L = R$ is a real closed field containing K and $C := R[i]$ denotes the algebraic closure of R.*

Let us specify some notations introduced in Definition 1.5.1 to the case $L = E = R$. For every $a = (a_1, \dots, a_n) \in R^n := (\overline{E}^r)^n$ denote by

$$\mathfrak{n}_a := \mathfrak{n}_{R,a} = \{f \in R[x] \mid f(a) = 0\} = (x - a_1, \dots, x - a_n).$$

Observe that the latter equality holds since $L = R = E$. Then, as explained in Remark 1.5.2, we denote by:

$$\begin{aligned} \mathcal{R}_{X,a} &:= \mathcal{R}^R_{X,a} = R[x]_{\mathfrak{n}_a}/\mathcal{I}_R(X)R[x]_{\mathfrak{n}_a}, \\ \operatorname{Reg}(X) &:= \operatorname{Reg}^R(X), \\ \operatorname{Sing}(X) &:= \operatorname{Sing}^R(X) = X \setminus \operatorname{Reg}(X). \end{aligned}$$

In addition, let us shorten the notation of $R|K$-local ring, $R|K$-regular and $R|K$-singular points as follows:

$$\begin{aligned} \mathcal{R}^*_{X,a} &:= \mathcal{R}^{R|K}_{X,a} = R[x]_{\mathfrak{n}_a}/\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}, \\ \operatorname{Reg}^*(X) &:= \operatorname{Reg}^{R|K}(X), \\ \operatorname{Sing}^*(X) &:= \operatorname{Sing}^{R|K}(X) = X \setminus \operatorname{Reg}^*(X). \end{aligned}$$

Let $X_1, \dots, X_s$ be the K-irreducible components of X and let $X_1, \dots, X_r$ be those K-irreducible components of X of maximal dimension, that is $\dim(X_i) = \dim(X)$, for every $i \in \{1, \dots, r\}$, and $\dim(X_i) < \dim(X)$, for every $i \in r+1, \dots, s$. Since regular rings are in particular integral domains and the dimensions of localizations of a Noetherian ring is smaller than or equal to the dimension of the ring, we get that:

$$\operatorname{Sing}(X) = \bigcup_{i=1}^{r} \operatorname{Sing}(X_i) \cup \Big(\bigcup_{i,j \in \{1,\dots,r\},\, i \neq j} (X_i \cap X_j) \Big) \cup \bigcup_{j=r+1}^{s} X_j, \tag{1.5.1}$$

$$\operatorname{Sing}^*(X) = \bigcup_{i=1}^{r} \operatorname{Sing}^*(X_i) \cup \Big(\bigcup_{i,j \in \{1,\dots,r\},\, i \neq j} (X_i \cap X_j) \Big) \cup \bigcup_{j=r+1}^{s} X_j, \tag{1.5.2}$$

where $\bigcup_{j=r+1}^{s} X_i = \varnothing$ if $r = s$. Thus, to compare $\operatorname{Sing}(X)$ and $\operatorname{Sing}^*(X)$ we can assume that X is K-irreducible. In next result we see how the set of bad points $B_K(X)$ (see Definition 1.3.8) comes into play.

THEOREM 1.5.3. *Let $X \subset R^n$ be a K-irreducible K-algebraic set of dimension d. Then:*

(i) $\operatorname{Reg}^*(X)$ *is a non-empty Zariski open subset of* X, $\operatorname{Reg}^*(X) \subset \operatorname{Reg}(X)$ *and*
$$\operatorname{Reg}^*(X) = \{a \in X \mid \mathcal{R}^*_{X,a} \textit{ is a regular local ring}\}.$$

(ii) $\operatorname{Sing}^*(X)$ *is an algebraic subset of R^n of dimension $< d$. In addition,* $\operatorname{Sing}^*(X) = \operatorname{Sing}(X) \cup B_K(X)$ *and both* $\operatorname{Sing}^*(X)$ *and* $B_K(X)$ *are $\overline{K}^r$-algebraic subsets of R^n.*

PROOF. We divide the proof into several steps for sake of clarity.

STEP I. *Galois presentation of* X. Let $X = \bigcup_{\sigma\in G'} Z^\sigma(R)$ be a Galois presentation of $X \subset R^n$ with start $Y \subset R^n$. Denote by $T := \bigcup_{\sigma\in G'} Z^\sigma \subset C^n$ and $T(R) \subset R^n$ the complex and the real Galois completions of Y, respectively. By Lemma 1.3.6(ii), we have that $X = T(R)$.

STEP II. *Construction of the set S of 'all bad points'.* Choose a subset F of G' such that $\{Z^\sigma\}_{\sigma\in F} = \{Z^\sigma\}_{\sigma\in G'}$ and $Z^\sigma \neq Z^\tau$ for every $\sigma,\tau \in F$ with $\sigma \neq \tau$. By Theorem 1.3.4(iv)(v)(vi), we know that $\mathcal{I}_R(T) = \mathcal{I}_K(T(R))R[x] = \mathcal{I}_K(X)R[x]$, thus

$$\mathcal{I}_K(X)R[x] = \bigcap_{\sigma\in F} \mathcal{I}_R(Z^\sigma), \tag{1.5.3}$$

$$\mathcal{I}_R(X) = \bigcap_{\sigma\in F} \mathcal{I}_R(Z^\sigma(R)). \tag{1.5.4}$$

Recall that, by Lemma 1.3.6(iii)(iv), $Z^\sigma \subset C^n$ is an irreducible $\overline{K}$-algebraic set and $Z^\sigma(R) \subset R^n$ is $\overline{K}^r$-algebraic for every $\sigma \in G'$. In particular, the complexification $Z := \operatorname{Zcl}_{C^n}(Y) = Z^e$ of Y is irreducible, where $e \in G'$ denotes the identity, and $\mathcal{I}_C(Z^\sigma)$ is a prime ideal of $C[x]$ for every $\sigma \in G'$. Hence, the ideal $\mathcal{I}_R(Z^\sigma) = \mathcal{I}_C(Z^\sigma) \cap R[x]$ of $R[x]$ is prime as well for every $\sigma \in G'$.

Define $F^* := \{\sigma \in F \mid \dim(Z^\sigma(R)) < \dim_C(Z^\sigma)(= \dim(X))\}$. Then, $\mathcal{I}_R(Z^\sigma) \subsetneq \mathcal{I}_R(Z^\sigma(R))$, for every $\sigma \in F^*$. On the other hand, if $\sigma \in F \setminus F^*$, then

$$\dim(Z^\sigma(R)) = \dim_C(Z^\sigma) = \dim_C(Z) = \dim(Y) = d,$$

thus, Corollary 1.1.16 and Proposition 1.2.15 ensure that $Z^\sigma(R) \subset R^n$ is an irreducible algebraic set of dimension d such that $Z^\sigma = \operatorname{Zcl}_C(Z^\sigma(R))$ and $\mathcal{I}_R(Z^\sigma) = \mathcal{I}_R(Z^\sigma(R))$. In particular, $Z^\sigma(R) \neq Z^\tau(R)$ for every $\sigma,\tau \in F \setminus F^*$ with $\sigma \neq \tau$. As $X = \bigcup_{\sigma\in G'} Z^\sigma(R)$ and $\dim(Z^\sigma(R)) < \dim(Z^\sigma) = d$ for every $\sigma \in F^*$, Lemmas 1.1.4(ii) & 1.1.7 ensure that $\{Z^\sigma(R)\}_{\sigma\in F\setminus F^*}$ is the set of all irreducible components of $X \subset R^n$ of dimension d.

Define:

$$\begin{aligned} S_0 &:= B_K(X) = \bigcup_{\sigma\in F^*} Z^\sigma(R),\\ S_1 &:= \bigcup_{\sigma,\tau\in F\setminus F^*,\, \sigma\neq\tau} (Z^\sigma \cap Z^\tau)(R),\\ S &:= S_0 \cup S_1. \end{aligned}$$

Observe that $S_0, S_1, S \subset R^n$ are $\overline{K}^r$-algebraic sets.

STEP III. *The local rings $\mathcal{R}^*_{X,a}$ and $\mathcal{R}_{X,a}$ are not regular for every $a \in S_1$.* Let $a \in S^1$ and let $\sigma,\tau \in F \setminus F^*$, with $\sigma \neq \tau$, be such that $a \in (Z^\sigma \cap Z^\tau)(R) = Z^\sigma(R) \cap Z^\tau(R)$. Choose $f \in R[x]$ so that $\mathcal{Z}_R(f) = Z^\sigma(R)$. We construct $g \in (\bigcup_{\xi\in F\setminus\{\sigma\}} \mathcal{I}_R(Z^\xi)) \setminus \mathcal{I}_R(Z^\sigma)$. Let $x \in Z^\sigma(R) \setminus \bigcup_{\xi\in F\setminus\{\sigma\}} Z^\xi(R)$. By Corollary 1.2.16(i), for every $\xi \in F \setminus \{\sigma\}$ there is $g_\xi \in \mathcal{I}_R(Z^\xi)$ such that $g_\xi(x) \neq 0$. Define $g := \prod_{\xi\in F\setminus\{\sigma\}} g_\xi$.

Since $Z^\sigma(R) = \mathcal{Z}_R(f)$ and Z^τ are distinct irreducible components of X of the same dimension d and $Z^\tau(R) \subset \mathcal{Z}_R(g)$, we deduce that $f \notin \mathcal{I}_R(Z^\tau(R)) = \mathcal{I}_R(Z^\tau)$ and $g \notin \mathcal{I}_R(Z^\sigma(R)) = \mathcal{I}_R(Z^\sigma)$. Thus, $f, g \notin \mathcal{I}_R(X)$ and $f, g \notin \mathcal{I}_K(X)R[x]$. Then, (1.5.3) & (1.5.4) ensure that $\frac{f}{1}, \frac{g}{1} \notin \mathcal{I}_R(X)R[x]_{\mathfrak{n}_a}$ and $\frac{f}{1}, \frac{g}{1} \notin \mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$. However, $fg \in \mathcal{I}_R(X)R[x]_{\mathfrak{n}_a}$ and $fg \in \mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$, thus both $\mathcal{R}_{X,a}$ and $\mathcal{R}^*_{X,a}$ are not integral domains, hence, a fortiori, they are not regular local rings.

As a consequence, $S_1 \subset \operatorname{Sing}(X) \cap \operatorname{Sing}^*(X)$.

STEP IV. *The local ring $\mathcal{R}^*_{X,a}$ is not regular for every $a \in S_0 = B_K(X)$.* Let $a = (a_1, \ldots, a_n) \in S_0$ and suppose that $\mathcal{R}^*_{X,a}$ is a regular local ring. Let $\sigma \in F^*$ so that $a \in Z^\sigma(R)$. Since $\dim(Z^\sigma(R)) < \dim_C(Z^\sigma)$, Proposition 1.2.15 ensures that the ideal $\mathcal{I}_R(Z^\sigma) \subset \mathfrak{n}_a$ of $R[x]$ is non-real, so the prime ideal $\mathcal{I}_R(Z^\sigma)R[x]_{\mathfrak{n}_a}$ of $R[x]_{\mathfrak{n}_a}$ is non-real as well. As $\mathcal{R}^*_{X,a} := R[x]_{\mathfrak{n}_a}/\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$ is a regular local ring, $\mathcal{R}^*_{X,a}$ is in particular an integral domain, thus the ideal $\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$ of $R[x]_{\mathfrak{n}_a}$ is prime. As a consequence, there is a unique prime ideal $\mathfrak{p}$ of $R[x]$ associated to $\mathcal{I}_K(X)R[x]$ that is contained in $\mathfrak{n}_a$, thus

$$\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a} = \mathfrak{p}R[x]_{\mathfrak{n}_a}.$$

Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset R[x]$ be the minimal prime ideals associated to $\sqrt[r]{\mathfrak{p}}$. By [BCR98, Lemma 4.1.5], $\mathfrak{q}_k$ is a real ideal of $R[x]$ for every $k \in \{1, \ldots, s\}$, $\sqrt[r]{\mathfrak{p}}$ is a radical ideal and $\sqrt[r]{\mathfrak{p}} = \bigcap_{k=1}^s \mathfrak{q}_k$. Up to reorder the ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset R[x]$, let $t \in \{1, \ldots, s\}$ such that exactly $\mathfrak{q}_1, \ldots, \mathfrak{q}_t \subset \mathfrak{n}_a$ and $\operatorname{ht}(\bigcap_{k=1}^t \mathfrak{q}_k) = \operatorname{ht}(\mathfrak{q}_1)$. Then, the prime ideal $\mathfrak{q}_k R[x]_{\mathfrak{n}_a}$ is real and $\operatorname{ht}(\bigcap_{k=1}^t \mathfrak{q}_k R[x]_{\mathfrak{n}_a}) = \operatorname{ht}(\mathfrak{q}_k)$ for every $k \in \{1, \ldots, t\}$. Moreover, $\sqrt[r]{\mathfrak{p}}R[x]_{\mathfrak{n}_a} = \bigcap_{k=1}^t \mathfrak{q}_k R[x]_{\mathfrak{n}_a}$ and $\operatorname{ht}(\sqrt[r]{\mathfrak{p}}R[x]_{\mathfrak{n}_a}) = \operatorname{ht}(\mathfrak{q}_1 R[x]_{\mathfrak{n}_a}) = \operatorname{ht}(\mathfrak{q}_1)$. In particular, the Zariski open neighborhood $U := R^n \setminus \bigcup_{k=t+1}^s \mathcal{Z}_R(\mathfrak{q}_k)$ of a in R^n satisfies

$$\mathcal{Z}_R(\sqrt[r]{p}) \cap U = \mathcal{Z}_R(\bigcap_{k=1}^t \mathfrak{q}_k) \cap U. \tag{1.5.5}$$

Since $\mathcal{R}^*_{X,a}$ is a regular local ring, [ZS75, Theorem 26, p. 303] ensures that there is a system of generators $\{f_1, \ldots, f_\ell\}$ of $\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$ in $R[x]_{\mathfrak{n}_a}$ whose classes modulo $\mathfrak{n}_a^2$ are linearly independent and $\ell = \operatorname{ht}(\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}) = \operatorname{ht}(\mathfrak{p}R[x]_{\mathfrak{n}_a})$. We may also assume that

$$\det\Big(\frac{\partial f_i}{\partial x_j}(a)\Big)_{i,j \in \{1,\ldots,\ell\}} \neq 0,$$

so the K-polynomial map $R^n \to R^n$ defined by

$$x := (x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_\ell(x), x_{\ell+1} - a_{\ell+1}, \ldots, x_n - a_n)$$

defines, by the inverse function theorem [BCR98, Proposition 2.9.7], a Nash diffeomorphism $\varphi : V \to U'$ between semialgebraic open neighborhoods $V \subset R^n$ and $U' \subset R^n$ of the origin of R^n and of a in R^n, respectively. Up to shrink V and U' if necessary, we may suppose that $U' \subset U$,

$$\varphi((\{0\} \times R^{n-\ell}) \cap V) = \mathcal{Z}_R(f_1, \ldots, f_\ell) \cap U' = \mathcal{Z}_R(\sqrt[r]{p}) \cap U'$$

and $n - \ell = \dim(\mathcal{Z}_R(f_1, \ldots, f_\ell) \cap U') = \dim(\mathcal{Z}_R(\sqrt[r]{p}) \cap U')$. In addition, (1.5.5) implies that

$$\begin{aligned} n - \operatorname{ht}(\mathfrak{p}R[x]_{\mathfrak{n}_a}) = n - \ell &= \dim(\mathcal{Z}_R(\sqrt[r]{p}) \cap U') \leq \dim(\mathcal{Z}_R(\sqrt[r]{p}) \cap U) \\ &= \dim(\mathcal{Z}_R(\bigcap_{k=1}^t \mathfrak{q}_k) \cap U) = \dim(\mathcal{Z}_R(\bigcap_{k=1}^t \mathfrak{q}_k)) \end{aligned}$$

$$
\begin{aligned}
&= n - \operatorname{ht}\Big(\bigcap_{k=1}^{t} \mathfrak{q}_k\Big) = n - \operatorname{ht}(\mathfrak{q}_1) = n - \operatorname{ht}(\mathfrak{q}_1 R[x]_{\mathfrak{n}_a})\\
&= n - \operatorname{ht}(\sqrt[r]{\mathfrak{p}} R[x]_{\mathfrak{n}_a}) \leq n - \operatorname{ht}(\mathfrak{p} R[x]_{\mathfrak{n}_a}),
\end{aligned}
$$

that is, $\operatorname{ht}(\mathfrak{p}R[x]_{\mathfrak{n}_a}) = \operatorname{ht}(\mathfrak{q}_1 R[x]_{\mathfrak{n}_a})$ and $\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a} = \mathfrak{p}R[x]_{\mathfrak{n}_a} = \mathfrak{q}_1 R[x]_{\mathfrak{n}_a}$ is a prime ideal of $R[x]_{\mathfrak{n}_a}$. As $\bigcap_{\tau\in F} \mathcal{I}_R(Z^\tau)R[x]_{\mathfrak{n}_a} = \mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$, there exists $\tau_0 \in F\setminus\{\sigma\}$ so that $\mathcal{I}_R(Z^{\tau_0})R[x]_{\mathfrak{n}_a} = \mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$. In addition, since $\mathcal{I}_R(Z^\sigma)R[x]_{\mathfrak{n}_a}$ is a non-real ideal of $R[x]_{\mathfrak{n}_a}$, we get that $\mathcal{I}_R(Z^{\tau_0})R[x]_{\mathfrak{n}_a} \subsetneq \mathcal{I}_R(Z^\sigma)R[x]_{\mathfrak{n}_a}$, so in particular $\mathcal{I}_R(Z^{\tau_0}) \subsetneq \mathcal{I}_R(Z^\sigma)$. Recall that $\mathcal{I}_R(Z^{\tau_0})$ is a prime ideal of $R[x]$, thus Proposition 1.2.15 ensures that

$$\mathcal{I}_C(Z^{\tau_0}) = \mathcal{I}_R(Z^{\tau_0})C[x] \subset \mathcal{I}_R(Z^\sigma)C[x] \subset \mathcal{I}_C(Z^\sigma),$$

that is, $Z^\sigma \subset Z^{\tau_0}$. Recall that, by Lemma 1.2.12(iii), $\dim_C(Z^\sigma) = d = \dim_C(Z^{\tau_0})$, thus, being Z^σ irreducible, we deduce that $Z^\sigma = Z^{\tau_0}$. This leads into a contradiction since $\mathcal{I}_R(Z^{\tau_0})$ is a real ideal of $R[x]$, whereas $\mathcal{I}_R(Z^\sigma)$ is not. We conclude that $\mathcal{R}^*_{X,a}$ is not regular.

STEP V. *The local rings $\mathcal{R}^*_{X,a}$ and $\mathcal{R}_{X,a}$ coincide and are regular of dimension d for every $a \in X\setminus S$.* Let $a \in X \setminus S$. By definition of S_0, S_1 and $S := S_0 \cup S_1$, we have that $a \in X\setminus S_0 \subset \bigcup_{\sigma\in F\setminus F^*} Z^\sigma(R)$ and $a \in X\setminus S_1$, thus there is a unique $\sigma \in F\setminus F^*$ such that $a \in Z^\sigma(R)\setminus S$. In particular, $\mathcal{I}_R(Z^\sigma) = \mathcal{I}_R(Z^\sigma(R))$. Let $\tau \in F\setminus\{\sigma\}$. As $a \notin Z^\tau(R)$, there exists a polynomial $h \in \mathcal{I}_C(Z^\tau)$ such that $h(a) \neq 0$. Let $h_1, h_2 \in R[x]$ so that $h = h_1 + ih_2$ and consider $h' := h_1 - ih_2 \in C[x]$. As $a \in R^n$, also $h'(a) \neq 0$, thus $hh'(a) \neq 0$ and $hh' \in \mathcal{I}_R(Z^\tau)$. Latter properties prove that $\mathcal{I}_R(Z^\tau) \not\subset \mathfrak{n}_a$. As a consequence, also $\mathcal{I}_R(Z^\tau(R)) \not\subset \mathfrak{n}_a$, for every $\tau \in F\setminus\{\sigma\}$. In addition, by (1.5.3) & (1.5.4) we have that

$$\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a} = \mathcal{I}_R(Z^\sigma)R[x]_{\mathfrak{n}_a} = \mathcal{I}_R(Z^\sigma(R))R[x]_{\mathfrak{n}_a} = \mathcal{I}_R(X)R[x]_{\mathfrak{n}_a},$$

that is, $\mathcal{R}^*_{X,a} = \mathcal{R}_{X,a}$.

Since $\sigma \in F\setminus F^*$, we have that $Z^\sigma(R) \subset R^n$ is an irreducible algebraic set of dimension d, thus the ideal $\mathcal{I}_R(Z^\sigma(R))R[x]_{\mathfrak{n}_a} = \mathcal{I}_R(Z^\sigma)R[x]_{\mathfrak{n}_a}$ of $R[x]_{\mathfrak{n}_a}$ is prime. Thus, by [AM69, Corollary 3.13], we have $\operatorname{ht}(\mathcal{I}_R(Z^\sigma(R))) = \operatorname{ht}(\mathcal{I}_R(Z^\sigma(R)))R[x]_{\mathfrak{n}_a}$ and

$$
\begin{aligned}
d &= \dim(Z^\sigma(R)) = \dim(R[x]/\mathcal{I}_R(Z^\sigma(R))) = n - \operatorname{ht}(\mathcal{I}_R(Z^\sigma(R)))\\
&= n - \operatorname{ht}(\mathcal{I}_R(Z^\sigma(R))R[x]_{\mathfrak{n}_a}) = n - \operatorname{ht}(\mathcal{I}_R(X)R[x]_{\mathfrak{n}_a}) = \dim(\mathcal{R}_{X,a}).
\end{aligned}
$$

STEP VI. *Proof of statements* (i) & (ii). Evidently, we have

$$\operatorname{Reg}^*(X) \subset \{a \in X \,|\, \mathcal{R}^*_{X,a} \text{ is a regular local ring}\}$$

and, by STEP V, also $X\setminus S \subset \operatorname{Reg}^*(X)$. Let $a \in X$ so that $\mathcal{R}^*_{X,a}$ is a regular local ring. By STEPS III & IV, $a \in X\setminus S$, and then by STEP V again the ring $\mathcal{R}^*_{X,a}$ has dimension d, that is, $a \in \operatorname{Reg}^*(X)$ if and only if $a \in \operatorname{Reg}(X)$. This proves that $\operatorname{Reg}^*(X) = \{a \in X \,|\, \mathcal{R}^*_{X,a} \text{ is a regular local ring}\} = \operatorname{Reg}(X)\setminus S = X\setminus S$. As a consequence, $\operatorname{Sing}^*(X) = \operatorname{Sing}(X)\cup S$ and, since $S_1 \subset \operatorname{Sing}(X)$, we have that $\operatorname{Sing}^*(X) = \operatorname{Sing}(X)\cup S_0 = \operatorname{Sing}(X)\cup B_K(X)$. Observe that $X \subset R^n$ is K-algebraic set thus, Proposition 1.1.19 ensures that $\mathcal{I}_R(X) = \mathcal{I}_{\overline{K}^r}(X)R[x]$. As a consequence, $\operatorname{Sing}(X) \subset R^n$ is a $\overline{K}^r$-algebraic set by [BCR98, Proposition 3.3.10]. Recall that, by STEP II, $S \subset R^n$ is a $\overline{K}^r$-algebraic set, thus also $\operatorname{Sing}^*(X) \subset R^n$ is a $\overline{K}^r$-algebraic set. We are only left to prove that $\mathrm{Reg}^*(X)$ is non-empty. This follows since both S and $\mathrm{Sing}(X)$ are $\overline{K}^r$-algebraic subsets of R^n of dimension $< d$. $\square$

Last theorem is sharp, that is, there are K-algebraic subsets X of R^n such that $\mathrm{Sing}(X) \subsetneq \mathrm{Sing}^*(X)$, as explained by next example.

Example 1.5.4. Consider the field extension $R|\mathbb{Q}$, where R is any real closed field. Denote by $C := R[i]$ the algebraic closure of R. Let $X \subset R^2$ be the nonsingular $\mathbb{Q}$-algebraic set defined as $X := \{(x_1, x_2) \in R^2 \,|\, x_2^3 - 2x_1^3 = 0\}$. Observe that $\mathcal{I}_R(X) = (x_2 - \sqrt[3]{2}x_1)$, by [BCR98, Theorem 4.5.1], thus $X \subset R^2$ is irreducible and in particular it is also $\mathbb{Q}$-irreducible. Let $Z := \mathrm{Zcl}_{C^n}(X) \subset C^2$. By Proposition 1.1.19, we have that $\mathcal{I}_C(Z) = \mathcal{I}_R(X)C[x] = (x_2 - \sqrt[3]{2}x_1)C[x]$, thus $Z = \{(x_1, x_2) \in C^2 \,|\, x_2 - \sqrt[3]{2}x_1 = 0\}$. Let E be a finite Galois extension of $\mathbb{Q}$ containing $\sqrt[3]{2}$. For instance, fix $E := \mathbb{Q}(\sqrt[3]{2}, \omega)$, where $\omega \in C \setminus R$ is a solution of the polynomial $z^3 = 1$. Observe that $\deg(\mathbb{Q}(\sqrt[3]{2}, \omega)|\mathbb{Q}) = \deg(\mathbb{Q}(\sqrt[3]{2}, \omega)|\mathbb{Q}(\sqrt[3]{2})) \cdot \deg(\mathbb{Q}(\sqrt[3]{2})|\mathbb{Q}) = 2 \cdot 3 = 6$ and $G' := G(\mathbb{Q}(\sqrt[3]{2}, \omega)|\mathbb{Q})$ coincides with the symmetric group S_3 of order 6.

Then, an application of Algorithm 1.2.3 with input $Z \subset C^2$ gives as an output the complex and real Galois completions of X, respectively:

$$T = \bigcup_{k=0}^{2} \{(x, y) \in C^2 \,|\, y - \sqrt[3]{2}\omega^k = 0\} = \{(x, y) \in C^2 \,|\, y^3 - 2x^3 = 0\},$$

$$T(R) = \bigcup_{k=0}^{2} \{(x, y) \in R^2 \,|\, y - \sqrt[3]{2}\omega^k = 0\} = X \cup \{(0, 0)\} \cup \{(0, 0)\}$$

Observe that in above definitions of T and $T(R)$ we omitted the repetitions occurring by the action of G'. This proves that $B_{\mathbb{Q}}(X) = \{(0, 0)\}$, thus by Theorem 1.5.3(ii), we have that

$$\mathrm{Sing}(X) = \varnothing \subsetneq \{(0, 0)\} = B_K(X) = B_K(X) \cup \mathrm{Sing}(X) = \mathrm{Sing}^*(X). \blacksquare$$

1.6. K-Determined K-algebraic sets

Throughout this section $R|K$ denotes a field extension in which R is a real closed field containing K. In this section we study those K-algebraic subsets $X \subset R^n$ such that $\mathrm{Reg}^*(X) = \mathrm{Reg}(X)$ or, equivalently, $\mathrm{Sing}^*(X) = \mathrm{Sing}(X)$. Those K-algebraic sets are particularly interesting when considering the field extension $\mathbb{R}|\mathbb{Q}$ in next chapters.

Definition 1.6.1. Let $X \subset R^n$ be a K-algebraic set. We say that X is *K-determined* if $\mathrm{Reg}^*(X) = \mathrm{Reg}(X)$ or, equivalently, $\mathrm{Sing}^*(X) = \mathrm{Sing}(X)$. If in addition X is nonsingular, that is, $\mathrm{Reg}^*(X) = \mathrm{Reg}(X) = X$, then we say that X is *K-nonsingular*. $\blacksquare$

Remark 1.6.2. Let $X \subset R^n$ be a K-algebraic set. By Theorem 1.5.3(ii), we have that $\mathrm{Sing}^*(X) = \mathrm{Sing}(X) \cup B_K(X)$, thus $X \subset R^n$ is K-determined if and only if $B_K(X) \subset \mathrm{Sing}(X)$. In particular, by Corollary 1.4.10, if $X \subset R^n$ is defined over K, then X is K-determined. In particular, if K is real closed, Remark 1.3.9 enures that $X \subset R^n$ is also K-determined. $\blacksquare$

First we relate the property of a K-algebraic set $X \subset R^n$ to be K-determined with properties of its irreducible components. As we will see, K-determinacy of $X \subset R^n$ does not reduce to K-determinacy of its K-irreducible components.

LEMMA 1.6.3. *Let $X \subset R^n$ be a K-algebraic set of dimension d. Let $X_1, \ldots, X_s$ be the K-irreducible components of X. Let $X_1, \ldots, X_r$ be the K-irreducible components of X of dimension d. Then the following are equivalent:*

(i) *X is K-determined;*
(ii) $\operatorname{Reg}(X) \subset \bigcup_{i=1}^r \operatorname{Reg}^*(X_i)$.

PROOF. By equations (1.5.1) & (1.5.2), we have that $\operatorname{Sing}^*(X) = \operatorname{Sing}(X)$ if and only if

$$\operatorname{Sing}^*(X_i) \subset \operatorname{Sing}(X_i) \cup \Big(\bigcup_{j \in \{1,\ldots,r\} \setminus \{i\}} (X_i \cap X_j) \Big) \cup \bigcup_{j=r+1}^{s} X_j$$

for every $i \in \{1, \ldots, r\}$. In addition, the latter inclusion is equivalent to the following:

$$\begin{aligned} \operatorname{Reg}^*(X_i) &\supset \operatorname{Reg}(X_i) \setminus \Big(\bigcup_{j \in \{1,\ldots,r\} \setminus \{i\}} (X_i \cap X_j) \cup \bigcup_{j=r+1}^{s} X_j \Big) \\ &= \operatorname{Reg}(X_i) \setminus \operatorname{Sing}(X) \end{aligned}$$

for every $i \in \{1, \ldots, r\}$. By (1.5.1) & (1.5.2), latter inclusion satisfied for every $i \in \{1, \ldots, r\}$ is equivalent to

$$\begin{aligned} \operatorname{Reg}(X) &= \Big(\bigcup_{i=1}^{r} \operatorname{Reg}(X_i) \Big) \setminus \operatorname{Sing}(X) \\ &= \bigcup_{i=1}^{r} (\operatorname{Reg}(X_i) \setminus \operatorname{Sing}(X)) \subset \bigcup_{i=1}^{r} \operatorname{Reg}^*(X_i), \end{aligned}$$

as required. □

Previous lemma is sharp, in the sense that there are K-reducible K-determined K-algebraic sets $X \subset R^n$ having some K-irreducible components which are not K-determined.

EXAMPLE 1.6.4. Consider the field extension $R|\mathbb{Q}$, in which R is a real closed field. Let $X = X_1 \cup X_2 \subset R^2$ be the $\mathbb{Q}$-algebraic set defined as the union of the $\mathbb{Q}$-irreducible $\mathbb{Q}$-algebraic sets $X_1 := \{(x_1, x_2) \in R^2 \mid x_1 = 0\}$ and $X_2 := \{(x_1, x_2) \in R^2 \mid x_2^3 + 2x_1^3 = 0\}$. Observe that $X_1 \subset R^2$ is defined over $\mathbb{Q}$, indeed $\mathcal{I}_R(X_1) = (x_1)R[x]$, thus is $\mathbb{Q}$-determined by Remark 1.6.2. Recall that, by Example 1.5.4, $X_2 \subset R^2$ is not $\mathbb{Q}$-determined. However,

$$\begin{aligned} \operatorname{Sing}^*(X) &= \operatorname{Sing}^*(X_1) \cup \operatorname{Sing}^*(X_2) \cup (X_1 \cap X_2) = \varnothing \cup \{(0,0)\} \cup \{(0,0)\} \\ &= \{(0,0)\} = \operatorname{Sing}(X), \end{aligned}$$

that is, X is $\mathbb{Q}$-determined, whereas X_2 is not. ■

1.6.1. Nash manifold structure of K-determined algebraic sets. Here we give some equivalent descriptions of the concept of K-determined K-algebraic set $X \subset R^n$ we introduced in Definition 1.6.1.

Let U be a Zariski open subset of R^n and let S be a subset of U. Denote by $\mathcal{R}(U)$ the ring of regular functions on U and by $\mathcal{I}^r_U(S)$ the ideal of regular functions on U vanishing on S. Next results establish some conditions equivalent of being K-determined via a $R|K$-jacobian criterion.

THEOREM 1.6.5. *Let $X \subset R^n$ be a K-algebraic set of dimension d and let $X_1, \dots, X_s$ be the K-irreducible components of X. Let U be a Zariski open neighborhood of* $\operatorname{Reg}(X)$ *in R^n such that* $\operatorname{Reg}(X) = X \cap U$ *(for instance $U = R^n \setminus \operatorname{Sing}(V)$). The following assertions are equivalent:*

(i) *X is K-determined.*
(ii) $B_K(X) \subset \operatorname{Sing}(X)$*, that is, if $Y_i \subset R^n$ is an irreducible component of $X_i \subset R^n$ of dimension* $\dim(X_i)$*, for every $i \in \{1, \dots, s\}$, E is a finite Galois extension of K containing all the coefficients of the equations of each Y_i, $G' := G(E|K)$ and $X = \bigcup_{i=1}^s \bigcup_{\sigma \in G'} Z_i^\sigma(R)$ is a Galois presentation of $X \subset R^n$ with start $(Y_1, \dots, Y_s)$, then*
$$Z_i^\sigma(R) \subset \operatorname{Sing}(X),$$
for every $\sigma \in G'$ and $i \in \{1, \dots, s\}$ such that $\dim(Z_i^\sigma) < \dim(X_i)$.
(iii) *For every $a \in$* $\operatorname{Reg}(X)$*, there exist an Euclidean open neighborhood Ω of a in R^n and polynomials $f_1, \dots, f_{n-d} \in \mathcal{I}_K(X)$ such that the gradients $\nabla f_1(a), \dots, \nabla f_{n-d}(a)$ are linearly independent over R and*
$$X \cap \Omega = \mathcal{Z}_R(f_1, \dots, f_{n-d}) \cap \Omega.$$
(iv) *For every $a \in$* $\operatorname{Reg}(X)$ *there are $f_1, \dots, f_{n-d} \in \mathcal{I}_K(X)$ and a Zariski open neighborhood U_a of a in R^n such that*
$$\mathcal{I}^r_{U_a}(\operatorname{Reg}(X) \cap U_a) = (f_1, \dots, f_{n-d})\mathcal{R}(U_a).$$
(v) $\mathcal{I}^r_U(\operatorname{Reg}(X)) = \mathcal{I}_K(X)\mathcal{R}(U)$.

PROOF. (i) $\iff$ (ii) Follows from Remark 1.6.2 and by definition of $B_K(X) := \bigcup_{i=1}^s \bigcup_{\sigma \in J_i} Z_i^\sigma(R)$, where $J_i := \{\sigma \in G' \mid \dim(Z_i^\sigma(R)) < \dim(X_i)\}$.

(i) $\implies$ (iii). We follow the proof of implication (i) $\implies$ (ii) in [FG, Theorem 4.1.10]. Assume that $\operatorname{Reg}(X) = \operatorname{Reg}^*(X)$, that is: for every $a \in \operatorname{Reg}(X)$ the local ring $\mathcal{R}^*_{X,a} := R[x]_{\mathfrak{n}_a}/\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$ is regular of dimension d. By [ZS75, Theorem 26, p. 303], there exists a system of generators $f_1, \dots, f_\ell$ of $\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}$ in $R[x]_{\mathfrak{n}_a}$ whose classes modulo $\mathfrak{n}_a^2$ are linearly independent and $\operatorname{ht}(\mathcal{I}_K(X)R[x]_{\mathfrak{n}_a}) = \ell$. In particular, we may assume that $f_1, \dots, f_r \in \mathcal{I}_K(X)$ and
$$\det\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j=1,\dots,\ell} \neq 0,$$
so the polynomial map $R^n \to R^n$ defined by
$$x := (x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_\ell(x), x_{\ell+1} - a_{\ell+1}, \dots, x_n - a_n)$$
provides a Nash diffeomorphism $\varphi : U_0 \to U_a$ between an open semialgebraic neighborhood $U_0 \subset R^n$ of the origin and an open semialgebraic neighborhood $U_a \subset R^n$

of a, by the inverse function theorem [BCR98, Proposition 2.9.7]. Shrinking U_0 and U_a if necessary, we can assume that

$$\varphi((\{0\} \times R^{n-\ell}) \cap U_0) = \mathcal{Z}_R(f_1, \dots, f_\ell) \cap U_a.$$

Hence, we deduce that $\ell = n - d$. Set $\Omega := U_a$ and we are done.

(iii) $\Longrightarrow$ (i). We follow the proof of implication (ii$'$) $\Longrightarrow$ (i) in [FG, Theorem 4.1.10]. Let $a \in \mathrm{Reg}(X)$ and $f_1, \dots, f_{n-d} \in \mathcal{I}_K(X)$ satisfying (iii). Since $a \in \mathrm{Reg}(X)$, the local ring $\mathcal{R}_{X,a}$ coincides with $R[x]_{\mathfrak{n}_a}/((f_1, \dots, f_{n-d})R[x]_{\mathfrak{n}_a})$, so it is regular of dimension d. In addition, by [ZS75, Theorem 26, p. 303], the local ring $R[x]_{\mathfrak{n}_a}/((f_1, \dots, f_{n-d})R[x]_{\mathfrak{n}_a})$ is regular of dimension d. Thus, since $(f_1, \dots, f_{n-d}) \subset \mathcal{I}_K(X) \subset \mathcal{I}_R(X)$, we deduce that $\mathcal{R}^*_{X,a}$ is regular of dimension d as well by [ZS75, Theorem 26, p. 303]. As previous property holds for every $a \in \mathrm{Reg}(X)$, we conclude that $X \subset R^n$ is K-determined.

(iii) $\Longleftrightarrow$ (iv) Assertion (iv) $\Longrightarrow$ (iii) is evident. The converse implication (iii) $\Longrightarrow$ (iv) follows directly by [AK92, Proposition 2.2.11], indeed in the proof of mentioned result any transcendental argument is applied, so [AK92, Proposition 2.2.11] holds true for every real closed field R.

(iv) $\Longleftrightarrow$ (v) Assertion (v) $\Longrightarrow$ (iv) is evident by [AK92, Proposition 2.2.11]. Let us prove implication (iv) $\Longrightarrow$ (v). Since X is K-algebraic and R contains K, there is $f \in \mathcal{I}_K(X)$ such that $\mathcal{Z}_K(f) = X$. By (iv), for each $a \in \mathrm{Reg}(X)$, there are polynomials $f_{a,1}, \dots, f_{a,n-d} \in \mathcal{I}_K(X)$ and a Zariski open neighborhood U_a of a in U such that $\mathcal{I}^r_{U_a}(\mathrm{Reg}(X) \cap U_a) = (f_{a,1}, \dots, f_{a,n-d})\mathcal{R}(U_a)$. The set $\mathrm{Reg}(X)$ is compact with respect to the Zariski topology, so there is $k \in \mathbb{N}$ and there are $a_1, \dots, a_k \in \mathrm{Reg}(X)$ such that $\mathrm{Reg}(X) \subset \bigcup_{j=1}^k U_{a_j}$. Let g be an arbitrary element of $\mathcal{I}^r_U(\mathrm{Reg}(X))$. There exist $\alpha, \beta \in R[x]$ such that $g = \frac{\alpha}{\beta}$ on U and $\mathcal{Z}_R(\beta) = R^n \setminus U$. Define the polynomials $\alpha_0, \beta_0 \in R[x]$ by $\alpha_0 := \alpha\beta f$ and $\beta_0 := \beta^2 f^2$. Notice that $\beta_0 \geq 0$ on R^n, $\mathcal{Z}_R(\beta_0) = X \cup (R^n \setminus U)$ and $\beta_0 g = \alpha_0 f$ on U. For each $j \in \{1, \dots, k\}$, there exist polynomials $\alpha_{j,1}, \dots, \alpha_{j,n-d}, \beta_j \in R[x]$ such that $g = \sum_{i=1}^{n-d} \frac{\alpha_{j,i}}{\beta_j} f_{j,i}$ on U_j and $\mathcal{Z}_R(\beta_j) = R^n \setminus U_j$, where $f_{j,i} := f_{a_j,i}$ and $U_j := U_{a_j}$. Replacing each $\alpha_{j,i}$ with $\alpha_{j,i}\beta_j$ and β_j with β_j^2, we can also assume that $\beta_j \geq 0$ on R^n. Since $\beta_0 g = \alpha_0 f$ and $\beta_j g = \sum_{i=1}^{n-d} \alpha_{j,i} f_{j,i}$ on the whole U for all $j \in \{1, \dots, k\}$, it follows that $(\sum_{j=0}^k \beta_j) g = \alpha_0 f + \sum_{j=1}^k \sum_{i=1}^{n-d} \alpha_{j,i} f_{j,i}$ on the whole U. On the other hand, the polynomial $\beta := \sum_{j=0}^k \beta_j \geq 0$ on R^n and $\mathcal{Z}_R(\beta) = \bigcap_{j=0}^k \mathcal{Z}_R(\beta_j) = (X \cup (R^n \setminus U)) \setminus \bigcup_{j=1}^k U_{x_j} = (X \setminus \bigcup_{j=1}^k U_{x_j}) \cup (R^n \setminus U) = R^n \setminus U$. As a consequence, β is invertible in $\mathcal{R}(U)$. Thus $\mathcal{I}^r_U(\mathrm{Reg}(X) \cap U) \subset \mathcal{I}_K(X)\mathcal{R}(U)$, as desired. $\square$

As an immediate consequence, we have:

Corollary 1.6.6. *Let $X \subset R^n$ be a nonsingular K-algebraic set of dimension d. The following assertions are equivalent:*

(i) *X is K-determined.*
(ii) *$B_K(X) = \varnothing$, that is, if $Y_i \subset R^n$ is an irreducible component of $X_i \subset R^n$, for every $i \in \{1, \dots, s\}$, E is a finite Galois extension of K containing all the coefficients of the equations of each Y_i and $X = \bigcup_{i=1}^s \bigcup_{\sigma \in G'} Z_i^\sigma(R)$ is a Galois presentation of $X \subset R^n$ with start $(Y_1, \dots, Y_s)$, then*

$$Z_i^\sigma(R) = \varnothing$$

for every $\sigma \in G'$ *and* $i \in \{1, \dots, s\}$ *such that* $\dim(Z_i^\sigma) < \dim(X_i)$.

(iii) *For every* $a \in X$*, there exist an Euclidean open neighborhood* Ω *of* a *in* R^n *and* $f_1, \dots, f_{n-d} \in \mathcal{I}_K(X)$ *such that the gradients* $\nabla f_1(a), \dots, \nabla f_{n-d}(a)$ *are linearly independent over* R *and*

$$X \cap \Omega = \mathcal{Z}_R(f_1, \dots, f_{n-d}) \cap \Omega.$$

(iv) *For every* $a \in X$ *there are* $f_1, \dots, f_{n-d} \in \mathcal{I}_K(X)$ *and a Zariski open neighborhood* U_a *of* a *in* R^n *such that*

$$\mathcal{I}^r_{U_a}(X \cap U_a) = (f_1, \dots, f_{n-d})\mathcal{R}(U_a).$$

(v) $\mathcal{I}^r_{R^n}(X) = \mathcal{I}_K(X)\mathcal{R}(R^n)$.

Algebraic sets defined over K **in** $\mathcal{R}(R^n)$**.** Corollary 1.6.6(v) suggests another variant of the concept of algebraic set $X \subset R^n$ to be 'defined over K'. In this subsection we introduce and study properties of algebraic sets $X \subset R^n$ 'defined over K in $\mathcal{R}(R^n)$'.

DEFINITION 1.6.7. Let $X \subset R^n$ be an algebraic set. We say that $X \subset R^n$ is *defined over* K *in* $\mathcal{R}(R^n)$ if $\mathcal{I}^r_{R^n}(X) = \mathcal{I}_K(X)\mathcal{R}(R^n)$. ■

REMARK 1.6.8. Every algebraic set $X \subset R^n$ defined over K in $\mathcal{R}(R^n)$ is K-algebraic, indeed if $\mathcal{I}_K(X) = (f_1, \dots, f_s)$, the equality $\mathcal{I}^r_{R^n}(X) = \mathcal{I}_K(X)\mathcal{R}(R^n)$ ensures that $X = \mathcal{Z}_R(\sum_{i=1}^s f_i^2)$. ■

Observe that Corollary 1.6.6(v) ensures that a nonsingular K-algebraic set $X \subset R^n$ is defined over K in $\mathcal{R}(R^n)$ if and only if it is K-nonsingular. Let us further investigate in general the relations of Definition 1.6.7 with other properties we have already introduced.

LEMMA 1.6.9. *Let* $X \subset R^n$ *be an algebraic set. Then the following assertions hold:*

(i) *If* $X \subset R^n$ *is defined over* K*, then* $X \subset R^n$ *is defined over* K *in* $\mathcal{R}(R^n)$*.*
(ii) *If* $X \subset R^n$ *is defined over* K *in* $\mathcal{R}(R^n)$*, then* $X \subset R^n$ *is a* K*-determined* K*-algebraic set.*

PROOF. If $\mathcal{I}_R(X) = \mathcal{I}_K(X)R[x]$, then a fortiori $\mathcal{I}^r_{R^n}(X) = \mathcal{I}_R(X)\mathcal{R}(R^n) = (\mathcal{I}_K(X)R[x])\mathcal{R}(R^n) = \mathcal{I}_K(X)\mathcal{R}(R^n)$. This proves (i). Assume X is defined over K in $\mathcal{R}(R^n)$, then Remark 1.6.2 ensures that X is K-algebraic. In addition, since $\mathcal{I}^r_{R^n}(X) = \mathcal{I}_K(X)\mathcal{R}(R^n)$, for every $a \in \operatorname{Reg}(X)$ there are $f_1, \dots, f_{n-d} \in \mathcal{I}_K(X)$ satisfying Theorem 1.6.5(iii), that is X is K-determined. This proves (ii). □

Observe that, in general, implications of Lemma 1.6.9 can not be reversed.

EXAMPLE 1.6.10. Consider the field extension $R|\mathbb{Q}$ in which R is a real closed field. Denote by $C := R[i]$ the algebraic closure of R.

(i) Let $X := \{\sqrt[3]{2}\} = \{x \in R \,|\, x^3 - 2 = 0\} \subset R$. Observe that $\mathcal{I}^r_R(X) = \mathcal{I}_R(X)\mathcal{R}(R) = (x - \sqrt[3]{2})\mathcal{R}(R)$ and $\mathcal{I}_\mathbb{Q}(X) = (x^3 - 2)\mathbb{Q}[x]$. As $x^3 - 2 = (x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}x + \sqrt[3]{4})$ and $\mathcal{Z}_R(x^2 + \sqrt[3]{2}x + \sqrt[3]{4}) = \varnothing$, we also have that $\mathcal{I}^r_R(X) = (x^3 - 2)\mathcal{R}(R)$. Thus, $\mathcal{I}^r_R(X) = \mathcal{I}_\mathbb{Q}(X)\mathcal{R}(R)$, so X is defined over $\mathbb{Q}$ in $\mathcal{R}(R)$. On the other hand, $x - \sqrt[3]{2} \in \mathcal{I}_R(X) \setminus \mathcal{I}_\mathbb{Q}(X)R[x]$, so X is not defined over $\mathbb{Q}$.

(ii) Let $X := \{x \in R^2 \,|\, x_2^6 - 2x_1^9 = 0\} \subset R^2$ with $x = (x_1, x_2)$. Let $f(x) := x_2^2 - \sqrt[3]{2}x_1^3 \in R[x]$ and $g(x) := x_2^6 - 2x_1^9 \in \mathbb{Q}[x]$. As $X = \mathcal{Z}_R(f)$, f is irreducible in $R[x]$ and changes sign, by [BCR98, Theorem 4.5.1], $\mathcal{I}_R(X) = (f)R[x]$, so $\mathcal{I}^r_{R^2}(X) = (f)\mathcal{R}(R^2)$ and $\operatorname{Sing}(X) = \{x \in X : \nabla f(x) = 0\} = \{(0,0)\}$. Observe that

$$\begin{aligned} X &= \mathcal{Z}_R(f) \cup \mathcal{Z}_R(x_2^2 - \sqrt[3]{2}\omega x_1^3) \cup \mathcal{Z}_R(x_2^2 + \sqrt[3]{2}\omega x_1^3) \\ &= X \cup \{(0,0)\} \cup \{(0,0)\} \end{aligned}$$

is a real Galois presentation of X with start $X = Y := \mathcal{Z}_R(f)$, where $\omega \in C \setminus R$ denotes a solution of the polynomial $z^3 - 1$. So $B_{\mathbb{Q}}(X) = \{(0,0)\} = \operatorname{Sing}(X)$, hence Theorem 1.6.5 ensures that $X \subset R^2$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set. In addition, by Theorem 1.3.4(iv), we have that $\mathcal{I}_{\mathbb{Q}}(X) = \mathcal{I}_{\mathbb{Q}}(\operatorname{Zcl}^{\mathbb{Q}}_{C^n}(X)) = \mathcal{I}_{\mathbb{Q}}(\mathcal{Z}_C(g)) = \mathcal{I}_C(\mathcal{Z}_C(g)) \cap \mathbb{Q}[x] = (g)C[x] \cap \mathbb{Q}[x] = (g)$. Define $f'(x) := x_2^4 + \sqrt[3]{2}x_2^2x_1^3 + \sqrt[3]{4}x_1^6 \in R[x]$. As $g(x) = f(x)f'(x)$ and $\mathcal{Z}_R(f') = \{(0,0)\} \subset X$, we have that $f \notin (g)\mathcal{R}(R^2) = \mathcal{I}_{\mathbb{Q}}(X)\mathcal{R}(\mathbb{R}^2)$ so $f \in \mathcal{I}^r_{R^2}(X) \setminus \mathcal{I}_{\mathbb{Q}}(X)\mathcal{R}(R^2)$. This proves that X is not defined over $\mathbb{Q}$ in $\mathcal{R}(R^2)$. ∎

Two complete characterizations via real Galois completion. The aim of this subsection is to develop two special cases in which we can provide a complete characterization of a K-algebraic set $X \subset R^n$ to be K-determined or defined over K in $\mathcal{R}(R^n)$ or defined over K via properties of the real Galois completion of $X \subset R^n$. Those special cases are nonsingular K-algebraic sets and K-algebraic hypersurfaces $X \subset R^n$.

Nonsingular K-algebraic sets. As a direct consequence of Theorem 1.4.8 and Corollary 1.6.6 we get the following result. Recall that a nonsingular K-determined K-algebraic set $X \subset R^n$ is called K-nonsingular.

COROLLARY 1.6.11. *Let $X \subset R^n$ be a nonsingular K-algebraic set of dimension d, let $X_1, \ldots, X_s \subset R^n$ be the K-irreducible components of X and let Y_i be an irreducible component of X_i, for every $i \in \{1, \ldots, s\}$. Let $X = \bigcup_{i=1}^s \bigcup_{\sigma \in G'} Z_i^\sigma(R)$ be a Galois presentation of $X \subset R^n$ with start $(Y_1, \ldots, Y_s)$. Then:*

(i) *$X \subset R^n$ is K-nonsingular if and only if $X \subset R^n$ is defined over K in $\mathcal{R}(R^n)$. In particular, $X \subset R^n$ is K-nonsingular if and only if $Z_i^\sigma(R) \subset \operatorname{Sing}(X) = \varnothing$ for every $\sigma \in G'$ and $i \in \{1, \ldots, s\}$ such that $\dim(Z_i^\sigma(R)) < d$.*
(ii) *$X \subset R^n$ is defined over K if and only if $\dim(Z_i^\sigma(R)) = d$ for every $\sigma \in G'$ and $i \in \{1, \ldots, s\}$.*

K-geometric hypersurfaces. The case of K-geometric hypersurfaces $X \subset R^n$ is very interesting, indeed K-determined K-geometric hypersurfaces, K-geometric hypersurfaces defined over K in $\mathcal{R}(R^n)$ and K-geometric hypersurfaces defined over K are distinct concepts that can be characterized by means of the real Galois completion of $X \subset R^n$. Let us start by recalling the notion of K-geometric hypersurface originally introduced in [FG, Definition 3.1.8].

DEFINITION 1.6.12. Let $f \in K[x]$. We say that f is *K-geometric in R^n* if $\mathcal{I}_K(\mathcal{Z}_R(f)) = (f)K[x]$. If a polynomial $f \in R[x]$ is *R-geometric in R^n* we say for short that f is *geometric in R^n*.

A set $X \subset R^n$ is a *K-geometric (algebraic) hypersurface of R^n* if $X = \mathcal{Z}_R(f)$ for some K-geometric polynomial f in R^n. If a set $X \subset R^n$ is a R-geometric hypersurface of R^n we say for short that $X \subset R^n$ is a *geometric (algebraic) hypersurface of R^n*. ∎

As every ideal of zeros is radical, we get that, if $f \in K[x]$ is K-geometric in R^n, then f is square-free. In addition, a K-geometric hypersurface $X \subset R^n$ is K-irreducible if and only if there is a K-geometric polynomial $f \in K[x]$ in R^n such that f is irreducible in $K[x]$ and $\mathcal{Z}_R(f) = X$. As a consequence, if $g \in K[x]$ is K-geometric in R^n and $\mathcal{Z}_R(f) = \mathcal{Z}_R(g)$, then f and g are associated. For more details about K-geometric polynomials in R^n and K-geometric hypersurfaces of R^n we refer to [FG, Sections 3 & 4].

Following result is our characterization of different notions of being 'defined over K' for a K-geometric hypersurface $X \subset R^n$ by means of real Galois completion.

THEOREM 1.6.13. *Let $X \subset R^n$ be a K-geometric hypersurface, let $X_1, \dots, X_s \subset R^n$ be the K-irreducible components of X and let Y_i be an irreducible component of X_i for every $i \in \{1, \dots, s\}$. Let $X = \bigcup_{i=1}^s \bigcup_{\sigma \in G'} Z_i^\sigma(R)$ be a Galois presentation of $X \subset R^n$ with start $(Y_1, \dots, Y_s)$. Then:*

(i) *$X \subset R^n$ is K-determined if and only if $Z_i^\sigma(R) \subset \operatorname{Sing}(X)$ for every $\sigma \in G'$ and $i \in \{1, \dots, s\}$ such that $\dim(Z_i^\sigma(R)) < n-1$.*
(ii) *$X \subset R^n$ is defined over K in $\mathcal{R}(R^n)$ if and only if $Z_i^\sigma(R) = \varnothing$ for every $\sigma \in G'$ and $i \in \{1, \dots, s\}$ such that $\dim(Z_i^\sigma(R)) < n-1$.*
(iii) *$X \subset R^n$ is defined over K if and only if $\dim(Z_i^\sigma(R)) = n-1$ for every $\sigma \in G'$ and $i \in \{1, \dots, s\}$.*

PROOF. Observe that (i) & (iii) derive directly from Theorems 1.6.5 & 1.4.8 and [FG, Proposition 3.1.10].

Let us prove (ii). Let $f \in K[x]$ be a K-geometric polynomial in R^n such that $X = \mathcal{Z}_R(f)$ and let $f = f_1 \dots f_s \in K[x]$ be its factorization in $K[x]$. By [FG, Proposition 3.1.10], we have that f is K-geometric in R^n if and only if f_i is so for every $i \in \{1, \dots, s\}$. Then, since $\mathcal{I}_K(X) = (f)K[x] = \bigcap_{i=1}^s (f_i)K[x]$ and $\mathcal{I}^r_{R^n}(X) = \bigcap_{i=1}^s \mathcal{I}^r_{R^n}(\mathcal{Z}_R(f_i))$, we may reduce to the case $s = 1$ with f a K-irreducible K-geometric polynomial in R^n such that $X = \mathcal{Z}_R(f)$.

So, assume f is a K-irreducible K-geometric polynomial in R^n such that $X = \mathcal{Z}_R(f)$ and let $X = \bigcup_{\sigma \in G'} Z^\sigma(R)$ be a Galois presentation of $X \subset R^n$. We prove that $X \subset R^n$ is defined over K in $\mathcal{R}(R^n)$ if and only if $Z^\sigma(R) = \varnothing$ for every $\sigma \in G'$ such that $\dim(Z_i^\sigma(R)) < n-1$.

Let $f = f_1 \dots f_s$ be the irreducible decomposition of f in $R[x]$ (actually in $\overline{K}^r$[x]). Observe that f_i is not associated to any f_j for every $i, j \in \{1, \dots, r\}$ with $i \neq j$. Indeed, assume there is f_i which is associated to f_j for some $i, j \in \{1, \dots, r\}$ and $i \neq j$. Then, $\nabla f(x) = 0$ for every $x \in \mathcal{Z}_R(f_i) = \mathcal{Z}_R(f_j)$, that is, $\mathcal{Z}_R(f_i) \subset \operatorname{Sing}^*(X)$ by Theorem 1.6.5 since $\mathcal{I}_K(X) = (f)K[x]$. This leads to a contradiction because f_i is K-geometric, thus $\dim(\mathcal{Z}_R(f_i)) = n-1$ by [FG, Proposition 3.1.10], but $\dim(\operatorname{Sing}^*(X)) < \dim(X) = n-1$ by Theorem 1.5.3(ii). On the other hand, [FG, Proposition 3.1.10], we also have that $\dim(\mathcal{Z}_R(f_i)) < n-1$ for every $i \in \{r+1, \dots, s\}$. This means that, if we choose $Y = \mathcal{Z}_R(f_1)$ as the start of above Galois presentation of $X \subset \mathbb{R}^n$ we have that $Z^e(R) = \mathcal{Z}_R(f_1)$, $\dim(Z^\sigma(R)) = n-1$ if and only if

$Z^\sigma(R) = \mathcal{Z}_R(f_i)$ fir some $i \in \{1,\dots,r\}$ and $\dim(Z^\sigma(R)) < n-1$ if and only if $Z^\sigma(R) \subset \mathcal{Z}_R(f_i)$ for some $i \in \{r+1,\dots,s\}$. Let us prove above implications.

Assume $Z^\sigma(R) = \varnothing$ for every $\sigma \in G'$ such that $\dim(Z_i^\sigma(R)) < n-1$, that is, $\mathcal{Z}_R(f_j) = \varnothing$ for every $j \in \{r+1,\dots,s\}$. Then, we have $f_1 \cdots f_r = f \cdot \frac{1}{f_{r+1}\cdots f_s} \in \mathcal{R}(R^n)$. Since each f_i with $i \in \{1,\dots,r\}$ is geometric, we also have $\mathcal{I}_R(X) = (f_1 \cdots f_r)R[x]$, thus

$$\begin{aligned}\mathcal{I}^r_{R^n}(X) &= (\mathcal{I}_R(X)R[x])\mathcal{R}(R^n) = ((f_1\cdots f_r)R[x])\mathcal{R}(R^n) = (f_1\cdots f_r)\mathcal{R}(R^n)\\ &= (f)\mathcal{R}(R^n) = \mathcal{I}_K(X)\mathcal{R}(R^n).\end{aligned}$$

On the other hand, assume X is defined over K in $\mathcal{R}(R^n)$, that is, $\mathcal{I}^r_{R^n}(X) = \mathcal{I}_K(X)\mathcal{R}(R^n) = (f)\mathcal{R}(R^n)$. Let $X_1,\dots,X_\ell$ with $\ell \leq s$ be the irreducible components of X, we prove that X is an algebraic hypersurface of R^n. Let $i \in \{1,\dots,\ell\}$ and X_i be of dimension $e \leq n-1$. Let $a \in \operatorname{Reg}(X_i) \setminus \left(\bigcup_{k\in\{1,\dots,\ell\}\setminus\{i\}} X_k\right)$ and let $h_1,\dots,h_{n-e} \in \mathcal{I}_R(X_i)$ be such that $\nabla h_1(a),\dots,\nabla h_{n-e}(a)$ are linearly independent over R. Observe that, up to multiply each h_j by a polynomial $h \in R[x]$ such that $\mathcal{Z}_R(h) = \bigcup_{k\in\{1,\dots,\ell\}\setminus\{i\}} X_k$, we may suppose that above polynomials $h_1,\dots,h_{n-e} \in \mathcal{I}_R(X)$. Being $\mathcal{I}^r_{R^n}(X) = \mathcal{I}_K(X)\mathcal{R}(R^n)$, we deduce that each $\nabla h_j(a)$ is a multiple of $\nabla f(a)$, that is, $e = n-1$ and X_i is an algebraic hipersurface for every $i \in \{1,\dots,\ell\}$. Let $g_1,\dots,g_\ell \in R[x]$ be irreducible polynomials such that $\mathcal{I}_R(X_i) = (g_i)R[x]$ for every $i \in \{1,\dots,\ell\}$, then $\mathcal{I}_R(X) = (g_1\cdots g_\ell)R[x]$ and $\mathcal{I}^r_{R^n}(X) = (g_1\cdots g_\ell)\mathcal{R}(R^n) = (f)\mathcal{R}(R^n)$. Then, there are $h,k \in R[x]$ such that $\mathcal{Z}_R(k) = \varnothing$ and $f = f_1\cdots f_s = g_1\cdots g_\ell \cdot \frac{h}{k}$, that is, $f_1\cdots f_s \cdot k = g_1\cdots g_\ell \cdot h$. Since $\mathcal{Z}_R(k) = \varnothing$, we have that any of the $g_1,\dots,g_\ell$ divides k, thus up to the order, we get that $\ell \leq r$ and g_i is associated to f_i in $R[x]$ for every $i \in \{1,\dots,\ell\}$. On the other way round, we also have that there are $h',k' \in R[x]$ such that $\mathcal{Z}_R(k') = \varnothing$ and $\frac{h'}{k'} \cdot f_1\cdots f_s = g_1\cdots g_\ell$, that is, $f_1\cdots f_s \cdot h' = g_1\cdots g_\ell \cdot k'$. Since $\mathcal{Z}_R(k') = \varnothing$ and f_i is associated with g_i for every $i \in \{1,\dots,\ell\}$, we have that $h'' f_{\ell+1}\cdots f_s = k'$ for some $h'' \in R[x]$. Since $\mathcal{Z}_R(k) = \varnothing$, any of the $f_{\ell+1},\dots,f_r$ divides k', thus we get that $\ell = r$ and f_i is invertible in $\mathcal{R}(R^n)$ for every $i \in \{r+1,\dots,s\}$, that is, $\mathcal{Z}_R(f_i) = \varnothing$ for every $i \in \{r+1,\dots,s\}$. As explained above, last condition is equivalent to the property that $Z^\sigma(R) = \varnothing$ for every $\sigma \in G'$ such that $\dim(Z^\sigma(R)) < n-1$. □

K-Nonsingular K-algebraic sets & K-irreducible components. We conclude this section with a consequence of Corollary 1.6.6 that will prove of crucial importance later on in the proof of Nash-Tognoli theorem and its relative version 'over $\mathbb{Q}$' in Sections 3.2 & 4.1, respectively. We outline that next result is a version with respect to the notion of $R|K$-nonsingularity of [FG, Corollary 4.2.2].

Lemma 1.6.14. *Let $X \subset R^n$ and $Z \subset R^n$ be two K-nonsingular K-algebraic sets of the same dimension d such that $Z \subsetneq X$. Then $X \setminus Z \subset R^n$ is a K-nonsingular K-algebraic set of dimension d as well.*

Proof. If $Z = \varnothing$, then $X \setminus Z = X$. Suppose $Z \neq \varnothing$. Let $X_1,\dots,X_s$ be the K-irreducible components of X. Denote by Z_i the K-algebraic set $Z \cap X_i$ for every $i \in \{1,\dots,s\}$, and denote by I the set of indices i such that $Z_i \neq \varnothing$. Let $i \in I$. We have to show that $Z_i = X_i$. Since $Z \subset X = \operatorname{Reg}^*(X)$, the local ring $\mathcal{R}^*_{X,a}$ is regular, so it is an integral domain. It follows that $Z_i \cap X_j = \varnothing$ for all $j \in \{1,\dots,s\}\setminus\{i\}$, and $Z_i = Z \setminus \bigcup_{j\in\{1,\dots,s\}\setminus\{i\}} X_j$. By Corollary 1.6.6, Z_i is a K-nonsingular K-algebraic set of dimension d. As $\dim_K(Z_i) = d = \dim_K(X_i)$ and $X_i \subset R^n$ is K-irreducible, Lemma 1.1.4(i) ensures that $Z_i = X_i$, as required. $\square$

CHAPTER 2

$\mathbb{Q}$-Nonsingular $\mathbb{Q}$-algebraic sets

Abstract. In this chapter we study $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of $\mathbb{R}^n$. In Section 2.1 we extend classical properties of algebraic sets and regular maps to the case of $\mathbb{Q}$-algebraic subsets of $\mathbb{R}^n$ and $\mathbb{Q}$-regular maps. The remaining part of the chapter is devoted to study $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic representatives of particular $\mathscr{C}^\infty$ manifolds and algebraic sets. In Section 2.2 we study (real) Grassmannians, classical algebraic bundles over Grasmannians and unoriented cobordism classes of compact $\mathscr{C}^\infty$ manifolds. In Section 2.3 we provide an explicit desingularization procedure for real Schubert varieties, thus we represent every $\mathbb{Z}/2\mathbb{Z}$-homology class of each Grassmannian as the pushforward of the fundamental class of a compact $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. In section 2.4 we study the relations between $\mathbb{Q}$-algebraic homology and unoriented $\mathbb{Q}$-bordism classes. Finally, in Section 2.5 we provide relative $\mathbb{Q}$-bordisms of smooth maps with respect to $\mathscr{C}^\infty$ submanifolds in general position.

The main references for this chapter are [GS23] and [Sav23].

In this chapter we specialize notions and results of Chapter 1 to the case of the field extension $\mathbb{R}|\mathbb{Q}$.

2.1. $\mathbb{Q}$-Regular maps & projectively $\mathbb{Q}$-closure

The aim of this section is to extend some classical properties of algebraic subsets of $\mathbb{R}^n$ and regular maps between algebraic sets to $\mathbb{Q}$-nonsingular algebraic subsets of $\mathbb{R}^n$ and the subclass of regular maps between $\mathbb{Q}$-algebraic sets that we will call '$\mathbb{Q}$-regular maps'.

2.1.1. $\mathbb{Q}$-regular maps. Let us start by introducing the notion of $\mathbb{Q}$-regular map we have already mentioned.

Definition 2.1.1. Let $S \subset \mathbb{R}^n$ be a set and let $f : S \to \mathbb{R}$ be a function. We say that f is $\mathbb{Q}$-*regular* if there exist $p, q \in \mathbb{Q}[x]$ such that $\mathcal{Z}_\mathbb{R}(q) \cap S = \varnothing$ and $f(x) = \frac{p(x)}{q(x)}$ for all $x \in S$. We denote by $\mathcal{R}^\mathbb{Q}(S)$ the set of $\mathbb{Q}$-regular functions on S, equipped with the ring structure induced by the usual pointwise addition and multiplication.

Let $T \subset \mathbb{R}^h$ be a set ad let $g : S \to T$ be a map. We say that g is $\mathbb{Q}$-*regular* if there exist $g_1, \dots, g_h \in \mathcal{R}^\mathbb{Q}(S)$ such that $g(x) = (g_1(x), \dots, g_h(x))$ for all $x \in S$. We denote by $\mathcal{R}^\mathbb{Q}(S, T)$ the set of $\mathbb{Q}$-regular maps from S to T. We say that the map $g : S \to T$ is a $\mathbb{Q}$-*biregular isomorphism* if g is bijective and both g and g^{-1} are $\mathbb{Q}$-regular. If there exists such a $\mathbb{Q}$-biregular isomorphism, we say that S is $\mathbb{Q}$-*biregularly isomorphic* to T. ■

The notion of $\mathbb{Q}$-regular function is local in the sense specified by next result.

LEMMA 2.1.2. *Let S be a subset of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be a function. The following assertions are equivalent:*

(i) $f \in \mathcal{R}^{\mathbb{Q}}(S)$.
(ii) *For each $a \in S$, there exist $p_a, q_a \in \mathbb{Q}[x]$ such that $q_a(a) \neq 0$ and $f(x) = \frac{p_a(x)}{q_a(x)}$ for all $x \in S \setminus \mathcal{Z}_{\mathbb{R}}(q_a)$.*

PROOF. A standard argument works. Implication (i) $\Longrightarrow$ (ii) is evident. Let us prove the converse implication (ii) $\Longrightarrow$ (i). Suppose (ii) is satisfied. Let Σ be the family of all subsets T of $\mathbb{R}^n$ with the following property: $S \not\subset T$ and there exist $p_T, q_T \in \mathbb{Q}[x]$ such that $\mathcal{Z}_{\mathbb{R}}(q_T) = T$ and $f(x) = \frac{p_T(x)}{q_T(x)}$ for all $x \in S \setminus T$. By (ii), we have $S \cap \bigcap_{T \in \Sigma} T = \varnothing$. Let us show that Σ is stable under finite intersections. Let $T, T' \in \Sigma$ and let p_T, q_T, $p_{T'}$ and $q_{T'}$ be polynomials in $\mathbb{Q}[x]$ with the above property. Define $p, q \in \mathbb{Q}[x]$ by $p := p_T q_T + p_{T'} q_{T'}$ and $q := q_T^2 + q_{T'}^2$. Note that $S \not\subset T \cap T' = \mathcal{Z}_{\mathbb{R}}(q)$ and $f(x) = \frac{p(x)}{q(x)}$ for all $x \in (S \setminus T) \cup (S \setminus T') = S \setminus (T \cap T')$. This proves the mentioned stability of Σ. By noetherianity, $\bigcap_{T \in \Sigma} T \in \Sigma$ and we are done. $\square$

The next lemma collects some basic properties of $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets and $\mathbb{Q}$-regular maps.

LEMMA 2.1.3. *Let $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^n$ be $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets of the same dimension d such that $V \cap V' = \varnothing$, let $W \subset \mathbb{R}^k$ be a $\mathbb{Q}$-algebraic set and let $f \in \mathcal{R}^{\mathbb{Q}}(V, W)$ and $f' \in \mathcal{R}^{\mathbb{Q}}(V', W)$. The following assertions hold.*

(i) *There exists $F \in \mathcal{R}^{\mathbb{Q}}(\mathbb{R}^n, \mathbb{R}^k)$ such that $F(x) = f(x)$ for all $x \in V$.*
(ii) *$V \sqcup V'$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^n$.*
(iii) *The map $f \sqcup f' : V \sqcup V' \to W$, defined as $(f \sqcup f')(x) := f(x)$ if $x \in V$ and $(f \sqcup f')(x) := f'(x)$ if $x \in V'$, is $\mathbb{Q}$-regular.*
(iv) *The graph $\Gamma_f = \{(x, y) \in V \times W \mid y = f(x)\}$ of f is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$.*
(v) *If $T \subset \mathbb{R}^h$ is a set and $g \in \mathcal{R}^{\mathbb{Q}}(W, T)$, then $g \circ f \in \mathcal{R}^{\mathbb{Q}}(V, T)$.*
(vi) *If V and W are $\mathbb{Q}$-nonsingular, then $V \times W \subset \mathbb{R}^{n+k}$ is $\mathbb{Q}$-nonsingular.*

PROOF. Let $p, p_1, \ldots, p_k, q, p', p'_1, \ldots, p'_k, q' \in \mathbb{Q}[x]$ such that $\mathcal{Z}_{\mathbb{R}}(p) = V$, $\mathcal{Z}_{\mathbb{R}}(q) \cap V = \varnothing$, $f(x) = \big(\frac{p_1(x)}{q(x)}, \ldots, \frac{p_k(x)}{q(x)}\big)$ for all $x \in V$, $\mathcal{Z}_{\mathbb{R}}(p') = V'$, $\mathcal{Z}_{\mathbb{R}}(q') \cap V' = \varnothing$ and $f'(x) = \big(\frac{p'_1(x)}{q'(x)}, \ldots, \frac{p'_k(x)}{q'(x)}\big)$ for all $x \in V'$. The regular map $F : \mathbb{R}^n \to \mathbb{R}^k$, defined by

$$F(x) := \big(\tfrac{p_1(x)q(x)}{p(x)^2 + q(x)^2}, \ldots, \tfrac{p_k(x)q(x)}{p(x)^2 + q(x)^2}\big)$$

for all $x \in \mathbb{R}^n$, proves (i). Consequently, we can assume that the preceding polynomials q and q' never vanish on $\mathbb{R}^n$. Define the polynomials $u_1, \ldots, u_k, v \in \mathbb{Q}[x]$ by

$$u_i(x) := p_i(x)p'(x)^2 q'(x) + p'_i(x)p(x)^2 q(x),$$
$$v(x) := (p(x)^2 + p'(x)^2) q(x) q'(x),$$

for every $i \in \{1, \ldots, k\}$. As v never vanishes on $\mathbb{R}^n$ and $(f \sqcup f')(x) = \big(\frac{u_1(x)}{v(x)}, \ldots, \frac{u_k(x)}{v(x)}\big)$ for all $x \in V \sqcup V'$, (iii) is proved.

The zero set of the polynomial $pp' \in \mathbb{Q}[x]$ is $V \sqcup V'$. Let $d = \dim(V \sqcup V') = \max(\dim(V), \dim(V'))$ and $a \in \mathrm{Reg}(V \sqcup V') = \mathrm{Reg}(V) \sqcup \mathrm{Reg}(V')$. We can assume that $a \in \mathrm{Reg}(V)$. Since $\mathcal{R}^*_{V,a}$ is a regular local ring by assumption and $V \cap V' = \varnothing$ we deduce that $\mathcal{R}^*_{V \sqcup V',a} \cong \mathcal{R}^*_{V,a}$ is a regular local ring of dimension d as well. This proves (ii).

Let us prove (iv). The graph Γ_f of f is the zero set in $\mathbb{R}^{n+k}$ of the polynomial $p(x)^2 + \sum_{i=1}^k (q(x)y_i - p_i(x))^2 \in \mathbb{Q}[x, y_1, \dots, y_k]$, so Γ_f is $\mathbb{Q}$-algebraic. The $\mathbb{Q}$-regular function $g \in \mathcal{R}^{\mathbb{Q}}(V, \Gamma_f)$ defined by $g(x) := (x, f(x))$ is a $\mathbb{Q}$-biregular isomorphism, so Γ_f is $\mathbb{Q}$-nonsingular.

Let $\xi_1, \dots, \xi_{h+1} \in \mathbb{Q}[y_1, \dots, y_k]$ be such that $\mathcal{Z}_{\mathbb{R}}(\xi_{h+1}) \cap W = \varnothing$ and $g(y) = \big(\frac{\xi_1(y)}{\xi_{h+1}(y)}, \dots, \frac{\xi_h(y)}{\xi_{h+1}(y)}\big)$ for all $y \in W$. For each $\alpha \in \{1, \dots, h+1\}$, write ξ_α as follows: $\xi_\alpha = \sum_{j=0}^{c_\alpha} \xi_{\alpha,j}$, where c_α is the degree of ξ_α and each $\xi_{\alpha,j}$ is a homogeneous polynomial of degree j. Let $c := \max\{c_1, \dots, c_{h+1}\}$ and, for each $\alpha \in \{1, \dots, h+1\}$, let $\xi^*_\alpha \in \mathbb{Q}[x]$ be the polynomial $\xi^*_\alpha(x) := q(x)^c \xi_\alpha(f(x)) = \sum_{j=0}^{c_\alpha} q(x)^{c-j} \xi_{\alpha,j}(p_1(x), \dots, p_h(x))$. By construction, we have $\mathcal{Z}_{\mathbb{R}}(\xi^*_{h+1}) \cap V = \varnothing$ and $(g \circ f)(x) = \big(\frac{\xi^*_1(x)}{\xi^*_{h+1}(x)}, \dots, \frac{\xi^*_h(x)}{\xi^*_{h+1}(x)}\big)$ for all $x \in V$. Thus, $g \circ f \in \mathcal{R}^{\mathbb{Q}}(V, T)$. Item (v) is proved.

Item (vi) follows immediately from Corollary 1.6.6. □

Remark 2.1.4. Preceding proof ensures that Lemma 2.1.3(i)(iii)(v) continue to hold when $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^n$ are $\mathbb{Q}$-algebraic sets. ■

The next result deals with transversality of $\mathbb{Q}$-regular maps.

Lemma 2.1.5. *Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^k$ and $Z \subset \mathbb{R}^k$ be $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets with $Z \subset W$, and let $f \in \mathcal{R}^{\mathbb{Q}}(V, W)$ be a $\mathbb{Q}$-regular map transverse to Z in W. Then, $f^{-1}(Z) \subset \mathbb{R}^n$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set such that*

$$\dim(f^{-1}(Z)) = \dim(V) - (\dim(W) - \dim(Z)). \tag{2.1.1}$$

Proof. By classical results about transversality, $f^{-1}(Z)$ is a nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^n$ satisfying (2.1.1). We are left to prove that $f^{-1}(Z) = \mathrm{Reg}^*(f^{-1}(Z))$. Let $a \in f^{-1}(Z)$ and $b = f(a) \in Z$. Let $s_1, \dots, s_k, t \in \mathbb{Q}[x]$ such that $\mathcal{Z}_{\mathbb{R}}(t) \cap V = \varnothing$ and $f(x) = \big(\frac{s_1(x)}{t(x)}, \dots, \frac{s_k(x)}{t(x)}\big)$ for all $x \in V$. Since W and Z are $\mathbb{Q}$-nonsingular, by Corollary 1.6.6 there exist an Euclidean open U_b of b in $\mathbb{R}^k$ and $p_1, \dots, p_{k-m_1} \in \mathbb{Q}[y]$ such that $\nabla p_1(b), \dots, \nabla p_{k-m_1}(b)$ are linearly independent in $\mathbb{R}^k$, $U_b \cap W = U_b \cap \mathcal{Z}_{\mathbb{R}}(p_1, \dots, p_{k-m_2})$ and $U_b \cap Z = U_b \cap \mathcal{Z}(p_1, \dots, p_{k-m_1})$, with $m_1 := \dim(Z)$ and $m_2 := \dim(W)$. Again, since V is $\mathbb{Q}$-nonsingular, by Corollary 1.6.6 there exist an euclidean open neighborhood U_a of a in $\mathbb{R}^n$ and $q_1, \dots, q_{n-\dim(V)} \in \mathbb{Q}[x]$ such that $\nabla q_1(a), \dots, \nabla q_{n-\dim(V)}(a)$ are linearly independent in $\mathbb{R}^n$ and $U_a \cap V = U_a \cap \mathcal{Z}_{\mathbb{R}}(q_1, \dots, q_{n-\dim(V)})$. Define the polynomials $r_i(x) := t(x)^{\deg(p_{k-m_2+i})} p_{k-m_2+i}(f(x)) \in \mathbb{Q}[x]$ for every $i \in \{1, \dots, \dim(W) - \dim(Z) = m_2 - m_1\}$, and the polynomial map $P : \mathbb{R}^n \to \mathbb{R}^{n-\dim(V)+\dim(W)-\dim(Z)}$ as

$$P := (q_1, \dots, q_{n-\dim(V)}, r_1, \dots, r_{\dim(W)-\dim(Z)}).$$

Transversality of f to W in Z implies that

$$\mathrm{rk}(J_P(a)) = n - \dim(V) + \dim(W) - \dim(Z).$$

Consequently, an application of Corollary 1.6.6 gives that $a \in \operatorname{Reg}^*(f^{-1}(Z))$, as desired. $\square$

Let us recall the definitions of overt polynomial and projectively closed real algebraic set, introduced in [AK81b, p. 427]. Let $p \in \mathbb{R}[x]$ be a nonconstant polynomial. Write p as follows: $p = \sum_{i=0}^{d} p_i$, where d is the degree of p and each polynomial p_i is homogeneous of degree i. The polynomial $p \in \mathbb{R}[x]$ is said to be *overt* if $\mathcal{Z}_{\mathbb{R}}(p_d) = \{0\}$. An algebraic set $V \subset \mathbb{R}^n$ is called *projectively closed* if there exists an overt polynomial $p \in \mathbb{R}[x]$ such that $V = \mathcal{Z}_{\mathbb{R}}(p)$. This notion has a simple geometric interpretation. Let $\theta : \mathbb{R}^n \to \mathbb{P}^n(\mathbb{R})$ be the affine chart $\theta(x_1, \dots, x_n) := [1, x_1, \dots, x_n]$. By elementary considerations concerning homogenization of polynomials, it is immediate to verify that the algebraic set $V \subset \mathbb{R}^n$ is projectively closed if and only if $\theta(V)$ is Zariski closed in $\mathbb{P}^n(\mathbb{R})$. As a consequence, if V is projectively closed, then it is compact in $\mathbb{R}^n$.

Note that a nonconstant overt polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ is proper. Indeed, if we write $p = \sum_{i=0}^{d} p_i$ as above, then there exists a real constant $C > 0$ such that $|p_d(x)| \geq 2C|x|_n^d$ for all $x \in \mathbb{R}^n$. Thus, $|p(x)| \geq C|x|_n^d$ for all $x \in \mathbb{R}^n$ with $|x|_n$ sufficiently large.

2.1.2. Projectively $\mathbb{Q}$-closure. Let us specialize 'over $\mathbb{Q}$' the notion of projectively closed (real) algebraic set as follows.

DEFINITION 2.1.6. We say that a $\mathbb{Q}$-algebraic set $V \subset \mathbb{R}^n$ is *projectively $\mathbb{Q}$-closed* if there exists an overt polynomial $p \in \mathbb{Q}[x]$ such that $V = \mathcal{Z}_{\mathbb{R}}(p)$.

Some basic properties of projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic sets are as follows.

LEMMA 2.1.7. *Let $V \subset \mathbb{R}^n$ be a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set. Then it hold:*

(i) *If $V' \subset \mathbb{R}^n$ is another projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set, $V \cup V' \subset \mathbb{R}^n$ is also a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set.*
(ii) *If $Z \subset \mathbb{R}^n$ is a $\mathbb{Q}$-algebraic set, $V \cap Z \subset \mathbb{R}^n$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set.*
(iii) *Given any $v \in \mathbb{Q}^n$, the translated set $V + v := \{x + v \in \mathbb{R}^n : x \in V\} \subset \mathbb{R}^n$ is projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic.*
(iv) *If $W \subset \mathbb{R}^k$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set, then the product $V \times W \subset \mathbb{R}^{n+k}$ is also a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set. In particular, for each $m \in \mathbb{N}$ with $m > n$, the set $V \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$ is projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic.*

PROOF. Let $p, p' \in \mathbb{Q}[x]$ and $s \in \mathbb{Q}[y]$ be overt polynomials such that $V = \mathcal{Z}_{\mathbb{R}}(p)$, $V' = \mathcal{Z}_{\mathbb{R}}(p')$ and $W = \mathcal{Z}_{\mathbb{R}}(s)$. Let $d, e \in \mathbb{N}^*$ be the degrees of p and s, respectively. Then $pp' \in \mathbb{Q}[x]$ and $p^{2e} + s^{2d} \in \mathbb{Q}[x, y]$ are overt polynomials, whose zero sets are $V \cup V'$ and $V \times W$, respectively. If $m \in \mathbb{N}$ with $m > n$, then $\{0\} \subset \mathbb{R}^{m-n}$ is projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic. Thus, $V \times \{0\} \subset \mathbb{R}^m$ is also projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic. Let $q \in \mathbb{Q}[x]$ be such that $Z = \mathcal{Z}_{\mathbb{R}}(q)$ and let ℓ be the degree of q. It follows that $p^{2\ell+2} + q^2 \in \mathbb{Q}[x]$ is overt and its zero set is $V \cap Z$. Given any $v \in \mathbb{Q}^n$, the polynomial $p(x - v) \in \mathbb{Q}[x]$ is overt and its zero set is $V + v$. $\square$

2.1.3. $\mathbb{R}|\mathbb{Q}$-Generic projection. We also have the following variant 'over $\mathbb{Q}$' of a classical generic projection lemma by preserving the $\mathbb{R}|\mathbb{Q}$-regularity at any point. Here we only remind the result and we refer to [FG] for a complete proof.

THEOREM 2.1.8. *Let $V \subset \mathbb{R}^n$ be a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set of dimension d. If $n > 2d+1$, then V is $\mathbb{Q}$-biregularly isomorphic to a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $V' \subset \mathbb{R}^{2d+1}$.*

2.2. $\mathbb{Q}$-algebraic embeddings of some special manifolds

Throughout this section, m and n denote two fixed positive natural numbers. The elements x of $\mathbb{R}^{m+n}$ are considered as column vectors. Thus, if $x_1, \dots, x_{m+n}$ are the entries of x, we write $x = (x_1, \dots, x_{m+n})^T$, where the superscript '^{T}' denotes the transpose operator.

Let $\mathbb{G}_{m,n}$ denote the (real) Grassmannian manifold of m-dimensional vector subspaces of $\mathbb{R}^{m+n}$. Identify $\mathbb{R}^{(m+n)^2}$ with the set of $(m+n) \times (m+n)$ real matrices. It is well known, see [BCR98, Theorem 3.4.4], that Grassmannians are biregular isomorphic to the following algebraic subsets of $\mathbb{R}^{(m+n)^2}$:

$$\mathbb{G}_{m,n} = \big\{X \in \mathbb{R}^{(m+n)^2} : X^T = X, X^2 = X, \operatorname{tr}(X) = m\big\}. \tag{2.2.1}$$

The biregular map assigns to each point p of the Grassmannian, corresponding to a m-dimensional vector subspace V_p of $\mathbb{R}^{m+n}$, the matrix $X_p \in \mathbb{R}^{(m+n)^2}$ of the orthogonal projection of $\mathbb{R}^{m+n}$ onto V_p with respect to the canonical basis of $\mathbb{R}^{m+n}$.

LEMMA 2.2.1. *Each Grassmannian $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.*

PROOF. Let $\phi : \mathbb{R}^{(m+n)^2} \to \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2}$ be the polynomial map $\phi(X) := (X^T - X, X^2 - X)$. We prove that the polynomial $\operatorname{tr}(X) - m$ and the polynomial components of ϕ suffice to describe nonsingular points of $\mathbb{G}_{m,n}$ in $\mathbb{R}^{(m+n)^2}$ via the $\mathbb{Q}$-jacobian criterion of Theorem 1.6.5. Since these polynomials have coefficients in $\mathbb{Q}$ and their common zero set is $\mathbb{G}_{m,n}$, bearing in mind that $\mathbb{G}_{m,n}$ has dimension mn, it suffices to show that, for each $A \in \mathbb{G}_{m,n}$, the rank of the jacobian matrix $J_\phi(A)$ of ϕ at A is greater than or equal to (and hence equal to) $(m+n)^2 - mn$, i.e. $\operatorname{rnk} J_\phi(A) \geq (m+n)^2 - mn$ for all $A \in \mathbb{G}_{m,n}$.

First, we prove that $\operatorname{rnk} J_\phi(D_m) \geq (m+n)^2 - mn$ if D_m is the diagonal matrix in $\mathbb{R}^{(m+n)^2}$ having 1 in the first m diagonal positions and 0 otherwise. For each $i, j \in \{1, \dots, m+n\}$, define the polynomial functions $f_{ij} : \mathbb{R}^{(m+n)^2} \to \mathbb{R}$ and $g_{ij} : \mathbb{R}^{(m+n)^2} \to \mathbb{R}$ by

$$f_{ij}(X) := x_{ij} - x_{ji} \qquad \text{and} \qquad g_{ij}(X) := \big(\textstyle\sum_{\ell=1}^{m+n} x_{i\ell}x_{\ell j}\big) - x_{ij}$$

for all $X = (x_{ij})_{i,j} \in \mathbb{R}^{(m+n)^2}$. Hence, $\phi(X) = ((f_{ij}(X))_{i,j}, (g_{ij}(X))_{i,j})$. Define:

$$\begin{aligned}
S_1 &:= \{(i,j) \in \{1, \dots, m+n\}^2 \,|\, i < j\},\\
S_2 &:= \{(i,j) \in \{1, \dots, m+n\}^2 \,|\, i \leq j \leq m\},\\
S_3 &:= \{(i,j) \in \{1, \dots, m+n\}^2 \,|\, m < i \leq j\}.
\end{aligned}$$

Notice that the sum of the cardinalities of S_1, S_2 and S_3 is equal to
$$\frac{(m+n-1)(m+n)}{2}+\frac{m(m+1)}{2}+\frac{n(n+1)}{2}=(m+n)^2-mn.$$
By a direct computation, we see that
$$\begin{array}{ll}\nabla f_{ij}(D_m)=E_{ij}-E_{ji} & \text{if } (i,j)\in S_1,\\ \nabla g_{ij}(D_m)=E_{ij} & \text{if } (i,j)\in S_2,\\ \nabla g_{ij}(D_m)=-E_{ij} & \text{if } (i,j)\in S_3,\end{array}$$
where E_{ij} is the matrix in $\mathbb{R}^{(m+n)^2}$ whose (i,j)-coefficient is equal to 1 and 0 otherwise. Consequently, we have that $\operatorname{rnk} J_\phi(D_m)\geq (m+n)^2-mn$. Let $A\in\mathbb{G}_{m,n}$ and let $G\in O(m+n)$ be such that $D_m=G^TAG$. Define the linear automorphism $\psi:\mathbb{R}^{(m+n)^2}\to\mathbb{R}^{(m+n)^2}$ by $\psi(X):=G^TXG$. Since $\psi(A)=D_m$ and $(\psi\times\psi)\circ\phi=\phi\circ\psi$, we have that $J_{\psi\times\psi}(\phi(A))J_\phi(A)=J_\phi(D_m)J_\psi(A)$. Bearing in mind that both matrices $J_{\psi\times\psi}(\phi(A))$ and $J_\psi(A)$ are invertible, it follows that $\operatorname{rnk} J_\phi(A)=\operatorname{rnk} J_\phi(D_m)\geq (m+n)^2-mn$, as desired. Finally, we note that $\operatorname{tr}(AA^T)$ equals the squared Euclidean norm of A in $\mathbb{R}^{(m+n)^2}$ and it holds $\operatorname{tr}(AA^T)=\operatorname{tr}(G^TAA^TG)=\operatorname{tr}(D_mD_m^T)=\operatorname{tr}(D_m)=m$. Since $\mathbb{G}_{m,n}\subset\mathbb{R}^{(m+n)^2}$ is the zero set of the polynomial $|\phi(X)|^2_{(m+n)^2}+(\operatorname{tr}(X)-m)^2\in\mathbb{Q}[(x_{ij})_{i,j}]$ and $\mathbb{G}_{m,n}$ is contained in the projectively $\mathbb{Q}$-closed sphere $\{\operatorname{tr}(XX^T)-m=0\}$ of $\mathbb{R}^{(m+n)^2}$, Lemma 2.1.7(ii) ensures that $\mathbb{G}_{m,n}$ is projectively $\mathbb{Q}$-closed in $\mathbb{R}^{(m+n)^2}$ as well. $\square$

Consider the special case $m=1$, thus $\mathbb{G}_{1,n}\subset\mathbb{R}^{(n+1)^2}$ is a $\mathbb{Q}$-algebraic embedding of the projective space $\mathbb{P}^n(\mathbb{R})$. Given a vector $x=(x_1,\dots,x_{n+1})^T\in\mathbb{R}^{n+1}\setminus\{0\}$, we denote by $[x]=[x_1,\dots,x_{n+1}]$ the corresponding element of $\mathbb{P}^n(\mathbb{R})$. We indicate by $\mu_n:\mathbb{P}^n(\mathbb{R})\to\mathbb{G}_{1,n}$ the $\mathscr{C}^\infty$ diffeomorphism given by
$$\mu_n([x]):=xx^T|x|^{-2}=(x_ix_j|x|^{-2})_{i,j}. \tag{2.2.2}$$
Note that, given any $x\in\mathbb{R}^{n+1}\setminus\{0\}$, $xx^T|x|^{-2}$ is the matrix associated to the orthogonal projection of $\mathbb{R}^{n+1}$ onto the vector line generated by x w.r.t. the canonical vector basis of $\mathbb{R}^{n+1}$.

Let $n'\in\mathbb{N}^*$ with $n\leq n'$. Given $x=(x_1,\dots,x_{n+1})^T\in\mathbb{R}^{n+1}$ and $y=(y_1,\dots,y_{n'+1})^T\in\mathbb{R}^{n'+1}$, we define $\langle x,y\rangle:=\sum_{i=1}^{n+1}x_iy_i$. Denote by $H_{n,n'}$ the nonsingular (real) algebraic hypersurface of $\mathbb{P}^n(\mathbb{R})\times\mathbb{P}^{n'}(\mathbb{R})$ defined by
$$H_{n,n'}:=\big\{([x],[y])\in\mathbb{P}^n(\mathbb{R})\times\mathbb{P}^{n'}(\mathbb{R})\,\big|\,\langle x,y\rangle=0\big\} \tag{2.2.3}$$
and the $\mathscr{C}^\infty$ hypersurface $\mathbb{H}_{n,n'}$ of $\mathbb{G}_{1,n}\times\mathbb{G}_{1,n'}$ by
$$\mathbb{H}_{n,n'}:=(\mu_n\times\mu_{n'})(H_{n,n'}). \tag{2.2.4}$$

Lemma 2.2.2. *Each $\mathbb{H}_{n,n'}\subset\mathbb{R}^{(n+1)^2+(n'+1)^2}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.*

Proof. For every $k\in\{1,\dots,n+1\}$, define the sets $U_{n,k}$ and $\Omega_{n,k}$ by
$$U_{n,k}:=\{[x_1,\dots,x_{n+1}]\in\mathbb{P}^n(\mathbb{R})\,|\,x_k\neq 0\},$$
$$\Omega_{n,k}:=\{X\in\mathbb{G}_{1,n}\,|\,x_{kk}\neq 0\},$$
and the map $\xi_{n,k}:\Omega_{n,k}\to U_{n,k}$ by
$$\xi_{n,k}(X):=[x_{1k},\dots,x_{n+1,k}].$$

It is immediate to verify that

$$\mu_n(U_{n,k}) = \Omega_{n,k} \quad \text{and} \quad \mu_n^{-1}(X) = \xi_{n,k}(X) \text{ for all } X \in \Omega_{n,k}. \tag{2.2.5}$$

Since $\bigcup_{k=1}^{n+1} U_{n,k} = \mathbb{P}^n(\mathbb{R})$, the family $\{\Omega_{n,k}\}_{k=1}^{n+1}$ is a Zariski open cover of $\mathbb{G}_{1,n}$. As a consequence, the family $\{\Omega_{n,k} \times \Omega_{n',h}\}_{k\in\{1,\dots,n+1\},h\in\{1,\dots,n'+1\}}$ is a Zariski open cover of $\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'}$.

Let $k \in \{1,\dots,n+1\}$, $h \in \{1,\dots,n'+1\}$ and $p_{k,h} \in \mathbb{Q}[X,Y]$ be the polynomial defined by

$$p_{k,h}(X,Y) := \textstyle\sum_{i=1}^{n+1} x_{ik}y_{ih}.$$

Let us show that

$$\mathbb{H}_{n,n'} \cap (\Omega_{n,k} \times \Omega_{n',h}) = \mathcal{Z}_{\mathbb{R}}(p_{k,h}) \cap (\Omega_{n,k} \times \Omega_{n',h}). \tag{2.2.6}$$

Indeed, given $(X,Y) = ((x_{ij})_{i,j},(y_{ij})_{i,j}) \in \Omega_{n,k} \times \Omega_{n',h}$, by (2.2.5), we have that $(X,Y) \in \mathbb{H}_{n,n'}$ if, and only if, $(\xi_{n,k}(X),\xi_{n',h}(Y)) \in H_{n,n'}$. Since $\xi_{n,k}(X) = [x_{1k},\dots,x_{n+1,k}]$ and $\xi_{n',h}(Y) = [y_{1h},\dots,y_{n'+1,h}]$, by definition (2.2.3) of $H_{n,n'}$, the latter condition is in turn equivalent to assert that $p_{k,h}(X,Y) = 0$. This proves (2.2.6). We claim that

$$\mathbb{H}_{n,n'} = \textstyle\bigcap_{h=1}^{n'+1}\bigcap_{k=1}^{n+1} \mathcal{Z}_{\mathbb{R}}(p_{k,h}) \cap (\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'}). \tag{2.2.7}$$

We can prove this equality as follows. Let $(X,Y) \in \bigcap_{h=1}^{n'+1}\bigcap_{k=1}^{n+1} \mathcal{Z}_{\mathbb{R}}(p_{k,h}) \cap (\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'})$. Since $(X,Y) \in \Omega_{n,k}\times\Omega_{n',h}$ for some $k \in \{1,\dots,n+1\}$ and $h \in \{1,\dots,n'+1\}$, it follows that $(X,Y) \in \mathcal{Z}_{\mathbb{R}}(p_{k,h}) \cap (\Omega_{n,k} \times \Omega_{n',h})$. By (2.2.6), we deduce that $(X,Y) \in \mathbb{H}_{n,n'} \cap (\Omega_{n,k} \times \Omega_{n',h}) \subset \mathbb{H}_{n,n'}$. Consider now $(X,Y) \in \mathbb{H}_{n,n'}$. Let $x \in \mathbb{R}^{n+1}$ and $y \in \mathbb{R}^{n'+1}$ be such that $([x],[y]) \in H_{n,n'}$, $X = xx^T|x|^{-2}$ and $Y = yy^T|y|^{-2}$. For every $k \in \{1,\dots,n+1\}$ and $h \in \{1,\dots,n'+1\}$, we have:

$$p_{k,h}(X,Y) = \textstyle\sum_{i=1}^{n+1}(x_ix_k|x|^{-2})(y_iy_h|y|^{-2}) = x_ky_h|x|^{-2}|y|^{-2}\langle x,y\rangle = 0.$$

This proves equality (2.2.7). By Lemmas 2.1.3(v) and 2.2.1, we know that $\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^{(n+1)^2+(n'+1)^2} = \mathbb{R}^{(n+1)^2} \times \mathbb{R}^{(n'+1)^2}$. Since $\mathbb{H}_{n,n'}$ is a $\mathscr{C}^\infty$ hypersurface of $\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'}$, in order to prove that $\mathbb{H}_{n,n'}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set it suffices to show that, for each $(X_0,Y_0) \in \mathbb{H}_{n,n'}$, there exist $k \in \{1,\dots,n+1\}$ and $h \in \{1,\dots,n'+1\}$ such that $\nabla p_{k,h}(X_0,Y_0)$ is not orthogonal to the tangent space of $\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'}$ at (X_0,Y_0) in $\mathbb{R}^{(n+1)^2} \times \mathbb{R}^{(n'+1)^2}$, i.e.

$$\nabla p_{k,h}(X_0,Y_0) \notin T_{(X_0,Y_0)}(\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'})^\perp. \tag{2.2.8}$$

Fix $(X_0,Y_0) \in \mathbb{H}_{n,n'}$. Choose $k \in \{1,\dots,n+1\}$ and $h \in \{1,\dots,n'+1\}$ in such a way that $(X_0,Y_0) \in \Omega_{n,k} \times \Omega_{n',h}$. Consider the coordinates $x' = (x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1})$ in $\mathbb{R}^n$ and $y' = (y_1,\dots,y_{h-1},y_{h+1},\dots,y_{n'+1})$ in $\mathbb{R}^{n'}$. Set $x'' = (x_1,\dots,x_{k-1},1,x_{k+1},\dots,x_{n+1}) \in \mathbb{R}^{n+1}$ and $y'' = (y_1,\dots,y_{h-1},1,y_{h+1},\dots,y_{n'+1}) \in \mathbb{R}^{n'+1}$. Define the $\mathscr{C}^\infty$ diffeomorphism $\chi_{k,h} : \mathbb{R}^n \times \mathbb{R}^{n'} \to U_{n,k} \times U_{n',h}$ and the $\mathscr{C}^\infty$ embedding $\mu_{k,h} : U_{n,k} \times U_{n',h} \to \mathbb{R}^{(n+1)^2} \times \mathbb{R}^{(n'+1)^2}$ by

$$\chi_{k,h}(x',y') := ([x''],[y'']) \quad \text{and} \quad \mu_{k,h}([x],[y]) := (\mu_{n,k}([x]),\mu_{n',h}([y])),$$

and the regular function $q_{k,h} : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}$ by

$$q_{k,h}(x',y') := (p_{k,h} \circ \mu_{k,h} \circ \chi_{k,h})(x',y') = (1+|x'|^2)^{-1}(1+|y'|^2)^{-1}\langle x'',y''\rangle.$$

Let (x'_0, y'_0) be the unique point in $\mathbb{R}^n \times \mathbb{R}^{n'}$ such that $(X_0, Y_0) = (\mu_{k,h} \circ \chi_{k,h})(x'_0, y'_0)$. Since $d_{(x'_0,y'_0)}(\mu_{k,h} \circ \chi_{k,h})$ maps isomorphically $T_{(x'_0,y'_0)}(\mathbb{R}^n \times \mathbb{R}^{n'})$ onto $T_{(X_0,Y_0)}(\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'})$, condition (2.2.8) is equivalent to the following:

$$\nabla q_{k,h}(x'_0, y'_0) \neq 0. \tag{2.2.9}$$

Notice that $q_{k,h} = rs$, where $r(x', y') := (1 + |x'|^2)^{-1}(1 + |y'|^2)^{-1}$ and $s(x', y') := \langle x'', y''\rangle$. Since $q_{k,h}(x'_0, y'_0) = 0$, $r(x'_0, y'_0) \neq 0$ and $\nabla q_{k,h} = s\nabla r + r\nabla s$, we deduce that $s(x'_0, y'_0) = 0$ and $\nabla q_{k,h}(x'_0, y'_0) = r(x'_0, y'_0)\nabla s(x'_0, y'_0)$. Hence we have to prove that $\nabla s(x'_0, y'_0) \neq 0$. It holds:

$$\nabla s(x'_0, y'_0) = (y^0_1, \dots, y^0_{k-1}, y^0_{k+1}, \dots, y^0_{n+1}, x^0_1, \dots, x^0_{h-1}, x^0_{h+1}, \dots, x^0_{n+1}),$$

with $x'_0 = (x^0_1, \dots, x^0_{k-1}, x^0_{k+1}, \dots, x^0_{n+1})$, $y'_0 = (y^0_1, \dots, y^0_{h-1}, y^0_{h+1}, \dots, y^0_{n'+1})$, $x^0_k := 1$ and $y^0_h := 1$. If $k \neq h$, then one of the components of $\nabla s(x'_0, y'_0)$ is $x^0_k = 1$, so $\nabla s(x'_0, y'_0) \neq 0$. Suppose that $k = h$. In this case, $\nabla s(x', y') = 0$ implies that $x' = 0$ and $y' = 0$. Consequently, $s(x', y') = \langle x', y'\rangle + 1 = 1$. Since $s(x'_0, y'_0) = 0$, it follows that $\nabla s(x'_0, y'_0) \neq 0$. This proves (2.2.9).

By Lemmas 2.1.7(iv) and 2.2.1, we have that $\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'}$ is projectively $\mathbb{Q}$-closed in $\mathbb{R}^{(n+1)^2+(n'+1)^2}$. By (2.2.7), we have that $\mathbb{H}_{n,n'} = \mathcal{Z}_\mathbb{R}(p) \cap (\mathbb{G}_{1,n} \times \mathbb{G}_{1,n'})$, where $p := \sum_{h=1}^{n'+1} \sum_{k=1}^{n+1} p^2_{k,h} \in \mathbb{Q}[X, Y]$. Lemma 2.1.7(ii) ensures that $\mathbb{H}_{n,n'}$ is projectively $\mathbb{Q}$-closed in $\mathbb{R}^{(n+1)^2+(n'+1)^2}$. This completes the proof. $\square$

Let $\mathbb{E}_{m,n}$ denote the (total space of the) universal vector bundle over $\mathbb{G}_{m,n}$ as the following algebraic subsets of $\mathbb{R}^{(m+n)^2+m+n} = \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n}$:

$$\mathbb{E}_{m,n} := \{(X, y) \in \mathbb{G}_{m,n} \times \mathbb{R}^{m+n} \,:\, Xy = y\}.$$

It is well-known that $\mathbb{E}_{m,n}$ is a connected $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^{(m+n)^2+m+n}$ of dimension $m(n + 1)$.

Lemma 2.2.3. *Each universal vector bundle $\mathbb{E}_{m,n} \subset \mathbb{R}^{(m+n)^2+m+n}$ over $\mathbb{G}_{m,n}$ is a $\mathbb{Q}$-nonsingular projectively $\mathbb{Q}$-closed algebraic set.*

Proof. Let $\phi : \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \to \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n}$ be the polynomial map defined by

$$\phi(X, y) := (X^T - X, X^2 - X, Xy - y).$$

We prove that the polynomial $\mathrm{tr}(X) - m$ and the polynomial components of ϕ do suffice to describe nonsingular points of $\mathbb{E}_{m,n} \subset \mathbb{R}^{m+n^2} \times \mathbb{R}^{m+n}$ via the $\mathbb{Q}$-jacobian criterion of Theorem 1.6.5. As in the proof of Lemma 2.2.1, it suffices to show that $\mathrm{rnk}\, J_\phi(A, b) \geq (m+n)^2 + m+n-m(n+1) = (m+n)^2 - mn+n$ for all $(A, b) \in \mathbb{E}_{m,n}$.

First, we prove that $\mathrm{rnk}\, J_\phi(D_m, v) \geq (m + n)^2 - mn + n$ if D_m is the diagonal matrix in $\mathbb{R}^{(m+n)^2}$ having 1 in the first m diagonal positions and 0 otherwise, and $v = (v_1, \dots, v_{m+n})^T$ is a vector of $\mathbb{R}^{m+n}$ such that $(D_m, v) \in \mathbb{E}_{m,n}$. For each $\ell \in \{1, \dots, m + n\}$, define the polynomial functions $h_\ell : \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)} \to \mathbb{R}$ by

$$h_\ell(X, y) := \big(\textstyle\sum_{j=1}^{m+n} x_{\ell j} y_j\big) - y_\ell$$

for all $X = (x_{ij})_{i,j} \in \mathbb{R}^{(m+n)^2}$ and $y = (y_1, \dots, y_{m+n})^T \in \mathbb{R}^{m+n}$. Thus, with the same notation used in the proof of Lemma 2.2.1, it follows that $\phi(X, y) = ((f_{ij}(X))_{i,j}, (g_{ij}(X))_{i,j}, (h_\ell(X, y))_\ell)$. Thanks to the proof of the mentioned Lemma

2.2.1, we already know that the rank of the jacobian matrix at (D_m, v) of the map $X \mapsto ((f_{ij}(X))_{i,j}, (g_{ij}(X))_{i,j})$ is $\geq (m+n)^2 - mn$. Thus, we only have to look at the components $(h_\ell(X,y))_\ell$. By a direct computation we see that

$$\nabla h_\ell(D_m, v) = \left(\textstyle\sum_{j=1}^n v_j E_{\ell j}, -e_\ell\right) \in \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n},$$

if $\ell \in \{m+1, \dots, m+n\}$,where $E_{\ell j}$ is the matrix in $\mathbb{R}^{(m+n)^2}$ whose (ℓ, j)-coefficient is equal to 1 and 0 otherwise, and $\{e_1, \dots, e_{m+n}\}$ is the canonical vector basis of $\mathbb{R}^{m+n}$. Consequently, we obtain that $\operatorname{rnk} J_\phi(D_m, v) \geq (m+n)^2 - mn + n$ for every $v \in \mathbb{R}^{m+n}$ such that $(D_m, v) \in \mathbb{E}_{m,n}$.

Let us complete the proof. Let $(A,b) \in \mathbb{E}_{m,n}$, let $G \in O(m+n)$ be such that $D_m = G^T A G$ and let $v := G^T b$. Note that $D_m v = G^T A G G^T b = G^T A b = G^T b = v$, i.e., $(D_m, v) \in \mathbb{E}_{m,n}$. Define the linear automorphisms $\psi : \mathbb{R}^{(m+n)^2} \to \mathbb{R}^{(m+n)^2}$ and $\tau : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ by $\psi(X) := G^T X G$ and $\tau(y) = G^T y$. Since $(\psi \times \tau)(A,b) = (D_m, v)$ and $(\psi \times \psi \times \tau) \circ \phi = \phi \circ (\psi \times \tau)$, we have that $J_{\psi\times\psi\times\tau}(\phi(A,b)) J_\phi(A,b) = J_\phi(D_m, v) J_{\psi\times\tau}(A,b)$. Bearing in mind that both matrices $J_{\psi\times\psi\times\tau}(\phi(A,b))$ and $J_{\psi\times\tau}(A,b)$ are invertible, it follows that $\operatorname{rnk} J_\phi(A,b) = \operatorname{rnk} J_\phi(D_m, d) \geq (m+n)^2 - mn + n$, as desired. Finally, we note that $\mathbb{E}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ is the zero set of the polynomial $|\phi(X,y)|^2_{(m+n)^2} + (\operatorname{tr}(X) - m)^2 \in \mathbb{Q}[(x_{ij})_{i,j}, (y_k)_k]$, thus we deduce that $\mathbb{E}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ is $\mathbb{Q}$-algebraic. □

Let $\mathbb{E}^*_{m,n}$ denote the (total space of the) universal sphere bundle over $\mathbb{G}_{m,n}$ as the following real algebraic subset of $\mathbb{R}^{(m+n)^2+m+n+1} := \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R}$:

$$\mathbb{E}^*_{m,n} = \{(X,y,t) \in \mathbb{G}_{m,n} \times \mathbb{R}^{m+n} \times \mathbb{R} \mid Xy = y,\ |y|^2_n + t^2 = t\}.$$

It is well-known that $\mathbb{E}^*_{m,n}$ is a connected $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R}$ of dimension $m(n+1)$.

Lemma 2.2.4. *Each universal sphere bundle $\mathbb{E}^*_{m,n} \subset \mathbb{R}^{(m+n)^2+m+n+1}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.*

Proof. Let $\phi : \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R} \to \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R}$ be the polynomial map defined by

$$\phi(X,y,t) := (X^T - X, X^2 - X, Xy - y, |y|^2_{m+n} + t^2 - t).$$

We prove that the polynomial $\operatorname{tr}(X) - m$ and the polynomial components of ϕ do suffice to describe nonsingular points of $\mathbb{E}^*_{m,n} \subset \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R}$ via the $\mathbb{Q}$-jacobian criterion of Theorem 1.6.5. As in the proof of Lemma 2.2.1, it suffices to show that $\operatorname{rnk} J_\phi(A,b,c) \geq (m+n)^2 + m + n + 1 - m(n+1) - 1 = (m+n)^2 - mn + n$ for all $(A,b,c) \in \mathbb{E}^*_{m,n}$.

First, we prove that $\operatorname{rnk} J_\phi(D_m, v, c) \geq (m+n)^2 - mn + n$ if D is the diagonal matrix in $\mathbb{R}^{(m+n)^2}$ having 1 in the first m diagonal positions and 0 otherwise, $v = (v_1, \dots, v_{m+n})^T$ is a vector of $\mathbb{R}^{m+n}$ and $c \in \mathbb{R}$ such that $(D_m, v, c) \in \mathbb{E}^*_{m,n}$. For each $\ell \in \{1, \dots, m+n+1\}$, define the polynomial functions $h_\ell : \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)} \times \mathbb{R} \to \mathbb{R}$

by

$$\begin{aligned} h_\ell(X,y,t) &:= \Big(\sum_{j=1}^{m+n} x_{\ell j}y_j\Big) - y_\ell \quad \text{if } \ell \neq m+n+1 \\ h_{m+n+1}(X,y,t) &:= |y|^2_{m+n} + t^2 - t, \end{aligned}$$

for all $X = (x_{ij})_{i,j} \in \mathbb{R}^{(m+n)^2}$, $y = (y_1,\ldots,y_{m+n}) \in \mathbb{R}^{m+n}$ and $t \in \mathbb{R}$. Thus, with the same notation of the proof of Lemma 2.2.3, it follows that $\phi(X,y,t) = ((f_{ij}(X))_{i,j},(g_{ij}(X))_{i,j},(h_\ell(X,y,t))_\ell)$. Thanks to the proof of mentioned Lemma 2.2.3, we already know that the rank of the jacobian matrix at (D_m,v,c) of the map

$$(X,y,t) \mapsto ((f_{ij}(X))_{i,j},(g_{ij}(X))_{i,j},(h_\ell(X,y,t))_\ell)$$

is $\geq (m+n)^2 - mn + n$. Thus, we only have to look at the components $(h_\ell(X,y))_\ell$ in order to prove that h_{m+n+1} always produces an additional independent gradient with respect to the gradients of $(h_\ell(X,y))_{\ell\neq m+n+1}$. By a direct computation we see that

$$\begin{aligned} \nabla h_\ell(D_m,v,c) &= \Big(\sum_{j=1}^{m+n} v_j E_{\ell j}, -e_\ell, 0\Big) \quad \text{if } \ell \in \{m+1,\ldots,m+n\}, \\ \nabla h_{m+n+1}(D_m,v,c) &= \big(0, 2v, 2c-1\big) \end{aligned}$$

where $E_{\ell j}$ is the matrix in $\mathbb{R}^{(m+n)^2}$ whose (ℓ,j)-coefficient is equal to 1 and 0 otherwise, and $\{e_1,\ldots,e_{m+n}\}$ is the canonical vector basis of $\mathbb{R}^{m+n}$. Observe that $\nabla h_{m+n+1}(D_m,v,c)$ is linearly independent with respect to $(\nabla h_\ell(X,y))_{\ell\neq m+n+1}$ when $c \neq 1/2$, otherwise, if $c = 1/2$, then

$$\nabla h_{m+n+1}(D_m,v,c) = (0,2v,0),$$

so it is contained in the m-plane satisfying $D_m v = v$, hence it is linearly independent with respect to $(\nabla h_\ell(X,y))_{\ell\neq m+n+1}$ as well. Consequently, we obtain that $\operatorname{rnk} J_\phi(D_m,v,c) \geq (m+n)^2 - mn + n$ for every $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $(D_m,v,c) \in \mathbb{E}^*_{m,n}$.

Let us complete the proof. Let $(A,b,c) \in \mathbb{E}^*_{m,n}$, let $G \in O(m+n)$ be such that $D_m = G^TAG$ and let $v := G^Tb$. By the choice of G we see that $|v|^2_{m+n} = |G^Tv|^2_{m+n} = |b|^2_{m+n}$, hence c satisfies $|v|^2_{m+n} + c^2 - c = 0$ as well. Note that $D_m v = G^TAGG^Tb = G^TAb = G^Tb = v$, i.e., $(D_m,v,c) \in \mathbb{E}^*_{m,n}$. Define the linear automorphisms $\psi : \mathbb{R}^{(m+n)^2} \to \mathbb{R}^{(m+n)^2}$ and $\tau : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ by $\psi(X) := G^TXG$ and $\tau(y) = G^Ty$. Since $(\psi\times\tau\times\mathrm{id}_\mathbb{R})(A,b,c) = (D_m,v,c)$ and $(\psi\times\psi\times\tau\times\mathrm{id}_\mathbb{R})\circ\phi = \phi\circ(\psi\times\tau\times\mathrm{id}_\mathbb{R})$, we have that $J_{\psi\times\psi\times\tau\times\mathrm{id}_\mathbb{R}}(\phi(A,b,c))J_\phi(A,b) = J_\phi(D_m,v,c)J_{\psi\times\tau\times\mathrm{id}_\mathbb{R}}(A,b,c)$. Bearing in mind that both matrices $J_{\psi\times\psi\times\tau\times\mathrm{id}_\mathbb{R}}(\phi(A,b,c))$ and $J_{\psi\times\tau\times\mathrm{id}_\mathbb{R}}(A,b,c)$ are invertible, it follows that $\operatorname{rnk} J_\phi(A,b,c) = \operatorname{rnk} J_\phi(D_m,v,c) \geq (m+n)^2 - mn + n + 1$, as desired.

Let $\mathbb{S}^{m+n} \subset \mathbb{R}^{m+n+1}$ be the standard unit sphere. Since $\mathbb{E}^*_{m,n} \subset \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R}$ is the zero set of $|\phi(X,y,t)|^2_{(m+n)^2} + (\mathrm{tr}(X) - m)^2 \in \mathbb{Q}[(x_{ij})_{i,j},(y_k)_k,t]$ and $\mathbb{E}^*_{m,n} \subset \mathbb{G}_{m,n} \times \mathbb{S}^{m+n}$, which is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic subset of $\mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R}$ by Lemmas 2.2.1 & 2.1.7(iv), we have that $\mathbb{E}^*_{m,n}$ is projectively $\mathbb{Q}$-closed in $\mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R}$ as well by Lemma 2.1.7(ii). $\square$

In Section 2.5 and later on in Chapter 3 we will also need the following refinement of a well known result.

LEMMA 2.2.5. *Let $V \subset \mathbb{R}^k$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set of dimension d, and let $\beta : V \to \mathbb{G}_{k-d,d}$ be the normal bundle map of V in $\mathbb{R}^k$ (also called the Gauss mapping of V in $\mathbb{R}^k$). Then $\beta \in \mathcal{R}^{\mathbb{Q}}(V, \mathbb{G}_{k-d,k})$.*

PROOF. By Lemma 2.1.2, it suffices to show that, for each $a \in V$, there exist polynomials $p_{i,j} \in \mathbb{Q}[x]$ for every $i, j \in \{1, \dots, k\}$ and $q \in \mathbb{Q}[x]$ such that $q(a) \neq 0$ and $\beta(x) = \frac{P(x)}{q(x)}$ for all $x \in V \setminus \mathcal{Z}_{\mathbb{R}}(q)$, where $P : \mathbb{R}^k \to \mathbb{R}^{k^2}$ is defined by $P(x) := (p_{i,j}(x))_{i,j}$. In order to prove this, we follow the argument used in the proof of [AK92, Proposition 2.4.3]. Let $a \in V$. By Corollary 1.6.6, there are $p_1, \dots, p_{k-d} \in \mathcal{I}_{\mathbb{Q}}(V)$ whose gradients at a are linearly independent in $\mathbb{R}^k$. Let $A : \mathbb{R}^k \to \mathbb{R}^{k(k-d)}$ and $M : \mathbb{R}^k \to \mathbb{R}^{(k-d)^2}$ be the polynomial maps defined as follows: given $x \in \mathbb{R}^k$, $A(x)$ is the $k \times (k-d)$-matrix whose columns are the gradients of $p_1, \dots, p_{k-d}$ at x and $M(x)$ is the $(k-d) \times (k-d)$-matrix defined by $M(x) := A(x)^T A(x)$. Define $q(x) := \det(M(x)) \in \mathbb{Q}[x]$. Note that $A(x)$ and $M(x)$ have the same rank for all $x \in \mathbb{R}^k$. Thus, $q(x) \neq 0$ if and only if the rank of $A(x)$ is $k-d$. It follows that $q(a) \neq 0$. By elementary considerations from linear algebra, we have that $\beta(x) = A(x)M(x)^{-1}A(x)^T$ for all $x \in V \setminus \mathcal{Z}_{\mathbb{R}}(q)$. Since all the entries of M are polynomials in $\mathbb{Q}[x]$, by Cramer's rule, there is a map $C = (c_{ij})_{i,j} : \mathbb{R}^k \to \mathbb{R}^{(k-d)^2}$ such that each entry $c_{ij} \in \mathbb{Q}[x]$ and $M(x)^{-1} = \frac{C(x)}{q(x)}$ for all $x \in \mathbb{R}^k \setminus \mathcal{Z}_{\mathbb{R}}(q)$. The map $P : \mathbb{R}^k \to \mathbb{R}^{k^2}$, defined by $P(x) := A(x)C(x)A(x)^T$, has the desired properties. $\square$

Let W be a nonsingular algebraic subset of $\mathbb{R}^k$ of dimension d. Let $\mathbb{G} := \prod_{i=1}^{\ell} \mathbb{G}_{m_i,n_i}$, let $\mathbb{E}^* := \prod_{i=1}^{\ell} \mathbb{E}^*_{m_i,n_i}$ and le $\mu : W \to \mathbb{G}$ be a regular map. Let $\pi_i := \mathbb{G} \to \mathbb{G}_{m_i,n_i}$ be the projection onto the i-th factor and let $\mu_i : W \to \mathbb{G}_{m_i,n_i}$ be defined as $\mu_i := \pi_i \circ \mu$ for every $i \in \{1, \dots, \ell\}$. We define the pull-back sphere bundle $\mu^*(\mathbb{E}^*)$ over W of $\mathbb{E}^*$ via μ as the following algebraic subset of $\mathbb{R}^k \times \prod_{i=1}^{\ell}(\mathbb{R}^{m_i+n_i} \times \mathbb{R})$:

$$\mu^*(\mathbb{E}^*) := \{(x, y^1, t_1, \dots, y^\ell, t_\ell) \in W \times \prod_{i=1}^{\ell}(\mathbb{R}^{m_i+n_i} \times \mathbb{R}) \mid \mu_i(x)y^i = y^i,\ |y^i|^2_{m+n} + t_i^2 = t_i \text{ for } i = 1, \dots, \ell\}.$$

It is well-known that $\mu^*(\mathbb{E}^*)$ is a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^k \times (\mathbb{R}^{m+n} \times \mathbb{R})^\ell$ of dimension $d + \sum_{i=1}^{\ell} m_i$.

LEMMA 2.2.6. *Let W be a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^k$ of dimension d. Let $\mu : W \to \mathbb{G}$ be a $\mathbb{Q}$-regular map. Then $\mu^*(\mathbb{E}^*) \subset \mathbb{R}^k \times \prod_{i=1}^{\ell}(\mathbb{R}^{m_i+n_i} \times \mathbb{R})$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.*

PROOF. For simplifying the notation we only prove the case $\ell = 1$, in the general case the proof works in the same way. Let $\ell = 1$, $\mathbb{G} = \mathbb{G}_{m,n}$ and $\mathbb{E} = \mathbb{E}^*_{m,n}$. There are $s \in \mathbb{N}^*$ and $p_1, \dots, p_s \in \mathbb{Q}[x] = \mathbb{Q}[x_1, \dots, x_k]$ such that $\mathcal{I}_{\mathbb{R}^k}(W) = (p_1, \dots, p_s)$. Let $\phi : \mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R} \to \mathbb{R}^s \times \mathbb{R}^{m+n} \times \mathbb{R}$ be the regular map defined by

$$\phi(x, y, t) := (p_1(x), \dots, p_s(x), \mu(x)y - y, |y|^2_{m+n} + t^2 - t),$$

where $\mu(x) \in \mathbb{G} \subset \mathbb{R}^{(m+n)^2}$ is in matrix form. We prove that the polynomial components of ϕ do suffice to describe nonsingular points of $\mu^*(\mathbb{E}^*) \subset \mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R}$

via the $\mathbb{Q}$-jacobian criterion of Theorem 1.6.5. As in the proves of previous lemmas, it suffices to show that $\operatorname{rnk} J_\phi(a,b,c) \geq k-d+n+1$ for all $(a,b,c) \in \mu^*(\mathbb{E}^*)$.

As in the proof of Lemma 2.2.4, for every $r \in \{1,\dots,m+n+1\}$, define the polynomial functions $h_r : \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{m+n} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} h_r(X,y,t) &:= \Big(\sum_{j=1}^{m+n} x_{rj}y_j\Big) - y_s \quad \text{if } r \neq m+n+1 \\ h_{m+n+1}(X,y,t) &:= |y|^2_{m+n} + t^2 - t, \end{aligned}$$

for all $X = (x_{ij})_{ij} \in \mathbb{R}^{(m+n)^2}$, $y = (y_1,\dots,y_{m+n}) \in \mathbb{R}^{m+n}$ and $t \in \mathbb{R}$. Thus, with the same notation of the proof of Lemma 2.2.4, it follows that

$$\phi(x,y,t) = (p_1(x),\dots,p_s(x),h_1(\mu(x),y,t),\dots,h_{m+n+1}(\mu(x),y,t)).$$

Let $\nu : \mathbb{R}^k \to \mathbb{G}$, defined as $\nu(x) := (\nu(x)_{ij})_{ij}$, be any regular function such that $D_m \in \nu(W)$. Define $h'_r : \mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R} \to \mathbb{R}$ as $h'_r := h_r \circ (\nu \times \mathrm{id}_{\mathbb{R}^{m+n}} \times \mathrm{id}_{\mathbb{R}})$ for every $r \in \{1,\dots,m+n+1\}$. Thanks to the proof of Lemma 2.2.3 and being W nonsingular of dimension d, we get that the rank of the jacobian matrix of the map $(x,y,t) \mapsto (p_1(x),\dots,p_s(x),h'_1(x,y,t),\dots,h'_{m+n+1}(x,y,t))$ at every $(a,v,c) \in \nu^*(\mathbb{E}^*)$ such that $\nu(a) = D_m$ is $\geq k-d+n+1$, hence equal to $k-d+n+1$. Indeed, denote by $\nu_r : \mathbb{R}^{m+n} \to \mathbb{R}$ be the regular map defined as $\nu_r(x) := (\nu_{r1}(x),\dots,\nu_{r\,m+n}(x))$, that is, the map associated to the r-th row of ν. Then we have:

$$\begin{aligned} \nabla p_i(a,v,c) &= (\nabla p_i(a),0,0) \quad \text{for every } i = 1,\dots,s; \\ \nabla h'_r(a,v,c) &= \Big(\nabla \nu_r(a)\cdot v^T, -e_r, 0\Big) \quad \text{if } r \in \{m+1,\dots,m+n\}, \\ \nabla h_{m+n+1}(a,v,c) &= \big(0, 2v, 2c-1\big), \end{aligned}$$

where $x = (x_1,\dots,x_k)$ and $\{e_1,\dots,e_{m+n}\}$ denotes the canonical vector basis of $\mathbb{R}^{m+n}$.

Let us complete the proof. Let $(a,b,c) \in \mu^*(\mathbb{E}^*)$, let $G \in O(m+n)$ be such that $D_m = G^T\mu(a)G$ and let $v := G^Tb$. By the choice of G we see that $|v|^2_{m+n} = |G^Tv|^2_{m+n} = |b|^2_{m+n}$, hence c satisfies $|v|^2_{m+n} + c^2 - c = 0$ as well. Note that $D_m v = G^TAGG^Tb = G^TAb = G^Tb = v$, i.e., $(D_m,v,c) \in \nu^*(E^*)$ with $\nu : W \to \mathbb{G}$ defined as $\nu(a) := G^T\mu(a)G$. Define the regular function $\psi : \mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R} \to \mathbb{R}^s \times \mathbb{R}^{m+n} \times \mathbb{R}$ by

$$\psi(x,y,t) := (p_1(x),\dots,p_s(x),\nu(x)y - y, |y|^2_{m+n} + t^2 - t),$$

and the linear automorphism $\tau : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ by $\tau(y) = G^Ty$. Since $(\mathrm{id}_{\mathbb{R}^k} \times \tau \times \mathrm{id}_{\mathbb{R}})(a,b,c) = (a,v,c)$ and $(\mathrm{id}_{\mathbb{R}^s} \times \tau \times \mathrm{id}_{\mathbb{R}}) \circ \phi = \phi \circ (\mathrm{id}_{\mathbb{R}^k} \times \tau \times \mathrm{id}_{\mathbb{R}})$ we have that $J_{\mathrm{id}_{\mathbb{R}^s} \times\tau\times\mathrm{id}_{\mathbb{R}}}(\phi(a,b,c))J_\phi(a,b,c) = J_\psi(a,v,c)J_{\mathrm{id}_{\mathbb{R}^k} \times\tau\times\mathrm{id}_{\mathbb{R}}}(a,b,c)$. Since both matrices $J_{\mathrm{id}_{\mathbb{R}^s} \times\tau\times\mathrm{id}_{\mathbb{R}}}(\phi(a,v,c))$ and $J_{\mathrm{id}_{\mathbb{R}^k} \times\tau\times\mathrm{id}_{\mathbb{R}}}(a,b,c)$ are invertible and $\nu(a) = D_m$, it follows that $\operatorname{rnk} J_\phi(a,b,c) = \operatorname{rnk} J_\psi(a,v,c) = k-d+n+1$, as desired.

Since $\mu^*(\mathbb{E}^*) \subset \mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R}$ is the zero set of $|\phi(x,y,t)|^2_{s+m+n+1} \in \mathbb{Q}[x,y,t]$ and $\mu^*(\mathbb{E}^*)$ is contained in $W \times S^{m+n}$, which is a projectively $\mathbb{Q}$-closed algebraic subset of $\mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R}$ by Lemmas 2.2.1 & 2.1.7(iv), we have that $\mu^*(\mathbb{E}^*)$ is projectively $\mathbb{Q}$-closed in $\mathbb{R}^k \times \mathbb{R}^{m+n} \times \mathbb{R}$ as well by Lemma 2.1.7(ii). $\square$

2.3. $\mathbb{Q}$-Desingularization of real embedded Schubert varieties

As in Section 2.2, in what follows m and n denote two fixed positive natural numbers. The elements x of $\mathbb{R}^{m+n}$ are considered as column vectors. Thus, if $x_1,\dots,x_{m+n}$ are the entries of x, we write $x=(x_1,\dots,x_{m+n})^T$, where the superscript '^{T}' denotes the transpose operator.

2.3.1. Embedded Schubert varieties. Let $\mathbb{G}_{m,n}\subset\mathbb{R}^{(m+n)^2}$ be the embedded Grassmannian manifold of m-planes in $\mathbb{R}^{m+n}$. Let us construct an embedded version of Schubert varieties inducing a cellular decomposition of $\mathbb{G}_{m,n}$. Consider the complete flag of $\mathbb{R}^{m+n}$ consisting of the strictly increasing sequence of each $\mathbb{R}^k$, with $k\leq m+n$, spanned by the first k elements of the canonical basis of $\mathbb{R}^{m+n}$. That is:

$$0\subset\mathbb{R}\subset\cdots\subset\mathbb{R}^i\subset\cdots\subset\mathbb{R}^{m+n}.$$

We will refer to the previous complete flag as the *canonical complete flag of* $\mathbb{R}^{m+n}$. Let us define the *Schubert varieties* of $\mathbb{G}_{m,n}$ with respect to the above complete flag by following the convention in [Man01, §3]. Define a *partition* $\lambda=(\lambda_1,\dots,\lambda_m)$ as a decreasing sequence of integers such that $n\geq\lambda_1\geq\cdots\geq\lambda_m\geq 0$. Hence, λ corresponds uniquely to a Young diagram in a $(m\times n)$-rectangle. Denote by D_ℓ the $(m+n)^2$ matrix associated to the orthogonal projection $\mathbb{R}^{m+n}\to\mathbb{R}^\ell$ sending $(x_1,\dots,x_{m+n})\mapsto(x_1,\dots,x_\ell)$ with respect to the canonical basis of $\mathbb{R}^{m+n}$ for every $\ell\in\{1,\dots,m+n\}$. Hence, D_ℓ is the diagonal matrix in $\mathbb{R}^{(m+n)^2}$ having 1 in the first ℓ diagonal positions and 0 otherwise. Define the *Schubert open cell* of $\mathbb{G}_{m,n}$ associated to λ with respect to the canonical complete flag as

$$\Omega_\lambda:=\big\{X\in\mathbb{G}_{m,n}\,|\,\mathrm{rnk}(XD_\ell)=k\quad\text{if } n+k-\lambda_k\leq\ell\leq n+k-\lambda_{k+1}\big\}.$$

Define the *Schubert variety* of $\mathbb{G}_{m,n}$ associated to the partition λ with respect to the canonical complete flag as

$$\sigma_\lambda:=\big\{X\in\mathbb{G}_{m,n}\,|\,\mathrm{rnk}(XD_{n+k-\lambda_k})\geq k,\ \text{for } k=1,\dots,m\big\}. \tag{2.3.1}$$

The partition λ is uniquely determined and determinates uniquely a sequence of incidence conditions with respect to the above canonical complete flag of $\mathbb{R}^{m+n}$. In addition, the matrix $XD_\ell=(x'_{ij})_{i,j}\in\mathbb{R}^{(m+n)^2}$ satisfies the following relations with respect to $X=(x_{ij})_{i,j}\in\mathbb{R}^{(m+n)^2}$:

$$x'_{ij}=x_{ij}\text{ if } j\leq\ell \text{ and } x'_{ij}=0\text{ otherwise.}$$

Here we summarize some general properties of Schubert varieties translated in our embedded construction. For more details see [MS74, §6] and [Man01, Section 3.2].

Lemma 2.3.1. *Let $\mathbb{G}_{m,n}\subset\mathbb{R}^{(m+n)^2}$ be an embedded Grassmannian manifold and let λ be a partition of the $(m\times n)$-rectangle. Let σ_λ be the Schubert variety in $\mathbb{G}_{m,n}$ defined by the incidence conditions prescribed by λ with respect to the canonical complete flag of $\mathbb{R}^{m+n}$. Then:*

(i) *σ_λ is an algebraic subset of $\mathbb{R}^{(m+n)^2}$ and $\Omega_\lambda\subset\mathrm{Reg}(\sigma_\lambda)$.*
(ii) *Ω_λ is biregular isomorphic to $\mathbb{R}^{mn-|\lambda|}$, where $|\lambda|:=\sum_{k=1}^m\lambda_k$.*
(iii) *σ_λ coincides with the Euclidean closure of Ω_λ.*
(iv) *$\sigma_\lambda=\bigcup_{\mu\geq\lambda}\Omega_\mu$, where $\mu\geq\lambda$ if and only if $\mu_k\geq\lambda_k$ for every $k\in\{1,\dots,m\}$.*

(v) $\sigma_\lambda \supset \sigma_\mu$ *if and only if* $\lambda \leq \mu$, *where* $\lambda \leq \mu$ *means* $\lambda_i \leq \mu_i$ *for every* $i \in \{1, \ldots, m\}$.

The choice of the canonical complete flag allows as to obtain $\mathbb{Q}$-algebraic equations of Schubert varieties, as explained by next result.

LEMMA 2.3.2. *Let* $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ *be a Grassmannian manifold and let* λ *be a partition of the rectangle* $m \times n$. *Then the Schubert variety* σ_λ *defined by the incidence conditions prescribed by* λ *with respect to the canonical flag of* $\mathbb{R}^{m+n}$ *is a projectively* $\mathbb{Q}$*-closed algebraic subset of* $\mathbb{R}^{(m+n)^2}$.

PROOF. We want to prove that σ_λ is $\mathbb{Q}$-algebraic, namely we prove that conditions in (2.3.1) are $\mathbb{Q}$-algebraic. Recall that $X \in \mathbb{G}_{m,n}$ is the matrix of the orthogonal projection of $\mathbb{R}^{m+n}$ onto an m-dimensional subspace W of $\mathbb{R}^{m+n}$, hence $\ker(X - \mathrm{id}_{\mathbb{R}^{m+n}}) = W$. This means that upper conditions on $\mathrm{rnk}(XD_\ell)$ correspond to lower conditions on $\mathrm{rnk}((X - \mathrm{id}_{\mathbb{R}^{m+n}})D_\ell)$, in particular for every $k \in \{1, \ldots, m\}$ the following hold:

$$\mathrm{rnk}(XD_{n+k-\lambda_k}) \geq k \quad \text{if and only if} \quad \mathrm{rnk}((X - \mathrm{id}_{\mathbb{R}^{m+n}})D_{n+k-\lambda_k}) \leq n - \lambda_k.$$

The latter condition is algebraic since it corresponds to the vanishing of the determinant of all $(n-\lambda_k+1)\times(n-\lambda_k+1)$-minors of the matrix $(X - \mathrm{id}_{\mathbb{R}^{m+n}})D_{n+k-\lambda_k}$. In particular, since $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ is $\mathbb{Q}$-algebraic, $\mathrm{id}_{\mathbb{R}^{m+n}}$ and $D_{n+k-\lambda_k}$ are matrices with rational coefficients and the determinant is a polynomial with rational coefficients with respect to the entries of the matrix X, the algebraic set σ_λ is $\mathbb{Q}$-algebraic. In addition, since $\mathbb{G}_{m,n}$ is projectively $\mathbb{Q}$-closed, σ_λ is projectively $\mathbb{Q}$-closed as well by Lemma 2.1.7(ii). □

2.3.2. $\mathbb{Q}$-Desingularization procedure. Let us introduce the notion of $\mathbb{Q}$-desingularization of a $\mathbb{Q}$-algebraic set $V \subset \mathbb{R}^n$.

DEFINITION 2.3.3. Let $V \subset \mathbb{R}^m$ be a $\mathbb{Q}$-algebraic set of dimension d. We say that $V' \subset \mathbb{R}^m \times \mathbb{R}^n$, for some $n \in \mathbb{N}$, is a *desingularization* of V if V' is a nonsingular algebraic subset of $\mathbb{R}^{m+n}$ of dimension d and $\pi|_{V'} : V' \to V$ is a birational map, where $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the projection onto the first factor. If, in addition, V' is a $\mathbb{Q}$-nonsingular projectively $\mathbb{Q}$-closed algebraic subset of $\mathbb{R}^{m+n}$ we say that V' is a $\mathbb{Q}$-*desingularization* of V. ■

The goal of this subsection is to find $\mathbb{Q}$-desingularizations of embedded Schubert's varieties defined by incidence conditions with respect to the canonical complete flag of $\mathbb{R}^{(m+n)^2}$, that is, to prove next result.

THEOREM 2.3.4. *Let* $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ *be a Grassmannian manifold and let* σ_λ *be any Schubert variety of* $\mathbb{G}_{m,n}$ *defined by incidence conditions, prescribed by* λ, *with respect to the canonical complete flag of* $\mathbb{R}^{m+n}$, *that is*

$$0 \subset \mathbb{R} \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{m+n}.$$

Then, σ_λ *admits a* $\mathbb{Q}$*-desingularization.*

Previous desingularization theorem will play a crucial role in Section 2.4 and then in Section 3.2.1 for the proof of the relative Nash-Tognoli theorem 'over $\mathbb{Q}$'. Let us provide a complete proof of Theorem 2.3.4.

Let $m, n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition together with its associated Young diagram in a $(m \times n)$-rectangle. Then, there are $c \in \mathbb{N}^*$, $a_1, \dots, a_{c-1} \in \mathbb{N}^*$, $a_c, b_0 \in \mathbb{N}$ and $b_1, \dots, b_{c-1} \in \mathbb{N}^*$, uniquely determined by λ, such that:

(a) $a_1 + \dots + a_c = m \quad$ and $\quad b_0 + \dots + b_{c-1} = n$,
(b) $\lambda_j = \sum_{k=i}^{c} b_k$ for every $j \leq a_i$ and for every $i = 1, \dots, c$.

The interpretation of the previous integers with respect to the Young diagram associated to the partition λ is explained in Figure 2.3.1.

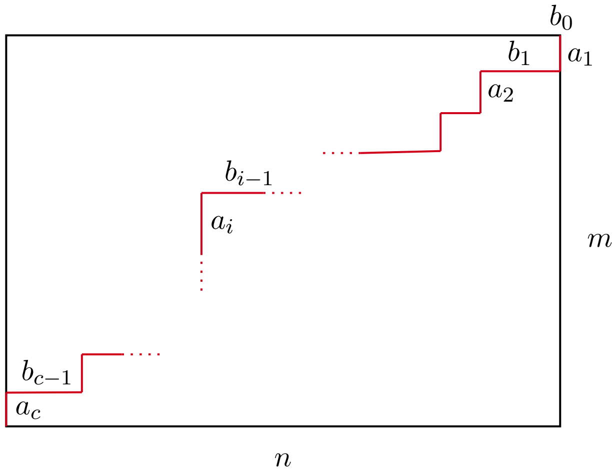

FIGURE 2.3.1. Disposition of the a_i's and b_i's with respect to the partition λ.

REMARK 2.3.5. Let $m', n' \in \mathbb{N}$ such that $m' \leq m$ and $n' \leq n$. Consider the Schubert variety of $\mathbb{G}_{m,n}$ associated to the partition $\lambda = (\lambda_1, \dots, \lambda_m)$ where:

$$\lambda_k = \begin{cases} n & \text{if } k \leq m - m'; \\ n - n' & \text{if } k > m - m'. \end{cases}$$

If $m = m'$ and $n = n'$ the Schubert variety σ_λ corresponds to the whole $\mathbb{G}_{m,n}$, otherwise σ_λ is given by the equations:

$$\sigma_\lambda = \{X \in \mathbb{G}_{m,n} \,|\, \operatorname{rnk}(XD_{m-m'}) = m - m', \operatorname{rnk}(XD_{m+n'}) = m\}.$$

Clearly σ_λ is biregular isomorphic to $\mathbb{G}_{m',n'}$. In our embedded version the $\mathbb{Q}$-biregular isomorphism $\varphi : \mathbb{G}_{m',n'} \to \sigma_\lambda \subset \mathbb{G}_{m,n}$ can be defined as follows: let $X' := (x'_{ij})_{i,j=1,\dots,m'+n'}$, then $\varphi(X') = (x_{i,j})_{i,j=1,\dots,m+n}$ is defined as

$$x_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i \leq m - m'; \\ x'_{st} & \text{if } m - m' < i, j < m + n' \text{ and } s = i - m + m',\ t = j - m + m'; \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathbb{G}_{m',n'} \subset \mathbb{R}^{(m'+n')^2}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. Let $\operatorname{graph}(\varphi) \subset \mathbb{G}_{m',n'} \times \mathbb{G}_{m,n}$ be the graph of φ. Then, we have that $\operatorname{graph}(\varphi) \subset \mathbb{R}^{(m'+n')^2+(m+n)^2}$ is a $\mathbb{Q}$-algebraic set contained in $\mathbb{G}_{m',n'} \times \mathbb{G}_{m,n}$, hence projectively $\mathbb{Q}$-closed by Lemma 2.1.7(ii) and $\mathbb{Q}$-nonsingular since φ is a $\mathbb{Q}$-biregular isomorphism. Thus, $\operatorname{graph}(\varphi) \subset \mathbb{R}^{(m'+n')^2+(m+n)^2}$ is a $\mathbb{Q}$-desingularization of σ_λ. ∎

By the above Remark 2.3.5 we are left to find $\mathbb{Q}$-desingularizations of Schubert varieties σ_λ of $\mathbb{G}_{m,n}$ defined by incidence conditions with respect to the canonical complete flag such that a_c and b_0 are non-null. Indeed, if λ is a partition with $a_c, b_0 = 0$, then σ_λ is $\mathbb{Q}$-biregularly isomorphic to a Schubert variety σ_μ of $\mathbb{G}_{m-a_1,n-b_{c-1}}$ with $\mu_i := \lambda_{i+a_1} - b_{c-1}$ for every $i \in \{1,\dots,m-a_1\}$. Hence, $\mu_1 := \lambda_{1+a_1} - b_{c-1} = n - b_1 - b_{c-1} < n - b_{c-1}$ and $\mu_{m-a_1} := b_{c-1} - b_{c-1} = 0$, as desired.

We define the *depressions* of the partition λ, with $a_c, b_0 > 0$, as the elements of the Young diagram whose coordinates, with respect to the upper corner on the left, are:

$$(a_1 + \cdots + a_i + 1, b_i + \cdots + b_{c-1} + 1), \quad i = 1, \dots, c-1.$$

Here we provide an inductive desingularization of the Schubert variety σ_λ with respect to the number $c-1 \in \mathbb{N}$ of depressions of the partition λ.

In next result we adapt to our real embedded setting a desingularization technique developed by Zelevinsky in [Zel83].

Lemma 2.3.6. *Let λ be a partition of the $(m \times n)$-rectangle such that a_c and b_0 are non-null. Let σ_λ be the Schubert variety of $\mathbb{G}_{m,n}$ defined by incidence conditions, prescribed by λ, with respect to the canonical complete flag of $\mathbb{R}^{(m+n)^2}$. Let $m_k := \sum_{i=1}^k a_i$, $n_k := m + n - m_k$ and $d_k := m_k + \sum_{i=1}^k b_{i-1}$ for every $k = 1, \dots, c$.*

Then the algebraic set:

$$\begin{aligned} Z_\lambda := \{(X, Y_{c-1}, \dots, Y_1) \in \mathbb{G}_{m,n} \times \mathbb{G}_{m_{c-1},n_{c-1}} \times \cdots \times \mathbb{G}_{m_1,n_1} \mid \\ Y_i D_{d_i} = Y_i, \quad \textit{for every } i = 1, \dots, c-1, \\ Y_{i+1} Y_i = Y_i, \quad \textit{for every } i = 1, \dots, c-2, \\ X Y_{c-1} = Y_{c-1}\}. \end{aligned}$$

is a desingularization of σ_λ.

Proof. Let us prove by induction on $c \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Let $c = 1$, that is $a_1, b_0 > 0$ and λ has no depressions, so λ is the null partition. Thus, $\sigma_\lambda = \mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$, which is a nonsingular algebraic set, thus there is nothing to prove.

Let $c > 1$ and λ be a partition with $c-1$ depressions such that $a_c, b_0 > 0$. Recall that the Schubert variety σ_λ defined by the incidence conditions, prescribed by λ, with respect to the canonical complete flag of $\mathbb{R}^{m+n}$ is defined as:

$$\sigma_\lambda = \{X \in \mathbb{G}_{m,n} \mid \operatorname{rnk}(X D_{d_k}) \geq m_k \quad \text{for } k = 1, \dots, c.\}$$

Consider the algebraic set $Z_\lambda \subset \mathbb{G}_{m,n} \times \mathbb{G}_{m_{c-1},n_{c-1}} \times \cdots \times \mathbb{G}_{m_1,n_1}$ as in the statement of Lemma 2.3.6. Define $\pi_i : Z_\lambda \to \mathbb{G}_{m_{c-i},n_{c-i}}$ for $i \in \{1, \dots, c\}$ be the restriction over Z_λ of the projection from $\mathbb{G}_{m,n} \times \mathbb{G}_{m_{c-1},n_{c-1}} \times \cdots \times \mathbb{G}_{m_1,n_1}$ onto the $(c-i+1)$-component.

Observe $\pi_1(Z_\lambda) = \{Y_1 \in \mathbb{G}_{m_1,n_1} \mid Y_1 D_{d_1} = Y_1\}$ is biregular isomorphic to $\mathbb{G}_{a_1,b_0} = \mathbb{G}_{m_1,d_1-m_1}$. Let μ be a partition of the $\big((m - m_1) \times n\big)$-rectangle defined as: $\mu = (\mu_1, \dots, \mu_{m-a_1})$ with

$$\mu_k = \lambda_{k+a_1} \text{ for every } k = 1, \dots, m - a_1.$$

Then, for every $B_1 \in \pi_1(Z_\lambda)$, we observe that $\pi_1^{-1}(B_1)$ is biregular isomorphic to the set Z_μ. Indeed, define the biregular isomorphism $\phi : Z_\mu \to (\pi_1)^{-1}(D_{m_1})$ as follows:

let $(A, B_{c-1}, \dots, B_2) \in Z_\mu$ then define

$$\phi(A, B_{c-1}, \dots, B_2) := (\varphi(A), \varphi(B_{c-1}), \dots, \varphi(B_2), D_{m_1}),$$

where $\varphi : \mathbb{R}^{(m-m_1+n)^2} \to \mathbb{R}^{(m+n)^2}$ is defined as $\varphi((x_{ij})_{ij}) = (x'_{ij})$, with

$$x'_{ij} := \begin{cases} 1 & \text{if } i = j \text{ and } i \leq m_1, \\ x_{st} & \text{if } m - m_1 < i, j, \text{ with } s = i - m + m_1 \text{ and } t = j - m + m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for every $B_1 \in \pi_1(Z_\lambda)$, then $(\pi_1)^{-1}(B_1)$ is biregularly isomorphic to $(\pi_1)^{-1}(D_{m_1})$, indeed it suffices to chose $G \in O(m+n)$ such that $D_{m_1} = G^T B_1 G$ and apply G to every factor of $(\pi_1)^{-1}(D_{m_1})$ to produce the wondered isomorphism. Observe that the partition μ has exactly $(c-2)$-depressions, indeed it is constructed by erasing the first depression $(a_1+1, n-b_0+1)$ of λ, thus by inductive assumption the algebraic set Z_μ is a desingularization of σ_μ. In particular:

$$\dim(Z_\mu) = \dim(\sigma_\mu) = \dim(\sigma_\lambda) - a_1 b_0.$$

Hence, $\pi_1 : Z_\lambda \to \mathbb{G}_{a_1,b_0}$ is an algebraic fibre bundle of dimension $\dim(\sigma_\lambda)$, thus Z_λ is a nonsingular algebraic subset of $\mathbb{R}^{(m+n)^2 c}$ of dimension $\dim(\sigma_\lambda)$. Moreover, Z_λ is a desingularization of σ_λ indeed, if $A \in \Omega_\lambda$, then $(A, B_{c-1}, \dots, B_1) \in Z_\lambda$ if and only if $B_i = AD_{d_i}$ for every $i \in \{1, \dots, c-1\}$. Hence, the map $\pi_c : Z_\lambda \to \sigma_\lambda$ is birational by Lemma 2.3.1(i). $\square$

By Remark 2.3.5, in order to prove Theorem 2.3.4 we are only left to prove next result.

LEMMA 2.3.7. *Each algebraic fibre bundle $Z_\lambda \subset \mathbb{R}^{(m+n)^2 c}$ as in Lemma 2.3.6 is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.*

PROOF. By definition, Z_λ is a $\mathbb{Q}$-algebraic subset of $\mathbb{R}^{(m+n)^2 c}$ defined by the following equations in the variables $X := (x_{i,j})_{i,j=1,\dots,m+n}$ and $Y_k := (y^{(k)}_{i,j})_{i,j=1,\dots,m+n}$, for $k = 1, \dots, c-1$:

$$\begin{aligned} X = X^T, \quad X^2 = X, \quad \operatorname{tr}(X) = m; \\ Y_k = Y_k^T, \quad Y_k^2 = Y_k, \quad \operatorname{tr}(Y_k) = m_k \quad \text{for every } k = 1, \dots, c-1; \\ Y_k D_{d_k} = Y_k \quad \text{for every } k = 1, \dots, c-1; \\ Y_{k+1} Y_k = Y_k \quad \text{for every } k = 1, \dots, c-2; \\ X Y_{c-1} = Y_{c-1}. \end{aligned}$$

Let $\varphi_k : \mathbb{R}^{(m+n)^2 c} \to \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2}$, for every $k = 1, \dots, c-2$, $\varphi_{c-1} : \mathbb{R}^{(m+n)^2 c} \to \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2}$ and $\varphi_c : \mathbb{R}^{(m+n)^2 c} \to \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2}$ be defined as:

$$\begin{aligned} \varphi_k(X, Y_{c-1}, \dots, Y_1) &:= (Y_k - Y_k^T, Y_k^2 - Y_k, Y_k D_{d_k} - Y_k, Y_{k+1} Y_k - Y_k) \\ \varphi_{c-1}(X, Y_{c-1}, \dots, Y_1) &:= (Y_{c-1} - Y_{c-1}^T, Y_{c-1}^2 - Y_{c-1}, Y_{c-1} D_{d_{c-1}} - Y_{c-1}, \\ &\qquad X Y_{c-1} - Y_{c-1}) \\ \varphi_c(X, Y_{c-1}, \dots, Y_1) &:= (X - X^T, X^2 - X). \end{aligned}$$

Define $\phi : \mathbb{R}^{(m+n)^2 c} \to (\mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2})^{c-1} \times \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2}$ be the polynomial map:

$$\begin{aligned}\phi(X, Y_{c-1}, \dots, Y_1) := (\varphi_1(X, Y_{c-1}, \dots, Y_1), \dots, \varphi_{c-1}(X, Y_{c-1}, \dots, Y_1),\\ \varphi_c(X, Y_{c-1}, \dots, Y_1)).\end{aligned}$$

We prove that the polynomials $\operatorname{tr}(X) - m$, $\operatorname{tr}(Y^k) - m_k$, for every $k = 1, \dots, c-1$, and the polynomial components of ϕ do suffice to describe the local structure of nonsingular points of Z_λ in $\mathbb{R}^{(m+n)^2 c}$. Since these polynomials have coefficients in $\mathbb{Q}$ and their common zero set is Z_λ, bearing in mind that

$$\dim(Z_\lambda) = \sum_{k=1}^{c} \dim(\mathbb{G}_{a_k, n - \sum_{i=k}^{c-1} b_k}) = \sum_{k=1}^{c} a_k \left(n - \sum_{i=k}^{c-1} b_k \right) = \dim(\sigma_\lambda),$$

it suffices to show that, for each $(A, B_{c-1}, \dots, B_1) \in Z_\lambda$, the rank of the jacobian matrix $J_\phi(A, B_{c-1}, \dots, B_1)$ of ϕ at $(A, B_{c-1}, \dots, B_1)$ is greater than or equal to (and hence equal to)

$$\begin{aligned} c(m+n)^2 - \dim(\sigma_\lambda) &= \sum_{k=1}^{c} (m+n)^2 - \dim(\mathbb{G}_{a_k, n - \sum_{i=k}^{c-1} b_k}) \\ &= \sum_{k=1}^{c} (m+n)^2 - a_k(d_k - m_k), \end{aligned}$$

i.e. $\operatorname{rnk} J_\phi(A, B_{c-1}, \dots, B_1) \geq c(m+n)^2 - \sum_{k=1}^{c} a_k(d_k - m_k)$ for all $(A, B_{c-1}, \dots, B_1) \in Z_\lambda$.

First, we prove that $\operatorname{rnk} J_\phi(D_m, D_{m_{c-1}}, \dots, D_{m_1}) \geq c(m+n)^2 - \dim(\sigma_\lambda)$ if $D_m = D_{m_c}$ and D_{m_k} are the diagonal matrices in $\mathbb{R}^{(m+n)^2}$ having 1 in the first m_k diagonal positions and 0 otherwise, for every $k = 1, \dots, c$. Observe that $(D_m, D_{m_{c-1}}, \dots, D_{m_1}) \in Z_\lambda$ since $D_{m_{k+1}} D_{m_k} = D_{m_k}$, for every $k = 1, \dots, c-1$.

For each $i, j \in \{1, \dots, m+n\}$ and $k \in \{1, \dots, c\}$, define the polynomial functions $f_{ij}^{(k)}, g_{ij}^{(k)}, p_{ij}^{(k)}, q_{ij}^{(k)} : \mathbb{R}^{(m+n)^2} \times \mathbb{R}^{(m+n)^2} \to \mathbb{R}$ by

$$\begin{aligned}
f_{ij}^{(c)}(X, Y_{c-1}, \dots, Y_1) &:= x_{ij} - x_{ji}, \\
g_{ij}^{(c)}(X, Y_{c-1}, \dots, Y_1) &:= \left(\textstyle\sum_{\ell=1}^{n} x_{i\ell} x_{\ell j} \right) - x_{ij}, \\
f_{ij}^{(k)}(X, Y_{c-1}, \dots, Y_1) &:= y_{ij}^{(k)} - y_{ji}^{(k)}, \\
g_{ij}^{(k)}(X, Y_{c-1}, \dots, Y_1) &:= \left(\textstyle\sum_{\ell=1}^{n} y_{i\ell}^{(k)} y_{\ell j}^{(k)} \right) - y_{ij}^{k}, \\
p_{ij}^{(k)}(X, Y_{c-1}, \dots, Y_1) &:= \begin{cases} 0 & \text{if } i, j \leq d_k = \sum_{\ell=1}^{k} (a_\ell + b_{\ell-1}); \\ -y_{ij}^{(k)} & \text{otherwise,} \end{cases} \\
q_{ij}^{(c)}(X, Y_{c-1}, \dots, Y_1) &:= y_{ij}^{(c-1)} - \sum_{\ell=1}^{m+n} x_{i\ell} y_{\ell j}^{(c-1)}, \\
q_{ij}^{(k)}(X, Y_{c-1}, \dots, Y_1) &:= y_{ij}^{(k)} - \sum_{\ell=1}^{m+n} y_{i\ell}^{(k+1)} y_{\ell j}^{(k)} \quad \text{with } k \neq 1, c.
\end{aligned}$$

for all $(X, Y_{c-1}, \ldots, Y_1) = ((x_{ij})_{i,j}, (y_{ij}^{(c-1)})_{i,j}, \ldots, (y_{ij}^{(1)})_{i,j}) \in \mathbb{R}^{(m+n)^2 c}$. It follows that

$$\begin{aligned}\phi(X, Y_{c-1}, \ldots, Y_1) = \Big(&(f_{ij}^{(1)}(X, Y_{c-1}, \ldots, Y_1))_{i,j}, (g_{ij}^{(1)}(X, Y_{c-1}, \ldots, Y_1))_{i,j},\\ &(p_{ij}^{(1)}(X, Y_{c-1}, \ldots, Y_1))_{i,j}, (q_{ij}^{(2)}(X, Y_{c-1}, \ldots, Y_1))_{i,j},\\ &\ldots,\\ &(f_{ij}^{(c-1)}(X, Y_{c-1}, \ldots, Y_1))_{i,j}, (g_{ij}^{(c-1)}(X, Y_{c-1}, \ldots, Y_1))_{i,j},\\ &(p_{ij}^{(c-1)}(X, Y_{c-1}, \ldots, Y_1))_{i,j}, (q_{ij}^{(c)}(X, Y_{c-1}, \ldots, Y_1))_{i,j},\\ &(f_{ij}^{(c)}(X, Y_{c-1}, \ldots, Y_1))_{i,j}, (g_{ij}^{(c)}(X, Y_{c-1}, \ldots, Y_1))_{i,j}\Big).\end{aligned}$$

Define, for every $k \in \{1, \ldots, c\}$:

$$\begin{aligned}S_1^{(k)} &:= \{(i,j) \in \{1, \ldots, m+n\}^2 \,|\, i < j \leq d_k\},\\ S_2^{(k)} &:= \{(i,j) \in \{1, \ldots, m+n\}^2 \,|\, i \leq j \leq m_k\},\\ S_3^{(k)} &:= \{(i,j) \in \{1, \ldots, m+n\}^2 \,|\, m_k < i \leq j \leq d_k\},\\ S_4^{(k)} &:= \{(i,j) \in \{1, \ldots, m+n\}^2 \,|\, d_k < i \text{ or } d_k < j\},\\ T^{(1)} &:= \varnothing,\\ T^{(k)} &:= \{(i,j) \in \{1, \ldots, m+n\}^2 \,|\, m_k < i \leq d_k,\, j \leq m_{k-1}\}.\end{aligned}$$

Notice that the sum of the cardinalities of $S_1^{(k)}$, $S_2^{(k)}$, $S_3^{(k)}$ and $S_4^{(k)}$ equals

$$\begin{aligned}\frac{(d_k-1)d_k}{2} + \frac{m_k(m_k+1)}{2} + &\frac{(d_k-m_k)(d_k-m_k+1)}{2} + (m+n)^2 - d_k^2\\ &= (m+n)^2 - m_k(d_k - m_k),\end{aligned}$$

for every $k \in \{1, \ldots, c\}$. In particular, the sum of the cardinalities of $S_1^{(1)}$, $S_2^{(1)}$, $S_3^{(1)}$ and $S_4^{(1)}$ is equal to $a_1 b_0$. In addition, the cardinality of $T^{(k)}$ is equal to $m_{k-1}(d_k - m_k)$, for every $k \in \{2, \ldots, c\}$. Hence the sum of the cardinalities of $S_1^{(k)}$, $S_2^{(k)}$, $S_3^{(k)}$, $S_4^{(k)}$ and $T^{(k)}$ equals $(m+n)^2 - a_k(d_k - m_k)$, for every $k \in \{2, \ldots, c\}$.

By a direct computation, we see that

$$\begin{aligned}&\nabla f_{ij}^{(1)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, E_{ij}^{(1)} - E_{ji}^{(1)}) && \text{if } (i,j) \in S_1^{(1)},\\ &\nabla g_{ij}^{(1)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, E_{ij}^{(1)}) && \text{if } (i,j) \in S_2^{(1)},\\ &\nabla g_{ij}^{(1)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, -E_{ij}^{(1)}) && \text{if } (i,j) \in S_3^{(1)},\\ &\nabla p_{ij}^{(1)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, E_{ij}^{(1)}) && \text{if } (i,j) \in S_4^{(1)},\end{aligned}$$

and, for every $k \in \{2, \ldots, c\}$

$$\begin{aligned}&\nabla f_{ij}^{(k)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, E_{ij}^{(k)} - E_{ji}^{(k)}, 0, \ldots, 0) && \text{if } (i,j) \in S_1^{(k)},\\ &\nabla g_{ij}^{(k)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, E_{ij}^{(k)}, 0, \ldots, 0) && \text{if } (i,j) \in S_2^{(k)},\\ &\nabla g_{ij}^{(k)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, -E_{ij}^{(k)}, 0, \ldots, 0) && \text{if } (i,j) \in S_3^{(k)},\\ &\nabla p_{ij}^{(k)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, E_{ij}^{(k)}, 0, \ldots, 0) && \text{if } (i,j) \in S_4^{(k)},\\ &\nabla q_{ij}^{(k)}(D_m, D_{m_{c-1}}, \ldots, D_{m_1}) = (0, \ldots, 0, -E_{ij}^{(k)}, 0, \ldots, 0) && \text{if } (i,j) \in T^{(k)},\end{aligned}$$

where $E_{ij}^{(k)}$ is the matrix in $\mathbb{R}^{(m+n)^2}$ whose (i,j)-coefficient equals 1 and the other coefficients are 0 holding the $(c-k+1)$-position in the vector $(X, Y_{c-1}, \dots, Y_1) \in \mathbb{R}^{(m+n)^2 c}$, for every $k \in \{1, \dots, c\}$. Consequently, we have that

$$\begin{aligned}\operatorname{rnk} J_\phi(D_m, D_{m_{c-1}}, \dots, D_{m_1}) &\geq \sum_{k=1}^{c}((m+n)^2 - a_k(d_k - m_k)) \\ &= c(m+n)^2 - \dim(\sigma_\lambda).\end{aligned}$$

Let $(A, B_{c-1}, \dots, B_1) \in Z_\lambda$ and let $G \in O(m+n)$ be such that $D_m = G^T A G$ and $D_{m_k} = G^T B_k G$, for every $k \in \{1, \dots, c-1\}$. Define the linear automorphisms $\psi : \mathbb{R}^{(m+n)^2} \to \mathbb{R}^{(m+n)^2}$ by $\psi(X) := G^T X G$ and $\psi^{\times k} : \mathbb{R}^{(m+n)^2 k} \to \mathbb{R}^{(m+n)^2 k}$ by $\psi^{\times k}(X_1, \dots, X_k) := (\psi(X_1), \dots, \psi(X_k))$, for $k \in \mathbb{N}^*$. Since $\psi(A) = D_m$ and $(\psi^{\times(4c-2)}) \circ \phi = \phi \circ (\psi^{\times c})$, we have that

$$\begin{aligned}&J_{\psi^{\times(4c-2)}}(\phi(A, B_{c-1}, \dots, B_1)) J_\phi(A, B_{c-1}, \dots, B_1) = \\ &\qquad J_\phi(D_m, D_{m_{c-1}}, \dots, D_{m_1}) J_{\psi^{\times c}}(A, B_{c-1}, \dots, B_1).\end{aligned}$$

Bearing in mind that both matrices $J_{\psi^{\times(4c-2)}}(\phi(A, B_{c-1}, \dots, B_1))$ and $J_{\psi^{\times c}}(A, B_{c-1}, \dots, B_1)$ are invertible, it follows that

$$\begin{aligned}\operatorname{rnk} J_\phi(A, B_{c-1}, \dots, B_1) &= \operatorname{rnk} J_\phi(D_m, D_{m_{c-1}}, \dots, D_{m_1}) \\ &\geq c(m+n)^2 - \dim(\sigma_\lambda),\end{aligned}$$

as desired. Since $Z_\lambda \subset \mathbb{R}^{(m+n)^2 c}$ is $\mathbb{Q}$-algebraic and is contained in the projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set $\mathbb{G}_{m,n} \times \mathbb{G}_{m_{c-1},n_{c-1}} \times \dots \times \mathbb{G}_{m_1,n_1} \subset \mathbb{R}^{(m+n)^2 c}$, Lemma 2.1.7(ii) ensures that $Z_\lambda \subset \mathbb{R}^{(m+n)^2 c}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set as well. This proves that $Z_\lambda \subset \mathbb{R}^{(m+n)^2 c}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set, as desired. □

A combination of Remark 2.3.5 and Lemmas 2.3.6 & 2.3.7 provides a complete proof of Theorem 2.3.4.

2.4. Unoriented (co)bordism and homology over $\mathbb{Q}$

In this section we introduce the notions of $\mathbb{Q}$-algebraic unoriented bordism and $\mathbb{Q}$-algebraic homology, thus we study their deep interplay.

Let $W \subset \mathbb{R}^k$ be a set. Given a compact $\mathscr{C}^\infty$ manifold P and a $\mathscr{C}^\infty$ map $f : P \to W$, we say that the unoriented bordism class of f is *projectively* $\mathbb{Q}$*-algebraic* if there exist a compact $\mathscr{C}^\infty$ manifold T with boundary, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Y \subset \mathbb{R}^h$, a $\mathscr{C}^\infty$ diffeomorphism $\psi : P \sqcup Y \to \partial T$ and a $\mathscr{C}^\infty$ map $F : T \to W$ such that $F \circ \jmath \circ (\psi|_P) = f$ and $F \circ \jmath \circ (\psi|_Y)$ is a $\mathbb{Q}$-regular map, where $\jmath : \partial T \hookrightarrow T$ is the inclusion map.

Definition 2.4.1. Given $d \in \mathbb{N}$, we say that W has *projectively* $\mathbb{Q}$*-algebraic unoriented bordism* if, for all $p \in \mathbb{N}$, for all p-dimensional compact $\mathscr{C}^\infty$ manifold P and for all $\mathscr{C}^\infty$ map $f : P \to W$, the unoriented bordism class of f is projectively $\mathbb{Q}$-algebraic. ∎

Let Z be a subset of $\mathbb{R}^h$. For every $k \leq h$, denote by $H_k(Z, \mathbb{Z}/2\mathbb{Z})$ the k-th homology group of V with coefficients in $\mathbb{Z}/2\mathbb{Z}$. It is well known that, if $Z \subset \mathbb{R}^h$ is a compact algebraic set of dimension d, the fundamental class $[Z]$ of Z in $H_d(Z, \mathbb{Z}/2\mathbb{Z})$ is a non trivial homology class. More details about fundamental classes of algebraic sets can be found in [BCR98, §11, Section 3] via triangulations. In an alternative way, the existence of fundamental classes of compact real algebraic sets is a consequence of Hironaka's desingularization theorem and the existence of fundamental classes for compact $\mathscr{C}^\infty$ manifolds.

Let $W \subset \mathbb{R}^k$ be a set. Given $p \in \mathbb{N}$ and $\alpha \in H_p(W, \mathbb{Z}/2\mathbb{Z})$, we say that α is *projectively $\mathbb{Q}$-algebraic* if there exist a p-dimensional projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Z \subset \mathbb{R}^h$ and a $\mathbb{Q}$-regular map $g : Z \to W$ such that $g_*([Z]) = \alpha$, where $[Z]$ is the fundamental class of Z in $H_p(Z, \mathbb{Z}/2\mathbb{Z})$.

DEFINITION 2.4.2. Given $d \in \mathbb{N}$, we say that W has *projectively $\mathbb{Q}$-algebraic homology* if, for all $p \in \{0, \dots, d\}$ and for all $\alpha \in H_p(W, \mathbb{Z}/2\mathbb{Z})$, the homology class α is projectively $\mathbb{Q}$-algebraic. ∎

In [Mil65] Milnor proved that the unoriented cobordism group $\mathfrak{N}_* = \bigoplus_{d\in\mathbb{N}} \mathfrak{N}_d$ is generated by disjoint unions of compact $\mathscr{C}^\infty$ manifolds of the form $Y = \mathbb{P}^{n_1}(\mathbb{R}) \times \cdots \times \mathbb{P}^{n_\alpha}(\mathbb{R}) \times H_{a_1,b_1} \times \cdots \times H_{a_\beta,b_\beta}$, where $Y = H_{a_1,b_1} \times \cdots \times H_{a_\beta,b_\beta}$ if $\alpha = 0$ and $\beta > 0$, $Y = \mathbb{P}^{n_1}(\mathbb{R}) \times \cdots \times \mathbb{P}^{n_\alpha}(\mathbb{R})$ if $\alpha > 0$ and $\beta = 0$ and Y is a singleton if $\alpha = \beta = 0$.

We need the following version of this remarkable result of Milnor.

THEOREM 2.4.3. *For each $d \in \mathbb{N}$, the unoriented cobordism group $\mathfrak{N}_d$ of d-dimensional compact $\mathscr{C}^\infty$ manifolds is generated by projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^{(2d+1)^2}$, obtained as the finite disjoint union of projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets of the form $Y + v \subset \mathbb{R}^{(2d+1)^2}$, where v belongs to $\mathbb{Q}^{(2d+1)^2}$ and*

$$Y = \mathbb{G}_{n_1+1,1} \times \ldots \times \mathbb{G}_{n_\alpha+1,1} \times \mathbb{H}_{a_1,b_1} \times \ldots \times \mathbb{H}_{a_\beta,b_\beta} \tag{2.4.1}$$

for some $\alpha, \beta \in \mathbb{N}$ and $n_1, \dots, n_\alpha, a_1, b_1, \dots, a_\beta, b_\beta \in \mathbb{N}^*$.

Before giving the proof of this result, we observe that,it is immediate to prove the following inequality:

$$\textstyle\sum_{s=1}^k (c_s+1)^2 \leq \left(1 + \sum_{s=1}^k c_s\right)^2 \text{ for all } k \in \mathbb{N}^* \text{ and } (c_1, \dots, c_k) \in (\mathbb{N}^*)^k. \tag{2.4.2}$$

PROOF OF THEOREM 2.4.3. Let M be a compact $\mathscr{C}^\infty$ manifold of dimension d. Combining the above result of Milnor with Lemmas 2.1.3(c)(e), 2.1.7(i)(iii)(iv), 2.2.1 & 2.2.2, we have that M is unoriented cobordant to the disjoint union $\bigsqcup_{h=1}^\ell Y_h$, where $\{Y_h\}_{h=1}^\ell$ is a finite family of projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets of the form

$$Y_h = \mathbb{G}_{n_{h,1}+1,1} \times \cdots \times \mathbb{G}_{n_{h,\alpha_h}+1,1} \times \mathbb{H}_{a_{h,1},b_{h,1}} \times \cdots \times \mathbb{H}_{a_{h,\beta_h},b_{h,\beta_h}}$$

for some $\alpha_h, \beta_h \in \mathbb{N}$ and $n_{h,1}, \dots, n_{h,\alpha_h}, a_{h,1}, b_{h,1}, \dots, a_{h,\beta_h}, b_{h,\beta_h} \in \mathbb{N}^*$. Note that $\sum_{i=1}^{\alpha_h} n_{h,i} + \sum_{j=1}^{\beta_h} (a_{h,j} + b_{h,j} - 1) = d$ and Y_h is contained in $\mathbb{R}^{N_h}$, where

$$N_h = \textstyle\sum_{i=1}^{\alpha_h} (n_{h,i}+1)^2 + \sum_{j=1}^{\beta_h} ((a_{h,j}+1)^2 + (b_{h,j}+1)^2).$$

Thanks to (2.4.2), we have

$$N_h \leq \big(1 + \textstyle\sum_{i=1}^{\alpha_h} n_{h,i} + \sum_{j=1}^{\beta_h}(a_{h,j} + b_{h,j})\big)^2 = (1 + d + \beta_h)^2 \leq (2d + 1)^2.$$

As a consequence, if we set $N := (2d + 1)^2$, then each Y_h is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^N$. For each $h \in \{1, \dots, \ell\}$, choose a vector $v_h \in \mathbb{Q}^N$ such that the sets $\{Y_h + v_h\}_{h=1}^{\ell}$ are pairwise disjoint. It follows that $\bigsqcup_{h=1}^{\ell}(Y_h + v_h) \subset \mathbb{R}^N$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set, which is unoriented cobordant to M. $\square$

As a consequence of Lemma 2.1.8, we have the following corollary of Theorem 2.4.3.

COROLLARY 2.4.4. *For each $d \in \mathbb{N}$, the unoriented cobordism group $\mathfrak{N}_d$ of compact $\mathscr{C}^\infty$ manifolds of dimension d is generated by projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of $\mathbb{R}^{2d+1}$.*

Let us explain the interplay between the properties of having projectively $\mathbb{Q}$-algebraic unoriented bordism and having projectively $\mathbb{Q}$-algebraic homology for $W \subset \mathbb{R}^k$.

THEOREM 2.4.5. *Let $W \subset \mathbb{R}^k$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. The following assertions are equivalent.*

(i) *W has projectively $\mathbb{Q}$-algebraic unoriented bordism.*
(ii) *W has projectively $\mathbb{Q}$-algebraic homology.*

PROOF. We adapt the argument used in [AK92, Lemma 2.7.1] to the present situation.

Let $\mathfrak{N}_*(W)$ be the unoriented bordism group of W and let $ev : \mathfrak{N}_*(W) \to H_*(W, \mathbb{Z}/2\mathbb{Z})$ be the evaluation map defined by $ev([f : P \to W]) := f_*([P])$. Since ev is surjective by [Tho54], implication (i) $\Longrightarrow$ (ii) follows immediately from Definitions 2.4.1 & 2.4.2.

Let us prove implication (ii) $\Longrightarrow$ (i). Suppose that (ii) holds. Let $\{Y_i \subset \mathbb{R}^{D_i}\}_{i\in I}$ be the generators of $\mathfrak{N}_*$ described in (2.4.1), where D_i is a sufficiently large natural number. Let $\{Z_j \subset \mathbb{R}^{h_j}\}_{j\in J}$ be projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets and let $\{g_j : Z_j \to W\}_{j\in J}$ be $\mathbb{Q}$-regular maps such that J is finite and the homology classes $\{ev([g_j : Z_j \to W]) = (g_j)_*([Z_j])\}_{j\in J}$ generate $H_*(W, \mathbb{Z}/2\mathbb{Z})$. Let $D := \max_{i\in I} D_i$ and let $h := \max_{j\in J} h_j$. Note that each Y_i is contained in $\mathbb{R}^D$ and each Z_j in $\mathbb{R}^h$; in particular, each $Z_j \times Y_i$ is contained in $\mathbb{R}^{D+h} = \mathbb{R}^D \times \mathbb{R}^h$. Choose vectors $v_{ij} \in \mathbb{Q}^{D+h}$ in such a way that the translated sets $\{(Z_j \times Y_i) + v_{ij}\}_{i\in I, j\in J}$ are pairwise disjoint. For each $i \in I$ and $j \in J$, denote by $\pi_{ij} : (Z_j \times Y_i) + v_{ij} \to Z_j$ the (translated) projection onto the first factor, sending $(z, y) + v_{ij}$ to z. Note that π_{ij} is a $\mathbb{Q}$-regular map. By Lemmas 2.1.3(v)(vi) & 2.1.7(iv), we know that each translated product $(Z_j \times Y_i) + v_{ij} \subset \mathbb{R}^{D+h}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set and each composition $g_j \circ \pi_{ij} : (Z_j \times Y_i) + v_{ij} \to W$ is a $\mathbb{Q}$-regular map. By [Tho54] and [CF64], the maps $g_j \circ \pi_{ij}$ generate $\mathfrak{N}_*(W)$. This proves (i). $\square$

REMARK 2.4.6. Observe that, by Lemma 2.1.8, in previous proof we may fix $D_i := 2\dim(Y_i) + 1$ for every $i \in I$. ■

We have the following result as a direct consequence of Künneth formula.

LEMMA 2.4.7. *Let $\ell \in \mathbb{N}^*$, let $W_1 \subset \mathbb{R}^{k_1}, \dots, W_\ell \subset \mathbb{R}^{k_\ell}$ and let $W := W_1 \times \dots \times W_\ell \subset \mathbb{R}^{k_1}, \times \dots \times \mathbb{R}^{k^\ell}$. If $W_i \subset \mathbb{R}^{k_i}$ has projectively $\mathbb{Q}$-algebraic homology for every $i \in \{1, \dots, \ell\}$, then W has projectively $\mathbb{Q}$-algebraic homology.*

2.4.1. Real embedded Grassmannians have totally $\mathbb{Q}$-algebraic homology. Let us fix some notation about CW complexes. Let X be a topological space endowed by a finite CW complex structure $\mathcal{S}$ of dimension d. We denote by $\mathcal{S}^{(k)}$ the set of open k-cells of $\mathcal{S}$, for every $k \in \{0, \dots, d\}$. Denote by $X_k := \bigcup_{\Omega \in \mathcal{S}^{(k)}} \overline{\Omega}$ the k-skeleton of X for every

$$k \in \{0, \dots, d\}$$

, and $X_{-1} := \varnothing$. Define $C_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z}) := H_k(X_k, X_{k-1})$ the group of unoriented cellular k-chains of $\mathcal{S}$ for every $k \in \{1, \dots, d\}$. Let $\partial_k^{\mathcal{S}} : C_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z}) \to C_{k-1}(\mathcal{S}, \mathbb{Z}/2\mathbb{Z})$ denote the boundary operator in cellular homology for every $k \in \{1, \dots, d\}$. Define the *k-cellular homology group of X (with coefficients in $\mathbb{Z}/2\mathbb{Z}$)* as $H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z}) := \ker(\partial_k^{\mathcal{S}}) / \operatorname{im}(\partial_{k+1}^{\mathcal{S}})$. For more details about CW complexes and their homological theory we refer to [LW69].

LEMMA 2.4.8. *Let $W \subset \mathbb{R}^n$ be a compact algebraic subset of dimension d. Suppose that W admits a finite CW complex structure $\mathcal{S}$ such that the closure of each open cell $\Omega \in \mathcal{S}^{(k)}$ is algebraic for every $k \in \{0, \dots, d\}$. Then,*

$$H_k(W, \mathbb{Z}/2\mathbb{Z}) = \operatorname{Span}\big(\{[\overline{\Omega}] \in H_k(W, \mathbb{Z}/2\mathbb{Z}) \,|\, \Omega \in \mathcal{S}^{(k)}\}\big).$$

and $\{[\overline{\Omega}] \in H_k(W, \mathbb{Z}/2\mathbb{Z}) \,|\, \Omega \in \mathcal{S}^{(k)}\}$ is a basis of $H_k(W, \mathbb{Z}/2\mathbb{Z})$ for every $k \in \{0, \dots, d\}$.

PROOF. By classical arguments about cellular and simplicial homology, $\{[\overline{\Omega}] \in H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z}) \,|\, \Omega \in \mathcal{S}^{(k)}\}$ constitutes a system of generators of $H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z})$, for every $k = 0, \dots, d$. We are only left to prove that $\{[\overline{\Omega}] \in H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z}) \,|\, \Omega \in \mathcal{S}^{(k)}\}$ is linearly independent over $\mathbb{Z}/2\mathbb{Z}$. Since $\overline{\Omega}$ is algebraic for every open cell $\Omega \in \mathcal{S}^{(k)}$ for every $k \in \{0, \dots, d\}$, the fundamental class $[\overline{\Omega}]$ of $\overline{\Omega}$ is a well defined homology class in $H_k(W, \mathbb{Z}/2\mathbb{Z})$. Suppose $\Omega \in \mathcal{S}^{(k)}$, then for every $\Omega' \in \mathcal{S}^{(k+1)}$ we have $\partial_{k+1}^{\mathcal{S}}(\overline{\Omega}') = 0$, since $\overline{\Omega}'$ is algebraic as well. Hence, we get that $[\overline{\Omega}] \in H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z})$ is non-null and linearly independent with respect to $\{[\overline{\Omega}'] \in H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z}) \,|\, \Omega' \in \mathcal{S}^{(k)} \text{ and } \Omega' \neq \Omega\}$ for every choice of $\Omega \in \mathcal{S}^{(k)}$ and $k \in \{0, \dots, d\}$. This proves that $\{[\overline{\Omega}] \in H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z}) \,|\, \Omega \in \mathcal{S}^{(k)}\}$ is a basis of $H_k(\mathcal{S}, \mathbb{Z}/2\mathbb{Z})$, then $\{[\overline{\Omega}] \in H_k(W, \mathbb{Z}/2\mathbb{Z}) \,|\, \Omega \in \mathcal{S}^{(k)}\}$ it is also a basis of $H_k(W, \mathbb{Z}/2\mathbb{Z})$, as desired. □

Following the notation of Section 2.3 we refer to embedded Schubert varieties σ_λ of $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ defined by incidence conditions, prescribed by λ, with respect to the canonical complete flag of $\mathbb{R}^{m+n}$. Denote by $|\lambda| := \sum_{i=1}^m \lambda_i$.

COROLLARY 2.4.9. *Let $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$. Then:*

$$H_k(\mathbb{G}_{m,n}, \mathbb{Z}/2\mathbb{Z}) = \operatorname{Span}(\{[\sigma_\lambda] \in H_k(\mathbb{G}_{m,n}, \mathbb{Z}/2\mathbb{Z}) \,|\, |\lambda| = mn - k\})$$

for every $k \in \{0, \dots, mn\}$, where λ is a partition of the $(m \times n)$-rectangle, σ_λ is the Schubert variety of $\mathbb{G}_{m,n}$ defined by the incidence conditions, prescribed by λ, with respect to the canonical complete flag.

In particular, $\{[\sigma_\lambda] \in H_k(\mathbb{G}_{m,n}, \mathbb{Z}/2\mathbb{Z}) \,|\, |\lambda| = mn - k\}$ as above is a basis of $H_k(\mathbb{G}_{m,n}, \mathbb{Z}/2\mathbb{Z})$ for every $k \in \{1, \dots, mn\}$.

PROOF. By Lemma 2.3.1 the family of Ω_λ such that λ is a partition of the $(m \times n)$-rectangle constitutes the cells of a finite CW-complex whose underlying topological space is $\mathbb{G}_{m,n}$ such that $\sigma_\lambda = \overline{\Omega}_\lambda$ is algebraic for every partition λ of the $(m \times n)$-rectangle. Hence, the thesis follows by Lemma 2.4.8. $\square$

THEOREM 2.4.10. *Each $\mathbb{G}_{m,n} \subset \mathbb{R}^{(m+n)^2}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set having projectively $\mathbb{Q}$-algebraic homology.*

PROOF. By Corollary 2.4.9, for every $k \in \{0, \dots, mn\}$:

$$H_k(\mathbb{G}_{m,n}, \mathbb{Z}/2\mathbb{Z}) = \langle \{[\sigma_\lambda] \in H_k(\mathbb{G}_{m,n}, \mathbb{Z}/2\mathbb{Z}) \,|\, |\lambda| = mn - k\} \rangle,$$

where each σ_λ is a Schubert variety of $\mathbb{G}_{m,n}$ defined by the incidence conditions, prescribed by λ, with respect to the canonical complete flag of $\mathbb{R}^{m+n}$. By Theorem 2.3.4, each Schubert variety σ_λ as above admits a $\mathbb{Q}$-desingularization, that is: there exists a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Z_\lambda \subset \mathbb{R}^{(m+n)^2} \times \mathbb{R}^p$ of dimension $\dim(\sigma_\lambda)$, for some $p \in \mathbb{N}$, such that $\pi_1 : Z_\lambda \to \sigma_\lambda$ is a birational map. Observe that, since $\pi_1 : Z_\lambda \to \sigma_\lambda$ is surjective, injective onto the Zariski open subset Ω_λ such that $\overline{\Omega}_\lambda = \sigma_\lambda$ and $\dim(Z_\lambda) = \dim(\sigma_\lambda)$, we get that $\pi_{1*}([Z_\lambda]) = [\sigma_\lambda]$, as desired. $\square$

2.5. Unoriented relative bordisms over $\mathbb{Q}$

Let us specify 'over $\mathbb{Q}$' the construction of the algebraic unoriented relative bordisms by Akbulut and King in [AK81a, Lemma 4.1].

LEMMA 2.5.1. *Let M be a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$ of dimension d and let M_i, for $i = 1, \dots, \ell$, be closed $\mathscr{C}^\infty$ submanifolds of M of codimension c_i in general position. Then there are a compact $\mathscr{C}^\infty$ manifold with boundary T and proper $\mathscr{C}^\infty$ submanifolds with boundary T_i, for $i = 1, \dots, \ell$, in general position, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset Y of $\mathbb{R}^h$ for some $h \in \mathbb{N}$, and a $\mathscr{C}^\infty$ diffeomorphism $\psi : M \sqcup Y \to \partial T$ such that:*

(i) *Y is the disjoint union of projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets $Y^\alpha \subset \mathbb{R}^h$ for every $\alpha \subset \{1, \dots, \ell\}$ such that $\bigcap_{i \in \alpha} M_i \neq \varnothing$.*

(ii) *$\partial T \cap T_i = \partial T_i$, $\psi(M) \cap T_i = \psi(M_i)$ and $\psi(Y^\alpha) \cap T_i = \psi(Y_i^\alpha)$ where Y_i^α, for $i = 1, \dots, \ell$, are projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of Y^α in general position with $Y_i^\alpha = \varnothing$ whenever $i \notin \alpha$.*

(iii) *For every $\alpha \subset \{1, \dots, \ell\}$ and $i \in \alpha$, there is a $\mathbb{Q}$-regular function $\mu_i^\alpha : Y_i^\alpha \to \mathbb{G}_{c_i, n-c_i}$ such that*

$$Y^\alpha = (\mu_i^\alpha)^*(\mathbb{E}^*_{c_i, n-c_i}).$$

In particular, μ_i^α is the Gauss mapping of Y_i^α in Y^α.

PROOF. For every $\alpha \subset \{1, \dots, \ell\}$ we denote by $M_\alpha := \bigcap_{i \in \alpha} M_i$, if $\alpha \neq \varnothing$, and $M_\varnothing := M$. We argue by induction on the subsets α of $\{1, \dots, \ell\}$ so that $M_\alpha \neq \varnothing$. The case in which all $M_\alpha = \varnothing$, for every $\alpha \subset \{1, \dots, \ell\}$, means that $M = M_\varnothing = \varnothing$, thus the theorem follows by taking $T = \varnothing$. Suppose the set of $\alpha \subset \{1, \dots, \ell\}$ so that $M_\alpha \neq \varnothing$ is non-empty. Let α be such that $M_\alpha \neq \varnothing$ and $M_{\alpha'} = \varnothing$ for every $\alpha' \subset \{1, \dots, \ell\}$ so that $\alpha \subsetneq \alpha'$. Let $\beta_i : M_i \to \mathbb{G}_{c_i, n-c_i}$ be the Gauss mapping of M_i in M for every $i \in \alpha$. Let $\mathbb{G}_\alpha := \prod_{i \in \alpha} \mathbb{G}_{c_i, n-c_i}$. By Theorem 2.4.10 and Lemma

2.4.7, $\mathbb{G}_\alpha \subset \mathbb{R}^{n^2|\alpha|}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set having projectively $\mathbb{Q}$-algebraic homology. Let $\beta_\alpha : M_\alpha \to \mathbb{G}_\alpha$ be the $\mathscr{C}^\infty$ function defined as $\beta_\alpha := \prod_{i\in\alpha} \beta_i$. Thus, Theorem 2.4.5 ensures the existence of $k_\alpha \in \mathbb{N}$, a compact $\mathscr{C}^\infty$ manifold with boundary T_α, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Y_\alpha \subset \mathbb{R}^{k_\alpha}$, a $\mathscr{C}^\infty$ diffeomorphism $\psi_\alpha : M_\alpha \sqcup Y_\alpha \to \partial T_\alpha$ and a $\mathscr{C}^\infty$ map $\mu^\alpha : T_\alpha \to \mathbb{G}_\alpha$ such that $\mu^\alpha \circ \jmath_\alpha \circ (\psi_\alpha|_{M_\alpha}) = \beta_\alpha$ and $g_\alpha := \mu^\alpha \circ \jmath_\alpha \circ (\psi_\alpha|_Y) \in \mathcal{R}^{\mathbb{Q}}(Y, \mathbb{G}_\alpha)$, that is, g_α is $\mathbb{Q}$-regular, where $\jmath_\alpha : \partial T_\alpha \hookrightarrow T_\alpha$ denotes the inclusion map.

Let $\mathbb{E}^*_\alpha := \prod_{i\in\alpha} \mathbb{E}^*_{c_i,n-c_i}$. Define the pullback bundle of $\mathbb{E}^*_\alpha$ via μ^α as $S^\alpha := (\mu^\alpha)^*(\mathbb{E}^*_\alpha)$ and the $\mathscr{C}^\infty$ submanifolds S^α_i of S^α as follows

$$\begin{aligned} S^\alpha &:= \{(x, y_1, t_1, \dots, y_{|\alpha|}, t_{|\alpha|}) \in T_\alpha \times (\mathbb{R}^n \times \mathbb{R})^{|\alpha|} \mid \\ &\qquad\qquad (\mu^\alpha(x), y_1, t_1, \dots, y_{|\alpha|}, t_{|\alpha|}) \in \mathbb{E}^*_\alpha\} \\ S^\alpha_i &:= \{(x, y_1, t_1, \dots, y_{|\alpha|}, t_{|\alpha|}) \in S^\alpha \mid y_i = 0,\, t_i = 0\}, \end{aligned}$$

for every $i \in \alpha$. By definition, the S^α_i, for $i \in \alpha$, are in general position and $\bigcap_{i\in\alpha} S^\alpha_i = T_\alpha \times \{0\} \subset T_\alpha \times (\mathbb{R}^n \times \mathbb{R})^{|\alpha|}$. In addition, considering the projections $\pi^i_0 : S^\alpha_i \to T_\alpha$ and $\pi_i : \mathbb{G}_\alpha \to \mathbb{G}_{c_i,n-c_i}$, we define $\mu^\alpha_i : S^\alpha_i \to \mathbb{G}_{c_i,n-c_i}$ as $\mu^\alpha_i = \pi_i \circ \mu^\alpha \circ \pi^i_0$. Thus, we deduce that S^α is the pullback sphere bundle of $\mathbb{E}^*_{c_i,n-c_i}$ by μ^α_i, i.e. $S^\alpha = (\mu^\alpha_i)^*(\mathbb{E}^*_{c_i,n-c_i})$, where

$$\begin{aligned} (\mu^\alpha_i)^*(\mathbb{E}^*_{c_i,n-c_i}) &:= \{(x, y_1, t_1, \dots, y_\ell, t_\ell, y_{\ell+1}, t_{\ell+1}) \in S^\alpha_i \times \mathbb{R}^n \times \mathbb{R} \mid \\ &\qquad\qquad (\mu^\alpha_i(x), y_{|\alpha|+1}, t_{|\alpha|+1}) \in \mathbb{E}^*_{c_i,n-c_i}\}. \end{aligned}$$

Thus, S^α and the S^α_i, for every $i \in \alpha$, are $\mathscr{C}^\infty$ manifolds with boundary satisfying $\partial S^\alpha_i \subset \partial S^\alpha$. Define:

$$\begin{aligned} M^\alpha &:= \beta^*_\alpha(\mathbb{E}^*_\alpha) = (\mu^\alpha \circ \jmath_\alpha \circ \psi_\alpha)|^*_{M_\alpha}(\mathbb{E}^*_\alpha) \subset M_\alpha \times \mathbb{R}^{(n+1)|\alpha|}, \\ Y^\alpha &:= g^*_\alpha(\mathbb{E}^*_\alpha) = (\mu^\alpha \circ \jmath_\alpha \circ \psi_\alpha)|^*_{Y_\alpha}(\mathbb{E}^*_\alpha) \subset \mathbb{R}^{k_\alpha} \times \mathbb{R}^{(n+1)|\alpha|}. \end{aligned}$$

Observe that, by Lemma 2.2.6, we deduce that $Y^\alpha \subset \mathbb{R}^{k_\alpha} \times \mathbb{R}^{(n+1)|\alpha|}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. Since $\psi_\alpha : M_\alpha \sqcup Y_\alpha \to \partial T_\alpha$ is a diffeomorphism, we deduce that $\Psi_\alpha : M^\alpha \sqcup Y^\alpha \to \partial S^\alpha$ defined as $\Psi_\alpha(x, y_1, t_1, \dots, y_{|\alpha|}, t_{|\alpha|}) = (\psi_\alpha(x), y_1, t_1, \dots, y_{|\alpha|}, t_{|\alpha|})$ is a diffeomorphism as well. Hence, define

$$Y^\alpha_i := Y^\alpha \cap \Psi^{-1}_\alpha(\partial S^\alpha_i)$$

for every $i \in \alpha$. Observe that $Y^\alpha_i = ((\mu^\alpha_{\alpha\setminus\{i\}} \circ \Psi_\alpha)|_{Y_\alpha})^*(\mathbb{E}^*_{\alpha\setminus\{i\}})$, where $\mu^\alpha_{\alpha\setminus\{i\}} : T_\alpha \to \mathbb{G}_{c_1,n-c_1} \times \dots \times \mathbb{G}_{c_{i-1},n-c_{i-1}} \times \{0\} \times \mathbb{G}_{c_{i+1},n-c_{i+1}} \times \dots \times \mathbb{G}_{c_{|\alpha|},n-c_{|\alpha|}}$ defined as $\mu^\alpha_{\alpha\setminus\{i\}}(x) := (\mu^\alpha_1(x), \dots, \mu^\alpha_{i-1}(x), 0, \mu^\alpha_{i+1}(x), \dots, \mu^\alpha_{|\alpha|}(x))$ and

$$\mathbb{E}^*_{\alpha\setminus\{i\}} := \{(y_1, t_1, \dots, y_{|\alpha|}, t_{|\alpha|}) \in \mathbb{E}^*_\alpha \mid y_i = 0,\, t_i = 0\}, \tag{2.5.1}$$

which is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sphere bundle by Lemma 2.2.4. Observe that $(\mu^\alpha_{\alpha\setminus\{i\}} \circ \Psi_\alpha)|_{Y_\alpha}$ is $\mathbb{Q}$-regular since $(\mu^\alpha \circ \psi_\alpha)|_{Y_\alpha}$ is so. Thus, $Y^\alpha_i \subset \mathbb{R}^{k_\alpha} \times \mathbb{R}^{(n+1)|\alpha|}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set by Lemma 2.2.6, for every $i \in \alpha$.

Since $\mu^\alpha|_{M_\alpha}$ is the Gauss mapping of M_α in each M_i with $i \in \alpha$, we can select two sufficiently small closed tubular neighborhoods U_α and V_α of M_α in M^α and in M, respectively, which are diffeomorphic via a diffeomorphism $h_\alpha : U_\alpha \to V_\alpha$ satisfying $h_\alpha(U_\alpha \cap S^\alpha_i) = V_\alpha \cap M_i$, for every $i \in \alpha$. Consider the $\mathscr{C}^\infty$ manifold with boundary S defined as $S^\alpha \cup (M \times [0,1])$ identifying U_α and $V_\alpha \times \{1\}$ via $h_\alpha \times \{1\} : U_\alpha \to V_\alpha \times \{1\}$

defined as $(h_\alpha \times \{1\})(a) = (h_\alpha(a), 1)$, after smoothing corners. In the same way define the $\mathscr{C}^\infty$ submanifolds with boundary S_i as $S_i^\alpha \cup (M_i \times [0,1])$ identifying $U_\alpha \cap S_i^\alpha$ with $(V_\alpha \cap M_i) \times \{1\}$ via $h_\alpha \times \{1\}$. Observe that the $\mathscr{C}^\infty$ submanifolds S_i of S, with $i \in \alpha$, are in general position.

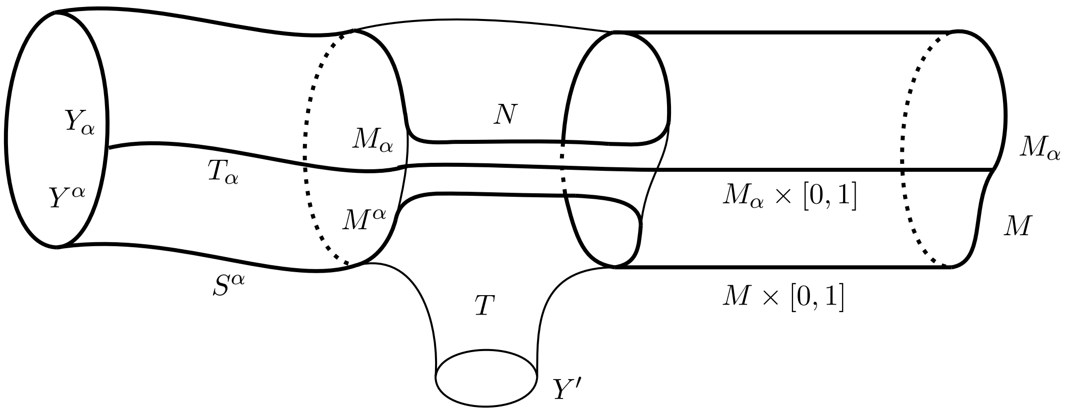

FIGURE 2.5.1. Inductive step constructing a relative bordism.

Define the $\mathscr{C}^\infty$ manifold N with $\mathscr{C}^\infty$ submanifolds in general position N_i, for every $i \in \{1, \dots, \ell\}$, as follows:

$$N := (M^\alpha \setminus \operatorname{Int}(U_\alpha)) \cup_{h_\alpha} (M \setminus \operatorname{Int}(V_\alpha)),$$

$$N_i := \begin{cases} N \cap S_i & \text{if } i \in \alpha, \\ M_i \times \{1\} & \text{otherwise.} \end{cases}$$

Observe that, by construction, $\partial(S^\alpha \cup_{h_\alpha} (M \times [0,1])) = N \sqcup Y^\alpha \sqcup M$, with M identified with $M \times \{0\}$, $\partial(S_i^\alpha \cup_{h_\alpha} (M_i \times [0,1])) = N_i \sqcup Y_i^\alpha \sqcup M_i$ for every $i \in \alpha$, and $\partial(M_i \times [0,1]) = N_i \sqcup M_i$ for every $i \notin \alpha$. In particular, it holds that $N_\alpha := \bigcap_{i \in \alpha} N_i = \varnothing$. By Whitney $\mathscr{C}^\infty$ embedding theorem, there is a $\mathscr{C}^\infty$ manifold $M' \subset \mathbb{R}^{2d+1}$ with $\mathscr{C}^\infty$ submanifolds M_i' of codimension c_i in general position for $i \in \{1, \dots, \ell\}$, which is diffeomorphic to N via a diffeomorphism $\varphi : M' \to N$ such that $\varphi(M_i') = N_i$ for every $i \in \{1, \dots, \ell\}$. Thus, by inductive assumption on $M' \subset \mathbb{R}^{2d+1}$, there exist $k' \in \mathbb{N}$, a $\mathscr{C}^\infty$ manifold with boundary T' and $\mathscr{C}^\infty$ submanifolds with boundary T_i' for every $i \in \{1, \dots, \ell\}$, with transverse intersection, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset Y' of $\mathbb{R}^{k'}$ for some $k' \in \mathbb{N}$, a $\mathscr{C}^\infty$ diffeomorphism $\psi' : M' \sqcup Y' \to \partial T'$ (without lost of generality we can assume $\psi'(M') = N$ and $\psi'(M_i') = N_i$) such that:

(i′) Y' is the disjoint union of a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets $Y'^\alpha \subset \mathbb{R}^n \times \mathbb{R}^{k'}$, for every $\alpha \subset \{1, \dots, \ell\}$ such that $\bigcap_{i \in \alpha} M_i' \neq \varnothing$.

(ii′) $\partial T' \cap T_i' = \partial T_i'$, $N \cap T_i' = \psi'(M') \cap T_i' = \psi(M_i') = N_i$ and $\psi'(Y'^\alpha) \cap T_i' = \psi'(Y_i'^\alpha)$ where $Y_i'^\alpha$, for $i \in \{1, \dots, \ell\}$, are projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of Y'^α transverse to each other with $Y_i'^\alpha = \varnothing$ whenever $i \notin \alpha$.

(iii′) For every $\alpha \subset \{1,\dots,\ell\}$ and $i \in \alpha$, there is a $\mathbb{Q}$-regular function $\mu_i'^{\alpha} : Y_i'^{\alpha} \to \mathbb{G}_{c_i,2d+1-c_i}$ such that
$$Y'^{\alpha} = (\mu_i'^{\alpha})^*(\mathbb{E}^*_{c_i,2d+1-c_i}).$$
In particular, $\mu_i'^{\alpha}$ is the Gauss mapping of Y^i_α in Y_i.

Define $T := S \cup T'$ and $T_i := S_i \cup T_i'$, after smoothing corners. Let $k := \max(k_\alpha, k')$ and consider $\iota_\alpha : \mathbb{R}^{k_\alpha} \to \mathbb{R}^k$ and $\iota' : \mathbb{R}^{k'} \to \mathbb{R}^k$ be the inclusion mappings. Then, after a translation of a rational factor $v \in \mathbb{Q}^k$ if necessary, we may assume that $(\iota'(Y')+v) \cap \iota_\alpha(Y^\alpha) = \varnothing$, thus $Y := \iota_\alpha(Y^\alpha) \sqcup (\iota'(Y')+v) \subset \mathbb{R}^k$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set by Lemmas 2.1.3(ii) & 2.1.7(i)(iii). Let $\psi : M \sqcup Y \to \partial T$ defined as follows $\psi|_M := \psi_\alpha|_M$, $\psi|_{\iota_\alpha(Y_\alpha)}(x) := \psi_\alpha(\iota_\alpha^{-1}(x))$ and $\psi|_{\iota'(Y')+v}(x) := \psi'((\iota')^{-1}(x-v))$. □

Here we provide an embedded version of Lemma 2.5.1 and we ‘double the relative bordism over $\mathbb{Q}$’ following the strategy used by Tognoli in [Tog73, § b), pp. 176-177].

THEOREM 2.5.2. *Let M be a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$ of dimension d, let M_i for $i = 1,\dots,\ell$, be $\mathscr{C}^\infty$ submanifolds of M of codimension c_i in general position. Then there exist $s \in \mathbb{N}$ with $s \geq n$, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Y \subset \mathbb{R}^s = \mathbb{R}^n \times \mathbb{R}^{s-n}$ of dimension d, $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets Y_i, for $i \in \{1,\dots,\ell\}$, of Y in general position, a compact $\mathscr{C}^\infty$ submanifold S of $\mathbb{R}^{s+1} = \mathbb{R}^s \times \mathbb{R}$ of dimension $d+1$ and compact $\mathscr{C}^\infty$ submanifolds S_i of S of codimension c_i, for $i = 1,\dots,\ell$, in general position with the following properties:*

(i) *$M \cap Y = \varnothing$.*
(ii) *$S \cap (\mathbb{R}^s \times (-1,1)) = (M \sqcup Y) \times (-1,1)$ and $S_i \cap (\mathbb{R}^s \times (-1,1)) = (M_i \sqcup Y_i) \times (-1,1)$, for every $i \in \{1,\dots,\ell\}$.*
(iii) *Y is the finite disjoint union $\bigsqcup_{\alpha \in A}(Y^\alpha + v_\alpha)$ of projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets of the form $Y^\alpha + v_\alpha \subset \mathbb{R}^s$, where v_α belongs to $\mathbb{Q}^s$, Y^α is inductively defined as in the proof of Lemma* 2.5.1 *and*
$$A := \Big\{\alpha \subset \{1,\dots,\ell\} \,|\, \bigcap_{j \in \alpha} M_j \neq \varnothing\Big\}.$$
In addition, there are projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset $Y_\alpha \subset \mathbb{R}^s$ and $\mathbb{Q}$-regular functions $\mu_\alpha : Y_\alpha \to \mathbb{G}^*_\alpha$ such that $Y^\alpha := \mu^*_\alpha(\mathbb{E}^*_\alpha)$, with $\mathbb{G}^*_\alpha := \prod_{i \in \alpha} \mathbb{G}^*_{c_i,n-c_i}$ and $\mathbb{E}^*_\alpha := \prod_{i \in \alpha} \mathbb{E}^*_{c_i,n-c_i}$.
(iv) *Let $i \in \{1,\dots,\ell\}$. Then, Y_i is the finite disjoint union $\bigsqcup_{\alpha \in A_i}(Y_i^\alpha + v_\alpha)$ of projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets of the form $Y_i^\alpha + v_\alpha \subset \mathbb{R}^s$, where v_α belongs to $\mathbb{Q}^s$ as above, Y_i^α is inductively defined as in the proof of Lemma 2.5.1 and*
$$A_i := \{\alpha \in A \,|\, i \in \alpha\}.$$
In addition, there is a $\mathbb{Q}$-regular map $\mu_i^\alpha : Y_i^\alpha \to \mathbb{G}_{c_i,n-c_i}$ such that, if $\beta_i : S_i \to \mathbb{G}_{c_i,n-c_i}$ denotes the Gauss mapping of S_i in S, then $\beta_i|_{Y_i} = \bigsqcup_{\alpha \in A_i} \mu_i^\alpha$ is a $\mathbb{Q}$-regular map.

PROOF. Thanks to the proof of Lemma 2.5.1, for $s \geq n$ sufficiently large, we know that there exist a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Y = \bigsqcup_{\alpha \in A}(Y^\alpha + v_\alpha) \subset \mathbb{R}^s$ and $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets $Y_i = \bigsqcup_{\alpha \in A_i}(Y_i^\alpha + v_\alpha) \subset$

$\mathbb{R}^s$, with $i \in \{1,\dots,\ell\}$, in general position with above properties (i) (changing the vectors $v_\alpha \in \mathbb{Q}^s$ if necessary), (ii) & (iii), compact $\mathscr{C}^\infty$ manifolds T and T_i with boundary ∂T and ∂T_i so that $T_i \subset T$ and $\partial T_i \subset \partial T$, for every $i = 1,\dots,\ell$.

Let us construct the desired compact $\mathscr{C}^\infty$ submanifold S of $\mathbb{R}^{s+1} = \mathbb{R}^s \times \mathbb{R}$, following the strategy used by Tognoli in [Tog73, § b), pp. 176-177]. By the collaring theorem (see [Hir94, Theorem 6.1, p. 113]), there exist an open neighborhood U of ∂T in T and a $\mathscr{C}^\infty$ diffeomorphism $\phi' : U \to \partial T \times [0,1)$ such that $\phi'(t) = (t,0)$ for all $t \in \partial T$ and $\phi'|_{T_i \cap U} : T_i \cap U \to \partial T_i \times [0,1)$ is a diffeomorphism as well, for every $i = 1,\dots,\ell$. Let $\phi : U \to (M \sqcup Y) \times [0,1)$ be the $\mathscr{C}^\infty$ diffeomorphism $\phi := (\psi^{-1} \times \mathrm{id}_{[0,1)}) \circ \phi'$. Note that $\phi(t) = (\psi^{-1}(t), 0)$ for all $t \in \partial T$. Set $A := T \setminus \partial T$, $B := \phi^{-1}((M \sqcup Y) \times (0, \frac{1}{2}]) \subset A$, $N := \mathbb{R}^s \times (0, +\infty)$ and define the map $\theta : B \to N$ by $\theta(x, x_{s+1}) := \phi(x, x_{s+1})$. Since we can safely assume $s+1 \geq 2(d+1)+1$, Tietze's theorem ensures the existence of a continuous extension of θ from the whole A to N, we can apply to θ the extension theorem [Whi36, Theorem 5(f)] of Whitney, obtaining a $\mathscr{C}^\infty$ embedding $\Theta : A \to N$ extending θ. Let $R : \mathbb{R}^{s+1} = \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^{s+1}$ be the reflection $R(x, x_{s+1}) := (x, -x_{s+1})$ and let S' and S'_i be the compact $\mathscr{C}^\infty$ submanifolds $\Theta(A) \sqcup ((M \sqcup Y) \times \{0\}) \sqcup R(\Theta(A))$ and $\Theta(A \cap T_i) \sqcup ((M_i \sqcup Y_i) \times \{0\}) \sqcup R(\Theta(A \cap T_i))$ of $\mathbb{R}^{s+1}$, for every $i = 1,\dots,\ell$, respectively. Thanks to the compactness of T and of each T_i, there exists $\epsilon > 0$ such that $S' \cap (\mathbb{R}^s \times (-\epsilon, \epsilon)) = (M \sqcup Y) \times (-\epsilon, \epsilon)$ and $S'_i \cap (\mathbb{R}^s \times (-\epsilon, \epsilon)) = (M_i \sqcup Y_i) \times (-\epsilon, \epsilon)$. Let $L : \mathbb{R}^{s+1} \to \mathbb{R}^{s+1}$ be the linear isomorphism $L(x, x_{s+1}) := (x, \epsilon^{-1} x_{s+1})$. The compact $\mathscr{C}^\infty$ submanifold $S := L(S')$ with $\mathscr{C}^\infty$ submanifolds $S_i := L(S'_i)$, for every $i = 1,\dots,\ell$, in general position of $\mathbb{R}^{s+1}$ have the desired properties (ii) & (iv). $\square$

CHAPTER 3

$\mathbb{Q}$-algebraic approximations à la Akbulut-King

ABSTRACT. In this chapter we extend 'over $\mathbb{Q}$' classical approximation techniques developed by Nash, Tognoli, Akbulut and King. The notion of $\mathbb{R}|\mathbb{Q}$-regularity of $\mathbb{Q}$-algebraic sets proves its importance here. In section 3.1 introduce the concept of approximable pair 'over $\mathbb{Q}$' and we give results on the relative approximation of $\mathscr{C}^\infty$ functions vanishing on $\mathbb{Q}$-approximable pairs. In Section 3.2 we extend 'over $\mathbb{Q}$' some Akbulut-King algebraic approximation results. In particular, we prove a relative version 'over $\mathbb{Q}$' (with respect to a finite set of $\mathscr{C}^\infty$ hypersurfaces in general position) of Nash-Tognoli theorem. Finally, in Section 3.3 we prove a version 'over $\mathbb{Q}$' with approximation of Akbulut-King blowing down lemma.

The main reference for this chapter is [GS23].

Throughout this chapter we consider $\mathbb{R}^n$ endowed with the euclidean topology.

3.1. $\mathbb{Q}$-Approximable pairs

The aim of this section is to generalize 'over $\mathbb{Q}$' the notions of nice algebraic set and approximable pair (see [AK92, §8, Definition p. 57-58]) and to produce useful examples. Let $P \subset \mathbb{R}^n$. We denote by $\operatorname{int}_{\mathbb{R}^n}(P)$ the interior of P in $\mathbb{R}^n$ and by $\mathcal{I}^\infty_{\mathbb{R}^n}(P)$ the ideal in $\mathscr{C}^\infty(\mathbb{R}^n)$ of those smooth functions vanishing on P.

DEFINITION 3.1.1. Let $L \subset \mathbb{R}^n$ be a $\mathbb{Q}$-algebraic set and let P be a subset of $\mathbb{R}^n$ containing L. We say that the pair (P, L) is a $\mathbb{Q}$-*approximable pair of* $\mathbb{R}^n$ if for each $a \in L \setminus \operatorname{int}_{\mathbb{R}^n}(P)$, there exists an open neighborhood U_a of a in $\mathbb{R}^n$ such that

$$\mathcal{I}^\infty_{\mathbb{R}^n}(P)\mathscr{C}^\infty(U_a) \subset \mathcal{I}_{\mathbb{Q}}(L)\mathscr{C}^\infty(U_a), \tag{3.1.1}$$

i.e., for each $f \in \mathcal{I}^\infty_{\mathbb{R}^n}(P)$, we have $f|_{U_a} = \sum_{i=1}^{\ell} u_i \cdot p_i|_{U_a}$ for some $u_1, \dots, u_\ell \in \mathscr{C}^\infty(U_a)$ and $p_1, \dots, p_\ell$ generators of $\mathcal{I}_{\mathbb{Q}}(L)$.

If (L, L) is a $\mathbb{Q}$-approximable pair, then we say that L is $\mathbb{Q}$-*nice*. ∎

The reader observes that condition (3.1.1) remains valid if we replace U_a with a smaller open neighborhood of a in $\mathbb{R}^n$. In addition, it is evident that the disjoint union of finitely many $\mathbb{Q}$-nice algebraic subsets of $\mathbb{R}^n$ is again a $\mathbb{Q}$-nice algebraic subset of $\mathbb{R}^n$.

Let us give some sufficient conditions to have a $\mathbb{Q}$-approximable pair.

LEMMA 3.1.2. *Let $L \subset \mathbb{R}^n$ be a $\mathbb{Q}$-algebraic set of dimension $d < n$, let $\mathrm{R}(L) := \bigsqcup_{e=0}^{d} \operatorname{Reg}^*(L, e)$ (see Definition 1.5.1) and let P be a subset of $\mathbb{R}^n$ such that $L \subset P$ and $L \setminus \operatorname{int}_{\mathbb{R}^n}(P) \subset \mathrm{R}(L)$. Then, (P, L) is a $\mathbb{Q}$-approximable pair.*

PROOF. Let $p_1,\dots,p_{n-d}\in\mathbb{Q}[x]$ be generators of $\mathcal{I}_\mathbb{Q}(L)$. By Theorem 1.6.5, for every $a\in L\setminus\mathrm{int}_{\mathbb{R}^n}(P)$ there exists a subset I_a of $\{1,\dots,\ell\}$ of cardinality $n-\dim_a(L)$ such that the vectors $\{\nabla p_i(a)\}_{i\in I_a}$ of $\mathbb{R}^n$ are linearly independent over $\mathbb{R}$. By [AK92, Lemma 2.5.4], the existence of a subset I_a of cardinality $n-\dim_a(L)$ such that the vectors $\{\nabla p_i(a)\}_{i\in I_a}$ of $\mathbb{R}^n$ are linearly independent over $\mathbb{R}$ implies property (3.1.1) at a. □

As an immediate consequence, we obtain:

COROLLARY 3.1.3. *Every disjoint union of finitely many $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of $\mathbb{R}^n$ (of possibly different dimensions) is $\mathbb{Q}$-nice.*

Another useful result is the following.

LEMMA 3.1.4. *Let $L\subset\mathbb{R}^n$ be a $\mathbb{Q}$-nice $\mathbb{Q}$-algebraic set and let $h\in\mathbb{N}^*$. Then $L\times\{0\}$ is a $\mathbb{Q}$-nice algebraic subset of $\mathbb{R}^{n+h}=\mathbb{R}^n\times\mathbb{R}^h$.*

PROOF. Let $\mathbb{R}[x,y]$ be the polynomial ring of $\mathbb{R}^{n+h}$. Since L is $\mathbb{Q}$-algebraic, we deduce that $L\times\{0\}=\mathcal{Z}_\mathbb{R}(\mathcal{I}_\mathbb{Q}(L)+(y_1,\dots,y_h)\mathbb{Q}[x])$, thus $L\times\{0\}$ is $\mathbb{Q}$-algebraic as well. Since $L\times\{0\}$ is a product we have that

$$\begin{aligned}\mathcal{I}^\infty_{\mathbb{R}^{n+h}}(L\times\{0\})&=\mathcal{I}^\infty_{\mathbb{R}^n}(L)\mathscr{C}^\infty(\mathbb{R}^{n+h})+\mathcal{I}^\infty_{\mathbb{R}^{n+h}}(\mathbb{R}^n\times\{0\})\\&=\mathcal{I}^\infty_{\mathbb{R}^n}(L)\mathscr{C}^\infty(\mathbb{R}^{n+h})+(y_1,\dots,y_h)\mathscr{C}^\infty(\mathbb{R}^{n+h}).\end{aligned}$$

Since L is $\mathbb{Q}$-nice, for every $a\in L$ there is a neighborhood U_a of a in L such that $\mathcal{I}^\infty_{\mathbb{R}^n}(L)\mathscr{C}^\infty(U_a)\subset\mathcal{I}_\mathbb{Q}(L)\mathscr{C}^\infty(U_a)$, hence by fixing the neighborhood $U_a\times\mathbb{R}^h$ of $(a,0)$ in $\mathbb{R}^{n+h}$ we get that:

$$\begin{aligned}\mathcal{I}^\infty_{\mathbb{R}^{n+h}}(L\times\{0\})\mathscr{C}^\infty(U_a\times\mathbb{R}^h)&=\mathcal{I}^\infty_{\mathbb{R}^n}(L)\mathscr{C}^\infty(U_a\times\mathbb{R}^h)+\mathcal{I}^\infty_{U_a\times\mathbb{R}^h}(U_a\times\{0\})\\&\subset\mathcal{I}_\mathbb{Q}(L)\mathscr{C}^\infty(U_a\times\mathbb{R}^h)+(y_1,\dots,y_h)\mathscr{C}^\infty(U_a\times\mathbb{R}^h)\\&\subset\mathcal{I}_\mathbb{Q}(L\times\{0\})\mathscr{C}^\infty(U_a\times\mathbb{R}^h).\end{aligned}$$

□

Let $M\subset\mathbb{R}^n$ be a $\mathscr{C}^\infty$ manifold of dimension d. We say that $Y\subset M$ is an *algebraic hypersurface of M* if $Y\subset\mathbb{R}^n$ is an algebraic set whose irreducible components have dimension $d-1$. We say that $Y\subset M$ is a *$\mathbb{Q}$-algebraic hypersurface of M* if $Y\subset\mathbb{R}^n$ is a $\mathbb{Q}$-algebraic set which is an algebraic hypersurface of M. Next lemma will play a crucial role in the proof of a relative version of Nash-Tognoli theorem 'over $\mathbb{Q}$' in Section 3.2.1.

LEMMA 3.1.5. *Let $M\subset\mathbb{R}^n$ be a compact $\mathscr{C}^\infty$ manifold of dimension d. Let $X\subset M$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^n$ of codimension c and let $Y\subset M$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic hypersurface of M. If the germ $(M,X\cup Y)$ of M at $X\cup Y$ is the germ of a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set, then $X\cup Y$ is $\mathbb{Q}$-nice.*

PROOF. Without lost of generality we may assume that none of the irreducible components of X is contained in Y. Let $a\in(X\cup Y)\setminus(X\cap Y)=(X\setminus Y)\sqcup(Y\setminus X)$. Since both $X\subset\mathbb{R}^n$ and $Y\subset\mathbb{R}^n$ are $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets, up to shrink the neighborhood U_a, we deduce property (3.1.1) by Corollary 1.6.6. Let $a\in X\cap Y$ and let $f\in\mathcal{I}^\infty_{\mathbb{R}^n}(X\cup Y)$. Let $V\subset\mathbb{R}^n$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set such that the germ $(M,X\cup Y)$ of M at $X\cup Y$ coincides to the germ

$(V, X\cup Y)$ of V at $X\cup Y$. Since Y is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic hypersurface of $V\subset\mathbb{R}^n$ there are $p_1,\dots,p_{n-d}\in\mathcal{I}_\mathbb{Q}(V)$ and $p\in\mathcal{I}_\mathbb{Q}(Y)$ whose gradients at a are linearly independent over $\mathbb{R}$ and there is a neighborhood U_a of a in $\mathbb{R}^n$ such that $Y\cap U_a=V\cap\mathcal{Z}_{\mathbb{R}^n}(p)\cap U_a=\mathcal{Z}_{\mathbb{R}^n}(p,p_1,\dots,p_{n-d})\cap U_a$. Hence, by [AK92, Lemma 2.5.4], there are $u,u_1,\dots,u_{n-d}\in\mathscr{C}^\infty(U_a)$ such that $f|_{U_a}=u\cdot p|_{U_a}+\sum_{i=1}^{n-d}u_i\cdot p_i|_{U_a}$, up to shrink the neighborhood U_a of a in $\mathbb{R}^n$ if necessary. Since none of the irreducible components of X is contained in Y, we deduce that $Y\cap U_a=V\cap\mathcal{Z}_{\mathbb{R}^n}(p)\cap U_a\subsetneq(X\cup Y)\cap U_a$. Thus $\mathcal{Z}_{\mathbb{R}^n}(p)\cap U_a\cap X\subset Y$, up to shrink the neighborhood U_a of a in $\mathbb{R}^n$ if necessary. In addition, since $f|_{U_a}=u\cdot p|_{U_a}+\sum_{i=1}^{n-d}u_i\cdot p_i|_{U_a}$, $p_1,\dots,p_{n-d}\in\mathcal{I}_\mathbb{Q}(V)$ and $\mathcal{Z}_{\mathbb{R}^n}(p)\cap U_a\cap X\subset Y$, we deduce that $X\cap U_a\subset\mathcal{Z}_{\mathbb{R}^n}(u)$. Now, let $U_a'\subset U_a$ be a neighborhood of a in $\mathbb{R}^n$ such that $\overline{U_a'}\subsetneq U_a$. An explicit construction via partitions of unity subordinated to the open cover $\{U_a, X\setminus\overline{U_a'}\}$ of $\mathbb{R}^n$ ensures the existence of $g\in\mathscr{C}^\infty(\mathbb{R}^n)$ such that $g|_{U_a'}=u|_{U_a'}$ and $g\in\mathcal{I}^\infty_{\mathbb{R}^n}(X)$. Since $X\subset\mathbb{R}^n$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set of codimension c in $V\subset\mathbb{R}^n$, which is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set as well, there are $q_1,\dots,q_c\in\mathcal{I}_\mathbb{Q}(X)$ such that $\nabla p_1(a),\dots,\nabla p_{n-d}(a),\nabla q_1(a),\dots,\nabla q_c(a)$ are linearly independent over $\mathbb{R}$ and there exists a neighborhood V_a of a in $\mathbb{R}^n$ such that $X\cap V_a=\mathcal{Z}_{\mathbb{R}^n}(p_1,\dots,p_{n-d},q_1,\dots,q_c)\cap V_a$. Thus, by [AK92, Lemma 2.5.4], there are $u_1',\dots,u_{n-d}',v_1,\dots,v_c\in\mathscr{C}^\infty(V_a)$ such that $g|_{V_a}=\sum_{i=1}^c v_i\cdot q_i|_{V_a}+\sum_{i=1}^{n-d}u_i'\cdot p_i|_{V_a}$, up to shrink the neighborhood V_a of a in $\mathbb{R}^n$ if necessary. Thus, fixing $V_a':=U_a'\cap V_a$, we have:

$$\begin{aligned}
f|_{V_a'}&=g|_{V_a'}\cdot p|_{V_a'}+\sum_{i=1}^{n-d}u_i|_{V_a'}\cdot p_i|_{V_a'}\\
&=\left(\sum_{i=1}^{c}v_i|_{V_a'}\cdot q_i|_{V_a'}+\sum_{i=1}^{n-d}u_i'|_{V_a'}\cdot p_i|_{V_a'}\right)\cdot p|_{V_a'}+\sum_{i=1}^{n-d}u_i|_{V_a'}\cdot p_i|_{V_a'}.\\
&=\sum_{i=1}^{c}v_i|_{V_a'}\cdot(p\cdot q_i)|_{V_a'}+\sum_{i=1}^{n-d}\big(u_i|_{V_a'}+u_i'|_{V_a'}\cdot p|_{V_a'}\big)\cdot p_i|_{V_a'},
\end{aligned}$$

where $p_1,\dots,p_{n-d},p\cdot q_1,\dots,p\cdot q_c\in\mathcal{I}_\mathbb{Q}(X\cup Y)$, as desired. □

Next lemma will prove its importance in the proof of Theorem 4.1.6, namely, in the proof of the relative $\mathbb{Q}$-algebrization of nonsingular algebraic sets of Section 4.1.

LEMMA 3.1.6. *Let $X\subset\mathbb{R}^n$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set of dimension d and let $\{X_i\}_{i=1}^\ell$ be a family of $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic hypersurfaces of X in general position, then $\bigcup_{i=1}^\ell X_i$ is $\mathbb{Q}$-nice.*

PROOF. Let $a\in\bigcup_{i=1}^\ell X_i$. At first, let us prove the following CLAIM:

CLAIM: *Denote by $J_a:=\{j\in\{1,\dots,\ell\}\,|\,a\in X_j\}$. Let $p_1,\dots,p_{n-d}\in\mathcal{I}_\mathbb{Q}(X)$ and $f_j\in\mathcal{I}_\mathbb{Q}(X_j)$, for every $j\in J_a$, such that $\nabla p_1(a),\dots,\nabla p_{n-d}(a)$, $\{\nabla f_j(a)\}_{j\in J_a}$ are linearly independent over $\mathbb{R}$. Then, there are a neighborhood U_a of a in $\mathbb{R}^n$ such that, for every $f\in\mathcal{I}^\infty_{\mathbb{R}^n}(\bigcup_{i=1}^\ell X_i)$, there are $\mathscr{C}^\infty$ functions $u_1,\dots,u_{n-d},v\in\mathscr{C}^\infty(U_a)$, such that*

$$f|_{U_a}=\sum_{i=1}^{n-d}u_ip_i|_{U_a}+v\cdot\prod_{j\in J_a}q_j|_{U_a}.$$

Let us prove the Claim by induction on $|J|\in\mathbb{N}^*$. The case $|J|=\ell=1$ follows by Corollary 3.1.3. Fix $j_0\in J_a$. By inductive assumption on $a\in\bigcup_{i=1,\,i\neq j_0}^{\ell}X_i\subset X$ there is a neighborhood U'_a of a in $\mathbb{R}^n$ such that, for every $f'\in\mathcal{I}^\infty_{\mathbb{R}^n}(\bigcup_{i=1,\,i\neq j_0}^{\ell}X_i)$, there are $u'_1,\dots,u'_{n-d},v'\in\mathscr{C}^\infty(V_a)$ such that

$$f'|_{U'_a}=\sum_{i=1}^{n-d}u'_ip_i|_{U'_a}+v'\cdot\prod_{j\in J_a\setminus\{j_0\}}q_j|_{U'_a}. \tag{3.1.2}$$

Since $f\in\mathcal{I}^\infty_{\mathbb{R}^n}(\bigcup_{i=1}^{\ell}X_i)\subset\mathcal{I}^\infty_{\mathbb{R}^n}(\bigcup_{i=1,\,i\neq j_0}^{\ell}X_i)$, we have above local structure (3.1.2) with '$f'|_{U'_a}$':$=f|_{U'_a}$. However, since $f\in\mathcal{I}^\infty_{\mathbb{R}^n}(X_{j_0})$ and $\{\nabla q_j(a)\}_{j\in J_a}$ are linearly independent over $\mathbb{R}$, it follows that

$$\mathcal{Z}_\mathbb{R}(q_{j_0})\cap U'_a\not\subset\bigcup_{j\in J_a\setminus\{J_0\}}\mathcal{Z}_\mathbb{R}(q_j)\cap U'_a.$$

Thus, (3.1.2) implies that $v'\in\mathcal{I}^\infty_{U'_a}(X_{j_0}\cap U'_a)$. Let $U_a\subset U'_a$ be a neighborhood of a in $\mathbb{R}^n$ such that $\overline{U_a}\subset U'_a$. By a partition of unity argument, there is $g\in\mathscr{C}^\infty(\mathbb{R}^n)$ such that $g\in\mathcal{I}^\infty_{\mathbb{R}^n}(X_{j_0})$ and $g|_{U_a}=v'|_{U_a}$. Then, there are $u''_1,\dots,u''_2,v''\in\mathscr{C}^\infty(U_a)$ such that:

$$v'|_{U_a}=g|_{U_a}=v''q_{j_0}|_{U_a}+\sum_{i=1}^{n-d}u''_ip_i|_{U_a}. \tag{3.1.3}$$

Then, (3.1.2) & (3.1.3) imply that:

$$\begin{aligned}f|_{U_a}&=\sum_{i=1}^{n-d}u'_i|_{U_a}p_i|_{U_a}+v'|_{U_a}\cdot\prod_{j\in J_a\setminus\{j_0\}}q_j|_{U_a}\\&=\sum_{i=1}^{n-d}u'_i|_{U_a}p_i|_{U_a}+\Big(v''q_{j_0}|_{U_a}+\sum_{i=1}^{n-d}u''_ip_i|_{U_a}\Big)\cdot\prod_{j\in J_a\setminus\{j_0\}}q_j|_{U_a}\\&=\sum_{i=1}^{n-d}\Big(u'_i|_{U_a}+u''_i\cdot\prod_{j\in J_a}q_j|_{U_a}\Big)p_i|_{U_a}+v''\cdot\prod_{j\in J_a}q_j|_{U_a}.\end{aligned}$$

Thus, to conclude the proof of the Claim it suffices to fix 'u_i':$=u'_i|_{U_a}+u''_i\cdot\prod_{j\in J_a}q_j|_{U_a}$, for every $i\in\{1,\dots,n-d\}$, and 'v':$=v''$. ∎

To actually conclude the proof of Lemma 3.1.6 it suffices to observe that, up to shrink the neighborhood U_a of a in $\mathbb{R}^n$, we may suppose that $X_j\cap U_a=\varnothing$, for every $j\in\{1,\dots,\ell\}\setminus J_a$. Thus, for every $j\in\{1,\dots,\ell\}\setminus J_a$, there is a polynomial $q_j\in\mathcal{I}_\mathbb{Q}(X_j)$ such that $\mathcal{Z}_\mathbb{R}(q_j)\cap U_a=\varnothing$. Thus, for every $f\in\mathcal{I}^\infty_{\mathbb{R}^n}(\bigcup_{i=1}^{\ell}X_i)$, by the Claim we have:

$$\begin{aligned}f|_{U_a}&=\sum_{i=1}^{n-d}u_ip_i|_{U_a}+v\cdot\prod_{j\in J_a}q_j|_{U_a}\\&=\sum_{i=1}^{n-d}u_ip_i|_{U_a}+\frac{v}{\prod_{j\notin J_a}q_j|_{U_a}}\cdot\prod_{j=1}^{\ell}q_j|_{U_a},\end{aligned}$$

as required by Definition 3.1.1. □

The importance of the concept of $\mathbb{Q}$-approximable pair is described by the next elementary, but crucial, result.

LEMMA 3.1.7. *If (P,L) is a $\mathbb{Q}$-approximable pair of $\mathbb{R}^n$, then*
$$\mathcal{I}^\infty_{\mathbb{R}^n}(P) \subset \mathcal{I}_{\mathbb{Q}}(L)\mathscr{C}^\infty(\mathbb{R}^n), \tag{3.1.4}$$
that is, for every $f \in \mathcal{I}^\infty_{\mathbb{R}^n}(P)$, there are $u_1,\dots,u_\ell \in \mathscr{C}^\infty(\mathbb{R}^n)$ and $p_1,\dots,p_\ell$ generators of $\mathcal{I}_{\mathbb{Q}}(L)$ such that $f=\sum_{i=1}^{\ell} u_i \cdot p_i$.

PROOF. Similarly to the proof of [AK92, Assertions 2.8.1.1 & 2.8.1.2, pp. 58-59], it suffices to proceed as follows: first, construct the u_i's locally as in Definition 3.1.1; then, define the u_i's globally via a $\mathscr{C}^\infty$ partition of unity. $\square$

Given $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we set $|\alpha|:=\sum_{i=1}^n \alpha_i$ and we denote by D_α the partial derivative operator $\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$.

LEMMA 3.1.8. *Let $L\subset\mathbb{R}^n$ be a compact $\mathbb{Q}$-algebraic set and let P be a subset of $\mathbb{R}^n$ such that (P,L) is a $\mathbb{Q}$-approximable pair of $\mathbb{R}^n$. Let K be a compact neighborhood of L in $\mathbb{R}^n$ and let $f\in\mathcal{I}^\infty_{\mathbb{R}^n}(P)$. Then, for each fixed $\varepsilon>0$ and $h\in\mathbb{N}$, there exists a polynomial $s\in\mathcal{I}_{\mathbb{Q}}(L)$ such that*
$$\max\nolimits_{x\in K}|D_\alpha f(x)-D_\alpha s(x)|<\varepsilon \ \text{ for all } \alpha\in\mathbb{N}^n \text{ with } |\alpha|\le h. \tag{3.1.5}$$

PROOF. Let $p_1,\dots,p_\ell\in\mathbb{Q}[x]$ be generators of $\mathcal{I}_{\mathbb{Q}}(L)$. Thanks to Lemma 3.1.7, there exist $u_1\dots,u_\ell\in\mathscr{C}^\infty(\mathbb{R}^n)$ such that $f=\sum_{i=1}^{\ell}u_ip_i$ on $\mathbb{R}^n$. By the Weierstrass approximation theorem, for each $i\in\{1,\dots,\ell\}$, there exists $v_i\in\mathbb{R}[x]$ arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to u_i. Thus, if $s\in\mathbb{R}[x]$ is the polynomial $s:=\sum_{i=1}^{\ell}v_ip_i$, then we can assume that (3.1.5) holds. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can slightly modify the coefficients of the v_i's in such a way that each v_i belongs to $\mathbb{Q}[x]$ and (3.1.5) holds as well. Evidently, $s\in\mathcal{I}_{\mathbb{Q}}(L)$. $\square$

Recall that $|x|_n$ denotes the Euclidean norm of a vector x of $\mathbb{R}^n$, and we identify $\mathbb{R}^n$ with the vector subspace $\mathbb{R}^n\times\{0\}$ of $\mathbb{R}^{n+k}=\mathbb{R}^n\times\mathbb{R}^k$. We denote $\mathscr{C}^\nu_{\mathrm{w}}(\mathbb{R}^n)$ the set $\mathscr{C}^\nu(\mathbb{R}^n)$ equipped with the usual *weak $\mathscr{C}^\infty$ topology*.

LEMMA 3.1.9. *Let $L\subset\mathbb{R}^n$ be compact $\mathbb{Q}$-nice $\mathbb{Q}$-algebraic set, let K be a compact neighborhood of L in $\mathbb{R}^n$ and let $f\in\mathcal{I}^\infty_{\mathbb{R}^n}(L\cup(\mathbb{R}^n\setminus K))$. Then, for each fixed $\varepsilon>0$ and $h\in\mathbb{N}$, there exists a $\mathbb{Q}$-regular function $g\in\mathcal{R}^{\mathbb{Q}}(\mathbb{R}^n)$ with the following four properties:*

(i) *There exist $e\in\mathbb{N}$ and $p\in\mathbb{Q}[x]$ such that $\deg(p)\le 2e$ and $g(x)=p(x)(1+|x|_n^2)^{-e}$ for all $x\in\mathbb{R}^n$.*
(ii) *$g\in\mathcal{I}_{\mathbb{Q}}(L)$.*
(iii) *$\sup_{x\in\mathbb{R}^n}|f(x)-g(x)|<\varepsilon$.*
(iv) *$\max_{x\in K}|D_\alpha f(x)-D_\alpha g(x)|<\varepsilon$ for all $\alpha\in\mathbb{N}^n$ with $|\alpha|\le h$.*

PROOF. Let us assume $L\neq\varnothing$. Otherwise, next proof, suitably simplified, continues to work also in the case $L=\varnothing$. We adapt the strategy used in [AK92, Lemma 2.8.1] to the present situation. Let $\mathbb{S}^n$ be the standard unit sphere of $\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R}$, let $N=(0,\dots,0,1)$ be its north pole and let $\theta:\mathbb{S}^n\setminus\{N\}\to\mathbb{R}^n$ be its stereographic projection from N. Recall that $\theta(x,x_{n+1})=\frac{x}{1-x_{n+1}}$ and $\theta^{-1}(x)=\big(\frac{2x}{1+|x|_n^2},\frac{|x|_n^2-1}{1+|x|_n^2}\big)$.

Set $d := \dim(L)$. Note that $d < n$, because L is compact and hence $L \neq \mathbb{R}^n$. Since the $\mathbb{Q}$-algebraic set $L \subset \mathbb{R}^n$ is $\mathbb{Q}$-nice, for each $a \in L$ there is an open neighborhood U_a of a in $\mathbb{R}^n$ such that $\mathcal{I}^\infty_{\mathbb{R}^n}(L)\mathscr{C}^\infty(U_a) \subset \mathcal{I}_\mathbb{Q}(L)\mathscr{C}^\infty(U_a)$. Choose generators $p_1, \dots, p_\ell$ of $\mathcal{I}_\mathbb{Q}(L)$ and let $i \in \{1, \dots, \ell\}$. Write p_i as follows: $p_i = \sum_{j=0}^{d_i} p_{i,j}$, where $p_{i,j}$ is a homogeneous polynomial in $\mathbb{Q}[x]$ of degree j, and d_i is the degree of p_i. Since $L \neq \varnothing$, we can assume that each d_i is positive. Define the polynomial $P_i \in \mathbb{Q}[x, x_{n+1}]$ by $P_i(x, x_{n+1}) := \sum_{j=0}^{d_i}(1 - x_{n+1})^{d_i - j} p_{i,j}(x)$. Note that

$$P_i(N) = p_{i,d_i}(0) = 0, \tag{3.1.6}$$

$$(p_i \circ \theta)(x, x_{n+1}) = (1 - x_{n+1})^{-d_i} P_i(x, x_{n+1}), \tag{3.1.7}$$

for every $(x, x_{n+1}) \in \mathbb{S}^n \setminus \{N\}$.

Let $\rho : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$ be the standard retraction $\rho(x, x_{n+1}) := (x, x_{n+1})(|x|_n^2 + x_{n+1}^2)^{-1/2}$. Choose a $\mathscr{C}^\infty$ function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi \geq 0$ on $\mathbb{R}$, the support of ψ is contained in $\left[\frac{1}{2}, \frac{3}{2}\right]$ and $\psi(1) = 1$. Let N_+ be the half line $\{tN \in \mathbb{R}^{n+1} \mid t \geq 0\}$ and let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be the $\mathscr{C}^\infty$ extension of $f \circ \theta : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}$ defined by $F(x, x_{n+1}) := \psi(|x|_n^2 + x_{n+1}^2) f(\theta(\rho(x, x_{n+1})))$ if $(x, x_{n+1}) \notin N_+$ and $F(x, x_{n+1}) := 0$ if $(x, x_{n+1}) \in N_+$. Note that N is an interior point of $F^{-1}(0)$. Choose a compact neighborhood H of N in $\mathbb{R}^{n+1}$ such that F vanishes on H and $H \cap \theta^{-1}(L) = \varnothing$. Define the subsets L' and P' of $\mathbb{R}^{n+1}$ by

$$L' := \theta^{-1}(L) \cup \{N\},$$
$$P' := H \cup L' = H \sqcup \theta^{-1}(L)$$

and the polynomial $P_{\ell+1} \in \mathbb{Q}[x, x_{n+1}]$ by $P_{\ell+1}(x, x_{n+1}) := |x|_n^2 + x_{n+1}^2 - 1$.

Let us show that (P', L') is a $\mathbb{Q}$-approximable pair of $\mathbb{R}^{n+1}$. By (3.1.6) and (3.1.7), we know that $L' = \bigcap_{i=1}^{\ell+1} P_i^{-1}(0)$, so L' is $\mathbb{Q}$-algebraic. Let $b \in L' \setminus \operatorname{int}_{\mathbb{R}^{n+1}}(P') = \theta^{-1}(L)$, let $a := \theta(b) \in L$ and let V_b be the open neighborhood of b in $\mathbb{R}^{n+1}$ defined by $V_b := \rho^{-1}(\theta^{-1}(U_a)) \setminus (H \cup \{x_{n+1} \geq 1\})$. Note that $V_b \cap \mathbb{S}^n \subset \theta^{-1}(U_a)$, $\rho(V_b) \subset \theta^{-1}(U_a)$ and $V_b \cap (H \cup \{x_{n+1} \geq 1\}) = \varnothing$. Shrinking U_a around a if necessary, we can assume that $V_b \cap \mathbb{S}^n = \theta^{-1}(U_a)$. By [AK92, Lemma 2.5.4], there exists an open neighborhood V_b^* of b in V_b such that $\mathcal{I}^\infty_{V_b^*}(V_b^* \cap \mathbb{S}^n) \subset (p_{\ell+1})\mathscr{C}^\infty(V_b^*)$. Let $g \in \mathcal{I}^\infty_{\mathbb{R}^{n+1}}(P')$ and let $g' : \mathbb{R}^n \to \mathbb{R}$ be the $\mathscr{C}^\infty$ function defined by $g'(x) := g(\theta^{-1}(x))$. Since $g' \in \mathcal{I}^\infty_{\mathbb{R}^n}(L)$, there exist $u_1, \dots, u_\ell \in \mathscr{C}^\infty(U_a)$ such that $g'(x) = \sum_{i=1}^\ell u_i(x) p_i(x)$ for all $x \in U_a$. As a consequence, bearing in mind (3.1.7), we obtain:

$$g(y) = g'(\theta(y)) = \textstyle\sum_{i=1}^\ell u_i(\theta(y))(1 - y_{n+1})^{-d_i} P_i(y),$$

for all $y = (y_1, \dots, y_{n+1}) \in V_b \cap \mathbb{S}^n$. For each $i \in \{1, \dots, \ell\}$, we define the $\mathscr{C}^\infty$ function $v_i : V_b \to \mathbb{R}$ by setting $v_i(y) := u_i(\theta(\rho(y)))(1 - y_{n+1})^{-d_i}$. It follows that the $\mathscr{C}^\infty$ function $G := \sum_{i=1}^\ell v_i \cdot P_i|_{V_b} \in \mathscr{C}^\infty(V_b)$ coincides with g on $V_b \cap \mathbb{S}^n$. Thus, $g - G = v_{\ell+1} P_{\ell+1}$ on V_b^* for some $v_{\ell+1} \in \mathscr{C}^\infty(V_b^*)$, and hence $g = \sum_{i=1}^{\ell+1} v_i P_i$ on V_b^*. This proves that $\mathcal{I}^\infty_{\mathbb{R}^{n+1}}(P')\mathscr{C}^\infty(V_b^*) \subset (P_1, \dots, P_{\ell+1})\mathscr{C}^\infty(V_b^*)$. Hence (P', L') is a $\mathbb{Q}$-approximable pair.

By Lemma 3.1.8, there exists a polynomial $S \in \mathcal{I}_\mathbb{Q}(L')$ arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to F. In particular, we can assume that $|F(y) - S(y)| < \varepsilon$ for all $y \in \mathbb{S}^n$. Define the regular function $g : \mathbb{R}^n \to \mathbb{R}$ by setting $g(x) := S(\theta^{-1}(x))$. Evidently, (ii) & (iii) hold, because $S \in \mathcal{I}_\mathbb{Q}(\theta^{-1}(L))$ and $|f(x) - g(x)| = |F(\theta^{-1}(x)) - S(\theta^{-1}(x))| < \varepsilon$ for all $x \in \mathbb{R}^n$. Write S as follows: $S = \sum_{i=0}^{e} S_i$, where $S_i \in \mathbb{Q}[x, x_{n+1}]$ is homogeneous of degree i, and e is the degree of S. It follows that $g(x) = (1 + |x|_n^2)^{-e} p(x)$, where $p(x) := \sum_{i=0}^{e} (1 + |x|_n^2)^{e-i} S_i(2x, -1 + |x|_n^2) \in \mathbb{Q}[x]$ is a polynomial of degree $\leq 2e$. This proves (i). It remains to prove (iv). Let $\Psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the $\mathscr{C}^\infty$ map $\Psi(x) := \theta^{-1}(x)$. Since the pullback map $\Psi^* : \mathscr{C}^\infty_{\mathrm{w}}(\mathbb{R}^{n+1}) \to \mathscr{C}^\infty_{\mathrm{w}}(\mathbb{R}^n)$ is continuous, it follows that $\Psi^*(S) = S \circ \Psi = g$ can be chosen arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\Psi^*(F) = F \circ \Psi = f$, as desired. $\square$

3.2. The workhorse theorem over $\mathbb{Q}$ and applications

Let U be a non-empty open subset of $\mathbb{R}^n$, let $x \in U$, let $\alpha \in \mathbb{N}^n$, let M be a $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^m$, let $j : M \hookrightarrow \mathbb{R}^m$ be the inclusion map and let $f : U \to M$ be a $\mathscr{C}^\infty$ map. To abbreviate notations, we write $D_\alpha f(x)$ in place $D_\alpha(j \circ f)(x)$.

The following is a version 'over $\mathbb{Q}$' of the workhorse approximation theorem of Akbulut and King, see [AK92, Theorems 2.8.3, pp. 63-64].

Lemma 3.2.1. *Let $L \subset \mathbb{R}^n$ be a compact $\mathbb{Q}$-algebraic set, let U be an open neighborhood of L in $\mathbb{R}^n$ whose closure $\overline{U}$ in $\mathbb{R}^n$ is compact, let $W \subset \mathbb{R}^k$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set and let $f : U \to W$ be a $\mathscr{C}^\infty$ map such that the restriction $f|_L : L \to W$ of f to L is a regular map. Suppose that L is $\mathbb{Q}$-nice and $f|_L$ is $\mathbb{Q}$-regular. Choose $\varepsilon > 0$, $h \in \mathbb{N}$ and an open neighborhood U' of L in $\mathbb{R}^n$ such that $\overline{U'} \subset U$. Then there exist an algebraic subset Z of $\mathbb{R}^{n+k}$, an open subset Z_0 of Z and a regular map $\eta : Z \to W$ with the following four properties:*

(i) *Z is $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic and η is $\mathbb{Q}$-regular.*
(ii) *Let $\iota : U' \hookrightarrow \mathbb{R}^{n+k}$ be the inclusion map and let $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the natural projection onto the first n coordinates, i.e., $\iota(x) := (x, 0)$ for all $x \in U'$ and $\pi(x, y) := x$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$. Then $L \times \{0\} \subset Z_0$, $\pi(Z_0) = U'$, the restriction $\pi|_{Z_0} : Z_0 \to U'$ is a $\mathscr{C}^\infty$ diffeomorphism, and the $\mathscr{C}^\infty$ map $\sigma : U' \to Z$, defined by $\sigma(x) := (\pi|_{Z_0})^{-1}(x)$ for all $x \in U'$, satisfies the following inequalities:*
$$\sup_{x \in U'} |D_\alpha \sigma(x) - D_\alpha \iota(x)|_n < \varepsilon$$
for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq h$.
(iii) *$\eta(x, 0) = f(x)$ for all $x \in L$.*
(iv) *$\sup_{x \in U'} ||D_\alpha f(x) - D_\alpha(\eta \circ \sigma)(x)|_k < \varepsilon$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq h$.*

Proof. Let T be an open tubular neighborhood of W in $\mathbb{R}^k$ and let $\rho : T \to W$ the closest point map, which is a $\mathscr{C}^\infty$ map. By Lemma 2.1.3(i) and Remark 2.1.4, there exists a $\mathbb{Q}$-regular map $F : \mathbb{R}^n \to \mathbb{R}^k$ such that $F(x) = f(x)$ for all $x \in L$. Let U'' and U''' be open neighborhoods of L in $\mathbb{R}^n$ such that $\overline{U'} \subset U''$, $\overline{U''} \subset U'''$ and $\overline{U'''} \subset U$. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function such that $\psi = 1$ on U''' and $\psi = 0$ on $\mathbb{R}^n \setminus U$ in $\mathbb{R}^n$. Define the $\mathscr{C}^\infty$ map $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^k$ by $\widetilde{f}(x) := \psi(x)(f(x) - F(x))$ for all $x \in U$ and $\widetilde{f}(x) := 0$ for all $x \in \mathbb{R}^n \setminus U$. Note that $\widetilde{f} = (\jmath \circ f) - F$ on U'' and $\widetilde{f}(x) = 0$ for all $x \in L \cup (\mathbb{R}^n \setminus \overline{U})$. Applying Lemma 3.1.9 to each component of $\widetilde{f}$, we obtain a $\mathbb{Q}$-regular map $\widetilde{g} : \mathbb{R}^n \to \mathbb{R}^k$ such that $\widetilde{g} = 0$ on L and $\widetilde{g}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\widetilde{f}$. Define the regular map $g : \mathbb{R}^n \to \mathbb{R}^k$ as $g := \widetilde{g} + F$. Note that g is $\mathbb{Q}$-regular,

$g = \jmath \circ f$ on L and $g|_{U'''}$ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $(\jmath \circ f)|_{U'''}$. In particular, we can assume that $g(\overline{U''}) \subset T$. Choose $\delta > 0$ such that $\max_{x \in \overline{U''}} |g(x) - \rho(g(x))|_k < \delta$ and $\{y \in \mathbb{R}^k : \mathrm{dist}_{\mathbb{R}^k}(y, \rho(g(\overline{U''}))) < \delta\} \subset T$. Consider $g|_{U''}$ and $f|_{U''}$ as $\mathscr{C}^\infty$ maps with from U'' to T. We know that $g|_{U''} = f|_{U''}$ on L and $g|_{U''}$ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $f|_{U'''}$.

Consider the normal bundle $N = \{(w, y) \in W \times \mathbb{R}^k : y \in T_w(W)^\perp\}$ of W in $\mathbb{R}^k$. Observe that $N = \beta^*(\mathbb{E}_{k,k-\dim(W)}) \subset \mathbb{R}^{2k}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set of dimension k since β is the mapping classifying the normal bundle of W in $\mathbb{R}^k$, which is $\mathbb{Q}$-regular by Lemma 2.2.5, and $\mathbb{E}_{k,k-\dim(W)}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set by Lemma 2.2.3. Define the regular map $\gamma : \mathbb{R}^{n+k} \to \mathbb{R}^{2k} = \mathbb{R}^k \times \mathbb{R}^k$ and the algebraic subset Z of $\mathbb{R}^{n+k}$ as $\gamma(x, y) := (g(x) + y, y)$ and $Z := \gamma^{-1}(N)$. Note that γ is $\mathbb{Q}$-regular, since g is. Moreover, it turns out that γ is transverse to N in $\mathbb{R}^{2k}$. Indeed, given any $(x, y) \in Z$ and set $w := g(x) + y \in W$, we have: $T_{(w,y)}(N) = T_w(W) \times T_w(W)^\perp$, $d\gamma_{(x,y)}(\mathbb{R}^{n+k})$ contains the diagonal $\Delta = \{(y, y) \in \mathbb{R}^{2k} : y \in \mathbb{R}^k\}$ of $\mathbb{R}^{2k}$, $\Delta \cap (T_w(W) \times T^\perp_w(W)) = \{(0, 0)\}$ and hence $d\gamma_{(x,y)}(\mathbb{R}^{n+k}) + T_{(w,y)}(N) = \mathbb{R}^{2k}$. By Lemmas 2.1.3(v) & 2.1.5, it follows that $Z \subset \mathbb{R}^{n+k}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.

Let $\eta : Z \to W$ be the $\mathbb{Q}$-regular map given by $\eta(x, y) := g(x) + y$, let Z_1 be the open subset $\{(x, y) \in Z : x \in U'', |y| < \delta\}$ of Z and let $v : U'' \to \mathbb{R}^k$ be the $\mathscr{C}^\infty$ map defined by $v(x) := \rho(g(x)) - g(x)$ for all $x \in U''$. By construction, Z_1 is the graph of v and $v(x) = 0$ for all $x \in L$. Thus, $L \times \{0\} \subset Z_1$, $\pi|_{Z_1} : Z_1 \to U''$ is a $\mathscr{C}^\infty$ diffeomorphism. Moreover, if $\sigma_1 : U'' \to Z_1$ denotes $(\pi|_{Z_1})^{-1}$ and x is a point of U'', then $\sigma_1(x) = (x, v(x))$ and hence $(\eta \circ \sigma_1)(x) = g(x) + v(x) = \rho(g(x))$. In particular, if $x \in L$, then $(\eta \circ \sigma_1)(x) = \rho(g(x)) = \rho(f(x)) = f(x)$. Since the push-forward $\rho_* : \mathscr{C}^\infty_\mathrm{w}(U'', T) \to \mathscr{C}^\infty_\mathrm{w}(U'', W)$ is continuous and $g|_{U''}$ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $f|_{U''}$, we can assume that $\rho_*(g|_{U''}) = \rho \circ g|_{U''}$ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\rho_*(f|_{U''}) = \rho \circ f|_{U''} = f|_{U''}$. In particular, $\rho \circ g|_{U''}$ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $g|_{U''}$, which is equivalent to say that σ_1 is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to the inclusion map $U'' \hookrightarrow \mathbb{R}^{n+k}$. Now it suffices to set $Z_0 := \{(x, y) \in Z : x \in U', |y| < \delta\}$. The proof is complete. $\square$

The following is a strong version 'over $\mathbb{Q}$' of [AK81b, Proposition 2.8], see also [AK92, Theorem 2.8.4, pp. 65-66].

THEOREM 3.2.2. *Let M be a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^s$ of dimension d, let L be a $\mathbb{Q}$-algebraic subset of M, let $\beta : M \to \mathbb{G}_{d,s-d}$ be the normal bundle map of M in $\mathbb{R}^s$, let $W \subset \mathbb{R}^k$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set and let $f : M \to W$ be a $\mathscr{C}^\infty$ map. Suppose that the following three conditions are verified.*

(1) *$L \subset \mathbb{R}^s$ is a $\mathbb{Q}$-nice projectively $\mathbb{Q}$-closed $\mathbb{Q}$-algebraic set.*
(2) *W has projectively $\mathbb{Q}$-algebraic homology.*
(3) *If $L \neq \varnothing$, then the restrictions $\beta|_L : L \to \mathbb{G}_{s,s-d}$ and $f|_L : L \to W$ are $\mathbb{Q}$-regular maps.*

Then there exist a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $V \subset \mathbb{R}^{s+t}$ for some $t \in \mathbb{N}$, a $\mathscr{C}^\infty$ diffeomorphism $\varphi : M \to V$ and a $\mathbb{Q}$-regular map $\xi : V \to W$ with the following three properties:

(i) *$L \times \{0\} \subset V$.*

(ii) $\varphi(x) = (x, 0)$ *for all* $x \in L$, *and* φ *is arbitrarily* $\mathscr{C}^\infty_{\mathrm{w}}$*-close to the inclusion map* $M \hookrightarrow \mathbb{R}^{s+t}$.
(iii) $\xi(x, 0) = f(x)$ *for all* $x \in L$, *and* $\xi \circ \varphi$ *is arbitrarily* $\mathscr{C}^\infty_{\mathrm{w}}$*-close to* f.

Proof. Let us give the proof in the case $L \neq \varnothing$. In the case $L = \varnothing$, the proof, suitably simplified, works as well. Thanks to Theorem 2.4.5, by increasing s if necessary, there exist a d-dimensional projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $X \subset \mathbb{R}^s$, a $\mathbb{Q}$-regular map $g : X \to W$, a compact $\mathscr{C}^\infty$ submanifold S of $\mathbb{R}^{s+1} = \mathbb{R}^s \times \mathbb{R}$ and a $\mathscr{C}^\infty$ map $G : S \to W$ such that

(iv) $M \cap X = \varnothing$,
(v) $S \cap (\mathbb{R}^s \times (-1, 1)) = (M \sqcup X) \times (-1, 1)$,
(vi) $G(x, 0) = f(x)$ for all $x \in M$ and $G(x, 0) = g(x)$ for all $x \in X$.

Let $\mathbb{G} := \mathbb{G}_{s+1,s-d}$ and let $\mathcal{B} : S \to \mathbb{G}$ be the normal bundle map of S in $\mathbb{R}^{s+1} = \mathbb{R}^s \times \mathbb{R}$. Using (iv), (v), Lemmas 2.2.5 & 2.1.3(iii) and Remark 2.1.4, we obtain that the restriction of $\mathcal{B}$ to $(L \times \{0\}) \sqcup (X \times \{0\}) = (L \sqcup X) \times \{0\}$ is a $\mathbb{Q}$-regular map.

Let us recall the construction of a tubular neighborhood U of S in $\mathbb{R}^{s+1}$ and the closest point map $\rho : U \to S$. Let $E := E_{s+1,s-d} = \{(N, y) \in \mathbb{G} \times \mathbb{R}^{s+1} : Ny = y\} \subset \mathbb{R}^{(s+1)^2+s+1} = \mathbb{R}^{(s+1)(s+2)}$ be the universal vector bundle over $\mathbb{G}$, let $\mathcal{B}^*(E) := \{(x, y) \in S \times \mathbb{R}^{s+1} : \mathcal{B}(x)y = y\}$ be the total space of the corresponding pullback bundle with projection map $\Pi : \mathcal{B}^*(E) \to S$ given by $\Pi(x, y) := x$, and let $\theta : \mathcal{B}^*(E) \to \mathbb{R}^{s+1}$ be the $\mathscr{C}^\infty$ map defined by $\theta(x, y) := x + y$. By the inverse function theorem, there exists an open neighborhood U_0 in $\mathcal{B}^*(E)$ of the zero section $S \times \{0\}$ of $\mathcal{B}^*(E)$ such that $U := \theta(U_0)$ is an open neighborhood of S in $\mathbb{R}^{s+1}$ and the restriction $\theta' : U_0 \to U$ of θ from U_0 to U is a $\mathscr{C}^\infty$ diffeomorphism. Define $\rho : U \to S$ by $\rho(z) := \Pi((\theta')^{-1}(z))$. Restricting U around S in $\mathbb{R}^{s+1}$ if necessary, we can assume that ρ is the closest point map. Since S is compact, we can also assume that the closure $\overline{U}$ of U in $\mathbb{R}^{s+1}$ is compact as well.

Let us define the $\mathscr{C}^\infty$ map $\Theta : U \to E$ by $\Theta(z) := (\mathcal{B}(\rho(z)), z - \rho(z))$. The $\mathscr{C}^\infty$ map Θ is transverse in E to the zero section $\mathbb{G} \times \{0\}$ of E, and $\Theta^{-1}(\mathbb{G} \times \{0\}) = S$. Note that $\Theta(z) = (\mathcal{B}(z), 0)$ for all $z \in (L \sqcup X) \times \{0\}$; thus, $\Theta|_{(L \sqcup X) \times \{0\}}$ is a $\mathbb{Q}$-regular map. Moreover, by Corollary 3.1.3 and Lemma 3.1.4, $X \times \{0\}$ and $L \times \{0\}$ are $\mathbb{Q}$-nice. Evidently, the same is true for their disjoint union $(L \sqcup X) \times \{0\}$.

Let $F : U \to W$ be the $\mathscr{C}^\infty$ map defined by $F := G \circ \rho$. By (3), (vi), Lemma 2.1.3(iii) and Remark 2.1.4, we have that $F|_{(L \sqcup X) \times \{0\}}$ is a $\mathbb{Q}$-regular map. Consider the product map $\Theta \times F : U \to E \times W$. Note that $(\Theta \times F)|_{(L \sqcup X) \times \{0\}}$ is a $\mathbb{Q}$-regular, because both $\Theta|_{(L \sqcup X) \times \{0\}}$ and $F|_{(L \sqcup X) \times \{0\}}$ are. By Lemma 2.1.3(vi), $E \times W$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^{(s+1)(s+2)+k}$.

Thanks to the properties of $\Theta \times F$ just described, we can apply Lemma 3.2.1 to $\Theta \times F$. Chosen an open neighborhood U' of S in $\mathbb{R}^{s+1}$ with $\overline{U'} \subset U$, there exist a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset Z of $\mathbb{R}^{s+1} \times \mathbb{R}^{(s+1)(s+2)+k} = \mathbb{R}^{s+t}$, an open subset Z_0 of Z and a $\mathbb{Q}$-regular map $\eta : Z \to E \times W$ satisfying the following properties:

(vii) Let $\iota : U' \hookrightarrow \mathbb{R}^{s+t}$ be the inclusion map and let $\pi : \mathbb{R}^{s+t} = \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^{(s+1)(s+2)+k} \to \mathbb{R}^s \times \mathbb{R}$ be the natural projection onto the first $s+1$ coordinates, i.e., $\iota(x, x_{s+1}) := (x, x_{s+1}, 0)$ and $\pi(x, x_{s+1}, y) := (x, x_{s+1})$. Then $(L \sqcup X) \times \{0\} \times \{0\} \subset Z_0$, $\pi(Z_0) = U'$, the restriction $\pi|_{Z_0} : Z_0 \to U'$

is a $\mathscr{C}^\infty$ diffeomorphism, and the $\mathscr{C}^\infty$ map $\sigma : U' \to \mathbb{R}^{s+t}$, defined by $\sigma(x, x_{s+1}) := (\pi|_{Z_0})^{-1}(x, x_{s+1})$ for all $(x, x_{s+1}) \in U'$, is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to ι.

(viii) $\eta(x, 0, 0) = (\Theta \times F)(x, 0)$ for all $x \in L \sqcup X$.

(ix) The $\mathscr{C}^\infty$ map $\widehat{\eta} : U' \to E \times W$, defined by $\widehat{\eta}(x, x_{s+1}) := \eta(\sigma(x, x_{s+1}))$, is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $(\Theta \times F)|_{U'}$.

Choose an open neighborhood U'' of S in $\mathbb{R}^{s+1}$ such that $\overline{U''} \subset U'$ (actually, we can also assume that $\overline{U''}$ is a compact $\mathscr{C}^\infty$ manifold with boundary). Set $Z_1 := (\pi|_{Z_0})^{-1}(U'')$. Since $\Theta \times F$ is transverse to $(\mathbb{G} \times \{0\}) \times W$ in $E \times W$, by (vii), (viii), (ix) and [BCR98, Theorem 14.1.1], we have that $S' := \widehat{\eta}^{-1}((\mathbb{G} \times \{0\}) \times W)$ is a compact $\mathscr{C}^\infty$ submanifold of U'' containing $(L \sqcup X) \times \{0\}$ and there exists a $\mathscr{C}^\infty$ diffeomorphism $\psi_1 : U'' \to U''$ arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\mathrm{id}_{U''}$ such that $\psi_1(S) = S'$ and $\psi = \mathrm{id}_{U''}$ on $(L \sqcup X) \times \{0\}$. Moreover, bearing in mind (vii) and Lemma 2.1.5, we have that $S'' := \eta^{-1}((\mathbb{G} \times \{0\}) \times W) \subset \mathbb{R}^{s+t}$ is a $\mathbb{Q}$-algebraic set such that $S''_1 := S'' \cap Z_1 = (\pi|_{Z_1})^{-1}(S') \subset \mathrm{Reg}^{\mathbb{R}|\mathbb{Q}}(S'')$. In addition, the $\mathscr{C}^\infty$ embedding $\psi_2 : S \to \mathbb{R}^{s+t}$, sending (x, x_{s+1}) to $(\pi|_{Z_1})^{-1}(\psi_1(x, x_{s+1}))$, is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to the inclusion map $j_S : S \hookrightarrow \mathbb{R}^{s+t}$ sending (x, x_{s+1}) to $(x, x_{s+1}, 0)$, $\psi_2 = j_S$ on $(L \sqcup X) \times \{0\}$ and $\psi_2(S) = S''_1$. Note that the set S''_1 is both compact and open in S''; thus, S''_1 is the union of some connected components of S'' and $S''_2 := S'' \setminus S''_1$ is a closed subset of $\mathbb{R}^{s+t}$ (recall that an algebraic set, as S'', has only finitely many connected components). Since ψ_2 is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to j_S, the coordinate hyperplane $\{x_{s+1} = 0\}$ of $\mathbb{R}^{s+t} = \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^{(s+1)(s+2)+k}$ is transverse to S''_1 in $\mathbb{R}^{s+t}$, $S''_1 \cap \{x_{s+1} = 0\} = M' \sqcup X$ for some compact $\mathscr{C}^\infty$ submanifold M' of $\mathbb{R}^{s+t}$ containing $L \times \{0\}$ and there exists a $\mathscr{C}^\infty$ embedding $\psi_3 : M \to \mathbb{R}^{s+t}$ arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to the inclusion map $j_M : M \hookrightarrow \mathbb{R}^{s+t}$, i.e., $j_M(x) := (x, 0, 0)$, such that $M' = \psi_3(M)$ and $\psi_3 = j_M$ on $L \times \{0\}$.

Let K be a compact neighborhood of S''_1 in $\mathbb{R}^{s+t}$ such that $K \cap S''_2 = \varnothing$ and let $\pi_{s+1} : \mathbb{R}^{s+t} = \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^{(s+1)(s+2)+k} \to \mathbb{R}$ be the projection $\pi_{s+1}(x, x_{s+1}, y) := x_{s+1}$. By Lemma 2.1.7(i)(iv), we know that the algebraic set $(L \sqcup X) \times \{0\} \times \{0\} \subset \mathbb{R}^{s+t}$ is projectively $\mathbb{Q}$-closed; thus, there exists an overt polynomial $q \in \mathbb{Q}[x, x_{s+1}, y]$ such that $\mathcal{Z}_{\mathbb{R}}(q) = (L \sqcup X) \times \{0\} \times \{0\}$. Since q is a proper function, replacing q with Cq^2 for some rational number $C > 0$ if necessary, we can assume that q is overt, $\mathcal{Z}_{\mathbb{R}}(q) = (L \sqcup X) \times \{0\} \times \{0\}$, $q \geq 0$ on $\mathbb{R}^{s+t}$ and $q \geq 2$ on $\mathbb{R}^{s+t} \setminus K$. Let K' be a compact neighborhood of S''_1 in $\mathrm{int}_{\mathbb{R}^{s+t}}(K)$. Using a $\mathscr{C}^\infty$ partition of unity subordinated to $\{\mathrm{int}_{\mathbb{R}^{s+t}}(K), \mathbb{R}^{s+t} \setminus K'\}$, we can define a $\mathscr{C}^\infty$ function $h : \mathbb{R}^{s+t} \to \mathbb{R}$ such that $h = \pi_{s+1}$ on K' and $h = q$ on $\mathbb{R}^{s+t} \setminus K$. Apply Lemma 3.1.9 to $h - q$, obtaining a $\mathbb{Q}$-regular function $u' : \mathbb{R}^{s+t} \to \mathbb{R}$ with the following properties:

(x) There exist $e \in \mathbb{N}$ and a polynomial $p \in \mathbb{Q}[x_1, \ldots, x_{s+t}]$ of degree $\leq 2e$ such that $u'(x) = p(x)(1 + |x|^2_{s+t})^{-e}$ for all $x \in \mathbb{R}^{s+t}$.

(xi) $(L \sqcup X) \times \{0\} \times \{0\} \subset \mathcal{Z}_{\mathbb{R}}(u')$.

(xii) $\sup_{x \in \mathbb{R}^{s+t}} |h(x) - q(x) - u'(x)| < 1$.

(xiii) u' is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\pi_{s+1} - q$ on $\mathrm{int}_{\mathbb{R}^{s+t}}(K')$.

Let $u : \mathbb{R}^{s+t} \to \mathbb{R}$ be the $\mathbb{Q}$-regular map given by $u := u' + q$, and let $v \in \mathbb{Q}[x_1, \ldots, x_{s+t}]$ be the polynomial $v(x) := q(x)(1 + |x|^2_{s+t})^e + p(x)$. Combining (x) with the fact that q is non-constant and overt, we immediately deduce that $u(x) = (1 + |x|^2_{s+t})^{-e} v(x)$ and v is $\mathbb{Q}$-overt. By (xi), (xii) and (xiii), we know that $(L \sqcup$

$X) \times \{0\} \times \{0\} \subset \mathcal{Z}_\mathbb{R}(u)$, $u > 1$ on $\mathbb{R}^{s+t} \setminus K$ and u is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to π_{s+1} on $\mathrm{int}_{\mathbb{R}^{s+t}}(K')$. In particular, 0 is a regular value of the restriction $u|_{S''_1}$ of u to S''_1, $S''_1 \cap u^{-1}(0) = (u|_{S''_1})^{-1}(0)$ is equal to $M'' \sqcup X$ for some compact $\mathscr{C}^\infty$ submanifold M'' of $\mathbb{R}^{s+t}$ containing $L \times \{0\} \times \{0\}$ and there exists a $\mathscr{C}^\infty$ embedding $\psi_4 : M' \to \mathbb{R}^{s+t}$ arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to the inclusion map $j_{M'} : M' \hookrightarrow \mathbb{R}^{s+t}$ such that $M'' = \psi_4(M')$ and $\psi_4 = j_{M'}$ on $L \times \{0\} \times \{0\}$. Since $M'' \sqcup X = S'' \cap u^{-1}(0)$, Lemma 2.1.5 ensures that $M'' \sqcup X \subset \mathbb{R}^{s+t}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. On the other hand, we also have that $M'' \sqcup X = S'' \cap u^{-1}(0) = S'' \cap v^{-1}(0)$; thus, Lemma 2.1.7(ii) implies that $M'' \sqcup X$ is projectively $\mathbb{Q}$-closed.

Let $T''' := T' \cap \{g'' = 0\}$. By the above properties of g'', it follows that: $T''' = M'' \sqcup Y$ where M'' is an approximation of M' in T' fixing L, T''' is an algebraic set because it is equal to $T'' \cap \{g'' = 0\}$ and T''' is nonsingular in fact $\pi_{n+1}|_{T'}$ (and hence $g''|_{T'}$) is transverse to 0 in $\mathbb{R}$. By Lemma 2.1.3(v), we deduce that T''' is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set, hence M'' is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set as well by Lemma 1.6.14. Since M'' is a $\mathbb{Q}$-algebraic subset of $M'' \sqcup X$ which is projectively $\mathbb{Q}$-closed, Lemma 2.1.7(ii) ensures that M'' is projectively $\mathbb{Q}$-closed as well.

Finally let $\pi_W : E \times W \to W$ be the natural projection and let $\xi := \pi_W \circ \eta|_{M''}$. Using the first part of b), it is easy to see that M'' and P satisfy the conclusion of the theorem. $\square$

3.2.1. A relative Nash-Tognoli theorem 'over $\mathbb{Q}$'. In the next result we prove a version 'over $\mathbb{Q}$' of [AK81b, Theorem 2.10].

THEOREM 3.2.3. *Let M be a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$ and let M_i for $i = 1, \dots, \ell$, be $\mathscr{C}^\infty$ hypersurfaces of M in general position. Then there exist a $\mathscr{C}^\infty$-diffeotopy $\{h_t\}_{t\in[0,1]}$ of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ (for some $k \in \mathbb{N}$) arbitrarily $\mathscr{C}^\infty_\mathrm{w}$-close to $\mathrm{id}_{\mathbb{R}^{n+k}}$ which simultaneously takes $M \times \{0\}$ and each $M_i \times \{0\}$ to projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets.*

In particular, there exist projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets $M', M'_1, \dots, M'_\ell$ and a $\mathscr{C}^\infty$ diffeomorphism $h : M \to M'$ such that $M'_i \subset M'$ and $h(M_i) = M'_i$ for all $i \in \{1, \dots, \ell\}$.

PROOF. Since the canonical line bundle over $\mathbb{G}_{1,n-2}$ classifies line bundles and $\mathbb{G}_{1,n-1}$ is the Thom space of the canonical bundle over $\mathbb{G}_{1,n-2}$, for every $i \in \{1, \dots, \ell\}$, increasing n if necessary, there is a smooth map $f_i : M \to \mathbb{G}_{1,n-1}$ that is transverse to $\mathbb{G}_{1,n-2} \subset \mathbb{G}_{1,n-1}$ and such that $f_i^{-1}(\mathbb{G}_{1,n-2}) = M_i$. Let $W = \mathbb{G}_{1,n-1} \times \cdots \times \mathbb{G}_{1,n-1} \subset \mathbb{R}^{n^2\ell}$ be the ℓ-times product of $\mathbb{G}_{1,n-1}$ with itself and let $f : M \to W$ be the smooth map having f_i as the i-th coordinate. Recall that, by Lemma 2.1.3 (f) and Lemma 2.2.1 W is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. Moreover, since each $\mathbb{G}_{1,n-1}$ has projectively $\mathbb{Q}$-algebraic homology, Lemma 2.4.7 implies that W has projectively $\mathbb{Q}$-algebraic homology as well. Hence, by Theorem 3.2.2, there is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $M' \subset \mathbb{R}^n \times \mathbb{R}^k$, for some $k \in \mathbb{N}$, a $\mathbb{Q}$-regular map $r : M' \to W$ and a diffeotopy $\{h'_t\}_{t\in[0,1]}$ of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ such that $h'_1(M) = M'$ and $r \circ h'_1$ is a $\mathscr{C}^\infty$-approximation of f. For every $i \in \{1, \dots, \ell\}$, define $W_i = \mathbb{G}_{1,n-1} \times \cdots \times \mathbb{G}_{1,n-2} \times \cdots \times \mathbb{G}_{1,n-1}$ with $\mathbb{G}_{1,n-2}$ at the i-th place. Observe that if we choose the approximations sufficiently $\mathscr{C}^\infty_\mathrm{w}$ close then $r \circ h_1$ is transverse to W_i, since f is so. In addition, there is a diffeotopy $\{h^i_t\}_{t\in[0,1]}$ of M' arbitrarily $\mathscr{C}^\infty$ close to $\mathrm{id}_{M'}$ such that $h^i_1(h'_1(M_i)) = r^{-1}(W_i)$. Moreover, by Lemma 2.1.3 we deduce that $r^{-1}(W_i) \subset M'$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. By [AK81b, Lemma 2.9] there is a diffeomorphism $h'' : M' \to M'$ such that $(h'' \circ h')(M_i) = r^{-1}(W_i)$ for every $i \in \{1, \dots, l\}$. Setting $h_t := h''_t \circ h'_t$ and $M'_i := r^{-1}(W_i)$, we are done. $\square$

We will see further deep generalizations of latter result in Section 4.1.

3.3. $\mathbb{Q}$-Algebraic blowing down with approximation

Here we propose a version 'over $\mathbb{Q}$' of the blowing down lemma by Akbulut-King. With respect to the original result [AK92, Lemma 2.6.1] we are also able to produce approximating algebraic sets.

Definition 3.3.1. Let V be a $\mathbb{Q}$-algebraic subset of $\mathbb{R}^n$ and let U be a Zariski open subset of V. We say that V is $\mathbb{Q}$-*determined on* U if $U \subset \mathrm{Reg}^*(V)$.

Remark 3.3.2. Let V be a $\mathbb{Q}$-algebraic subset of $\mathbb{R}^n$. Then, in terms of Definition 3.3.1, V is $\mathbb{Q}$-determined means that V is $\mathbb{Q}$-determined on $\mathrm{Reg}(V)$.

Lemma 3.3.3. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be $\mathbb{Q}$-algebraic sets. Let A be a $\mathbb{Q}$-algebraic subset of X and let $p : X \to Y$ be a $\mathbb{Q}$-regular map. Suppose that $X \subset \mathbb{R}^n$ is projectively $\mathbb{Q}$-closed and $\mathbb{Q}$-determined. Let $X \cup_p Y$ be the adjunction space of X and Y via p, namely the quotient topological space obtained from the disjoint union $X \sqcup Y$ by identifying every $x \in A$ with $p(x) \in Y$, and let $\pi : X \sqcup Y \to X \cup_p Y$ be the quotient map. Then, there are a $\mathbb{Q}$-algebraic set $V \subset \mathbb{R}^{n+m+1}$, $\mathbb{Q}$-regular maps $f : X \to V$ and $g : Y \to V$ and a homeomorphism $h : V \to X \cup_p Y$ such that:*

(i) *$h \circ f = \pi|_X$ and $h \circ g = \pi|_Y$; in other words, the following diagram commutes.*

$$\begin{tikzcd}
X \arrow[dr, hook] \arrow[rrr, bend left, "f"] & & & \\
& X \sqcup Y \arrow[r, "\pi"] & X \cup_p Y & V \arrow[l, "h"'] \\
Y \arrow[ur, hook] \arrow[rrr, bend right, "g"'] & & &
\end{tikzcd}$$

(ii) *g coincides with the inclusion map $\jmath|_Y : Y \hookrightarrow \mathbb{R}^{n+m+1}$, thus $Y_0 := g(Y) \subset \mathbb{R}^{n+m+1}$ is a $\mathbb{Q}$-algebraic subset of V.*

(iii) *$f| : X \setminus A \to V \setminus Y_0$ is a $\mathbb{Q}$-biregular isomorphism; in particular, $V = f(X \setminus A) \sqcup Y_0$.*

(iv) *If $\dim(Y) < \dim(X \setminus A)$ then $\mathrm{Sing}(V) \subset Y_0 \cup f(\mathrm{Sing}(X))$ and $V \subset \mathbb{R}^{n+m+1}$ is $\mathbb{Q}$-determined on the Zariski open subset $\mathrm{Reg}(V) \setminus Y_0$.*

(v) *$\jmath_V \circ f$ is arbitrarily $\mathscr{C}^0_{\mathrm{w}}$ close to $\jmath_Y \circ p$, where $\jmath_Y : Y \hookrightarrow \mathbb{R}^{n+m+1} = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ as $y \to (0, y, 0)$ and $\jmath_V : V \hookrightarrow \mathbb{R}^{n+m+1}$ denote the inclusion maps.*

(vi) *$(\jmath_V \circ f)|_{\mathrm{Reg}(X)}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $(\jmath_Y \circ p)|_{\mathrm{Reg}(X)}$.*

Proof. Let $a, b \in \mathbb{Q}[x]$ such that $A = \mathcal{Z}_\mathbb{R}(a)$ and $X = \mathcal{Z}_\mathbb{R}(b)$. Since $p : A \to Y$ is $\mathbb{Q}$-regular, let $p_1, \dots, p_m, q \in \mathbb{Q}[x]$ such that $\mathcal{Z}_\mathbb{R}(q) = \varnothing$ and $p(x) =$

$\big(\frac{p_1}{q}(x),\dots,\frac{p_m}{q}(x)\big)$ for every $x\in A$ by Lemma 2.1.3(i). Let $C_1\in\mathbb{Q}\setminus\{0\}$ arbitrarily small. Define the $\mathbb{Q}$-regular maps $p':A\to Y\times\{0\}$ and $P':A\to\mathbb{R}^m\times\mathbb{R}$ by $p'(x)=(p(x),0)$ and $P'(x)=(P(x),C_1a(x))$, respectively. Denote by $Y':=Y\times\{0\}\subset\mathbb{R}^m\times\mathbb{R}$. Observe that P' extends p' and the topological space $X\sqcup_{p'}Y'$ is homeomorphic to $X\sqcup_p Y$. Thus it suffices to prove the following claim:

Claim: *Let $X\subset\mathbb{R}^n$ be a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set, let $Y\subset\mathbb{R}^m$ be a $\mathbb{Q}$-algebraic set and let $p:X\to\mathbb{R}^m$ be a $\mathbb{Q}$-regular map. Let $X\cup_p Y$ be the quotient topological space obtained from the disjoint union $X\sqcup Y$ by identifying every $y\in Y$ with the points of the fiber $p^{-1}(y)$. Then, there are a $\mathbb{Q}$-algebraic set $V\subset\mathbb{R}^{n+m}$, $\mathbb{Q}$-regular maps $f:X\to V$ and $g:Y\to V$ and a homeomorphism $h:X\cup_p Y\to V$ satisfying conditions* (i)-(vi).

Since X is projectively $\mathbb{Q}$-closed, hence compact, we may assume that $0\notin X$, up to perform a small translation of a rational vector. Let $s\in\mathbb{Q}[x]$ and $t\in\mathbb{Q}[y]$ such that $X=\mathcal{Z}_\mathbb{R}(s)$ and $Y=\mathcal{Z}_\mathbb{R}(t)$, with s overt. Since p is $\mathbb{Q}$-regular, there are $p_1,\dots,p_m,q\in\mathbb{Q}[x]$ such that $p(x)=\big(\frac{p_1}{q}(x),\dots,\frac{p_m}{q}(x)\big)$ for every $x\in X$ and $\mathcal{Z}_\mathbb{R}(q)=\varnothing$. Define a $\mathbb{Q}$-regular extension $p':\mathbb{R}^n\to\mathbb{R}^m$ of p defined by $p'=\big(\frac{p_1q}{q^2+s^{2b}},\dots,\frac{p_mq}{q^2+s^{2b}}\big)$, where b is chosen in such a way that the degree of q^2+s^{2b} is greater than the degree of p_iq, for every $i\in\{1,\dots,m\}$. Let d be the degree of s and $C_2\in\mathbb{Q}\setminus\{0\}$. Let $r:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ be the $\mathbb{Q}$-regular function defined as

$$r(x,y)=t(y)^{2d}\Big(|y-p(x/C_2t(y))|^2_m+s(x/C_2t(y))^2\Big).$$

Observe that the choices of b and d ensure that the regular map r is globally defined as a polynomial $r\in\mathbb{Q}[x,y]$ by erasing the denominators. Let $V:=\mathcal{Z}_\mathbb{R}(r)$. Define $f:X\to\mathbb{R}^n\times\mathbb{R}^m$ by $f(x):=(x\cdot C_2t(p(x)),p(x))$, $g:Y\to\mathbb{R}^n\times\mathbb{R}^m$ by $g(y):=(0,y)$ and let $Y_0:=g(Y)=\{0\}\times Y\subset\mathbb{R}^{n+m}$. Thus, $Y_0\subset\mathbb{R}^{n+m}$ is a $\mathbb{Q}$-algebraic set. In addition, $g=\jmath|_Y$ and, if $C_2\in\mathbb{Q}\setminus\{0\}$ is chosen sufficiently small, $\jmath_V\circ f$ is arbitrarily $\mathscr{C}^0_\mathrm{w}$ close to $\jmath_Y\circ p$ and $(\jmath_V\circ f)|_{\mathrm{Reg}(X)}$ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $(\jmath_Y\circ p)|_{\mathrm{Reg}(X)}$. This proves (ii),(v) & (vi).

Let us prove that $V=f(X)\cup Y_0$. Let $(x,y)\in V$. If $y\in Y$ then $t(y)=0$, thus $r(x,y)=s_d(x)^2$, where s_d is the homogeneous part of s of highest degree equal to d. Since s is supposed to be overt, the only root of $s_d(x)$ is $x=0\in\mathbb{R}^n$. Hence $V\cap(\mathbb{R}^n\times Y)=\{0\}\times Y=Y_0$. Now suppose $y\in\mathbb{R}^m\setminus Y$. Since $t(y)\neq 0$, if $r(x,y)=0$ for some $x\in\mathbb{R}^n$, it follows that $y=p(x/C_2t(y))$ and $s(x/C_2t(y))=0$, that is, $x/C_2t(y)\in V$. We get $(x,y)=f(x/C_2t(y))$, thus $V\setminus(\mathbb{R}^n\times Y)\subset f(X\setminus p^{-1}(Y))$. On the other hand, $r(f(x))=0$ for every $x\in X$ since $X=\mathcal{Z}_\mathbb{R}(s)$, namely $f(X)\subset V$.

Now we apply the universal property of the quotient topology to find an homeomorphism $h:X\cup_p Y\to V$ so that $h^{-1}\circ f$ and $h^{-1}\circ g$ are the quotient maps. Consider the following diagram:

$$\begin{tikzcd} X\sqcup Y \arrow[d, "\pi"] \arrow[dr, "f\sqcup g"] & \\ X\cup_p Y \arrow[r, "h"] & V \end{tikzcd}$$

We want to prove that for every $z,z'\in X\sqcup Y$ such that $p(z)=p(z')\in Y$, then $(f\sqcup g)(z)=(f\sqcup g)(z')$. Let $z,z'\in X$, with $z\neq z'$, such that $p(z)=p(z')\in Y$, then $t(p(z))=0=t(p(z'))$ and $f(z)=(z\cdot t(p(z)),p(z))=(0,p(z))=(0,p(z'))=(z\cdot r(p(z')),p(z'))=f(z')$. Let $z\in X$ and $z'\in Y$ so that $p(z)=z'$, then $f(z)=$

$(z\cdot r(p(z)),p(z))=(0,z')=g(z')$, since $Y=\mathcal{Z}_\mathbb{R}(t)$. This proves, by the universal property of the quotient topology, that there is a unique continuous map h that makes the previous diagram commute. To get that h is actually an homeomorphism we are left to prove that $f\sqcup g$ is closed. Let A be a closed subset of $X\sqcup Y$, then it splits into the union of the two closed sets $A\cap X$ and $A\cap Y$, thus it suffices to prove that the images of both these sets are closed. Since X is compact, $A\cap X$ is compact and $f(A\cap X)$ is compact as well, hence $f(A\cap X)\subset V$ is closed in V. On the other hand, $g(Y)=\{0\}\times Y=V\cap(\mathbb{R}^n\times Y)$, hence $g(Y)$ is closed in V. Thus, $g(A\cap Y)=\{0\}\times(A\cap Y)$ is closed in V as well. This proves that $f\sqcup g$ is a quotient map, then, by the universal property of the quotient topology, we conclude that h is an homeomorphism satisfying (i).

It remains to show that, if $\dim(Y)<\dim(X\setminus p^{-1}(Y))$, then $\operatorname{Sing}(V)\subset Y_0\cup f(\operatorname{Sing}(X))$ and V is $\mathbb{Q}$-determined on $\operatorname{Reg}(V)\setminus Y_0$, that is $\operatorname{Reg}(V)\setminus Y_0\subset\operatorname{Reg}^*(V)$. Observe that the $\mathbb{Q}$-regular function $f':V\setminus Y_0\to X\setminus p^{-1}(Y)$ defined as $f'(x,y)=x/C_2t(y)$ is the inverse of $f|_{X\setminus p^{-1}(Y)}$, thus $\operatorname{Sing}(V\setminus Y_0)=f(\operatorname{Sing}(X\setminus p^{-1}(Y)))\subset f(\operatorname{Sing}(X))$ and $\dim(V\setminus Y_0)=\dim(X\setminus p^{-1}(Y))$. In particular, (iii) holds. Let $a:=(x,y)\in\operatorname{Reg}(V)\setminus Y_0$. Since $f:X\setminus p^{-1}(Y)\to V\setminus Y_0$ is a $\mathbb{Q}$-biregular map we deduce that $\mathcal{R}^*_{V,a}\equiv\mathcal{R}^*_{X,f'(a)}$. This proves that V is $\mathbb{Q}$-determined on $\operatorname{Reg}(V)\setminus Y_0$, thus also (iv) holds and the Claim follows. $\square$

CHAPTER 4

$\mathbb{Q}$-Algebrization results

Abstract. This chapter contains our main $\mathbb{Q}$-approximation results. In Section 4.1 we extend some results of [BFR14] from the $\mathscr{C}^\infty$ to the Nash category. Then, we prove a relative Nash-Tognoli theorem 'over $\mathbb{Q}$' improving a previous version already presented in Section 3.2. In particular, we prove that, in case the starting data are actually Nash than the diffeomorphism in the statement of the theorem can be produced Nash. After dealing with a singularity at infinity, we prove a general relative $\mathbb{Q}$-algebrization theorem in the case of nonsingular algebraic sets with a finite set of nonsingular algebraic subsets in general position. In Section 4.2 we prove a $\mathbb{Q}$-algebrization theorem for Nash manifolds over any real closed field. The main ingredients are a deep result in [CS92], model completeness of the theory of real closed fields and our $\mathbb{Q}$-algebrization results over $\mathbb{R}$. Section 4.3 is devoted to the proof of a $\mathbb{Q}$-algebrization result in the case of algebraic sets with isolated singularities at first in the compact case and then in general by applying algebraic compactification. The results explained so far provide a complete positive answer of [Par21, Open problem 1, p. 199] in the case of nonsingular algebraic sets and singular algebraic sets with isolated singularities. In Section 4.4 we prove a consequence of our $\mathbb{Q}$-algebrization theorems to provide a complete positive answer of [Par21, Open problem 2, p. 200] in the case of algebraic set germs with an isolated singularity.

The main references for this chapter are [GS23] and [Sav23].

4.1. Relative $\mathbb{Q}$-algebrization problem for nonsingular algebraic sets

This section is devoted to provide a complete positive answer to the Relative $\mathbb{Q}$-algebrization of nonsingular algebraic sets. We will divide the compact and the non-compact cases.

4.1.1. Relative Nash approximation of $\mathscr{C}^\infty$ diffeomorphisms. A subset of $\mathbb{R}^a$ is *semialgebraic* if it is a Boolean combination of subsets of $\mathbb{R}^a$ defined by polynomial equations and polynomial strict inequalities. A locally closed semialgebraic set $M \subset \mathbb{R}^a$ is called (affine) *Nash manifold* if it is also a $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^a$. Let $M \subset \mathbb{R}^a$ be a Nash manifold, let $X \subset M$ be a (non-empty) semialgebraic subset of $\mathbb{R}^a$ contained in M, and let $Y \subset \mathbb{R}^b$ be a (non-empty) semialgebraic set. We denote $\mathscr{C}^0_{\mathrm{w}}(X,Y)$ the set $\mathscr{C}^0(X,Y)$ of continuous maps from X to Y, equipped with the compact-open topology. Let $\nu \in \mathbb{N}^* \cup \{\infty\}$ and let $f : X \to Y$ be a map. We say that f is a $\mathscr{C}^\nu$ *map* if there exist an open (not necessarily semialgebraic) neighborhood U of X in M and a map $F : U \to \mathbb{R}^b$ such that F is of class $\mathscr{C}^\nu$ in the usual sense of $\mathscr{C}^\infty$ (and hence $\mathscr{C}^\nu$) manifolds and $F(x) = f(x)$ for all $x \in X$. We denote $\mathscr{C}^\nu(X,Y)$ the set of $\mathscr{C}^\nu$ maps from X to Y. The map $f : X \to Y$ is called *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{a+b} = \mathbb{R}^a \times \mathbb{R}^b$. The map $f : X \to Y$ is said to be a *Nash map* if there exist an open semialgebraic neighborhood U of X in M and a $\mathscr{C}^\infty$ map $F : U \to \mathbb{R}^b$ such that F is semialgebraic and $F(x) = f(x)$ for all $x \in X$. We denote $\mathcal{N}(X, Y)$ the set of Nash maps from X to Y. Note that $\mathcal{N}(X, Y) \subset \mathscr{C}^\infty(X, Y) \subset \mathscr{C}^\nu(X, Y) \subset \mathscr{C}^0(X, Y)$. We denote $\mathscr{C}^\nu_{\mathrm{w}}(M, \mathbb{R}^b)$ the set $\mathscr{C}^\nu(M, \mathbb{R}^b)$ equipped with the usual *weak $\mathscr{C}^\nu$ topology*, see [Hir94, §2]. Similarly, we denote $\mathcal{N}_{\mathrm{w}}(M, \mathbb{R}^b)$ the set $\mathcal{N}(M, \mathbb{R}^b)$ equipped with the relative topology of $\mathscr{C}^\nu_{\mathrm{w}}(M, \mathbb{R}^b)$.When $Y = \mathbb{R}$, we often speak about $\mathscr{C}^\nu$ (Nash) functions instead of $\mathscr{C}^\nu$ (Nash) maps, and we set $\mathscr{C}^\nu(X) := \mathscr{C}^\nu(X, \mathbb{R})$ and $\mathcal{N}(X) := \mathcal{N}(X, \mathbb{R})$. Note that $f = (f_1, \dots, f_b) : X \to Y \subset \mathbb{R}^b$ is a $\mathscr{C}^\nu$ (Nash) map if, and only if, each component $f_i : X \to \mathbb{R}$ of f is a function in $\mathscr{C}^\nu(X)$ (in $\mathcal{N}(X)$). The set $X \subset M$ is a *Nash set* if it is the common zero set of a finite family of Nash functions defined on M. Each Nash set $X \subset M$ is closed in M, and it decomposes into the finite union of its Nash irreducible components, see [BCR98, Corollary 8.6.8]. For further information on semialgebraic and Nash sets, we refer to [BCR98] (see also [BR90; Shi87]).

Let us recall the concepts of Nash set with monomial singularities and of Nash monomial crossings.

Definition 4.1.1 ([BFR14, Definitions 1.1, 1.3 & p. 63]). A set $X \subset M$ is called *Nash set with monomial singularities* if it is a Nash set and, for each $x \in X$, there exist a semialgebraic open neighborhood U of x in M and a Nash diffeomorphism $u : U \to \mathbb{R}^m$, where $m = \dim(M)$, such that $u(x) = 0$ and $u(X \cap U)$ is equal to a union of coordinate linear varieties of $\mathbb{R}^m$. If in addition the Nash irreducible components of X are Nash manifolds, then $X \subset M$ is called *Nash monomial crossings*.

Given a Nash set $M' \subset M$, we say that M' is a *Nash submanifold of M* if $M' \subset \mathbb{R}^a$ is a Nash manifold. If $\{M_i\}_{i=1}^\ell$ is a finite family of Nash submanifolds of M in general position, then their union $\bigcup_{i=1}^\ell M_i \subset M$ is an example of Nash monomial crossings.

Here we state a theorem which is a variant of results originally proved in [BFR14]. The interested reader will find the complete proof in the Appendix A. Next result will play an important role in the proof of Theorems 4.1.4 & 4.3.4.

Theorem 4.1.2. *Let M and N be compact Nash manifolds, let $\ell \in \mathbb{N}^*$, let $\{M_i\}_{i=1}^\ell$ be a family of Nash submanifolds of M in general position, let $\{N_i\}_{i=1}^\ell$ be a family of Nash submanifolds of N in general position, let $X' \subset M$ be a Nash set with monomial singularities and let $\phi : M \to N$ be a $\mathscr{C}^\infty$ diffeomorphism such that $\phi(M_i) = N_i$ for all $i \in \{1, \dots, \ell\}$, $X' \cap \left(\bigcup_{i=1}^\ell M_i\right) = \varnothing$ and $\phi|_{X'}$ is a Nash map. Then there exists a Nash diffeomorphism $\psi : M \to N$ arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to ψ such that $\psi(M_i) = N_i$ for all $i \in \{1, \dots, \ell\}$ and $\psi|_{X'} = \phi|_{X'}$.*

As a direct consequence of Lemma 2.1.8 and Theorems 3.2.3 & 4.1.2 we have the following result.

Corollary 4.1.3. *Let M be a compact Nash submanifold of $\mathbb{R}^n$ of dimension d and let $\{M_i\}_{i=1}^\ell$ be a family of Nash hypersurfaces of M in general position. Then there exist a $\mathscr{C}^\infty$-diffeotopy $\{h_t\}_{t\in[0,1]}$ of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ (for some $k \in \mathbb{N}$) arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$-close to $\mathrm{id}_{\mathbb{R}^{n+k}}$ which simultaneously take $M \times \{0\}$ and each $M_i \times \{0\}$ to a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets.*

In particular, there exist projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets $M', M'_1, \dots, M'_\ell \subset \mathbb{R}^{2d+1}$ and a Nash diffeomorphism $h : M \to M'$ such that $M'_i \subset M'$ and $h(M_i) = M'_i$ for all $i \in \{1, \dots, \ell\}$.

4.1.2. The compact case: the relative Nash-Tognoli theorem 'over $\mathbb{Q}$'. In order to prove the version 'over $\mathbb{Q}$' of [AK81a, Theorem 2.10] presented below, the main ingredients will be our relative bordisms 'over $\mathbb{Q}$' constructed in Theorem 2.5.2 combined with approximation techniques 'over $\mathbb{Q}$' of Chapter 3, Lemma 1.6.14 and generic projection 'over $\mathbb{Q}$' (see Lemma 2.1.8).

THEOREM 4.1.4 (Relative Nash-Tognoli theorem 'over $\mathbb{Q}$'). *Let M be a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$ of dimension d and let M_i for $i = 1, \dots, \ell$, be $\mathscr{C}^\infty$ submanifolds of M in general position. Set $m := \max\{n, 2d+1\}$. Then, for every neighborhood $\mathcal{U}$ of the inclusion map $\iota : M \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^\infty_{\mathrm{w}}(M, \mathbb{R}^m)$ and for every neighborhood $\mathcal{U}_i$ of the inclusion map $\iota|_{M_i} : M_i \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^\infty_{\mathrm{w}}(M_i, \mathbb{R}^m)$, for every $i \in \{1, \dots, \ell\}$, there exist a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $M' \subset \mathbb{R}^m$, $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets M'_i of M' for $i = 1, \dots, \ell$, in general position and a $\mathscr{C}^\infty$ diffeomorphism $h : M \to M'$ which simultaneously takes each M_i to M'_i such that, if $\jmath : M' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map, then $\jmath \circ h \in \mathcal{U}$ and $\jmath \circ h|_{M_i} \in \mathcal{U}_i$, for every $i \in \{1, \dots, \ell\}$.*

If in addition M and each M_i are compact Nash manifolds, then we can assume that $h : M \to M'$ is a Nash diffeomorphism and h extends to a semialgebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$.

PROOF. Let c_i be the codimension of M_i in M for $i \in \{1, \dots, \ell\}$. An application of Theorem 2.5.2 gives $s \in \mathbb{N}$, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Y \subset \mathbb{R}^s := \mathbb{R}^n \times \mathbb{R}^{s-n}$ of dimension d, $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets Y_i, for $i \in \{1, \dots, \ell\}$, of Y in general position, a compact $\mathscr{C}^\infty$ submanifold S of $\mathbb{R}^{s+1} = \mathbb{R}^s \times \mathbb{R}$ of dimension $m+1$ and compact $\mathscr{C}^\infty$ submanifolds S_i of S of codimension c_i, for $i \in \{1, \dots, \ell\}$, in general position satisfying Theorem 2.5.2(i)-(iv).

Consider the map $\beta_i : S_i \to \mathbb{G}_{c_i, s+1-c_i}$ classifying the normal bundle of S_i in S. By Theorem 2.5.2(iv) we have that $\beta_i|_{Y_i}$ is a $\mathbb{Q}$-regular map extending the codomain from $\mathbb{G}_{c_i, n-c_i}$ to $\mathbb{G}_{c_i, s+1-c_i}$. An application of Theorem 3.2.2, with "L" $:= Y_i \times \{0\}$, "M" $:= S_i$, "W" $:= \mathbb{G}_{c_i, s+1-c_i}$ and "f" $:= \beta_i$ gives $t \in \mathbb{N}$, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset X_i of $\mathbb{R}^{s+1+t}$, a diffeomorphism $\rho_i : S_i \to X_i$ and a $\mathbb{Q}$-regular map $\gamma_i : X_i \to \mathbb{G}_{c_i, s+1-c_i}$ satisfying Theorem 3.2.2(i)-(iii). In particular, $(Y_i \times \{0\}) \times \{0\} \subset X_i$, $\rho_i(x) = (x, 0)$ and $\gamma_i(x, 0) = \beta_i|_{Y_i \times \{0\}}(x)$ for every $x \in Y_i \times \{0\}$.

Consider the pullback bundle $Z_i := \gamma_i^*(\mathbb{E}^*_{c_i, s+1-c_i})$. By Lemma 2.2.6, Z_i is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^{s+1+t} \times \mathbb{R}^{s+1+t+1}$ and it contains those subsets Y^α of Y such that $i \in \alpha$, by Theorem 2.5.2(iii)(iv) and Lemma 2.5.1(iii). More precisely, following the notations of Theorem 2.5.2, we have that

$$Y'^\alpha := (\gamma_i|_{(Y_i^\alpha + v_\alpha) \times \{0\} \times \{0\}})^*(\mathbb{E}^*_{c_i, s+1-c_i})$$

is contained in Z_i, for every $\alpha \in A_i$, and is $\mathbb{Q}$-biregularly isomorphic to Y^α fixing each $x \in Y_\alpha$. Let $Y'_i \subset \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^t \times \mathbb{R}^{s+1+t+1}$ be defined as

$$Y'_i := \Big(\bigsqcup_{\alpha \in A_i} Y'^\alpha \Big) \sqcup \Big(\bigsqcup_{\alpha \notin A_i} (Y^\alpha + v_\alpha) \times \{0\} \times \{0\} \times \{0\} \Big).$$

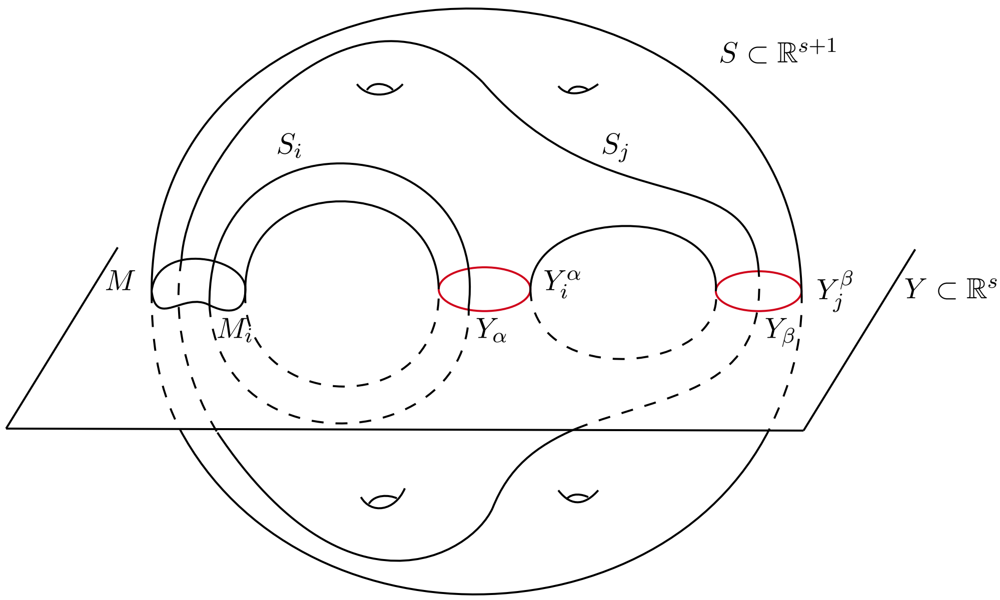

FIGURE 4.1.1. Starting situation after the application of Theorem 2.5.2.

Since γ_i can be chosen such that $\gamma_i \circ \rho$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to β_i, those maps are homotopic, thus the normal bundle of S_i in S and the normal bundle of $X_i \times \{0\} \times \{0\}$ in Z_i are equivalent. Hence, the germ $(S, S_i \cup (Y \times \{0\}))$ of S at $S_i \cup (Y \times \{0\})$ is diffeomorphic to the germ

$$\Big(Z_i \cup \big(\bigsqcup_{\alpha \notin A_i} (Y^\alpha + v_\alpha) \times \mathbb{R} \times \{0\} \times \{0\}\big), (X_i \times \{0\}) \cup Y_i'\Big)$$

of the $\mathbb{Q}$-algebraic set $Z_i \cup (\bigsqcup_{\alpha \notin A_i} (Y^\alpha + v_\alpha) \times \mathbb{R} \times \{0\} \times \{0\})$ $\mathbb{Q}$-nonsingular locally at $(X_i \times \{0\}) \cup Y_i'$.

Let $\phi_i : U_i \to V_i$ be the above $\mathscr{C}^\infty$ diffeomorphism between a neighborhood U_i of $S_i \cup (Y \times \{0\})$ in S and a neighborhood V_i of $(X_i \times \{0\}) \cup Y_i'$ in $Z_i \cup (\bigsqcup_{\alpha \notin A_i} (Y^\alpha + v_\alpha) \times \mathbb{R} \times \{0\} \times \{0\})$ such that $\phi_i|_{S_i} = \rho_i \times \{0\}$ and $\phi_i|_{Y^\alpha}$ is the above $\mathbb{Q}$-biregular isomorphism for every $\alpha \in A_i$, and $\phi_i|_{Y^\alpha}$ is the inclusion map for every $\alpha \notin A_i$. Let $V_i' \subset V_i$ be a neighborhood of $(X_i \times \{0\}) \cup Y_i'$ in $Z_i \cup (\bigsqcup_{\alpha \notin A_i} (Y^\alpha + v_\alpha) \times \mathbb{R} \times \{0\} \times \{0\})$ such that $\overline{V_i'} \subsetneq V_i$. Set $A_i := \phi_i^{-1}(\overline{V_i'}) \subset U_i$ closed neighborhood of $S_i \cup (Y \times \{0\})$ in S and consider the map $\phi_i|_{A_i} : A_i \to \mathbb{R}^{s+1+t} \times \mathbb{R}^{s+1+t+1}$. Since $2(s+1+t)+1 \geq 2(d+1)+1$, Tietze's theorem ensures the existence of a continuous extension of ϕ_i from the whole S to $\mathbb{R}^{s+1+t} \times \mathbb{R}^{s+1+t+1}$, we can apply to $\phi_i|_{A_i}$ the extension theorem [Whi36, Theorem 5(f)] of Whitney, obtaining a $\mathscr{C}^\infty$ embedding $\phi_i' : S \to \mathbb{R}^{s+1+t} \times \mathbb{R}^{s+1+t+1}$ extending $\phi_i|_{A_i}$. Thus, there exists a $\mathscr{C}^\infty$ manifold $N_i \subset \mathbb{R}^{s+1+t} \times \mathbb{R}^{s+1+t+1}$ which is $\mathscr{C}^\infty$ diffeomorphic to S via ϕ_i' and, by construction, the following properties are satisfied:

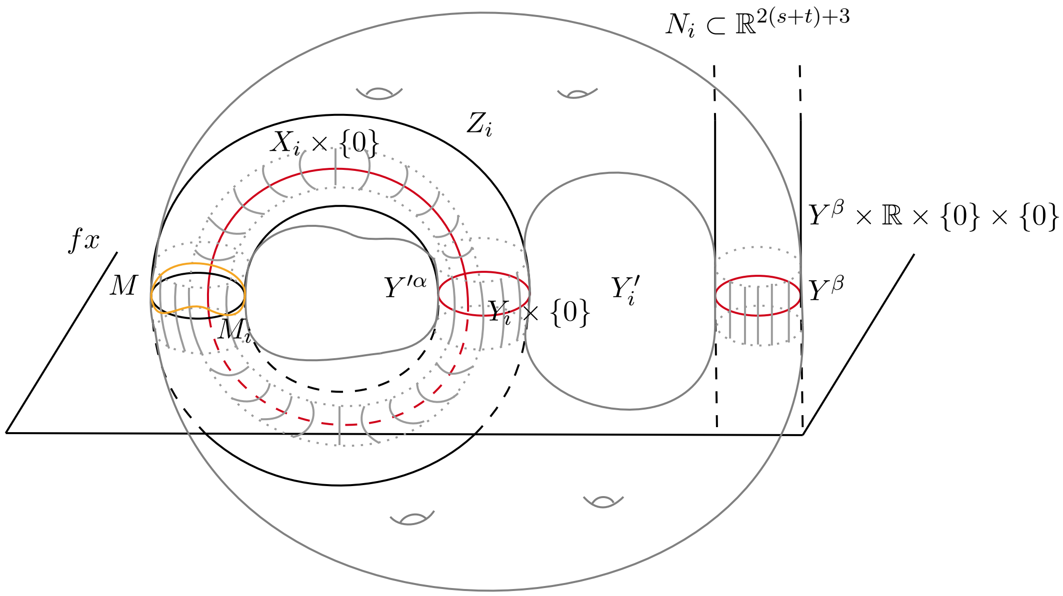

FIGURE 4.1.2. Topological construction of N_i with $i \in \alpha$ and $i \notin \beta$.

(i) $(X_i \times \{0\}) \cup Y'_i \subset N_i$;
(ii) the germ of N_i at $(X_i \times \{0\}) \cup Y'_i$ is the germ of a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set.

Since $X_i \times \{0\}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of N_i and Y'_i is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic hypersurface of N_i satisfying above property (ii), Lemma 3.1.5 ensures that $(X_i \times \{0\}) \cup Y'_i \subset \mathbb{R}^{s+1+t} \times \mathbb{R}^{s+1+t+1}$ is a $\mathbb{Q}$-nice $\mathbb{Q}$-algebraic set. An application of Theorem 3.2.2 with "L" $:= (X_i \times \{0\}) \cup Y'_i$, "$M$" $:= N_i$ and "W" $:= \{0\}$ gives $t_i \in \mathbb{N}$, with $t_i \geq 2(s+1+t)+1$, a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $X^i \subset \mathbb{R}^{t_i} = \mathbb{R}^{s+1+t} \times \mathbb{R}^{s+1+t+1} \times \mathbb{R}^{t_i - 2(s+1+t)+1}$ such that $((X_i \times \{0\}) \cup Y'_i) \times \{0\} \subset X^i$ and a $\mathscr{C}^\infty$ diffeomorphism $\tau_i : N_i \to X^i$ such that $\tau_i(x, 0) = x$ for every $x \in (X_i \times \{0\}) \cup Y'_i$. Define the diffeomorphism $\varphi_i : S \to X^i$ as $\varphi_i := \tau_i \circ \phi'_i$, for every $i \in \{1, \dots, \ell\}$. Let $t := \max_{i \in \{1,\dots,\ell\}} t_i$ and consider $X^i \subset \mathbb{R}^t$, for every $i \in \{1, \dots, \ell\}$.

Here we adapt part of the proof of Treorem 3.2.2 to the present situation.

Let $G := \mathbb{G}_{s-d,d+1}$ and let $\beta : S \to G$ be the Gauss mapping of S in $\mathbb{R}^{s+1}$. Recall that $Y \subset \mathbb{R}^s$ and, by Theorem 2.5.2(iii)(iv), $\beta|_{Y \times \{0\}}$ coincides with the Gauss mapping of the $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $Y \times \mathbb{R}$ in $\mathbb{R}^{s+1}$. Hence, $\beta|_{Y \times \{0\}}$ is $\mathbb{Q}$-regular since the Gauss mapping of $Y \times \mathbb{R}$ in $\mathbb{R}^{s+1}$ is so by Lemma 2.2.5.

Let $E := \mathbb{E}_{s-d,d+1} = \{(A, b) \in G \times \mathbb{R}^{s+1} \mid Ab = b\}$ be the universal vector bundle over the grassmannian G. Let $\beta^*(E) := \{(x, y) \in S \times \mathbb{R}^{s+1} \mid \beta(x)y = y\}$ be the pullback bundle and let $\theta : \beta^*(E) \to \mathbb{R}^{s+1}$ defined by $\theta(x, y) := x + y$. By the Implicit Function Theorem, there exists an open neighborhood U_0 in $\beta^*(E)$ of the zero section $S \times 0$ of $\beta^*(E)$ and an open neighborhood U of S in $\mathbb{R}^{s+1}$ such that $\theta|_{U_0} : U_0 \to U$ is a diffeomorphism.

Define a $\mathscr{C}^\infty$ map $\widetilde{\beta} : U \to E$ and a smooth map $\widetilde{\varrho} : U \to S$ in the following way: for every $x \in U$, let $(z_x, y_x) := (\theta|_{U_0})^{-1}(x)$ and let $N_x := \beta(z_x)$, then define $\widetilde{\beta}(x) := (N_x, y_x)$ and $\widetilde{\varrho}(x) := z_x$. Since $(\theta|_{U_0})^{-1}(S) = G \times \{0\}$, we have that $\widetilde{\beta}^{-1}(G \times \{0\}) = S$; moreover if $x \in S$ then $\widetilde{\beta}(x) = (\beta(x), 0)$, so $\widetilde{\beta}|_Y$ is $\mathbb{Q}$-regular. Now we prove that $\widetilde{\beta}$ is transverse to $G \times \{0\}$ in E. Fix $x \in S$ and let $N_x := \beta(x)$. Let $\widetilde{N}_x$ be the $(s-d)$-dimensional vector subspace of $\mathbb{R}^{s+1}$ corresponding to N_x and let $y \in \widetilde{N}_x$ so close to the origin 0 of $\mathbb{R}^{s+1}$ that $(x, y) \in U_0$. We have that $\theta|_{U_0}(x, y) = x + y$, so $(\theta|_{U_0})^{-1}(x + y) = (x, y)$ and $\widetilde{\beta}(x + y) = (N_x, y)$. It follows that $d\widetilde{\beta}_x(\widetilde{N}_x) = \widetilde{N}_x$, hence $d\widetilde{\beta}_x(\mathbb{R}^{s+1})$ contains the vector subspace $\{0\} \times \mathbb{R}^{s-d}$ of $T_{N_x}(G) \times \mathbb{R}^{s-d} = T_{(N_x,0)}(E)$ and so $\widetilde{\beta}$ is transverse to $G \times \{0\}$ at x.

Recall that, by Lemma 2.1.7(iv), $X^1 \times \cdots \times X^\ell \subset (\mathbb{R}^s)^\ell$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. Let $\varphi : S \to X^1 \times \cdots \times X^\ell$ be the $\mathscr{C}^\infty$ map defined as $\varphi = (\varphi_1, \dots, \varphi_\ell)$. Let $\widetilde{\varphi} : U \to X^1 \times \cdots \times X^\ell$ be the smooth map defined by $\widetilde{\varphi} := \varphi \circ \widetilde{\varrho}$. The smooth map $\widetilde{\beta} \times \widetilde{\varphi} : U \to E \times X^1 \times \cdots \times X^\ell$ satisfies the following properties:

(iii) $\widetilde{\beta} \times \widetilde{\varphi}$ is transverse to $(G \times \{0\}) \times X^1 \times \cdots \times X^\ell$ and $(\widetilde{\beta} \times \widetilde{\varphi})^{-1}((G \times \{0\}) \times X^1 \times \cdots \times X^\ell) = S$,

(iv) $(\widetilde{\beta} \times \widetilde{\varphi})|_Y$ is $\mathbb{Q}$-regular.

Apply Lemma 3.2.1 with the following substitutions: "W":$= E \times X^1 \times \cdots \times X^\ell$, "$L$":$= Y \times \{0\}$, "$f$":$= \widetilde{\beta} \times \widetilde{\varphi}$ and "U" equal to some open neighborhood U' of S in $\mathbb{R}^{s+1}$ relatively compact in U, obtaining a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset Z of $\mathbb{R}^{s+1} \times \mathbb{R}^k$, for some integer k, an open subset Z_0 of Z and a $\mathbb{Q}$-regular map $\eta : Z \to E \times X^1 \times \cdots \times X^\ell$ such that, if $\pi : \mathbb{R}^{s+1} \times \mathbb{R}^k \to \mathbb{R}^{s+1}$ is the natural projection and $\iota : U' \hookrightarrow \mathbb{R}^{s+1} \times \mathbb{R}^k$ is the inclusion map, the following conditions hold:

(v) $Y \times \{0\} \times \{0\} \subset Z_0$, $\pi(Z_0) = U'$, the restriction $\pi|_{Z_0} : Z_0 \to U'$ is a $\mathscr{C}^\infty$ diffeomorphism, and the $\mathscr{C}^\infty$ map $\sigma : U' \to \mathbb{R}^{s+1+k}$, defined by $\sigma(x, x_{s+1}) := (\pi|_{Z_0})^{-1}(x, x_{s+1})$ for all $(x, x_{s+1}) \in U'$, is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to ι.

(vi) $\eta(x, x_{s+1}, 0) = (\widetilde{\beta} \times \widetilde{\varphi})(x, x_{s+1})$ for all $(x, x_{s+1}) \in Y \times \{0\}$.

(vii) The $\mathscr{C}^\infty$ map $\widehat{\eta} : U' \to E \times X^1 \times \cdots \times X^\ell$, defined by $\widehat{\eta}(x, x_{s+1}) := \eta(\sigma(x, x_{s+1}))$, is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $(\widetilde{\beta} \times \widetilde{\varphi})|_{U'}$.

Choose an open neighborhood U'' of S in $\mathbb{R}^{s+1}$ such that $\overline{U''} \subset U'$. Set $Z_1 := (\pi|_{Z_0})^{-1}(U'')$. Since $\widetilde{\beta} \times \widetilde{\varphi}$ is transverse to $(G \times \{0\}) \times X^1 \times \cdots \times X^\ell$ in $E \times X^1 \times \cdots \times X^\ell$, by (v), (vi), (vii) and [BCR98, Theorem 14.1.1], we have that $S' := \widehat{\eta}^{-1}((G \times \{0\}) \times X^1 \times \cdots \times X^\ell)$ is a compact $\mathscr{C}^\infty$ submanifold of U'' containing $Y \times \{0\} \times \{0\}$ and there exists a $\mathscr{C}^\infty$ diffeomorphism $\psi_1 : U'' \to U''$ arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\mathrm{id}_{U''}$ such that $\psi_1(S) = S'$ and $\psi = \mathrm{id}_{U''}$ on $Y \times \{0\} \times \{0\}$. Moreover, Lemma 2.1.5 ensures that $S'' := \eta^{-1}((G \times \{0\}) \times X^1 \times \cdots \times X^\ell) \subset \mathbb{R}^{s+1+k}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set of dimension $d + 1$. In addition, the $\mathscr{C}^\infty$ embedding $\psi_2 : S \to \mathbb{R}^{s+1+k}$ defined by $\psi(x, x_{s+1}) := (\pi|_{Z_1})^{-1}(\psi_1(x, x_{s+1}))$, is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to the inclusion map $j_S : S \hookrightarrow \mathbb{R}^{s+1+k}$, $\psi_2 = j_S$ on $Y \times \{0\}$ and $\psi_2(S) = S''_1$. Note that the set S''_1 is both compact and open in S''; thus, S''_1 is the union of some connected components of S'' and $S''_2 := S'' \setminus S''_1$ is a closed subset of $\mathbb{R}^{s+1+k}$ (recall that an algebraic set, as S'' is, only has finitely many connected components). Since ψ_2 is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to j_S, the coordinate hyperplane $\{x_{s+1}=0\}$ of $\mathbb{R}^{s+1+k}$ is transverse to S''_1 in $\mathbb{R}^{s+1+k}$, $S''_1 \cap \{x_{s+1}=0\} = M' \sqcup (Y \times \{0\} \times \{0\})$ for some compact $\mathscr{C}^\infty$ submanifold M' of $\mathbb{R}^{s+1+k}$ and there exists a $\mathscr{C}^\infty$ embedding $\psi_3 : M \to \mathbb{R}^{s+1+k}$ arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to the inclusion map $j_M : M \hookrightarrow \mathbb{R}^{s+1+k}$ such that $M' = \psi_3(M)$.

Let K be a compact neighborhood of S''_1 in $\mathbb{R}^{s+1+k}$ such that $K \cap S''_2 = \varnothing$ and let $\pi_{s+1} : \mathbb{R}^{s+1+k} = \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be the projection $\pi_{s+1}(x, x_{s+1}, y) := x_{s+1}$. By Lemma 2.1.7(i)(iv), the algebraic set $Y \times \{0\} \times \{0\} \subset \mathbb{R}^{s+1+k}$ is projectively $\mathbb{Q}$-closed. Let $q \in \mathbb{Q}[x_1, \dots, x_{s+1+k}]$ be an overt polynomial such that $\mathcal{Z}_\mathbb{R}(q) = Y \times \{0\} \times \{0\}$. Since q is a proper function, replacing q with Cq^2 for some rational number $C > 0$ if necessary, we can assume that $q \in \mathbb{Q}[x_1, \dots, x_{s+1+k}]$ is overt, $\mathcal{Z}_\mathbb{R}(q) = Y \times \{0\} \times \{0\}$, $q \geq 0$ on $\mathbb{R}^{s+1+k}$ and $q \geq 2$ on $\mathbb{R}^{s+1+k} \setminus K$. Let K' be a compact neighborhood of S''_1 in $\mathrm{int}_{\mathbb{R}^{s+1+k}}(K)$. Using a $\mathscr{C}^\infty$ partition of unity subordinated to $\{\mathrm{int}_{\mathbb{R}^{s+1+k}}(K), \mathbb{R}^{s+1+k} \setminus K'\}$, we can define a $\mathscr{C}^\infty$ function $h : \mathbb{R}^{s+1+k} \to \mathbb{R}$ such that $h = \pi_{s+1}$ on K' and $h = q$ on $\mathbb{R}^{s+1+k} \setminus K$. Apply Lemma 3.1.9 to $h - q$, obtaining a $\mathbb{Q}$-regular function $u' : \mathbb{R}^{s+1+k} \to \mathbb{R}$ with the following properties:

(viii) There exist $e \in \mathbb{N}$ and a polynomial $p \in \mathbb{Q}[x_1, \dots, x_{s+1+k}]$ of degree $\leq 2e$ such that $u'(x) = p(x)(1 + |x|^2_{s+t})^{-e}$ for all $x \in \mathbb{R}^{s+1+k}$.
(ix) $Y \times \{0\} \times \{0\} \subset \mathcal{Z}_\mathbb{R}(u')$.
(x) $\sup_{x \in \mathbb{R}^{s+1+k}} |h(x) - q(x) - u'(x)| < 1$.
(xi) u' is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\pi_{s+1} - q$ on $\mathrm{int}_{\mathbb{R}^{s+1+k}}(K')$.

Let $u : \mathbb{R}^{s+1+k} \to \mathbb{R}$ be the $\mathbb{Q}$-regular map given by $u := u' + q$, and let $v \in \mathbb{Q}[x]$, with $x = (x_1, \dots, x_{s+1+k})$, be the polynomial $v(x) := q(x)(1 + |x|^2_{s+1+k})^e + p(x)$. Combining (viii) with the fact that $q \in \mathbb{Q}[x]$ is non-constant and overt, we immediately deduce that $u(x) = (1 + |x|^2_{s+1+k})^{-e} v(x)$ and $v \in \mathbb{Q}[x]$ is overt. By (ix), (x) & (xi), we know that $Y \times \{0\} \times \{0\} \subset \mathcal{Z}_\mathbb{R}(u)$, $u > 1$ on $\mathbb{R}^{s+1+k} \setminus K$ and u is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to π_{s+1} on $\mathrm{int}_{\mathbb{R}^{s+1+k}}(K')$. In particular, 0 is a regular value of the restriction $u|_{S''_1}$ of u to S''_1, $S''_1 \cap \mathcal{Z}_\mathbb{R}(u) = M'' \sqcup X$ for some compact $\mathscr{C}^\infty$ submanifold M'' of $\mathbb{R}^{s+1+k}$ and there exists a $\mathscr{C}^\infty$ embedding $\psi_4 : M' \to \mathbb{R}^{s+1+k}$ arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to the inclusion map $j_{M'} : M' \hookrightarrow \mathbb{R}^{s+1+k}$ such that $M'' = \psi_4(M')$. Since $M'' \sqcup X = S'' \cap \mathcal{Z}_\mathbb{R}(u)$, Lemma 2.1.5 ensures that $M'' \sqcup X \subset \mathbb{R}^{s+1+k}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. On the other hand, we also have that $M'' \sqcup X = S'' \cap \mathcal{Z}_\mathbb{R}(u) = S'' \cap \mathcal{Z}_\mathbb{R}(v)$; thus, Lemma 2.1.7(ii) implies that $M'' \sqcup X$ is projectively $\mathbb{Q}$-closed. In addition, by Lemma 1.6.14 we deduce that $M'' \subset \mathbb{R}^{s+1+k}$ is a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. Consider the embedding $\psi : M \to \mathbb{R}^{s+1+k}$ defined as $\psi := \psi_4 \circ \psi_3$. Then, ψ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to j_M, $\psi(M) = M''$ and consider the $\mathscr{C}^\infty$ submanifolds $\psi(M_i)$, for every $i \in \{1, \dots, \ell\}$, of M'' in general position.

Consider $\pi_i : E \times X^1 \times \cdots \times X^\ell \to X^i$ the projection on the i-th component of $X^1 \times \cdots \times X^\ell$, thus $\pi_i \circ (\widetilde{\beta} \times \widetilde{\varphi}) = \varphi_i \circ \widetilde{\rho}$. Let $X'_i := X_i \times \{0\} \times \{0\} \subset X^i$, for every $i \in \{1, \dots, \ell\}$. By (vii), we know that $\pi_i \circ \widehat{\eta}$ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\varphi_i \circ \widetilde{\rho}$, thus $\pi_i \circ \widehat{\eta}$ is transverse to X'_i in X^i for every $i \in \{1, \dots, \ell\}$. By (v), (vi), (vii) and [BCR98, Theorem 14.1.1], we have that $S'_i := \widehat{\eta}^{-1}((G \times \{0\}) \times X^1 \times \cdots \times X'_i \times \cdots \times X^\ell) = S' \cap (\pi_i \circ \widehat{\eta})^{-1}(X'_i)$ is a compact $\mathscr{C}^\infty$ submanifold of $S \subset U''$ containing $Y_i \times \{0\} \times \{0\}$ and there exists a $\mathscr{C}^\infty$ diffeomorphism $\psi_1^i : U'' \to U''$ arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\mathrm{id}_{U''}$ such that $\psi_1^i(S_i) = S'_i$ and $\psi_1^i = \mathrm{id}_{U''}$ on $Y_i \times \{0\} \times \{0\}$. Moreover, by (v), Lemma 2.1.5 ensures that $S''_i := \eta^{-1}((G \times \{0\}) \times X^1 \times \cdots \times X'_i \times \cdots \times X^\ell) = S'' \cap (\pi_i \circ \eta)^{-1}(X'_i) \subset \mathbb{R}^{s+1+k}$ is a $\mathbb{Q}$-algebraic set such that $S''_1 := S'' \cap Z_1 = (\pi|_{Z_1})^{-1}(S') \subset \mathrm{Reg}^{\mathbb{R}|\mathbb{Q}}(S'')$ In addition, the $\mathscr{C}^\infty$ embedding $\psi_2^i : S_i \to \mathbb{R}^{s+1+k}$ defined by $\psi_2^i(x, x_{s+1}) := (\pi|_{Z_1})^{-1}(\psi_1^i(x, x_{s+1}))$, is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to the inclusion map $\jmath_{S_i} : S_i \hookrightarrow \mathbb{R}^{s+1+k}$, $\psi_2^i = \jmath_{S_i}$ on $Y_i \times \{0\}$ and $\psi_2^i(S_i) = S''_{i1}$. Note that the set S''_{i1} is both compact and open in S''_i; thus, S''_{i1} is the union of some connected components of S''_i and $S''_{i2} := S''_i \setminus S''_{i1}$ is a closed subset of $\mathbb{R}^{s+1+k}$. Since ψ_2^i is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\jmath_{S_i}$, the coordinate hyperplane $\{x_{s+1} = 0\}$ of $\mathbb{R}^{s+1+k}$ is transverse to S''_{i1} in $\mathbb{R}^{s+1+k}$, $S''_{i1} \cap \{x_{s+1} = 0\} = M'_i \sqcup (Y_i \times \{0\} \times \{0\})$ for some compact $\mathscr{C}^\infty$ submanifold M'_i of $\mathbb{R}^{s+1+k}$ and there exists a $\mathscr{C}^\infty$ embedding $\psi_3^i : M_i \to \mathbb{R}^{s+1+k}$ arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to the inclusion map $\jmath_{M_i} : M_i \hookrightarrow \mathbb{R}^{s+1+k}$ such that $M'_i = \psi_3^i(M_i)$. Observe that, by construction $M'_i \subset M'$, for every $i \in \{1, \ldots, \ell\}$, are in general position. Define $M''_i := M'' \cap S''_i$, for every $i \in \{1, \ldots, \ell\}$. By (ix), (x) & (xi), we deduce that M''_i, for every $i \in \{1, \ldots, \ell\}$, are $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of M'' in general position and there exists a $\mathscr{C}^\infty$ embedding $\psi_4^i : M_i \to \mathbb{R}^{s+1+k}$ arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to the inclusion map $\jmath_{M'_i} : M'_i \hookrightarrow \mathbb{R}^{s+1+k}$ such that $M''_i = \psi_4^i(M'_i)$, for every $i \in \{1, \ldots, \ell\}$. Consider the embeddings $\psi_i : M_i \to \mathbb{R}^{s+1+k}$ defined as $\psi_i := \psi_3^i \circ \psi_4^i$. Then, ψ_i is $\mathscr{C}^\infty_\mathrm{w}$ close to $\jmath_{M_i}$ and $\psi(M_i) = M''_i$, for every $i \in \{1, \ldots, \ell\}$. As a consequence, $\psi_i \circ (\psi|)^{-1}| : \psi(M_i) \to M''_i \subset M''$ is a $\mathscr{C}^\infty_\mathrm{w}$ diffeomorphism $\mathscr{C}^\infty_\mathrm{w}$ close to $\jmath_{\psi(M_i)} : \psi(M_i) \hookrightarrow \mathbb{R}^{s+1+k}$, for every $i \in \{1, \ldots, \ell\}$. Thus, [AK81b, Lemma 2.9] ensures the existence of a $\mathscr{C}^\infty$ diffeomorphism $\psi_5 : M'' \to M''$ such that $\psi_5(\psi(M_i)) = M''_i$ and ψ_5 is $\mathscr{C}^\infty_\mathrm{w}$ close to $\jmath_{M'} : M' \hookrightarrow \mathbb{R}^{s+1+k}$.

Hence, by setting "M'" $:= M''$ and "M'_i" $:= M''_i$, for every $i \in \{1, \ldots, \ell\}$, and the $\mathscr{C}^\infty$ diffeomorphism $h : M \to M'$ as "h" $:= \psi_5 \circ \psi_4 \circ \psi_3$ we get the wondered projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic model $M' \subset \mathbb{R}^{s+1+k}$ of M with $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets $\{M'_i\}_{i=1}^\ell$ in general position such that there exists a $\mathscr{C}^\infty$ diffeomorphism $h : M \to M'$ satisfying $h(M_i) = M'_i$, for every $i \in \{1, \ldots, \ell\}$, $\jmath \circ h \in \mathcal{U}$ and $\jmath \circ h|_{M_i} \in \mathcal{U}_i$, for every $i \in \{1, \ldots, \ell\}$, where $\jmath : M' \hookrightarrow \mathbb{R}^{s+1+k}$ denotes the inclusion map. Finally, applying $\mathbb{Q}$-generic projection (see Lemma 2.1.8) we may suppose that the projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets $M', M'_1, \ldots, M'_\ell \subset \mathbb{R}^m$, with $m := \max(n, 2d + 1)$.

Assume in addition that M and each M_i are Nash manifolds, for every $i \in \{1, \ldots, \ell\}$. By Theorem 4.1.2 we can assume that $h : M \to M'$ is a Nash diffeomorphism such that $h(M_i) = M'_i$, for every $i \in \{1, \ldots, \ell\}$. Moreover, an application of [Jel09] provides a semi-algebraic homeomorphism $\mathbb{R}^m \to \mathbb{R}^m$ extending h, as desired. $\square$

Remark 4.1.5. In the statement of Theorem 4.1.4 we can add the following requirement: " $M' \subset \mathbb{R}^m$ contains an hypersurface of rational points, that is, $\dim(\mathrm{Zcl}_{\mathbb{R}^m}(M'(\mathbb{Q}))) \geq d - 1$".

Indeed, up to perform a small translation and rotation we may suppose that there is $a \in (M \setminus \bigcup_{i=1}^\ell M_i) \cap \mathbb{Q}^n$ and the tangent space T_aM of M at a has equation over the rationals. Then, consider a sphere $\mathbb{S}^{n-1}(a, r)(a, r)$ centred at a of radius $r \in \mathbb{Q}$ such that $\mathbb{S}^{n-1}(a, r) \cap \bigcup_{i=1}^\ell M_i = \varnothing$. Observe that $\mathbb{S}^{n-1}(a, r) \cap T_aM$ is a

$\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set of dimension $d-1$ having Zariski dense (actually Euclidean dense) rational points. Choose neighborhoods U'_a and U_a of a in M such that $\mathbb{S}^{n-1}(a,r)\cap T_aM \subset U'_a$ and $\overline{U'_a} \subset U_a$ and neighborhoods V and V' of $\bigcup_{i=1}^{\ell} M_i$ in M such that $\overline{V'} \subset V$. By a partition of unity argument, we may find a $\mathscr{C}^\infty$ manifold $\widetilde{M} \subset \mathbb{R}^n$ such that:

(i) $M_{\ell+1} := \mathbb{S}^{n-1}(a,r) \cap T_aM \subset \widetilde{M}$ and $\{M_i\}_{i=1}^{\ell+1}$ are $\mathscr{C}^\infty$ submanifolds of $\widetilde{M}$ in general position.

(ii) $\mathbb{S}^{n-1}(a,r) \cap T_aM \subset \widetilde{M}(\mathbb{Q})$, thus $\dim(\mathrm{Zcl}_{\mathbb{R}^m}(\widetilde{M}(\mathbb{Q}))) \geq d-1$.

(iii) We may choose $\widetilde{M}$ to be diffeomorphic to M, in addition, by [AK81b, Lemma 2.9], there exists a diffeomorphism $\widetilde{\phi} : M \to \widetilde{M}$ such that $\widetilde{\phi}|_{M_i} = \mathrm{id}_{M_i}$, for every $i \in \{1,\dots,\ell\}$, and $\jmath_{\widetilde{M}} \circ \widetilde{\phi}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\jmath_M$, where $\jmath_M : M \hookrightarrow \mathbb{R}^n$ and $\jmath_{\widetilde{M}} : \widetilde{M} \hookrightarrow \mathbb{R}^n$ denote the inclusion maps.

(iv) Suppose that in addition $M, M_1, \dots, M_\ell \subset \mathbb{R}^n$ are Nash manifolds. By Theorem 4.1.2, we may suppose that above diffeomorphism $\widetilde{\phi} : M \to \widetilde{M}$ is actually a Nash diffeomorphism such that $\widetilde{\phi}|_{M_i} = \mathrm{id}_{M_i}$ and $\jmath_{\widetilde{M}} \circ \widetilde{\phi}$ is arbitrarily $\mathcal{N}_{\mathrm{w}}$ close to $\jmath_M$.

Then, it suffices to substitute "M" $:= \widetilde{M}$, "ℓ" $:= \ell+1$ and fix "$M_{\ell+1}$" $:= \mathbb{S}^{n-1}(a,r)\cap T_aM$ in the proof of Theorem 4.1.4 and observe that, being $M_{\ell+1}$ a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subset of $\mathbb{R}^n$ contained in M such that $M_{\ell+1} \cap \bigcup_{i=1}^{\ell} M_i = \varnothing$ and the Gauss mapping of M restricted to $M_{\ell+1}$ is $\mathbb{Q}$-regular, we can keep $M_{\ell+1}$ fixed during the approximation steps. This ensures that $M_{\ell+1} = \mathbb{S}^{n-1}(a,r)\cap T_aM \subset M'(\mathbb{Q})$, thus $\dim(\mathrm{Zcl}_{\mathbb{R}^m}(M'(\mathbb{Q}))) \geq d-1$, as desired. ∎

4.1.3. The non-compact case. The relative Nash-Tognoli theorem 'over $\mathbb{Q}$', namely above Theorem 4.1.4, can be extended also to the non-compact case when $M \subset \mathbb{R}^n$ and each $M_i \subset \mathbb{R}^n$ are nonsingular algebraic sets. The strategy is to apply algebraic compactification getting a compact algebraic set with (eventually) only one isolated singularity, apply Hironaka's desingularization theorem, apply our approximation results 'over $\mathbb{Q}$' of Chapter 3 and our blowing down lemma 'over $\mathbb{Q}$', namely Lemma 3.3.3. Hence, next theorem provides a complete positive answer to the RELATIVE $\mathbb{Q}$-ALGEBRIZATION PROBLEM FOR NONSINGULAR ALGEBRAIC SETS.

THEOREM 4.1.6. *Let V be a nonsingular algebraic subset of $\mathbb{R}^n$ of dimension d and let $\{V_i\}_{i=1}^{\ell}$ be a finite family of nonsingular algebraic subsets of V in general position. Set $m := n+2d+3$. Then, for every neighborhood $\mathcal{U}$ of the inclusion map $\iota : V \hookrightarrow \mathbb{R}^m$ in $\mathcal{N}_{\mathrm{w}}(V,\mathbb{R}^m)$ and for every neighborhood $\mathcal{U}_i$ of the inclusion map $\iota|_{V_i} : V_i \hookrightarrow \mathbb{R}^m$ in $\mathcal{N}_{\mathrm{w}}(V_i,\mathbb{R}^m)$ for every $i \in \{1,\dots,\ell\}$, there exist a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $V' \subset \mathbb{R}^m$, a family $\{V'_i\}_{i=1}^{\ell}$ of $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets of V' in general position and a Nash diffeomorphism $h : V \to V'$ which simultaneously takes each V_i to V'_i such that, if $\jmath : V' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map, then $\jmath \circ h \in \mathcal{U}$ and $\jmath \circ h|_{M_i} \in \mathcal{U}_i$ for every $i \in \{1,\dots,\ell\}$. Moreover, h extends to a semialgebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$.*

PROOF. Let c_i be the codimension of M_i in M for every $i \in \{1,\dots,\ell\}$. We can assume V is noncompact. If $V = \mathbb{R}^n$, then it suffices to identify V with the algebraic set $V \times \{0\} \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and next proof continues to work with the same estimate $m = n+2d+3$. Up to translate V and each V_i with $i \in \{1,\dots,\ell\}$, of a very small vector we may suppose that the origin 0 of $\mathbb{R}^n$ is not contained in V. Let $s, s_1, \dots, s_\ell \in \mathbb{R}[x]$ such that $\mathcal{Z}_\mathbb{R}(s) = V$ and $\mathcal{Z}_\mathbb{R}(s_i) = V_i$, for every $i \in \{1, \dots, \ell\}$. Let $\mathbb{S}^{n-1}$ be the standard unit sphere of $\mathbb{R}^n$ and let $\theta : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ as $\theta(x) = \frac{x}{|x|_n^2}$ be the inversion with respect to $\mathbb{S}^{n-1}$. Recall that $\theta \circ \theta = \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$. Let $e \geq \max\{\deg(s), \deg(s_1), \dots, \deg(s_\ell)\}$. Define the polynomials $t := |x|_n^{2e} \cdot (s \circ \theta)(x) \in \mathbb{R}[x]$, $t_i := |x|_n^{2e}(s_i \circ \theta(x)) \in \mathbb{R}[x]$, the compact algebraic sets $\widetilde{V} := \mathcal{Z}_\mathbb{R}(t)$ and $\widetilde{V}_i := \mathcal{Z}_\mathbb{R}(t_i)$, for every $i \in \{1, \dots, \ell\}$. By construction, $\widetilde{V} = \theta(V) \sqcup \{0\}$, $\widetilde{V}_i = \theta(V)_i \sqcup \{0\}$, for every $i \in \{1, \dots, \ell\}$, and $\theta : V \to \widetilde{V} \setminus \{0\}$ is a $\mathbb{Q}$-biregular map between the algebraic set V and the Zariski open subset $\widetilde{V} \setminus \{0\}$ of $\widetilde{V}$. In general, 0 may be a singular point of $\widetilde{V}$ and $\widetilde{V}_i$ for $i \in \{1, \dots, \ell\}$.

By a relative version of Hironaka's desingularization theorem (see [AK92, Lemma 6.2.3]) there are a finite set $J \subset \mathbb{N} \setminus \{1, \dots, \ell\}$, nonsingular algebraic sets X, X_i and E_j, for every $i \in \{1, \dots, \ell\}$ and $j \in J$, and a regular map $p : X \to \widetilde{V}$ satisfying the following properties:

(i) E_j is an algebraic hypersurface of X for every $j \in J$ and $\bigcup_{j \in J} E_j = p^{-1}(0)$;
(ii) the nonsingular algebraic sets $\{X_i\}_{i=1}^\ell \sqcup \{E_j\}_{j \in J}$ are in general position;
(iii) $p|_{X \setminus \bigcup_{j \in J} E_j} : X \setminus \bigcup_{j \in J} E_j \to \widetilde{V}$ is biregular.
(iv) $p(X_i) = V_i$ for every $i \in \{1, \dots, \ell\}$.

An application of Theorem 4.1.4 with the following substitutions: "M":$= X$, "ℓ":$= \ell + |J|$,"M_i":$= X_i$ for every $i \in \{1, \dots, \ell\}$, "M_j":$= E_j$ for every $j \in J$, gives a projectively $\mathbb{Q}$-closed $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $X' \subset \mathbb{R}^{2d+1}$ of dimension d, $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets X'_i for $i \in \{1, \dots, \ell\}$, and $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic hypersurfaces E'_j, for $j \in J$, of X' in general position and a Nash diffeomorphism $\phi : X \to X'$ such that $\phi(X_i) = X'_i$ for every $i \in \{1, \dots, \ell\}$, and $\phi(E_j) = E'_j$, for every $j \in J$.

Consider the Nash map $p' := p \circ \phi^{-1} : X' \to \widetilde{V}$ such that $(p')^{-1}(0) = \bigcup_{j \in J}^\ell E'_i$. By Lemma 3.1.6, $\bigcup_{j \in J}^\ell E'_i \subset \mathbb{R}^{2d+1}$ is $\mathbb{Q}$-nice, thus we can apply Lemma 3.1.8 with "L" $=$ "P" $:= \bigcup_{j \in J} E'_j$ to each entry of any smooth extension $\mathbb{R}^{2d+1} \to \mathbb{R}^n$ of $p' : X' \to \mathbb{R}^n$ getting a polynomial map $q := (q_1, \dots, q_n) : \mathbb{R}^{2d+1} \to \mathbb{R}^n$ such that $q|_{X'}$ is arbitrarily $\mathcal{N}_\mathrm{w}$ close to p' and $q_i \in \mathcal{I}_\mathbb{Q}(\bigcup_{j \in J} E'_j)$.

Finally, an application of Lemma 3.3.3 with the following substitutions: "X":$= X'$, "Y":$= \{0\}$, "A":$= \bigcup_{j \in J} E'_j$, "p":$= q|_{\bigcup_{j \in J} E'_j}$ and "P":$= q$ gives a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set $\widetilde{V}' \subset \mathbb{R}^{2d+1} \times \mathbb{R}^n \times \mathbb{R}$ of dimension d with (eventually) only an isolated singularity at the origin 0 of $\mathbb{R}^{2d+1} \times \mathbb{R}^n \times \mathbb{R}$, such that $\widetilde{V}'_i := f(\widetilde{V}_i) \cup \{0\}$, for every $i \in \{1, \dots, \ell\}$, is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic subset of $\widetilde{V}'$ of codimension c_i with (eventually) only an isolated singularity at the origin 0 of $\mathbb{R}^{2d+1} \times \mathbb{R}^n \times \mathbb{R}$, where $f : X'' \to \widetilde{V'}$ denotes the $\mathbb{Q}$-regular map of Lemma 3.3.3, and a semialgebraic homeomorphism $\widetilde{h} : \widetilde{V} \to \widetilde{V}'$ defined as:

$$\widetilde{h}(x) = \begin{cases} 0 & \text{if } x = 0 \in \mathbb{R}^n, \\ f \circ \phi \circ p^{-1}(x) & \text{otherwise.} \end{cases}$$

Let $m' := 2(d+1)+n$. Observe that $\widetilde{h}|_{\widetilde{V}\setminus\{\overline{0}\}} : \widetilde{V}\setminus\{\overline{0}\} \to \widetilde{V}'\setminus\{\overline{0}\}$ is a Nash diffeomorphism and $\widetilde{h}|_{\widetilde{V}_i} : \widetilde{V}_i \to \widetilde{V}_i'$ is a semialgebraic homeomorphism satisfying the following approximation properties:

(iv) $\widetilde{h}$ is arbitrarily $\mathscr{C}^0_{\mathrm{w}}$ close to $\iota_{\widetilde{V}}$ and $\widetilde{h}|_{V\setminus\{0\}}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\iota_{\widetilde{V}}|_{\widetilde{V}\setminus\{\overline{0}\}}$,

(v) $\widetilde{h}_{\widetilde{V}_i}$ is arbitrarily $\mathscr{C}^0_{\mathrm{w}}$ close to $\iota_{\widetilde{V}}|_{\widetilde{V}_i}$ and $\widetilde{h}_{\widetilde{V}_i\setminus\{0\}}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\iota_{\widetilde{V}}|_{\widetilde{V}_i\setminus\{\overline{0}\}}$,

where $\iota_{\widetilde{V}} : \widetilde{V} \hookrightarrow \mathbb{R}^{m'}$ denotes the inclusion map.

Let $t', t_1', \dots, t_\ell' \in \mathbb{Q}[x_1,\dots,x_{m'}]$ such that $\mathcal{Z}_\mathbb{R}(t') = \widetilde{V'}$ and $\mathcal{Z}_\mathbb{R}(t_i') = \widetilde{V_i'}$ for every $i \in \{1,\dots,\ell\}$. Let $\mathbb{S}^{m-1}$ be the standard unit sphere of $\mathbb{R}^{m'}$ and let $\theta' : \mathbb{R}^{m'}\setminus\{0\} \to \mathbb{R}^{m'}\setminus\{0\}$ as $\theta'(x) = \frac{x}{|x|^2_{m'}}$ be the inversion with respect to $\mathbb{S}^{m'-1}$. Recall that $\theta'\circ\theta' = \mathrm{id}_{\mathbb{R}^{m'}\setminus\{0\}}$. Let $e' > \max\{\deg(t'), \deg(t_1'), \dots, \deg(t_\ell')\}$. Define the polynomials $s' := |x|^{2e'}_{m'}\cdot(t'\circ\theta')(x) \in \mathbb{Q}[x]$, $s_i' := |x|^{2e'}_{\mathbb{R}^{m'}}(t_i'\circ\theta')(x) \in \mathbb{Q}[x]$, the algebraic sets $V' := \mathcal{Z}_\mathbb{R}(s')$ and $V_i := \mathcal{Z}_\mathbb{R}(s_i')$, for every $i \in \{1,\dots,\ell\}$. By construction,

$$V' = \theta'(\widetilde{V}'\setminus\{0\})\cup\{0\} \quad \text{and} \quad V_i' = \theta'(\widetilde{V}_i'\setminus\{0\})\cup\{0\},$$

for every $i \in \{1,\dots,\ell\}$, and $\theta' : \widetilde{V}'\setminus\{0\} \to V'\cup\{0\}$ is a $\mathbb{Q}$-biregular map between Zariski open subsets of $\mathbb{Q}$-algebraic sets. Moreover, $\theta'(\widetilde{V}_i'\setminus\{0\}) = V_i'\cup\{0\}$ for every $i \in \{1,\dots,\ell\}$. Observe that, by construction, the $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets $\{V_i'\}_{i=1}^\ell$ are in general position. Let $C \in \mathbb{Q}\setminus\{0\}$ and define the $\mathbb{Q}$-algebraic sets

$$V'' := \Big\{(x,y)\in\mathbb{R}^{m'}\times\mathbb{R} \,\Big|\, y\sum_{k=1}^{m'} x_k^2 = C,\, s'(x) = 0\Big\},$$
$$V_i'' := \Big\{(x,y)\in\mathbb{R}^{m'}\times\mathbb{R} \,\Big|\, y\sum_{k=1}^{m'} x_k^2 = C,\, s_i'(x) = 0\Big\}$$

for every $i \in \{1,\dots,\ell\}$. By construction, V'' and V_i'' are $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets, for every $i \in \{1,\dots,\ell\}$, $V'\setminus\{0\}$ and V'' are $\mathbb{Q}$-biregularly isomorphic via projection $\pi : \mathbb{R}^{m'}\times\mathbb{R} \to \mathbb{R}^{m'}$, $\pi|_{V_i''} : V_i'' \to V_i'\setminus\{0\}$, for every $i \in \{1,\dots,\ell\}$, and the $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic sets $\{V_i''\}_{i=1}^\ell$ are in general position.

Define the Nash diffeomorphism $h : V \to V''$ as

$$h := (\pi|_{V''})^{-1}\circ\theta'|_{\widetilde{V}'\setminus\{\overline{0}\}}\circ\widetilde{h}|_{\widetilde{V}\setminus\{\overline{0}\}}\circ\theta|_V.$$

Let $m := m'+1 = 2d+n+3$. If we fix $C \in \mathbb{Q}\setminus\{0\}$ be sufficiently small, by (iv), (v) and the choice of $\widetilde{h}$ as above, we deduce that $h|_{V_i} : V_i \to V_i''$ is a Nash diffeomorphism, $\jmath_{V''}\circ h$ is $\mathscr{C}^\infty_{\mathrm{w}}$ close to the inclusion $\iota_V : V \hookrightarrow \mathbb{R}^m$ and $h|_{V_i}$ is $\mathscr{C}^\infty_{\mathrm{w}}$ close to the inclusion map $\iota_{V_i} : V_i \hookrightarrow \mathbb{R}^m$, where $\jmath_{V'} : V' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map. Moreover, an application of [Jel09] provides a semialgebraic homeomorphism $\mathbb{R}^m \to \mathbb{R}^m$ extending h, as desired. $\square$

Remark 4.1.7. In the statement of Theorem 4.1.6 we can add the following requirement: " $V' \subset \mathbb{R}^m$ contains an hypersurface of rational points, that is, $\dim(\mathrm{Zcl}_{\mathbb{R}^m}(V'(\mathbb{Q}))) \geq d-1$".

By Remark 4.1.5 we may suppose that $X' \subset \mathbb{R}^{2d+1}$ in the proof of Theorem 4.1.6 is such that $\dim(\mathrm{Zcl}_{\mathbb{R}^{2d+1}}(X'(\mathbb{Q}))) \geq n-1$. In addition, since $\mathbb{Q}$-biregular maps send rational points to rational points, as $f|_{X'\setminus(\bigcup_{j\in J} E'_j)} : X' \setminus (\bigcup_{j\in J} E'_j) \to \widetilde{V'} \setminus \{0\}$ and $\theta' : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$ are, we get that $\dim(\mathrm{Zcl}_{\mathbb{R}^{2d+1}}(X'(\mathbb{Q}) \setminus (\bigcup_{j\in J} E'_j))) \geq d-1$ and $(\theta' \circ f)(X'(\mathbb{Q}) \setminus (\bigcup_{j\in J} E'_j)) = V'(\mathbb{Q})$, hence, being both f and θ' biregular, $\dim(\mathrm{Zcl}_{\mathbb{R}^m}(V'(\mathbb{Q}))) \geq d-1$, as desired. ∎

4.2. $\mathbb{Q}$-Algebrization of Nash manifolds over real closed fields

Throughout this section R denotes a real closed field. Let us generalize the concept of semialgebraic and Nash sets and functions over R. Our main result is the $\mathbb{Q}$-algebrization theorem for Nash manifolds $M \subset R^n$ improving [CS92, Corollary 3.9].

A subset S of R^n is *semialgebraic* if it is a Boolean combination of subsets of R^n defined by polynomial equations and polynomial strict inequalities. Observe that, by quantifier elimination of the theory of real closed fields (see [BCR98, Theorem 1.4.2 & Proposition 5.2.2]), $S \subset R^a$ is semialgebraic if and only if it is described by a first-order formula in the language of ordered fields. A locally closed semialgebraic set $M \subset R^n$ is called (affine) *Nash manifold* if it is also a $\mathscr{C}^\infty$ submanifold of R^n. Let $M \subset R^n$ be a Nash manifold, let $X \subset M$ be a (non-empty) semialgebraic subset of R^n contained in M, and let $Y \subset R^m$ be a (non-empty) semialgebraic set. Let $\nu \in \mathbb{N}^* \cup \{\infty\}$ and let $f : X \to Y$ be a map. We say that f is a *$\mathscr{C}^\nu$ map* if there exist an open (not necessarily semialgebraic) neighborhood U of X in M and a map $F : U \to R^m$ such that F is of class $\mathscr{C}^\nu$ in the usual sense of $\mathscr{C}^\infty$ (and hence $\mathscr{C}^\nu$) manifolds and $F(x) = f(x)$ for all $x \in X$. We denote $\mathscr{C}^\nu(X, Y)$ the set of $\mathscr{C}^\nu$ maps from X to Y. The map $f : X \to Y$ is called *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$. The map $f : X \to Y$ is said to be a *Nash map* if there exist an open semialgebraic neighborhood U of X in M and a $\mathscr{C}^\infty$ map $F : U \to \mathbb{R}^m$ such that F is semialgebraic and $F(x) = f(x)$ for all $x \in X$. The map $f : X \to Y$ is said to be a *Nash diffeomorphism* if f is a bijective Nash map and f^{-1} is a Nash map too.

Let $R_2|R_1$ be a field extension such that both R_1 and R_2 are real closed fields. We present a version of Definition 1.1.18 in the case of semialgebraic sets.

Definition 4.2.1. Let $S \subset R_1^n$ be an algebraic set and let ϕ be a first-order formula in the language of ordered fields (with coefficients in R_1) such that $S = \{x \in R_1^n \mid \phi(x)\}$. We say that $S_{R_2} := \{x \in R_2^n \mid \phi(x)\}$ is the extension of coefficients of S to R_2. ∎

Observe that, by model completeness of the theory of real closed fields, the semialgebraic set $S_{R_2} \subset R_2^n$ in Definition 4.2.1 only depends on $S \subset R_1^n$, so the definition is well posed. For more details about extension of coefficients and model theoretical properties of the theory of real closed fields we refer to [BCR98, §5].

Let us introduce the main result of this section.

Theorem 4.2.2. *Let $M \subset R^n$ be a Nash manifold of dimension d. Then, there exists a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $M' \subset R^m$ and a Nash diffeomorphism h :*

$M \to M'$, for some $m \in \mathbb{N}$ with $m \geq n$. In particular, $M' \subset R^m$ can be chosen in such a way that $\dim(\mathrm{Zcl}_{R^m}(M'(\mathbb{Q}))) \geq d-1$.

Proof. Denote by $\overline{\mathbb{Q}}^r$ the real closure of $\mathbb{Q}$. By [CS92, Corollary 3.3], there exists a nonsingular algebraic set $X \subset (\overline{\mathbb{Q}}^r)^{n'}$ and a Nash diffeomorphism $h' : M \to X_R$, for some $n' \in \mathbb{N}$ with $n' \geq n$. Consider $X_\mathbb{R} \subset \mathbb{R}^{n'}$ as a $\overline{\mathbb{Q}}^r$-algebraic set (actually $X_\mathbb{R} \subset \mathbb{R}^{n'}$ is defined over $\overline{\mathbb{Q}}^r$ by Proposition1.1.19(ii)). An application of Theorem 4.1.4 ensures the existence of a $\mathbb{Q}$-algebraic set $X' \subset (\overline{\mathbb{Q}}^r)^m$ and a Nash diffeomorphism $\varphi : X_\mathbb{R} \to X'_\mathbb{R}$ such that $X'_\mathbb{R} \subset \mathbb{R}^m$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set, for some $m \in \mathbb{N}$ with $m \geq n'$. Moreover, the Nash diffeomorphism $\varphi : X_\mathbb{R} \to X'_\mathbb{R}$ can be approximated by a Nash diffeomorphism described by a first order formula with coefficients in $\overline{\mathbb{Q}}^r$. Let us prove this assertion. The normal bundle of $X'_\mathbb{R}$ in $\mathbb{R}^m$ is the extension over $\mathbb{R}$ of the normal bundle of X' in $(\overline{\mathbb{Q}}^r)^m$, thus it is a Nash submanifold of $\mathbb{R}^m$ defined by a first order formula with coefficients in $\overline{\mathbb{Q}}^r$ whose retraction on $X'_\mathbb{R}$ is a Nash map defined by a first order formula with coefficients in $\overline{\mathbb{Q}}^r$ as well. Hence, it suffices to approximate by Weierstrass approximation the diffeomorphism φ with a polynomial map whose coefficients lie over $\overline{\mathbb{Q}}^r$ and compose it with the retraction. Thus, the resulting Nash diffeomorphism $\psi_\mathbb{R} : X_\mathbb{R} \to X'_\mathbb{R}$ is the extension over $\mathbb{R}$ of a Nash diffeomorphism $\psi : X \to X'$.

Let $\mathcal{I}_\mathbb{Q}(X'_\mathbb{R}) = (q_1, \dots, q_s) \subset \mathbb{Q}[x]$. Observe that the property of the $\mathbb{Q}$-algebraic set $X'_\mathbb{R} \subset \mathbb{R}^m$ of being $\mathbb{Q}$-nonsingular can be expressed by a first-order sentence with coefficients over $\mathbb{Q}$ corresponding to the following assertion: *"For every $a \in X'_\mathbb{R}$ the rank of the Jacobian matrix $J_Q(a)$ of $Q := (q_1, \dots, q_s) : \mathbb{R}^m \to \mathbb{R}^s$ at a is $m-d$ (with $d = \dim_R(M) = \dim_\mathbb{R}(X_\mathbb{R})$) and there are $q_{i_1,a}, \dots, q_{i_{m-d},a}$ polynomials among the $q_i's$ such that*

$$\mathcal{Z}_{\mathbb{R}^m}(q_{i_1,a}, \dots, q_{i_{m-d},a}) \cap U = V_\mathbb{R} \cap U,$$

where $U = \mathbb{R}^m \setminus \{y \in \mathbb{R}^m \mid \dim(J_{(q_{i_1,x},\dots,q_{i_{m-d},x})}(y) < m-d)\}$".

Since above first-order sentence can be expressed with coefficients over $\mathbb{Q}$ (i.e. without coefficients), $X' \subset (\overline{\mathbb{Q}}^r)^m$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set by model completeness of the theory of real closed fields. Again, by model completeness of the theory of real closed fields, we get that $\psi_R : X_R \to X'_R$ is a Nash diffeomorphism and $X'_R \subset R^m$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. So it suffices to fix "M'" $:= X'_R$ and "h" $:= \psi_R \circ h'$. In addition, since the $\mathbb{Q}$-algebraic set $X'_\mathbb{R} \subset \mathbb{R}^m$ can be chosen such that $\dim(\mathrm{Zcl}_{\mathbb{R}^m}(X'_\mathbb{R}(\mathbb{Q}))) \geq d-1$ and $X'_\mathbb{R}(\mathbb{Q}) = X'(\mathbb{Q}) = X'_R(\mathbb{Q})$, Proposition 1.1.19(iv) ensures that

$$\begin{aligned}\dim_R(\mathrm{Zcl}_{R^m}(M'(\mathbb{Q}))) &= \dim_R(\mathrm{Zcl}_{R^m}(X'_R(\mathbb{Q}))) = \dim_{\overline{Q}^r}(\mathrm{Zcl}_{(\overline{Q}^r)^m}(X'(\mathbb{Q}))) \\ &= \dim_\mathbb{R}(\mathrm{Zcl}_{\mathbb{R}^m}(X'_\mathbb{R}(\mathbb{Q}))) \geq d-1,\end{aligned}$$

as desired. □

4.3. Global $\mathbb{Q}$-algebrization of isolated singularities

This section is devoted to provide a complete positive answer to [Par21, Open problem 1, p. 199] in the case of algebraic sets $V \subset \mathbb{R}^n$ with isolated singularities. First we deal with the compact case and then we extend the proof to the general one. Let us start by proving some preparatory results.

4.3.1. Small Lagrange-type interpolations. In this subsection we prove a technical result which will allow us to apply our blowing down lemma 'over $\mathbb{Q}$', namely Lemma 3.3.3, in the case of algebraic sets with isolated singularities in Subsection 4.3.2.

Let $\beta \in \mathbb{N}^*$ and let $c = (c_1, \dots, c_\beta) \in \mathbb{R}^\beta$ be such that $c_j \neq c_{j'}$ for all $j, j' \in \{1, \dots, \beta\}$ with $j \neq j'$. We denote $\mathcal{L}_{c,1}, \dots, \mathcal{L}_{c,\beta} \in \mathbb{R}[x]$ the *Lagrange basis polynomials associated to* c, i.e.,

$$\mathcal{L}_{c,j}(x) := \prod\nolimits_{s\in\{1,\dots,\beta\}\setminus\{j\}} \left(\frac{x - c_s}{c_j - c_s}\right). \tag{4.3.1}$$

If $d = (d_1, \dots, d_\beta) \in \mathbb{R}^\beta$, then $\sum_{j=1}^\beta d_j \mathcal{L}_{c,j}$ is the Lagrange interpolation polynomial associated to c and d. This polynomial coincides with the unique polynomial $\mathcal{L} \in \mathbb{R}[x]$ of degree $< \beta$ such that $\mathcal{L}(c_j) = d_j$ for all $j \in \{1, \dots, \beta\}$. Furthermore, if each d_j is sufficiently small then $\mathcal{L}$ is arbitrarily $\mathscr{C}^\infty$ small on compact subsets of $\mathbb{R}$. We need a variant of this result in which we permit that the interpolating function $\mathcal{L}$ is regular (not only polynomial), it vanishes on a finite set disjoint from $\{c_1, \dots, c_\beta\}$ and its smallness is also controlled at infinity. The mentioned variant is as follows.

PROPOSITION 4.3.1. *Let A be a finite subset of $\mathbb{R}$, let $\beta \in \mathbb{N}^*$ and let $b_1, \dots, b_\beta \in \mathbb{R}$ be real numbers such that $b_j \neq b_{j'}$ for all $j, j' \in \{1, \dots, \beta\}$ with $j \neq j'$, and $A \cap \{b_1, \dots, b_\beta\} = \varnothing$. Let $k, m \in \mathbb{N}$ and let $\epsilon \in \mathbb{R}^+$. Then there exists $\delta \in \mathbb{R}^+$ with the following property: for each $c = (c_1, \dots, c_\beta) \in \mathbb{R}^\beta$ such that $|b_j - c_j| < \delta$ for all $j \in \{1, \dots, \beta\}$, there exists a regular function $L_c : \mathbb{R} \to \mathbb{R}$ such that:*

(a) *$L_c(a) = 0$ for all $a \in A$.*
(b) *$L_c(c_j) = b_j - c_j$ for all $j \in \{1, \dots, \beta\}$.*
(c) *$|D_h L_c(x)| < \epsilon(1 + x^2)^{-k}$ for all $h \in \{0, \dots, m\}$ and for all $x \in \mathbb{R}$, where D_h denotes the h^{th} derivative operator.*

PROOF. Let us assume that $A \neq \varnothing$. If $A = \varnothing$, the proof we present below (suitably simplified) continues to work. Let $a_1, \dots, a_\alpha$ be the elements of A. Choose a natural number ℓ such that

$$\alpha + \beta - 1 - 2\ell \leq -2k$$

and define:

$$\begin{aligned}
\delta_1 &:= \tfrac{1}{3} \min\nolimits_{i,j\in\{1,\dots,\beta\}, i\neq j} |b_i - b_j| > 0,\\
\delta_2 &:= \tfrac{1}{2} \min\nolimits_{i\in\{1,\dots,\alpha\}, j\in\{1,\dots,\beta\}} |a_i - b_j| > 0,\\
\delta_3 &:= \min\{\delta_1, \delta_2\} > 0,\\
K &:= \bigcup\nolimits_{j=1}^\beta [b_j - \delta_3, b_j + \delta_3],\\
H &:= \prod\nolimits_{j=1}^\beta [b_j - \delta_3, b_j + \delta_3],\\
p(x) &:= \prod\nolimits_{i=1}^\alpha (x - a_i) \in \mathbb{R}[x].
\end{aligned}$$

Note that $\mathrm{dist}(A, K) \geq 2\delta_2 - \delta_3 \geq \delta_2 > 0$; thus, p never vanishes on the compact subset K of $\mathbb{R}$ and hence $M := \max_{x\in K} |p(x)|^{-1}(1 + x^2)^\ell$ is finite (and positive). It follows that

$$|p(x)|^{-1}(1 + x^2)^\ell \leq M \quad \text{for all } x \in K. \tag{4.3.2}$$

For each $j \in \{1,\dots,\beta\}$ and $w \in \{1,\dots,\alpha+\beta-1\}$, let $q_{j,w}(y) \in \mathbb{R}[y] = \mathbb{R}[y_1,\dots,y_\beta]$ be the (unique) polynomials such that

$$p(x)\prod_{s\in\{1,\dots,\beta\}\setminus\{j\}}(x-y_s) = \sum_{w=0}^{\alpha+\beta-1} q_{j,w}(y)x^w \tag{4.3.3}$$

for all $(x,y) \in \mathbb{R}\times\mathbb{R}^\beta$. Note that, for each $j,j' \in \{1,\dots,\beta\}$ with $j \neq j'$, the intervals $[b_j-\delta_3, b_j+\delta_3]$ and $[b_{j'}-\delta_3, b_{j'}+\delta_3]$ are disjoint; indeed, their distance is $\geq 3\delta_1 - 2\delta_3 \geq \delta_1 > 0$. As a consequence,

$$N_{j,w} := \max_{(y_1,\dots,y_n)\in H}\left|q_{j,w}(y)\prod_{s\in\{1,\dots,\beta\}\setminus\{j\}}(y_j-y_s)^{-1}\right|$$

is finite (and positive) for all $j \in \{1,\dots,\beta\}$ and $w \in \{1,\dots,\alpha+\beta-1\}$. Set

$$N := \max_{j\in\{1,\dots,\beta\},w\in\{1,\dots,\alpha+\beta-1\}} N_{j,w} > 0.$$

It follows that

$$\left|q_{j,w}(y)\prod_{s\in\{1,\dots,\beta\}\setminus\{j\}}(y_j-y_s)^{-1}\right| \leq N \tag{4.3.4}$$

for all $j \in \{1,\dots,\beta\}$, $w \in \{1,\dots,\alpha+\beta-1\}$ and $y \in H$. Given $w \in \{1,\dots,\alpha+\beta-1\}$, let $g_w : \mathbb{R}\to\mathbb{R}$ be the $\mathscr{C}^\infty$ function defined by $g_w(x) := x^w(1+x^2)^{-\ell}$. By elementary considerations from calculus, for each $h \in \{0,\dots,m\}$, there exists a constant $L_{w,h} > 0$ such that $|D_h g_w(x)| \leq L_{w,h}(1+x^2)^{(w-2\ell-h)/2}$ for all $x \in \mathbb{R}$. Set

$$L := \max_{w\in\{1,\dots,\alpha+\beta-1\},h\in\{0,\dots,m\}} L_{w,h} > 0.$$

Since $\alpha+\beta-1-2\ell \leq -2k$, we have that

$$|D_h g_w(x)| \leq L_{w,h}(1+x^2)^{(w-2\ell-h)/2} \leq L(1+x^2)^{(\alpha+\beta-1-2\ell)/2} \leq L(1+x^2)^{-k}$$

for all $x \in \mathbb{R}$. It follows that

$$|D_h g_w(x)| \leq L(1+x^2)^{-k} \tag{4.3.5}$$

for all $w \in \{1,\dots,\alpha+\beta-1\}$, $h \in \{0,\dots,m\}$ and $x \in \mathbb{R}$. Set

$$\delta := \min\{\delta_3, (2\beta(\alpha+\beta-1)MNL)^{-1}\epsilon\} > 0.$$

Let $c_j \in \mathbb{R}$ such that

$$|b_j - c_j| < \delta \text{ for each } j \in \{1,\dots,\beta\}. \tag{4.3.6}$$

Note that

$$c_j \in K \text{ for all } j \in \{1,\dots,\beta\}, \text{ and } c := (c_1,\dots,c_\beta) \text{ belongs to } H, \tag{4.3.7}$$

because $\delta \leq \delta_3$. Define the regular function $L_c : \mathbb{R}\to\mathbb{R}$ by setting

$$L_c(x) := \frac{p(x)}{(1+x^2)^\ell}\left(\sum_{j=1}^\beta \frac{(1+c_j^2)^\ell}{p(c_j)}(b_j-c_j)\mathcal{L}_{c,j}(x)\right).$$

Evidently, L_c satisfies items (a) and (b). Let us prove point (c). By (4.3.1) and (4.3.3), for each $x \in \mathbb{R}$, we have

$$\begin{aligned} L_c(x) &= \sum_{j=1}^\beta (b_j-c_j)\frac{(1+c_j^2)^\ell}{p(c_j)}\frac{1}{\prod_{s\neq j}(c_j-c_s)}\cdot\left(p(x)\prod_{s\neq j}(x-c_s)\right)\frac{1}{(1+x^2)^\ell} \\ &= \sum_{j=1}^\beta\sum_{w=1}^{\alpha+\beta-1}(b_j-c_j)\frac{(1+c_j^2)^\ell}{p(c_j)}\frac{q_{j,w}(c)}{\prod_{s\in\{1,\dots,\beta\}\setminus\{j\}}(c_j-c_s)}x^w(1+x^2)^{-\ell}. \end{aligned}$$

As a consequence, by (4.3.2), (4.3.4), (4.3.5), (4.3.6) and (4.3.7), it follows that

$$\begin{aligned} |D_h L_c(x)| &\leq \sum_{j=1}^\beta\sum_{w=1}^{\alpha+\beta-1}\delta MNL(1+x^2)^{-k} = \\ &= \beta(\alpha+\beta-1)MNL\delta(1+x^2)^{-k} \leq \tfrac{\epsilon}{2}(1+x^2)^{-k} < \epsilon(1+x^2)^{-k} \end{aligned}$$

for all $h \in \{0, \dots, m\}$ and for all $x \in \mathbb{R}$, as desired. $\square$

Let τ be the topology of $\mathcal{N}(\mathbb{R}) = \mathcal{N}(\mathbb{R}, \mathbb{R})$ for which a fundamental system of neighborhoods of $f \in \mathcal{N}(\mathbb{R})$ is given by the sets

$$\mathrm{U}_{k,m,\epsilon}(f) := \left\{ g \in \mathcal{N}(\mathbb{R}) \,\middle|\, \left| D_h(g - f)(x) \right| < \tfrac{\epsilon}{(1+x^2)^k}, \forall h \in \{0, \dots, m\}, x \in \mathbb{R} \right\},$$

where $k, m \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$. Recall that D_h denotes the h^{th} derivative operator. The topology τ on $\mathcal{N}(\mathbb{R})$ coincides with the "$\mathscr{C}^\infty$ topology" on $\mathrm{N}^\omega(\mathbb{R}) = \mathcal{N}(\mathbb{R})$ defined in [Shi87, §II.1]. Denote $\mathcal{N}_\tau(\mathbb{R})$ the set $\mathcal{N}(\mathbb{R})$ equipped with the topology τ. Let $\mathscr{D}$ be the subset of $\mathcal{N}(\mathbb{R})$ of all Nash diffeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. By [Shi87, Lemma II.1.7], $\mathscr{D}$ is open in $\mathcal{N}_\tau(\mathbb{R})$ and the map $\mathrm{Inv} : \mathscr{D} \to \mathscr{D}$, sending f into f^{-1}, is continuous with respect to the relative topology induced by τ on $\mathscr{D}$.

Two consequences of Proposition 4.3.1 are as follows.

Corollary 4.3.2. *Let A and B be two finite subsets of $\mathbb{R}$ such that $A \subset \mathbb{Q}$ and $B \subset \mathbb{R} \setminus \mathbb{Q}$. Then, for each neighborhood $\mathcal{U}$ of $\mathrm{id}_\mathbb{R}$ in $\mathcal{N}_\tau(\mathbb{R})$, there exists a Nash diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \in \mathcal{U}$, $\varphi(a) = a$ for all $a \in A$, $\varphi(B) \subset \mathbb{Q}$ and $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$ is a regular map.*

Proof. If $B = \varnothing$, then it suffices to set $\varphi := \mathrm{id}_\mathbb{R}$. Suppose that $B \neq \varnothing$. Let $b_1, \dots, b_\beta$ be the elements of B. Let $\mathcal{U}$ be an arbitrary neighborhood of $\mathrm{id}_\mathbb{R}$ in $\mathcal{N}_\tau(\mathbb{R})$. Choose a neighborhood $\mathcal{V}$ of $\mathrm{id}_\mathbb{R}$ in $\mathcal{N}_\tau(\mathbb{R})$ such that $\mathcal{V} \subset \mathscr{D}$ and $\mathrm{Inv}(\mathcal{V}) \subset \mathcal{U}$. Shrinking $\mathcal{V}$ if necessary, we can assume that there exist $k, m \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$ such that $\mathcal{V} = \mathrm{U}_{k,m,\epsilon}(\mathrm{id}_\mathbb{R})$. Let $\delta \in \mathbb{R}^+$ be a positive real number with the properties (a), (b) and (c) described in Proposition 4.3.1. Choose $c = (c_1, \dots, c_\beta) \in \mathbb{Q}^\beta$ in such a way that $|b_j - c_j| < \delta$ for all $j \in \{1, \dots, \beta\}$, and consider the regular function $L_c : \mathbb{R} \to \mathbb{R}$ given by the mentioned Proposition 4.3.1. We know that $L_c(a) = 0$ for all $a \in A$, $L_c(c_j) = b_j - c_j$ for all $j \in \{1, \dots, \beta\}$ and $L_c \in \mathcal{V}$. Define the regular function $\psi : \mathbb{R} \to \mathbb{R}$ by setting $\psi := \mathrm{id}_\mathbb{R} + L_c$. Note that $\psi(a) = a$ for all $a \in A$ and $\psi(c_j) = b_j$ for all $j \in \{1, \dots, \beta\}$. Moreover, since $\mathcal{V} \subset \mathscr{D}$ and $\mathrm{Inv}(\mathcal{V}) \subset \mathcal{U}$, we have that ψ is a Nash diffeomorphism such that $\varphi := \psi^{-1} \in \mathcal{U}$. The Nash diffeomorphism φ has all the desired properties. $\square$

Corollary 4.3.3. *Let $n \in \mathbb{N}^*$, and let A and B be two finite subsets of $\mathbb{R}^n$ such that $A \subset \mathbb{Q}^n$ and $B \subset \mathbb{R}^n \setminus \mathbb{Q}^n$. Then there exists a Nash diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\mathrm{id}_{\mathbb{R}^n}$ such that $\varphi(a) = a$ for all $a \in A$, $\varphi(B) \subset \mathbb{Q}^n$ and $\varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a regular map. More precisely, for each neighborhood $\mathcal{U}$ of $\mathrm{id}_{\mathbb{R}^n}$ in $\mathscr{C}^\infty_\mathrm{w}(\mathbb{R}^n, \mathbb{R}^n)$, there exists a Nash diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi \in \mathcal{U}$, $\varphi(a) = a$ for all $a \in A$, $\varphi(B) \subset \mathbb{Q}^n$ and $\varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a regular map.*

Proof. For each $i \in \{1, \dots, n\}$, let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the projection $\pi_i(x) := x_i$, let $A_i := \pi_i(A \cup B) \cap \mathbb{Q}$ and let $B_i := \pi_i(A \cup B) \setminus \mathbb{Q}$. By Corollary 4.3.2, there exists a Nash diffeomorphism $\varphi_i : \mathbb{R} \to \mathbb{R}$ arbitrarily close to $\mathrm{id}_\mathbb{R}$ in $\mathcal{N}_\tau(\mathbb{R})$ such that $\varphi_i(a) = a$ for all $a \in A_i$, $\varphi_i(B_i) \subset \mathbb{Q}$ and φ_i^{-1} is a regular function. Note that τ is finer than the relative topology induced by $\mathscr{C}^\infty_\mathrm{w}(\mathbb{R})$ on $\mathcal{N}(\mathbb{R})$; thus, we can assume that each φ_i is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\mathrm{id}_\mathbb{R}$. Define the Nash diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by setting $\varphi(x_1, \dots, x_n) := (\varphi_1(x_1), \dots, \varphi_n(x_n))$. Note that $\varphi(a) = a$ for all $a \in A$ and $\varphi(B) \subset \mathbb{Q}^n$. Furthermore, the i^{th} component of φ equals $(\pi_i)^*(\varphi_i) = \varphi_i \circ \pi_i$, where $(\pi_i)^* : \mathscr{C}^\infty_\mathrm{w}(\mathbb{R}) \to \mathscr{C}^\infty_\mathrm{w}(\mathbb{R}^n)$ is the pullback map associated to π_i. Since each pullback map $(\pi_i)^*$ is continuous (with respect to the weak $\mathscr{C}^\infty$ topology, see [Hir94, §2]), we can assume that each component of φ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\mathrm{id}_\mathbb{R}$, which is equivalent to assume that φ is arbitrarily $\mathscr{C}^\infty_\mathrm{w}$ close to $\mathrm{id}_{\mathbb{R}^n}$. $\square$

4.3.2. The compact case. The aim of this subsection is to provide a proof of the following $\mathbb{Q}$-algebrization result.

THEOREM 4.3.4. *Let $V \subset \mathbb{R}^n$ be a compact algebraic set with isolated singularities. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ with the following properties:*

(i) *$V' \subset \mathbb{R}^m$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set.*
(ii) *$\phi(\mathrm{Reg}(V)) = \mathrm{Reg}(V')$ and $\phi| : \mathrm{Reg}(V) \to \mathrm{Reg}(V')$ is a Nash diffeomorphism. In particular, V' is $\mathbb{Q}$-nonsingular if V is nonsingular.*

More precisely, the following is true. Denote by d the dimension of V and set $m := n + 2d + 3$. Choose a neighborhood $\mathcal{U}$ of the inclusion map $V \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^0_\mathrm{w}(V, \mathbb{R}^m)$, and a neighborhood $\mathcal{V}$ of the inclusion map $\mathrm{Reg}(V) \hookrightarrow \mathbb{R}^m$ in $\mathcal{N}_\mathrm{w}(\mathrm{Reg}(V), \mathbb{R}^m)$. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ that have both the preceding properties (i) *and* (ii) *and the following:*

(iii) *The Zariski closure of $V'(\mathbb{Q})$ in $\mathbb{R}^m$ has dimension at least $d - 1$.*
(iv) *ϕ extends to a semialgebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$.*
(v) *ϕ fixes $\mathrm{Sing}(V) \cap \mathbb{Q}^n$, that is, $\phi(x) = x$ for all $x \in \mathrm{Sing}(V) \cap \mathbb{Q}^n$. In particular, $V'(\mathbb{Q})$ contains $\mathrm{Sing}(V) \cap \mathbb{Q}^n$.*
(vi) *If $\jmath : V' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map, then $\jmath \circ \phi \in \mathcal{U}$ and $(\jmath \circ \phi)|_{\mathrm{Reg}(V)} \in \mathcal{V}$.*

PROOF. Let $\mathrm{Sing}(V) = \{a_1, \dots, a_s\}$ with $a_1, \dots, a_r \in \mathbb{Q}^n$ and $a_{r+1}, \dots, a_s \in \mathbb{R}^n \setminus \mathbb{Q}^n$. By Corollary 4.3.3 for every neighborhood $\mathcal{U}$ of $\mathrm{id}_{\mathbb{R}^n}$ in $\mathscr{C}^\infty_\mathrm{w}(\mathbb{R}^n, \mathbb{R}^n)$, there exists a Nash diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi \in \mathcal{U}$, $\varphi(a_i) = a_i$ for all $i = 1, \dots, r$, $\varphi(a_i) = b_i \in \mathbb{Q}^n$ for every $i = r + 1, \dots, s$ and $\varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a regular map. Then $\widetilde{V} := (\varphi^{-1})^{-1}(V) \subset \mathbb{R}^n$ is an algebraic set such that $\mathrm{Sing}(\widetilde{V}) \subset \{a_1, \dots, a_r, b_{r+1}, \dots, b_s\} \subset \mathbb{Q}^n$, indeed $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Nash diffeomorphism of Nash manifolds and φ^{-1} is regular, hence $\varphi(\mathrm{Reg}(V)) = (\varphi^{-1})^{-1}(\mathrm{Reg}(V)) \subset \mathrm{Reg}(\widetilde{V})$. In particular, $\varphi|_V : V \to \widetilde{V}$ is a semialgebraic homeomorphism such that $\varphi|_{\mathrm{Reg}(V)} : \mathrm{Reg}(V) \to \widetilde{V} \setminus \{a'_1, \dots, a'_s\}$, where $a'_i := a_i$ if $i \leq r$ or $a'_i = b_i$ otherwise, is a Nash diffeomorphism of Nash manifolds and $(\varphi|_V)^{-1} = (\varphi^{-1})|_{\widetilde{V}}$ is a regular map.

By Hironaka's resolution theorem and generic projection, there are a compact nonsingular algebraic set $M \subset \mathbb{R}^{2d+1}$ with nonsingular algebraic hypersurfaces $\{M_{ij}\}_{j=1}^{\ell_i}$ in general position, for $i \in \{1, \dots, s\}$, and a regular map $p : M \to \widetilde{V}$ such that:

(vii) $A_i := p^{-1}(a'_i) = \bigcup_{j=1}^{\ell_i} M_{ij}$, for every $i \in \{1, \dots, s\}$.
(viii) Let $A := \bigcup_{i=1}^{s} A_i = \bigcup_{i=1}^{s} \bigcup_{j=1}^{\ell_i} M_{ij}$, then $p|_{M \setminus A} : M \setminus A \to \widetilde{V} \setminus \{a'_1, \dots, a'_s\}$ is biregular.

By Theorem 4.1.4 and Lemma 2.1.8 there are a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set $M' \subset \mathbb{R}^{2d+1}$, $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic subsets M'_{ij}, for $i \in \{1, \dots, s\}$ and $j = \{1, \dots, \ell_i\}$, of M' in general position and a Nash diffeomorphism $\varphi' : M \to M'$

such that $\varphi'(M_{ij}) = M'_{ij}$ for every $i \in \{1, \dots, s\}$ and $j = \{1, \dots, \ell_i\}$. Let $A' := \bigcup_{i=1}^{s} A_i = \bigcup_{i=1}^{s} \bigcup_{j=1}^{\ell_i} M'_{ij} \subset \mathbb{R}^{2d+1}$. By Remark 4.1.5, we may suppose that $M' \setminus A' \subset \mathbb{R}^{2d+1}$ contains a "hypersurface of rational points", that is, we may suppose that $\dim(\mathrm{Zcl}_{\mathbb{R}^{2d+1}}((M' \setminus A')(\mathbb{Q}))) \geq d - 1$. In particular, $A' \subset \mathbb{R}^{2d+1}$ is a $\mathbb{Q}$-nice $\mathbb{Q}$-algebraic set by Lemma 3.1.6 and $p' := p \circ (\varphi')^{-1}$ is a Nash map such that $p'|_{A'}$ is $\mathbb{Q}$-regular, since $p'|_{A_i} \equiv a'_i$ if $i \in \{1, \dots, s\}$.

Denote by $p' := (p'_1, \dots, p'_n) : M' \to \mathbb{R}^n$ and by $P' : \mathbb{R}^{2d+1} \to \mathbb{R}^n$ any smooth extension of p'. Since $a'_i \in \mathbb{Q}^n$ for every $i \in \{1, \dots, s\}$, we have that $p'_j|_{A'}$ is $\mathbb{Q}$-regular for every $j \in \{1, \dots, 2d+1\}$. Let $f_j \in \mathcal{R}_{\mathbb{Q}}(\mathbb{R}^{2d+1})$ be a $\mathbb{Q}$-regular function such that $p'_j|_{A'} = f_j|_{A'}$ for every $j \in \{1, \dots, n\}$. Then, apply Lemma 3.1.8 with "P" $=$ "L" $:= A'$, "K" be any compact neighborhood of M' in $\mathbb{R}^{2d+1}$ and "f" $:= p'_j - f_j$ getting a polynomial $s_j \in \mathcal{I}_{\mathbb{Q}}(A')$ arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $p'_j - f_j$, for every $j \in \{1, \dots, n\}$. Hence, the $\mathbb{Q}$-regular map $q \in \mathcal{R}_{\mathbb{Q}}(\mathbb{R}^{2d+1}, \mathbb{R}^n)$ defined as $q(y) := (s_1(y) + f_1(y), \dots, s_n(y) + f_n(y))$ for every $y \in \mathbb{R}^{2d+1}$ is such that $q|_{M'}$ is arbitrarily $\mathcal{N}_{\mathrm{w}}$ close to p' and $q|_{A'} = p'|_{A'}$.

Let $W := \{a'_1, \dots, a'_s\} \subset \mathbb{R}^n$ be a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set. An application of Lemma 3.3.3 with the following substitutions: "X" $:= M'$, "Y" $:= W \times \{0\}$, "A" $:= A'$ and "p" $:= q \times Ca$, with $a \in \mathbb{Q}[y]$ such that $A' = \mathcal{Z}_{\mathbb{R}}(a)$ and $C \in \mathbb{Q} \setminus \{0\}$ sufficiently small, gives $m' = n + 2d + 2$, a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set $V' \subset \mathbb{R}^{m'}$ on $V' \setminus (W \times \{0\})$, that is, $V' \setminus (W \times \{0\}) \subset \mathrm{Reg}^*(V')$, a homeomorphism $h : M' \cup_{q \times Ca} W \to V'$ and $\mathbb{Q}$-regular maps $f : M' \to V'$, $g : W \to V'$ such that $f|_{M' \setminus A'} : M' \setminus A' \to V' \setminus g(W)$ is $\mathbb{Q}$-biregular and $g = \jmath_W$ is the inclusion map satisfying the following diagram:

$$
\begin{tikzcd}
M' \arrow[dr, hook] \arrow[drrr, bend left, "f"] & & & \\
& M' \sqcup (W \times \{0\}) \arrow[r, "\pi"] & M' \cup_{q \times Ca} (W \times \{0\}) & V'. \arrow[l, "h"'] \\
W \arrow[ur, hook] \arrow[urrr, bend right, "g"'] & & &
\end{tikzcd}
$$

In addition, by Lemma 3.3.3(v)(vi), the next conditions are satisfied:

(ix) $\jmath_{(q \times Ca)(M')} \circ (q \times Ca)$ is arbitrarily $\mathscr{C}^0_{\mathrm{w}}$ close to f,

(x) $\jmath_{(q \times Ca)(M')}|_{X'' \setminus A''} \circ (q \times Ca)|_{X'' \setminus A''}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to f,

where $\jmath_{(q \times Ca)(M')} : (q \times Ca)(M') \hookrightarrow \mathbb{R}^{m'}$ denotes the inclusion map.

Let $m := m' + 1 = n + 2d + 3$. Let us construct an algebraic set $V'' \subset \mathbb{R}^m$ and a semialgebraic homeomorphism $\phi : V \to V''$ satisfying (i)-(vi). Recall that, by Remark 4.1.5, we may assume that $M' \setminus A'$ contains an algebraic hypersurface $S \subset \mathbb{R}^{m'}$ which is $\mathbb{Q}$-biregular isomorphic to the standard sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, thus $S \subset \mathbb{R}^{m'}$ is a $\mathbb{Q}$-nonsingular $\mathbb{Q}$-algebraic set and $S(\mathbb{Q})$ is Zariski dense in S. Let $s \in \mathcal{I}_{\mathbb{Q}}(S)$ such that $\mathcal{Z}_{\mathbb{R}}(s) = S$ and $s(x) \geq 0$ for every $x \in \mathbb{R}^{m'}$. Recall that $g(W) \subset \mathbb{R}^{m'}$ is a $\mathbb{Q}$-algebraic set consisting of a finite set of points, thus there exists

$t \in \mathbb{Q}[x]$ such that $\mathcal{Z}_R(t) = g(W)$ and $t(x) \geq 0$ for every $x \in \mathbb{R}^{m'}$. Let $C' \in \mathbb{Q} \setminus \{0\}$. Consider the algebraic set $V'' \subset \mathbb{R}^{m'+1}$ defined as:

$$V'' := \big\{(x, y) \in V' \times \mathbb{R} \,\big|\, (s(x)^2 + t(x)^2)y^3 = (C')^3 t(x)^2\big\}.$$

Observe that $V'' \subset \mathbb{R}^{m'+1}$ is a $\mathbb{Q}$-algebraic set such that $V'' \cap (S \times \mathbb{R}) = S \times \{C'\}$, $V'' \cap (g(W) \times \mathbb{R}) = g(W) \times \{0\} = \operatorname{Sing}(V'')$ and $V'' \subset \mathbb{R}^{m'+1}$ coincides with the graph of the function $y : V' \to \mathbb{R}$ defined as $y(x) = C' \sqrt[3]{\frac{t(x)^2}{(s(x)^2 + t(x)^2)}}$. Consider the projection $\Pi : \mathbb{R}^{m'+1} \to \mathbb{R}^{m'}$. Then, the restriction $\Pi|$ of Π from $\operatorname{Reg}(V'')$ to $V' \setminus g(W)$ is a Nash diffeomorphism.

$$\begin{tikzcd}
 & (M,A) \arrow[r, "\varphi'"] \arrow[d, "p"] & (M',A') \arrow[dl] \arrow[d, dashed, "q"] \arrow[dd, bend left, "f"] & \\
V \arrow[r, "\varphi"] & \widetilde{V} \arrow[r, hook] & \mathbb{R}^n & \\
\operatorname{Sing}(V) \arrow[u, hook] \arrow[r, "\varphi|"] & W \arrow[u, hook] \arrow[r, "g"] \arrow[d, hook, two heads] & V' \arrow[r, hook] & \mathbb{R}^{m'} \\
 & \operatorname{Sing}(V'') \arrow[r, hook] & V'' \arrow[u, "\Pi|"] \arrow[r, hook] & \mathbb{R}^m \arrow[u, "\Pi"]
\end{tikzcd}$$

FIGURE 4.3.1. Approximation steps performed during the proof.

Define the semialgebraic homeomorphism $\phi : V \to V''$ as follows:

$$\phi(x) = \begin{cases} (\Pi|)^{-1} \circ f \circ \varphi' \circ (p|_{X \setminus A})^{-1} \circ \varphi(x) & \text{if } x \in \operatorname{Reg}(V); \\ \jmath_{V'} \circ g \circ \varphi(x) & \text{if } x \in \operatorname{Sing}(V), \end{cases}$$

where $\jmath_{V'}$ denotes the inclusion map of $V' \subset \mathbb{R}^{m'}$ in $\mathbb{R}^{m'+1}$. Observe that $\phi|_{\operatorname{Reg}(V)} : \operatorname{Reg}(V) \to V'' \setminus (\jmath_{V'} \circ g)(W)$ is a Nash diffeomorphism of Nash manifolds since it is the composition of Nash diffeomorphisms of Nash manifolds. Let $(a, b) \in V''$. Since V' is $\mathbb{Q}$-determined on $V' \setminus g(W)$, that is, $V' \setminus g(W) \subset \operatorname{Reg}^*(V')$, there are $p_1, \dots, p_{m-1-d} \in \mathcal{I}_{\mathbb{Q}}(V')$ such that $\{\nabla p_i(a)\}_{i=1}^{m-1-d}$ are linearly independent over $\mathbb{R}$. Let $q_1, \dots, q_{m-d} \in \mathcal{I}_{\mathbb{Q}}(V'')$ be defined as $q_i(x, y) := p_i(x)$, for every $i \in \{1, \dots, m-1-d\}$, and $q_{m-d}(x, y) := (s(x)^2 + t(x)^2)y^3 - (C')^3 t(x)^2$. Hence, $\{\nabla q_i(a, b)\}_{i=1}^{m-d}$ are linearly independent over $\mathbb{R}$ as well since the last entry of $\nabla q_{m-d}(a, b)$ only vanishes if $b = 0$ since $S \cap g(W) = \varnothing$, that is, if $(a, b) \in \operatorname{Sing}(V'')$. This proves that $\operatorname{Reg}(V'') = \operatorname{Reg}^*(V'')$, that is, $V'' \subset \mathbb{R}^m$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set. This proves (i) & (ii) with "V'" := V'' and "ϕ" := ϕ. In addition, an application of [Jel09] provides a semialgebraic extension $\mathbb{R}^m \to \mathbb{R}^m$ of ϕ, thus (iv) holds. Observe that, by construction, $\jmath_{V'}(S \cup g(W)) \subset V''$, thus (iii) and (v) hold. Finally, since φ is arbitrarily $\mathscr{C}^0_{\mathrm{w}}$ close to $\mathrm{id}_{\mathbb{R}^n}$ and it restriction to $\operatorname{Reg}(V)$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to the inclusion $\operatorname{Reg}(V) \hookrightarrow \mathbb{R}^m$, by (ix), (x), by letting $C, C' \in \mathbb{Q} \setminus \{0\}$ sufficiently small and by the continuity of the pullback with respect to the $\mathscr{C}^0_{\mathrm{w}}$ and the $\mathscr{C}^\infty_{\mathrm{w}}$ topologies, (vi) follows. □

If we are willing to lose properties (v) & (vi), we can find a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic model V' of V as in Theorem 4.3.4 with an improvement on the estimate of m, namely, we can choose $m = 2d + 3$. Indeed, since $\mathrm{Sing}(V)$ is finite, we may consider "W":=$\{0, \dots, s\} \subset \mathbb{R}$, where s is the cardinality of $\mathrm{Sing}(V)$, and then follow the steps of the proof of Theorem 4.3.4.

THEOREM 4.3.5. *Let $V \subset \mathbb{R}^n$ be a compact algebraic set with isolated singularities of dimension d. Set $m := 2d + 3$. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ having properties* (i)-(iv) *of Theorem* 4.3.4.

4.3.3. The non-compact case. In this subsection we extend Theorem 4.3.4 to the non-compact case, that is, we provide a complete positive answer to [Par21, Open problem 1, p. 199] in the case of algebraic sets with isolated singularities.

THEOREM 4.3.6. *Let $V \subset \mathbb{R}^n$ be an algebraic set with isolated singularities. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ with the following properties:*

(i) *$V' \subset \mathbb{R}^m$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set.*
(ii) *$\phi(\mathrm{Reg}(V)) = \mathrm{Reg}(V')$ and $\phi| : \mathrm{Reg}(V) \to \mathrm{Reg}(V')$ is a Nash diffeomorphism. In particular, V' is $\mathbb{Q}$-nonsingular if V is nonsingular.*

More precisely, the following is true. Denote by d the dimension of V and set $m := n + 2d + 4$. Choose a neighborhood $\mathcal{U}$ of the inclusion map $V \hookrightarrow \mathbb{R}^m$ in $\mathscr{C}^0_{\mathrm{w}}(V, \mathbb{R}^m)$, and a neighborhood $\mathcal{V}$ of the inclusion map $\mathrm{Reg}(V) \hookrightarrow \mathbb{R}^m$ in $\mathcal{N}_{\mathrm{w}}(\mathrm{Reg}(V), \mathbb{R}^m)$. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ that have both the preceding properties (i) *and* (ii) *and the following:*

(iii) *The Zariski closure of $V'(\mathbb{Q})$ in $\mathbb{R}^m$ has dimension at least $d - 1$.*
(iv) *ϕ extends to a semialgebraic homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$.*
(v) *ϕ fixes $\mathrm{Sing}(V) \cap \mathbb{Q}^n$, that is, $\phi(x) = x$ for all $x \in \mathrm{Sing}(V) \cap \mathbb{Q}^n$. In particular, $V'(\mathbb{Q})$ contains $\mathrm{Sing}(V) \cap \mathbb{Q}^n$.*
(vi) *If $\jmath : V' \hookrightarrow \mathbb{R}^m$ denotes the inclusion map, then $\jmath \circ \phi \in \mathcal{U}$ and $(\jmath \circ \phi)|_{\mathrm{Reg}(V)} \in \mathcal{V}$.*

PROOF. If $V = \mathbb{R}^n$, it suffices to fix "V'":=V. If V is finite, we conclude by density of $\mathbb{Q}$ in $\mathbb{R}$. Suppose that V is infinite and different form $\mathbb{R}^n$. Suppose also V is non compact, otherwise we conclude by Theorem 4.3.4 with the improved estimate of $m := n + 2d + 3$ such that $V' \subset \mathbb{R}^m$. Let $\mathrm{Sing}(V) = \{a_1, \dots, a_s\}$ with $\{a_1, \dots, a_r\} = \mathrm{Sing}(V) \cap \mathbb{Q}^n$, for some $s, r \in \mathbb{N}$ with $r \leq s$. Up to perform a translation of a rational factor we may suppose that $0 \notin V$. Let $u \in \mathbb{R}[x]$ such that $V = \mathcal{Z}_\mathbb{R}(u)$. Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ be the standard unit sphere and let $\theta : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ as $\theta(x) = \frac{x}{|x|^2_n}$ be the inversion with respect to $\mathbb{S}^{n-1}$. Recall that $\theta \circ \theta = \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$. Let $d \geq \deg(u)$. Define the polynomial $v := |x|^{2d}_n \cdot (u \circ \theta)(x) \in \mathbb{R}[x]$ and the compact algebraic set $W := \mathcal{Z}_\mathbb{R}(v)$. By construction, $W = \theta(V) \sqcup \{0\}$ and $\theta : V \to W \setminus \{0\}$ is a biregular isomorphism of locally Zariski closed algebraic sets, thus $\mathrm{Reg}(W) \setminus \{0\} = \theta(\mathrm{Reg}(V))$, $\mathrm{Sing}(W) \setminus \{0\} = \theta(\mathrm{Sing}(V)) =: \{b_1, \dots, b_s\} \subset \mathbb{R}^n$ and $\theta(\mathrm{Sing}(V) \cap \mathbb{Q}) = \{b_1, \dots, b_r\} \subset \mathbb{Q}^n$. Consider the origin $0 \in \mathbb{R}^n$ as a singular point, that is in the application of Theorem 4.3.4 apply Hironaka's resolution of singularities to the set $\{0, b_1, \dots, b_s\} \subset W$. Hence, the application of mentioned

Theorem 4.3.4, gives $m' = n + 2d + 3$, a compact algebraic subset W' of $\mathbb{R}^{m'}$ and a semialgebraic homeomorphism $\phi' : W \to W'$ satisfying the following conditions:

(i′) $W' \subset \mathbb{R}^{m'}$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set.

(ii′) $\phi'(\operatorname{Reg}(W)) = \operatorname{Reg}(W')$ and $\phi'|_{\operatorname{Reg}(W)} : \operatorname{Reg}(W) \to \operatorname{Reg}(W')$ is a Nash diffeomorphism. In particular, W' is $\mathbb{Q}$-nonsingular if W is nonsingular.

(iii′) The Zariski closure of $W'(\mathbb{Q})$ in $\mathbb{R}^{m'}$ has dimension at least $d-1$.

(iv′) ϕ' extends to a semialgebraic homeomorphism from $\mathbb{R}^{m'}$ to $\mathbb{R}^{m'}$.

(v′) ϕ' fixes $(\operatorname{Sing}(W) \cup \{0\}) \cap \mathbb{Q}^n$, that is, $\phi(x) = x$ for every $x \in \{0, b_1, \dots, b_r\}$. In particular, $(\operatorname{Sing}(W) \cup \{0\}) \cap \mathbb{Q}^n \subset W'(\mathbb{Q})$ and $\operatorname{Sing}(W') = \phi'(\operatorname{Sing}(W)) \cup \{0\}$.

(vi′) If $\jmath_W : W \hookrightarrow \mathbb{R}^m$ and $\jmath_{W'} : W' \hookrightarrow \mathbb{R}^m$ denote the inclusion maps, then $\jmath_{W'} \circ \phi'$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\jmath_W$ and $(\jmath_{W'} \circ \phi)|_{\operatorname{Reg}(W)}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\jmath_W|_{\operatorname{Reg}(W)}$.

Let $W' = \mathcal{Z}_\mathbb{R}(u')$ for some $u' \in \mathbb{Q}[x, y]$, with $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_{m'-n})$. Let $\mathbb{S}^{m'-1} \subset \mathbb{R}^{m'}$ be the standard unit sphere and let $\theta' : \mathbb{R}^{m'} \setminus \{0\} \to \mathbb{R}^{m'} \setminus \{0\}$ as $\theta'(x, y) = \frac{(x,y)}{|(x,y)|^2_{m'}}$ be the inversion with respect to $\mathbb{S}^{m'-1}$. Recall that $\theta' \circ \theta' = \mathrm{id}_{\mathbb{R}^{m'} \setminus \{0\}}$. Let $e \geq \deg(u')$. Define the polynomial $v' := |(x, y)|^{2e}_{m'} \cdot (u' \circ \theta')(x, y) \in \mathbb{Q}[x, y]$ and the algebraic set $W'' := \mathcal{Z}_\mathbb{R}(v')$. By construction, $W'' = \theta'(W') \sqcup \{0\}$ is an algebraic set with $0 \in \operatorname{Sing}(W'')$ (since 0 is an isolated point of W'') and $\theta' : W' \setminus \{0\} \to W'' \setminus \{0\}$ is a $\mathbb{Q}$-biregular isomorphism of Zariski open subsets of W' and W'', respectively, thus $\operatorname{Reg}^*(W'') = \theta(\operatorname{Reg}^*(W'))$. In addition, if $S \subset W'$ denotes an hypersurface of W' contained in $\operatorname{Reg}(W')$ with dense rational points, then the dimension of $\operatorname{Zcl}_{\mathbb{R}^{m'}}(\theta'(S)(\mathbb{Q})) \subset W''$ is $d - 1$ since θ' is $\mathbb{Q}$-biregular, thus it sends rational points to rational ones and $\operatorname{Zcl}_{\mathbb{R}^{m'}}(\theta'(S)(\mathbb{Q})) = \operatorname{Zcl}_{\mathbb{R}^{m'}}(\theta'(S)) \subset \theta'(S) \cup \{0\}$. Observe that $(\theta' \circ \phi' \circ \theta)(a_i) = (a_i, 0) \in \mathbb{R}^{m'}$, for every $i \in \{1, \dots, r\}$, thus $\operatorname{Sing}(W'') \setminus \{(0, 0)\} = \theta(\{b_1, \dots, b_s\}) = \{(a_1, 0). \dots, (a_r, 0), a'_{r+1}, \dots, a'_s\} \subset \mathbb{Q}^{m'}$.

Let $a, s \in \mathbb{Q}[x, y]$ such that $a(x, y), s(x, y) \geq 0$ for every $(x, y) \in \mathbb{R}^{m'}$, $\mathcal{Z}_\mathbb{R}(a) = \{0\} \subset \mathbb{R}^{m'}$ and $\mathcal{Z}_\mathbb{R}(s) = \operatorname{Sing}(W'') \setminus \{0\} = \theta(\operatorname{Sing}(W') \setminus \{0\}) \subset \mathbb{R}^{m'}$. Let $m := m' + 1 = n + 2d + 4$ and $C \in \mathbb{Q} \setminus \{0\}$. Define the $\mathbb{Q}$-algebraic set $V' \subset \mathbb{R}^m$ as

$$V' := \big\{((x, y), z) \in \mathbb{R}^{m'} \times \mathbb{R} \,\big|\, v'(x, y) = 0,\, a(x, y)z = Cs(x, y)\big\}.$$

Observe that $V' \subset \mathbb{R}^m$ is $\mathbb{Q}$-biregularly isomorphic to $W'' \setminus \{0\}$ via projection $\pi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$, thus V' is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set. By construction, we also have that

$$\begin{aligned} V' \cap (\mathbb{R}^{m'} \times \{0\}) &= (\operatorname{Sing}(W'') \setminus \{0\}) \times \{0\} \\ &= \{(a_1, 0, 0). \dots, (a_r, 0, 0), (a'_{r+1}, 0), \dots, (a'_s, 0)\} \subset \mathbb{Q}^m. \end{aligned}$$

In addition, since $0 \in \mathbb{R}^{m'}$ is an isolated point of W'', up to chose $C \in \mathbb{Q} \setminus \{0\}$ sufficiently small, we may suppose that $\jmath_{V'} \circ (\pi|_{V'})^{-1}$ is arbitrarily $\mathscr{C}^0_{\mathrm{w}}$ close to $\jmath_{W'' \setminus \{0\}}$ and $\jmath_{V'}|_{\operatorname{Reg} V'} \circ (\pi|_{\operatorname{Reg} V'})^{-1}$ is arbitrarily $\mathscr{C}^\infty_{\mathrm{w}}$ close to $\jmath_{W'' \setminus \{0\}}|_{\operatorname{Reg}(W'') \setminus \{0\}}$, where $\jmath_{V'} : V' \hookrightarrow \mathbb{R}^m$ and $\jmath_{W'' \setminus \{0\}} : W'' \setminus \{0\} \hookrightarrow \mathbb{R}^m$ denote the inclusion maps. Finally, define $\phi : V \to V'$ as follows:

$$\phi := (\pi|_{W'' \setminus \{0\}})^{-1} \circ \theta'_{W'} \circ \phi' \circ \theta|_V$$

Observe that ϕ is a semialgebraic homeomorphism, since it is a composition of semialgebraic homeomorphisms, and $\phi|_{\operatorname{Reg}(V)} : \operatorname{Reg}(V) \to \operatorname{Reg}(V')$ is a Nash diffeomorphism, since it is a composition of Nash diffeomorphisms. This proves (i), (ii) & (v). In addition, [Jel09] ensures that ϕ can be extended to a semialgebraic homeomorphism $\mathbb{R}^m \to \mathbb{R}^m$, thus (iv) holds. As above, since $\pi|_{V'}$ is a $\mathbb{Q}$-biregular map, we deduce that $(\pi|_{V'}^{-1} \circ \theta')(S) \subset V'(\mathbb{Q})$ and $\operatorname{Zcl}_{\mathbb{R}^m}((\pi|_{V'}^{-1} \circ \theta')(S))$ has dimension $d-1$, thus (iii) follows. Finally, Theorem 4.3.4(vi), the fact that $\theta'|_{\mathbb{R}^n} = \theta$ and $C \in \mathbb{Q} \setminus \{0\}$ is chosen sufficiently small, imply (vi) by continuity of the pull-back with respect to the $\mathscr{C}^0_{\mathrm{w}}$ and the $\mathscr{C}^\infty_{\mathrm{w}}$ topologies. □

If we are willing to lose properties (v) & (vi), we can find a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic model V' of V as in Theorem 4.3.6 with an improvement on the estimate of m, namely, we can choose $m = 2d + 4$. Indeed, since $\operatorname{Sing}(V)$ is finite, we may consider "W":$=\{0, \dots, s\} \subset \mathbb{R}$, where s is the cardinality of $\operatorname{Sing}(V)$, and then follow the steps of the proof of Theorem 4.3.6.

THEOREM 4.3.7. *Let $V \subset \mathbb{R}^n$ be an algebraic set with isolated singularities of dimension d. Set $m := 2d + 4$. Then there exist an algebraic set $V' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi : V \to V'$ having properties* (i)-(iv) *of Theorem* "W":$=\{0, \dots, s\} \subset \mathbb{R}$.

4.4. Local $\mathbb{Q}$-algebrization of isolated singularities

This section is devoted to provide a complete positive answer to [Par21, Open problem 2, p. 200] in the case of germs $(V, 0)$ of an algebraic set $V \subset \mathbb{R}^n$ having an isolated singularity at $0 \in \mathbb{R}^n$. Our result appears as a consequence of Theorem 4.3.4.

THEOREM 4.4.1. *Let $(V, 0) \subset (\mathbb{R}^n, 0)$ be the germ of an isolated algebraic singularity. Then there exist a germ of an isolated algebraic singularity $(V', 0) \subset (\mathbb{R}^m, 0)$, semialgebraic nieghborhoods U of 0 in $\mathbb{R}^n$ and U' of 0 in $\mathbb{R}^m$ and a semialgebraic homeomorphism $\phi : V \cap U \to V' \cap U'$, with the following properties:*

(i) *$V' \subset \mathbb{R}^m$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set.*
(ii) *$\phi(\operatorname{Reg}(V) \cap U) = \operatorname{Reg}(V') \cap U'$ and $\phi| : \operatorname{Reg}(V) \cap U \to \operatorname{Reg}(V') \cap U'$ is a Nash diffeomorphism.*

PROOF. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the projection map. By [BCR98, Theorem 9.3.6], we can choose $U := B_n(0, \epsilon) \subset \mathbb{R}^n$ to be an open ball centered at 0 of radius $\epsilon > 0$, with $\epsilon > 0$ sufficiently small, such that $\operatorname{Sing}(V) \cap U = \{0\}$ and $V \cap \overline{U} = V \cap \overline{B}_n(0, \epsilon)$ is semialgebraically homeomorphic to the cone over $V \cap \mathbb{S}^{n-1}(0, \epsilon) \subset \mathbb{R}^n$ centered at 0 of radius $\epsilon > 0$. Consider the sphere $\mathbb{S}^n(0, \epsilon) \subset \mathbb{R}^{n+1}$ and denote by $\mathbb{S}^n_+(0, \epsilon) := \{(x, y) \in \mathbb{S}^n(0, \epsilon) \,|\, y > 0\} \subset \mathbb{R}^{n+1}$. Define the algebraic set $W := \pi^{-1}(V) \cap \mathbb{S}^n(0, \epsilon)$. Observe that, by above construction, W is a compact algebraic subset of $\mathbb{R}^{n+1}$ such that $\operatorname{Sing}(W) = \{(0, \dots, 0, 1), (0, \dots, 0, -1)\} \subset \mathbb{R}^{n+1}$ and, since $\pi| : \mathbb{S}^n_+(0, \epsilon) \to U$ is a Nash diffeomorphism, thus $\pi|_{W \cap \mathbb{S}^n_+(0,\epsilon)} : W \cap \mathbb{S}^n_+(0, \epsilon) \to V \cap U$ is a Nash diffeomorphism of Nash manifolds. Let $W' \subset \mathbb{R}^{n+1}$ be the algebraic set obtained by applying the translation $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined as $\varphi(x, y) = (x, y-1)$, for every $(x, y) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. An application of Theorem 4.3.4 provides a compact algebraic set $W'' \subset \mathbb{R}^m$ with isolated singularities and a semialgebraic homeomorphism $\phi' : W' \to W''$ such that:

(iii) $W'' \subset \mathbb{R}^m$ is a $\mathbb{Q}$-determined $\mathbb{Q}$-algebraic set.
(iv) $\phi'(\operatorname{Reg}(W')) = \operatorname{Reg}(W'')$ and $\phi'| : \operatorname{Reg}(W') \to \operatorname{Reg}(W'')$ is a Nash diffeomorphism.
(v) ϕ' fixes $\operatorname{Sing}(W')$, that is, $\phi'(x) = x$ for all $x \in \operatorname{Sing}(W')$. In particular, $0 \in W''$.

Let "m" $:= (n+1)+2d+3 = n+2d+4$, "V" $:= W'$, "U'" to be any open neighborhood of 0 in $\mathbb{R}^n$ such that $W' \cap U' = \phi'(W \cap \mathbb{S}^n_+(0,\epsilon))$ and $\phi := \phi' \circ \varphi \circ (\pi|_{W \cap \mathbb{S}^n_+(0,\epsilon)})^{-1} : V \cap U \to V' \cap U'$. Thus, (i) & (ii) hold. $\square$

APPENDIX A

Smooth variants of Baro-Fernando-Ruiz results

ABSTRACT. Here we propose some smooth variants of results originally proven in [BFR14] in order to give a complete proof of Theorem 4.1.2.

A.1. Relative Nash approximation of $\mathscr{C}^\infty$ diffeomorphisms

We refer to the notation of Subsection 4.1.1. Let $M \subset \mathbb{R}^a$ be a Nash manifold, let $X \subset M$ be a (non-empty) Nash subset of $\mathbb{R}^a$ contained in M and let $Y \subset \mathbb{R}^b$ be a (non-empty) semialgebraic set. Accordingly with [BFR14, Definition 1.5 and p. 72], we say that a map $f : X \to Y$ is c-*Nash* if the restriction of f to each Nash irreducible component of X is a Nash map. We denote ${}^{\mathsf{c}}\mathcal{N}(X,Y)$ the set of c-Nash maps from X to Y, and we set ${}^{\mathsf{c}}\mathcal{N}(X) := {}^{\mathsf{c}}\mathcal{N}(X,\mathbb{R})$. Let us introduce similar concepts for $\mathscr{C}^\nu$ maps.

DEFINITION A.1.1. Let $X \subset M$ be a Nash set and let $X_1, \dots, X_s$ be the Nash irreducible components of X. Given a map $f : X \to Y$, we say that f is a ${}^{\mathsf{c}}\mathscr{C}^\nu$ *map* if, for each $j \in \{1, \dots, s\}$, the restriction $f|_{X_j} : X_j \to Y$ of f to X_j is a $\mathscr{C}^\nu$ map. We denote ${}^{\mathsf{c}}\mathscr{C}^\nu(X,Y)$ the set of ${}^{\mathsf{c}}\mathscr{C}^\nu$ maps from X to Y, and we set ${}^{\mathsf{c}}\mathscr{C}^\nu(X) := {}^{\mathsf{c}}\mathscr{C}^\nu(X,\mathbb{R})$.

Once again, $f = (f_1, \dots, f_b) : X \to Y \subset \mathbb{R}^b$ is a ${}^{\mathsf{c}}\mathscr{C}^\nu$ (c-Nash) map if, and only if, each component $f_i : X \to \mathbb{R}$ of f is a function in ${}^{\mathsf{c}}\mathscr{C}^\nu(X)$ (in ${}^{\mathsf{c}}\mathcal{N}(X)$). Note that $\mathcal{N}(X,Y) \subset {}^{\mathsf{c}}\mathcal{N}(X,Y)$ and $\mathscr{C}^\nu(X,Y) \subset {}^{\mathsf{c}}\mathscr{C}^\nu(X,Y)$.

Consider again a Nash manifold $M \subset \mathbb{R}^a$, a Nash set $X \subset M$ with Nash irreducible components $X_1, \dots, X_s$, a semialgebraic set $Y \subset \mathbb{R}^b$ and $\nu \in \mathbb{N}^* \cup \{\infty\}$. Our next goal is to define suitable topologies on the sets $\mathscr{C}^\nu(X,Y)$ and ${}^{\mathsf{c}}\mathscr{C}^\nu(X,Y)$. First, we consider the set $\mathscr{C}^\nu(X,\mathbb{R}^b)$. We denote $\mathscr{C}^\nu_{\mathrm{w}}(M,\mathbb{R}^b)$ the set $\mathscr{C}^\nu(M,\mathbb{R}^b)$ equipped with the usual *weak $\mathscr{C}^\nu$ topology*, see [Hir94, §2]. This topology makes $\mathscr{C}^\nu_{\mathrm{w}}(M,\mathbb{R}^b)$ a topological real vector space, with the usual pointwise defined addition and multiplication by real scalars. Consider the restriction map $\rho : \mathscr{C}^\nu_{\mathrm{w}}(M,\mathbb{R}^b) \to \mathscr{C}^\nu(X,\mathbb{R}^b)$, i.e., $\rho(F) := F|_X$. Let $f \in \mathscr{C}^\nu(X,\mathbb{R}^b)$ and let $F : U \to \mathbb{R}^b$ be a $\mathscr{C}^\nu$ map extending f to an open neighborhood U of X in M. Choose a $\mathscr{C}^\nu$ partition of unity $\{\alpha,\beta\}$ subordinate to the open cover $\{U, M \setminus X\}$ of M, and define the $\mathscr{C}^\nu$ map $\widetilde{F} : M \to \mathbb{R}^b$ by $\widetilde{F}(x) := \alpha(x)F(x)$ if $x \in U$ and $\widetilde{F}(x) := 0$ otherwise; evidently, $\rho(\widetilde{F}) = f$. This proves that ρ is surjective. We equip $\mathscr{C}^\nu(X,\mathbb{R}^b)$ with the *quotient topology induced by* ρ, i.e., the finest topology of $\mathscr{C}^\nu(X,\mathbb{R}^b)$ which makes ρ continuous. We denote $\mathscr{C}^\nu_{\mathrm{w}}(X,\mathbb{R}^b)$ the set $\mathscr{C}^\nu(X,\mathbb{R}^b)$ equipped with such a quotient topology. An important property of $\mathscr{C}^\nu_{\mathrm{w}}(X,\mathbb{R}^b)$ is that $\rho : \mathscr{C}^\nu_{\mathrm{w}}(M,\mathbb{R}^b) \to \mathscr{C}^\nu_{\mathrm{w}}(X,\mathbb{R}^b)$ is an *open* map. Indeed, if $\mathcal{U} \subset \mathscr{C}^\nu_{\mathrm{w}}(M,\mathbb{R}^b)$ is open and $\mathcal{I} = \{F \in \mathscr{C}^\nu(M,\mathbb{R}^b) : \rho(F) = 0\}$, then $\rho^{-1}(\rho(\mathcal{U})) =$

$\bigcup_{F\in\mathcal{I}}(\mathcal{U}+F)$ is open as well, because the translations of $\mathscr{C}^\nu_{\mathrm{w}}(M,\mathbb{R}^b)$ are homeomorphisms. Identify $\mathscr{C}^\nu(X,Y)$ with the subset of $\mathscr{C}^\nu(X,\mathbb{R}^b)$ of all $\mathscr{C}^\nu$ maps $f:X\to\mathbb{R}^b$ such that $f(X)\subset Y$. We denote $\mathscr{C}^\nu_{\mathrm{w}}(X,Y)$ the set $\mathscr{C}^\nu(X,Y)$ equipped with the *relative topology induced by* $\mathscr{C}^\nu_{\mathrm{w}}(X,\mathbb{R}^b)$. Let us topologize ${}^{\mathsf{c}}\mathscr{C}^\nu(X,Y)$. Consider the topological product $\mathscr{C}^\nu_{\mathrm{w}}(X_1,Y)\times\cdots\times\mathscr{C}^\nu_{\mathrm{w}}(X_s,Y)$ and the multiple restriction map $\jmath:{}^{\mathsf{c}}\mathscr{C}^\nu(X,Y)\to\mathscr{C}^\nu_{\mathrm{w}}(X_1,Y)\times\cdots\times\mathscr{C}^\nu_{\mathrm{w}}(X_s,Y)$ defined by $\jmath(f):=(f|_{X_1},\ldots,f|_{X_s})$. Note that $\jmath$ is injective. We denote ${}^{\mathsf{c}}\mathscr{C}^\nu_{\mathrm{w}}(X,Y)$ the set ${}^{\mathsf{c}}\mathscr{C}^\nu(X,Y)$ equipped with the *topology induced by* $\jmath$, i.e., the topology making $\jmath$ a homeomorphism onto its image. Let $\gamma:\mathscr{C}^\nu(X,Y)\hookrightarrow{}^{\mathsf{c}}\mathscr{C}^\nu(X,Y)$ be the inclusion map. By the universal property of quotient topology, we know that each restriction map $\mathscr{C}^\nu_{\mathrm{w}}(X,\mathbb{R}^b)\to\mathscr{C}^\nu_{\mathrm{w}}(X_j,\mathbb{R}^b)$ is continuous. This implies at once that $\gamma:\mathscr{C}^\nu_{\mathrm{w}}(X,Y)\hookrightarrow{}^{\mathsf{c}}\mathscr{C}^\nu_{\mathrm{w}}(X,Y)$ is continuous as well. We denote $\mathrm{J}:\mathscr{C}^\nu_{\mathrm{w}}(X,Y)\to\mathscr{C}^\nu_{\mathrm{w}}(X_1,Y)\times\cdots\times\mathscr{C}^\nu_{\mathrm{w}}(X_s,Y)$ the composition map $\jmath\circ\gamma$.

The next result is a variant of [BFR14, Propositions 6.2 and 8.1].

PROPOSITION A.1.2. *Suppose that $X\subset M$ is a Nash set with monomial singularities. Then there exists a continuous linear map $\theta:\mathscr{C}^\nu_{\mathrm{w}}(X,\mathbb{R}^b)\to\mathscr{C}^\nu_{\mathrm{w}}(M,\mathbb{R}^b)$ such that θ is an extension map, i.e, $\theta(f)|_X=f$ for all $f\in\mathscr{C}^\nu_{\mathrm{w}}(X,\mathbb{R}^b)$. Moreover, the restriction map $\mathrm{J}:\mathscr{C}^\nu_{\mathrm{w}}(X,Y)\to\mathscr{C}^\nu_{\mathrm{w}}(X_1,Y)\times\cdots\times\mathscr{C}^\nu_{\mathrm{w}}(X_s,Y)$ is a homeomorphism onto its image.*

PROOF. As in the proof of [BFR14, Proposition 6.2], it suffices to consider the case $Y=\mathbb{R}^b=\mathbb{R}$ and to prove the existence of an extension continuous linear map ${}^{\mathsf{c}}\theta:{}^{\mathsf{c}}\mathscr{C}^\nu_{\mathrm{w}}(X)\to\mathscr{C}^\nu_{\mathrm{w}}(M)$. This implies at once that $\theta:={}^{\mathsf{c}}\theta\circ\gamma$ is the desired extension map and γ is a homeomorphism (hence J is a homeomorphism onto its image). The problem of constructing ${}^{\mathsf{c}}\theta$ is local in nature, because M admits $\mathscr{C}^\nu$ partitions of unity subordinate to each open cover of M. This fact and Definition 4.1.1 reduce the problem to the case in which $M=\mathbb{R}^m$ and $X=L_1\cup\ldots\cup L_s$ is a union of coordinate linear varieties of $\mathbb{R}^m$. In this situation, the proof of [BFR14, Proposition 4.C.1] gives an explicit formula for ${}^{\mathsf{c}}\theta$. Let $\mathcal{P}(s)$ be the power set of $\{1,\ldots,s\}$ and, for each $I\in\mathcal{P}(s)\setminus\{\varnothing\}$, let $L_I:=\bigcap_{i\in I}L_i$, let $\jmath_I:L_I\hookrightarrow X$ be the inclusion map and let $\pi_I:\mathbb{R}^m\to L_I$ be the orthogonal projection of $\mathbb{R}^m$ onto L_I. Set ${}^{\mathsf{c}}\theta(f):=-\sum_{I\in\mathcal{P}(s)\setminus\{\varnothing\}}(-1)^{|I|}(f\circ\jmath_I\circ\pi_I)$, where $|I|$ is the cardinality of I. It is immediate to verify that ${}^{\mathsf{c}}\theta(f)|_{L_i}=f|_{L_i}$ for all $i\in\{1,\ldots,s\}$, so ${}^{\mathsf{c}}\theta(f)|_X=f$. It is well-known that the composition operation is continuous in the weak $\mathscr{C}^\nu$ topology (see [Hir94, Exercise 10(a), p. 64]). Thus, ${}^{\mathsf{c}}\theta$ is continuous, as desired. $\square$

REMARK A.1.3. In [BFR14, Theorem 1.6], the authors prove the following remarkable fact: *if $X\subset M$ is a Nash set with monomial singularities, then ${}^{\mathsf{c}}\mathcal{N}(X,Y)=\mathcal{N}(X,Y)$.*

The next result is a variant of [BFR14, Proposition 8.2].

PROPOSITION A.1.4. *Let $X\subset M$ be a Nash set with monomial singularities, let $N\subset\mathbb{R}^b$ be a Nash manifold and let $F:M\to N$ be a $\mathscr{C}^\nu$ map. Suppose that $\nu\geq m$, where $m=\dim(M)$. Then each Nash map $h:X\to N$ which is sufficiently $\mathscr{C}^\nu_{\mathrm{w}}$ close to $F|_X$ has a Nash extension $H:M\to N$ which is arbitrarily $\mathscr{C}^{\nu-m}$ close to F. More precisely, for each neighborhood $\mathcal{U}$ of F in $\mathscr{C}^{\nu-m}(M,N)$, there exists a neighborhood $\mathcal{V}$ of $F|_X$ in $\mathscr{C}^\nu(X,N)$ with the following property: for each Nash map $h\in\mathcal{N}(X,N)\cap\mathcal{V}$, there exists a Nash map $H\in\mathcal{N}(M,N)\cap\mathcal{U}$ such that $H|_X=h$.*

Proof. Since N has a Nash tubular neighborhood in $\mathbb{R}^b$ (see [BCR98, Corollary 8.9.5] or [Shi87, §I.3]), it suffices to consider the case in which $N = \mathbb{R}^b$. Reasoning component by component, we can further assume that $N = \mathbb{R}$. We follow the strategy of the proof of [BFR14, Proposition 7.6]. The set $\mathcal{U}' = \mathcal{U} \cap \mathscr{C}^\nu(M)$ is a neighborhood of F in $\mathscr{C}^\nu_{\mathrm{w}}(M)$. The openness of the restriction map $\rho : \mathscr{C}^\nu_{\mathrm{w}}(M) \to \mathscr{C}^\nu_{\mathrm{w}}(X)$ implies that $\rho(\mathcal{U}')$ is a neighborhood of $F|_X$ in $\mathscr{C}^\nu_{\mathrm{w}}(X)$. Suppose that $h \in \rho(\mathcal{U}')$. Choose $G \in \mathcal{U}' \subset \mathcal{U}$ such that $\rho(G) = h$, i.e., $G|_X = h$. It remains to show that there exists a Nash function $H : M \to \mathbb{R}$ arbitrarily $\mathscr{C}^{\nu-m}_{\mathrm{w}}$ close to F (thus, we can assume $H \in \mathcal{U}$) such that $H|_X = G|_X$. By [BFR14, Theorem 1.4], there exist a finite family of open semialgebraic subsets $U_1, \ldots, U_\ell$ of M and Nash diffeomorphisms $\{u_i : U_i \to \mathbb{R}^m\}_{i=1}^\ell$ such that $X \subset U_1 \cup \ldots \cup U_\ell$ and, for each $i \in \{1, \ldots, \ell\}$, $u_i(X \cap U_i)$ is a union of coordinate linear varieties of $\mathbb{R}^m$. It follows that $\mathcal{I}^\nu_{\mathbb{R}^m}(u_i(X \cap U_i)) \subset \mathcal{I}^{\mathcal{N}}_{\mathbb{R}^m}(u_i(X \cap U_i))\mathscr{C}^{\nu-m}(\mathbb{R}^m)$ for each fixed i. The latter inclusion can be proven exactly as in [BFR14, Proposition 7.3 and Remark 7.4] (however, here the proof is slightly easier). As an immediate consequence, we have $\mathcal{I}^\nu_{U_i}(X \cap U_i) \subset \mathcal{I}^{\mathcal{N}}_{U_i}(X \cap U_i)\mathscr{C}^{\nu-m}(U_i)$. Since $X \subset M$ is coherent, it holds $\mathcal{I}^{\mathcal{N}}_{U_i}(X \cap U_i) = \mathcal{I}^{\mathcal{N}}_M(X)\mathcal{N}(U_i)$ (see equation (2.2) and Lemma 5.1 of [BFR14]). It follows that $\mathcal{I}^\nu_{U_i}(X \cap U_i) \subset \mathcal{I}^{\mathcal{N}}_M(X)\mathscr{C}^{\nu-m}(U_i)$ for all $i \in \{1, \ldots, \ell\}$. Making use of a $\mathscr{C}^\nu$ partition of unity subordinate to the open cover $\{U_1, \ldots, U_\ell, M \setminus X\}$ of M, we obtain at once that $\mathcal{I}^\nu_M(X) \subset \mathcal{I}^{\mathcal{N}}_M(X)\mathscr{C}^{\nu-m}(M)$. Since $G|_X = h \in \mathcal{N}(X)$, by [BCR98, Theorem 8.9.12], there exists $\widetilde{G} \in \mathcal{N}(M)$ such that $\widetilde{G}|_X = G|_X$, i.e., $G - \widetilde{G} \in \mathcal{I}^\nu_M(X)$. Thus, there exist $e \in \mathbb{N}^*$, functions $\{f_j\}_{j=1}^e$ in $\mathcal{I}^{\mathcal{N}}_M(X)$ and functions $\{\psi_j\}_{j=1}^e$ in $\mathscr{C}^{\nu-m}(M)$ such that $G - \widetilde{G} = \sum_{j=1}^e \psi_j f_j$ on M. Thanks to the Weierstrass approximation theorem, for each $j \in \{1, \ldots, e\}$, there exists a polynomial (and hence Nash) function $\widetilde{\psi}_j : M \to \mathbb{R}$ arbitrarily $\mathscr{C}^{\nu-m}_{\mathrm{w}}$ close to ψ_j on M. This proves that the Nash function $H \in \mathcal{N}(M)$ defined by $H := \widetilde{G} + \sum_{j=1} \widetilde{\psi}_j f_j$ is arbitrarily $\mathscr{C}^{\nu-m}_{\mathrm{w}}$ close to G and $H|_X = G|_X$. □

The following result is a variant of [BFR14, Theorem 1.7].

Theorem A.1.5. *Let $M \subset \mathbb{R}^a$ and $N \subset \mathbb{R}^b$ be Nash manifolds, let $X \subset M$ and $Y \subset N$ be two Nash monomial crossings, let $X' \subset M$ be a Nash set with monomial singularities, let $\nu \in \mathbb{N} \cup \{\infty\}$ and let $f : M \to N$ be a $\mathscr{C}^\nu$ map such that $X \cap X' = \varnothing$, $f|_{X'}$ is a Nash map, $f(X) \subset Y$ and $f|_X^Y$ preserves irreducible components. Set $m := \dim(M)$, $n := \dim(N)$ and $q := m\binom{n}{\lfloor n/2 \rfloor}$, where $\lfloor n/2 \rfloor$ is the integer part of $n/2$. Suppose that $\nu \geq q$. Then there exists a Nash map $g : M \to N$ arbitrarily $\mathscr{C}^{\nu-q}_{\mathrm{w}}$ close to f such that $g(X) \subset Y$, $g|_X^Y$ preserves irreducible components and $g|_{X'} = f|_{X'}$. More precisely, for each neighborhood $\mathcal{U}$ of f in $\mathscr{C}^{\nu-q}_{\mathrm{w}}(M, N)$, there exists a Nash map $g : M \to N$ such that $g \in \mathcal{U}$, $g(X) \subset Y$, $g|_X^Y$ preserves irreducible components and $g|_{X'} = f|_{X'}$.*

Proof. We divide the proof into two steps.

STEP I. Let us adapt the proof of [BFR14, Theorem 1.7] to the present situation. Let $X_1, \ldots, X_s$ be the Nash irreducible components of X and let $Y_1, \ldots, Y_t$ be the Nash irreducible components of Y. By hypothesis, all $X_i \subset M$ and $Y_j \subset N$ are Nash manifolds, and there exists a function $\kappa : \{1, \ldots, s\} \to \{1, \ldots, t\}$ such that $f(X_i) \subset Y_{\kappa(i)}$ for all $i \in \{1, \ldots, s\}$. For each $J \in \mathcal{P}(t)$ and for each $p \in \mathbb{N}^*$, we set $Y_J := \bigcap_{j\in J} Y_j$, $X_J := \bigcap_{i\in\kappa^{-1}(J)} X_i$, $\mathcal{P}(t,p) := \{J\in\mathcal{P}(t) : |J| = p\}$, $X^{(p)} := \bigcup_{J\in\mathcal{P}(t,p)} X_J$ and $Y^{(p)} := \bigcup_{J\in\mathcal{P}(t,p)} Y_J$. By construction, we have that $f(X_J)\subset Y_J$ and $f(X^{(p)})\subset Y^{(p)}$ for all $J\in\mathcal{P}(t)$ and for all $p\in\mathbb{N}^*$, and $X^{(1)} = X$ and $Y^{(1)} = Y$. By [BFR14, Lemma 5.1 and Proposition 8.3], we know that all $X_J\subset\mathbb{R}^a$ and $Y_J\subset\mathbb{R}^b$ are Nash manifolds, and all $X^{(p)}\subset M$ and $Y^{(p)}\subset N$ are Nash monomial crossings. Note that each of the sets X_J, $X^{(p)}$, Y_J and $Y^{(p)}$ may be empty. Let $r := \max\{p\in\mathbb{N}^* : X^{(p)}\neq\varnothing\}$. By [BFR14, Proof of Theorem 1.7, Final arrangement, p. 108], we know that $r\leq\binom{n}{\lfloor n/2\rfloor}$. Given any $p\in\{1,\dots,r\}$, we have that both $X^{(p)}$ and $Y^{(p)}$ are non-empty, because $X^{(p)}\supset X^{(r)}\neq\varnothing$ and $Y^{(p)}\supset f(X^{(p)})\neq\varnothing$; thus, we can define the $\mathscr{C}^\nu$ map $f_p : X^{(p)}\to Y^{(p)}$ as the restriction of f from $X^{(p)}$ to $Y^{(p)}$. Moreover, for each $J\in\mathcal{P}(t)$ such that $X_J\neq\varnothing$ (and hence $Y_J\supset f(X_J)\neq\varnothing$), we define the $\mathscr{C}^\nu$ map $f_J : X_J\to Y_J$ as the restriction of f from X_J to Y_J.

Let us prove the following

Claim: *For each $p\in\{1,\dots,r\}$, there exists a Nash map $g_p : X^{(p)}\to Y^{(p)}$ arbitrarily $\mathscr{C}^{\nu-m(r-p)}_{\mathrm{w}}$ close to f_p such that $g_p(X_J)\subset Y_J$ for all $J\in\mathcal{P}^*(t,p)$, where $\mathcal{P}^*(t,p) := \bigcup_{\ell=p}^r\mathcal{P}(t,\ell)$.*

Let us proceed by induction on $p = r, r-1,\dots,1$. Suppose that $p = r$. Note that $X^{(r)}$ is the disjoint union of the X_J's with J in $\mathcal{P}(t,r)$; otherwise, there would exist $J_1, J_2\in\mathcal{P}(t,r)$ with $J_1\neq J_2$ (and hence $|J_1\cup J_2| > r$) such that $\varnothing\neq X_{J_1}\cap X_{J_2} = X_{J_1\cup J_2}$, contradicting the maximality of r. Given any $J\in\mathcal{P}(t,r)$, we have that $f_r(X_J) = f(X_J)\subset Y_J$. Thus, if $X_J\neq\varnothing$ (and hence $Y_J\neq\varnothing$), the Weierstrass approximation theorem and the existence of Nash tubular neighborhoods of Y_J in $\mathbb{R}^b$ imply at once that there exists a Nash map $g_J : X_J\to Y_J$ arbitrarily $\mathscr{C}^\nu_{\mathrm{w}}$ close to f_J. On the other hand, we know that $X^{(r)}$ is the disjoint union of the X_J's with $J\in\mathcal{P}(t,r)$. Consequently, the Nash map $g_r : X^{(r)}\to Y^{(r)}$, defined as $g_r(x) := g_J(x)$ if $x\in X_J$ for some (unique) $J\in\mathcal{P}(t,r)$, is arbitrarily $\mathscr{C}^\nu_{\mathrm{w}}$ close to f_r and $g_r(X_J)\subset Y_J$ for all $J\in\mathcal{P}(t,r) = \mathcal{P}^*(t,r)$, as desired. Suppose that the assertion we want to prove is true for some $p\in\{1,\dots,r\}$. We can assume that $p\neq 1$; otherwise, we are done. Let $g_p : X^{(p)}\to Y^{(p)}$ be a Nash map arbitrarily $\mathscr{C}^{\nu-m(r-p)}_{\mathrm{w}}$ close to f_p such that $g_p(X_J)\subset Y_J$ for all $J\in\mathcal{P}(t,p)$. Let $K\in\mathcal{P}(t,p-1)$. If $K'\in\mathcal{P}(t,p-1)\setminus\{K\}$ then $X_K\cap X_{K'} = X_{K\cup K'}\subset X^{(p)}$. Note that $g_p(X_K\cap X^{(p)})\subset Y_K$; indeed, $X_K\cap X^{(p)} = \bigcup_{J\in\mathcal{P}(t,p)}(X_K\cap X_J) = \bigcup_{J\in\mathcal{P}(t,p)} X_{K\cup J}$ and hence $g_p(X_K\cap X^{(p)}) = \bigcup_{J\in\mathcal{P}(t,p)} g_p(X_{K\cup J})\subset\bigcup_{J\in\mathcal{P}(t,p)} Y_{K\cup J}\subset Y_K$; here we used the fact that, for all $J\in\mathcal{P}(t,p)$, it holds $K\cup J\in\mathcal{P}^*(t,p)$, so $g_p(X_{K\cup J})\subset Y_{K\cup J}$. If $X_K\cap X^{(p)}\neq\varnothing$, then we define the $\mathscr{C}^{\nu-m(r-p)}_{\mathrm{w}}$ maps $f_{p,K} : X_K\cap X^{(p)}\to Y_K$ and $g_{p,K} : X_K\cap X^{(p)}\to Y_K$ as the restrictions of f_p and g_p from $X_K\cap X^{(p)}$ to Y_K, respectively. Recall that $X_K\subset\mathbb{R}^a$ and $Y_K\subset\mathbb{R}^b$ are Nash manifolds, and $X_K\cap X^{(p)}\subset X_K$ is a Nash monomial crossings. If we choose g_p sufficiently $\mathscr{C}^{\nu-m(r-p)}$ close to f_p, then we can assume that, for each $K\in\mathcal{P}(t,p-1)$, $g_{p,K}$ is arbitrarily $\mathscr{C}^{\nu-m(r-p)}$ close to $f_{p,K} = f_K|_{X_K\cap X^{(p)}}$. By Proposition A.1.4, there exists a Nash map $\widetilde{g}_{p,K} : X_K\to Y_K$ arbitrarily $\mathscr{C}^{\nu-m(r-p+1)}$ close to f_K such that $\widetilde{g}_{p,K}|_{X_K\cap X^{(p)}} = g_{p,K} = g_p|_{X_K\cap X^{(p)}}$. Consider $K, K'\in\mathcal{P}(t,p-1)$ such that $K\neq K'$ and $X_K\cap X_{K'}\neq\varnothing$, and choose $x\in X_K\cap X_{K'}$. By construction, we know that $x\in X_K\cap X^{(p)}$ and $x\in X_{K'}\cap X^{(p)}$, so $\widetilde{g}_{p,K}(x) = g_p(x) = \widetilde{g}_{p,K'}(x)$. This proves that the map $g_{p-1} : X^{(p-1)}\to Y^{(p-1)}$, defined by $g_{p-1}(x) := \widetilde{g}_{p,K}(x)$ if $x \in X_K$ for some $K \in \mathcal{P}(t, p-1)$, is a well-defined c-Nash map, which is also a Nash map by Remark A.1.3. Note that $g_{p-1}(X_K) \subset Y_K$ for all $K \in \mathcal{P}(t, p-1)$ and g_{p-1} is an extension of g_p; indeed, it coincides with g_p on $\bigcup_{K\in\mathcal{P}(t,p-1)}(X_K \cap X^{(p)}) = \big(\bigcup_{K\in\mathcal{P}(t,p-1)} X_K\big) \cap X^{(p)} = X^{(p-1)} \cap X^{(p)} = X^{(p)}$. As a consequence, $g_{p-1}(X_K) \subset Y_K$ for all $K \in \mathcal{P}^*(t, p-1)$. Since $g_{p-1}|_{X_K}$ is arbitrarily $\mathscr{C}^{\nu-m(r-p+1)}$ close to $f_K = f|_{X_K}$ for all $K \in \mathcal{P}(t, p-1)$, by Proposition A.1.2, it follows that g_{p-1} is arbitrarily $\mathscr{C}^{\nu-m(r-p+1)}$ close to f_{p-1}. This proves the preceding Claim. In particular, we proved the existence of a Nash map $g_1 : X \to Y$ arbitrarily $\mathscr{C}^{\nu-m(r-1)}$ close to $f|_X^Y$.

STEP II. Let us complete the proof. Note that $X \cup X' \subset M$ is a Nash set with monomial singularities. We denote $g_0 : X \to N$ the Nash map defined by $g_0(x) := g_1(x)$ if $x \in X$ and $g_0(x) := f(x)$ if $x \in X'$. Since g_0 is arbitrarily $\mathscr{C}^{\nu-m(r-1)}$ close to $f|_{X\cup X'}$, using again Proposition A.1.4, we obtain a Nash map $g : M \to N$ arbitrarily $\mathscr{C}^{\nu-mr}$ close (and hence $\mathscr{C}^{\nu-q}$ close, being $q \geq mr$) to f such that $g|_{X\cup X'} = g_0|_{X\cup X'}$. The proof is complete. $\square$

Since $\infty - q = \infty$ and the subset of $\mathscr{C}^\infty_{\mathrm{w}}(M, N)$ of all $\mathscr{C}^\infty$ diffeomorphisms is open when M and N are compact, Theorem 4.1.2 is an immediate consequence of Theorem A.1.5.

APPENDIX B

On the degree of global smoothing mappings of subanalytic sets

Abstract. Let $X \subset \mathbb{R}^n$ be a subanalytic set of dimension k, let U be an open subset of the smooth part of X of dimension k and let W be a connected component of U. In this work we present a criterion for any global smoothing section $\Gamma := (X', \varphi, U)$ of X to have even degree over W. This appendix is based on [Sav22].

B.1. Global smoothings of subanalytic sets

In [BP18] Bierstone and Parusiński proved the following two remarkable global smoothing results for subanalytic sets. The term 'analytic' means 'real analytic'. *Let V be an analytic manifold of dimension n, and let X be a closed subanalytic subset of V of dimension k.*

Theorem A ([BP18, Theorem 1.1] Non-embedded global smoothing). *There exist an analytic manifold X' of pure dimension k, a proper analytic mapping $\varphi : X' \to V$, and a smooth open subanalytic subset U of X such that:*

(i) $\varphi(X') \subset X$.
(ii) $\dim(X \setminus U) < k$ *and* $\varphi^{-1}(X \setminus U)$ *is a simple normal crossings hypersurface* B' *of* X'.
(iii) *For each connected component W of U, $\varphi^{-1}(W)$ is a finite union of subsets open and closed in $\varphi^{-1}(U)$, each mapped isomorphically onto W by φ.*

Theorem B ([BP18, Theorem 1.2] Embedded global smoothing). *There exist an analytic manifold V', a smooth closed analytic subset $X' \subset V'$ of dimension k, a simple normal crossings hypersurface $B' \subset V'$ transverse to X', and a proper analytic mapping $\varphi : V' \to V$ such that:*

(i) $\dim(\varphi(B')) < k$.
(ii) *The restriction $\varphi|_{V' \setminus B'}$ is finite-to-one and of constant rank n;*
(iii) *φ induces an isomorphism from a union of connected components of $X' \setminus B'$ to a smooth open subanalytic subset $U \subset X$ such that* $\dim(X \setminus U) < k$.

We give a couple of remarks motivating our study. Although the previous results are global, the techniques involved in their proves are local. Indeed, in [BP18, Section 2.3] the authors provide a partition of the analytic manifold V into a countable number of semianalytic cells in general position with respect to X and then they develop explicit desingularization techniques for these cells with respect to the global behaviour of V. More in detail, in [BP18, Section 2.2] the authors develop a desingularizing procedure for a semianalytic n-cell C of V by explicitly finding an analytic subset Z_C of $V \times \mathbb{R}^m$, for some $m \in \mathbb{N}$ depending on the number of inequalities defining C, a map $\varphi_C : Z_C \to C$ and an open semianalytic subset U_C of C such that $\varphi_C^{-1}(U_C)$ is a 2^m covering of U_C and $\dim(C \setminus U_C) < k$. Then the authors apply desingularitazion techniques in the sense of [BM97] to Z_C finding a smoothing of the cell C. Thus, we see that the smoothing map φ_C of a single cell C of V is even-to-one over U_C. Since the global maps φ in Theorem A and Theorem B are constructed in terms of the local maps φ_C, we deduce that φ is even-to-one over each open set U_C, hence, in particular, φ is even-to-one over each intersection $U_C \cap X$.

Let us give a definition.

Definition B.1.1. Let X', φ, U and W be as in the previous *Theorem A*, that is, X' is an analytic manifold of pure dimension k, $\varphi : X' \to V$ is a proper analytic mapping, U is an open subset of the smooth part of X of dimension k and W is a connected component of U such that $\varphi(X') \subset X$, $\dim(X \setminus U) < k$, $\varphi^{-1}(X \setminus U)$ is a simple normal crossings hypersuface of X' and $\varphi^{-1}(W)$ is a finite union of subsets open and closed in $\varphi^{-1}(U)$, each mapped isomorphically onto W by φ. We call the triple $\Gamma := (X', \varphi, U)$ *global smoothing section of* $X \subset V$ and the finite positive number of subsets open and closed in $\varphi^{-1}(U)$, each mapped isomorphically onto W by φ, as the *degree of* Γ *over* W.

Theorem A asserts that global smoothing sections of $X \subset V$ always exist.

Thanks to [BP18, Remark 2.6], if $V = \mathbb{R}^n$ and X is a closed semialgebraic subset of $\mathbb{R}^n$, then Theorem B can be strengthened by requiring the mapping in (2) to be injective. On the other hand, in the setting of Theorem A, it is not possible in general to choose a global smoothing section (X', φ, U) of $X \subset \mathbb{R}^n$ whose degree in each connected component of U is equal to 1, as it happens in the case of Hironaka's resolution of singularities, see Example B.2.5 below.

The aim of this note is to give a criterion for the evenness of the degree of global smoothing sections on the connected components over an arbitrary open subset U of the smooth part of X of dimension k. This criterion aims to be useful, somehow, in producing counterexamples about the existence of a global smoothing section with U to be the entire smooth part of X of dimension k, as Bierstone and Parusiński assert to believe in [BP18, p. 3117] without explicit examples.

B.2. The evenness criterion, consequences and examples

By Whitney's embedding theorem we can assume that the analytic manifold V coincides with $\mathbb{R}^n$. Let X be a subanalytic subset of $\mathbb{R}^n$ and let $k \in \mathbb{N}$. Recall that a point $x \in X$ is *smooth of dimension* k if there exists an open neighborhood N of x in $\mathbb{R}^n$ such that $X \cap N$ is an analytic submanifold of $\mathbb{R}^n$ of dimension k, see [BM88, Definition 3.3]. The set of all points of X of dimension k is an open subset of X and an analytic submanifold of $\mathbb{R}^n$ of pure dimension k.

Let us introduce the concept of nonbounding equator for subanalytic sets.

Definition B.2.1. Let X be a closed subanalytic subset of $\mathbb{R}^n$ of dimension k, let W be an open subset of the smooth part of X of dimension k and let Y be a subset of W. We say that Y is a *nonbounding equator of* W *in* X if it satisfies the following properties:

(i) Y is a compact $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^n$ of dimension $k-1$.
(ii) Y does not bound, that is, it is not the boundary of a compact $\mathscr{C}^\infty$ manifold with boundary.
(iii) Y has a collar in W, that is, there exists a $\mathscr{C}^\infty$ map $\psi : Y \times (-1,1) \to W$ such that the image $T := \psi(Y \times (-1,1))$ of ψ is an open neighborhood of Y in W, the restriction $\psi : Y \times (-1,1) \to T$ is a $\mathscr{C}^\infty$ diffeomorphism and $\psi(Y \times \{0\}) = Y$.
(iv) There exists a relatively compact open subset K of X such that $\partial K := \overline{K} \setminus K = Y$ and $K \cap T = \psi(Y \times (-1,0))$. Here $\overline{K}$ denotes the closure of K in X.

If such a Y exists, we say that W *has a nonbounding equator in* X.

The next lemma gives an alternative description of the notion of nonbounding equator. We keep the notations of Definition B.2.1.

LEMMA B.2.2. *The set Y is a nonbounding equator of W in X if and only if there exists a continuous function $h : X \to \mathbb{R}$ with the following properties:*

(i) *There exist an open neighborhood Z of Y in W and $\epsilon > 0$ such that the restriction $h' := h|_Z : Z \to \mathbb{R}$ is a $\mathscr{C}^\infty$ function, $h^{-1}([-\epsilon,\epsilon])$ is a compact neighborhood of Y in Z containing no critical points of h' and $h^{-1}(0) = Y$.*
(ii) *Y does not bound.*
(iii) *The subset $h^{-1}((-\infty,0])$ of X is compact.*

PROOF. Let X, k, W, Y, $\psi : Y \times (-1,1) \to W$ and K be as in Definition B.2.1 and let $\pi : Y \times (-1,1) \to (-1,1)$ be the projection onto the second factor. Let us prove that Lemma B.2.2(i)-(iii) are satisfied. Define $Z := \psi(Y \times (-1/2,1/2))$ and $h' : Z \to \mathbb{R}$ as $h'(x) := (\pi \circ \psi)^{-1}(x)$. Then extend h' to the whole X as follows: define $h : X \to \mathbb{R}$ as $h(x) := -1/2$ if $x \in K \setminus Z$, $h(x) := h'(x)$ if $x \in Z$ and $h(x) := 1/2$ otherwise. Fix $\epsilon := 1/4$. Observe that $h|_Z = (\pi \circ \psi^{-1})|_Z$, thus $h|_Z$ has no critical points, $h^{-1}([-1/4,1/4]) = \psi(Y \times [-1/4,1/4])$, which is compact and contains Y, and $h^{-1}((-\infty,0]) = K \cup Y = \overline{K}$.

On the other hand, assume that X, Y, W, Z and h satisfy conditions Lemma B.2.2(i)-(iii). By Lemma B.2.2(i) and [Hir94, Corollary 2.3, p. 154], $h|_{h^{-1}([-\varepsilon,\varepsilon])}$ induces the existence of a collar of Y in W, as in Definition B.2.1(iii). Moreover, by Lemma B.2.2(i)(iii), we have that $K := h^{-1}((-\infty,0))$ satisfies Definition B.2.1(iv). □

Our evenness criterion reads as follows.

THEOREM B.2.3. *Let X be a closed subanalytic subset of $\mathbb{R}^n$, let $\Gamma := (X',\varphi,U)$ be a global smoothing section of $X \subset \mathbb{R}^n$ and let W be a connected component of U. If W has a nonbounding equator in X then the degree of Γ over W is even.*

PROOF. Let $Y \subset W$ be a nonbounding equator of W in X. By Definition B.2.1, there is an open neighborhood T of Y in W, a diffeomorphism $\psi : Y \times (-1,1) \to T$ such that $\psi(Y \times \{0\}) = Y$ and a relatively compact open subset K of X such that $\partial K = Y$ and $K \cap T = \psi(Y \times (-1,0))$. Since Γ is a global smoothing section, $\varphi^{-1}(W)$ consists of a finite disjoint union of open and closed subsets of $\varphi^{-1}(U)$, each mapped isomorphically onto W. Hence, each connected component of $\varphi^{-1}(W)$ contains a copy of Y and a copy of the collar T of Y in W. By Definition B.1.1, the map φ is proper, hence $\varphi^{-1}(\overline{K})$ is a compact subset of X'. Moreover, since $\partial K = Y$, $K \cap T = \psi(Y \times (-1,0))$ and φ is a diffeomorphism when restricted to each connected component of $\varphi^{-1}(W)$, we have that $\varphi^{-1}(\overline{K})$ is a manifold with boundary whose boundary is the disjoint union of d copies of Y, where d denotes the degree of Γ over W. Since Y is nonbounding, we deduce that d is even since the Stiefel-Whitney numbers of $\bigsqcup_1^d Y$ must be all zero [MS74, Theorem 4.9, p. 52]. $\square$

As a consequence, the nonexistence of nonbounding equators of the smooth part of X of dimemension k is a necessary condition to have global one-to-one smoothings similar to Hironaka's resolution of singularities.

Corollary B.2.4. *Let X be a closed subanalytic subset of $\mathbb{R}^n$. If the degree of a global smoothing section of $X \subset \mathbb{R}^n$ over W is 1, then W does not have any nonbounding equator in X.*

Here we present some examples of semialgebraic sets concerning our Theorem B.2.3.

Example B.2.5. Let $X := \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \,|\, x \geq 0\}$. There is a global smoothing section of the whole smooth part of X, that is $\Gamma := (X', \varphi, U)$ with $U := \mathbb{R}_{>0} = \{x \in \mathbb{R} \,|\, x > 0\}$, $X' := \{(x,y) \in \mathbb{R}^2 \,|\, x = y^2\}$ and $\varphi : X' \to X$ defined as the projection onto the first factor. According to our Theorem B.2.3, the degree of the above smoothing section over the whole smooth part of X is 2. But our result says something more, indeed any global smoothing section $\Gamma := (X', \varphi, U)$ of X, with U any open subset of the smooth part of X, has even degree over any connected component of U. Indeed, since U is an open subset of $\mathbb{R}_{>0}$, every connected component of U has a nonbounding equator Y consisting of a singleton $\{p\}$, with $K := [0,p)$ and the collar $(p-\varepsilon, p+\varepsilon) \subset U$ of p in W, for $\epsilon > 0$ sufficiently small.

Examples B.2.6. Let M be a connected compact $\mathscr{C}^\infty$ manifold of dimension $k-1$, which does not bound (so $k-1 \geq 2$): for instance, the real projective plane $\mathbb{P}^2(\mathbb{R})$. By the Nash-Tognoli theorem, [Nas52] and [Tog73], we can assume that M is a compact nonsingular real algebraic subset of some $\mathbb{R}^n$.

(1) Consider the standard circumference $\mathbb{S}^1 := \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$, the compact nonsingular real algebraic set $X' := M \times \mathbb{S}^1 \subset \mathbb{R}^{n+2}$, and the polynomial maps $\pi_1 : X' \to \mathbb{R}^{n+2}$ and $\pi_2 : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ defined as follows:
$$\pi_1(x,a,b) := (bx, a, b) \quad \text{and} \quad \pi_2(x,a,b) := (x,a,b^2),$$
where $x = (x_1, x_2, \ldots, x_n)$. The set $\pi_1(X')$ is equal to X' with $M \times \{(-1,0)\}$ crushed to the point $p := (0,\ldots,0,-1,0)$ and $M \times \{(1,0)\}$ crushed to the point $q := (0,\ldots,0,1,0)$. The set $X := \pi_2(\pi_1(X'))$ is a semialgebraic subset of $\mathbb{R}^{n+2}$ homeomorphic to the suspension of M. Define $X'_\pm := X' \cap \{\pm b > 0\}$ and the polynomial map $\varphi : X' \to \mathbb{R}^{n+2}$ by $\varphi(x,a,b) := \pi_2(\pi_1(x,a,b))$. Observe that $\varphi(X') = X$, $\varphi^{-1}(p) = M \times \{(-1,0)\}$, $\varphi^{-1}(q) = M \times \{(1,0)\}$, and the restriction of φ from $X'_\pm$ to $U := X \setminus \{p,q\}$, namely to the whole smooth part of X, is a Nash diffeomorphism between connected Nash manifolds. For more details about Nash functions and Nash manifolds we refer to [BCR98, § 8]. The triple $\Gamma := (X', \varphi, U)$ is a global smoothing section of $X \subset \mathbb{R}^{n+2}$ and $\varphi(M \times \{(0,1)\})$ is a nonbounding equator of $W := U$ in X. The degree of Γ over W is two, in accordance with our Theorem B.2.3.

(2) Let $Z' := M \times [-1,1] \subset \mathbb{R}^{n+1}$, let $\phi : Z' \to \mathbb{R}^{n+1}$ be the polynomial map

$$\phi(x,a) := (x(1-a^2), a)$$

and let X be the semialgebraic subset $\phi(Z')$ of $\mathbb{R}^{n+1}$. Observe that X is homeomorphic to the suspension of M, $\phi^{-1}(z_\pm) = M \times \{\pm 1\}$, where $z_\pm := (0,\dots,0,\pm 1)$, the restriction of ϕ from $Z' \setminus (M \times \{-1,1\}) = M \times (-1,1)$ to $U := X \setminus \{z_-, z_+\}$ is a Nash diffeomorphism between connected Nash manifolds (so ϕ has degree one over U), and $\phi(M \times \{0\})$ is a nonbounding equator of $W := U$ in X. However, the triple (Z', ϕ, U) is not a global smoothing section of $X \subset \mathbb{R}^{n+1}$, because Z' is not an analytic manifold: it has the nonempty boundary $M \times \{-1,1\}$.

Nevertheless, the previous construction arises as an explicit case of Theorem B. Let $V := \mathbb{R}^{n+1}$. By [AK92, Corollary 2.5.14, p. 50] we may assume in addition that M is projectively closed, that is M is the zero set $\mathcal{Z}_{\mathbb{R}^n}(p)$ in $\mathbb{R}^n$ of some overt polynomial $p \in \mathbb{R}[x_1,\dots,x_n]$. Write p as follows: $p = \sum_{i=0}^d p_i$, where p_i is an homogeneous polynomial of degree i. Recall that $\mathcal{Z}_{\mathbb{R}^n}(p_d) = \{0\}$ as p is overt. Thus, if $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the polynomial map $(x,a) \mapsto ((1-a^2)x, a)$, $Z := \varphi(M \times \mathbb{R})$ and $q(x,a) \in \mathbb{R}[x_1,\dots,x_n,a]$ is the polynomial $q(x,a) := \sum_{i=0}^d (1-a^2)^{d-i} p_i(x)$, then $M \times \mathbb{R} = \mathcal{Z}_{\mathbb{R}^{n+1}}(p)$ and $q(\varphi(x,a)) = (1-a^2)^d p(x) = 0$ for all $(x,a) \in M \times \mathbb{R}$. It follows that

$$Z = \mathcal{Z}_{\mathbb{R}^{n+1}}(q).$$

This proves that Z is algebraic and irreducible, so Z is the Zariski closure of X in $\mathbb{R}^{n+1}$. Thus, we deduce that X, Z, $Y := \{z_-, z_+\}$, U, Z' and X' constitute an explicit embedded global smoothing as in [BP18, Remark 2.6].

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