The Hodge decomposition theorem for general three–dimensional vector fields, without cuts

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Introduction. The Hodge decomposition of the space $L^2(\Omega)^3$ of square sommable vector fields defined on a domain Ω of the ordinary three–dimensional space is a tool used in a wide theoretic and applied literature concerning Electromagnetism and Fluid dynamics. In order to obtain an effective Hodge decomposition, suitable also for numerical computations, the *classical strategy*, which goes back to Helmholtz and which is used nowadays as well, is to consider only bounded domains Ω with sufficiently regular boundary, which contain connected and pairwise disjoint surfaces, called cut surfaces of Ω, whose complement C in Ω is simply connected (see for example [2, 3, 6, 7, 9, 10, 11, 13]).

The advantage of this situation is that the union of such cut surfaces does not have singularities, every curl–free vector field in Ω admits potential in C and hence it is possible to apply standard variational methods to obtain the Hodge decomposition via scalar potential formulations.

Recently, R. Benedetti, R. Frigerio and the author [5] gave an exhaustive topological description of bounded domains, called *Helmholtz domains*, which admit such cut surfaces. We proved that the Helmholtz domains are truly special three–dimensional domains. In fact, their topology is elementary. An evidence of this claim is given, for example, by the fact that the complement of any non–trivial thickened knot in a box domain is not a Helmholtz domain. In this way, we realize that the range of application of the *classical strategy* is quite limited.

In this paper, we announce several results concerning an effective Hodge decomposition of $L^2(\Omega)^3$ valid for every bounded domain Ω with Lipschitz–regular boundary, where Ω describes a spatial medium formed by a possible inhomogeneous and/or anisotropic material.

In the contest of Hodge decomposition theorem used in applications, a crucial problem is to understand the topological natural of the space of harmonic vector fields and to give explicit bases of such space. In Section 4, we settle completely this problem, including the affermative solution of Auchmuty–Alexander conjectures (see [4]). Such a solution is obtained by combining some basic facts of Algebraic Topology with a new tool, called tubular integral.

In the final version of the paper (which will appear soon), we shall present applications of our general version of Hodge decomposition theorem to Electrostatic, to Magnetostatic and to Navier-Stokes equations. Moreover, we will give an explicit algorithm to compute the Hodge decomposition of $L^2(\Omega)^3$ for every bounded triangulated domain Ω of $\mathbb{R}^3$. Part of the proofs of results presented here will appear in the final version of the paper as well.

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1 Functional spaces: preliminaries and notations

1.1 Preliminary analytic notions

Lipschitz domains of $\mathbb{R}^3$. Let $\mathbb{R}^3$ be the standard 3–dimensional real vector space. Equip $\mathbb{R}^3$ and each of its subsets with the usual euclidean metric and with the usual euclidean topology. Given a subset S of $\mathbb{R}^3$, we denote by $\overline{S}$ the closure of S in $\mathbb{R}^3$, by $\mathrm{int}(S)$ the interior of S in $\mathbb{R}^3$ and by ∂S the topological boundary of S in $\mathbb{R}^3$, which is equal to $\overline{S} \setminus \mathrm{int(S)}$. We call ∂S simply *boundary of S in* $\mathbb{R}^3$. These notions have also a relative version. We remind the reader that, if A is a subset of S, then the closure of A in S, the interior of A in S and the boundary of A in S coincide with $S \cap \overline{A}$, $A \setminus \overline{S \setminus A}$ and $S \cap \overline{A} \cap \overline{S \setminus A}$, respectively. The subset S is said to be *bounded* if there exists a positive real number r such that $|v|_3 \leq r$ for each $v \in S$, where $|v|_3$ denotes the usual euclidean norm $(v_1^2 + v_2^2 + v_3^2)^{1/2}$ of a vector $v = (v_1, v_2, v_3)$ of $\mathbb{R}^3$.

By a *domain of* $\mathbb{R}^3$, we mean a non–empty connected open subset of $\mathbb{R}^3$.

Let Ω be a domain of $\mathbb{R}^3$. The open subset Ω of $\mathbb{R}^3$ is called *Lipschitz* if either $\partial\Omega$ is empty or $\partial\Omega$ is non–empty and each point x of $\partial\Omega$ has the following property: there exist positive constant a, b and c, an open neighborhood U of x in $\mathbb{R}^3$, an isometric homeomorphism Ψ from the pluri–interval $I_{a,b,c} := (-a, a) \times (-b, b) \times (-c, c)$ of $\mathbb{R}^3$ to U and a Lipschitz map $f : (-a, a) \times (-b, b) \longrightarrow (-c, c)$ such that $\Psi(0) = x$ and

$$\Psi^{-1}(U \cap \Omega) = \big\{(y, z, s) \in I_{a,b,c} \,\big|\, s > f(y, z)\big\}.$$

In particular, $\Psi^{-1}(U \cap \partial\Omega)$ is equal to $\{(y, z, s) \in I_{a,b,c} \,|\, s = f(y, z)\}$.

Let A be an open subset of $\partial\Omega$ and let $\hat{\partial}A$ be its boundary in $\partial\Omega$, which is equal to $\overline{A} \setminus A$. We say that A is a *Lipschitz open subset of* $\partial\Omega$ if either $\hat{\partial}A$ is empty or $\hat{\partial}A$ is non–empty and, for each $x \in \hat{\partial}A$, it is possible to find a, b, c, U, Ψ, f with the preceding property and, in addition, a Lipschitz map $g : (-a, a) \longrightarrow (-b, b)$ such that

$$\Psi^{-1}(U \cap A) = \big\{(y, z, s) \in I_{a,b,c} \,\big|\, s = f(y, z), z > g(y)\big\}.$$

In particular, $\Psi^{-1}(U \cap \hat{\partial} A)$ is equal to $\{(y,z,s) \in I_{a,b,c} \,|\, s = f(y,z), z = g(y)\}$.

In order to make the paper more readable, we introduce the notion of Lipschitz device of $\mathbb{R}^3$.

Definition 1.1 *Let Ω be a Lipschitz domain of $\mathbb{R}^3$ and let A be a Lipschitz open subset of its boundary. We say that the pair (Ω, A) is a* Lipschitz device of $\mathbb{R}^3$ *if $\partial\Omega$ is bounded. Such a Lipschitz device (Ω, A) is said to be* bounded *if Ω is bounded. Otherwise, it is called* unbounded.

Basic functional spaces. In what follows, the notions of measurability, integrability and of integral are understood in the sense of Lebesgue. Let E be a non–empty measurable subset of $\mathbb{R}^3$. As is usual, for each $p \in [1, +\infty)$, we denote by $L^p(E)$ the Banach space of all measurable functions $f : E \longrightarrow \mathbb{R}$ such that $|f|^p$ is integrable, whose norm is given by $\|f\|_{L^p(E)} := (\int_E |f(x)|^p \, dx)^{1/p}$. Similarly, we denote by $L^\infty(E)$ the vector space of all measurable functions $f : E \longrightarrow \mathbb{R}$ such that $|f(x)| \leq C$ for some constant C and for almost every $x \in E$. This is a Banach space with respect to the norm given by setting $\|f\|_{L^\infty(E)} := \inf\big\{C \in \mathbb{R} \,\big|\, |f(x)| \leq C \text{ for almost every } x \in E\big\}$. Let now $p \in [1, +\infty)$ or $p = \infty$. Two functions in $L^p(E)$ are identified if they are equal almost everywhere on E so, to be precise, $L^p(E)$ consists of "equal almost everywhere" equivalence classes of functions. However, for simplicity, we abuse notation by confusing a function with its "equal almost everywhere" equivalence class. We denote by $L^p(E)^3$ the Banach space of all vector fields $V = (V_1, V_2, V_3) : E \longrightarrow \mathbb{R}^3$ whose components V_i belong to $L^p(E)$ and whose norm $\|V\|_{L^p(E)^3}$ is equal to $\|(V_1^2 + V_2^2 + V_3^2)^{1/2}\|_{L^p(E)}$.

Fix a bounded Lipschitz domain Ω of $\mathbb{R}^3$. By using the surface measure induced on $\partial\Omega$ by Lebesgue one of $\mathbb{R}^2$ via Lipschitz charts, one can define the Banach spaces $L^p(A)$ and $L^p(A)^3$ for each non–empty open subset A of $\partial\Omega$ (see [14]).

Now we focus our attention to the case $p = 2$. The space $L^2(\Omega)$ is a Hilbert space with respect to the scalar product which sends $(f,g) \in L^2(\Omega) \times L^2(\Omega)$ into $\int_\Omega fg \, dx \in \mathbb{R}$. It follows that $L^2(\Omega)^3$ is a Hilbert space as well. In fact, given $V = (V_1, V_2, V_3)$ and $W = (W_1, W_2, W_3)$ in $L^2(\Omega)^3$, the scalar product (V, W) between V and W in $L^2(\Omega)^3$ is given by

$$(V, W) := \int_\Omega V \bullet W \, dx,$$

where $V \bullet W := \sum_{i=1}^3 V_i \cdot W_i \in L^1(\Omega)$. As is usual, we denote by $H^1(\Omega)$ the Hilbert space of all elements f of $L^2(\Omega)$ having weak gradient ∇f in $L^2(\Omega)^3$, whose norm $\|f\|_{H^1(\Omega)}$ is equal to $(\|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)^3})^{1/2}$.

Let $\mathscr{C}^\infty(\Omega)$ be the set of all real valued smooth functions on Ω. By the adjective "smooth", we always mean "differentiable of class $\mathscr{C}^\infty$". We denote by $\mathscr{D}(\Omega)$ the set of all functions in $\mathscr{C}^\infty(\Omega)$ with compact support and by $\mathscr{C}^\infty(\overline{\Omega})$ the set of restrictions to Ω of all functions in $\mathscr{C}^\infty(\mathbb{R}^3)$.

Let $V \in L^2(\Omega)^3$. We remind the reader that V has weak curl in $L^2(\Omega)^3$, denoted by $\mathrm{curl}(V)$, if $\mathrm{curl}(V)$ is a vector field in $L^2(\Omega)^3$ such that

$$\int_\Omega \mathrm{curl}(V) \bullet \Phi \; dx = \int_\Omega V \bullet \mathrm{curl}(\Phi) \, dx \quad \text{for each } \Phi \in \mathscr{D}(\Omega)^3 := (\mathscr{D}(\Omega))^3.$$

Moreover, V has weak divergence $\mathrm{div}(V)$ in $L^2(\Omega)$ if $\mathrm{div}(V)$ is a function in $L^2(\Omega)$ such that

$$\int_\Omega \mathrm{div}(V) \cdot \varphi \; dx = -\int_\Omega V \bullet \nabla\varphi \, dx \quad \text{for each } \varphi \in \mathscr{D}(\Omega).$$

Let $H(\mathrm{curl}, \Omega)$ be the vector space of all vector fields V in $L^2(\Omega)^3$ with weak curl in $L^2(\Omega)^3$ and let $H(\mathrm{div}, P)$ be the vector space of all vector fields V in $L^2(\Omega)^3$ with weak divergence in $L^2(\Omega)$. They are Hilbert spaces with respect to the norms

$$\|V\|_{H(\mathrm{curl},\Omega)} := \big(\|V\|^2_{L^2(\Omega)^3} + \|\mathrm{curl}(V)\|^2_{L^2(\Omega)^3}\big)^{1/2}$$

and

$$\|V\|_{H(\mathrm{div},\Omega)} := \left(\|V\|^2_{L^2(\Omega)^3} + \|\mathrm{div}(V)\|^2_{L^2(\Omega)}\right)^{1/2},$$

respectively. The vector field V in $L^2(\Omega)^3$ is said to be *curl–free* if $\mathrm{curl}(V) = 0$ and *divergence–free* if $\mathrm{div}(V) = 0$.

We denote by $\mathbf{n}_{\partial\Omega} : \partial\Omega \longrightarrow \mathbb{R}^3$ the element of $L^\infty(\partial\Omega)^3$ defined as the unit normal vector field of $\partial\Omega$, oriented towards $\mathbb{R}^3 \setminus \overline{\Omega}$. This definition is consistent. In fact, the Alexander duality theorem ensures that the compact Lipschitz surface $\partial\Omega$ of $\mathbb{R}^3$ is orientable and Ω is one of the connected components of $\mathbb{R}^3 \setminus \partial\Omega$. This fact and Rademacher's theorem imply the existence and the uniqueness of the mentioned element $\mathbf{n}_{\partial\Omega}$ of $L^\infty(\partial\Omega)$. Let $H^{1/2}(\partial\Omega)$ be the Hilbert space of traces of functions in $H^1(\Omega)$ on $\partial\Omega$ (see [14] for a definition) and let $H^{-1/2}(\partial\Omega)$ be the dual space of $H^{1/2}(\partial\Omega)$. Identify the dual space of the Hilbert space $H^{1/2}(\partial\Omega)^3 := (H^{1/2}(\partial\Omega))^3$ with $H^{-1/2}(\partial\Omega)^3 := (H^{-1/2}(\partial\Omega))^3$ in the natural way. It is well–known that there exist, and are unique, two continuous linear maps $\tau_{\partial\Omega} : H(\mathrm{curl},\Omega) \longrightarrow H^{-1/2}(\partial\Omega)^3$ and $\nu_{\partial\Omega} : H(\mathrm{div},\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ such that, for each $V \in \mathscr{C}^\infty(\overline{\Omega})^3 := (\mathscr{C}^\infty(\overline{\Omega}))^3$, $\tau_{\partial\Omega}(V) = V|_{\partial\Omega} \times \mathbf{n}_{\partial\Omega}$ and $\nu_{\partial\Omega}(V) = V|_{\partial\Omega} \bullet \mathbf{n}_{\partial\Omega}$, where $V|_{\partial\Omega}$ indicates the restriction to $\partial\Omega$ of any element of $\mathscr{C}^\infty(\mathbb{R}^3)^3$ extending V and $V|_{\partial\Omega} \times \mathbf{n}_{\partial\Omega}$ indicates the usual vector product between $V|_{\partial\Omega}$ and $\mathbf{n}_{\partial\Omega}$. If $V \in H(\mathrm{curl},\Omega)$, then $\tau_{\partial\Omega}(V)$ is said to be the *tangential component of V on* $\partial\Omega$ and, for convenience, it will be indicated by $V \times \mathbf{n}_{\partial\Omega}$. The following Green formula holds:

$$\int_\Omega \mathrm{curl}(V) \bullet W \, dx = \int_\Omega V \bullet \mathrm{curl}(W) \, dx - < V \times \mathbf{n}_{\partial\Omega} \, , \, W|_{\partial\Omega} >_{H^{1/2}(\partial\Omega)^3} \qquad (1)$$

for each $V \in H(\mathrm{curl},\Omega)$ and for each $W \in H^1(\Omega)^3$, where $W|_{\partial\Omega}$ denotes the trace of W on $\partial\Omega$ and $< \cdot \, , \, \cdot >_{H^{1/2}(\partial\Omega)^3}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)^3$ and $H^{1/2}(\partial\Omega)^3$. If $V \in H(\mathrm{div},\Omega)$, then $\nu_{\partial\Omega}(V)$ is called *normal component of V on* $\partial\Omega$ and, for convenience, it will be denoted by $V \bullet \mathbf{n}_{\partial\Omega}$. The following Green formula holds as well:

$$\int_\Omega \mathrm{div}(V) \cdot u \, dx = -\int_\Omega V \bullet \nabla u \, dx + < V \bullet \mathbf{n}_{\partial\Omega} \, , \, u|_{\partial\Omega} >_{H^{1/2}(\partial\Omega)} \qquad (2)$$

for each $V \in H(\mathrm{div},\Omega)$ and for each $u \in H^1(\Omega)$, where $u|_{\partial\Omega}$ denotes the trace of u on $\partial\Omega$ and $< \cdot \, , \, \cdot >_{H^{1/2}(\partial\Omega)}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

Let A be a Lipschitz open subset of $\partial\Omega$. Suppose that A is non–empty. For each $v \in L^2(A)$, let $\widetilde{v}$ be the element of $L^2(\partial\Omega)$, which coincides with v on A and is constantly null on $(\partial\Omega) \setminus A$. Indicate by $H^{1/2}_{00}(A)$ the vector subspace of $L^2(A)$ consisting of all functions v such that $\widetilde{v} \in H^{1/2}(\partial\Omega)$. Equip $H^{1/2}_{00}(A)$ with the norm $\|v\|_{H^{1/2}_{00}(A)} := \|\widetilde{v}\|_{H^{1/2}(\partial\Omega)}$, which makes it a Hilbert space. The reader observes that, if A is a union of connected components of $\partial\Omega$, then $H^{1/2}_{00}(A)$ coincides with $H^{1/2}(A)$. We denote by $H^{-1/2}_{00}(A)$ the dual of $H^{1/2}_{00}(A)$ and identify the dual space of the Hilbert space $H^{1/2}_{00}(A)^3 := (H^{1/2}_{00}(A))^3$ with $H^{-1/2}_{00}(A)^3 := (H^{-1/2}_{00}(A))^3$.

Let $\theta \in H^{-1/2}(\partial\Omega)$. The *restriction $\theta|_A$ of θ to A* is the element of $H^{-1/2}_{00}(A)$ defined as follows:

$$< \theta|_A \, , \, v >_{H^{1/2}_{00}(A)} := < \theta \, , \, \widetilde{v} >_{H^{1/2}(\partial\Omega)} \qquad \text{for each } v \in H^{1/2}_{00}(A).$$

Given $u \in H^1(\Omega)$, we indicate by $u|_A$ the trace of u on A, which coincides with the restriction to A of the trace of u on $\partial\Omega$. In this way, we can define:

$$H^1_{0,A}(\Omega) := \left\{ u \in H^1(\Omega) \,\middle|\, u|_A = 0 \right\}.$$

Since Ω is assumed to be non–empty, bounded and Lipschitz, $H^1_{0,\partial\Omega}(\Omega)$ is equal to the closure $H^1_0(\Omega)$ of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$.

In the case in which A is empty, we fix the following conventions: the symbols $H^{1/2}(A)$, $H_{00}^{1/2}(A)$ and $H_{00}^{-1/2}(A)$ indicate the Hilbert space consisting of the null element only, $\theta|_A$ is the null element of $H_{00}^{-1/2}(A)$ for each $\theta \in H^{-1/2}(\partial\Omega)$, $u|_A$ is the null element of $H^{1/2}(A)$ for each $u \in H^1(\Omega)$ and hence $H^1_{0,A}(\Omega)$ is equal to $H^1(\Omega)$.

Suppose now that A is a, possibly empty, Lipschitz open subset of $\partial\Omega$. Let $\complement A$ be the interior of $(\partial\Omega) \setminus A$ in $\partial\Omega$. The set $\complement A$ is a Lipschitz open subset of $\partial\Omega$, its boundary in $\partial\Omega$ is equal to the boundary $\hat{\partial} A$ of A in $\partial\Omega$ and $\partial\Omega$ is the disjoint union of A, of $\complement A$ and of $\hat{\partial} A$. By the trace theorem (see [14]), we have that

$$H_{00}^{1/2}(A) = \big\{ u|_A \in L^2(A) \,\big|\, u \in H^1_{0,\complement A}(\Omega) \big\}.$$

The latter equality ensures that

$$\int_\Omega \mathrm{curl}(V) \bullet W \, dx = \int_\Omega V \bullet \mathrm{curl}(W)\, dx - < V \times \mathbf{n}_{\partial\Omega}|_A \,,\, W|_A >_{H_{00}^{1/2}(A)^3}$$

for each $V \in H(\mathrm{curl}, \Omega)$ and for each $W \in H^1_{0,\complement A}(\Omega)^3$, where $V \times \mathbf{n}_{\partial\Omega}|_A$ indicates the restriction of $V \times \mathbf{n}_{\partial\Omega}$ to A and $< \cdot \,, \cdot >_{H_{00}^{1/2}(A)^3}$ denotes the duality pairing between $H_{00}^{-1/2}(A)^3$ and $H_{00}^{1/2}(A)^3$. Moreover, it holds:

$$\int_\Omega \mathrm{div}(V) \cdot u \, dx = - \int_\Omega V \bullet \nabla u \, dx + < V \bullet \mathbf{n}_{\partial\Omega}|_A \,,\, u|_A >_{H_{00}^{1/2}(A)}$$

for each $V \in H(\mathrm{div}, \Omega)$ and for each $u \in H^1_{0,\complement A}(\Omega)$, where $V \bullet \mathbf{n}_{\partial\Omega}|_A$ indicates the restriction of $V \bullet \mathbf{n}_{\partial\Omega}$ to A and $< \cdot \,, \cdot >_{H_{00}^{1/2}(A)}$ denotes the duality pairing between $H_{00}^{-1/2}(A)$ and $H_{00}^{1/2}(A)$. In particular, given $V \in H(\mathrm{curl}, \Omega)$, we have:

$$V \times \mathbf{n}_{\partial\Omega}|_A = 0 \text{ if and only if } \textstyle\int_\Omega \mathrm{curl}(V) \bullet W \, dx = \int_\Omega V \bullet \mathrm{curl}(W)\, dx$$

for each $W \in H^1_{0,\complement A}(\Omega)^3$. Similarly, given $V \in H(\mathrm{div}, \Omega)$, it holds:

$$V \bullet \mathbf{n}_{\partial\Omega}|_A = 0 \text{ if and only if } \textstyle\int_\Omega \mathrm{div}(V) \cdot u \, dx = - \int_\Omega V \bullet \nabla u \, dx$$

for each $u \in H^1_{0,\complement A}(\Omega)$.

Material matrix. Let $L^\infty(\Omega)^{3\times 3}$ be the set of all (3×3)–matrices, whose coefficients are elements of $L^\infty(\Omega)$. A matrix $\omega = (\omega_{ij})_{i,j\in\{1,2,3\}}$ in $L^\infty(\Omega)^{3\times 3}$ is said to be *uniformly elliptic* if there exists a positive constant C such that

$$\textstyle\sum_{i=1}^3 \omega_{ij}(x) v_i v_j \geq C |v|_3^2 \tag{3}$$

for almost every $x \in \Omega$ and for each $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, where $|c|$ denotes the absolute value of the real number c and, as we have just said, $|v|_3 = (\sum_{i=1}^3 v_i^2)^{1/2}$. It is immediate to verify that a uniformly elliptic matrix is invertible in $L^\infty(\Omega)^{3\times 3}$ and its inverse is uniformly elliptic too.

Given $V \in L^2(\Omega)^3$, we denote by $\omega \cdot V$, or simply by ωV, the vector field in $L^2(\Omega)^3$, which sends $x \in \Omega$ into the standard product $\omega(x)V(x)$ between the 3×3 real matrix $\omega(x)$ and the vector $V(x)$ of $\mathbb{R}^3$. Given a subset S of $L^2(\Omega)^3$, we define the subset $\omega \cdot S$ of $L^2(\Omega)^3$, denoted by ωS also, as follows

$$\omega \cdot S := \big\{ \omega V \in L^2(\Omega)^3 \,\big|\, V \in S \big\}.$$

Given a map $\mathcal{L} : G \longrightarrow K$ between Hilbert spaces G and K, we say that $\mathcal{L}$ is a *topological linear isomorphism* if it is both a linear isomorphism and a homeomorphism. If G is equal to K then we call such a map $\mathcal{L}$ *topological linear automorphism of* G. The map $\mathcal{L}_\omega : L^2(\Omega)^3 \longrightarrow L^2(\Omega)^3$, which sends $V \in L^2(\Omega)^3$ into $\omega V \in L^2(\Omega)^3$, is an example of topological linear automorphism of $L^2(\Omega)^3$.

Definition 1.2 *We call* material matrix of Ω *a matrix in* $L^\infty(\Omega)^{3\times 3}$, *which is both uniformly elliptic and symmetric.*

Suppose that ω is a material matrix of Ω. Consider the bilinear form $\mathcal{B}_\omega : L^2(\Omega)^3 \times L^2(\Omega)^3 \longrightarrow \mathbb{R}$ defined by $\mathcal{B}_\omega(V,W) := \int_\Omega (\omega V) \bullet W\,dx$. Thanks to condition (3), it follows immediately that $\mathcal{B}_\omega$ is a scalar product on $L^2(\Omega)^3$, which makes it a Hilbert space whose norm is equivalent to the usual one. Finally, we define $\mathbf{1}_{3\times 3}$ as the element of $L^\infty(\Omega)^{3\times 3}$, which is constantly equal to the 3×3 identity matrix. Evidently, $\mathbf{1}_{3\times 3}$ is a material matrix of Ω with ellipticity constant equal to 1. We call $\mathbf{1}_{3\times 3}$ *homogeneous/isotropic material matrix of* Ω.

1.2 Special functional spaces: notations and terminology

We make the following assumptions:

- (Ω, A) is a bounded Lipschitz device of $\mathbb{R}^3$.
- Γ is the boundary of Ω in $\mathbb{R}^3$, $\complement A$ is the (possibly empty) Lipschitz open subset of Γ equal to the interior of $\Gamma \setminus A$ in Γ, $\overline{\Omega}$ is the closure of Ω in $\mathbb{R}^3$ and the element $\mathbf{n}$ of $L^\infty(\Gamma)^3$ is the unit normal vector field of Γ, oriented towards $\mathbb{R}^3 \setminus \overline{\Omega}$.
- ω is a material matrix of Ω; that is, a uniformly elliptic symmetric matrix in $L^\infty(\Omega)^{3\times 3}$.
- $\mathcal{L}_{\omega^{-1}} : L^2(\Omega)^3 \longrightarrow L^2(\Omega)^3$ is the topological linear automorphism, which sends $V \in L^2(\Omega)^3$ into $\omega^{-1}V \in L^2(\Omega)^3$.
- $(\cdot\,,\cdot)_\omega : L^2(\Omega)^3 \times L^2(\Omega)^3 \longrightarrow \mathbb{R}$ is the scalar product of $L^2(\Omega)^3$ defined by setting $(V,W)_\omega := \int_\Omega (\omega V) \bullet W\,dx$.
- $\mathbf{1}_{3\times 3}$ is the homogeneous/isotropic material matrix of Ω.

We need the following definition.

Definition 1.3 *Let X be a vector subspace of $L^2(\Omega)^3$. We define the ω–*orthogonal $X^{\perp_\omega}$ of X in $L^2(\Omega)^3$ *by setting*

$$X^{\perp_\omega} := \big\{ V \in L^2(\Omega)^3 \,\big|\, (V,W)_\omega = 0 \text{ for each } W \in X \big\}.$$

*Given another vector subspace Y of $L^2(\Omega)^3$, we say that X and Y are ω–*orthogonal *if $(V,W)_\omega = 0$ for each $V \in X$ and for each $W \in Y$. If $\{X_i\}_{i=1}^k$ is a finite family of vector subspaces of $L^2(\Omega)^3$, then we write the equality*

$$X = X_1 \overset{\perp}{\oplus}_\omega X_2 \overset{\perp}{\oplus}_\omega \cdots \overset{\perp}{\oplus}_\omega X_k$$

*meaning that X is the direct sum of the X_i's and the X_i's are mutually ω–orthogonal. If $\omega = \mathbf{1}_{3\times 3}$, then we write $X^\perp$ in place of $X^{\perp_\omega}$, $X = X_1 \overset{\perp}{\oplus} X_2 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} X_k$ in place of $X = X_1 \overset{\perp}{\oplus}_\omega X_2 \overset{\perp}{\oplus}_\omega \cdots \overset{\perp}{\oplus}_\omega X_k$ and we use the term "*orthogonal*" in place of "$\mathbf{1}_{3\times 3}$–orthogonal" also.*

We need the following vector subspaces of $L^2(\Omega)^3$ as well (see [10]):

- $H(\mathrm{curl}\,0, \Omega) := \big\{ V \in H(\mathrm{curl}, \Omega) \,\big|\, \mathrm{curl}(V) = 0 \big\}$;
- $H_{0,A}(\mathrm{curl}, \Omega) := \big\{ V \in H(\mathrm{curl}, \Omega) \,\big|\, V \times \mathbf{n}|_A = 0 \big\}$;
- $H_{0,A}(\mathrm{curl}\,0, \Omega) := \big\{ V \in H(\mathrm{curl}\,0, \Omega) \,\big|\, V \times \mathbf{n}|_A = 0 \big\}$;

▷ $H(\operatorname{div},\Omega;\omega) := \big\{V \in L^2(\Omega)^3 \,\big|\, \omega V \in H(\operatorname{div},\Omega)\big\}$;

▷ $H(\operatorname{div} 0,\Omega;\omega) := \big\{V \in H(\operatorname{div},\Omega;\omega) \,\big|\, \operatorname{div}(\omega V) = 0\big\}$;

▷ $H_{0,A}(\operatorname{div},\Omega;\omega) := \big\{V \in H(\operatorname{div},\Omega;\omega) \,\big|\, (\omega V) \bullet \mathbf{n}|_A = 0\big\}$;

▷ $H_{0,A}(\operatorname{div} 0,\Omega;\omega) := \big\{V \in H(\operatorname{div} 0,\Omega;\omega) \,\big|\, (\omega V) \bullet \mathbf{n}|_A = 0\big\}$;

▷ $H(\operatorname{div} 0,\Omega) := H(\operatorname{div} 0,\Omega;\mathbf{1}_{3\times 3})$, $H_{0,A}(\operatorname{div},\Omega) := H_{0,A}(\operatorname{div},\Omega;\mathbf{1}_{3\times 3})$, $H_{0,A}(\operatorname{div} 0,\Omega) := H_{0,A}(\operatorname{div} 0,\Omega;\mathbf{1}_{3\times 3})$.

The reader observes that it hold:

▷ $\mathcal{L}_{\omega^{-1}}(H(\operatorname{div},\Omega)) = H(\operatorname{div},\Omega;\omega)$, $\mathcal{L}_{\omega^{-1}}(H(\operatorname{div} 0,\Omega)) = H(\operatorname{div} 0,\Omega;\omega)$, $\mathcal{L}_{\omega^{-1}}(H_{0,A}(\operatorname{div},\Omega)) = H_{0,A}(\operatorname{div},\Omega;\omega)$, $\mathcal{L}_{\omega^{-1}}(H_{0,A}(\operatorname{div} 0,\Omega)) = H_{0,A}(\operatorname{div} 0,\Omega;\omega)$.

Given a vector field V in $L^2(\Omega)^3$, we say that:

▷ V is *normal to* A if $V \in H_{0,A}(\operatorname{curl},\Omega)$;

▷ V is ω*–tangent to* A if $V \in H_{0,A}(\operatorname{div},\Omega;\omega)$ and V is *tangent to* A if $V \in H_{0,A}(\operatorname{div},\Omega)$ or, equivalently, if it is $\mathbf{1}_{3\times 3}$–tangent to A.

Morever, we define:

▷ $\operatorname{grad}(H^1_{0,A}(\Omega)) := \big\{\nabla u \in L^2(\Omega)^3 \,\big|\, u \in H^1_{0,A}(\Omega)\big\}$;

▷ $\operatorname{curl}(H_{0,A}(\operatorname{curl},\Omega)) := \big\{\operatorname{curl}(V) \in L^2(\Omega)^3 \,\big|\, V \in H_{0,A}(\operatorname{curl},\Omega)\big\}$.

In the case in which A is empty, we have:

▷ $H_{0,\emptyset}(\operatorname{curl},\Omega) = H(\operatorname{curl},\Omega)$, $H_{0,\emptyset}(\operatorname{curl} 0,\Omega) = H(\operatorname{curl} 0,\Omega)$, $H_{0,\emptyset}(\operatorname{div},\Omega;\omega) = H(\operatorname{div},\Omega;\omega)$, $H_{0,\emptyset}(\operatorname{div} 0,\Omega;\omega) = H(\operatorname{div} 0,\Omega;\omega)$, $H_{0,\emptyset}(\operatorname{div},\Omega) = H(\operatorname{div},\Omega)$, $H_{0,\emptyset}(\operatorname{div} 0,\Omega) = H(\operatorname{div} 0,\Omega)$.

We recall also that $H^1_{0,\emptyset}(\Omega) = H^1(\Omega)$ and hence it hold:

▷ $\operatorname{grad}(H^1(\Omega)) := \big\{\nabla u \in L^2(\Omega)^3 \,\big|\, u \in H^1(\Omega)\big\}$;

▷ $\operatorname{curl}(H(\operatorname{curl},\Omega)) := \big\{\operatorname{curl}(V) \in L^2(\Omega)^3 \,\big|\, V \in H(\operatorname{curl},\Omega)\big\}$.

Given a real vector space M and one of its vector subspaces N, we denote by M/N (or sometimes by $\frac{M}{N}$) the quotient vector space of M modulo N (see Subsection 2.3 below also).

By elementary considerations, it is easy to see that $\operatorname{grad}(H^1_{0,A}(\Omega)) \subset H_{0,A}(\operatorname{curl} 0,\Omega)$ and $\operatorname{curl}(H_{0,A}(\operatorname{curl},\Omega)) \subset H_{0,A}(\operatorname{div} 0,\Omega)$. The interpretation of gradient, curl and divergence operators as differential operators between forms of suitable degrees suggests to give the following definition.

Definition 1.4 *We call* first de Rham cohomology vector space of (Ω, A)*, denoted by* $\mathbb{H}^1_{\mathrm{DR}}(\Omega, A)$*, and* second de Rham cohomology vector space of (Ω, A)*, denoted by* $\mathbb{H}^2_{\mathrm{DR}}(\Omega, A)$*, the following quotient vector spaces:*

$$\mathbb{H}^1_{\mathrm{DR}}(\Omega, A) := \frac{H_{0,A}(\operatorname{curl} 0,\Omega)}{\operatorname{grad}(H^1_{0,A}(\Omega))} \quad \text{and} \quad \mathbb{H}^2_{\mathrm{DR}}(\Omega, A) := \frac{H_{0,A}(\operatorname{div} 0,\Omega)}{\operatorname{curl}(H_{0,A}(\operatorname{curl},\Omega))}.$$

For simplicity, if A is empty, we use also the symbols $\mathbb{H}^1_{\mathrm{DR}}(\Omega)$ and $\mathbb{H}^2_{\mathrm{DR}}(\Omega)$ in place of $\mathbb{H}^1_{\mathrm{DR}}(\Omega, A)$ and $\mathbb{H}^2_{\mathrm{DR}}(\Omega, A)$, respectively. In other words, we define:

$$\mathbb{H}^1_{\mathrm{DR}}(\Omega) := \frac{H(\operatorname{curl} 0,\Omega)}{\operatorname{grad}(H^1(\Omega))} \quad \text{and} \quad \mathbb{H}^2_{\mathrm{DR}}(\Omega) := \frac{H(\operatorname{div} 0,\Omega)}{\operatorname{curl}(H(\operatorname{curl},\Omega))}.$$

Given a vector field V of $L^2(\Omega)^3$, we say that:

▷ V is a *gradient vector field of* Ω or, simply, a *gradient* if $V \in \mathrm{grad}(H^1(\Omega))$;

▷ V is a *primary vector field of* Ω if $\mathrm{curl}(V) = 0$, but $V \notin \mathrm{grad}(H^1(\Omega))$; that is, if it belongs to $H(\mathrm{curl}\,0, \Omega) \setminus \mathrm{grad}(H^1(\Omega))$.

Let us specify the meaning we give to the notion of ω–harmonic vector field.

Definition 1.5 *Given a vector field V in $L^2(\Omega)^3$, we say that V is ω–*harmonic *if $\mathrm{curl}(V) = 0$ and $\mathrm{div}(\omega V) = 0$; that is, if $V \in H(\mathrm{curl}\,0, \Omega) \cap H(\mathrm{div}\,0, \Omega; \omega)$.*

A $\mathbf{1}_{3\times3}$–harmonic vector field of Ω is simply called harmonic.

We denote by $\mathcal{H}(\Omega, A; \omega)$ the vector subspace of $L^2(\Omega)^3$ consisting of all ω–harmonic vector fields of Ω normal to A and ω–tangent to $\complement A$. In other words, we define:

$$\mathcal{H}(\Omega, A; \omega) := H_{0,A}(\mathrm{curl}\,0, \Omega) \cap H_{0,\complement A}(\mathrm{div}\,0, \Omega; \omega).$$

Moreover, we denote by $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ the vector subspace of $L^2(\Omega)^3$ consisting of all gradient ω–harmonic vector fields of Ω normal to A and ω–tangent to $\complement A$. In other words, we define:

$$\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega) := \mathcal{H}(\Omega, A; \omega) \cap \mathrm{grad}(H^1(\Omega)).$$

If A is empty and/or ω is equal to $\mathbf{1}_{3\times3}$, then A and/or ω can be omitted from the preceding notations. In fact, we define:

▷ $\mathcal{H}(\Omega; \omega) := \mathcal{H}(\Omega, \emptyset; \omega)$, $\mathcal{H}(\Omega, A) := \mathcal{H}(\Omega, A; \mathbf{1}_{3\times3})$, $\mathcal{H}(\Omega) := \mathcal{H}(\Omega, \emptyset; \mathbf{1}_{3\times3})$;

▷ $\mathcal{H}_{\mathrm{grad}}(\Omega; \omega) := \mathcal{H}_{\mathrm{grad}}(\Omega, \emptyset; \omega)$, $\mathcal{H}_{\mathrm{grad}}(\Omega, A) := \mathcal{H}_{\mathrm{grad}}(\Omega, A; \mathbf{1}_{3\times3})$, $\mathcal{H}_{\mathrm{grad}}(\Omega) := \mathcal{H}_{\mathrm{grad}}(\Omega, \emptyset; \mathbf{1}_{3\times3})$.

The reader observes that a vector field V in $L^2(\Omega)^3$ is a ω–harmonic vector field of Ω normal to A and ω–tangent to $\complement A$ if and only if it belongs to $H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega; \omega)$ and satisfies the following system:

$$\begin{cases} \mathrm{curl}(V) = 0 \\ \mathrm{div}(\omega V) = 0 \\ V \times \mathbf{n}|_A = 0 \\ (\omega V) \bullet \mathbf{n}|_{\complement A} = 0. \end{cases} \tag{4}$$

Remark 1.6 The notions of gradient, primary and harmonic vector field can be repeated for smooth vector fields defined on a generic open subset of $\mathbb{R}^3$ by using the standard gradient, curl and divergence. Let (x_1, x_2, x_3) be the coordinates of $\mathbb{R}^3$, let O be a non–empty open subset of $\mathbb{R}^3$ and let $V = (V_1, V_2, V_3) \in \mathscr{C}^\infty(O)^3$. The vector field V is said to be: *gradient* if there exists $f \in \mathscr{C}^\infty(O)$ such that $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$ coincides with V on O, *primary* if $\mathrm{curl}(V) = \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}, \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}, \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)$ vanishes on O, but V is not gradient and *harmonic* if $\mathrm{curl}(V)$ and $\mathrm{div}(V) = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}$ vanish on O. We recall that, if $V \in \mathscr{C}^\infty(O)^3$ is harmonic, then it is locally the gradient of harmonic functions. In particular, it is analytic. The same is true if $V \in \mathcal{H}(\mathrm{curl}\,0, \Omega) \cap \mathcal{H}(\mathrm{div}\,0, \Omega)$.

We now introduce some notions, which will prove to be of crucial importance later. First, we denote by $\Pi : \mathcal{H}(\Omega, A; \omega) \longrightarrow \mathcal{H}(\Omega, A; \omega)/\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ the natural projection of $\mathcal{H}(\Omega, A; \omega)$ onto the quotient vector space $\mathcal{H}(\Omega, A; \omega)/\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$.

Definition 1.7 *We call* primary dimension of $\mathcal{H}(\Omega, A; \omega)$, *denoted by* $\rho \dim \mathcal{H}(\Omega, A; \omega)$, *the dimension of the real vector space* $\mathcal{H}(\Omega, A; \omega) \big/ \mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$.

Let $p := \rho \dim \mathcal{H}(\Omega, A; \omega)$. *Suppose that* p *is finite and non–null; that is,* p *is a positive integer. Let* $\mathcal{V} = (V_1, \ldots, V_p)$ *be a* p*–uple of primary* ω*–harmonic vector fields of* Ω *normal to* A *and* ω*–tangent to* $\complement A$. *We say that* $\mathcal{V}$ *is a* primary system of $\mathcal{H}(\Omega, A; \omega)$ *if the set* $\{\Pi(V_1), \ldots, \Pi(V_p)\}$ *is a vector base of* $\mathcal{H}(\Omega, A; \omega) \big/ \mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$.

We remark that, if a primary system of $\mathcal{H}(\Omega, A; \omega)$ exists, then the real vector space $\mathcal{H}(\Omega, A; \omega) / \mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ must have finite and positive dimension.

2 A new tool: the tubular integral

Throughout this section, we shall make the following assumptions: (Ω, A) *is a bounded Lipschitz device of* $\mathbb{R}^3$, Γ *is the boundary of* Ω *and* $\complement A$ *is the interior of* $\Gamma \setminus A$ *in* Γ.

2.1 Tubes, tubular integrals and tubular loops

Let (X, d) and (Y, e) be metric spaces and let $F : X \longrightarrow Y$ be a Lipschitz map between them. The *Lipschitz constant* $\mathrm{Lip}(F)$ *of* F is the real number defined as follows:

$$\mathrm{Lip}(F) := \inf \big\{ c \in \mathbb{R} \,\big|\, e(F(x), F(x')) \leq c \cdot d(x, x') \text{ for each } x, x' \in X \big\}.$$

If F is bijective and both F and F^{-1} are Lipschitz, then F is called *bi–Lipschitz isomorphism.* We say that F is *locally bi–Lipschitz onto its image* if, for each $x \in X$, there exists an open neighborhood U of x in X such that the restriction of F from the metric subspace U of X to the metric subspace $F(U)$ of Y is a bi–Lipschitz isomorphism.

Indicate by t the coordinate of $\mathbb{R}$ and by $x = (x_1, x_2)$ the coordinates of $\mathbb{R}^2$. Let $\mathbb{B}$ be the open ball of $\mathbb{R}^2$ centered at the origin with radius equal to $1/\sqrt{\pi}$ and hence with area equal to 1, and let $\mathbb{D}$ be its closure in $\mathbb{R}^2$. Identify $\mathbb{R} \times \mathbb{R}^2$ with $\mathbb{R}^3$ and consider $I \times \mathbb{D}$ as a metric subspace of $\mathbb{R}^3$. We denote by I the closed interval $[0, 1]$ of $\mathbb{R}$. Given $t \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$, $|t|$ is the absolute value of t and $|(t, x)|_3$ is the usual norm $(t^2 + x_1^2 + x_2^2)^{1/2}$ of (t, x) in $\mathbb{R}^3$. Let E be a non–empty measurable subset of $\mathbb{R}^3$ with finite measure $\mathrm{meas}(E)$.

Definition 2.1 *We call* tube of E *a map* $T : I \times \mathbb{D} \longrightarrow E$ *from* $I \times \mathbb{D}$ *to* E*, which is locally bi–Lipschitz onto its image. If* T *is a tube, then we define the subsets* D_T, $D_{T,0}$ *and* $D_{T,1}$ *of* $\mathrm{Image}(T)$ *by setting* $D_T := T((0, 1) \times \mathbb{B})$, $D_{T,0} := T(\{0\} \times \mathbb{B})$ *and* $D_{T,1} := T(\{1\} \times \mathbb{B})$.

The reader observes that every map $T : I \times \mathbb{D} \longrightarrow E$, which can be extended to a local $\mathscr{C}^1$–diffeomorphism from an open neighborhood of $I \times \mathbb{D}$ in $\mathbb{R}^3$ to an open neighborhood of $\mathrm{Image}(T)$ in $\mathbb{R}^3$, is a tube of E.

Let $T : I \times \mathbb{D} \longrightarrow E$ be a tube. By the theorem of invariance of domain (see [12, p. 82]), the set D_T is open in $\mathbb{R}^3$ and hence it is a domain of $\mathbb{R}^3$. Moreover, thanks to the fact that $I \times \mathbb{D}$ is a compact convex subset of $\mathbb{R}^3$, it follows that T is a Lipschitz map. In this way, we have that

$$\frac{\partial T}{\partial t} \in L^\infty(I \times \mathbb{D})^3 \quad \text{and} \quad \left\| \frac{\partial T}{\partial t} \right\|_{L^\infty(I \times \mathbb{D})^3} \leq \mathrm{Lip}(T), \tag{5}$$

where $\partial T / \partial t$ indicates the weak partial derivative, with respect to the first coordinate t of $\mathbb{R}^3$, of the map T, viewed as a Lipschitz map from $I \times \mathbb{D}$ to $\mathbb{R}^3$.

Theorem-Definition 2.2 *Let* $T : I \times \mathbb{D} \longrightarrow E$ *be a tube of* E *and let* $V \in L^2(E)^3$.

The following assertions hold:

(1) *The composition* $V \circ T : I \times \mathbb{D} \longrightarrow \mathbb{R}^3$ *is a well–defined vector field in* $L^2(I \times \mathbb{D})^3$. *In this way, we can define the* tubular integral $\int_T V$ of V along T *by setting*

$$\int_T V := \int_{I \times \mathbb{D}} (V \circ T)(s,x) \bullet \frac{\partial T}{\partial t}(s,x)\, ds\, dx.$$

(2) *There exists a positive constant* C_T, *depending only on* T *and on* $\mathrm{meas}(E)$, *such that*

$$\left| \int_T V \right| \leq C_T \|V\|_{L^2(E)^3}.$$

In other words, the linear functional $\int_T : L^2(E)^3 \longrightarrow \mathbb{R}$, *which sends* $V \in L^2(E)^3$ *into* $\int_T V \in \mathbb{R}$, *is continuous.*

Remark 2.3 The notion of tubular integral given above extends to the case in which the vector field $V : E \longrightarrow \mathbb{R}^3$ is solely locally integrable; that is, for each $x \in E$, there exists an open neighborhood U of x in $\mathbb{R}^3$ such that the restriction of V to $E \cap U$ is integrable. However, we do not need such an extended notion here.

Definition 2.4 *Let* $T : I \times \mathbb{D} \longrightarrow E$ *be a tube of* E. *We say that* T *is a* regular tube of E *if the following hold:*

(1) T *is a homeomorphism onto its image or, equivalently,* T *is injective.*

(2) *The domain* D_T *is Lipschitz.*

(3) $D_{T,0}$ *and* $D_{T,1}$ *are Lipschitz open subsets of* ∂D_T.

Suppose that T *is such a regular tube. Given a function* f *in* $H^1(D_T)$, *we define the functions* $f_{T,0}$ *in* $H^{1/2}(D_{T,0})$ *and* $f_{T,1}$ *in* $H^{1/2}(D_{T,1})$ *as the restrictions of* f *to* $D_{T,0}$ *and to* $D_{T,1}$, *respectively.*

Let us introduce the notion of tubular loop.

Definition 2.5 *A tube* $T : I \times \mathbb{D} \longrightarrow E$ *is said to be a* tubular loop *if* $T(0,p) = T(1,p)$ *for each* $p \in \mathbb{D}$.

The reader observes that, if T is a tubular loop, then the sets $D_{T,0}$ and $D_{T,1}$ coincide.

Let S^1 be the standard circle $\{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 = 1\}$ of $\mathbb{R}^2$. Given a tubular loop $T : I \times \mathbb{D} \longrightarrow E$, we denote by $\widetilde{T} : S^1 \times \mathbb{D} \longrightarrow E$ the unique map from $S^1 \times \mathbb{D}$ to E such that $T(t,x) = \widetilde{T}\big((\cos(2\pi t), \sin(2\pi t)), x\big)$ for each $(t,x) \in I \times \mathbb{D}$. Evidently, such a map $\widetilde{T}$ is continuous.

Definition 2.6 *A tubular loop* $T : I \times \mathbb{D} \longrightarrow E$ *is said to be* regular *if the associated map* $\widetilde{T} : S^1 \times \mathbb{D} \longrightarrow E$ *has the following properties:*

(1) $\widetilde{T}$ *is a homeomorphism onto its image or, equivalently,* $\widetilde{T}$ *is injective.*

(2) *The open subset* $\widetilde{T}(S^1 \times \mathbb{B}) = T\big([0,1] \times \mathbb{B}\big)$ *of* $\mathbb{R}^3$ *is Lipschitz.*

(3) *There exists* $\delta_0 \in (0,1)$ *such that* $\Delta_0 := T\big((0,\delta_0) \times \mathbb{B})\big)$ *is a Lipschitz domain of* $\mathbb{R}^3$ *and* $D_{T,0}$ *is a Lipschitz open subset* $\partial\Delta_0$. *Similarly, there exists* $\delta_1 \in (0,1)$ *such that* $\Delta_1 := T\big((\delta_1,1) \times \mathbb{B})\big)$ *is a Lipschitz domain of* $\mathbb{R}^3$ *and* $D_{T,0} = D_{T,1}$ *is a Lipschitz open subset* $\partial\Delta_1$.

Suppose that T is such a regular tubular loop, and let δ_0, Δ_0, δ_1 and Δ_1 be as above. Let g be a function in $H^1(D_T)$. Indicate by $g_0 \in H^{1/2}(\partial\Delta_0)$ the trace of $g|_{\Delta_0}$ on $\partial\Delta_0$ and by $g_1 \in H^{1/2}(\partial\Delta_1)$ the trace of $g|_{\Delta_1}$ on $\partial\Delta_1$. We define the functions $g_{T,0} \in H^{1/2}(D_{T,0})$ and $g_{T,1} \in H^{1/2}(D_{T,0})$ as the restrictions of g_0 and of g_1 to $D_{T,0}$, respectively. Moreover, we call jump function of g along T *the function $[g]_T \in H^{1/2}(D_{T,0})$ defined by $[g]_T := g_{T,1} - g_{T,0}$.*

Remark 2.7 (1) A tubular loop is a tube of a particular kind. On the contrary, a regular tubular loop is not injective, so it is never a regular tube. In this way, the notions of regular tube and of regular tubular loop are distinct.

(2) Fix a regular tubular loop T. Let $\widetilde{T}$, δ_0 and δ_1 be as in the preceding definition. It is immediate to verify that the boundary of D_T is equal to the union of $\widetilde{T}\big(S^1 \times \partial\mathbb{D}\big)$ and of $D_{T,0} = D_{T,1}$, the set $T\big(\partial((0,\delta_0) \times \mathbb{B})\big)$ is equal to $\partial\Delta_0$ and the set $T\big(\partial((\delta_1, 1) \times \mathbb{B})\big)$ is equal to $\partial\Delta_1$. Moreover, $D_{T,0}$ is a Lipschitz open subset both of $\partial\Delta_0$ and of $\partial\Delta_1$. Finally, the reader observes that the definitions of $g_{T,0}$, of $g_{T,1}$ and of $[g]_T$ do not depend on the choise of δ_0 and of δ_1.

(3) In Definitions 2.1, 2.4, 2.5 and 2.6, the codomain E of the map T is not important in the sense that one can always view such a map T as a map from $I \times \mathbb{D}$ to $\mathbb{R}^3$ whose image is contained in E.

2.2 Potential jumps and circulations of curl-free vector fields

Let P be a non–empty subset of $\mathbb{R}^3$. A continuous map from I to P is called *path of P*. Let $\gamma : I \longrightarrow P$ be a path of P. We say that γ is an *embedded smooth path of P* if there exists a positive real number ε and a smooth embedding $\psi : (-\varepsilon, 1+\varepsilon) \longrightarrow \mathbb{R}^3$ such that $\gamma(t) = \psi(t)$ for each $t \in I$. If $\gamma(0) = \gamma(1)$, then the path γ is called *loop of P*. Suppose that γ is a loop. We say that γ is an *embedded smooth loop of P* if there exists a smooth embedding $\phi : S^1 \longrightarrow \mathbb{R}^3$ from S^1 to $\mathbb{R}^3$, equipped with their natural structures of smooth manifold, such that $\gamma(t) = \phi(\cos(2\pi t), \sin(2\pi t))$ for each $t \in I$.

Let $\lambda : I \longrightarrow \Omega \cup A$ be a path of $\Omega \cup A$. We say that λ is a *path of $\Omega \cup A$ modulo A* if $\{\lambda(0), \lambda(1)\} \subset A$. Let us specialize such a notion as follows.

Definition 2.8 *Let $\lambda : I \longrightarrow \Omega \cup A$ be a path of $\Omega \cup A$ modulo A. A* tubular extension of *λ is a tube $\Lambda : I \times \mathbb{D} \longrightarrow \Omega \cup A$ such that $\Lambda(t, 0) = \lambda(t)$ for each $t \in I$, $\Lambda((0,1) \times \mathbb{D}) \subset \Omega$ and $\Lambda(\{0,1\} \times \mathbb{D}) \subset A$. If, in addition, Λ is a regular tube, then Λ is said to be a* regular tubular extension of λ*. The path λ is called* extendable *if it admits a tubular extension and it is called* regularly extendable *if it admits a regular tubular extension.*

The reader observes that, if a path $\lambda : I \longrightarrow \Omega \cup A$ is extendable, then, in addition to the condition $\{\lambda(0), \lambda(1)\} \subset A$, we have that λ is Lipschitz and $\lambda((0,1)) \subset \Omega$.

Lemma 2.9 *Every embedded smooth path of $\Omega \cup A$ modulo A transverse to Γ is regularly extendable.*

Given a Lipschitz path λ of $\Omega \cup A$ modulo A and a continuous vector field $W : \Omega \cup A \longrightarrow \mathbb{R}^3$, one can define the line integral $\int_\lambda W$ of W along λ by the usual formula

$$\int_\lambda W = \int_0^1 W(\lambda(s)) \bullet \frac{d\lambda}{dt}(s)\, ds.$$

In the following results, we extend such a classical notion.

Theorem 2.10 *Let λ be an extendable path of $\Omega \cup A$ modulo A. The following statements hold:*

(1) *Let $W : \Omega \cup A \longrightarrow \mathbb{R}^3$ be a continuous vector field whose restriction to Ω belongs to $H_{0,A}(\mathrm{curl}\, 0, \Omega)$. Then, for each tubular extension Λ of λ, we have that*

$$\int_\Lambda W = \int_\lambda W.$$

(2) *Let $V \in H_{0,A}(\mathrm{curl}\, 0, \Omega)$ and let Λ' and Λ'' be two tubular extensions of λ. Then we have that*

$$\int_{\Lambda'} V = \int_{\Lambda''} V.$$

Point (2) of the preceding result permits to give the following definition.

Definition 2.11 *Given an extendable path λ of $\Omega \cup A$ modulo A and $V \in H_{0,A}(\mathrm{curl}\, 0, \Omega)$, we define the* line integral $\int_\lambda V$ of V along λ *by setting*

$$\int_\lambda V := \int_\Lambda V,$$

where Λ is any tubular extension of λ. We call the line integral $\int_\lambda V$ just defined potential jump of V along λ *also.*

The latter terminology is justified by the following result.

Theorem 2.12 *Let λ be a regularly extendable path of $\Omega \cup A$ modulo A and let $V \in H_{0,A}(\mathrm{curl}\, 0, \Omega)$. Choose a regular tubular extension Λ^* of λ. Then there exists a unique function f in $H^1(D_{\Lambda^*})$ such that $\nabla f = V$, $f_{\Lambda^*,0}$ is constantly null and $f_{\Lambda^*,1}$ is constantly equal to $\int_\lambda V$. In particular, if there exist $h \in H^1(\Omega)$ and $d_0, d_1 \in \mathbb{R}$ such that $\nabla h = V$, the trace of h on $D_{\Lambda^*,0}$ is constantly equal to d_0 and the trace of h on $D_{\Lambda^*,1}$ is constantly equal to d_1, then $\int_\lambda V$ is equal to $d_1 - d_0$.*

We now introduce the notion of extendable loop.

Definition 2.13 *Let $\ell : I \longrightarrow \Omega$ be a loop. A* tubular loop extension of ℓ *is a tubular extension $L : I \times \mathbb{D} \longrightarrow \Omega$ of ℓ, which is also a tubular loop. If, in addition, Λ is a regular tubular loop, then it is called* regular tubular loop extension of ℓ. *The loop ℓ is said to be* extendable *if it admits a tubular loop extension and* regularly extendable *if it admits a regular tubular loop extension.*

Evidently, if a loop of Ω is extendable, that it is Lipschitz as well.

Lemma 2.14 *Every embedded smooth loop of $\mathbb{R}^3$ is regularly extendable.*

We have:

Theorem 2.15 *Let ℓ be an extendable loop of Ω. The following statements hold:*

(1) *Let $W : \Omega \longrightarrow \mathbb{R}^3$ be a continuous vector field belonging to $H(\mathrm{curl}\, 0, \Omega)$. Then, for each tubular extension L of ℓ, we have that*

$$\int_L W = \int_\ell W.$$

(2) *Let $V \in H(\mathrm{curl}\, 0, \Omega)$ and let L' and L'' be two tubular extensions of λ. Then we have that*

$$\int_{L'} V = \int_{L''} V.$$

By point (2) of this result, we can give the following definition.

Definition 2.16 *Given an extendable loop of* Ω *and* $V \in H(\operatorname{curl} 0, \Omega)$*, we define the* line integral $\int_\lambda V$ of V along λ *by setting*

$$\int_\ell V := \int_L V,$$

where L *is any tubular loop extension of* ℓ*. We call such a line integral* circulation $\int_\ell V$ of V along ℓ *also.*

As above, the latter terminology is justified by the next result.

Theorem 2.17 *Let* ℓ *be a regularly extendable loop of* Ω *and let* $V \in H(\operatorname{curl} 0, \Omega)$*. Choose a regular tubular loop extension* L^* *of* ℓ*. Then there exists a unique function* g *in* $H^1(D_{L^*})$ *such that* $\nabla g = V$*,* $g_{L^*,0}$ *is constantly null and* $g_{L^*,1}$ *is constantly equal to* $\int_\ell V$*. In particular, the jump function* $[g]_{L^*}$ *of* g *along* L^* *is constantly equal to* $\int_\ell V$*. Moreover, if there exists* $h \in H^1(\Omega)$ *such that* $\nabla h = V$*, then* $\int_\ell V$ *is null.*

2.3 Some basic notions of Algebraic Topology

Abelian groups and vector spaces. We briefly recall some basic definitions and constructions concerning abelian groups and vector spaces over the rational and real numbers. In order to unify the presentation, we use the notion of module over a ring.

An *abelian group* is a non–empty set R, equipped with a binary operation $+ : R \times R \longrightarrow R$, called addition, such that, for each $a, b, c \in R$, it hold:

- $\triangleright$ $(a + b) + c = a + (b + c)$;
- $\triangleright$ there exist an element 0 of R, called null element, such that $a + 0 = 0 + a = a$;
- $\triangleright$ there exists $d \in R$, called additive inverse of a, such that $a + d = d + a = 0$;
- $\triangleright$ $a + b = b + a$.

The abelian group R is said to be *ring* if R is equipped with another binary operation $* : R \times R \longrightarrow R$, called multiplication, such that, for each $a, b, c \in R$, it hold:

- $\triangleright$ $(a * b) * c = a * (b * c)$;
- $\triangleright$ $a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$.

The ring R is said to be *commutative* if $a * b = b * a$ for each $a, b \in R$. An element 1 of R is called *identity of* R if $a * 1 = 1 * a = a$ for each $a \in R$. The null element, the additive inverse of any element of R and the identity (if it exists) are unique. Given $a, b \in R$, we denote the additive inverse of a by $-a$ and the element $b + (-a)$ by $b - a$.

The set $\mathbb{Z}$ of the integers, equipped with the usual addition and multiplication, is an example of commutative ring with 1.

Suppose that R is commutative and has identity 1. Then R is a *field* if each element u of $R \setminus \{0\}$ has a multiplicative inverse; that is, there exists $v \in R$ such that $u * v = 1$.

As is usual, we denote by $\mathbb{Q}$ the field of rational numbers and by $\mathbb{R}$ the field of real numbers, equipped with the usual additions and multiplications.

Fix a commutative ring R *with identity* 1. Let M be an abelian group and let $\boldsymbol{+} : M \times M \longrightarrow M$ be its addition. M is called R*–module* or *module over* R if it is equipped with the another operation $\star : R \times M \longrightarrow M$, called scalar multiplication, such that, for each $a, b \in R$ and for each $x, y \in M$, it hold:

- $\triangleright$ $a \star (x \boldsymbol{+} y) = (a \star x) \boldsymbol{+} (a \star y)$;

▷ $(a+b)\star x = (a\star x) \boldsymbol{+} (b\star x)$;

▷ $a\star(b\star x) = (a*b)\star x$;

▷ $1\star x = x$.

For simplicity, given $a, b \in R$ and $x, y \in M$, we abuse notation by writing ab, ax and $x+y$ in place of $a*b$, $a\star x$ and $x \boldsymbol{+} y$, respectively. A R–module is said to be *null* if it consists of the null element only. The ring R is a module over itself if we consider its multiplication also as a scalar multiplication.

In the case in which R is a field, the notion of R–module becomes more familiar. In fact, if R is a field, then a R–module is called *vector space over* R.

The abelian group M has a unique scalar multiplication $\mathbb{Z}\times M \ni (n,x) \longmapsto nx \in M$, which coincides with the natural one defined as follows: nx is the null element of M if $n=0$, nx is equal to the n–fold sum $x+\ldots+x$ if $n>0$ and nx is equal to $(-n)(-x)$ if $n<0$. For this reason, from now on, we will use the term "$\mathbb{Z}$–module" as a sinonimous of "abelian group".

Let R denote either the ring $\mathbb{Z}$ or the field $\mathbb{Q}$ or the field $\mathbb{R}$.

Let M be a R–module; that is, an abelian group if $R=\mathbb{Z}$ or a vector space over $\mathbb{Q}$ or over $\mathbb{R}$ if $R=\mathbb{Q}$ or $R=\mathbb{R}$, respectively. A non–empty subset $B=\{m_j\}_{j\in J}$ of M is a *base of* M if, for each element x of $M\setminus\{0\}$, there exist, and are unique, a finite subset J' of J and, for each $j\in J'$, an element a_j of R (depending on x) such that $x=\sum_{j\in J'} a_j m_j$. If M admits a base, then M is said to be *free*. A vector space is always free in this sense, so M is free if $R=\mathbb{Q}$ and $R=\mathbb{R}$. If M is free and B and B' are two bases of M, then the cardinality of B and of B' coincide. Such a cardinality is said to be the *rank of* M. Usually, if $R=\mathbb{Q}$ or $R=\mathbb{R}$, then such a rank is called *dimension of* M. For convention, we consider the null abelian group as a free abelian group of rank zero and we assume that the dimension of the null vector space is zero.

Let K be a set and, for each $k\in K$, let a_k be an element of R and let x_k be an element of M. We write the symbol $\sum_{k\in K} a_k x_k$ understanding that the set $K':=\{k\in K \,|\, a_k x_k \neq 0\}$ is finite and $\sum_{k\in K} a_k x_k$ is either the null element of R if $K'=\emptyset$ or the element $\sum_{k\in K'} a_k x_k$ of M otherwise.

Let us present an important construction. Let S be a non–empty set. Denote by M_S the set of all functions $\phi : S \longrightarrow R$ such that $\phi^{-1}(R\setminus\{0\})$ is finite. Define an addition and a scalar multiplication on M_S in the usual way as follows: if $\phi,\psi\in M_S$ and $a\in R$, then $(\phi+\psi)(\sigma) := \phi(\sigma)+\psi(\sigma)$ and $(a\phi)(\sigma) := a\phi(\sigma)$ for each $\sigma\in S$. The set M_S, equipped with these two operations, is called *free R–module generated by S*. For each $\sigma\in S$, let ϕ_σ be the element of M_S such that $\phi_\sigma(\sigma)=1$ and $\phi_\sigma(\xi)=0$ for each $\xi\in S\setminus\{\sigma\}$. Evidently, $\{\phi_\sigma\}_{\sigma\in S}$ is a base of M_S. In fact, each element ϕ of M_S is equal to $\sum_{\sigma\in S}\phi(\sigma)\phi_\sigma$ and $\sum_{\sigma\in S} a_\sigma\phi_\sigma$ is the null function on S if and only if every a_σ is the null element of R. Usually, for simplicity, one identifies each function ϕ_σ with σ itself and writes a generic element ϕ of M_S as a "finite formal linear combination" $\phi=\sum_{\sigma\in S} a_\sigma\sigma$ of elements of S with coefficients in R. We adopt this simplified notation. The reader observes that, if $\phi\in M_S\setminus\{0\}$, then there exist, and are unique, a positive integer h, elements $\{a_i\}_{i=1}^h$ of $R\setminus\{0\}$ and elements $\{\sigma_i\}_{i=1}^h$ of S such that $\sigma_i\neq\sigma_j$ for each $i,j\in\{1,\ldots,h\}$ with $i\neq j$ and $\phi=\sum_{i=1}^h a_i\sigma_i$.

It is worth to give a definition concerning the construction just presented.

Definition 2.18 *Let S be a non–empty set, let M_S be the free R–module generated by S and let $\phi=\sum_{\sigma\in S} a_\sigma\sigma$ be an element of M_S. We call* coefficients of ϕ *the elements of the set $\{a_\sigma\in R \,|\, \sigma\in S\}$. Suppose now that ϕ belongs to $M_S\setminus\{0\}$. Let h be a positive integer and, for each $i\in\{1,\ldots,h\}$, let a_i be an element of R and let σ_i be an element of S. We say that $\phi=\sum_{i=1}^h a_i\sigma_i$ is the* finite representation of ϕ in base *if $a_i\neq 0$ for each $i\in\{1,\ldots,h\}$, $\sigma_i\neq\sigma_j$ for each $i,j\in\{1,\ldots,h\}$ with $i\neq j$ and ϕ is equal to $\sum_{i=1}^h a_i\sigma_i$.*

Let N be a non–empty subset of the R–module M. Suppose that $x - y \in N$ for each $x, y \in N$ and $az \in N$ for each $a \in R$ and for each $z \in N$. Under these conditions, the restrictions to N of the addition and of the scalar multiplication of M induce a structure of R–module on N. The set N, equipped with such a structure, is called *submodule of M*. More specifically, a submodule of M is called *subgroup of M* if $R = \mathbb{Z}$ and *vector subspace of M* if $R = \mathbb{Q}$ or $R = \mathbb{R}$. The submodule N of M defines an equivalence relation $\mathcal{R}_N$ on M as follows: $x\mathcal{R}_N y$ in M if $x - y \in N$. Denote the quotient set $M/\mathcal{R}_N$ by M/N (or sometimes by $\frac{M}{N}$) and, for each $x \in M$, indicate the equivalence class corresponding to x in M/N by $x + N$. The set M/N inherits a natural structure of R–module from M. Given $x, y \in M$ and $a \in R$, it suffices to define $(x+N)+(y+N) := (x+y)+N$ and $a(x+N) := ax+N$. The set M/N, equipped with such addition and scalar multiplication, is called *quotient R–module of M modulo N.*

Let $\{M_i\}_{i=1}^k$ be a non–empty finite family of submodules of M. If each element x of M can be written uniquely as a sum $x = x_1 + \ldots + x_k$, where x_i belongs M_i for each $i \in \{1, \ldots, k\}$, then M is said to be the *direct sum of the M_i's*. In this case, we write $M = M_1 \oplus \ldots \oplus M_h$.

Let P be another R–module. A map $\varphi : M \longrightarrow P$ is a *module homorphism* if, for each $x, y \in M$ and for each $a \in R$, $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(ax) = a\varphi(x)$. As is usual, the kernel $\ker(\varphi)$ of φ is defined as $\varphi^{-1}(0_P)$, where 0_P is the null element of P. The reader observes that $\ker(\varphi)$ is a submodule of M and the image $\mathrm{Image}(\varphi)$ of φ is a submodule of P. If φ is a bijective module homomorphism, and hence φ^{-1} is a module homomorphism as well, then φ is called *module isomorphism.* If there exists an isomorphism between M and P, then M and P are said to be *module isomorphic.* More specifically, in place of the terms "module homomorphism", "module isomorphism" and "module isomorphic", one uses, respectively, the terms "homomorphism", "isomorphism" and "isomorphic" if $R = \mathbb{Z}$ and the terms "linear map", "linear isomorphism" and "linearly isomorphic" if $R = \mathbb{Q}$ or $R = \mathbb{R}$.

Given a submodule N of M, the map $\pi : M \longrightarrow M/N$, which sends $x \in M$ into $x+N \in M/N$, is an example of module homomorphism, which we call *natural projection of M onto M/N.*

Singular homology modules. Let $q \in \mathbb{N}$. Indicate by $e_0^{(q)}$ the origin of $\mathbb{R}^q$ and by $e_1^{(q)}, \ldots, e_q^{(q)}$ the vectors of the canonical base of $\mathbb{R}^q$. We have:

$$\begin{array}{rcl} e_0^{(q)} & = & (0, 0, \ldots, 0, 0), \\ e_1^{(q)} & = & (1, 0, \ldots, 0, 0), \\ e_2^{(q)} & = & (0, 1, \ldots, 0, 0), \\ \vdots & \vdots & \vdots \\ e_q^{(q)} & = & (0, 0, \ldots, 0, 1). \end{array}$$

The *standard geometric q–simplex* Δ_q is defined as the smallest convex subset of $\mathbb{R}^q$ containing $e_0^{(q)}, e_1^{(q)}, \ldots, e_q^{(q)}$. It is immediate to verify that

$$\Delta_q = \big\{(x_1, \ldots, x_q) \in \mathbb{R}^q \,\big|\, x_i \geq 0 \text{ for each } i = 1, \ldots, q \text{ and } \textstyle\sum_{i=0}^q x_i \leq 1\big\}.$$

For $q \leq 3$, Δ_q is very easy to visualize. In fact, Δ_0 consists of a single point $\{0\}$, Δ_1 coincides with the closed interval $I = [0, 1]$ of $\mathbb{R}$, Δ_2 is the union of the triangle of $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and of its interior, and Δ_3 is the tetrahedron of $\mathbb{R}^3$ with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

Fix a non–empty topological space X. The reader reminds that R denotes either $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$. A *singular q–simplex of X* is a continuous map from Δ_q to X; that is an element of the set $\mathscr{C}^0(\Delta_q, X)$. In this way, a singular 0–simplex $\sigma : \Delta_0 = \{0\} \longrightarrow X$ can be identified with the point $\sigma(0)$ of X and a singular 1–simplex of X is simply a path of X. Denote by $S_q(X;R)$ the free R–module generated by the set $\mathscr{C}^0(\Delta_q, X)$. The elements of $S_q(X;R)$ are called *singular q–chain of X over R*. Thus we can say that a singular q–chain of X over R is a finite formal linear combination $\sum_{\sigma\in\mathscr{C}^0(\Delta_q,X)} a_\sigma\sigma$ of singular q–simplexes σ of X with coefficients a_σ in R. For convention, we define $S_{-1}(X;R)$ as the null R–module.

Definition 2.19 *Let $c = \sum_{\sigma\in\mathscr{C}^0(\Delta_q,X)} a_\sigma\sigma$ be a singular q–chain of X over R and let $J_c := \{\sigma \in \mathscr{C}^0(\Delta_q, X)\,|\,a_\sigma \neq 0\}$. We define the* support $|c|$ of c *as follows: $|c| = \emptyset$ if c is null (or, equivalently, if $J_c = \emptyset$) and $|c| := \bigcup_{\sigma\in J_c}$ Image(σ) otherwise. If $c \in S_{-1}(X;R)$, then we consider the support $|c|$ of c equal to the empty set.*

In the case in which $c \in S_q(X;R)\setminus\{0\}$ and $c = \sum_{i=1}^h a_i\sigma_i$ is the finite representation of c in base, we have that $|c| = \bigcup_{i=1}^h |\sigma_i|$, where $|\sigma_i| = \mathrm{Image}(\sigma_i)$ for each $i \in \{1,\ldots,h\}$.

Let us introduce the boundary operators. We recall that a map from $\mathbb{R}^q$ to $\mathbb{R}^n$ is said to be *affine* if it is the composition of a linear map from $\mathbb{R}^q$ to $\mathbb{R}^n$ with a translation of $\mathbb{R}^n$. Evidently, an affine map is continuous. Given points $y_0, y_1, \ldots, y_q$ of $\mathbb{R}^n$, there exists a unique affine map $\varphi : \mathbb{R}^q \longrightarrow \mathbb{R}^n$, which sends $e_i^{(q)}$ into y_i for each $i \in \{0,1,\ldots,q\}$. It is given explicitly by the following equation

$$\varphi(x_1,\ldots,x_q) = y_0 + \textstyle\sum_{i=1}^q x_i(y_i - y_0).$$

Suppose $q \geq 1$. Let $i \in \{0,1,\ldots,q\}$. We indicate by $F_q^i : \Delta_{q-1} \longrightarrow \Delta_q$ the restriction from Δ_{q-1} to Δ_q of the unique affine map from $\mathbb{R}^{q-1}$ to $\mathbb{R}^q$, which sends $e_j^{(q-1)}$ into $e_j^{(q)}$ for each $j \in \{0,1,\ldots,q-1\}$ with $j < i$ and $e_j^{(q-1)}$ into $e_{j+1}^{(q)}$ for each $j \in \{0,1,\ldots,q-1\}$ with $j \geq i$. It is easy to verify that each map F_q^i induces a homeomorphism from Δ_{q-1} onto its image. Let σ be a singular q–chain of X. The singular $(q-1)$–simplex $\sigma^{(i)} : \Delta_{q-1} \longrightarrow X$, defined as the composition $\sigma \circ F_q^i$, is called i^{th}*–face of σ*, while the singular $(q-1)$–chain $\partial_q(\sigma)$ in $S_{q-1}(X;R)$, defined by

$$\partial_q(c) := \textstyle\sum_{i=0}^q (-1)^i\sigma^{(i)},$$

is called *boundary of σ*. The reader observes that, if $q = 1$, then $\partial_1(\sigma) = \sigma(1) - \sigma(0)$. The latter definition gives a map from $\mathscr{C}^0(\Delta_q, X)$ to $S_{q-1}(X;R)$. Such a map extends uniquely to a module homomorphism from $S_q(X;R)$ to $S_{q-1}(X;R)$, denoted by ∂_q again, by setting

$$\partial_q\big(\textstyle\sum_{\sigma\in\mathscr{C}^0(\Delta_q,X)} a_\sigma\sigma\big) := \textstyle\sum_{\sigma\in\mathscr{C}^0(\Delta_q,X)} a_\sigma\partial_q(\sigma).$$

If $q = 0$, then $\partial_0 : S_0(X;R) \longrightarrow \{0\} = S_{-1}(X;R)$ denotes the null module homomorphism. A simple computation ensures that, for each $q \in \mathbb{N}$, it holds:

$$\mathrm{Image}(\partial_{q+1}) \subset \ker(\partial_q) \quad \text{or, equivalently,} \quad \partial_q \circ \partial_{q+1} \equiv 0.$$

The submodule $\ker(\partial_q)$ of $S_q(X;R)$ is denoted by $Z_q(X;R)$ and its elements are called singular q–cycles of X over R or, simply, *q–cycles of X over R*. The submodule $\mathrm{Image}(\partial_{q+1})$ of $Z_q(X;R)$ is denoted by $B_q(X;R)$ and its elements are called singular q–boundaries of X over R or, simply, *q–boundaries of X over R*.

The quotient R–module $Z_q(X;R)/B_q(X;R)$ is denoted by $H_q(X;R)$ and it is called *singular q–homology module of X over R*. If c is a q–cycle of X over R, then we denote by $[c]_{(X;R)}$ the corresponding element $c + B_q(X;R)$ of $H_q(X;R)$. Given two q–cycles c and c' of X over R, c is said to be *homologous to c' in X over R* if $[c]_{(X;R)} = [c']_{(X;R)}$ or, equivalently, if there exists a singular $(q+1)$–chain d of X over R such that $c - c' = \partial_{q+1}(d)$.

If $R = \mathbb{R}$, then the dimension of $H_q(X;R)$ as a real vector space, denoted by $\beta_q(X)$, is called q^{th}*–Betti number of X*. It is well–known that, if X is a Lipschitz domain of $\mathbb{R}^3$

with bounded boundary, then all its Betti numbers are finite. The same is true if X is equal to the union $\Omega \cup A$, where (Ω, A) is a Lipschitz device of $\mathbb{R}^3$.

Let us define the useful notion of augmented 0–homology module. We begin by defining the module homomorphism $\partial_0^\sharp : S_0(X;R) \longrightarrow R$ as follows:

$$\partial_0^\sharp \big(\textstyle\sum_{x\in X} a_x x \big) := \textstyle\sum_{x\in X} a_x.$$

It is immediate to see that $B_0(X;R) = \mathrm{Image}(\partial_1)$ is contained in $\ker(\partial_0^\sharp)$. In this way, we can define the *augmented* 0*–homology module* $H_0^\sharp(X;R)$ *of* X *over* R as the quotient R–module $\ker(\partial_0^\sharp)/B_0(X;R)$. If $c \in \ker(\partial_0^\sharp)$, then we denote by $[c]^\sharp_{(X;R)}$ the corresponding element $c + B_0(X;R)$ of $H_0^\sharp(X;R)$.

The geometric nature of $H_0^\sharp(X;R)$ is described in the following elementary result (see (10.7) in [12]).

Theorem 2.20 *The following statements hold.*

(1) *If X is path–connected, then $H_0^\sharp(X;R)$ is null.*

(2) *Suppose that X has at least two path–connected components. Let $\{X_j\}_{j\in J}$ be the family of all path–connected components of X and, for each $j \in J$, choose a point x_j of X_j and view such a point as a singular 0–simplex of X. Fix $j_0 \in J$. Then $H_0^\sharp(X;R)$ is a free R–module, having $\{[x_j - x_{j_0}]^\sharp_{(X;R)}\}_{j\in J\setminus\{j_0\}}$ as a base. In particular, if X has a finite number r of path–connected components, then the rank of $H_0^\sharp(X;R)$ is equal to $r-1$.*

Let now Y be another topological space and let $f : X \longrightarrow Y$ be a continuous map. Fix $q \in \mathbb{N}$. The map f induces a well–defined module homomorphism $H_q(f) : H_q(X;R) \longrightarrow H_q(Y;R)$ as follows: given any q–cycle $c = \sum_{\sigma\in\mathscr{C}^0(\Delta_q,X)} a_\sigma \sigma$ of X over R, we define

$$H_q(f)([c]_{(X;R)}) := \big[\textstyle\sum_{\sigma\in\mathscr{C}^0(\Delta_q,X)} a_\sigma (f\circ\sigma) \big]_{(Y;R)}.$$

The same map f defines also a module homomorphism $H_0^\sharp(f) : H_0^\sharp(X;R) \longrightarrow H_0^\sharp(Y;R)$ by setting

$$H_0^\sharp(f)\big([\textstyle\sum_{x\in X} a_x x]^\sharp_{(X;R)}\big) := \big[\textstyle\sum_{x\in X} a_x f(x) \big]^\sharp_{(Y;R)}.$$

for each element $\sum_{x\in X} a_x x$ of $\ker(\partial_0^\sharp)$. It is an elementary fact that, if f is a homeomorphism, then the maps $H_q(f)$ and the map $H_0^\sharp(f)$ are module isomorphism (see Section 11 of [12] for a more general result).

Relative singular homology modules. *Fix a subset A of the topological space X.* A *relative* (*singular*) *q–cycle of X over R modulo A* is a singular q–chain c of X over R such that the support of $\partial_q(c)$ is contained in A. The submodule of $S_q(X;R)$ consisting of all relative q–cycles of X over R modulo A is denoted by $Z_q(X, A;R)$.

An important case for us is the one of paths. Let $\gamma : I \longrightarrow X$ be a path of X, viewed as a singular 1–simplex of X. Then γ is a relative 1–cycle of X over R modulo A if and only if the points $\gamma(0)$ and $\gamma(1)$ belong to A.

A *relative* (*singular*) *q–boundary of X over R modulo A* is a singular q–chain c of X over R, which is homologous to a singular q–chain of X over R having support contained in A. More explicitly, c is such a relative q–boundary if there exist $c' \in S_q(X;R)$ and $d \in S_{q+1}(X;R)$ such that $|c'| \subset A$ and $c - c' = \partial_{q+1}(d)$. Applying ∂_q to both members of the latter equation, we infer that $\partial_q(c) = \partial_q(c')$ and hence $|\partial_q(c)| \subset A$. It follows that every relative q–boundary of X over R modulo A is also a relative q–cycle of X over R modulo A. We denote by $B_q(X, A;R)$ the submodule of $Z_q(X, A;R)$ consisting of all relative q–boundaries of X over R module A.

The quotient R–module $Z_q(X,A;R)/B_q(X,A;R)$ is denoted by $H_q(X,A;R)$ and is called *relative singular q–homology module of X over R modulo A*. If $c \in Z_q(X,A;R)$, then we indicate by $[c]_{(X,A;R)}$ the corresponding element $c+B_q(X,A;R)$ of $H_q(X,A;R)$. Given $c,c' \in Z_q(X,A;R)$, c is said to be *homologous to c' in X over R modulo A* if $[c]_{(X,A;R)} = [c']_{(X,A;R)}$ or, equivalently, if there exist $c'' \in S_q(X;R)$ and $d \in S_{q+1}(X;R)$ such that $|c''| \subset A$ and $c - c' = c'' + \partial_{q+1}(d)$.

The reader observes that, if $A = \emptyset$, then $Z_q(X,A;R) = Z_q(X;R)$, $B_q(X,A;R) = B_q(X;R)$ and $H_q(X,A;R) = H_q(X;R)$.

Let $g : X \longrightarrow Y$ be a continuous map between the topological spaces X and Y, and let B be a subset of Y. If $g(A)$ is contained in B, then g defines a module homomorphism $H_q(g) : H_q(X,A;R) \longrightarrow H_q(Y,B;R)$ by setting

$$H_q(g)\big([\textstyle\sum_{\sigma \in \mathscr{C}^0(\Delta_q,X)} a_\sigma \sigma]_{(X,A;R)}\big) := \big[\textstyle\sum_{\sigma \in \mathscr{C}^0(\Delta_q,X)} a_\sigma (g \circ \sigma)\big]_{(Y,B;R)}$$

for each $\sum_{\sigma \in \mathscr{C}^0(\Delta_q,X)} a_\sigma \sigma \in Z_q(X,A;R)$.

We conclude this section with a construction of crucial importance for us.

Consider the following finite sequence of module homomorphisms

$$H_1(A;R) \xrightarrow{i_*} H_1(X;R) \xrightarrow{\pi_*} H_1(X,A;R) \xrightarrow{\partial_*} H_0^\sharp(A;R) \xrightarrow{i_\sharp} H_0^\sharp(X;R), \tag{6}$$

where i_* and $i_\sharp$ are induced by the inclusion map $A \hookrightarrow X$, π_* is induced by the identity map of X (that is, $\pi_*([c]_{(X;R)}) = [c]_{(X,A;R)}$ for each $c \in Z_1(X;R)$) and ∂_* is defined as follows:

$$\partial_*\big([c]_{(X,A;R)}\big) := [\partial_1(c)]^\sharp_{(A;R)}$$

for each $c \in Z_1(X,A;R)$. By a direct and elementary verification, it follows that the sequence (6) is exact; that is, it hold:

$$\mathrm{Image}(i_*) = \ker(\pi_*),\ \mathrm{Image}(\pi_*) = \ker(\partial_*) \text{ and } \mathrm{Image}(\partial_*) = \ker(i_\sharp). \tag{7}$$

The sequence (6) is the final part of the so–called long exact homology sequence of the topological pair (X,A) over R (see [12, Section 14]).

Regularization of 1-cycles. Let us state two regularization results for 1–cycles.

Lemma 2.21 *Let O be a non–empty open subset of $\mathbb{R}^3$ and let $z \in Z_1(O;R) \setminus \{0\}$. Then there exist a positive integer h, embedded smooth loops $\gamma_1,\ldots,\gamma_h$ of O and elements $\alpha_1,\ldots,\alpha_h$ of R such that z is homologous to $\sum_{i=1}^h \alpha_i\gamma_i$ in O over R and the supports of the γ_i's are pairwise disjoint.*

Let X be a non–empty subset of $\mathbb{R}^3$. Let z be a 1–cycle of X over $\mathbb{Z}$. We say that z is a *Lipschitz 1–cycle of X over $\mathbb{Z}$* if either z is null or z is not null and, denoted by $z = \sum_{i=1}^h a_i\gamma_i$ its finite representation in base, each singular 1–simplex γ_i of X is a Lipschitz path of X. We denote by $Z_1^{(lip)}(X;\mathbb{Z})$ the abelian subgroup of $Z_1(X;\mathbb{Z})$ consisting of all Lipschitz 1–cycles of X over $\mathbb{Z}$. The reader observes that the Lipschitz loops of X and the finite formal sum of it are elements of $Z_1^{(lip)}(X;\mathbb{Z})$.

Lemma 2.22 *Let O be a non–empty open subset of $\mathbb{R}^3$, let $z \in Z_1^{(lip)}(O;\mathbb{Z}) \setminus \{0\}$ and let r be the number of path–connected components of $|z|$. Then there exist Lipschitz loops $\ell_1,\ldots,\ell_r$ of O and, for each $i \in \{1,\ldots,r\}$, a sequence $\{\gamma_{i,n}\}_{n\in\mathbb{N}}$ of embedded smooth loops of O such that*

1. *For each $W \in \mathscr{C}^0(O)^3$, $\int_z W$ is equal to $\int_{\sum_{i=1}^r \ell_i} W$.*
2. *For each $i \in \{1,\ldots,r\}$, the sequence $\{\gamma_{i,n}\}_{n\in\mathbb{N}}$ converges uniformly to ℓ_i on I and the sequence $\{d\gamma_{i,n}/dt\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(I)^3$ and it converges to $d\ell_i/dt$ almost everywhere on I.*

(3) *For each* $n \in \mathbb{N}$, $\sum_{i=1}^{r} \gamma_{i,n}$ *is homologous to* z *in* O *over* $\mathbb{Z}$.

(4) *For each* $n \in \mathbb{N}$ *and for each* $i, j \in \{1, \ldots, r\}$ *with* $i \neq j$, $|\gamma_{i,n}|$ *and* $|\gamma_{j,n}|$ *are disjoint.*

Let us give the proof of the preceding two lemmas.

Proof of Lemmas 2.21 and 2.22. Step I. Let O be a non–empty open subset of $\mathbb{R}^3$ and let $z \in Z_1(O; R) \setminus \{0\}$, let r be the number of path–connected components of $|z|$ and let $C_1, \ldots, C_r$ be the path–connected components of $|z|$. Since the C_i's are compact and pairwise disjoint, there exists a positive real number ε with the following property: for each $i, j \in \{1, \ldots, r\}$ with $i \neq j$ and for each $(x, y) \in C_i \times C_j$, the open balls $B(x, \varepsilon)$ and $B(y, \varepsilon)$ of $\mathbb{R}^3$ are disjoint and contained in O. For each $i \in \{1, \ldots, r\}$, define the path–connected open neighborhood O_i of C_i in O by setting $O_i := \bigcup_{x \in C_i} B(x, \varepsilon)$. The O_i's are pairwise disjoint so z is equal to the sum $z_1 + \ldots + z_r$, where each z_i is a 1–cycle of O whose support is contained in O_i. It is now suffices to show that, for each $i \in \{1, \ldots, r\}$, Lemma 2.21 and Lemma 2.22 (when $R = \mathbb{Z}$ and $z \in Z_1^{(lip)}(O; \mathbb{Z}) \setminus \{0\}$) are true with $z = z_i$ and $O = O_i$. This is equivalent to say that it suffices to prove such lemmas in the special case in which O is path–connected.

Step II. For this reason, in the remainder of this proof, we assume that O is path–connected. Let $z = \sum_{j=1}^{k} a_j \sigma_j$ be the finite representation of $z \in Z_1(O; R) \setminus \{0\}$ in base. We have:

$$0 = \partial_1(z) = \textstyle\sum_{j=1}^{k} a_j(\sigma_j(1) - \sigma_j(0)) \text{ in } S_1(O; R). \tag{8}$$

Define $Z := \bigcup_{j=1}^{k} \{\sigma_j(0), \sigma_j(1)\}$ and, for each $x \in Z$, $I_{0,x} := \{j \in \{1, \ldots, k\} \,|\, \sigma_j(0) = x\}$ and $I_{1,x} := \{j \in \{1, \ldots, k\} \,|\, \sigma_j(1) = x\}$. After collecting terms in the right member of (8), we obtain

$$0 = \textstyle\sum_{x \in Z} \big(\sum_{j \in I_{1,x}} a_j - \sum_{j \in I_{0,x}} a_j \big) x \text{ in } S_1(O; R). \tag{9}$$

For each $x \in Z$, define the element a_x of R by setting

$$a_x := \textstyle\sum_{j \in I_{1,x}} a_j - \sum_{j \in I_{0,x}} a_j.$$

Equation (9) is equivalent to say that $a_x = 0$ for each $x \in Z$. Fix a point x_0 of O. For each $x \in Z$, choose a *Lipschitz* path η_x of O from x_0 to x. For each $j \in \{1, \ldots, k\}$, define the *Lipschitz* path $\eta_{j0} : I \longrightarrow O$ and $\eta_{j1} : I \longrightarrow O$ by setting $\eta_{j0} := \eta_{\sigma_j(0)}$ and $\eta_{j1} := \eta_{\sigma_j(1)}$. We obtain:

$$\textstyle\sum_{j=1}^{k} a_j(\eta_{j1} - \eta_{j0}) = \sum_{x \in Z} a_x \eta_x = 0 \text{ in } S_1(O; R).$$

It follows that z can be written as follows:

$$z = z - \textstyle\sum_{j=1}^{k} a_j(\eta_{j1} - \eta_{j0}) = \sum_{j=1}^{k} a_j(\eta_{j0} + \sigma_j - \eta_{j1}). \tag{10}$$

The reader observes that each sum $\eta_{j0} + \sigma_j - \eta_{j1}$ is a 1–cycle of O over R. For each $j \in \{1, \ldots, k\}$, define the loop ξ_j of O at x_0 by setting

$$\xi_j := (\eta_{j0} * \sigma_j) * \eta_{j1}^{-1}, \tag{11}$$

where "$*$" denotes the usual product between paths. By an elementary construction (see the first part of the proof of Theorem 12.1 of [12, p. 48]), it follows that each ξ_j is homologous to $\eta_{j0} + \sigma_j - \eta_{j1}$ in O over R and hence

$$z \text{ is homologous to } \textstyle\sum_{j=1}^{k} a_j \xi_j \text{ in } O \text{ over } R. \tag{12}$$

Step III. Let us complete the proof of Lemma 2.21. Fix $j \in \{1,\ldots,k\}$. Let $\widetilde{\xi}_j : S^1 \longrightarrow \mathbb{R}^3$ be the unique continuous map from S^1 to $\mathbb{R}^3$ such that $\xi_j(t) = \widetilde{\xi}_j(\cos(2\pi t),\sin(2\pi t))$ for each $t \in I$. Equip the set $\mathscr{C}^0(S^1,\mathbb{R}^3)$ of all continuous maps from S^1 to $\mathbb{R}^3$ with the topology of uniform convergence and S^1 and $\mathbb{R}^3$ with their natural structure of smooth manifold. It is well–known that the subset of $\mathscr{C}^0(S^1,\mathbb{R}^3)$ of all smooth embeddings of S^1 into $\mathbb{R}^3$ is dense in $\mathscr{C}^0(S^1,\mathbb{R}^3)$. In this way, there exists a smooth embedding $\widetilde{\gamma}_j : S^1 \longrightarrow \mathbb{R}^3$ arbitrarily close to $\widetilde{\xi}_j$ in $\mathscr{C}^0(S^1,\mathbb{R}^3)$. Composing the $\widetilde{\gamma}_j$'s with translations of $\mathbb{R}^3$ along suitable small vectors, we may also suppose that the $|\widetilde{\gamma}_j|$'s are pairwise disjoint. Define $\widetilde{F}_j : S^1 \times I \longrightarrow \mathbb{R}^3$ by setting $\widetilde{F}_j(y,t) := \widetilde{\xi}_j(y) + t(\widetilde{\gamma}_j(y) - \widetilde{\xi}_j(y))$. Choosing $\widetilde{\gamma}_j$ sufficiently close to $\widetilde{\xi}_j$, we may suppose that the image of $\widetilde{F}_j$ is contained in O. Define the embedded smooth loop $\gamma_j : I \longrightarrow O$ and the continuous map $F_j : I \times I \longrightarrow O$ by setting

$$\gamma_j(s) := \widetilde{\gamma}_j(\cos(2\pi s),\sin(2\pi s))$$

and

$$F_j(s,t) := \widetilde{F}_j\big((\cos(2\pi s),\sin(2\pi s)),t\big)$$

The reader observes that F_j is a homotopy from ξ_j and γ_j and $F_j(0,t) = F_j(1,t)$ for each $t \in I$. Let $\tau_j : I \longrightarrow O$ and $\kappa_j : I \longrightarrow O$ be the paths and let $d_1^{(j)} : \Delta_2 \longrightarrow O$ and let $d_2^{(j)} : \Delta_2 \longrightarrow O$ be the singular 2–simplexes defined as follows:

$$\tau_j(s) := F_j(0,s) = F_j(1,s),\ \kappa_j(s) := F_j(1-s,s),\ d_1^{(j)}(s,t) := F_j(s,t)$$

and

$$d_2^{(j)}(s,t) := F_j(1-t,s+t).$$

We have that

$$\partial_2(d_1^{(j)} + d_2^{(j)}) = (\xi_j + \kappa_j - \tau_j) + (\tau_j - \gamma_j - \kappa_j) = \xi_j - \gamma_j.$$

It follows that each γ_j is homologous to ξ_j in O over R and hence, by (12), z is homologous to $\sum_{j=1}^k a_j\gamma_j$ on O over R, as desired.

Step IV. Let us complete the proof of Lemma 2.22. Suppose $R = \mathbb{Z}$ and $z \in Z_1^{(lip)}(O;\mathbb{Z}) \setminus \{0\}$. The reader reminds that O is assumed to be path–connected and the paths η_{j0} and η_{j1} have been chosen to be Lipschitz. By hypothesis, z is Lipschitz; that is, each path σ_j is Lipschitz. By (11), it follows that each ξ_j is Lipschitz as well. Define the Lipschitz loop ℓ of O by setting $\ell := \xi_1^{a_1} * \cdots * \xi_k^{a_k}$. Given any $W \in \mathscr{C}^0(O)^3$, it holds:

$$\int_z W = \sum\nolimits_{j=1}^k a_j \int_{\eta_{j0}+\sigma_j-\eta_{j1}} W = \sum\nolimits_{j=1}^k a_j \int_{\xi_j} W = \int_\ell W.$$

For each $t \in \mathbb{R}$, indicate by $\lfloor t \rfloor$ the real number $t - \max\{s \in \mathbb{Z} \,|\, s \leq t\}$ in $[0,1)$. Define the map $\overline{\ell} : \mathbb{R} \longrightarrow \mathbb{R}^3$ by setting $\overline{\ell}(t) := \ell(\lfloor t \rfloor)$. It is evident that $\overline{\ell}$ is Lipschitz, extends ℓ and is 1–periodic. Let $\{\rho_n : \mathbb{R} \longrightarrow \mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of mollifiers in $\mathbb{R}$ and, for each $n \in \mathbb{N}$, define $\overline{\ell}_n : \mathbb{R} \longrightarrow \mathbb{R}^3$ and $\ell_n : I \longrightarrow \mathbb{R}^3$ by $\overline{\ell}_n := \rho_n * \overline{\ell}$ and $\ell_n := \overline{\ell}_n|_I$, respectively. By elementary properties of convolution, we know that each $\overline{\ell}_n$ is a 1–periodic smooth map, the sequence $\{\ell_n\}_{n\in\mathbb{N}}$ converges uniformly to ℓ in I and the sequence $\{d\ell_n/dt\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(I)^3$ and converges to $d\ell/dt$ in $L^1(I)^3$. Extracting a subsequence if needed, we may also suppose that $\{d\ell_n/dt\}_{n\in\mathbb{N}}$ converges to $d\ell/dt$ almost everywhere on I. Fix $n \in \mathbb{N}$. Let $\widetilde{\ell}_n : S^1 \longrightarrow \mathbb{R}^3$ be the unique smooth map such that $\overline{\ell}_n(s) = \widetilde{\ell}_n(\cos(2\pi s),\sin(2\pi s))$ for each $s \in \mathbb{R}$. Equip the set $\mathscr{C}^1(S^1,\mathbb{R}^3)$ of all maps from S^1 to $\mathbb{R}^3$ of class $\mathscr{C}^1$ with the $\mathscr{C}^1$–topology. Since the subset of $\mathscr{C}^1(S^1,\mathbb{R}^3)$ of all smooth embeddings of S^1 to $\mathbb{R}^3$ is dense in $\mathscr{C}^1(S^1,\mathbb{R}^3)$, for each $n \in \mathbb{N}$, there exists a smooth embedding $\widetilde{\gamma}_n : S^1 \longrightarrow \mathbb{R}^3$ arbitrarily close to $\widetilde{\ell}_n$ in $\mathscr{C}^1(S^1,\mathbb{R}^3)$. In this way, if

$\gamma_n : I \longrightarrow \mathbb{R}^3$ is the embedding smooth loop defined by $\gamma_n(s) := \widetilde{\gamma}_n(\cos(2\pi s), \sin(2\pi s))$, then we may suppose that

$$\max\left\{\|\gamma_n - \ell_n\|_{L^\infty(I)^3}\,, \left\|\frac{d\gamma_n}{dt} - \frac{d\ell_n}{dt}\right\|_{L^\infty(I)^3}\right\} < 2^{-n}.$$

It follows that $\{\gamma_n\}_{n\in\mathbb{N}}$ converges uniformly to ℓ on I and the sequence $\{d\gamma_n/dt\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(I)^3$ and it converges to $d\ell/dt$ almost everywhere on I. Finally, using the argument of *Step III*, we obtain that, up to consider a subsequence of $\{\gamma_n\}_{n\in\mathbb{N}}$ if needed, each γ_n is homologous to z in O over $\mathbb{Z}$. $\square$

An immediate by–product of the preceding proof is as follows:

Corollary 2.23 *Let $z \in Z_1(\mathbb{R}^3;\mathbb{Z})$ and let U be an open neighborhood of $|z|$ in $\mathbb{R}^3$. Then there exists $z' \in Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z})$ and $d \in S_2(\mathbb{R}^3;\mathbb{Z})$ such that $|z'| \cup |d| \subset U$ and $z - z' = \partial_2(d)$.*

2.4 Line integrals of curl-free vector fields along relative 1-cycles and periods of first de Rham cohomology classes

Let us introduce the notion of integral of a vector field in $H_{0,A}(\mathrm{curl}\,0,\Omega)$ along an extendable relative 1–cycle in $Z_1(\Omega\cup A, A;\mathbb{R})$. First, we need some preparations.

Definition 2.24 *Let z be a relative 1–cycle of $\Omega \cup A$ over $\mathbb{R}$ modulo A; that is, an element of $Z_1(\Omega\cup A, A;\mathbb{R})$. We say that z is* extendable *if there exist extendable paths $\lambda_1,\ldots,\lambda_r$ of $\Omega\cup A$ modulo A, extendable loops $\ell_1,\ldots,\ell_s$ of Ω and real numbers $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s$ such that*

$$z = \sum\nolimits_{i=1}^{r}\alpha_i\lambda_i + \sum\nolimits_{j=1}^{s}\beta_j\ell_j. \tag{13}$$

in $Z_1(\Omega\cup A, A;\mathbb{R})$. We denote by $Z_1^{(ext)}(\Omega\cup A, A;\mathbb{R})$ the vector subspace of $Z_1(\Omega \cup A, A;\mathbb{R})$ consisting of all extendable relative 1–cycles of $\Omega\cup A$ over $\mathbb{R}$ modulo A. If $z \in Z_1^{(ext)}(\Omega\cup A, A;\mathbb{R})$ and has the form (13), then, for each $V \in H_{0,A}(\mathrm{curl}\,0,\Omega)$, we define the line integral $\int_z V$ of V along z *as follows*

$$\int_z V := \sum\nolimits_{i=1}^{r}\alpha_i\int_{\lambda_i} V + \sum\nolimits_{j=1}^{s}\beta_j\int_{\ell_j} V.$$

Lemma 2.25 *The following statements hold.*

1. *For each $z \in Z_1(\Omega\cup A, A;\mathbb{R})$, there exists $z' \in Z_1^{(ext)}(\Omega\cup A, A;\mathbb{R})$, which is homologous to z in $\Omega\cup A$ over $\mathbb{R}$ modulo A; that is, $[z']_{(\Omega\cup A,A;\mathbb{R})} = [z]_{(\Omega\cup A,A;\mathbb{R})}$.*
2. *Let $V \in H_{0,A}(\mathrm{curl}\,0,\Omega)$ and let $z', z'' \in Z_1^{(ext)}(\Omega\cup A, A;\mathbb{R})$ such that z' is homologous to z'' in $\Omega\cup A$ over $\mathbb{R}$ modulo A; that is, $[z']_{(\Omega\cup A,A;\mathbb{R})} = [z'']_{(\Omega\cup A,A;\mathbb{R})}$. Then it holds:*
$$\int_{z'} V = \int_{z''} V.$$

Thanks to the latter result, we can give the following definition.

Definition 2.26 *Given $V \in H_{0,A}(\mathrm{curl}\,0,\Omega)$ and $z \in Z_1(\Omega\cup A, A;\mathbb{R})$, we define the* line integral $\int_z V$ of V along z *by setting*

$$\int_z V := \int_{z'} V,$$

where z' is any element of $Z_1^{(ext)}(\Omega\cup A, A;\mathbb{R})$ such that $[z']_{(\Omega\cup A,A;\mathbb{R})} = [z]_{(\Omega\cup A,A;\mathbb{R})}$.

In the next result, we present an important situation in which the line integral of a vector field in $H_{0,A}(\mathrm{curl}\,0,\Omega)$ along a relative 1–cycle in $Z_1(\Omega\cup A, A;\mathbb{R})$ can be computed directly.

Lemma 2.27 *Let $W:\Omega\cup A\longrightarrow\mathbb{R}^3$ be a continuous vector field whose restriction to Ω belongs to $H_{0,A}(\mathrm{curl}\,0,\Omega)$ and let $z\in Z_1(\Omega\cup A, A;\mathbb{R})$. Suppose that there exist a positive integer k and, for each $i\in\{1,\dots,k\}$, $a_i\in\mathbb{R}$ and a Lipschitz path $\gamma_i:I\longrightarrow\Omega\cup A$ such that $z=\sum_{i=1}^k a_i\gamma_i$. Then it holds*

$$\int_z W=\sum\nolimits_{i=1}^k a_i\int_{\gamma_i}W.$$

Our next aim is to define the notion of period of first de Rham cohomology classes in $\mathbb{H}^1_{\mathrm{DR}}(\Omega,A)$ along relative homology classes in $H_1(\Omega\cup A,A;\mathbb{R})$.

Lemma 2.28 *Let $V\in H_{0,A}(\mathrm{curl}\,0,\Omega)$, let $c\in B_1(\Omega\cup A,A;\mathbb{R})$, let $h\in H^1_{0,A}(\Omega)$ and let $z\in Z_1(\Omega\cup A,A;\mathbb{R})$. It hold:*

$$\int_c V=0\quad\text{and}\quad\int_z\nabla h=0.\tag{14}$$

The latter lemma permits to give the following definition.

Definition 2.29 *Given $V\in H_{0,A}(\mathrm{curl}\,0,\Omega)$ and $\alpha\in H_1(\Omega\cup A,A;\mathbb{R})$, we denote by $\int_\alpha V$ the real number defined as follows*

$$\int_\alpha V:=\int_z V,$$

where z is an element of $Z_1(\Omega\cup A,A;\mathbb{R})$ with $[z]_{(\Omega\cup A,A;\mathbb{R})}=\alpha$.

We denote by $\boldsymbol{\int}_{(\Omega,A;\omega)}:\mathcal{H}(\Omega,A;\omega)\times H_1(\Omega\cup A,A;\mathbb{R})\longrightarrow\mathbb{R}$ the bilinear pairing defined as follows

$$\boldsymbol{\int}_{(\Omega,A;\omega)}(V,\alpha):=\int_\alpha V$$

for each $(V,\alpha)\in\mathcal{H}(\Omega,A;\omega)\times H_1(\Omega\cup A,A;\mathbb{R})$.

Given $\upsilon\in\mathbb{H}^1_{\mathrm{DR}}(\Omega,A)$ and $\alpha\in H_1(\Omega\cup A,A;\mathbb{R})$, we define the period $\int_\alpha\upsilon$ of υ along *α by setting*

$$\int_\alpha\upsilon:=\int_\alpha V,$$

where V is an element of $H_{0,A}(\mathrm{curl}\,0,\Omega)$ whose de Rham cohomology class $V+\mathrm{grad}(H^1_{0,A}(\Omega))$ is equal to υ.

Finally, we denote by $\boldsymbol{\int}^{\mathrm{DR}}_{(\Omega,A)}:\mathbb{H}^1_{\mathrm{DR}}(\Omega,A)\times H_1(\Omega\cup A,A;\mathbb{R})\longrightarrow\mathbb{R}$ the bilinear pairing defined as follows

$$\boldsymbol{\int}^{\mathrm{DR}}_{(\Omega,A)}(\upsilon,\alpha):=\int_\alpha\upsilon$$

for each $(\upsilon,\alpha)\in\mathbb{H}^1_{\mathrm{DR}}(\Omega,A)\times H_1(\Omega\cup A,A;\mathbb{R})$.

3 The Hodge decomposition theorem

Throughout this section, we shall make the following assumptions: (Ω,A) *is a bounded Lipschitz device of* $\mathbb{R}^3$, Γ *is the boundary of* Ω, $\complement A$ *is the interior of* $\Gamma\setminus A$ *in* Γ *and* ω *is a material matrix of* Ω.

Main decomposition. Let us present the main Hodge–type decomposition theorem contained in this paper. Such a decomposition was proved by P. Fernandes and G.

Gilardi in the special setting of Helmholtz domains (see [10, p. 968]). However, as we see in the final version of the paper, their proof works with minor changes in the general case as well.

Theorem 3.1 *The Hilbert space $L^2(\Omega)^3$ can be expressed as the following direct sum of mutually ω–orthogonal closed vector subspaces:*

$$L^2(\Omega)^3 = \mathrm{grad}(H^1_{0,A}(\Omega)) \overset{\perp}{\oplus}_\omega \mathcal{H}(\Omega, A; \omega) \overset{\perp}{\oplus}_\omega \omega^{-1}\mathrm{curl}(H_{0,\complement A}(\mathrm{curl}, \Omega)).$$

Moreover, it hold:

$$H_{0,A}(\mathrm{curl}\, 0, \Omega) = \mathrm{grad}(H^1_{0,A}(\Omega)) \overset{\perp}{\oplus}_\omega \mathcal{H}(\Omega, A; \omega),$$
$$H_{0,\complement A}(\mathrm{div}\, 0, \Omega; \omega) = \mathcal{H}(\Omega, A; \omega) \overset{\perp}{\oplus}_\omega \omega^{-1}\mathrm{curl}(H_{0,\complement A}(\mathrm{curl}, \Omega)).$$

This statement is summarized in the following ω–decomposition diagram:

<table>
<tr><td colspan="3">$L^2(\Omega)^3$</td></tr>
<tr><td>$\mathrm{grad}(H^1_{0,A}(\Omega))$</td><td>$\mathcal{H}(\Omega, A; \omega)$</td><td>$\omega^{-1}\mathrm{curl}(H_{0,\complement A}(\mathrm{curl}, \Omega))$</td></tr>
<tr><td colspan="2">$H_{0,A}(\mathrm{curl}\, 0, \Omega)$</td><td>$\omega^{-1}\mathrm{curl}(H_{0,\complement A}(\mathrm{curl}, \Omega))$</td></tr>
<tr><td>$\mathrm{grad}(H^1_{0,A}(\Omega))$</td><td colspan="2">$H_{0,\complement A}(\mathrm{div}\, 0, \Omega; \omega)$</td></tr>
</table>

Five-subspaces decomposition. Let us deduce from Theorem 3.1 a more precise decomposition. First, we need some preparations.

Definition 3.2 *Let D and E be Lipschitz open subsets of Γ. We say that (D, E) is a* curl–gradient boundary pair of Ω *if one of the following two equivalent conditions is satisfied:*

$$H_{0,D}(\mathrm{curl}\, 0, \Omega) \subset \mathrm{grad}(H^1_{0,E}(\Omega)), \tag{15}$$
$$H_{0,\complement E}(\mathrm{div}\, 0, \Omega) \subset \mathrm{curl}(H_{0,\complement D}(\mathrm{curl}, \Omega)). \tag{16}$$

In the next result, we give a simple topological characterization of curl–gradient boundary pairs of Ω.

Lemma 3.3 *Let D and E be Lipschitz open subsets of Γ. The following assertions are equivalent:*

(1) *(D, E) is a curl–gradient boundary pair of Ω.*

(2) *E is contained in a connected component of D and the linear map from $H_1(D; \mathbb{R})$ to $H_1(\overline{\Omega}; \mathbb{R})$ induced by the inclusion $D \hookrightarrow \overline{\Omega}$ is surjective.*

In particular, if (1) holds, then E is contained in D.

Let us give some examples of curl–gradient boundary pairs.

Corollary 3.4 *Let D and E be Lipschitz open subsets of Γ. In each of the following cases, (D, E) is a curl–gradient boundary pair of Ω:*

(1) *E is contained in a connected component of D and the linear map from $H_1(D; \mathbb{R})$ to $H_1(\Gamma; \mathbb{R})$ induced by the inclusion $D \hookrightarrow \Gamma$ is surjective.*

(2) *E is contained in a connected component of D and $\Gamma \setminus D$ is the union of some connected components of Γ, each of which is homeomorphic to the standard 2–sphere.*

(3) *$D = \Gamma$ and $E = \emptyset$.*

Let us give another definition.

Definition 3.5 *Let (D, E) be a curl–gradient boundary pair of Ω. Given a vector field V in $L^2(\Omega)^3$, we say that V is a ω–*curly gradient vector field of Ω with respect to (D, E) *if there exist $h \in H^1_{0,E}(\Omega)$ and $Z \in H_{0,\complement D}(\mathrm{curl}, \Omega)$ such that $V = \nabla h$ and $\omega V = \mathrm{curl}(Z)$; that is, $V \in \mathrm{grad}(H^1_{0,E}(\Omega)) \cap \omega^{-1}\mathrm{curl}(H_{0,\complement D}(\mathrm{curl}, \Omega))$.*

A $\mathbf{1}_{3\times 3}$–curly gradient vector field of Ω with respect to (D, E) is simply called curly gradient vector field of Ω with respect to (D, E).

We denote by $H_{\mathrm{cg}}(\Omega, D, E; \omega)$ the vector subspace of $L^2(\Omega)^3$ consisting of all ω–curly gradient vector fields of Ω with respect to (D, E). In other words, we define:

$$H_{\mathrm{cg}}(\Omega, D, E; \omega) := \mathrm{grad}(H^1_{0,E}(\Omega)) \cap \omega^{-1}\mathrm{curl}(H_{0,\complement D}(\mathrm{curl}, \Omega)).$$

If ω is equal to $\mathbf{1}_{3\times 3}$ and/or $(D, E) = (\Gamma, \emptyset)$, then, for simplicity, we omit ω and/or the letters D and E from the symbol $H_{\mathrm{cg}}(\Omega, D, E; \omega)$. In fact, we define:

- ▷ $H_{\mathrm{cg}}(\Omega, D, E) := H_{\mathrm{cg}}(\Omega, D, E; \mathbf{1}_{3\times 3})$, $H_{\mathrm{cg}}(\Omega; \omega) := H_{\mathrm{cg}}(\Omega, \Gamma, \emptyset; \omega)$, $H_{\mathrm{cg}}(\Omega) := H_{\mathrm{cg}}(\Omega, \Gamma, \emptyset; \mathbf{1}_{3\times 3})$.

*Finally, if $V \in H_{\mathrm{cg}}(\Omega; \omega)$, then we call V also ω–*curly gradient vector field of Ω. *Similarly, if $V \in H_{\mathrm{cg}}(\Omega)$, then we call V* curly gradient vector field of Ω.

The reader observes that a curly gradient vector field of Ω is an element of $L^2(\Omega)^3$, which admit both scalar and vector potential. In fact, we have that

$$H_{\mathrm{cg}}(\Omega) = \mathrm{grad}(H^1(\Omega)) \cap \mathrm{curl}(H(\mathrm{curl}, \Omega)).$$

Theorem 3.6 *Let (D, E) be a curl–gradient boundary pair of Ω. Then the Hilbert space $L^2(\Omega)^3$ can be expressed as the following direct sum of mutually ω–orthogonal closed vector subspaces:*

$$L^2(\Omega)^3 = \mathbb{L}_1 \overset{\perp}{\oplus}_\omega \mathbb{L}_2 \overset{\perp}{\oplus}_\omega \mathbb{L}_3 \overset{\perp}{\oplus}_\omega \mathbb{L}_4 \overset{\perp}{\oplus}_\omega \mathbb{L}_5,$$

where $\mathbb{L}_1 := \mathrm{grad}(H^1_{0,D}(\Omega))$, $\mathbb{L}_2 := \mathcal{H}(\Omega, D; \omega)$, $\mathbb{L}_3 := H_{\mathrm{cg}}(\Omega, D, E; \omega)$, $\mathbb{L}_4 := \mathcal{H}(\Omega, E; \omega)$ and $\mathbb{L}_5 := \omega^{-1}\mathrm{curl}(H_{0,\complement E}(\mathrm{curl}, \Omega))$. Moreover, it hold:

$$\begin{aligned}
H_{0,E}(\mathrm{curl}\,0, \Omega) &= \mathbb{L}_1 \overset{\perp}{\oplus}_\omega \mathbb{L}_2 \overset{\perp}{\oplus}_\omega \mathbb{L}_3 \overset{\perp}{\oplus}_\omega \mathbb{L}_4,\\
\mathrm{grad}(H^1_{0,E}(\Omega)) &= \mathbb{L}_1 \overset{\perp}{\oplus}_\omega \mathbb{L}_2 \overset{\perp}{\oplus}_\omega \mathbb{L}_3,\\
H_{0,D}(\mathrm{curl}\,0, \Omega) &= \mathbb{L}_1 \overset{\perp}{\oplus}_\omega \mathbb{L}_2,\\
H_{0,\complement E}(\mathrm{div}\,0, \Omega; \omega) &= \mathbb{L}_4 \overset{\perp}{\oplus}_\omega \mathbb{L}_5,\\
\omega^{-1}\mathrm{curl}(H_{0,\complement D}(\mathrm{curl}, \Omega)) &= \mathbb{L}_3 \overset{\perp}{\oplus}_\omega \mathbb{L}_4 \overset{\perp}{\oplus}_\omega \mathbb{L}_5,\\
H_{0,\complement D}(\mathrm{div}\,0, \Omega; \omega) &= \mathbb{L}_2 \overset{\perp}{\oplus}_\omega \mathbb{L}_3 \overset{\perp}{\oplus}_\omega \mathbb{L}_4 \overset{\perp}{\oplus}_\omega \mathbb{L}_5.
\end{aligned}$$

This statement is summarized in the following ω–decomposition diagram:

<table>
<tr><td colspan="5">$L^2(\Omega)^3$</td></tr>
<tr><td>$\mathrm{grad}(H^1_{0,D}(\Omega))$</td><td>$\mathcal{H}(\Omega, D; \omega)$</td><td>$H_{\mathrm{cg}}(\Omega, D, E; \omega)$</td><td>$\mathcal{H}(\Omega, E; \omega)$</td><td>$\omega^{-1}\mathrm{curl}(H_{0,\complement E}(\mathrm{curl}, \Omega))$</td></tr>
<tr><td colspan="4">$H_{0,E}(\mathrm{curl}\, 0, \Omega)$</td><td>$\omega^{-1}\mathrm{curl}(H_{0,\complement E}(\mathrm{curl}, \Omega))$</td></tr>
<tr><td colspan="3">$\mathrm{grad}(H^1_{0,E}(\Omega))$</td><td colspan="2">$H_{0,\complement E}(\mathrm{div}\, 0, \Omega; \omega)$</td></tr>
<tr><td colspan="2">$H_{0,D}(\mathrm{curl}\, 0, \Omega)$</td><td colspan="3">$\omega^{-1}\mathrm{curl}(H_{0,\complement D}(\mathrm{curl}, \Omega))$</td></tr>
<tr><td>$\mathrm{grad}(H^1_{0,D}(\Omega))$</td><td colspan="4">$H_{0,\complement D}(\mathrm{div}\, 0, \Omega; \omega)$</td></tr>
</table>

Combining point (3) of Corollary 3.4 with Theorem 3.6, we obtain a simple generalization of the classical five–subspaces Hodge decomposition (see [9, p. 314]).

Corollary 3.7 (classical five-subspaces decomposition) *The Hilbert space $L^2(\Omega)^3$ can be expressed as the following direct sum of mutually ω–orthogonal closed vector subspaces:*

$$L^2(\Omega)^3 = \mathrm{grad}(H^1_0(\Omega)) \overset{\perp}{\oplus}_\omega \mathcal{H}(\Omega, \Gamma; \omega) \overset{\perp}{\oplus}_\omega H_{\mathrm{cg}}(\Omega; \omega) \overset{\perp}{\oplus}_\omega \mathcal{H}(\Omega, \omega) \overset{\perp}{\oplus}_\omega \omega^{-1}\mathrm{curl}(H^1_0(\Omega)^3).$$

More precisely, the following ω–decomposition diagram holds:

<table>
<tr><td colspan="5">$L^2(\Omega)^3$</td></tr>
<tr><td>$\mathrm{grad}(H^1_0(\Omega))$</td><td>$\mathcal{H}(\Omega, \Gamma; \omega)$</td><td>$H_{\mathrm{cg}}(\Omega; \omega)$</td><td>$\mathcal{H}(\Omega; \omega)$</td><td>$\omega^{-1}\mathrm{curl}(H^1_0(\Omega)^3)$</td></tr>
<tr><td colspan="4">$H(\mathrm{curl}\, 0, \Omega)$</td><td>$\omega^{-1}\mathrm{curl}(H^1_0(\Omega)^3)$</td></tr>
<tr><td colspan="3">$\mathrm{grad}(H^1(\Omega))$</td><td colspan="2">$H_{0,\Gamma}(\mathrm{div}\, 0, \Omega; \omega)$</td></tr>
<tr><td colspan="2">$H_{0,\Gamma}(\mathrm{curl}\, 0, \Omega)$</td><td colspan="3">$\omega^{-1}\mathrm{curl}(H^1(\Omega)^3)$</td></tr>
<tr><td>$\mathrm{grad}(H^1_0(\Omega))$</td><td colspan="4">$H(\mathrm{div}\, 0, \Omega; \omega)$</td></tr>
</table>

Principle of vector invariance. In what follows, given two real vector spaces G and K, we write $G \simeq K$ if there exists a linear isomorphism between G and K.

As an immediate consequence of the preceding two theorems, we infer the vector invariance of the (preceding) Hodge decompositions from the material matrix.

Theorem 3.8 *The following statements hold:*

(1) *The linear isomorphism class of the real vector space $\mathcal{H}(\Omega, A; \omega)$ do not depend on ω. More precisely, for every material matrix ω in $L^\infty(\Omega)^{3\times 3}$, we have that*

$$\mathcal{H}(\Omega, A; \omega) \simeq \mathbb{H}^1_{\mathrm{DR}}(\Omega, A) \simeq \mathbb{H}^2_{\mathrm{DR}}(\Omega, \complement A).$$

In particular, it holds

$$\mathcal{H}(\Omega, A; \omega) \simeq \mathcal{H}(\Omega, A)$$

and hence $\mathcal{H}(\Omega; \omega) \simeq \mathcal{H}(\Omega)$ if $A = \emptyset$.

(2) *Given a curl–gradient boundary pair (D, E) of Ω, the linear isomorphism class of the real vector space $\mathcal{H}_{\mathrm{cg}}(\Omega, D, E; \omega)$ does not depend on ω. More precisely, for every material matrix ω in $L^\infty(\Omega)^{3\times 3}$, we have that*

$$\mathcal{H}_{\mathrm{cg}}(\Omega, D, E; \omega) \simeq \frac{\mathrm{grad}(H^1_{0,E}(\Omega))}{H_{0,D}(\mathrm{curl}\, 0, \Omega)} \simeq \frac{\mathrm{curl}(H_{0,\complement D}(\mathrm{curl}, \Omega))}{H_{0,\complement E}(\mathrm{div}\, 0, \Omega)}.$$

In particular, it holds

$$\mathcal{H}_{\rm cg}(\Omega, D, E; \omega) \simeq \mathcal{H}_{\rm cg}(\Omega, D, E) \tag{17}$$

and hence $\mathcal{H}_{\rm cg}(\Omega; \omega) \simeq \mathcal{H}_{\rm cg}(\Omega)$ *if* $(D, E) = (\Gamma, \emptyset)$.

Moreover, we obtain:

Corollary 3.9 *The following statements are verified:*

(1) *There exists a linear automorphism* $M_\omega : L^2(\Omega)^3 \longrightarrow L^2(\Omega)^3$, *which sends the* ω*–decomposition of Theorem 3.1 into the corresponding* $\mathbf{1}_{3\times 3}$*–decomposition. More precisely,* M_ω *can be chosen in such a way that it fixes* $\mathrm{grad}(H^1_{0,A}(\Omega))$ *and it hold:*

$$\begin{aligned} &M_\omega\big(\mathcal{H}(\Omega, A; \omega)\big) = \mathcal{H}(\Omega, A), \\ &M_\omega\big(\omega^{-1}\mathrm{curl}(H_{0,\complement A}(\mathrm{curl}, \Omega))\big) = \mathrm{curl}(H_{0,\complement A}(\mathrm{curl}, \Omega)). \end{aligned}$$

(2) *Given a curl–gradient boundary pair* (D, E) *of* Ω, *there exists a linear automorphism* $N_\omega : L^2(\Omega)^3 \longrightarrow L^2(\Omega)^3$, *which sends the* ω*–decomposition of Theorem 3.6* (*and hence of Corollary 3.7*) *into the corresponding* $\mathbf{1}_{3\times 3}$*–decomposition. More precisely,* N_ω *can be chosen in such a way that it fixes* $\mathrm{grad}(H^1_{0,D}(\Omega))$ *and it hold:*

$$\begin{aligned} &N_\omega\big(\mathcal{H}(\Omega, D; \omega)\big) = \mathcal{H}(\Omega, D), \\ &N_\omega\big(H_{\rm cg}(\Omega, D, E; \omega)\big) = H_{\rm cg}(\Omega, D, E), \\ &N_\omega\big(\mathcal{H}(\Omega, E; \omega)\big) = \mathcal{H}(\Omega, E), \\ &N_\omega\big(\omega^{-1}\mathrm{curl}(H_{0,\complement E}(\mathrm{curl}, \Omega))\big) = \mathrm{curl}(H_{0,\complement E}(\mathrm{curl}, \Omega)). \end{aligned}$$

Informally, the preceding two results can be rephrased as follows:

Principle of vector invariance. *From the vectorial point of view, the* (*preceding*) *Hodge decompositions of the space of square sommable vector fields on a bounded Lipschitz three–dimensional domain, describing a certain medium, are independent from the material matrix of the domain, describing the inhomogeneity/anisotropy of the medium.*

4 The summand subspace of harmonic vector fields and its topological nature

Throughout this section, we shall make the following assumptions: (Ω, A) *is a bounded Lipschitz device of* $\mathbb{R}^3$, Γ *is the boundary of* Ω, $\complement A$ *is the interior of* $\Gamma \setminus A$ *in* Γ *and* ω *is a material matrix of* Ω.

4.1 Preparation of a Lipschitz device of $\mathbb{R}^3$

Systems of primary 1-cycles. We begin by identifying the 1–cycles of Ω over $\mathbb{Z}$ with the 1–cycles of $\Omega \cup A$ over $\mathbb{R}$, whose supports are in Ω and whose coefficients are in $\mathbb{Z}$. Under this identification, $Z_1(\Omega; \mathbb{Z})$ and hence $Z_1^{(lip)}(\Omega; \mathbb{Z})$ become subsets of $Z_1(\Omega \cup A; \mathbb{R})$.

Consider the linear map

$$i_* : H_1(A; \mathbb{R}) \longrightarrow H_1(\Omega \cup A; \mathbb{R})$$

induced by the inclusion map $A \hookrightarrow \Omega \cup A$.

Definition 4.1 *Let $z \in Z_1(\Omega \cup A; \mathbb{R})$ be a 1–cycle of $\Omega \cup A$ over $\mathbb{R}$. We say that z is a* secondary cycle of (Ω, A) *if $[z]_{(\Omega\cup A;\mathbb{R})} \in \mathrm{Image}(i_*)$ or, equivalently, if there exists a 1–cycle z' of $\Omega \cup A$ over $\mathbb{R}$ such that $|z'| \subset A$ and z is homologous to z' in $\Omega \cup A$ over $\mathbb{R}$. If $[z]_{(\Omega\cup A;\mathbb{R})} \not\in \mathrm{Image}(i_*)$ and $z \in Z_1^{(lip)}(\Omega; \mathbb{Z})$, then we say that z is a* primary cycle of (Ω, A). *We call* (first) secondary Betti number of (Ω, A), *denoted by $\sigma\beta_1(\Omega, A)$, the dimension of $\mathrm{Image}(i_*)$ as vector subspace of $H_1(\Omega \cup A; \mathbb{R})$, and we call* (first) primary Betti number of (Ω, A), *denoted by $\rho\beta_1(\Omega, A)$, the non–negative integer defined by setting*

$$\rho\beta_1(\Omega, A) := \beta_1(\Omega \cup A) - \sigma\beta_1(\Omega, A),$$

which coincides with the dimension of the quotient vector space $H_1(\Omega\cup A; \mathbb{R})/\mathrm{Image}(i_*)$.

Lemma 4.2 *Let $\chi_* : H_1(A; \mathbb{R}) \longrightarrow H_1(\overline{\Omega}; \mathbb{R})$ be the linear map induced by the inclusion $A \hookrightarrow \overline{\Omega}$ and let $\varrho_*^{(1)} : H_1(\overline{\Omega}; \mathbb{R}) \longrightarrow H_1(\overline{\Omega}, A; \mathbb{R})$ and $\varrho_*^{(2)} : H_2(\overline{\Omega}, \complement A; \mathbb{R}) \longrightarrow H_2(\overline{\Omega}, \Gamma; \mathbb{R})$ be the linear maps induced by the identity map of $\overline{\Omega}$. Then it holds:*

$$\sigma\beta_1(\Omega, A) = \dim \mathrm{Image}(\chi_*), \tag{18}$$

$$\rho\beta_1(\Omega, A) = \dim \mathrm{Image}(\varrho_*^{(1)}) = \dim \mathrm{Image}(\rho_*^{(2)}). \tag{19}$$

In the remainder of this subsection, we will denote by p the primary Betti number $\rho\beta_1(\Omega, A)$ of the Lipschitz device (Ω, A).

Definition 4.3 *A p–uple $(z_1, \ldots, z_p)$ of primary cycles of (Ω, A) is said to be a* system of primary cycles of (Ω, A) *if $\{[z_1]_{(\Omega\cup A;\mathbb{R})} + \mathrm{Image}(i_*), \ldots, [z_p]_{(\Omega;\mathbb{R})} + \mathrm{Image}(i_*)\}$ is a base of $H_1(\Omega \cup A; \mathbb{R})/\mathrm{Image}(i_*)$ or, equivalently, if, fixed a base K of $\mathrm{Image}(i_*)$, then $\{[z_1]_{(\Omega\cup A;\mathbb{R})}, \ldots, [z_p]_{(\Omega\cup A;\mathbb{R})}\} \cup K$ is a base of $H_1(\Omega \cup A; \mathbb{R})$.*

Systems of jump paths. Let a be a non–negative integer and let K_a be the subset of $\mathbb{Z}^{a+1}$ consisting of elements $(n_0, n_1, \ldots, n_a)$ such that $\sum_{i=0}^{a} n_i = 0$. The set K_a is a free abelian subgroup of $\mathbb{Z}^{a+1}$ of rank a. Let $v_0^{(a)} = (1, 0, \ldots, 0), v_1^{(a)} = (0, 1, \ldots, 0), \ldots, v_a^{(a)} = (0, \ldots, 0, 1)$ be the elements of the canonical base of $\mathbb{Z}^{a+1}$. For each $k, h \in \{0, 1, \ldots, a\}$ with $k \neq h$, we denote by $v^{(a)}(k, h)$ the element of K_a defined by $v^{(a)}(k, h) := v_h^{(a)} - v_k^{(a)}$.

Definition 4.4 *We call a–*jump *an element of $\mathbb{Z}^{a+1}$ of the form $v^{(a)}(k, h)$ and* system of a–jumps *an ordered base $(v^{(a)}(k_1, h_1), \ldots, v^{(a)}(k_a, h_a))$ of K_a formed by a–jumps.*

By definition, there do not exist systems of 0–jumps. On the contrary, if a is positive, then the a–uple $\mathcal{S}_a = (v^{(a)}(0, 1), \ldots, v^{(a)}(0, a))$ is always a system of a–jumps. We call $\mathcal{S}_a$ *standard system of a–jumps.*

It is very easy to characterize systems of a–jumps via Linear Algebra.

Let $\mathcal{V} := (v^{(a)}(k_1, h_1), \ldots, v^{(a)}(k_a, h_a))$ be an a–uple of a–jumps. We call *matrix of* $\mathcal{V}$ the $(a \times a)$–matrix $M_{\mathcal{V}} = (m_{ij})_{i,j\in\{1,\ldots,a\}} \in \{-1, 0, 1\}^{a\times a}$ defined by

$$m_{ij} := \begin{cases} 1 & \text{if } i = h_j, \\ -1 & \text{if } i = k_j, \\ 0 & \text{if } i \in \{1, \ldots, a\} \setminus \{k_j, h_j\}. \end{cases} \tag{20}$$

Lemma 4.5 *The a–uple $\mathcal{V}$ is a system of a–jumps if and only if the rank of $M_{\mathcal{V}}$ is a. Moreover, if such a rank is a, then the coefficients of its inverse $M_{\mathcal{V}}^{-1}$ in $\mathbb{Q}^{a\times a}$ belongs to $\{-1, 0, 1\}$; that is, $M_{\mathcal{V}}^{-1} \in \{-1, 0, 1\}^{a\times a}$.*

Proof. Let $\mathcal{S}_a = (v^{(a)}(0, 1), \ldots, v^{(a)}(0, a))$ be the standard system of a–jumps. Suppose that $\mathcal{V}$ is a system of a–jumps. It is immediate to verify that $M_{\mathcal{V}}$ is the matrix associated with the change of bases of K_a from $\mathcal{V}$ to $\mathcal{S}_a$. It follows that $M_{\mathcal{V}}$ is invertible in $\mathbb{Z}^{a\times a}$. In particular, its rank is a, its determinant is either 1 or -1 and hence $M_{\mathcal{V}}^{-1} \in \{-1, 0, 1\}^{a \times a}$. Suppose now that the rank of $M_{\mathcal{V}}$ is a or, equivalently, $\det(M_{\mathcal{V}}) \neq 0$. It is easy to prove, by induction on a, that the determinant of a matrix N in $\{-1, 0, 1\}^{a \times a}$ belongs to $\{-1, 0, 1\}$, provided N satisfies the following property: the components of the columns of N are either all null or null expect for one component belonging to $\{-1, 1\}$ or null except for two components, one equal to 1 and one equal to -1. It follows that $\det M_{\mathcal{V}}$ is equal to either 1 or -1 and hence $\mathcal{V}$ is an ordered base of K_a. $\square$

Let us introduce the notion of system of jump paths. Let $A_0, A_1, \ldots, A_a$ be the connected components of A.

Definition 4.6 *Let $\lambda : I \longrightarrow \Omega \cup A$ be a path of $\Omega \cup A$ modulo A; that is, a path of $\Omega \cup A$ such that $\{\lambda(0), \lambda(1)\} \subset A$. If, in addition, $\lambda((0,1)) \subset \Omega$ and there exist two different connected components A' and A'' of A such that $\lambda(0) \in A'$ and $\lambda(1) \in A''$, then we call λ* jump path *of (Ω, A).*

Given a jump path λ of (Ω, A), we denote by $s(\lambda)$ and $e(\lambda)$ the distinct integers contained in $\{0, 1, \ldots, a\}$ such that $\lambda(0) \in A_{s(\lambda)}$ and $\lambda(1) \in A_{e(\lambda)}$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_a)$ be an a–uple of jump paths of (Ω, A). We define the a–uple $\mathcal{V}_{\boldsymbol{\lambda}}$ of a–jumps, called a–uple of a–jumps relative to $\boldsymbol{\lambda}$, *by setting $\mathcal{V}_{\boldsymbol{\lambda}} := \big(v^{(a)}(s(\lambda_1), e(\lambda_1)), \ldots, v^{(a)}(s(\lambda_a), e(\lambda_a))\big)$ and the* matrix $M_{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ *as the matrix of $\mathcal{V}_{\boldsymbol{\lambda}}$. We say that $\boldsymbol{\lambda}$ is a* system of jump paths of (Ω, A) *if each path λ_i is regularly extendable and $\mathcal{V}_{\boldsymbol{\lambda}}$ is a system of a–jumps.*

We remark that, given an a–uple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_a)$ of jump paths of (Ω, A), for each $i, j \in \{1, \ldots, a\}$, the (i, j)–coefficient λ_{ij} of $M_{\boldsymbol{\lambda}}$ is equal to

$$\lambda_{ij} = \begin{cases} 1 & \text{if } i = e(\lambda_j), \\ -1 & \text{if } i = s(\lambda_j), \\ 0 & \text{if } i \in \{1, \ldots, a\} \setminus \{s(\lambda_j), e(\lambda_j)\}. \end{cases}$$

We have:

Lemma 4.7 *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_a)$ be an a–uple of regularly extendable jump paths of (Ω, A); that is, each λ_i is a jump path of (Ω, A) and a regularly extendable path of $\Omega \cup A$ modulo A. The following assertions are equivalent:*

(1) *$\boldsymbol{\lambda}$ is a system of jump paths of (Ω, A).*

(2) *The matrix $M_{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ has rank a.*

(3) *The set $\big\{[\lambda_j(1) - \lambda_j(0)]^{\sharp}_{(A;\mathbb{R})}\big\}_{j \in \{1,\ldots,a\}}$ is a base of the vector space $H_0^{\sharp}(A; \mathbb{R})$.*

Proof. The equivalence between (1) and (2) follows immediately from Lemma 4.5. Let us prove that (1) is equivalent to (3). For each $i \in \{0, 1, \ldots, a\}$, choose a point x_i of A_i and view such a point x_i as a singular 0–simplex of A. Define:

$$\mathcal{A} := \big([x_1 - x_0]^{\sharp}_{(A;\mathbb{R})}, \ldots, [x_a - x_0]^{\sharp}_{(A;\mathbb{R})}\big)$$

and

$$\mathcal{B} := \big([\lambda_1(1) - \lambda_1(0)]^{\sharp}_{(A;\mathbb{R})}, \ldots, [\lambda_a(1) - \lambda_a(0)]^{\sharp}_{(A;\mathbb{R})}\big).$$

By Theorem 2.20, $\mathcal{A}$ is an ordered base of $H_0^{\sharp}(A; \mathbb{R})$. Suppose that $\mathcal{B}$ is an ordered base of $H_0^{\sharp}(A; \mathbb{R})$. The matrix $M_{\boldsymbol{\lambda}}$ coincides with the matrix associated with the change of bases of $H_0^{\sharp}(A; \mathbb{R})$ from $\mathcal{B}$ to $\mathcal{A}$. In particular, the rank of $M_{\boldsymbol{\lambda}}$ is a and hence Lemma 4.5 ensures that $\boldsymbol{\lambda}$ is a system of jumps paths of (Ω, A). Suppose now that $\boldsymbol{\lambda}$ is such a system; that is, $\mathcal{V}_{\boldsymbol{\lambda}}$ is a system of a–jumps. By Lemma 4.5, $M_{\boldsymbol{\lambda}}$ is invertible in $\mathbb{R}^{a \times a}$ and hence $\mathcal{B}$ is an ordered base of $H_0^{\sharp}(A; \mathbb{R})$. $\square$

If a is positive, then systems of jump paths of (Ω, A) always exist.

Corollary 4.8 *Suppose that a is positive. For each $i \in \{0, 1, \dots, a\}$, choose a point x_i of A_i. Then, for each $i \in \{1, \dots, a\}$, there exists a smooth embedded path λ_i of $\Omega \cup A$ modulo A transverse to Γ such that $\lambda_i(0) = x_0$ and $\lambda_i(1) = x_i$. The a–uple $(\lambda_1, \dots, \lambda_a)$ turns out to be a system of jump paths of (Ω, A).*

The next result describes the relation existing between the relative homology vector space $H_1(\Omega \cup A, A; \mathbb{R})$ and the notions of system of primary cycles and of system of jump paths of (Ω, A).

Theorem 4.9 *Let (Ω, A) be a Lipschitz device of $\mathbb{R}^3$, let $p = \rho\beta_1(\Omega, A)$ be its primary Betti number and let $A_0, A_1, \dots, A_a$ be the connected components of A. Then, if $\mathbf{z} = (z_1, \dots, z_p)$ is a system of primary cycles of (Ω, A) and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_a)$ is a system of jump paths of (Ω, A), the $(p + a)$–uple*

$$\big([z_1]_{(\Omega\cup A,A;\mathbb{R})}, \dots, [z_p]_{(\Omega\cup A,A;\mathbb{R})}, [\lambda_1]_{(\Omega\cup A,A;\mathbb{R})}, \dots, [\lambda_a]_{(\Omega\cup A,A;\mathbb{R})}\big)$$

is an ordered base of the vector space $H_1(\Omega \cup A, A; \mathbb{R})$, which we call ordered base of $H_1(\Omega \cup A, A; \mathbb{R})$ induced by $(\mathbf{z}, \boldsymbol{\lambda})$. *In particular, we have:*

$$\dim H_1(\Omega \cup A, A; \mathbb{R}) = p + a.$$

Proof. Let $\theta_i := [z_i]_{(\Omega\cup A;\mathbb{R})}$ for each $i \in \{1, \dots, p\}$ and let $\tau_j := [\lambda_j]_{(\Omega\cup A,A;\mathbb{R})}$ for each $j \in \{1, \dots, a\}$. Consider the following portion of the long exact homology sequence of the topological pair $(\Omega \cup A, A)$ over $\mathbb{R}$:

$$H_1(A;\mathbb{R}) \xrightarrow{i_*} H_1(\Omega \cup A;\mathbb{R}) \xrightarrow{\pi_*} H_1(\Omega \cup A, A;\mathbb{R}) \xrightarrow{\partial_*} H_0^\sharp(A,\mathbb{R}) \xrightarrow{i_\sharp} H_0^\sharp(\Omega \cup A;\mathbb{R}).$$

By the definition of system of primary cycles of (Ω, A) and by the equality $\mathrm{Image}(i_*) = \ker(\pi_*)$, it follows that $\{\pi_*(\theta_i)\}_{i=1}^p$ is a base of $\mathrm{Image}(\pi_*)$. Since Ω is path–connected, $\Omega\cup A$ is path–connected as well and hence $H_0^\sharp(\Omega\cup A;\mathbb{R})$ is null and hence ∂_* is surjective. The reader observes that $\partial_*(\tau_j)$ is equal to $[\lambda_j(1) - \lambda_j(0)]_0^\sharp$ for each $j \in \{1, \dots, a\}$. Combining the latter fact with Lemma 4.7, we infer that $\{\pi_*(\theta_1), \dots, \pi_*(\theta_p), \tau_1, \dots, \tau_a\}$ is a base of $H_1(\Omega \cup A, A; \mathbb{R})$. $\square$

Systems of fundamental vector fields and of their local potentials. Let us introduce some crucial notions.

Definition 4.10 *By a* fundamental vector field of (Ω, A), *we mean a smooth vector field B in $\mathscr{C}^\infty(\overline{\Omega})^3$, having the following two properties:*

1. *B is primary; that is,* $\mathrm{curl}(B) = 0$ *on Ω, but B is not a gradient on Ω.*
2. *B is a gradient locally at A. More precisely, there exists $g \in \mathscr{C}^\infty(\overline{\Omega})$ such that, for some open neighborhood U of A in $\mathbb{R}^3$, $B = \nabla g$ on $U \cap \Omega$. We call g* local potential of B at A.

Suppose that $p = \rho\beta_1(\Omega, A)$ is positive. Given a p–uple $\mathbf{B} = (B_1, \dots, B_p)$ of fundamental vector fields of (Ω, A), we say that $\mathbf{B}$ is a system of fundamental vector fields of (Ω, A) *if there exists a system $\mathbf{z} = (z_1, \dots, z_p)$ of primary cycles of (Ω, A) such that the rank of the following $(p \times p)$–matrix*

$$\left(\int_{z_j} B_i\right)_{i,j\in\{1,\dots,p\}}$$

is p. In this situation, we say that $\mathbf{B}$ is circulation–nonsingular with respect to $\mathbf{z}$. *Moreover, if $\int_{z_j} B_i = \delta_{ij}$ for each $i, j \in \{1, \dots, p\}$, then we say that $\mathbf{B}$ is* circulation–dual to $\mathbf{z}$.

Definition 4.11 *Let $B : \Omega \longrightarrow \mathbb{R}^3$ be a fundamental vector field of (Ω, A), let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_a)$ be a system of jump paths of (Ω, A) and let g be a local potential of B at A. Denote by $\overline{g} : \overline{\Omega} \longrightarrow \mathbb{R}$ the unique continuous extension of g on $\overline{\Omega}$. We say that the local potential g of B at A is* relative to $\boldsymbol{\lambda}$ *if $\overline{g}(\lambda_i(1)) - \overline{g}(\lambda_i(0)) = \int_{\lambda_i} B$ for each $i \in \{1, \ldots, a\}$.*

As we see in the next result, local potentials of fundamental vector fields relative to systems of jump paths always exist.

Lemma 4.12 *Given a fundamental vector field B of (Ω, A) and given a system $\boldsymbol{\lambda}$ of jump paths of (Ω, A), there exists a local potential of B at A relative to $\boldsymbol{\lambda}$.*

Proof. Let B be a fundamental vector field of (Ω, A), let $g \in \mathscr{C}^\infty(\overline{\Omega})$ be a local potential of B at A and let $\boldsymbol{\lambda}$ be a system of jump paths of (Ω, A). Denote by $A_0, A_1, \ldots, A_a$ the path–connected components of A. Since A is a Lipschitz open subset of Γ, the closures $\overline{A_i}$ of the A_i's in $\mathbb{R}^3$ are pairwise disjoint. In this way, there exists pairwise disjoint open neighborhoods $U_0, U_1, \ldots, U_a$ of $\overline{A_0}, \overline{A_1}, \ldots, \overline{A_a}$ in $\mathbb{R}^3$, respectively. Restricting each U_i around $\overline{A_i}$ if needed, we may suppose that the $\overline{U_i}$'s are pairwise disjoint as well. Define $\alpha_i := \overline{g}(\lambda_i(0)) - \overline{g}(\lambda_i(1)) + \int_{\lambda_i} B$ for each $i \in \{1, \ldots, a\}$, $\alpha := (\alpha_1, \ldots, \alpha_a) \in \mathbb{R}^a$ and denote by $M^t_{\boldsymbol{\lambda}}$ the traspose of the matrix $M_{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$. Consider the following linear system

$$M^t_{\boldsymbol{\lambda}} \cdot x = \alpha, \tag{21}$$

where $x = (x_1, \ldots, x_a) \in \mathbb{R}^a$ is the variable. Thanks to Definition 4.6 and Lemma 4.7, $M_{\boldsymbol{\lambda}}$ has rank a and hence (21) admits a unique solution $c = (c_1, \ldots, c_a)$. Let $c_0 := 0$. Using a smooth partition of unity of $\mathbb{R}^3$ subordinate to the open cover $\{U_1, \ldots, U_a, \mathbb{R}^3 \setminus \bigcup_{i=1}^a \overline{A_i}\}$, one can define easily a smooth function ψ in $\mathscr{C}^\infty(\mathbb{R}^3)$ such that $\psi|_{U_i}$ is constantly equal to c_i for each $i \in \{0, 1, \ldots, a\}$. Define $f \in \mathscr{C}^\infty(\overline{\Omega})$ by setting

$$f := g + \psi|_\Omega.$$

The fact that c is a solution of (21) is equivalent to say that

$$c_{\lambda_i(1)} - c_{\lambda_i(0)} = \overline{g}(\lambda_i(0)) - \overline{g}(\lambda_i(1)) + \int_{\lambda_i} B$$

for each $i \in \{1, \ldots, a\}$. In this way, fixed $i \in \{1, \ldots, a\}$, we have:

$$\begin{aligned} \overline{f}(\lambda_i(1)) - \overline{f}(\lambda_i(0)) &= \big(\overline{g}(\lambda_i(1)) + \psi(\lambda_i(1))\big) - \big(\overline{g}(\lambda_i(0)) + \psi(\lambda_i(0)\big) = \\ &= \overline{g}(\lambda_i(1)) - \overline{g}(\lambda_i(0)) + c_{\lambda_i(1)} - c_{\lambda_i(0)} = \\ &= \textstyle\int_{\lambda_i} B. \end{aligned}$$

This completes the proof. □

Let us give the definition of system of local potentials of a system of fundamental vector field.

Definition 4.13 *Suppose that $p = \rho\beta_1(\Omega, A)$ is positive. Let $\mathbf{B} = (B_1, \ldots, B_p)$ be a system of fundamental vector fields of (Ω, A) and, for each $i \in \{1, \ldots, p\}$, let b_i be a local potential of B_i at A. Denote the p–uple $(b_1, \ldots, b_p)$ by $\mathbf{b}$. We say that $\mathbf{b}$ is a* system of local potentials of $\mathbf{B}$ at A *if there exists a system of jump paths $\boldsymbol{\lambda}$ of (Ω, A) such that, for each $i \in \{1, \ldots, p\}$, the local potential g_i of B_i at A is relative to $\boldsymbol{\lambda}$. Moreover, if $\mathbf{b}$ and $\boldsymbol{\lambda}$ have such a property, then we say that $\mathbf{b}$ is a* system of local potentials of $\mathbf{B}$ at A relative to $\boldsymbol{\lambda}$.

Preparations of a Lipschitz device of $\mathbb{R}^3$. Let us introduce the notions of preparation and of complete preparation of the Lipschitz device (Ω, A) of $\mathbb{R}^3$.

Definition 4.14 *Given a system* $\mathbf{B}$ *of fundamental vector fields of* (Ω, A) *and a system* $\mathbf{b}$ *of local potentials of* $\mathbf{B}$ *at* A*, we say that* $[\![\mathbf{B}, \mathbf{b}]\!]$ *is a* preparation of (Ω, A)*. Moreover, given a system* $\mathbf{z}$ *of primary cycles of* (Ω, A) *and a system* $\boldsymbol{\lambda}$ *of jump paths of* (Ω, A)*, we say that* $[\![\mathbf{B}, \mathbf{b} \,|\, \mathbf{z}, \boldsymbol{\lambda}]\!]$ *is a* complete preparation of (Ω, A) *if* $\mathbf{B}$ *is circulation–dual to* $\mathbf{z}$ *and the system* $\mathbf{b}$ *of local potentials of* $\mathbf{B}$ *at* A *is relative to* $\boldsymbol{\lambda}$*.*

4.2 Existence of preparations

This subsection is devoted to prove the following crucial result.

Theorem 4.15 *Every Lipschitz device of* $\mathbb{R}^3$ *admits a complete preparation.*

We subdivide the proof into two steps.

Step I. Biot-Savare vector fields, linking number and Alexander duality theorem. The aim of this step is to formulate a particular version of the celebrated Alexander duality theorem in terms of the notion of Biot–Savare vector field generated by Lipschitz 1–cycle over $\mathbb{Z}$.

First, we define the latter notion.

Let X be a non–empty subset of $\mathbb{R}^3$. Identify the 1–cycles of X over $\mathbb{Z}$ with the 1–cycles of $\mathbb{R}^3$ over $\mathbb{Z}$, whose supports are contained in X. Under this identification, $Z_1(X;\mathbb{Z})$ becomes an abelian subgroup of $Z_1(\mathbb{R}^3;\mathbb{Z})$. Let $\gamma : I \longrightarrow X$ be a Lipschitz path of $\mathbb{R}^3$ and let $\mathrm{BS}(\gamma) : \mathbb{R}^3 \setminus |\gamma| \longrightarrow \mathbb{R}^3$ be the vector field defined by setting

$$\mathrm{BS}(\gamma)(y) := \frac{1}{4\pi} \int_0^1 \frac{d\gamma}{dt}(s) \times \frac{y - \gamma(s)}{|y - \gamma(s)|_3^3} \, ds$$

for each $y \in \mathbb{R}^3 \setminus |\gamma|$ (recall that $|\gamma| = \mathrm{Image}(\gamma)$ and $|(v_1, v_2, v_3)|_3 = (v_1^2 + v_2^2 + v_3^2)^{1/2}$), where $d\gamma/dt$ is the element of $L^\infty(I)^3$ equal to the weak derivative of the path γ, viewed as a map from I to $\mathbb{R}^3$.

Let us introduce the notion of Biot–Savare vector field generated by a Lipschitz 1–cycle over $\mathbb{Z}$. To the best of our knowledge, this notion does not exist in literature yet.

Definition 4.16 *Let* X *be a non–empty subset of* $\mathbb{R}^3$ *and let* z *be a Lipschitz* 1*–cycle* z *of* X *over* $\mathbb{Z}$*. We define the* Biot–Savare vector field generated by z*, denoted by* $\mathrm{BS}(z) : \mathbb{R}^3 \setminus |z| \longrightarrow \mathbb{R}^3$*, as follows. If* z *is null, then* $\mathrm{BS}(z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *is the null vector field. If* z *is not null and* $z = \sum_{i=1}^h a_i \gamma_i$ *is its finite representation in base, then* $\mathrm{BS}(z) : \mathbb{R}^3 \setminus |z| \longrightarrow \mathbb{R}^3$ *is given by setting*

$$\mathrm{BS}(z)(y) := \textstyle\sum_{i=1}^h a_i \cdot \mathrm{BS}(\gamma_i)(y)$$

for each $y \in \mathbb{R}^3 \setminus |z| = \mathbb{R}^3 \setminus \bigcup_{i=1}^h |\gamma_i|$*.*

Lemma 4.17 *The Biot–Savare vector field* $\mathrm{BS}(z) : \mathbb{R}^3 \setminus |z| \longrightarrow \mathbb{R}^3$ *generated by a Lipschitz* 1*–cycle* z *of* X *over* $\mathbb{Z}$ *is smooth and its curl and divergence vanish; that is, it is harmonic.*

Before proving Lemma 4.17, we need a technical result.

Lemma 4.18 *Let $v, w \in \mathbb{R}^3$. Define the smooth vector fields $G_v : \mathbb{R}^3 \setminus \{v\} \longrightarrow \mathbb{R}^3$ and $g(v,w) : \mathbb{R}^3 \setminus \{v\} \longrightarrow \mathbb{R}^3$ by setting*

$$G_v(y) := \frac{y-v}{|y-v|_3^3} \quad \textit{and} \quad g(v,w)(y) := w \times G_v(y)$$

for each $y \in \mathbb{R}^3 \setminus \{v\}$. Then, for each $y \in \mathbb{R}^3 \setminus \{v\}$, we have that

$$\mathrm{curl}(g(v,w))(y) = -J_{G_y}(v) \cdot w \tag{22}$$

and

$$\mathrm{div}(g(v,w))(y) = 0. \tag{23}$$

Proof. For each $i \in \{1,2,3\}$, let $G_v^{(i)} : \mathbb{R}^3 \setminus \{v\} \longrightarrow \mathbb{R}$ be the i^{th}–component of G_v; that is, $G_v^{(i)}(y) = (y_i - v_i)|y-v|_3^{-3}$ for each $y \in \mathbb{R}^3 \setminus \{v\}$, where $v = (v_1, v_2, v_3)$. Let $y \in \mathbb{R}^3 \setminus \{v\}$. For each $i, j \in \{1,2,3\}$ with $i \neq j$, we have that

$$\frac{\partial G_v^{(i)}}{\partial y_j}(y) = -3(y_i - v_i)(y_j - v_j)|y-v|_3^{-5}$$

and

$$\frac{\partial G_v^{(i)}}{\partial y_i}(y) = \big(|y-v|_3^2 - 3(y_i - v_i)^2\big)|y-v|_3^{-5}.$$

It follows immediately that

$$\mathrm{curl}(G_v)(y) = 0, \ \mathrm{div}(G_v)(y) = 0 \quad \text{and} \quad J_{G_v}(y) = J_{G_y}(v). \tag{24}$$

By direct computations, one easily obtain that

$$\mathrm{curl}(g(v,w))(y) = \mathrm{div}(G_v)(y) - J_{G_v}(y) \cdot w \quad \text{and} \quad \mathrm{div}(g(v,w))(y) = -w \bullet \mathrm{curl}(G_v)(y).$$

Combining the latter equalities with (24), we infer at once (22) and (23). $\square$

Proof of Lemma 4.17. If $z = 0$, then $\mathrm{BS}(z)$ is null and the lemma is evident. Let $z \neq 0$ and let $z = \sum_{i=1}^k a_i \gamma_i$ be the finite representation of z in base. Fix $i \in \{1, \ldots, k\}$ and $y \in \mathbb{R}^3 \setminus |\gamma_i|$. The reader observes that it holds:

$$\mathrm{BS}(\gamma_i)(y) = \frac{1}{4\pi} \int_0^1 g(\gamma_i(s), \dot{\gamma}_i(s))(y)\, ds,$$

where $\dot{\gamma}_i$ denotes the weak derivative $d\gamma_i/dt$ of γ_i and, for each $s \in I$, $g(\gamma_i(s), \dot{\gamma}_i(s))$ is the smooth vector field of $\mathbb{R}^3 \setminus \{\gamma_i(s)\}$ defined as in the statement of Lemma 4.18. By the theorem of derivation under the sign of integral, the vector field $\mathrm{BS}(\gamma_i)$ is smooth and it hold:

$$\mathrm{curl}(\mathrm{BS}(\gamma_i))(y) = \frac{1}{4\pi} \int_0^1 \mathrm{curl}\big(g(\gamma_i(s), \dot{\gamma}_i(s))\big)(y)\, ds \tag{25}$$

and

$$\mathrm{div}(\mathrm{BS}(\gamma_i))(y) = \frac{1}{4\pi} \int_0^1 \mathrm{div}\big(g(\gamma_i(s), \dot{\gamma}_i(s))\big)(y)\, ds. \tag{26}$$

Equations (23) and (26) imply that $\mathrm{div}(\mathrm{BS}(\gamma_i))$ is null on whole $\mathbb{R}^3 \setminus |\gamma_i|$. On the other hand, by equation (22), we infer that

$$\mathrm{curl}\big(g(\gamma_i(s), \dot{\gamma}_i(s))\big)(y) = -J_{G_y}(\gamma_i(s)) \cdot \dot{\gamma}_i(s)$$

for each $s \in I$, where $G_y : \mathbb{R}^3 \setminus \{y\} \longrightarrow \mathbb{R}^3$ is the smooth vector field defined as in the statement of Lemma 4.18. It follows immediately that

$$\mathrm{curl}\big(g(\gamma_i(s), \dot{\gamma}_i(s))\big)(y) = -\frac{d}{dt}(G_y \circ \gamma_i)(s)$$

for each $s \in I$, where $\frac{d}{dt}(G_y \circ \gamma_i)$ denotes the weak derivative of the Lipschitz path $G_y \circ \gamma_i$ of $\mathbb{R}^3$. Combining the latter equation with (25), we obtain that

$$\operatorname{curl}(\mathrm{BS}(\gamma_i))(y) = \frac{1}{4\pi}\big(G_y(\gamma_i(0)) - G_y(\gamma_i(1))\big). \tag{27}$$

We are now in position to complete the proof. Define $Z := \bigcup_{j=1}^k \{\gamma_i(0), \gamma_i(1)\}$ and, for each $x \in Z$, $I_{0,x} := \{i \in \{1, \ldots, k\} \,|\, \gamma_i(0) = x\}$, $I_{1,x} := \{i \in \{1, \ldots, k\} \,|\, \gamma_i(1) = x\}$ and $a_x \in \mathbb{Z}$ by setting

$$a_x := \textstyle\sum_{i \in I_{1,x}} a_i - \sum_{i \in I_{0,x}} a_i.$$

Since z is a 1–cycle of X over $\mathbb{Z}$, we have that

$$0 = \partial_1(z) = \textstyle\sum_{i=1}^k a_i(\gamma_i(1) - \gamma_i(0)) = \sum_{x \in Z} a_x x \ \text{ in } S_1(X;\mathbb{Z}).$$

In this way, a_x is null for each $x \in Z$. Fix $y \in \mathbb{R}^3 \setminus |z|$. By (27), we obtain:

$$\begin{aligned}\operatorname{curl}(\mathrm{BS}(z))(y) &= \frac{1}{4\pi}\sum_{i=1}^k a_i\big(G_y(\gamma_i(0)) - G_y(\gamma_i(1))\big) = \\ &= -\frac{1}{4\pi}\sum_{x \in Z} a_x G_y(x).\end{aligned}$$

Since each a_x is null, we infer that $\operatorname{curl}(\mathrm{BS}(z))$ is null on whole $\mathbb{R}^3 \setminus |z|$. $\square$

Define the set $Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$ by

$$Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z}) := \big\{(z,w) \in Z_1(\mathbb{R}^3;\mathbb{Z}) \times Z_1(\mathbb{R}^3;\mathbb{Z}) \,\big|\, |z| \cap |w| = \emptyset\big\}$$

and its subset $Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z}) \circ Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z})$ by

$$Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z}) \circ Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z}) := \big\{(z,w) \in Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z}) \,\big|\, z, w \in Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z})\big\}.$$

Given (z,w) and (z',w') in $Z_1(\mathbb{R}^3,\mathbb{Z}) \times Z_1(\mathbb{R}^3;\mathbb{Z})$, we say that (z,w) is *homology–connected to* (z',w') if there exist $d, e \in S_2(\mathbb{R}^3;\mathbb{Z})$ such that $\partial_2(d) = z - z'$, $\partial_2(e) = w - w'$ and

$$\big(|z| \cup |d| \cup |z'|\big) \cap \big(|w| \cup |e| \cup |w'|\big) = \emptyset.$$

Evidently, if this happens, then (z,w) and (z',w') belong to $Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$.

As an immediate consequence of Corollary 2.23, we obtain:

Lemma 4.19 *Each pair (z,w) in $Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$ is homology–connected to a pair of Lipschitz 1–cycles of $\mathbb{R}^3$ over $\mathbb{Z}$.*

Let us state an important result.

Theorem-Definition 4.20 *There exists, and is unique, a function ℓk from $Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$ to $\mathbb{Z}$ with the following two properties:*

(1) *For each pair (z',w') in $Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z}) \circ Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z})$, $\ell k(z',w')$ is equal to the line integral $\int_{z'} \mathrm{BS}(w')$ of $\mathrm{BS}(w')$ along z'.*

(2) *If (z,w) is a pair in $Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$ and (z',w') is a pair in $Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z}) \circ Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z})$, homology–connected to (z,w), then $\ell k(z,w)$ is equal to $\ell k(z',w')$.*

Moreover, we have that, given $(z,w) \in Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$, the value $\ell k(z,w)$ of such a function at (z,w) coincides with the linking number between the singular 1–cycles z and w of $\mathbb{R}^3$ over $\mathbb{Z}$, classically defined, for example, in Section 73 of [15]. For this reason, we call $\ell k(z,w)$ linking number between z and w.

We collect some properties of the function ℓk in the next result.

Lemma 4.21 *The function $\ell k : Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z}) \longrightarrow \mathbb{Z}$ has the following properties:*

(1) *If $(z,w) \in Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$ and $a \in \mathbb{Z}$, then*

$$\ell k(z,w) = \ell k(w,z) \quad \text{and} \quad \ell k(az,w) = a \cdot \ell k(z,w) = \ell k(z,aw).$$

(2) *Let $z,z',w,w' \in Z_1(\mathbb{R}^3,\mathbb{Z})$ such that (z,w), (z',w) and (z,w') belong to $Z_1(\mathbb{R}^3;\mathbb{Z})\circ Z_1(\mathbb{R}^3;\mathbb{Z})$. Then $(z+z',w)$ and $(z,w+w')$ belong to $Z_1(\mathbb{R}^3;\mathbb{Z})\circ Z_1(\mathbb{R}^3;\mathbb{Z})$ as well, and it hold:*

$$\ell k(z+z',w) = \ell k(z,w) + \ell k(z',w)$$

and

$$\ell k(z,w+w') = \ell k(z,w) + \ell k(z,w').$$

(3) *Let $(z,w) \in Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$. If $z' \in Z_1(\mathbb{R}^3 \setminus |w|,\mathbb{Z})$ is homologous to z in $\mathbb{R}^3 \setminus |w|$ over $\mathbb{Z}$, then*

$$\ell k(z,w) = \ell k(z',w).$$

Similarly, if $w' \in Z_1(\mathbb{R}^3 \setminus |z|,\mathbb{Z})$ is homologous to w in $\mathbb{R}^3 \setminus |z|$ over $\mathbb{Z}$, then

$$\ell k(z,w) = \ell k(z,w').$$

Let us prove Theorem–Definition 4.20 and Lemma 4.21 together.

Proof of Theorem–Definition 4.20 and Lemma 4.21. First, we observe that, thanks to Lemma 4.19, if there exists a function $\ell k : Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z}) \longrightarrow \mathbb{Z}$ with properties (1) and (2), then it is unique.

Let us prove the existence of such a function. Denote by $\mathrm{Lk} : Z_1(\mathbb{R}^3;\mathbb{Z})\circ Z_1(\mathbb{R}^3;\mathbb{Z}) \longrightarrow \mathbb{Z}$ the classical function, which sends $(z,w) \in Z_1(\mathbb{R}^3;\mathbb{Z})\circ Z_1(\mathbb{R}^3;\mathbb{Z})$ into the linking number between the singular 1–cycles z and w of $\mathbb{R}^3$ over $\mathbb{Z}$, defined as in Section 73 of [15]. Since the function Lk has properties (2) of Theorem–Definition 4.20 and (1), (2) and (3) of Lemma 4.21, it suffices to show that it satisfies (1) of Theorem–Definition 4.20. Let $(z',w') \in Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z}) \circ Z_1^{(lip)}(\mathbb{R}^3;\mathbb{Z})$.

If either $z'=0$ or $w'=0$, then both $\mathrm{Lk}(z',w')$ and $\int_{z'} \mathrm{BS}(w')$ are null and hence equal.

Suppose that $z' \neq 0$ and $w' \neq 0$. Let P be an open neighborhood of $|z'|$ in $\mathbb{R}^3$ whose closure in $\mathbb{R}^3$ is disjoint from $|w'|$. By Lemma 2.22, there exist a positive integer r, Lipschitz loops $\ell_1,\ldots,\ell_r$ of P and, for each $i \in \{1,\ldots,r\}$, a sequence $\{\gamma_{i,n}\}_{n\in\mathbb{N}}$ of embedded smooth loops of P such that:

(i) $\int_{z'} \mathrm{BS}(w') = \int_{\sum_{i=1}^r \ell_i} \mathrm{BS}(w')$.

(ii) For each $i \in \{1,\ldots,r\}$, the sequence $\{\gamma_{i,n}\}_{n\in\mathbb{N}}$ converges uniformly to ℓ_i on I and the sequence $\{d\gamma_{i,n}/dt\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(I)^3$ and it converges to $d\ell_i/dt$ almost everywhere on I.

(iii) For each $n \in \mathbb{N}$, $\sum_{i=1}^r \gamma_{i,n}$ is homologous to z' in P over $\mathbb{Z}$.

By (i), we have that

$$\int_{z'} \mathrm{BS}(w') = \sum\nolimits_{i=1}^r \int_{\ell_i} \mathrm{BS}(w') = \sum\nolimits_{i=1}^r \int_0^1 \mathrm{BS}(w')(\ell_i(s)) \bullet \frac{d\ell_i}{dt}(s)\, ds.$$

Moreover, for each $i \in \{1, \ldots, r\}$ and for each $n \in \mathbb{N}$, it holds:

$$\int_{\gamma_{i,n}} \mathrm{BS}(w') = \int_0^1 \mathrm{BS}(w')(\gamma_{i,n}(s)) \bullet \frac{d\gamma_{i,n}}{dt}(s)\, ds.$$

In this way, thanks to (ii), we obtain that

$$\left\{ \sum\nolimits_{i=1}^{r} \int_{\gamma_{i,n}} \mathrm{BS}(w') \right\}_{n \in \mathbb{N}} \longrightarrow \int_{z'} \mathrm{BS}(w'). \tag{28}$$

Let O be an open neighborhood of $|w'|$ in $\mathbb{R}^3$ disjoint from P. Let $i \in \{1, \ldots, r\}$, let $n \in \mathbb{N}$ and let $w' = \sum_{k=1}^{c} b_k \sigma_k$ be the finite representation of w' in base. It hold:

$$\begin{aligned}
\int_{\gamma_{i,n}} \mathrm{BS}(w') &= \sum\nolimits_{k=1}^{c} b_k \int_{\gamma_{i,n}} \mathrm{BS}(\sigma_k)(\gamma_{i,n}(s)) \bullet \frac{d\gamma_{i,n}}{dt}(s)\, ds = \\
&= \frac{1}{4\pi} \sum\nolimits_{k=1}^{c} b_k \int_0^1 \left(\int_0^1 \frac{d\sigma_k}{dt}(\xi) \times \frac{\gamma_{i,n}(s) - \sigma_k(\xi)}{|\gamma_{i,n}(s) - \sigma_k(\xi)|_3^3}\, d\xi \right) \bullet \frac{d\gamma_{i,n}}{dt}(s)\, ds = \\
&= \frac{1}{4\pi} \sum\nolimits_{k=1}^{c} b_k \int_0^1 \left(\int_0^1 \frac{d\gamma_{i,n}}{dt}(s) \times \frac{\sigma_k(\xi) - \gamma_{i,n}(s)}{|\sigma_k(\xi) - \gamma_{i,n}(s)|_3^3}\, ds \right) \bullet \frac{d\sigma_k}{dt}(\xi)\, d\xi = \\
&= \sum\nolimits_{k=1}^{c} b_k \int_{\sigma_k} \mathrm{BS}(\gamma_{i,n}) = \int_{w'} \mathrm{BS}(\gamma_{i,n}).
\end{aligned}$$

It follows that

$$\sum\nolimits_{i=1}^{r} \int_{\gamma_{i,n}} \mathrm{BS}(w') = \sum\nolimits_{i=1}^{r} \int_{w'} \mathrm{BS}(\gamma_{i,n}) \tag{29}$$

for each $n \in \mathbb{N}$. Applying Lemma 2.22 to w', we obtain a positive integer h, Lipschitz loops $L_1, \ldots, L_\ell$ of O and, for each $j \in \{1, \ldots, h\}$, a sequence $\{\Gamma_{j,m}\}_{m \in \mathbb{N}}$ of embedded smooth loops of O such that:

- (iv) $\int_{w'} \mathrm{BS}(\gamma_{i,n}) = \int_{\sum_{j=1}^{h} L_j} \mathrm{BS}(\gamma_{i,n})$ for each $i \in \{1, \ldots, r\}$ and for each $n \in \mathbb{N}$.
- (v) For each $j \in \{1, \ldots, h\}$, the sequence $\{\Gamma_{j,m}\}_{m \in \mathbb{N}}$ converges uniformly to L_j on I and the sequence $\{d\Gamma_{j,m}/dt\}_{m \in \mathbb{N}}$ is bounded in $L^\infty(I)^3$ and it converges to dL_j/dt almost everywhere on I.
- (vi) For each $m \in \mathbb{N}$, $\sum_{j=1}^{h} \Gamma_{j,m}$ is homologous to w' in O over $\mathbb{Z}$.

By (iv), it follows that

$$\begin{aligned}
\sum\nolimits_{i=1}^{r} \int_{\gamma_{i,n}} \mathrm{BS}(w') &= \sum\nolimits_{i=1}^{r} \int_{\sum_{j=1}^{h} L_j} \mathrm{BS}(\gamma_{i,n}) = \\
&= \sum\nolimits_{i=1}^{r} \sum\nolimits_{j=1}^{h} \int_{L_j} \mathrm{BS}(\gamma_{i,n}) = \\
&= \sum\nolimits_{i=1}^{r} \sum\nolimits_{j=1}^{h} \int_0^1 \mathrm{BS}(\gamma_{i,n})(L_j(s)) \bullet \frac{dL_j}{dt}(s)\, ds.
\end{aligned}$$

From (v), we infer that

$$\left\{ \sum\nolimits_{i=1}^{r} \sum\nolimits_{j=1}^{h} \int_{\Gamma_{j,m}} \mathrm{BS}(\gamma_{i,n}) \right\}_{m \in \mathbb{N}} \longrightarrow \sum\nolimits_{i=1}^{r} \int_{\gamma_{i,n}} \mathrm{BS}(w') \tag{30}$$

for each $n \in \mathbb{N}$. In this way, up to extract suitable subsequences of $\{\Gamma_{j,m}\}_{m\in\mathbb{N}}$ for each $j \in \{1,\dots,h\}$, (28) and (30) imply that

$$\left\{\sum\nolimits_{i=1}^{r}\sum\nolimits_{j=1}^{h}\int_{\Gamma_{j,n}}\mathrm{BS}(\gamma_{i,n})\right\}_{n\in\mathbb{N}} \longrightarrow \int_{z'}\mathrm{BS}(w') \tag{31}$$

Let $i \in \{1,\dots,r\}$, let $j \in \{1,\dots,h\}$ and let $n \in \mathbb{N}$. Since $\gamma_{i,n}$ and $\Gamma_{j,n}$ are embedded smooth loops of $\mathbb{R}^3$ with disjoint supports, the line integral $\int_{\Gamma_{j,n}}\mathrm{BS}(\gamma_{i,n})$ is precisely the Gauss formula for the linking number $\mathrm{Lk}(\gamma_{i,n},\Gamma_{j,n})$ (see [16, pp. 132–134]): that is, we have:

$$\int_{\Gamma_{j,n}}\mathrm{BS}(\gamma_{i,n}) = \mathrm{Lk}(\gamma_{i,n},\Gamma_{j,n}).$$

Bearing in mind (*iii*), (*vi*) and the condition $P \cap O = \emptyset$, we obtain that

$$\begin{aligned}\sum\nolimits_{i=1}^{r}\sum\nolimits_{j=1}^{h}\int_{\Gamma_{j,n}}\mathrm{BS}(\gamma_{i,n}) &= \sum\nolimits_{i=1}^{r}\sum\nolimits_{j=1}^{h}\mathrm{Lk}(\gamma_{i,n},\Gamma_{j,n}) = \\ &= \mathrm{Lk}(\textstyle\sum_{i=1}^{r}\gamma_{i,n},\sum_{j=1}^{h}\Gamma_{j,n}) = \mathrm{Lk}(z',w').\end{aligned} \tag{32}$$

By (31) and (32), it follows that $\mathrm{Lk}(z',w') = \int_{z'}\mathrm{BS}(w')$, as desired. $\square$

Let P and Q be two disjoint non–empty subsets of $\mathbb{R}^3$. Consider $Z_1(P;\mathbb{Z})$ and $Z_1(Q;\mathbb{Z})$ as abelian subgroup of $Z_1(\mathbb{R}^3;\mathbb{Z})$ in the natural way. It follows that, if $z \in Z_1(P;\mathbb{Z})$ and $w \in Z_1(Q;\mathbb{Z})$, then $(z,w) \in Z_1(\mathbb{R}^3;\mathbb{Z}) \circ Z_1(\mathbb{R}^3;\mathbb{Z})$ and hence the linking number $\ell k(z,w)$ is well–defined. Combining this fact and point (3) of Lemma 4.21, we obtain:

Corollary 4.22 *Let P and Q be two non–empty disjoint subsets P and Q of $\mathbb{R}^3$, let $z,z' \in Z_1(P;\mathbb{Z})$ and let $w,w' \in Z_1(Q;\mathbb{Z})$ such that z is homologous to z' in P over $\mathbb{Z}$ and w is homologous to w' in Q over $\mathbb{Z}$. Then we have:*

$$\ell k(z,w) = \ell k(z',w').$$

The latter result and points (1) and (2) of Lemma 4.21 ensures that the following definition is consistent.

Definition 4.23 *Given two non–empty disjoint subsets P and Q of $\mathbb{R}^3$, we define the bilinear form $\ell k_{P,Q} : H_1(P;\mathbb{Z}) \times H_1(Q;\mathbb{Z}) \longrightarrow \mathbb{Z}$ by setting*

$$\ell k_{P,Q}(\alpha,\beta) := \ell k(z_\alpha, w_\beta)$$

for each $(\alpha,\beta) \in H_1(P;\mathbb{Z})\times H_1(Q;\mathbb{Z})$, where z_α is an element of $Z_1(P;\mathbb{Z})$ with $[z_\alpha]_{(P;\mathbb{Z})} = \alpha$ and w_β is an element of $Z_1(Q;\mathbb{Z})$ with $[w_\beta]_{(Q;\mathbb{Z})} = \beta$.

We are now in position to present the desired version of the Alexander duality theorem.

Theorem 4.24 (Alexander duality theorem) *Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^3$, let $\overline{\Omega}$ be its closure in $\mathbb{R}^3$ and let $\Omega' := \mathbb{R}^3 \setminus \overline{\Omega}$. Then the abelian groups $H_1(\overline{\Omega};\mathbb{Z})$ and $H_1(\Omega';\mathbb{Z})$ are free, isomorphic and their rank is equal to the genus g of $\partial\Omega$. Furthermore, the bilinear form $\ell k_{\overline{\Omega},\Omega'} : H_1(\overline{\Omega};\mathbb{Z}) \times H_1(\Omega';\mathbb{Z}) \longrightarrow \mathbb{Z}$ is non–degenerate in the sense that there exist an ordered base $(\alpha_1,\dots,\alpha_g)$ of $H_1(\overline{\Omega};\mathbb{Z})$ and an ordered base $(\alpha'_1,\dots,\alpha'_g)$ of $H_1(\Omega';\mathbb{Z})$ such that $\ell k_{\overline{\Omega},\Omega'}(\alpha_i,\alpha'_j) = \delta_{ij}$ for each $i,j \in \{1,\dots,g\}$.*

Step II. Completing the proof (sketch). We will prove that the Lipschitz device (Ω, A) of $\mathbb{R}^3$ has a preparation. The existence of a complete preparation follows easily.

Thanks to the version of the Alexander duality theorem stated in Theorem 4.24, there exist ordered bases $\mathcal{A} = (\alpha_1, \ldots, \alpha_g)$ of $H_1(\overline{\Omega}; \mathbb{Z})$ and $\mathcal{A}' = (\alpha'_1, \ldots, \alpha'_g)$ of the free abelian groups $H_1(\overline{\Omega}; \mathbb{Z})$ and $H_1(\Omega'; \mathbb{Z})$, respectively, such that $\ell k_{\overline{\Omega}, \Omega'}(\alpha_i, \alpha'_j) = \delta_{ij}$ for each $i, j \in \{1, \ldots, g\}$. Consider the homomorphisms $\xi : H_1(\Omega; \mathbb{Z}) \longrightarrow H_1(\overline{\Omega}; \mathbb{Z})$ and $\zeta : H_1(\Omega \cup A; \mathbb{Z}) \longrightarrow H_1(\overline{\Omega}; \mathbb{Z})$ induced by the inclusion maps $\Omega \hookrightarrow \overline{\Omega}$ and $\Omega \cup A \hookrightarrow \overline{\Omega}$, respectively. Since $\Gamma = \partial\Omega$ is bi–collared in $\mathbb{R}^3$ (see [8, Theorem 3]), ξ and ζ are isomorphisms. Identify $H_1(\Omega; \mathbb{Z})$, $H_1(\Omega \cup A; \mathbb{Z})$ and $H_1(\overline{\Omega}; \mathbb{Z})$ via such isomorphisms.

The abelian group $H_1(A; \mathbb{Z})$ is free and of finite rank. Choose an ordered base $\mathcal{D} = \{\delta_1, \ldots, \delta_h\}$ of $H_1(A; \mathbb{Z})$. Denote by $i_{*,\mathbb{Z}} : H_1(A; \mathbb{Z}) \longrightarrow H_1(\Omega \cup A; \mathbb{Z})$ the homomorphism induced by the inclusion map $i : A \hookrightarrow \Omega \cup A$ and indicate by $M_{\mathcal{A},\mathcal{D}}(i_{*,\mathbb{Z}}) = (m_{ij})_{i,j} \in \mathbb{Z}^{g \times h}$ the matrix associated with the homomorphism $i_{*,\mathbb{Z}}$ with respect to the bases $\mathcal{D}$ of $H_1(A; \mathbb{Z})$ and $\mathcal{A}$ of $H_1(\Omega \cup A; \mathbb{Z})$. Evidently, it holds:

$$m_{ij} = \ell k(\delta_j, \alpha'_i)$$

for each $i \in \{1, \ldots, g\}$ and for each $j \in \{1, \ldots, h\}$. Let $\kappa := \sigma\beta_1(\Omega, A)$ and $p := \rho\beta_1(\Omega, A)$. Recall that $\kappa + p = g$. By the Universal Coefficient Theorem for Homology, we know that $H_1(A; \mathbb{R})$ is isomorphic to $H_1(A; \mathbb{Z}) \otimes \mathbb{R}$ and $H_1(\Omega \cup A; \mathbb{R})$ is isomorphic to $H_1(\Omega \cup A; \mathbb{Z}) \otimes \mathbb{R}$. In particular, thanks to such isomorphisms, we can consider $\mathcal{D}$ as an ordered base of $H_1(A; \mathbb{R})$ and $\mathcal{A}$ as an ordered base of $H_1(\Omega \cup A; \mathbb{R})$. Let $i_* : H_1(A; \mathbb{R}) \longrightarrow H_1(\Omega \cup A; \mathbb{R})$ be the homomorphism induced by $i : A \hookrightarrow \Omega \cup A$. It follows that $M_{\mathcal{A},\mathcal{D}}(i_{*,\mathbb{Z}})$ is equal to the matrix associated with the homomorphism i_* with respect to the bases $\mathcal{D}$ of $H_1(A; \mathbb{R})$ and $\mathcal{A}$ of $H_1(\Omega \cup A; \mathbb{R})$ and hence the rank of $M_{\mathcal{A},\mathcal{D}}(i_{*,\mathbb{Z}})$ is κ. Repeating similar considerations with $\mathbb{Q}$ in place of $\mathbb{R}$, one obtains that, denoting by $i_{*,\mathbb{Q}} : H_1(A; \mathbb{Q}) \longrightarrow H_1(\Omega \cup A; \mathbb{Q})$ the homomorphism induced by $i : A \hookrightarrow \Omega \cup A$, the matrix associated with the homomorphism $i_{*,\mathbb{Q}}$ with respect to the bases $\mathcal{D}$ of $H_1(A; \mathbb{Q})$ and $\mathcal{A}$ of $H_1(\Omega \cup A; \mathbb{Q})$ is equal to $M_{\mathcal{A},\mathcal{D}}(i_{*,\mathbb{Z}})$. Reordering the components of the ordered bases $\mathcal{A}$ and $\mathcal{D}$ if needed, we may suppose that

$$M_{\mathcal{A},\mathcal{D}}(i_{*,\mathbb{Z}}) = \left(\begin{array}{c|c} C & D \\ \hline E & F \end{array}\right)$$

for some $C \in \mathbb{Z}^{\kappa \times \kappa}$ with $\det(C) \neq 0$, $D \in \mathbb{Z}^{\kappa \times (h-\kappa)}$, $E \in \mathbb{Z}^{p \times \kappa}$ and $F \in \mathbb{Z}^{p \times (h-\kappa)}$. Let $\upsilon_i := i_{*,\mathbb{Z}}(\delta_i) \in H_1(\Omega \cup A; \mathbb{Z})$ for each $i \in \{1, \ldots, \kappa\}$ and let $\theta_j := \alpha_{\kappa+j}$ for each $j \in \{1, \ldots, p\}$. Viewing the υ_i's and the θ_j's as elements of $H_1(\Omega, A; \mathbb{Q})$ the g–uple $\mathcal{B} := (\upsilon_1, \ldots, \upsilon_\kappa, \theta_1, \ldots, \theta_p)$ is an ordered base of $H_1(\Omega \cup A; \mathbb{Q})$. Moreover, the elements $\upsilon_1, \ldots, \upsilon_\kappa$ generates $\mathrm{Image}(i_{*,\mathbb{Q}})$ in $H_1(\Omega \cup A; \mathbb{Q})$ and, viewed as elements of $H_1(\Omega \cup A; \mathbb{R})$, generates $\mathrm{Image}(i_*)$. Define the matrix $Q = (q_{\ell j})_{\ell j}$ in $\mathbb{Z}^{\kappa \times p}$ by setting

$$Q = (q_{\ell j})_{\ell j} := -\det(C)(C^t)^{-1}E^t$$

and, for each $j \in \{1, \ldots, p\}$, the element θ'_j of $H_1(\Omega'; \mathbb{Z})$ by setting

$$\theta'_j := \det(C)\alpha'_{\kappa+j} + \textstyle\sum_{\ell=1}^{\kappa} q_{\ell j}\alpha'_\ell.$$

It is immediate to verify that

$$\ell k(\theta_i, \theta'_j) = \det(C)\delta_{ij} \ \text{ for each } i, j \in \{1, \ldots, p\}, \tag{33}$$

$$\ell k(\upsilon_i, \theta'_j) = 0 \ \text{ for each } (i, j) \in \{1, \ldots, \kappa\} \times \{1, \ldots, p\}. \tag{34}$$

By Lemma 2.21, there exist 1–cycles $z_1, \ldots, z_p$ in $Z_1^{(lip)}(\Omega; \mathbb{Z})$ and 1–cycles $z'_1, \ldots, z'_p$ in $Z_1^{(lip)}(\Omega'; \mathbb{Z})$ such that $\theta_i = [z_i]_{(\Omega \cup A; \mathbb{Z})}$ and $\theta'_j = [z'_j]_{(\Omega'; \mathbb{Z})}$ for each $i, j \in \{1, \ldots, p\}$.

For each $j \in \{1, \dots, p\}$, we define the harmonic vector field $B_j : \mathbb{R}^3 \setminus |z'_j| \longrightarrow \mathbb{R}^3$ by setting

$$B_j := \mathrm{BS}(z'_j).$$

Equation (33) is equivalent to the following one:

$$\int_{z_i} B_j = \det(C)\delta_{ij} \ \text{ for each } i, j \in \{1, \dots, p\}. \tag{35}$$

Thanks to (35), Corollary 4.8 and Lemma 4.12, it remains to show that each B_j is a fundamental vector field of (Ω, A). Fix $j \in \{1, \dots, p\}$. As we have just recalled, Γ is bi–collared in $\mathbb{R}^3$; that is, there exists an open neighborhood O of Γ in $\mathbb{R}^3$ and a homeomorphism $\mu : \Gamma \times (-1, 1) \longrightarrow O$ such that $\mu(x, 0) = x$ for each $x \in \Gamma$ and $\mu^{-1}(\Omega \cap O) = \Gamma \times (-1, 0)$. We may also suppose that O does not intersect $|z'_j|$. Denote by $\pi : \Gamma \times (-1, 1) \longrightarrow \Gamma$ the natural projection, choose an open neighborhood W of $\overline{A}$ in Γ and define $U := \mu(W \times (-1, 1))$. Since the boundary of A in Γ is Lipschitz, it is possible to choose W and hence U in such a way that

$$\text{the inclusion map } A \hookrightarrow U \text{ is a homotopy equivalence.} \tag{36}$$

Thanks to de Rham's Theorem and to the existence of smooth partitions of unity subordinate to the open cover $\{U, \mathbb{R}^3 \setminus \overline{A}\}$ of $\mathbb{R}^3$, it suffices to prove that, for each Lipschitz loop γ of U, the line integral $\int_\gamma B_j$ is null. Let γ be such a loop. By (36), there exist a loop γ' of A and a loop γ'' of $\Omega \cap U$ such that γ is homologous to γ' and γ' is homologous to γ'' in U over $\mathbb{Z}$. Let $\eta'' := [\gamma'']_{(\Omega \cup A; \mathbb{Q})}$. It follows that

$$\int_\gamma B_j = \ell k(\gamma'', z'_j), \tag{37}$$

$$\eta'' \in \mathrm{Image}(i_{*,\mathbb{Q}}).$$

Thanks to the latter fact, there exist integers $m, a_1, \dots, a_\kappa$ such that $m \neq 0$ and

$$m\eta'' = \textstyle\sum_{i=1}^{\kappa} a_i \upsilon_i \ \text{ in } H_1(\Omega \cup A; \mathbb{Z}). \tag{38}$$

Bearing in mind (37), (38) and (34), we obtain:

$$\begin{aligned} \int_\gamma B_j &= \ell k(\gamma'', z'_j) = \ell k(\eta'', \theta'_j) = \frac{1}{m} \ell k(m\eta'', \theta'_j) = \\ &= \frac{1}{m} \sum\nolimits_{i=1}^{\kappa} a_i \ell k(\upsilon_i, \theta'_j) = 0. \end{aligned}$$

This completes the proof.

4.3 Topological nature and explicit bases of $\mathcal{H}(\Omega, A; \omega)$

In this crucial subsection, we shall make the followig assumptions: (Ω, A) *is a bounded Lipschitz device of* $\mathbb{R}^3$, ω *is a material matrix of* $\mathbb{R}^3$, $\{A_0, A_1, \dots, A_a\}$ *is the family of the connected components of* A *and* $p := \rho\beta_1(\Omega, A)$. Moreover, we shall use the terminology introduced in Subsections 1.2, 2.4 and 4.1.

The results of this and of the next subsection are (more or less) direct consequences of results presented above. The detailed proofs will appear in the final version of the paper.

The following two theorems give an exhaustive description of $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ and of $\mathcal{H}(\Omega, A; \omega)$.

Theorem 4.25 *The following statements hold:*

(1) *For each* $i \in \{0, 1, \dots, a\}$, *we define the vector field* G_i *in* $L^2(\Omega)^3$ *by setting*

$$G_i := \nabla g_i,$$

where g_i *is a function in* $H^1(\Omega)$ *satisfying the following system*

$$\begin{cases} \operatorname{div}(\omega \nabla g_i) = 0 \\ g_i = \begin{cases} 1 & \text{on } A_i \\ 0 & \text{on } A \setminus A_i \end{cases} \\ (\omega \nabla g_i) \bullet n|_{\complement A} = 0. \end{cases}$$

Then each G_i *is a uniquely determined gradient* ω*–harmonic vector field of* Ω *normal to* A *and* ω*–tangent to* $\complement A$, *the sum* $\sum_{i=0}^{a} G_i$ *is equal to the null vector field in* $L^2(\Omega)^3$ *and* $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ *is the vector subspace of* $\mathcal{H}(\Omega, A; \omega)$ *generated by* $G_0, G_1, \dots, G_a$. *In particular, we have that*

$$\dim \mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega) = a$$

and, if a *is positive, then* $\{G_1, \dots, G_a\}$ *is a base of* $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$.

(2) *Let* $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_a)$ *be a system of jump paths of* (Ω, A) *and let* $N_{\boldsymbol{\lambda}} = (n_{ij})_{i,j} \in \{-1, 0, 1\}^{a \times a}$ *be the inverse matrix of* $M_{\boldsymbol{\lambda}}$. *For each* $i \in \{1, \dots, a\}$, *we define the vector field* $G_i^{(\boldsymbol{\lambda})}$ *in* $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ *by setting*

$$G_i^{(\boldsymbol{\lambda})} := \textstyle\sum_{k=1}^{a} n_{ik} G_k.$$

Then the a*–uple* $(G_1^{(\boldsymbol{\lambda})}, \dots, G_a^{(\boldsymbol{\lambda})})$ *is the unique ordered base of* $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ *such that*

$$\int_{\lambda_j} G_i^{(\boldsymbol{\lambda})} = \delta_{ij}$$

for each $i, j \in \{1, \dots, a\}$. *We call such a base* ordered base of $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ induced by $\boldsymbol{\lambda}$.

(3) *Given* $V \in \mathcal{H}(\Omega, A; \omega)$, V *is a gradient if and only if there exists a system* $(z_1, \dots, z_p)$ *of primary cycles of* (Ω, A) *such that* $\int_{z_i} V = 0$ *for each* $i \in \{1, \dots, p\}$.

The following theorem is the deepest result of this paper.

Theorem 4.26 *The vector space* $\mathcal{H}(\Omega, A; \omega)$ *is finite dimensional and it hold:*

$$\rho \dim \mathcal{H}(\Omega, A; \omega) = p, \tag{39}$$

$$\dim \mathcal{H}(\Omega, A; \omega) = p + a = \dim H_1(\Omega \cup A, A; \mathbb{R}). \tag{40}$$

More precisely, the following statements are verified:

(1) $p = 0$ *if and only if* $\mathcal{H}(\Omega, A; \omega) = \mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$.

(2) *Suppose* p *positive. Let* $[\![\mathbf{B}, \mathbf{b}]\!]$ *be a preparation of* (Ω, A), *where* $\mathbf{B} = (B_1, \dots, B_p)$ *and* $\mathbf{b} = (b_1, \dots, b_p)$. *For each* $i \in \{1, \dots, p\}$, *we define the vector field* V_i *in* $L^2(\Omega)^3$ *by setting*

$$V_i := B_i - \nabla b_i - \nabla v_i,$$

where v_i *is a function in* $H^1_{0,A}(\Omega)$ *satisfying the following variational problem*

$$\begin{cases} \displaystyle\int_\Omega (\omega \nabla v_i) \bullet \nabla w \, dx = \int_\Omega (\omega (B_i - \nabla b_i)) \bullet \nabla w \, dx \\ \text{for each } w \in H^1_{0,A}(\Omega). \end{cases} \tag{41}$$

Then each V_i *is a uniquely determined primary* ω*–harmonic vector field of* Ω *normal to* A *and* ω*–orthogonal to* $\complement A$ *and* $(V_1, \dots, V_p)$ *is a primary system of* $\mathcal{H}(\Omega, A; \omega)$. *We call* $(V_1, \dots, V_p)$ primary system of $\mathcal{H}(\Omega, A; \omega)$ induced by $[\![\mathbf{B}, \mathbf{b}]\!]$.

(3) *Let* $[\![\mathbf{B}, \mathbf{b} \,|\, \mathbf{z}, \boldsymbol{\lambda}]\!]$ *be a complete presentation of* (Ω, A) *with* $\mathbf{z} = (z_1, \ldots, z_p)$ *and* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_a)$*, let* $(V_1, \ldots, V_p)$ *be the primary system of* $\mathcal{H}(\Omega, A; \omega)$ *induced by* $[\![\mathbf{B}, \mathbf{b}]\!]$ *and let* $(G_1, \ldots, G_a)$ *be the ordered base of* $\mathcal{H}_{\mathrm{grad}}(\Omega, A; \omega)$ *induced by* $\boldsymbol{\lambda}$*. Then the* $(p + a)$*–uple*

$$\big(V_1, \ldots, V_p, G_1, \ldots, G_a\big)$$

is an ordered base of $\mathcal{H}(\Omega, A; \omega)$*, called* ordered base of $\mathcal{H}(\Omega, A; \omega)$ induced by $[\![\mathbf{B}, \mathbf{b} \,|\, \mathbf{z}, \boldsymbol{\lambda}]\!]$*, having the following properties:*

$$\int_{z_j} V_i = \delta_{ij}, \quad \int_{\lambda_k} V_i = 0, \quad \int_{z_j} G_h = 0 \quad \textit{and} \quad \int_{\lambda_k} G_h = \delta_{hk}$$

for each $i, j \in \{1, \ldots, p\}$ *and for each* $h, k \in \{1, \ldots, a\}$*.*

As a consequence, we have:

Corollary 4.27 *The bilinear pairing* $\textstyle\int_{(\Omega, A; \omega)} : \mathcal{H}(\Omega, A; \omega) \times H_1(\Omega \cup A, A; \mathbb{R}) \longrightarrow \mathbb{R}$ *is non–degenerate. More precisely, if* $[\![\mathbf{B}, \mathbf{b} \,|\, \mathbf{z}, \boldsymbol{\lambda}]\!]$ *is a complete preparation of* (Ω, A)*,* $(\Pi_1, \ldots, \Pi_{p+a})$ *is an ordered base of* $\mathcal{H}(\Omega, A; \omega)$ *induced by* $[\![\mathbf{B}, \mathbf{b} \,|\, \mathbf{z}, \boldsymbol{\lambda}]\!]$ *and* $(\alpha_1, \ldots, \alpha_{p+a})$ *is the ordered base of* $H_1(\Omega \cup A, A; \mathbb{R})$ *induced by* $(\mathbf{z}, \boldsymbol{\lambda})$*, then we have that*

$$\textstyle\int_{(\Omega, A; \omega)} \big(\Pi_i, \alpha_j\big) = \delta_{ij}$$

for each $i, j \in \{1, \ldots, p + a\}$*. In particular, as we just know, it holds*

$$\mathcal{H}(\Omega, A; \omega) \simeq H_1(\Omega \cup A, A; \mathbb{R}).$$

Combining Theorem 3.8 with the preceding corollary, we obtain a version of de Rham theorem. To the best of our knowledge, such a version is new if A is different from a union of connected components of Γ.

Corollary 4.28 *The bilinear pairing* $\textstyle\int^{\mathrm{DR}}_{(\Omega, A)} : \mathbb{H}^1_{\mathrm{DR}}(\Omega, A) \times H_1(\Omega \cup A, A; \mathbb{R}) \longrightarrow \mathbb{R}$ *is non–degenerate. In particular, we have that*

$$\mathbb{H}^1_{\mathrm{DR}}(\Omega, A) \simeq H_1(\Omega \cup A, A; \mathbb{R}).$$

In 2006, G. Auchmuty and J. C. Alexander made three conjectures concerning the Hodge decomposition of $L^2(\Omega)^3$ in the case in which Ω is a bounded domain of $\mathbb{R}^3$ whose boundary is regular of class $\mathscr{C}^2$ (see the end of Section 10 and the end of Section 12 of [4]). We reformulate those conjectures in the more general setting of bounded Lipschitz domains of $\mathbb{R}^3$ using our terminology.

Generalized Auchmuty-Alexander conjectures 4.29 *Let* (Ω, A) *be a bounded Lipschitz device of* $\mathbb{R}^3$ *and let* ω *be a material matrix of* Ω*. The following assertions hold:*

(C1) $\mathcal{H}(\Omega, A; \omega)$ *has finite dimension.*

(C2) *The dimension of* $\mathcal{H}(\Omega, A; \omega)$ *does not depend on* ω*.*

(C3) *If* $a + 1$ *is the number of connected components of* A *and* $\varrho^{(2)}_* : H_2(\overline{\Omega}, \complement A; \mathbb{R}) \longrightarrow H_2(\overline{\Omega}, \Gamma; \mathbb{R})$ *is the linear maps induced by the identity map of* $\overline{\Omega}$*, then we have that*

$$\dim \mathcal{H}(\Omega, A; \omega) = a + \dim \mathrm{Image}(\rho^{(2)}_*).$$

The affermative solution of conjectures (C1) and (C2) is given in [10] under the special assumption that Ω is a Helmholtz domain. A direct application of Theorem 4.26 and of Lemma 4.2 furnishes immediately the complete solution.

Corollary 4.30 *The three preceding conjectures are true.*

5 The summand subspaces of gradients and of curls

Fix a bounded Lipschitz device (Ω, A) *of* $\mathbb{R}^3$. In this section, our task is to characterize the vector spaces $\mathrm{grad}(H^1_{0,A}(\Omega))$ and $\mathrm{curl}(H_{0,A}(\mathrm{curl}, \Omega))$ in terms of $\mathcal{H}(\Omega, A, \omega)$ and of $\mathcal{H}(\Omega, \complement A, \omega)$. The results presented below have a more or less classic statement. However, to the best of our knowledge, except for Corollary 5.4, their generality is completely new. We shall use the terminology introduced in Subsections 1.2, 2.4 and 4.1.

We begin with $\mathrm{grad}(H^1_{0,A}(\Omega))$.

Theorem 5.1 *Given* $V \in H_{0,A}(\mathrm{curl}\, 0, \Omega)$*, the following assertions are equivalent:*

(1) $V \in \mathrm{grad}(H^1_{0,A}(\Omega))$.

(2) $\int_\alpha V = 0$ *for each* $\alpha \in H_1(\Omega \cup A, A; \mathbb{R})$.

(3) *There exists a base* $\{\alpha_1, \ldots, \alpha_k\}$ *of* $H_1(\Omega \cup A, A; \mathbb{R})$*, for example the one induced by a system of primary cycles of* (Ω, A) *and by a system of jump paths of* (Ω, A)*, such that* $\int_{\alpha_i} V = 0$ *for each* $i \in \{1, \ldots, k\}$.

(4) $(V, Z)_\omega = 0$ *for each material matrix* ω *of* Ω *and for each* $Z \in \mathcal{H}(\Omega, A; \omega)$.

(5) *There exist a material matrix* ω *of* Ω *and a base* $\{\Pi_1, \ldots, \Pi_k\}$ *of* $\mathcal{H}(\Omega, A; \omega)$*, for example the one induced by a complete preparation of* (Ω, A)*, such that* $(V, \Pi_i)_\omega = 0$ *for each* $i \in \{1, \ldots, k\}$.

The case $A = \emptyset$ is quite interesting.

Corollary 5.2 *Given* $V \in H(\mathrm{curl}\, 0, \Omega)$*, the following assertions are equivalent:*

(1) V *is a gradient.*

(2) $\int_\alpha V = 0$ *for each* $\alpha \in H_1(\Omega; \mathbb{R})$.

(3) *There exists a base* $\{\alpha_1, \ldots, \alpha_g\}$ *of* $H_1(\Omega; \mathbb{R})$ *such that* $\int_{\alpha_i} V = 0$ *for each* $i \in \{1, \ldots, g\}$.

(4) $(V, Z)_\omega = 0$ *for each material matrix* ω *of* Ω *and for each* $Z \in \mathcal{H}(\Omega; \omega)$.

(5) *There exist a material matrix* ω *of* Ω *and a base* $\{V_1, \ldots, V_g\}$ *of* $\mathcal{H}(\Omega; \omega)$ *such that* $(V, V_i)_\omega = 0$ *for each* $i \in \{1, \ldots, g\}$.

Let us see some characterizations of $\mathrm{curl}(H_{0,A}(\mathrm{curl}, \Omega))$.

Theorem 5.3 *Let* ω *be a material matrix of* Ω. *Given* $V \in H_{0,A}(\mathrm{div}\, 0, \Omega; \omega)$*, the following assertions are equivalent:*

(1) $\omega V \in \mathrm{curl}(H_{0,A}(\mathrm{curl}, \Omega))$*; that is, there exists* $Z \in H_{0,A}(\mathrm{curl}, \Omega)$ *such that* $\omega V = \mathrm{curl}(Z)$.

(2) $(V, Z)_\omega = 0$ *for each* $Z \in \mathcal{H}(\Omega, \complement A; \omega)$.

(3) *There exists a base* $\{W_1, \ldots, W_k\}$ *of* $\mathcal{H}(\Omega, \complement A; \omega)$*, for example the one induced by a complete preparation of* $(\Omega, \complement A)$*, such that* $(V, W_i)_\omega = 0$ *for each* $i \in \{1, \ldots, k\}$.

(4) *Let* $A^*_0, A^*_1, \ldots, A^*_c$ *be the connected components of* $\complement A$. *Then the fluxes of* ωV *through* A^*_i *is null for each* $i \in \{1, \ldots, c\}$ *and there exists a primary system* $(W_1, \ldots, W_q)$ *of* $\mathcal{H}(\Omega, \complement A; \omega)$ *such that* $(V, W_j)_\omega = 0$ *for each* $j \in \{1, \ldots, q\}$.

In the case in which $\omega = \mathbf{1}_{3\times 3}$ and $A = \emptyset$, we rediscover a well–known result.

Corollary 5.4 *Given $V \in H(\operatorname{div} 0, \Omega)$, the following assertions are equivalent:*

(1) *V has a vector potential; that is, there exists $Z \in H(\operatorname{curl}, \Omega)$ such that $V = \operatorname{curl}(Z)$.*

(2) *$(V, G) = 0$ for each $G \in \mathcal{H}(\Omega, \Gamma) = \mathcal{H}_{\mathrm{grad}}(\Omega, \Gamma)$.*

(3) *There exists a base $\{G_1, \ldots, G_l\}$ of $\mathcal{H}(\Omega, \Gamma) = \mathcal{H}_{\mathrm{grad}}(\Omega, \Gamma)$, for example the one induced by a system of jump paths of (Ω, Γ), such that $(V, G_i) = 0$ for each $i \in \{1, \ldots, l\}$.*

(4) *The fluxes of V through each connected component of Γ is null.*

As an immediate consequence of Theorems 5.1 and 5.3, we obtain:

Corollary 5.5 *Let (D, E) be a curl–gradient boundary pair of Ω, let $D_1, \ldots, D_d$ be the connected components of D and let ω be a material matrix Ω. Given $V \in H_{0,E}(\operatorname{curl} 0, \Omega) \cap H_{0,\complement D}(\operatorname{div} 0, \Omega; \omega)$, the following statements are equivalent:*

(1) *V is a ω–curly gradient vector field of Ω with respect to (D, E).*

(2) *$(V, Z)_\omega = 0$ for each $Z \in \mathcal{H}(\Omega, D; \omega) \oplus \mathcal{H}(\Omega, E; \omega)$.*

(3) *$\int_\alpha V = 0$ for each $\alpha \in H_1(\Omega \cup E, E; \mathbb{R})$, the fluxes of ωV through D_i is null for each $i \in \{1, \ldots, d\}$ and there exists a primary system $(W_1, \ldots, W_\ell)$ of $\mathcal{H}(\Omega, D; \omega)$ such that $(V, W_j)_\omega = 0$ for each $j \in \{1, \ldots, \ell\}$.*

If $\omega = \mathbf{1}_{3\times 3}$ and $(D, E) = (\Gamma, \emptyset)$, then this corollary can be rephrased as follows.

Corollary 5.6 *Given $V \in H(\operatorname{curl} 0, \Omega) \cap H(\operatorname{div} 0, \Omega)$, the following statements are equivalent:*

(1) *V is a curly gradient vector field of Ω.*

(2) *$(V, Z) = 0$ for each $Z \in \mathcal{H}(\Omega, \Gamma) \oplus \mathcal{H}(\Omega)$.*

(3) *$\int_\alpha V = 0$ for each $\alpha \in H_1(\Omega; \mathbb{R})$ and the fluxes of V through each connected component of Γ is null.*

Acknowledgements. I wish to express my gratitude to Alberto Valli, for bringing these fascinating themes to my attention and for several useful discussions. A special thank goes to Ana Alonso Rodríguez for helpful conversations.

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